AN INTRODUCTION TO CLUSTER SET THEORY

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TABLE OF CONTENTS

Chapter | Page

LIST OF FIGURES

Figure		Page

CHAPTER I

INTRODUCTION

The theory of cluster sets has grown up around complex valued functions of a complex variable and is concerned primarily with the behavior of a function near the boundary of the domain on which it is defined. It is well known that a function which is analytic at a point can be expanded in a power series about that point. If the power series has a finite radius of convergence, then the concern arises as to the behavior of the function near the singular points on the circle of convergence. The classification of this behavior provides one example of a type of problem which the student of cluster set theory might investigate. It is difficult to give an all inclusive definition of the area of mathematics which is referred to as "Cluster Set Theory". In particular, one concern is the set of limit points of the function at a boundary point of its domain. This set is called the cluster set of the function at the point in question. A value w belongs to the cluster set of the function at the boundary point if there exists a sequence of points in the domain which converge to the boundary point with the function values of the sequence converging to w.

The study of the cluster set of a function at each point of the boundary gives some information about the boundary behavior of the function but only in a local sense. If one wants to investigate the boundary behavior in a global sense, it is of interest to consider what is called the global cluster set of the function. A value w belongs to the global cluster set if there exists a sequence which converges to the boundary with the function values converging to w. It is not required here that the sequence converge to an individual point of the boundary.

The two types of cluster sets described above do not describe completely the content of the theory of cluster sets but rather describe some notation which has been used in the development of the theory. Although some of the topics now included under the heading of cluster set theory were studied before 1900, it was not until the development of measure theory that significant advancement took place. During this century the study of cluster sets has gone through periods of great activity as well as periods of inactivity. In recent years there has been renewed interest in the field. The extent of this interest is evidenced by the great number of research articles published under the heading "Cluster Set Theory". It was not until the nineteen sixties that an organized account of the theory appeared. Two such works are: Cluster Sets by K. Noshiro and Theory of Cluster Sets by E. F. Collingwood and A. J. Lohwater. The Noshiro book is written for the accomplished complex analyst. The work of Collingwood and Lohwater, although requiring less background in complex analysis, is essentially a collection of research papers and does not provide a unified and detailed introductory work in the field.

It is therefore intended that this paper provide such an introduction to the theory of cluster sets. It is hoped that the reading of this paper will supply at least a partial background for continued study in the field. It is also hoped, however, that this paper will provide a source with which one can, in a relatively short time, gain insight into the field of cluster set theory. Because of the introductory nature of this paper, many results included here were originally established during the early and middle years in the historical development of the theory. For example, there is one class of functions, to which Chapter V is devoted, that was originally studied by Siedel in the 1930's. This study, however, illustrates well the beauty of the theory, and also provides a stepping stone to the current study of less restrictive classes of function.

It has been mentioned that the development of measure theory provided an important tool for the study of cluster sets. It is not intended that this paper be directed solely to the reader who has a thorough background in measure theory. In fact, much of the paper can be read with understanding by someone with an elementary knowledge of complex variables and a knowledge of the usual topics of advanced calculus. When theorems relating to measure theory are needed in this paper, they are stated and a reference given. For the most part, however, they are of such a nature that they closely parallel results from advanced calculus and therefore are not difficult to accept without proof.

A few general comments follow regarding notation and terminology. The domains considered are restricted to those which are bounded by a Jordan curve. By the Riemann mapping theorem, every simply connected domain Ω is conformally equivalent to the open unit disk.[1] The role

[1] W. Rudin, Real and Complex Analysis (New York, 1966), p. 274.

which this fact plays in this study is discussed below. First notation relating to the unit disk is introduced. Here U denotes the open unit disk and C the unit circle.

Definition 1.1.

$$U = \{z: |z| < 1\} \qquad D_r = \{z: |z| < r\}$$

$$C = \{z: |z| = 1\} \qquad C_r = \{z: |z| = r\}$$

For each simply connected domain Ω, there exists a one-to-one analytic map h from U onto Ω. Topologically speaking, h is a homeomorphism from U onto Ω. If, in addition, Ω is a bounded domain whose boundary is a Jordan curve, then h can be extended to a homeomorphism of U onto Ω.[2] Although h is analytic in U this extension need not be analytic on C the boundary of U. To each function f defined on Ω, there corresponds the function $f_1 = f \circ h$. Then f and f_1 have the same range of values. If $\{z_n\}$ is a sequence in Ω converging to a boundary point z of Ω, then $h^{-1}(z_n)$ is a sequence of points converging to the boundary point $h^{-1}(z)$ of U. In addition, the function values of f on the sequence $\{z_n\}$ are identical to the function values of f_1 on $h^{-1}(z_n)$. So the behavior of f at z can be analyzed by studying the behavior of f_1 at $h^{-1}(z)$. In this paper, then, the study of cluster sets will be restricted to functions defined on the open unit disk with the realization that in effect the study includes functions defined on any Jordan domain.

Much of the theory developed will be for functions which are bounded and analytic on U. The symbol H^∞ will be used to denote this

[2]Rudin, p. 281.

class of functions.

Definition 1.2. A function f belongs to H^∞ if and only if f is analytic in U and also bounded there.

It will become necessary to state that a function has a particular behavior at all points of the unit circle except possibly at a subset of C which is in a special sense small. The sense of smallness referred to here is termed "linear measure zero". Intuitively, a subset E of C has linear measure zero if it can be covered with a sequence of arcs the sum of whose lengths is less than ε where ε is an arbitrary preassigned positive number. As an example, consider any countable subset

$$E = \{e^{i\theta_n}\} \text{ of } C.$$

Each $e^{i\theta_n}$ is contained in an open arc I_n of length $\frac{\varepsilon}{2^n}$. Clearly, the sequence of arcs $\{I_n\}$ covers E and the sum of the lengths of the arcs is less than ε. Therefore, any countable subset of C has linear measure zero. It should be noted that E could have linear measure zero and yet be an uncountable set. The following definition gives a precise statement of this concept.

Definition 1.3. A subset E of C is said to have linear measure zero if for each ε there exists a sequence of arcs $\{I_k\}$ of C such that E is contained in the union of the arcs and the sum of the lengths of the arcs is less than ε.

In this paper the term "measure" will be understood to mean linear measure. A particular property is said to hold almost everywhere on C, if the exceptional set has measure zero.

CHAPTER II

BASIC LANGUAGE OF CLUSTER SET THEORY

Every area of mathematics has some language all its own and cluster set theory is no different. Since this investigation is limited to functions whose domain is the open unit disk U, the definitions given here will be so restricted although it should be kept in mind that they could be given with respect to a more general domain.

Cluster set theory is concerned primarily with behavior of a function near its boundary points, so here the investigation will be made of the behavior of a function near points of the unit circle. The major concern will be the limit points (cluster points) of the values a function assumes on a sequence which converges to a point of the unit circle. The set of all such limit points, resulting from all possible sequences which converge to the point of the unit circle, will be called the cluster set of the function at that boundary point.

Definition 2.1. Let f be defined in U; then $C(f,e^{i\theta})$ is the cluster set of f at $e^{i\theta}$ where

$$C(f,e^{i\theta}) = \{w \mid \text{there exists a sequence } \{z_n\} \subset U \text{ such that } \lim_{n\to\infty} z_n = e^{i\theta} \text{ and } \lim_{n\to\infty} f(z_n) = w\}$$

If the function f can be extended to a function which is continuous at $e^{i\theta}$, then any sequence of points in U which converges to $e^{i\theta}$

will have its function values converging to the unique value $f(e^{i\theta})$ which makes f continuous there. This implies that

$$C(f,e^{i\theta}) = \{f(e^{i\theta})\}.$$

Example 2.2. Let $f(z) = \frac{z+1}{z-1}$ for each $z\varepsilon U$. By the preceding argument

$$C(f,e^{i\theta}) = \left\{\frac{e^{i\theta}+1}{e^{i\theta}-1}\right\}$$

at each point $e^{i\theta}\varepsilon C$ except at $z = 1$. At $z = 1$ f can be extended so it has a pole there which implies $C(f,1) = \{\infty\}$.

$C(f,e^{i\theta})$ can be expressed in terms of neighborhoods but involves only that portion of the neighborhood which lies inside U. These neighborhoods might be referred to as partial neighborhoods and are defined as follows.

Definition 2.3.

$$N^*_\varepsilon(e^{i\theta}) = \{z| \ |z-e^{i\theta}|< \varepsilon\}\cap U \quad \text{(see Figure 1)}$$

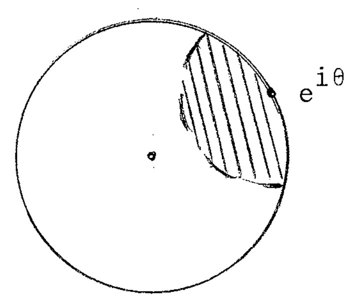

Figure 1

In terms of partial neighborhoods, a point w belongs to $C(f,e^{i\theta})$ if it is a limit point of the image of each partial neighborhood of $e^{i\theta}$.

Theorem 2.5

$$C(f,e^{i\theta}) = \bigcap_{\varepsilon>0} \overline{f(N^*_\varepsilon(e^{i\theta}))}$$

Proof: Let $w\varepsilon C(f,e^{i\theta})$ then there exists a sequence $\{z_n\}\subset U$ such that $z_n\to e^{i\theta}$ and $f(z_n)\to w$. Then for each $\varepsilon>0$ there exists N such that $z_n\varepsilon$ $N^*_\varepsilon(e^{i\theta})$ for $n>N$ which implies $f(z_n)\varepsilon$ $f(N^*_\varepsilon(e^{i\theta}))$ for $n>N$. Therefore w ε $\overline{f(N^*_\varepsilon(e^{i\theta}))}$ for each $\varepsilon>0$.

Now if w ε $f(N^*_\varepsilon(e^{i\theta}))$ for each $\varepsilon>0$ this implies for each n there exists $z_n\varepsilon$ $N^*_{1/n}(e^{i\theta})$ such that $|f(z_n)-w|<1/n$. So $z_n\to e^{i\theta}$ and $f(z_n)\to w$ which implies $w\varepsilon C(f,e^{i\theta})$.

Corollary 2.6. $C(f,e^{i\theta})$ is a non-empty closed set.

Proof: $C(f,e^{i\theta})$ is closed since by Theorem 2.5 it is the intersection of closed sets. The cluster set is non-empty since $\{\overline{f(N^*_\varepsilon(e^{i\theta})}\}$ forms a nested set of non-empty closed sets of which it is the intersection.

Some simple examples will be given which will serve as a basis for further discussion of the properties of cluster sets.

Example 2.7. Define

$$f(re^{i\theta}) = \begin{cases} 1 \text{ if } 0<r<1;\ r,\ \theta \text{ rational} \\ \\ 0 \text{ elsewhere in } U \end{cases}$$

It should be noted that on a radius to $e^{i\theta}$, θ irrational, f is constant of value 0; however on a radius to $e^{i\theta}$, θ rational, f assumes the values 0 and 1 according as r is irrational and rational. Because of the denseness of the rationals in the reals there are points within each $N_\varepsilon^*(e^{i\theta})$ where f assumes the values 0 and 1 which implies

$$C(f,e^{i\theta}) = \{0,1\}$$

for each $e^{i\theta}\varepsilon C$.

Example 2.8. Define

$$g(z) = e^{\frac{i}{|1-z|}}$$

and consider C(g,1). Now

$$|g(z)| = e^0 = 1$$

implies g(z) lies on the unit circle for each $z\varepsilon U$, and thus any cluster value, which is the limit point of a sequence of function values, must lie on the unit circle (i.e. $C(g,1)\subset C$). Now consider approach to z = 1 along its radius. In particular consider the interval

$$(1 - \frac{1}{2\pi k}, 1 - \frac{1}{2\pi(k+1)}]$$

which maps under $z' = \frac{1}{|1-z|}$ onto the interval $(2\pi k, 2\pi(k+1)]$ and $g(z) = e^{iz'}$ maps $(2\pi k, 2\pi(k+1)]$ onto the unit circle. For each $\varepsilon>0$ there exists an interval $(1 - \frac{1}{2\pi k}, 1 - \frac{1}{2\pi(k+1)}]$ contained in $N_\varepsilon^*(1)$. This implies $C\subset g(N_\varepsilon^*(1))$ and thus $C\subset C(g,1)$. Therefore $C(g,1) = C$.

Example 2.9. Define

$$h(re^{i\theta}) = \begin{cases} e^{\frac{i}{1-r}} & \text{if } r, \theta \text{ are rational, } 0\leq r<1 \\ 1 & \text{elsewhere in } U \end{cases}$$

Here $h(U) \subset C$ and each $N_\varepsilon^*(1)$ contains an interval $(1 - \frac{1}{2\pi k}, 1 - \frac{1}{2\pi(k+1)}]$ which h maps onto the points of the unit circle which have rational values of θ. Therefore $e^{i\theta} \varepsilon C(h,1)$ if θ is rational; but the cluster set is a closed set, so all of $C \subset C(h,1)$. Now it has been shown $C(h,1) = C$.

Example 2.7 points out that a cluster set need not be connected. It will be shown in the next theorem that continuity of the function is a sufficient condition for the cluster set to be connected. In Example 2.8 the function

$$g(z) = e^{\frac{i}{|1-z|}}$$

is continuous in the unit disk and $C(g,1)$ was seen to be connected. Continuity is not necessary, however, for in Example 2.9 the function is discontinuous at every point of the unit disk yet $C(h,1)$ is connected.

Definition 2.10. A continuum is a closed connected non-empty set.

Theorem 2.11. Let f be a continuous function defined on U then $C(f,e^{i\theta})$ is a continuum.

Proof: Corollary 2.6 established $C(f,e^{i\theta})$ as a non-empty closed set. Since $N_\varepsilon^*(e^{i\theta})$ is connected and f is continuous, $f(N_\varepsilon^*(e^{i\theta}))$ and thus $\overline{f(N_\varepsilon^*(e^{i\theta}))}$ are connected. Therefore $\bigcap \overline{f(N_\varepsilon^*(e^{i\theta}))}$ is connected and the proof is complete.

In Examples 2.8 and 2.9 every cluster value at $z = 1$ was obtainable from approach along the radius to $z = 1$, but in general this is not the case as may be seen in Example 2.7 if θ is irrational. In that example f assumes only the value 0 on the radius while $C(f,e^{i\theta}) = \{0,1\}$.

In this investigation it will be desirable to consider various approaches to the boundary as a help to an analysis of the boundary behavior of the function. There will be times when the approach will be along a continuous path such as a radius, but also times when a discontinuous approach will be considered. Following is the definition of what will be called a partial cluster set, but in reality it is just the cluster set of the function at the point $e^{i\theta}$ where a restriction is placed on the approach.

Definition 2.12. Let f be defined on U and $G \subset U$ where $e^{i\theta} \in \overline{G}$; then the partial cluster set of f at $e^{i\theta}$ with respect to G is given by

$$C_G(f, e^{i\theta}) = \left\{ w \,\middle|\, \begin{array}{l} \text{there exists } \{z_n\} \subset G \text{ such that} \\ \\ \lim_{n\to\infty} z_n = e^{i\theta} \text{ and } \lim_{n\to\infty} f(z_n) = w \end{array} \right\}$$

A case of special importance results when G is the radius. Such a partial cluster set will be called the radial cluster set at $e^{i\theta}$ and have its own special notation.

Definition 2.13. $C_\rho(f, e^{i\theta})$ is the radial cluster set of f at $e^{i\theta}$ and is defined by

$$C_\rho(f, e^{i\theta}) = C_G(f, e^{i\theta})$$

where G is the radius to $e^{i\theta}$.

Theorem 2.14.

$$C_G(f, e^{i\theta}) = \bigcap_{\varepsilon>0} \overline{f(N_\varepsilon^*(e^{i\theta}) \cap G}$$

Proof: Essentially the same as Theorem 2.5.

Corollary 2.15. $C_G(f,e^{i\theta})$ is a closed non-empty set and if G is connected, it is a continuum.

Proof: See Corollary 2.6.

Consider again Example 2.7 and expressed in terms of the new notation to give

$$C(f,e^{\sqrt{2}i}) = \{0, 1\}$$

and

$$C_\rho(f,e^{\sqrt{2}i}) = \{0\}.$$

The following example will further illustrate the definitions which have been introduced and in addition, will be used frequently in subsequent chapters to illustrate the theory. Therefore a thorough understanding of this example should prove helpful.

Example 2.16. Let

$$f(z) = e^{\frac{z+1}{z-1}}$$

and consider f as a composite function $f(z) = e^{g(z)}$ where

$$g(z) = \frac{z+1}{z-1} .$$

The following investigation is made in order to determine the cluster set of f at $z = 1$ as well as selected partial cluster sets at that point.

Since g is analytic in U, so is $f = e^g$. Also

$$|f(z)| = e^{Re(g)}$$

and

$$Re(g(z)) = \frac{|z|^2 - 1}{|z|^2-2Re(z)+1} < 0$$

for each $z \varepsilon U$ which implies

$$|f(z)| < e^0 = 1$$

for $z \varepsilon U$. Thus $f(U) \subset U$.

Consider the circle

$$G_r = \{z \mid |z-r| = 1-r\},\ 0 \le r < 1,$$

which is internally tangent to C at $z = 1$ for $0<r<1$ and note that for each $z \varepsilon U$, $z \neq 0$, there exists $0<r<1$ such that $z \varepsilon G_r$. Now

$$g(z) = \frac{z+1}{z-1} = z'$$

is a linear fraction and maps G_r onto the line $\mathrm{Re}(z') = \frac{r}{r-1}$ and $g(1) = \infty$ (see Figures 2 and 3). Now consider a circle of radius ε, $0<\varepsilon<2$ with center at $z = 1$,

$$H_\varepsilon = \{z \mid |z-1| = \varepsilon\},$$

which g maps onto the circle

$$\{z' \mid |z'-1| = \frac{2}{\varepsilon}\}$$

(see Figures 2 and 3). Since $g(1) = \infty$, g maps the disk whose boundary is the circle H_ε onto the exterior of the circle. The image of $N_\varepsilon^*(1)$ under g is shown as the shaded region in Figure 3. Because of the nature of $g(N_\varepsilon^*(1))$, for each $0<r<1$ there exists a k such that the segment on $\mathrm{Re}(z') = \frac{r}{r-1}$ between $\frac{r}{r-1} + 2k\pi i$ and $\frac{r}{r-1} + 2(k+1)\pi i$ lies in $g(N_\varepsilon^*(1))$. But $f(z) = e^{z'}$ maps this interval onto the circle with center at the origin and radius $\exp \frac{r}{r-1}$. This circle of radius $\exp \frac{r}{r-1}$ is contained in $f(N_\varepsilon^*(1))$ for each $\varepsilon>0$. So each circle with radius $\exp \frac{r}{r-1}$, $0<r<1$, and center at the origin is contained in $C(f,1)$. This shows that

$$(2.16.1) \qquad \bigcup_{0<r<1} \{w : |w| = \exp \frac{r}{r-1}\} = \{w:\ 0<|w|<1\} \subset C(f,1).$$

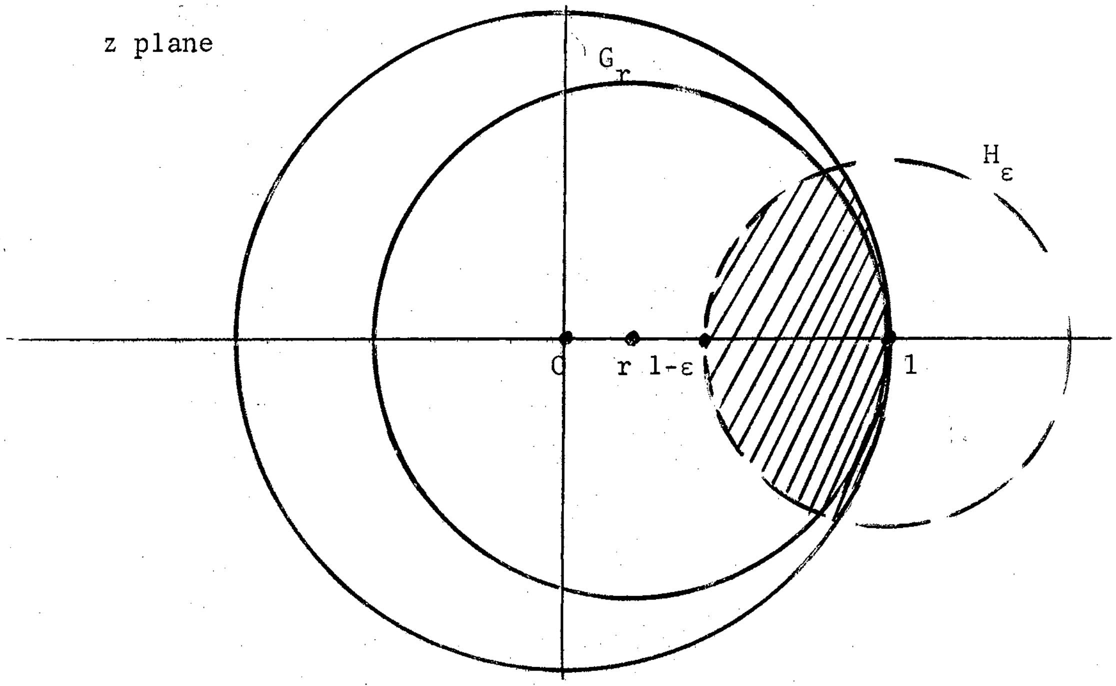

Figure 2: The First Illustration for Example 2.16
[The Shaded Area is $N_\varepsilon^*(1)$]

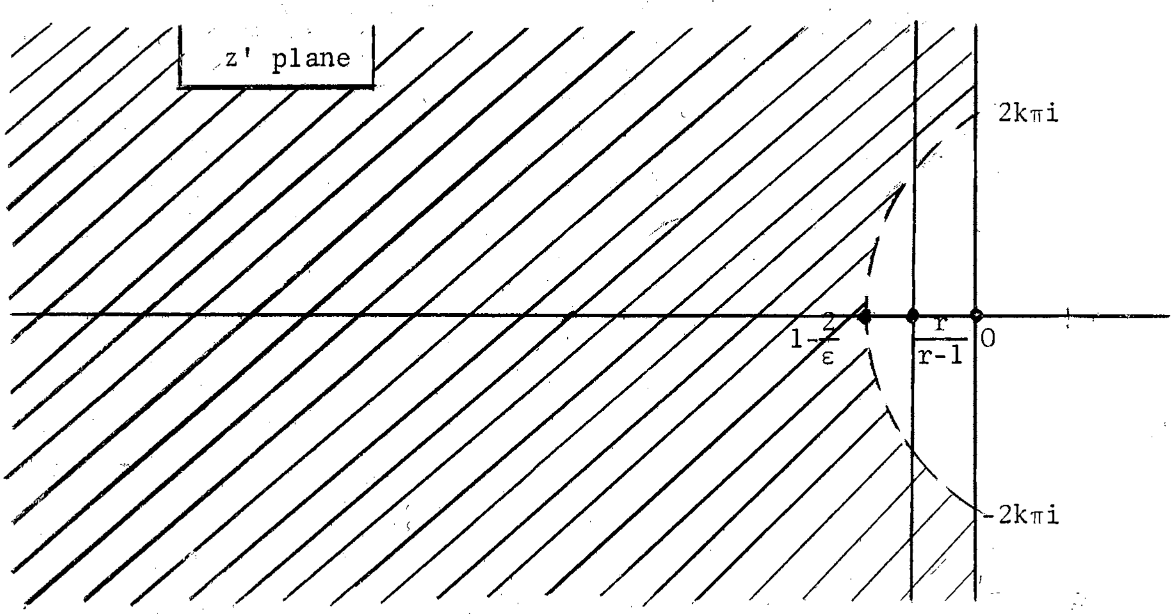

Figure 3. The Second Illustration for Example 2.16
[The Shaded Area is $g(N_\varepsilon^*(1))$]

It has already been shown that $f(U) \subset U$ thus $C(f,1) \subset U$. Corollary 2.6 states that the cluster set $C(f,1)$ is closed. Therefore Statement 2.16.1 implies

$$C(f,1) = \{w: |w| \leq 1\} = \overline{U}.$$

Note that $z \varepsilon G_r$ implies

$$|f(z)| = e^{Re(g(z))} = e^{\frac{r}{r-1}}$$

so

$$C_{G_r}(f,1) = \{w: |w| = \exp^{\frac{r}{r-1}}\}$$

Now consider approach to $z = 1$ along the radius. For $z = r \varepsilon U$,

$$f(r) = e^{\frac{r+1}{r-1}}$$

and

$$\lim_{r \to 1} f(r) = e^{-\infty} = 0$$

therefore

$$C_\rho(f,1) = \{0\}.$$

Consider an angle at $z = 1$ of opening $\pi-\delta$ which is bisected by the radius and whose sides are chords as shown in Figure 4. Let $A_{\pi-\delta}$ represent all those points of the unit disk which lie interior to this angle. Now $g(z) = \frac{z+1}{z-1}$ maps the endpoints B and D of the chords onto the points B' and D' of $Re(z') = 0$, and because $g(1) = -\infty$, it must map the chords onto straight line segments each of which meets $Re(z') = 0$ at an angle of $\delta/2$ (see Figure 5). Now for each $\frac{r_o}{1-r_o}$ there exists an ε sufficiently small such that $1 - 2/\varepsilon < \frac{r_o}{1-r_o}$ which implies (see Figure 5) that

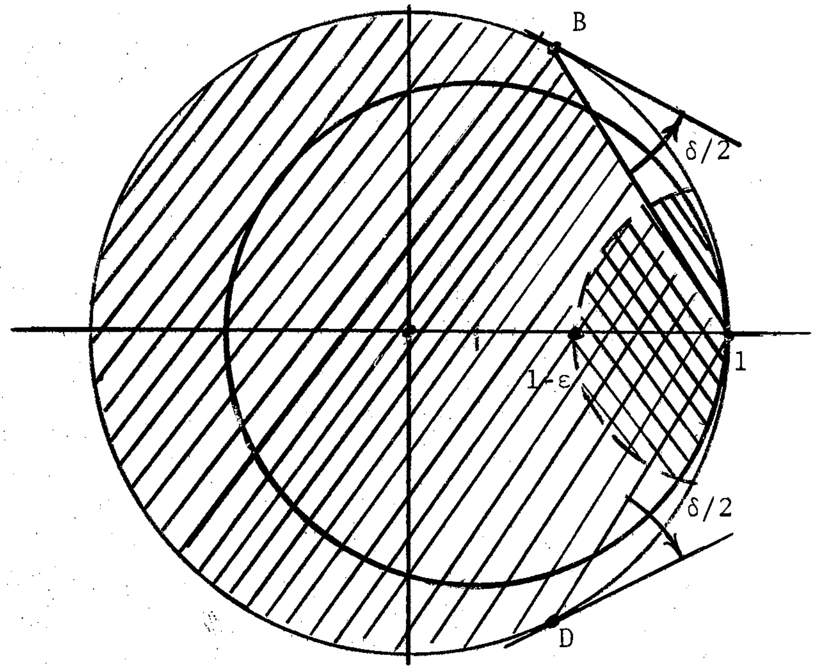

Figure 4. The Third Illustration of Example 2.16.
[The Double Shaded Area is $N_\varepsilon^*(1) \cap A_{\pi-\delta}$]

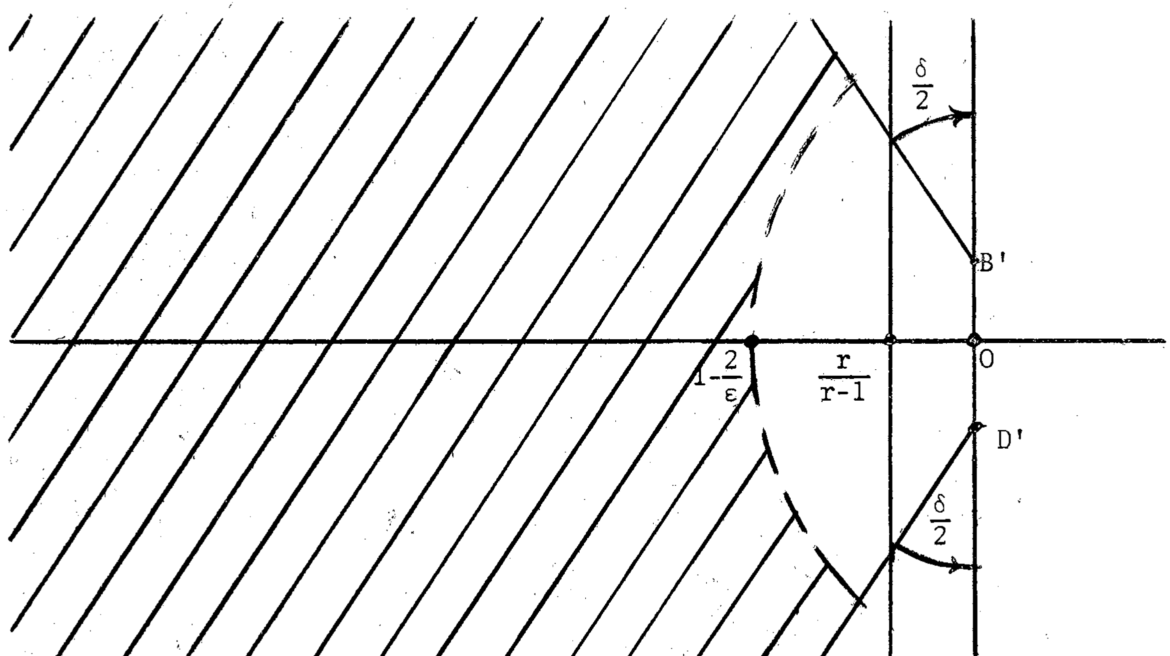

Figure 5. The Fourth Illustration for Example 2.16.
[The Shaded Area is $g(N_\varepsilon^*(1) \cap A_{\pi-\delta})$]

$$g(G_r) \cap g(N_\varepsilon^*(1) \cap A_{\pi-\delta}) = \phi$$

for each $0 < r \leq r_0$. Therefore

$$\{w: |w| = \exp\frac{r}{r-1}\} \cap f(N_\varepsilon^*(1) \cap A_{\pi-\delta}) = \phi$$

for each $0 < r \leq r_0$. Thus all the circles of radius greater than $e^{\frac{r_0}{r_0-1}}$ are not contained in $f(N_\varepsilon^*(1) \cap A_{\pi-\delta})$, which results in $\overline{f(N_\varepsilon^*(1) \cap A_{\pi-\delta})}$ being contained in the closed disk of radius $\exp\frac{r_0}{r_0-1}$. Since this is true for each $\frac{r_0}{1-r_0}$

$$C_{A_{\pi-\delta}}(f,1) = \{0\}.$$

The following cluster sets have now been calculated for $f(z) = e^{\frac{z+1}{z-1}}$.

(a) $C(f,1) = \overline{U}$

(b) $C_{G_r}(f,1) = \{w: |w| = \exp\frac{r}{r-1}\}$

(c) $C_\rho(f,1) = \{0\}$

(d) $C_{A_{\pi-\delta}}(f,1) = \{0\}$

It has also been shown that f is analytic and bounded by one in U.

In the previous examples, it was seen that some cluster values at a point $e^{i\theta}$ are assumed by the function in each neighborhood of the point, while other cluster values were never assumed by the function. A case in point is Example 2.16 where

$$f(z) = e^{\frac{z+1}{z-1}}$$

never assumes the value 0 yet $0 \varepsilon C(f,1)$. The values which are assumed in each neighborhood are of special interest and are said to belong to the range of f at $e^{i\theta}$.

Definition 2.17.

$$R(f,e^{i\theta}) = \left\{ \begin{array}{l} w \mid \text{ there exists } \{z_n\} \subset U \text{ such that} \\ \\ \lim\limits_{n\to\infty} z_n = e^{i\theta} \text{ and } f(z_n) = w \text{ for each } n \end{array} \right\}$$

is the range of f at $e^{i\theta}$.

Theorem 2.18.

$$R(f,e^{i\theta}) = \bigcap_{\varepsilon>0} f(N_\varepsilon^*(e^{i\theta})).$$

Proof: Similar to Theorem 2.5.

It is clear that the range may be empty as is the case for $f(z) = z$ at each $e^{i\theta}$. Example 2.16 where

$$R(e^{\frac{z+1}{z-1}},1) = \{w \mid 0<|w|<1\}$$

and Example 2.9 where

$$R(h,1) = \{e^{i\theta} \mid \theta \text{ is rational}\}$$

demonstrate that the range need not be closed.

Another type of cluster value of special importance is the type which can be obtained by continuous approach to the boundary. For example, the function

$$f(z) = e^{\frac{z+1}{z-1}}$$

has the cluster value 0 at $z = 1$ on continuous approach along the radius. Such a cluster value will be called an asymptotic value of the function at the boundary point.

Definition 2.19.

$$A(f,e^{i\theta}) = \left\{ \begin{array}{l} w \mid \text{there exists a continuous path } \Gamma \subset U \\ \text{which terminates at } e^{i\theta} \text{ and where} \\ f(z) \to w \text{ as } z \to e^{i\theta} \text{ along } \Gamma \end{array} \right\}$$

is the asymptotic cluster set of f at $e^{i\theta}$.

If f is non-constant and continuous at $e^{i\theta}$, then

$$C(f,e^{i\theta}) = A(f,e^{i\theta}) = \{f(e^{i\theta})\}$$

and

$$R(f,e^{i\theta}) = \phi.$$

So far some language has been given with which to discuss the boundary behavior at $e^{i\theta}$. Such considerations have been dependent upon the values the function assumes arbitrarily close to $e^{i\theta}$ in the open disk U. It is also of interest to consider the behavior by taking an approach not in U but on the unit circle C. This makes sense in Example 2.16 where

$$f(z) = e^{\frac{z+1}{z-1}},$$

and at each $e^{i\theta} \neq 1$ f is continuous. In this example

$$C_{G_r}(f,1) = \{w \mid |w| = e^{\frac{r}{r-1}}\}.$$

In the case r = 0, G_0 is the unit circle C. Therefore the approach to z = 1 along the unit circle would give the unit circle as the set of cluster values. However, if f were not continuous at each $e^{i\theta} \neq 1$, it would not be obvious how to take an approach along the boundary. As a generalization, a point w is a boundary cluster value if there exists a sequence of boundary points converging to $e^{i\theta}$ and a corresponding sequence selected from the cluster sets of the sequence of boundary points such that this second sequence converges to w.

Definition 2.20.

$$C_B(f,e^{i\theta}) = \left\{ \begin{array}{l} w \mid \text{there exists } \{e^{i\theta_n}\} \subset C \text{ such that} \\ \lim_{n\to\infty} e^{i\theta_n} = e^{i\theta} \text{ and there exists } w_n \in C(f,e^{i\theta_n}) \\ \text{such that } \lim_{n\to\infty} w_n = w \end{array} \right\}$$

is the boundary cluster set of f at $e^{i\theta}$.

Theorem 2.25.

$$C_B(f,e^{i\theta}) = \bigcap_{\eta>0} \overline{\bigcup_{0<|\theta-\alpha|<\eta} C(f,e^{i\alpha})}$$

Proof: See Theorem 2.5.

CHAPTER III

BLASCHKE PRODUCTS

The major purpose of this chapter is the introduction of a subclass of bounded analytic functions which will provide a useful source of examples to illustrate the theory. Members of this class are called Blaschke products and a few of their basic properties will be developed here. Other important properties of this class will be left for subsequent chapters in order that a logical, sequential development of the theory be maintained.

Definition 3.1. Let $\{a_n\} \subset U$ with $\sum_1^\infty (1-|a_n|) < \infty$ then the product

$$B(z) = e^{i\alpha} z^\lambda \prod_1^k \frac{a_n - z}{1-\bar{a}_n z} \frac{|a_n|}{a_n}$$

where α is real, λ is a non-negative integer and k is a positive integer or ∞, is a Blaschke product. If k is finite B(z) is a finite Blaschke product.

It will be shown next that a Blaschke product belongs to H^∞ where H^∞ is the class of all functions analytic and bounded in U.

Theorem 3.2. If $w_n(z)$ is analytic in U for each n and if $\sum_1^\infty |w_n(z)|$ converges uniformly on compact subsets of U then $\prod_1^\infty [1+w_n(z)]$ is analytic in U.

Proof: See Hille, Volume I, page 224.

Theorem 3.3. Let $\{a_n\}$ U, $a_n \neq 0$ and $\sum_1^\infty (1-|a_n|) < \infty$; then

$$B(z) = \prod_1^\infty \frac{a_n - z}{1-\bar{a}_n z} \frac{|a_n|}{a_n}$$

belongs to H^∞ and $B(z) \leq 1$ for each $z \varepsilon U$. It then follows that the more general form a Blaschke product of Definition 3.1 represents a function of H^∞ and has one as a bound.

Proof: It will be shown that $\sum_1^\infty \left| 1 - \frac{a_n - z}{1-\bar{a}_n z} \frac{|a_n|}{a_n} \right|$ converges uniformly on compact subsets of U which implies by Theorem 3.2 that B(z) is analytic on U. Now

$$\left| \frac{a_n - z}{1-\bar{a}_n z} \frac{|a_n|}{a_n} \right|^2 = 1 - \frac{(1-|a_n|)(1-|z|)}{(1-\bar{a}_n z)(1-a_n \bar{z})} < 1$$

which implies each factor of B(z) is less than one in modulus and therefore $|B(z)| \leq 1$ for each $z \varepsilon U$.

Consider the compact set

$$\bar{D}_r = \{z| \ |z| \leq r\}$$

where $0<r<1$. If

$$a_n = r_n e^{i\theta_n}$$

then

$$\frac{|a_n|}{a_n} = e^{-i\theta_n}.$$

Now note that for each $z \varepsilon \bar{D}_r$

$$\left|1 + \frac{|a_n|}{a_n} z\right| \leq 1 + r \text{ and } |1-\overline{a}_n z| \geq 1-r$$

Therefore

$$\left|1 - \frac{a_n - z}{1-\overline{a}_n z} \frac{|a_n|}{a_n}\right| = \left|\frac{a_n + |a_n| z}{(1-\overline{a}_n z) a_n}\right|(1-|a_n|) = \frac{\left|1+ \frac{|a_n|}{a_n} z\right|}{|1-\overline{a}_n z|}(1-|a_n|) \leq \frac{1+r}{1-r}(1-|a_n|)$$

and

$$\sum_1^{\infty}\left|1 - \frac{a_n - z}{1-\overline{a}_n z} \frac{|a_n|}{a_n}\right| \leq \frac{1+r}{1-r} \sum_1^{\infty}(1-|a_n|) < \infty$$

for each $z \in \overline{D}_r$. By the Weierstrass M-test the series converges uniformly on $\overline{D}_r$. Since every compact subset of U is a subset of some $\overline{D}_r$, it follows that the series converges uniformly on any compact subset of U.

Theorem 3.3 establishes infinite Blaschke products as bounded analytic functions on U. The finite Blaschke product is also analytic on the unit circle so that its boundary behavior is of little interest. Therefore examples of interest will come from among the infinite products where the boundary behavior is less obvious.

Theorem 3.5, by Jensen, is needed to help develop some of the later theory including a characterization for bounded analytic functions.

Theorem 3.4. (Lebesque Dominated Convergence Theorem).

Let $\{f_n\}$ be a sequence of integrable functions with $f_n(x) \to f(x)$ almost everywhere. Also let g be an integrable function such that $|f_n| \leq g$ for each n. Then f is integrable and

$$\int f_n d\mu \to \int f d\mu$$

Proof: See Royden, page 200.

Theorem 3.5. (Jensen's Formula)

Let f be bounded and analytic in U with $f(0) \neq 0$ and let $a_1, a_2, \ldots, a_N$ be the zeros of f in D_r; then

$$|f(0)| \prod_1^N \frac{r}{|a_n|} = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta\right) \tag{3.5.1}$$

Proof: Consider those values of r, $0<r<1$ where no zeros of f lie on the circle C_r. An analytic function g will be constructed such that

$$|g(0)| = |f(0)| \prod_1^N \frac{r}{|a_n|} \tag{3.5.2}$$

and

$$g(z) \neq 0 \text{ for } z \text{ in some domain } \Omega \text{ containing } \overline{D}_r \tag{3.5.3}$$

Then 3.5.3 implies there exists an analytic function $u + iv$ with

$$g(z) = e^{u+iv}.$$

Now

$$|g(z)| = e^u$$

so

$$\log|g(z)| = u$$

is harmonic in Ω. By Cauchy's integral formula

$$\log|g(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|g(re^{i\theta})|\,d\theta.$$

It will then be shown that

$$\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta = \int_{-\pi}^{\pi} \log|g(re^{i\theta})|\,d\theta \tag{3.5.4}$$

These results give

$$(3.5.5)\quad |f(0)|\prod_1^N \frac{r}{|a_n|} = |g(0)| = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(re^{i\theta})|d\theta\}$$

$$= \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta\}$$

for each $0<r<1$ such that no zeros of f lie on C_r. For those values of r where C_r contains zeros of f, a sequence $\{r_k\}$ can be selected such that $r_k \uparrow r$ and all zeros of f interior to D_r are also interior to D_{r_k} for each k. In addition, C_{r_k} contains no zeros of f; therefore

$$|f(0)|\prod_1^N \frac{r_k}{|a_n|} = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r_k e^{i\theta})|d\theta\} \text{ for each } k.$$

Since

$$\lim_{k\to\infty} \log|f(r_k e^{i\theta})| = \log|f(re^{i\theta})|,$$

Theorem 3.4 shows that

$$\lim_{k\to\infty}|f(0)| \prod_1^N \frac{r_k}{|a_n|} = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta\}$$

so

$$|f(0)|\prod_1^N \frac{r}{|a_n|} = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta\}$$

and 3.5.1 holds for each r, $0<r<1$.

In order to complete the details and establish 3.5.2, 3.5.3, and 3.5.4, consider the function

$$(3.5.6)\qquad g(z) = f(z) \prod_1^N \frac{r^2-\bar{a}_n z}{r(a_n-z)} .$$

Now g is analytic in U and has no zeros in D_r and since f has no zeros on C_r neither does g. Therefore there exists a domain Ω such that

$$\overline{D}_r = D_r \cup C_r \subset \Omega$$

and such that g is non-zero in Ω. Also

$$|g(0)| = |f(0)| \prod_1^N \frac{r}{|a_n|}$$

and

$$\log|g(z)| = \log|f(z)| + \sum_1^N \log \left|\frac{r^2 - \overline{a}_n z}{r(a_n - z)}\right|.$$

If $z = re^{i\theta}$ then

$$|g(z)| = |f(z)|$$

because each of the N factors in g has modulus one.

So

$$\int_{-\pi}^{\pi} \log|g(re^{i\theta}|d\theta = \int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta + \sum_1^N \int_{-\pi}^{\pi} \log \left|\frac{r^2 - \overline{a}_n z}{r(a_n - z)}\right| d\theta$$

$$\int_{-\pi}^{\pi} \log|g(re^{i\theta})|d\theta = \int_{-\pi}^{\pi} \log|f(re^{i\theta}|d\theta + 0$$

and the proof is complete.

A sequence $\{a_n\} \subset U$ with

$$\sum_1^\infty (1-|a_n|) < \infty$$

is called a Blaschke sequence, and it will be shown that the zeros of any function which belongs to H^∞ form such a sequence.

Theorem 3.6. Let $f \varepsilon H^\infty$ with $f(z) \not\equiv 0$ and $\{a_n\} \subset U$ be the set of zeros of f with multiplicities considered, then

$$\sum_1^\infty (1-|a_n|) < \infty.$$

Proof: It can be assumed that $f(0) \neq 0$, for otherwise consider $\frac{f(z)}{z^{\lambda}}$ where λ is the order of the zero at $z = 0$. Let $n(r)$ denote the number of zeros of f in D_r, $0<r<1$, and let $0<k<n(r)$. By Jensen's formula

$$|f(0)|\prod_{1}^{k} \frac{r}{|a_n|} \leq |f(0)|\prod_{1}^{n(r)} \frac{r}{|a_n|} = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta\} < M < \infty$$

where the last inequalities follows from f being bounded. Therefore

$$\prod_{1}^{n(r)} |a_n| \geq \frac{r^k|f(0)|}{M} \quad \text{for } k<n(r) \text{ and for } 0<r<1$$

which implies

$$\prod_{1}^{\infty}|a_n| \geq \frac{|f(0)|}{M} > 0$$

So

$$\sum_{1}^{\infty}(1-|a_n|) < \infty$$

Corollary 3.7. If $f \varepsilon H^{\infty}$ there exists a Blaschke product whose zeros are identical with those of f.

The following theorem gives a useful representation for functions of H^{∞}.

Theorem 3.8. If $f \varepsilon H^{\infty}$ there exists a Blaschke product $B(z)$ and a non-zero function $h(z) \varepsilon H^{\infty}$ such that

$$f(z) = B(z)h(z)$$

Proof: By the previous corollary there exists a Blaschke product $B(z)$ such that f and B have the same zeros. This implies

$$h(z) = \frac{f(z)}{B(z)}$$

is a non-zero analytic function.

Now consider $B_n(z)$ the partial product of $B(z)$ consisting of the first n factors. $B_n(z)$ is a finite Blaschke product and therefore is analytic on the boundary C and of modulus one there. Therefore there exists a circle C_r, $0<r<1$, such that $|B_n(z)|>1-\varepsilon$ for each $z\varepsilon C_r$ and such that all the zeros of $B_n(z)$ lie in D_r. Since $\frac{f(z)}{B_n(z)}$ is analytic in $\overline{D}_r$ it attains its maximum on C_r. Therefore

$$\frac{|f(z)|}{|B_n(z)|} < \frac{M}{1-\varepsilon}$$

for each $z\varepsilon\overline{D}_r$. Now $B_n(z)$ is analytic and non-zero in the closed annulus between C_r and C. Therefore $B_n(z)$ attains a minimum on C_r. This minimum must clearly be greater than $1-\varepsilon$ in modulus. Therefore

$$\frac{|f(z)|}{|B_n(z)|} < \frac{M}{1-\varepsilon} \text{ everywhere in the open annulus.}$$

and so

$$\frac{|f(z)|}{|B_n(z)|} < \frac{M}{1-\varepsilon} \text{ everywhere in U.}$$

Since ε is arbitrary

$$\frac{|f(z)|}{|B_n(z)|} \leq M \text{ for } z\varepsilon U \text{ and for every n.}$$

Thus

$$|h(z)| = \left|\frac{f(z)}{B(z)}\right| = \lim_{n\to\infty} \frac{|f(z)|}{|B_n(z)|} \leq M \text{ for } z\varepsilon U$$

and $h\varepsilon H^{\infty}$.

Before leaving this chapter, a partial investigation will be made of the boundary behavior of selected Blaschke products. Any point $e^{i\theta}$ which is a limit point of zeros will be of interest, since $B(z)$ cannot be extended so it is analytic there.

Example 3.9. Consider a Blaschke product with

$$a_n = 1 - \frac{1}{n^2}.$$

$B(a_n) = 0$ for each n implies $0 \varepsilon C_\rho(B,1)$. It will be shown that

$$C_\rho(B,1) = \{0\}.$$

Let $0<r<1$ then $ra_n<a_n<1$ and $ra_n<r$ for each n and $r-a_n<1-ra_n$. Therefore

$$\frac{r-a_n}{1-ra_n} < 1 \text{ for each n.}$$

Now

$$|B(r)| = \prod_1^\infty \frac{|a_n-r|}{1-a_n r} < \prod_1^{N-1} \frac{|a_n-r|}{1-a_n r}$$

If $a_{N-1}<r<a_N$ then $\frac{r-a_n}{1-ra_N} < \frac{a_N-a_n}{1-a_n}$ for $n < N$.

So

$$|B(r)| < \prod_1^{N-1} \frac{r-a_n}{1-a_n r} < \prod_1^{N-1} \frac{a_N-a_n}{1-a_n} = \prod_1^{N-1} \frac{N^2-n^2}{N^2} = \frac{(2N-1)!}{N^{2N-1}} < \frac{1}{N-1}$$

for $N = 2, 3, \ldots$.

Therefore

$$|B(r)| < \frac{1}{N-1}$$

for

$$a_{N-1}<r<a_N$$

which implies

$$\lim_{r\to 1} |B(r)| = 0$$

and gives the result that

$$C_\rho(B,1) = \{0\}.$$

Example 3.10. Consider the Blaschke product with

$$a_n = 1 - \frac{1}{e^{e^{2n}}} \qquad n = 2, 3, \ldots$$

again as in Example 3.9 $0 \varepsilon C_\rho(B,1)$, but it will be shown that B has other values in the radial cluster. This will then provide the first concrete example of a function which is analytic in U, yet fails to have a radial limit at z = 1. It will be shown that for

$$z_N = 1 - \frac{1}{e^{e^{2N+1}}} \quad N = 2, 3, \ldots \qquad \overline{\lim_{N\to\infty}} |B(z_N)| > \frac{1}{2}$$

which implies $C_\rho(B,1)$ fails to be degenerate.

Let k be a positive integer and note $e>2$, $2k^2<e^{e^{2k}}$ and $2(k+1)^2 < e^{e^{2k}}$ for each k. Thus

(3.10.1) $$e^{e^{2k+1}} = (e^{e^{2k}})^e > (e^{e^{2k}})^2 > 2k^2 e^{e^{2k}}$$

and

(3.10.2) $$e^{e^{2k+2}} = (e^{e^{2k+1}})^e > (e^{e^{2k+1}})^2 > 2(k+1)^2 e^{e^{2k+1}}$$

Statement 3.10.1 implies

(3.10.3) $$e^{e^{2N+1}} > 2N^2 e^{e^{2N}} \geq 2n^2 e^{e^{2n}} \quad \text{for } n \leq N$$

and 3.10.2 implies

(3.10.4) $$e^{e^{2n}} = e^{e^{2(n-1)+2}} > 2n^2 e^{e^{2n-1}} \geq 2n^2 e^{e^{2N+1}}$$

Statement 3.10.3 implies

$$e^{e^{2n+1}} - 2n^2 e^{e^{2n}} > 0 \text{ for } n \leq N$$

$$-n^2 e^{e^{2n}} > n^2 e^{e^{2n}} - n^2 - e^{e^{2n}} + 1$$

$$n^2 e^{e^{2N+1}} - n^2 e^{e^{2n}} > n^2 e^{e^{2N+1}} + n^2 e^{e^{2n}} - n^2 - e^{e^{2n}} - e^{e^{2N+1}} + 1$$

$$n^2 (e^{e^{2N+1}} - e^{e^{2n}}) > (n^2 - 1)(e^{e^{2N+1}} + e^{e^{2n}} - 1)$$

so

$$\text{(3.10.5)} \qquad \frac{e^{e^{2N+1}} - e^{e^{2n}}}{e^{e^{2N+1}} + e^{e^{2n}} - 1} > \frac{n^2-1}{n^2} = 1 - \frac{1}{n^2} .$$

Now for $n \leq N$ the left hand member of 3.10.5 is $\frac{z_N - a_n}{1 - a_n z_N}$ so

$$\text{(3.10.6)} \qquad \frac{z_N - a_n}{1 - a_n z_N} > 1 - \frac{1}{n^2} \text{ for } n \leq N.$$

Statement 3.10.4 implies for $n > N$

$$e^{e^{2n}} - (2n^2-1) e^{e^{2N+1}} > 0$$

$$-n^2 e^{e^{2N+1}} > n^2 e^{e^{2N+1}} - e^{e^{2N+1}} - e^{e^{2n}}$$

$$n^2 e^{e^{2n}} - n^2 e^{e^{2N+1}} > n^2 e^{e^{2n}} + n^2 e^{e^{2N+1}} - e^{e^{2N+1}} - e^{e^{2n}} + 1 - n^2$$

$$n^2 (e^{e^{2n}} - e^{e^{2N+1}}) > (n^2-1)(e^{e^{2n}} + e^{e^{2N+1}} - 1)$$

so

$$\text{(3.10.7)} \qquad \frac{e^{e^{2n}} - e^{e^{2N+1}}}{e^{e^{2n}} + e^{e^{2N+1}} - 1} > \frac{n^2-1}{n^2} = 1 - \frac{1}{n^2}$$

Now for $n > N$ the left hand member of 3.10.7 is $\frac{a_n - z_N}{1 - a_n Z_n}$ so

(3.10.8) $$\frac{a_n - z_N}{1-a_n z_N} > 1 - \frac{1}{n^2} \text{ for } n > N.$$

Combining 3.10.6 and 3.10.8 gives

$$|B(z_N)| = \prod_2^{\infty} \frac{|a_n - z_N|}{1-a_n z_N} = \prod_2^{N} \frac{z_N - a_n}{1-a_n z_N} \prod_{N+1}^{\infty} \frac{a_n - z_N}{1-a_n z_N} > \prod_2^{\infty}(1-\frac{1}{n^2}) = \frac{1}{2}$$

So

$$\overline{\lim_{N\to\infty}} |B(z_N)| > \frac{1}{2} .$$

Example 3.11. Consider any sequence $\{a_n\} \subset U$ such that

$$|a_n| = 1 - \frac{1}{n^2} .$$

It is clear that a Blaschke product can be constructed to have zeros at a_n since

$$\sum_1^{\infty}(1-|a_n|) < \infty.$$

Notice that nothing has been stated to indicate where a_n lies on the circle of radius $1 - \frac{1}{n^2}$. Since there are only countably many radii terminating at points of C which have rational arguments, it is possible to select $\{a_n\}$ such that infinitely many a_n's lie on each radius with a rational argument. Therefore the Blaschke product B(z) with zeros at $\{a_n\}$ has infinitely many zeros on each rational radius and so $0 \varepsilon C_\rho(f,e^{i\theta})$ for θ rational. Now B(z) has no zeros on the radius to a point $e^{i\theta}$, θ irrational; but each partial neighborhood of $e^{i\theta}$ contains infinitely many a_n's. Therefore $0 \varepsilon C(f,e^{i\theta})$ for all θ.

CHAPTER IV

THE FATOU BOUNDARY FUNCTION

In this chapter a foundation will be laid for the study of functions of the class H^∞ of bounded analytic function on U. This work is of great importance even in the study of functions which do not fall in this class. From the elementary theory of complex variables it is seen that an analytic function is, in some sense, described by its behavior on the boundary. For example, if a function f of H^∞ can be extended so it is continuous on the closed unit disk, then the maximum modulus theorem and Cauchy integral formula hold with respect to the unit circle. It is obvious, however, that for an arbitrary function of H^∞ such an extension may not be possible.

The key theorem in this study is Fatou's theorem which states that a function which belongs to H^∞ has a radial limit at almost every point of the unit circle. Using this theorem, a function can be defined on the unit circle which may not have the "nice" properties, such as providing a continuous or analytic extension for f, yet it does in some sense describe the behavior of f in U. This function will be called the Fatou boundary function of f.

In the development of the theory surrounding the Fatou boundary function, it will be necessary to introduce an abundance of material which is of importance as a background in many areas of investigation in analysis. Because of its basic nature it is included in detail here although its role in the present development is auxiliary.

The first theorem gives an integral representation for a function which is bounded and analytic on a closed disk in terms of the values the function assumes on the boundary. It really is a special form of the Cauchy integral representation and is referred to as the Poisson integral.

Theorem 4.1. Let f be bounded and analytic in a domain which contains $D_\rho \cup C_\rho$, $\rho > 0$ then

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it})\frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\theta - t) + r^2}\,dt \text{ where } 0 \leq r < \rho.$$

Proof: By Cauchy's integral formula

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(\xi)}{\xi - re^{i\theta}}d\xi = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f(\rho e^{it})i\rho e^{it}}{\rho e^{it} - re^{i\theta}}\,dt$$

if

$$\xi = \rho e^{it}.$$

Therefore

(4.1.1) $$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it})\frac{1}{1 - \frac{r}{\rho}e^{(\theta - t)}}\,dt.$$

The familiar equality

$$\frac{1}{1-x} = \sum_0^\infty x^n \text{ gives}$$

$$\frac{1}{1 - \frac{r}{\rho}e^{i(\theta - t)}} = \sum_0^\infty \left(\frac{r}{\rho}\right)^n e^{in(\theta - t)}.$$

Substitution in 4.1.1 gives

(4.1.2) $$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it})[1 + \sum_1^\infty \left(\frac{r}{\rho}\right)^n e^{in(\theta - t)}]dt.$$

Since f is analytic in $D_\rho \cup C_\rho$ so is

$$g_n(re^{i\theta}) = f(re^{i\theta})e^{i(n-1)\theta}$$

for n = 1, 2, 3, ... Therefore

$$\int_{C_\rho} g_n(z)\,dz = 0.$$

Note that

$$\int_{C_\rho} g_n(z)\,dz = \int_{-\pi}^{\pi} i\rho f(\rho e^{it})e^{int}dt = 0$$

and thus for fixed θ

(4.1.3) $$e^{-in\theta}\int_{-\pi}^{\pi} f(\rho e^{it})e^{int}\,dt = \int_{-\pi}^{\pi} f(\rho e^{it})e^{-in(\theta-t)}dt = 0$$

for n = 1, 2, 3, ...

Statements 4.1.2 and 4.1.3 yield

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it})[1 + \sum_1^\infty \left(\frac{r}{\rho}\right)^n [e^{in(\theta-t)}+e^{-in(\theta-t)}]]dt$$

and

(4.1.4) $$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it})[1+2\sum_1^\infty \left(\frac{r}{\rho}\right)^n \cos(n(\theta-t))]dt$$

It will be shown now that the integrand of 4.1.1 is equal to the desired one in the statement of the theorem. Consider the following quotient whose denominator will be made real.

$$\frac{\rho e^{it}+re^{i\theta}}{\rho e^{it}-re^{i\theta}} = \frac{\rho e^{it}+re^{i\theta}}{\rho e^{it}-re^{i\theta}} \cdot \frac{\rho e^{-it}-re^{-i\theta}}{\rho e^{-it}-re^{-i\theta}}$$

$$= \frac{\rho^2-r^2+r\rho(e^{i(\theta-t)}-e^{-i(\theta-t)})}{\rho^2-\rho r(e^{i(\theta-t)}+e^{-i(\theta-t)})+r^2}$$

(4.1.5) $$\frac{\rho e^{it}+re^{i\theta}}{\rho e^{it}-re^{i\theta}} = \frac{\rho^2-r^2+2i\ Re\ \sin(\theta-t)}{\rho^2-2\rho r\cos(\theta-t)+r^2}$$

Also

$$\frac{\rho e^{it}+re^{i\theta}}{\rho e^{it}-re^{i\theta}} = \frac{1+\frac{r}{\rho}e^{i(\theta-t)}}{1-\frac{r}{\rho}e^{i(\theta-t)}} = [1+\frac{r}{\rho}e^{i(\theta-t)}]\sum_0^\infty\left(\frac{r}{\rho}\right)^n e^{in(\theta-t)}$$

$$= 1 + 2\sum_1^\infty\left(\frac{r}{\rho}\right)^n e^{in(\theta-t)}$$

(4.1.6) $$= 1 + 2\sum_1^\infty\left(\frac{r}{\rho}\right)^n[\cos(n(\theta-t)+i\sin(n(\theta-t)].$$

Statements 4.1.4, 4.1.5 and 4.1.6 imply

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it})\ \frac{\rho^2-r^2}{\rho^2-2r\rho\cos(\theta-t)+r^2}\,dt$$

which is the desired conclusion.

<u>Theorem 4.2. (Schwarz's Lemma)</u>. Let $f\varepsilon H^\infty$ with $|f(z)|\leq 1$ for $z\varepsilon U$ and $f(0) = 0$; then $|f(z)|<|z|$ for $z\varepsilon U$.

<u>Proof</u>: See Hille, Volume II.

<u>Theorem 4.3</u>. Let f be a real valued monotonic increasing function on [a,b]; then f is differentiable almost everywhere in [a,b].

<u>Proof</u>: See Royden, Theorem 2, Page 82.

<u>Theorem 4.4</u>. Let f be a real valued function defined on [a,b] such that

$$\left|\frac{f(x+\Delta x)-f(x)}{\Delta x}\right| < M \text{ for } x,\ x+\Delta x\varepsilon[a,b];$$

then f is differentiable almost everywhere in [a,b].

Proof: Let

$$a = x_0 < x_1 < \ldots < x_n = b$$

be a partition of [a,b] then by hypothesis

$$\left|\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}\right| < M \text{ for } k = 1, 2, \ldots, n.$$

So

$$\sum_1^n |f(x_k)-f(x_{k-1})| < \sum_1^n |x_k - x_{k-1}| M = (b-a)M$$

therefore f is of bounded variation on [a,b]. Any function of bounded variation can be expressed as $f = f_1 - f_2$ where f_1 and f_2 are monotonic increasing real valued functions. By Theorem 4.3 f_1 and f_2 are differentiable almost everywhere on [a,b] and therefore the same is true of f.

The following theorem by Fatou is the one mentioned previously as being the key theorem of this study. It shows that a function of H^∞ has a radial limit at almost every point of the unit circle.

Theorem 4.5. (Fatou). Let $f \in H^\infty$ then $C_\rho(f,e^{i\theta})$ is degenerate for almost all θ, $-\pi \leq \theta \leq \pi$.

Proof: A complex valued function $F(\theta)$ will be constructed such that its difference quotients are bounded. This implies the real and imaginary parts of $F(\theta)$ have their difference quotients bounded. By Theorem 4.4 this implies the real and imaginary parts are differentiable almost everywhere in $[-\pi,\pi]$. Therefore $F(\theta)$ is differentiable almost everywhere there. It will then be shown that

$$\lim_{r \to 1} f(re^{i\theta}) = F'(\theta)$$

for those values of θ where the derivative exists. This implies that $C_\rho(f,e^{i\theta})$ is degenerate for almost all θ.

The assumption will be made that $f(0) = 0$. If this is not the case, the function $f(z) - f(0)$ can be considered. Define

$$(4.5.1)\quad F(\rho,\theta) = \int_0^\theta f(\rho e^{it})dt = \int_\rho^{\rho e^{i\theta}} \frac{f(z)}{iz}\, dz \text{ where } 0 < \rho < 1.$$

and the path of integration is taken along an arc of C_ρ. Now $f(0) = 0$ implies $\frac{f(z)}{iz}$ is analytic in U (i.e. has only a removable singularity). Therefore the last integral of 4.5.1 may be taken along any contour from $z = \rho$ to $z = \rho e^{i\theta}$ which lies entirely in U. In particular

$$(4.5.2)\qquad F(\rho,\pi) = F(\rho,-\pi)$$

Let M be the bound on f. Then

$$(4.5.3)\quad |F(\rho,\theta+\Delta\theta)-F(\rho,\theta)| = \left|\int_0^{\theta+\Delta\theta} f(\rho e^{it})dt - \int_0^\theta f(\rho e^{it})dt\right|$$

$$= \left|\int_\theta^{\theta+\Delta\theta} f(\rho e^{it})dt\right| < M|\Delta\theta|$$

where $|f(\rho e^{i\theta})|<M$ by Schwarz's lemma and the fact $|\rho e^{i\theta}|<1$. Consider the annular sector (see Figure 6) connecting the four points ρ, $\rho+\Delta\rho$, $(\rho+\Delta\rho)e^{i\theta}$, $\rho e^{i\theta}$. Integration around the boundary of this sector gives

$$(4.5.4)\quad \int_\rho^{\rho+\Delta\rho} \frac{f(z)}{iz}\, dz + \int_{\rho+\Delta\rho}^{(\rho+\Delta\rho)e^{i\theta}} \frac{f(z)}{iz}\, dz + \int_{(\rho+\Delta\rho)e^{i\theta}}^{\rho e^{i\theta}} \frac{f(z)}{iz}\, dz$$

$$+ \int_{\rho e^{i\theta}}^{\rho} \frac{f(z)}{iz}\, dz = 0$$

Therefore

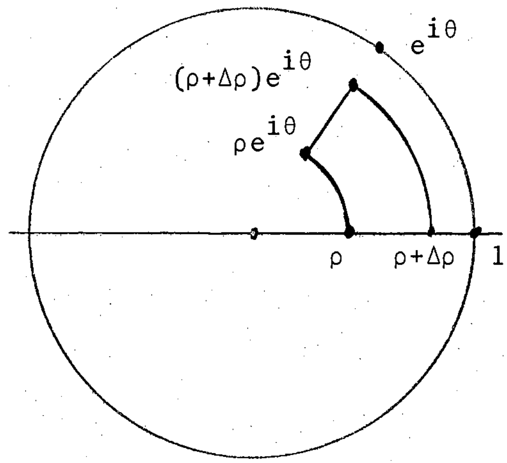

Figure 6

$$|F(\rho+\Delta\rho,\theta)-F(\rho,\theta)| = \left|\int_{\rho+\Delta\rho}^{(\rho+\Delta\rho)e^{i\theta}} \frac{f(z)}{iz}\,dz - \int_{\rho}^{\rho e^{i\theta}} \frac{f(z)}{iz}\,dz\right|$$

$$= \left|\int_{\rho e^{i\theta}}^{(\rho+\Delta\rho)e^{i\theta}} \frac{f(z)}{iz}\,dz - \int_{\rho}^{\rho+\Delta\rho} \frac{f(z)}{iz}\,dz\right|$$

$$\leq \int_{\rho}^{(\rho+\Delta\rho)e^{i\theta}} \left|\frac{f(z)}{iz}\right| dz + \int_{\rho}^{\rho+\Delta\rho} \left|\frac{f(z)}{iz}\right| dz < 2M|\Delta\rho|$$

so

(4.5.5) $$|F(\rho+\Delta\rho,\theta) - F(\rho,\theta)| < 2M|\Delta\rho|$$

Now 4.5.5 is essentially a Cauchy condition as $\rho\to 1$ so $F(\rho,\theta)$ converges uniformly to a continuous function $F(\theta)$, $-\pi \leq \theta < \pi$. Result 4.5.3 implies

$$\lim_{\rho\to 1} \left|\frac{F(\rho,\theta+\Delta\theta)-F(\rho,\theta)}{\Delta\theta}\right| = \left|\frac{F(\theta+\Delta\theta)-F(\theta)}{\Delta\theta}\right| \leq M$$

so $F(\theta)$ has its difference quotients bounded. By Theorem 4.4 $F'(\theta)$ exists almost everywhere on $(-\pi,\pi)$.

It remains to be shown that

$$\lim_{r\to 1} f(re^{i\theta}) = F'(\theta).$$

If $0\le r<\rho<1$ then

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it}) \frac{\rho^2-r^2}{\rho^2-2r\rho\cos(t-\theta)+r^2}\, dt.$$

For convenience let

$$P_r^{\rho}(t-\theta) = \frac{\rho^2-r^2}{\rho^2-2r\rho\cos(t-\theta)+r^2} .$$

Then integration by parts gives

$$f(re^{i\theta}) = F(\rho,t)P_r^{\rho}(t-\theta)\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\rho,t)\frac{d}{dt}(P_r^{\rho}(t-\theta)\,dt$$

But $F(\rho,\pi) = F(\rho,-\pi)$ and

$$P_r^{\rho}(\pi-\theta) = P_r^{\rho}(-\pi-\theta)$$

implies

$$(4.5.6) \qquad f(re^{i\theta}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\rho,t)\frac{d}{dt}(P_r^{\rho}(t-\theta))\,dt \text{ for } \rho > r.$$

Thus taking a limit as $\rho\to 1$ in 4.5.6 yields

$$f(re^{i\theta}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\frac{d}{dt}(P_r(t-\theta))\,dt$$

where

$$P_r(t-\theta) = P_r^{1}(t-\theta)$$

Let $\theta_\circ$ be such that $F'(\theta_\circ)$ exists. No generality is lost if $\theta_\circ$ is assumed to be 0 because all of the preceding arguments apply to the function $F(\theta-\theta_\circ)$ considering F as a periodic function.

Therefore the following work will deal with $F'(0)$. If z is real (i.e. $z = r$), then

$$f(r) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\frac{d}{dt}(P_1(t))dt \tag{4.5.7}$$

Consider the special case $f(z) = k$, a constant. Here

$$k = \frac{1}{2\pi}\int_{-\pi}^{\pi} k\, P_1(t)dt$$

and integration by parts gives

$$k = \frac{k(1-r^2)}{1+2r\cos\theta+r^2} - \frac{1}{2\pi}\int_{-\pi}^{\pi} kt\,\frac{d}{dt}\, P_1(t-\theta)dt.$$

Now let $k = 1$ and $\theta = 0$; then

$$1 = \frac{1-r^2}{1-2r+r^2} - \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,\frac{d}{dt}\, P_1(t)dt \tag{4.5.8}$$

and

$$-F'(0) = -F'(0)\,\frac{1+r}{1-r} + \frac{1}{2\pi}\int_{-\pi}^{\pi} F'(0)t\left(\frac{dP_1(t)}{dt}\right) dt. \tag{4.5.9}$$

Combining 4.5.7 and 4.5.9 gives

$$f(r) - F'(0) = -F'(0)\frac{1-r}{1+r} - \frac{1}{2\pi}\int_{-\pi}^{\pi}[F(t)-tF'(0)]\left(\frac{d}{dt}\,P_1(t)\right)dt. \tag{4.5.10}$$

The object now is to show

$$\lim_{r\to 1}|f(r)-F'(0)| = 0$$

and once this is shown the proof is complete. Let

$$H(r,\lambda) = -\frac{1}{2\pi}\int_{-\lambda}^{\lambda}[F(t)-tF'(0)]\left(\frac{dP_1(t)}{dt}\right) dt \tag{4.5.11}$$

$$(4.5.12) \quad J(r,\lambda) = -\frac{1}{2\pi}\int_{-\pi}^{-\lambda}[F(t)-tF'(0)]\left(\frac{dP_1(t)}{dt}\right)dt$$

$$-\frac{1}{2\pi}\int_{\lambda}^{\pi}[F(t)-tF'(0)]\left(\frac{dP_1(t)}{dt}\right)dt.$$

In this notation 4.5.10 becomes

$$(4.5.13) \qquad |f(r)-F'(0)| \leq |F'(0)|\frac{1-r}{1+r} + |H(r,\lambda)| + |J(r,\lambda)|$$

Note $F(\rho,0) = 0$ and

$$F(0) = \lim_{\rho\to 1} F(\rho,0) = 0$$

Thus

$$F'(0) = \frac{F(t)-F(0)}{t} + \eta(t) = \frac{F(t)}{t} + \eta(t)$$

where

$$\lim_{t\to 0} \eta(t) = 0.$$

Therefore

$$(4.5.14) \qquad F(t)-tF'(0) = t\eta(t)$$

Substitution of 4.5.14 into 4.5.11 gives

$$H(r,\lambda) = -\frac{1}{2\pi}\int_{-\lambda}^{\lambda} t\eta(t)\left(\frac{d}{dt}P_1(t)\right)dt$$

thus

$$|H(r,\lambda)| \leq \frac{1}{2\pi} \sup_{-\lambda<t<\lambda}\{|\eta(t)|\} \left|\int_{-\lambda}^{\lambda} t\left(\frac{d}{dt}P_1(t)\right)dt\right|$$

and 4.5.8 gives

$$|H(r,\lambda)| \leq \sup_{-\lambda<t<\lambda}\{|\eta(t)|\}\left(1 - \frac{1-r^2}{1-2r+r^2}\right)$$

(4.5.15) $$|H(r,\lambda)| \leq \frac{2}{1+\frac{1}{r}} \sup_{-\lambda<t<\lambda}\{|\eta(t)|\}$$

so

$$\lim_{\lambda\to 0} |H(r,\lambda)| = 0$$

independent of r.

Note

$$\frac{d}{dt}(P_1(t)) = \frac{d}{dt}\left(\frac{1-r^2}{1-2r\cos t+r^2}\right) = \frac{-2r(1-r^2)\sin t}{(1-2r\cos t+r^2)^2}$$

and for $\lambda<|t|<\pi$

$$\cos t < \cos \lambda$$

and

$$1-2r\cos t+r^2 > 1-r\cos\lambda+r^2 > \sin^2\lambda$$

where the last inequality follows because $1-2r\cos t+r^2$ attains its minimum as a function of r when $r = \cos\lambda$. Therefore

(4.5.16) $$\left|\frac{d}{dt} P_1(t)\right| < \frac{2(1-r^2)}{(\sin^2\lambda)^2}$$

Now

$$\left|\frac{F(\theta)-F(0)}{\theta}\right| \leq M \text{ and } F(0) = 0 \text{ implies}$$

$$|F(\theta)| \leq M |\theta| \text{ and therefore}$$

(4.5.17) $$|F(\theta)| \leq M\pi \text{ for } |\theta| < \pi$$

A combination of 4.5.17, 4.5.16 and 4.5.12 gives

$$|J(r,\lambda)| < \frac{1}{2\pi} \cdot 2\pi[M\pi+\pi F'(0)] \frac{2(1-r^2)}{(\sin^2\lambda)^2}$$

(4.5.18) $$|J(r,\lambda)| < \frac{2\pi[M+F'(0)]}{(\sin^2\lambda)^2} \quad (1-r^2).$$

Therefore by 4.5.18, 4.5.15, and 4.5.13

$$|f(r)-F'(0)|<|F'(0)|\frac{1-r}{1+r} + |H(r,\lambda)| + \frac{2\pi[M+F'(0)](1-r^2)}{(\sin^2\lambda)^2}$$

where

$$\lim_{\lambda\to 0} |H(r,\lambda)| = 0.$$

So for sufficiently small λ such that $|H(r,\lambda)|<\varepsilon/3$, choose $r_\circ$ such that

$$|F'(0)|\frac{1-r_\circ}{1+r_\circ} < \varepsilon/3 \text{ and } \frac{2\pi[M+F'(0)](1-r_\circ^2)}{(\sin^2\lambda)^2} < \varepsilon/3$$

Therefore for $r_\circ<r<1$

$$|f(r)-F'(0)| < \varepsilon$$

and the proof is completed.

With each function f belonging to H^∞, an auxiliary function will be associated. This function is defined on C and agrees with the radial limits of f at each point where the limit exists. This function will be referred to as the boundary function of f. It is easy to see that there may be many different functions which would qualify. For the purposes here any representative will do.

Definition 4.6. Let $f\varepsilon H^\infty$ and define a function f* on C such that

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

wherever the limit exists. f* is the Fatou boundary function.

Of course from Fatou's theorem, f* agrees with the radial limit for almost all θ. However, since it must only agree with radial limits, there is no guarantee that f* is in any way a continuous extension of f to the unit circle. In fact, it will be shown later that it is possible to have a function $f\varepsilon H^\infty$ such that $C(f,e^{i\theta})$ is non-degenerate for each θ. In spite of its possible lack of "nicer" properties it can be shown in general that f* is integrable and describes f.

The next two theorems could be called the integral formula and maximum modulus theorems of f with respect to f*.

<u>Theorem 4.7.</u> Let $f\varepsilon H^\infty$ and $z = re^{i\theta}\varepsilon U$. Then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f^*(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{it})\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\,dt.$$

<u>Proof:</u> $f\varepsilon H^\infty$ implies there exists an M>0 such that $|f(z)|<M$ for $z\varepsilon U$. f analytic in U gives the usual Cauchy integral representation for closed Jordan curves lying in U. So fix $z = re^{i\theta}$ and select $r<r_1<r_2<\ldots$ with $\lim_{n\to\infty} r_n = 1$. Then

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_n}} \frac{f(\xi)}{\xi - z}\,d\xi \quad \text{where } \xi = r_n e^{it}$$

Now

$$\lim_{n\to\infty} \frac{f(r_n e^{it})}{r_n e^{it} - z} = \frac{f^*(e^{it})}{e^{it}-z} \quad \text{for almost all } t\varepsilon[-\pi,\pi]$$

and

$$\left|\frac{f(r_n e^{it})}{r_n e^{it}-z}\right| < \frac{M}{r_1 - r}\ .$$

By the Lebesque dominated convergence theorem $\frac{f^*(e^{it})}{e^{it}-z}$ is integrable and

$$f(z) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{C_{r_n}} \frac{f(\xi)}{\xi-z}\,d\xi = \frac{1}{2\pi}\int_C \frac{f(\xi)}{\xi - z}\,d\xi$$

The Poisson representation then follows the same way as in Theorem 4.1.

It is worth noting that if $f(z) \equiv 1$, then $f^*(e^{i\theta}) = 1$ and

$$(4.7.1) \qquad 1 = f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, dt.$$

<u>Theorem 4.8</u>. If $f \varepsilon H^\infty$ and $|f^*(e^{i\theta})| \leq M$ for almost all $\theta \varepsilon [-\pi,\pi]$; then $|f(z)|<M$ or is constant of absolute value M everywhere in U.

<u>Proof</u>: By the previous theorem, the Poisson integral representation of f with respect to f* is

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f^*(e^{it})(1-r^2)}{1+r^2-2r\cos(\theta-t)}\, dt \qquad 0<r<1$$

This statement and 4.7.1 yield

$$|f(z)| \leq M \text{ for } z \varepsilon U.$$

Now consider any smaller disk D_r, $0<r<1$; then f is analytic on the closure of this disk and must therefore assume its maximum on the boundary C_r or be a constant of absolute value M in D_r. Since this is true for each $0<r<1$, $|f(z)| < M$ or is constant in U.

It was proven in Chapter III that a Blaschke product is a bounded analytic function and therefore has a Fatou boundary function. Theorem 4.11 shows that its boundary function $B^*(e^{i\theta})$ has modulus 1 almost everywhere on C. It will be noted, however, that an example has already been given such that $0 \varepsilon C_\rho(B,e^{i\theta})$ for θ rational (see Example 3.11). Therefore this Blaschke product fails to have a radial limit of modulus 1 at a countable number of points of C. The following two theorems are preliminary to Theorem 4.11 mentioned previously. The first of these is a theorem from measure theory called Fatou's Lemma. Although credited to the same person, it not intended that there be any direct connection between it and Fatou's theorem on radial limits. The second of these two theorems establishes a relationship between the integral of the logarithm of the absolute value of a function and the integral of the logarithm of the absolute value of its Fatou boundary function. It will be of later use in Chapter V also.

Theorem 4.9. (Fatou's Lemma). If $<f_n>$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E then

$$\int_E f d\mu \leq \underline{\lim} \int_E f_n d\mu .$$

Proof: See Royden, page 72.

Theorem 4.10. Let $f \varepsilon H^\infty$; then

$$\log|f(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})| dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f^*(e^{it})| dt;$$

and if f is such that $0<m<|f(z)| < M < \infty$ for each $z \varepsilon U$ then

$$\log|f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f^*(e^{it})| dt$$

Proof: By Jensen's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})| dt = \sum_1^N \log \frac{r}{|a_n|} + \log|f(0)| \quad \text{for } 0<r<1$$

But $|a_n|<r$ implies $\frac{r}{|a_n|} > 1$ and $\log \frac{r}{|a_n|} > 0$ therefore

$$\log|f(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})| dt. \tag{4.10.1}$$

Again by Jensen's theorem and $\frac{r}{|a_n|} > 1$ the integral in the right member of 4.10.1 is a non-decreasing function of r. Without loss of generality, assume $|f| \leq 1$ so $\frac{1}{|f|} \geq 1$ and $\log \frac{1}{|f|} \geq 0$. By Fatou's Lemma

$$\int_{-\pi}^{\pi} \log \frac{1}{|f^*(e^{it})|}\, dt \le \lim_{r\to 1} \int_{-\pi}^{\pi} \log \frac{1}{|f(re^{it})|}\, dt$$

This implies

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log|f(re^{it})|\,dt \le \int_{-\pi}^{\pi} \log|f^*(e^{it})|\,dt$$

and because the integral in the left member is a non-decreasing function of r

$$\int_{-\pi}^{\pi} \log|f(re^{it})|\,dt \le \int_{-\pi}^{\pi} \log|f^*(e^{it})|\,dt \text{ for } 0<r<1.$$

Now consider the second part of the conclusion of this theorem. If $0<m<|f(z)|<M$, then f is non-zero in U and as was shown in Jensen's theorem, $\log|f|$ is harmonic. Therefore

$$\log|f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})|\,dt \text{ for } 0<r<1.$$

Now

$$|\log|f(re^{it})|| \le \max\{|\log m|, |\log M|\}$$

and

$$\lim_{r\to 1} (\log|f(re^{it})|) = \log|f^*(e^{it})|$$

for almost all $t\varepsilon[-\pi,\pi]$. By the Lebesque dominated convergence theorem

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log|f(re^{it})|\,dt = \int_{-\pi}^{\pi} \log|f^*(e^{it})|\,dt$$

and the conclusion follows.

Theorem 4.11. If B(z) is a Blaschke product then $|B^*(e^{i\theta})| = 1$ for almost all $\theta\varepsilon[-\pi,\pi]$.

Proof: If B is a finite Blaschke product, then it is analytic at all points of $\overline{U}$ and each factor has modulus 1 on C. Therefore in this case B has modulus 1 everywhere on C.

If B is an infinite Blaschke product, arrange $\{a_n\}$ such that $0<a_n \leq a_{n+1}$ for n = 1, 2, ... It has been shown in Theorem 3.7 that $\prod_1^\infty |a_n|>0$ which implies $\lim_{n\to\infty}|a_n| = 1$. Consider

$$B_N(z) = \prod_N^\infty \frac{a_n - z}{1-\bar{a}_n z} \frac{|a_n|}{a_n}$$

Then $\frac{B(z)}{B_N(z)}$ is a finite Blaschke product. Therefore according to the statements about finite Blaschke products in the first paragraph of this proof

$$\lim_{r\to 1} \left[\int_{-\pi}^{\pi} \log|B(re^{it})|dt - \int_{-\pi}^{\pi} \log|B_N(e^{it})|dt\right] = \lim_{r\to 1} \int_{-\pi}^{\pi} \log\left|\frac{B(re^{it})}{B_N(re^{it})}\right| dt$$

$$= \int_{-\pi}^{\pi} \log\left|\lim_{r\to 1} \left(\frac{B(re^{it})}{B_N(re^{it})}\right)\right| dt = \int_{-\pi}^{\pi} \log 1 \, dt = 0.$$

Using this result and Theorem 4.10 gives

$$\log|B_N(0)| \leq \lim_{r\to 1} \int_{-\pi}^{\pi} \log|B_N(re^{it})|dt$$

$$= \lim_{r\to 1} \int_{-\pi}^{\pi} \log|B(re^{it})|dt \leq \int_{-\pi}^{\pi} \log|B^*(e^{it})|dt$$

Since $|B(z)| \leq 1$ the same is true of B* and the last integral is less than or equal to zero. The condition that $\lim_{n\to\infty}|a_n| = 1$ implies

$$\lim_{N\to\infty} \log|B_N(0)| = 0$$

Therefore

$$\int_{-\pi}^{\pi} \log|B^*(e^{it})|dt = 0$$

and this coupled with the fact $\log|B^*(e^{it})| \leq 0$ (at least almost everywhere) implies

$$\log|B^*(e^{it})| = 0$$

almost everywhere and

$$|B^*(e^{it})| = 1$$

almost everywhere.

The following theorem and its corollary establish a uniqueness theorem of F. and M. Reisz. The important result is that two distinct functions from the class of bounded analytic functions on U can have their Fatou boundary functions agreeing on at most a set of linear measure zero.

Theorem 4.12. Let $f \epsilon H^\infty$ and $f^*(e^{i\theta}) = 0$ on a set of positive measure then f is identically zero on U.

Proof: If f is not identically zero, then without loss of generality it can be assumed $f(0) \neq 0$, for if this is not the case, consider $\frac{f(z)}{z^k}$ where k is the order of the zero at z = 0. By Theorem 4.10 and this assumption

$$-\infty < \log|f(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f^*(e^{it})|dt \tag{4.12.1}$$

Under the hypothesis there is a set E of positive measure such that $f^*(e^{i\theta}) = 0$ for $\theta \epsilon E \subset [-\pi,\pi]$ so

$$\int_E \log|f^*(e^{it})|dt = -\infty$$

Also f bounded implies f* is bounded at least almost everywhere so

$$\int_{[-\pi,\pi]\setminus E} \log|f^*(e^{it})|dt < \infty$$

Therefore

$$\int_{-\pi}^{\pi} \log|f^*(e^{it})|dt = \int_{[-\pi,\pi]\setminus E} \log|f^*(e^{it})|dt + \int_E \log|f^*(e^{it})|dt = -\infty$$

which contradicts Statement 4.12.1. So $f \equiv 0$ on U.

Corollary 4.13. Let $f, g \in H^\infty$ with $f^* = g^*$ on a set of positive measure; then $f(z) = g(z)$ for all $z \in U$.

Proof: $f-g$ belongs to H^∞ and $f^*-g^*=0$ on a set of positive measure. By the previous theorem $f(z)-g(z)=0$ for $z \in U$.

The remaining portion of this chapter makes use of the Fatou Boundary Function to establish some interesting results regarding the cluster set of a function along an arc.

Consider again the function

$$f(z) = e^{\frac{z+1}{z-1}}$$

and recall that f approached zero within any approach between two chords at $z = 1$ even though $C(f,1) = \overline{U}$. This can be expressed another way, as follows: f approaches zero along any Jordan curve in U which terminates at $z = 1$ and is not tangent to C. It may be wondered if this is a special case, and the answer is no. It will be shown that a similar result holds for any function belonging to H^∞. That is, if $f \in H^\infty$ and has a radial limit at some point, then it has this limit along any non-tangential approach. This result requires a number of preliminary results which will be developed now.

Lemma 4.14. Consider the circle in Figure 7. There exists a function $\phi(z)$ such that

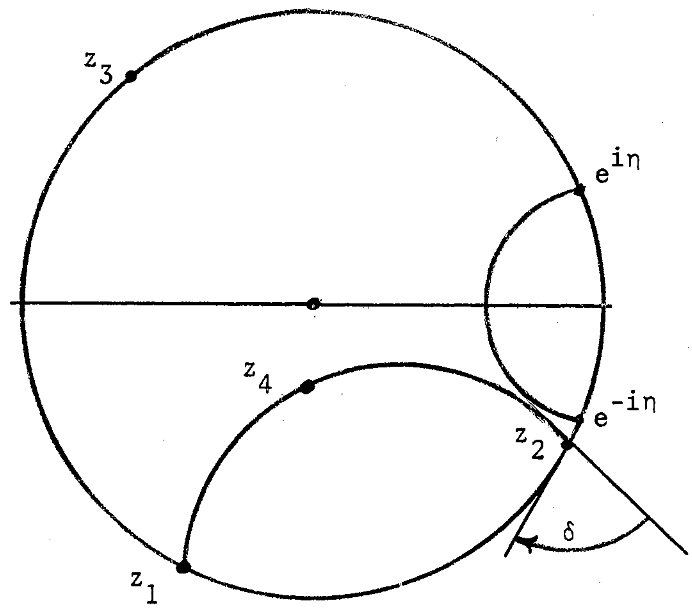

Figure 7. First Illustration for Lemma 4.14.

i) ϕ is a non-zero analytic function in the open disk which has the circle as its boundary.

ii) $|\phi|$ is continuous at all points of the closed disk except z_1 and z_2.

iii) $|\phi(z)| = 1$ on the open arc $\widehat{z_1 z_3 z_2}$.

iv) $|\phi(z)| = k>0$ on the open arc $\widehat{z_1 z_2}$.

Proof: A function g will be constructed which is analytic in the open disk and such that $Re(g) = u$ is continuous at all points of the closed disk except at z_1 and z_2. In addition $u(z) = 0$ on $\widehat{z_1 z_3 z_2}$

$$u(z) = \log k$$

on $\widehat{z_1 z_2}$ and $u(z)<\log k$ on $\widehat{z_1 z_4 z_2}$. Let $\phi(z) = e^{g(z)}$ then

$$|\phi(z)| = e^{u(z)}$$

and the four desired results follow at once.

Without any loss in generality, the circle will be assumed to be the unit circle C. Define

$$g(e^{it}) = \begin{cases} 0 \text{ on arc } \widehat{z_1 z_3 z_2} \\ \\ \log k \text{ on arc } \widehat{z_1 z_2} \end{cases} \quad \text{and}$$

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})\frac{e^{it}+z}{e^{it}-z}\, dt \text{ for } z\varepsilon U$$

Let $z_1 = e^{i\alpha_1}$ and $z_2 = e^{i\alpha_2}$ where $-\pi<\alpha_1<\alpha_2<\pi$, then

$$g(z) = \frac{\log k}{2\pi}\int_{\alpha_1}^{\alpha_2} \frac{e^{it}+z}{e^{it}-z}\, dt.$$

Now consider a fixed $z_\circ \varepsilon U$. Then there exists a $\eta>0$ and a neighborhood $N(z_\circ)$ such that $|z-e^{it}|>\eta$ for $z \varepsilon N(z_\circ)$ and $e^{it}\varepsilon C$. Let $(z_\circ+h) \varepsilon N(z_\circ)$ then

$$g(z_\circ+h)-g(z_\circ) = \frac{\log k}{2\pi}\int_{\alpha_1}^{\alpha_2} \frac{h(e^{it}+z_\circ)}{(e^{it}-z_\circ)(e^{it}-(z_\circ+h))}\, dt.$$

Now $$\frac{h(e^{it}+z_\circ)}{(e^{it}-z_\circ)(e^{it}-(z_\circ+h))} = \frac{h(e^{it}+z_\circ)}{(e^{it}-z_\circ)^2} + \frac{h^2(e^{it}+z_\circ)}{(e^{it}-z_\circ)^2(e^{it}-(z_\circ+h)}$$

implies

$$\frac{g(z_\circ+h)-g(z_\circ)}{h} = \frac{\log k}{2\pi}\left[\int_{\alpha_1}^{\alpha_2} \frac{e^{it}+z_\circ}{(e^{it}-z_\circ)^2}\, dt + \int_{\alpha_1}^{\alpha_2} \frac{h(e^{it}+z_\circ)}{(e^{it}-z_\circ)^2(e^{it}-(z_\circ+h)}dt\right].$$

Note that the absolute value of this last integral is less than

$$\frac{\eta|h|(\alpha_2-\alpha_1)\log k}{2\pi\eta^3} = \frac{|h|(\alpha_2-\alpha_1)\log k}{2\pi\eta^2}$$

which approaches zero as h approaches zero. Therefore

$$g'(z_0) = \frac{\log k}{2\pi} \int_{\alpha_1}^{\alpha_2} \frac{e^{it}+z_0}{(e^{it}-z_0)^2}\, dt$$

so g is analytic for $z_0 \varepsilon U$.

It has been shown in Theorem 4.1 that

$$\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}} = \frac{1-r^2+2irsin(\theta-t)}{1+r^2-2rcos(\theta-t)}$$

therefore

$$u(r,\theta) = Re(g(re^{i\theta})) = \frac{\log k}{2\pi} \int_{\alpha_1}^{\alpha_2} \frac{1-r^2}{1+r^2-2rcos(\theta-t)}\, dt.$$

Since u is harmonic in U, all that remains to be shown is that $u(r,\theta)$ is continuous as a function of two variables at each point $e^{it} \neq z_1, z_2$ and equals either 0 or log k as e^{it} lies on $\widehat{z_1 z_3 z_2}$ or $\widehat{z_1 z_2}$ respectively.

For case one consider a point $e^{i\theta_0}$ on the open arc $\widehat{z_1 z_3 z_2}$. No generality will be lost if it is assumed that $\theta_0 = 0$. Since it is an open arc, there exists a $\eta > 0$ such that $e^{i\theta} \varepsilon \widehat{z_1 z_3 z_2}$ for $|\theta| < \eta$. Now consider $re^{i\theta} \varepsilon N^*_\varepsilon(1) \subset N^*_\eta(1)$ (see Figure 7), then

(4.14.1) $$u(r,\theta) - 0 = \frac{\log k}{2\pi} \int_{\alpha_1}^{\alpha_2} \frac{1-r^2}{1+r^2-2rcos(\theta-t)}\, dt.$$

Now $\alpha_1 < t < \alpha_2$ and $|\theta| < \eta$ implies $|\theta - t| > \eta$ and

$$1+r^2-2rcos(\theta-t) > 1+r^2-2rcos\eta > sin^2\eta$$

therefore

$$|u(r,\theta)-0| < \frac{(\alpha_2-\alpha_1)(1-r^2)\log k}{2\pi\ sin^2\eta} < \frac{\log k(\alpha_2-\alpha)(1-r)}{2\pi sin^2\eta} < \frac{(\alpha_2-\alpha_1)\log k}{2\pi sin^2\eta}\varepsilon$$

This implies $u(r,\theta)$ is continuous at each point of the arc $\widehat{z_1 z_3 z_2}$ and equals 0 there.

For the second case consider a point $e^{i\theta_o}$ on the open arc $\widehat{z_1z_2}$ and again, for convenience sake but without loss of generality, let $e^{i\theta_o} = 1$ (i.e. $\theta_o = 0$). Now there exists a $\eta>0$ such that $|\theta| < \eta$ implies $e^{i\theta}\varepsilon\widehat{z_1z_2}$. Note this assumption implies $\alpha_1<0<\alpha_2$ and recall that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\,dt = 1$$

So for $re^{i\theta}\varepsilon\ N^*_\varepsilon(1)\subset N^*_\eta(1)$

$$\log k - u(r,\theta) = \frac{\log k}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}dt - \frac{\log k}{2\pi}\int_{\alpha_1}^{\alpha_2}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}dt$$

$$(4.14.2)\quad \log k-u(r,\theta) = \frac{\log k}{2\pi}\left[\int_{-\pi}^{\alpha_1}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}dt+\int_{\alpha_2}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}dt\right].$$

By the same type reasoning as in case 1

$$|\log k-u(r,\theta)| < \frac{\log k(\alpha_1+\pi)(1-r^2)}{2\pi\sin^2\eta} + \frac{\log k(\pi-\alpha_2)(1-r^2)}{2\pi\sin^2\eta}$$

$$|\log k-u(r,\theta)| < \frac{\log k(2\pi+\alpha_1-\alpha_2)}{2\pi\sin^2\eta}\,\varepsilon$$

which approaches zero as ε approaches zero.

<u>Lemma 4.15</u>. Consider the function ϕ of Lemma 4.14 and the arc $\widehat{z_1z_4z_2}$ lying in the disk; then $|\phi(z)| = k^{\delta/\pi}$ for $z\varepsilon\widehat{z_1z_4z_2}$ where δ is the angle between the arc $\widehat{z_1z_4z_2}$ and the circle as shown in Figure 7. Also $k<|\phi(z)|<1$ at each point z in the disk if $0<k<1$.

<u>Proof</u>: Without loss of generality, let $\alpha_1=0$ so that $z_1=1$. Let $z=re^{i\theta}$ lie on arc $\widehat{z_1z_4z_2}$ and e^{it}, $e^{i(t+\Delta t)}$ both lie on arc $\widehat{z_1z_2}$(i.e. $\alpha_1=0<t<\alpha_2$

and $0<t+\Delta t<\alpha_2$). Take note that on the unit circle the length of an arc is the same as the central angle at the arc. Therefore $\widehat{z_1 e^{it}}$ and $\widehat{z_1 e^{i(t+\Delta t)}}$ are arcs of length t and t+Δt respectively. Let ω and ω+Δω be the lengths of $\widehat{z_1' e^{it'}}$ and $\widehat{z_1' e^{i(t+\Delta t)'}}$ where z_1', $e^{it'}$ and $e^{i(t+\Delta t)'}$ are endpoints of chords as shown in Figure 8. Now consider triangles $\widehat{e^{it} z e^{i(t+\Delta t)}}$ and $\widehat{e^{it'} z e^{i(t+\Delta t)'}}$ which are similar and let $|e^{it}-z| = p$ and $|e^{it'}-z| = q$. Then

$$(4.15.1) \qquad \frac{\Delta\omega}{\Delta t} \simeq \frac{q}{p} \text{ so } \frac{d\omega}{dt} = \frac{q}{p}.$$

Now

$$p^2 = |e^{it}-z|^2 = |e^{it}-re^{i\theta}|^2 = 1-2r\cos(\theta-t)+r^2$$

and similar triangles as shown in Figure 8 imply

$$pq = (1-r)(1+r) = 1-r^2.$$

So

$$\frac{q}{p} = \frac{pq}{p^2} = \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}$$

Combine this result with 4.15.1 to find

$$\frac{d\omega}{dt} = \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}$$

Therefore

$$(4.15.2) \quad \frac{1}{2\pi}\int_{\alpha_1=0}^{\alpha_2} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\,dt = \frac{\omega(t)}{2\pi}\Bigg|_{t=0}^{t=\alpha_2} = \frac{\omega(\alpha_2)}{2\pi} - 0$$

Note here that $\omega(\alpha_2)$ is the length of arc $\widehat{z_1 z_2}$.

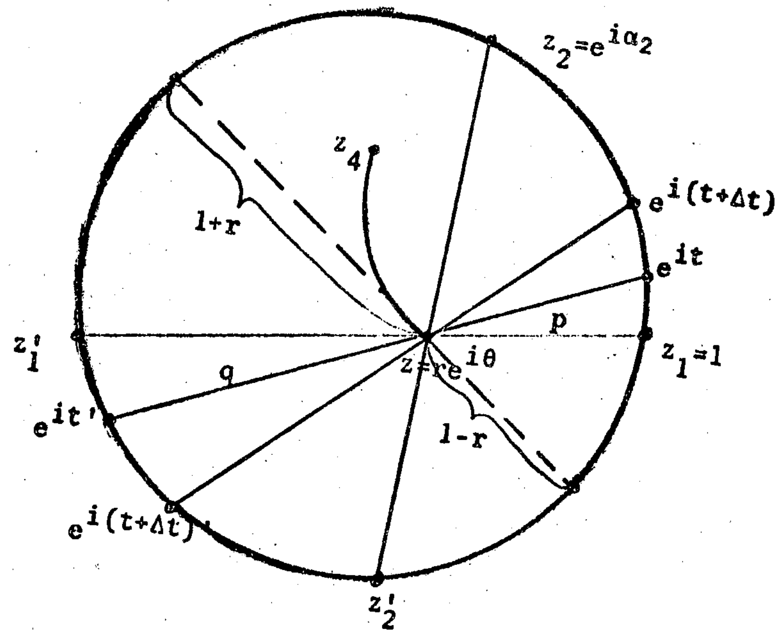

Figure 8

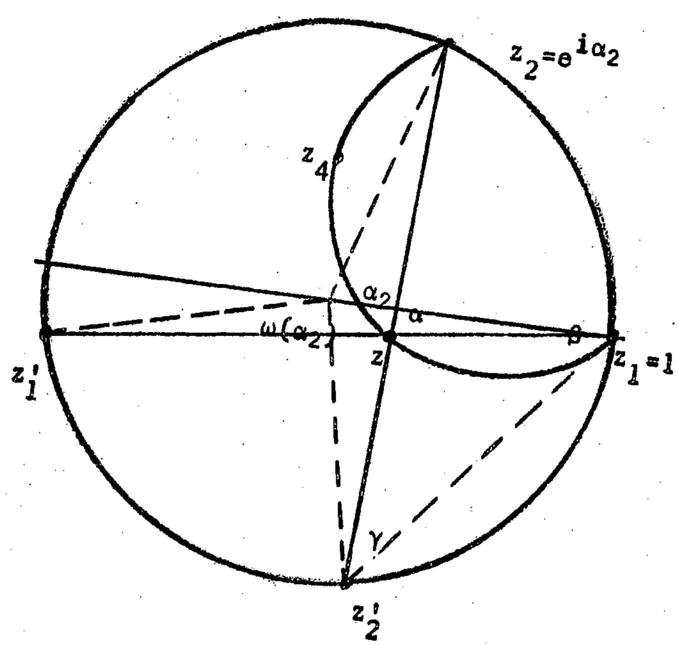

Figure 9

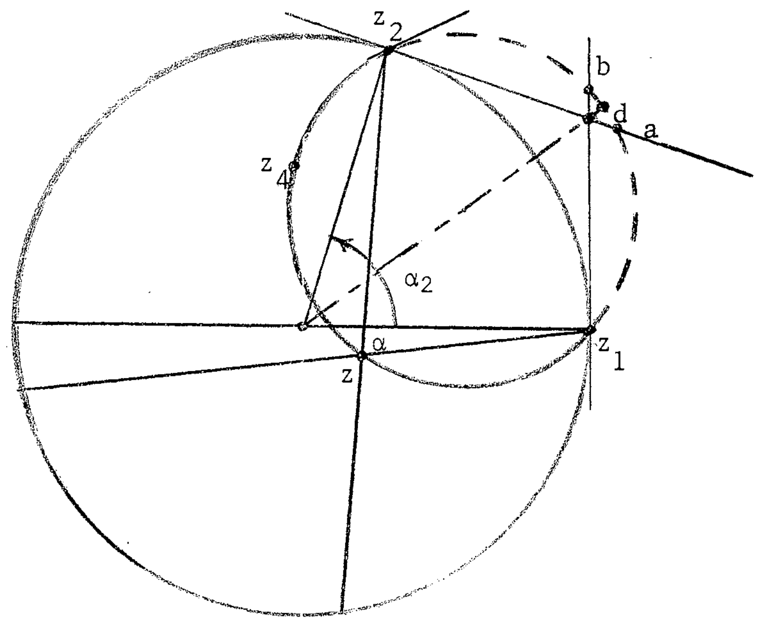

Figure 10

Now recall from plane geometry that an inscribed angle subtends an arc twice its size in angular measure. From Figure 9 it is seen that

$$\alpha_2 = 2\gamma, \ \omega(\alpha_2) = 2\beta$$

and $\alpha = \gamma+\beta$. In Figure 10 it is seen that arcs $\widehat{az_2}$, $\widehat{bz_2}$ and $\widehat{z_1dz_2}$ have lengths 2δ, α_2 and 2α respectively. Therefore

$$\alpha_2 = 2\alpha - 2\delta.$$

These results yield

(4.15.3) $\omega(\alpha_2) = 2\beta = 2(\alpha-\delta) = 2\alpha-2\gamma = 2\alpha-\alpha_2 = 2\alpha-(2\alpha-2\delta) = 2\delta$

Now Statements 4.15.2 and 4.15.3 give

$$(4.15.4)\qquad \frac{1}{2\pi}\int_0^{\alpha_2} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}dt = \frac{2\delta}{2\pi} = \frac{\delta}{\pi} \text{ for } re^{i\theta}\varepsilon\widehat{z_1z_4z_2}.$$

Therefore $u(r,\theta) = \frac{\delta\log k}{\pi}$ for all $z=re^{i\theta}\varepsilon\widehat{z_1z_4z_2}$ which implies u is constant on this arc. Thus

$$|\phi(z)| = e^u = e^{\frac{\delta\log k}{\pi}} = k^{\frac{\delta}{\pi}} \text{ for } z\varepsilon\widehat{z_1z_4z_2}$$

It needs to be noted that

$$\frac{1-r^2}{1+r^2-2r\cos(\theta-t)} \geq \frac{1-r^2}{1+r^2} \geq 0 \text{ for } 0\leq r\leq 1$$

Let $0<k<1$ and refer to 4.14.1 and 4.14.2 which give

$$u(r,\theta)<0$$

and

$$\log k - u(r,\theta)<0$$

Therefore

$$\log k < u(r,\theta) < 0 \text{ so}$$

$$k < |\phi| = e^u < 1$$

and the proof of this lemma is complete.

Lemma 4.16. Let f be bounded and analytic in a Jordan domain Ω and at all points of the boundary Γ except possibly a set of linear measure zero. Let g be a non-zero analytic function Ω with $0<m<g(z)<M$ for $z\varepsilon\Omega$. Also let $|g|$ be continuous at all points of $\overline{\Omega} = \Omega \cup \Gamma$ except possibly a set of linear measure zero on Γ. If $|g(z)|\geq|f(z)$ almost everywhere on Γ then $|g(z)|\geq|f(z)|$ everywhere in Ω.

Proof: Since Ω is a Jordan domain, it is conformally equivalent to the unit circle U and the conformal mapping can be extended to a homeomorphism of $\overline{U}$ onto $\overline{\Omega}$, which maps a set of linear measure zero of C onto a set of linear measure zero on Γ. In addition, if $z_1 \epsilon \Omega$, then the mapping may be chosen so it maps 0 onto z_1. Let $\psi: \overline{U} \to \overline{\Omega}$ be this function and define

$$f_1(z) = f(\psi(z))$$

$$g_1(z) = g(\psi(z))$$

Then all the conditions of the hypothesis apply to f_1 and g_1 with respect to the Jordan domain U. By Theorem 4.10

$$\log|g(z_1)| = \log|g_1(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g^*(e^{i\theta})|d\theta$$

and

$$\log|f(z_1)| = \log|f_1(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^*(e^{i\theta})|d\theta$$

Now f_1 is analytic almost everywhere on C and $|g_1|$ is continuous at all points of $U \cup C$ except possibly a set of linear measure zero on C. Therefore

$$f^*(e^{i\theta}) = f(e^{i\theta})$$

almost everywhere on C and

$$|g(e^{i\theta})| = |g^*(e^{i\theta})|$$

almost everywhere on C which implies from the hypothesis that

$$|f^*(e^{i\theta})| \leq |g^*(e^{i\theta})|$$

almost everywhere on C. Thus

$$(4.16.1)\quad \log|g(z_1)|-\log|f(z_1)|\geq \frac{1}{2\pi}\int_{-\pi}^{\pi}[\log|g^*(e^{i\theta})|-\log|f^*(e^{i\theta})|]d\theta$$

Since $|f^*| \leq |g^*|$ almost everywhere on C $\log|g^*| - \log|f^*|\geq 0$ almost everywhere on C. Therefore the integral in 4.16.1 is non-negative and so $|g(z_1)|\geq|f(z_1)|$. Since $z_1 \varepsilon \Omega$ is arbitrary, the conclusion of the theorem follows at once.

The next theorem, mentioned previously, shows that a function f belonging to H^∞ can have at most one non-tangential asymptotic value.

Theorem 4.17. Let $f \varepsilon H^\infty$, $z_\circ \varepsilon C$, and $Az_\circ$, $Bz_\circ$ be distinct chords at $z_\circ$, each of which makes an angle of $\frac{\pi-\varepsilon}{2}$ with the radius at $z_\circ$. Also let Δ_ε be the set of all points interior to the angle $Az_\circ B$ which lie in U. If γ is a Jordan arc such that $\gamma \subset \Delta_\varepsilon$ and $C_\gamma(f,z_\circ)$ is degenerate, then $C_{\Delta_\varepsilon}(f,z_\circ) = C_\gamma(f,z_\circ)$ (i.e. f has a unique limit as f approaches $z_\circ$ within Δ_ε).

Proof: It will be sufficient to complete the proof for $z_\circ = 1$ because $e^{i\theta}f(z)$ will always give a rotation to this situation if $\theta \neq 0$. Also without any loss of generality, it will be assumed $C_\gamma(f,1) = \{0\}$ because $f(z) - a$ can be considered if $C_\gamma(f,1) = \{a\}$. In addition assume

$$|f(z)| \leq 1 \text{ on } U.$$

Let A_1 and B_1 be midpoints of the chords $Az_\circ$ and $Bz_\circ$ respectively and construct circular arcs $\widehat{z_\circ A_1 B}$ and $\widehat{z_\circ B_1 A}$ as shown in Figure 11. Let δ be the angle each of these arcs makes with the unit circle. For each $\eta>0$ there exists a $k>0$ such that

$$(4.17.1)\qquad k^{\delta/\pi} < \eta$$

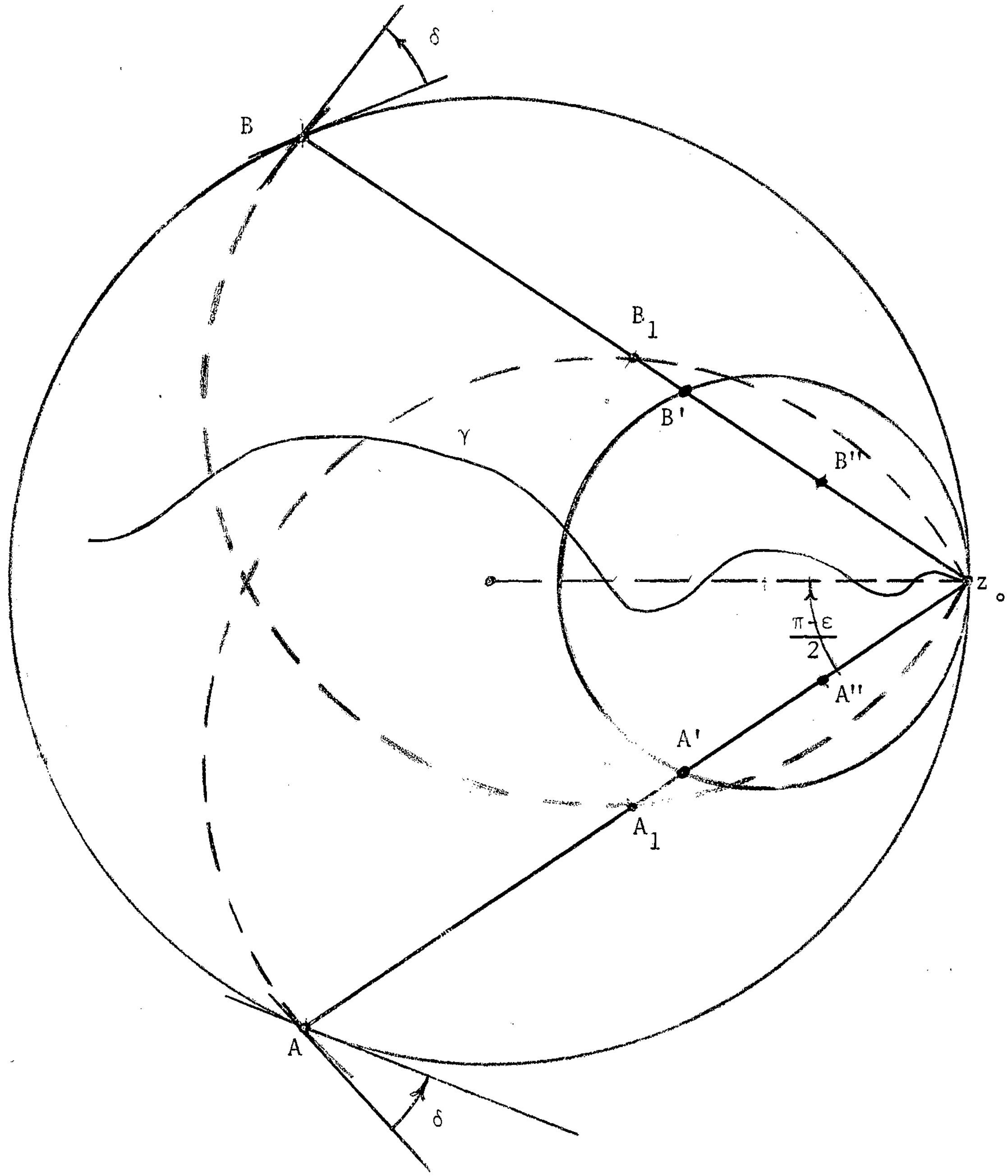

Figure 11

Also for each $\eta>0$ there exists a circle D internally tangent to C at $z = 1$ such that

(4.17.2) $$|f(z)| < k \text{ for } z\varepsilon\gamma \text{ and } z\varepsilon\Omega \cup D$$

where Ω is the open disk bounded by D. Let A' and B' be the points of intersection of Ω with the two chords and A", B" be the midpoints of $A'z_o$ and $B'z_o$ (see Figure 11). Now construct circular arcs $z_oA''B'$ and $z_oB''A'$. These arcs meet Ω in an angle of size δ (see Figure 12).

By Lemma 4.14 and 4.15 there exists a function $\phi_1(z)$ analytic in Ω with

(4.17.3) $$k < |\phi_1(z)| < 1 \text{ for } z\varepsilon\Omega$$

(4.17.4) $$|\phi_1(z)| = k \text{ on } \widehat{z_oB'}$$

(4.17.5) $$|\phi_1(z)| = 1 \text{ on } \widehat{z_oA'B'}$$

(4.17.6) $$|\phi_1(z)| = k^{\delta/\pi} \text{ on } \widehat{z_oA''B'}$$

Similarly there exists a function ϕ_2 analytic in Ω such that

(4.17.7) $$k < |\phi_2(z)| < 1 \text{ for } z\varepsilon\Omega$$

(4.17.8) $$|\phi_2(z)| = k \text{ on } \widehat{z_oA'}$$

(4.17.9) $$|\phi_2(z)| = 1 \text{ on } \widehat{z_oA''B'}$$

(4.17.10) $$|\phi_2(z)| = k^{\delta/\pi} \text{ on } \widehat{z_oB''A'}.$$

γ divides Ω into two domains G_1 and G_2 (see Figure 13) each of which is bounded by a Jordan curve Γ_1 and Γ_2. Statements 4.17.1, 4.17.3 and 4.17.5 together with $|f(z)| \leq 1$ in U imply

$$|f(z)| \leq |\phi_1(z)| \text{ for all } z\varepsilon\Gamma_1.$$

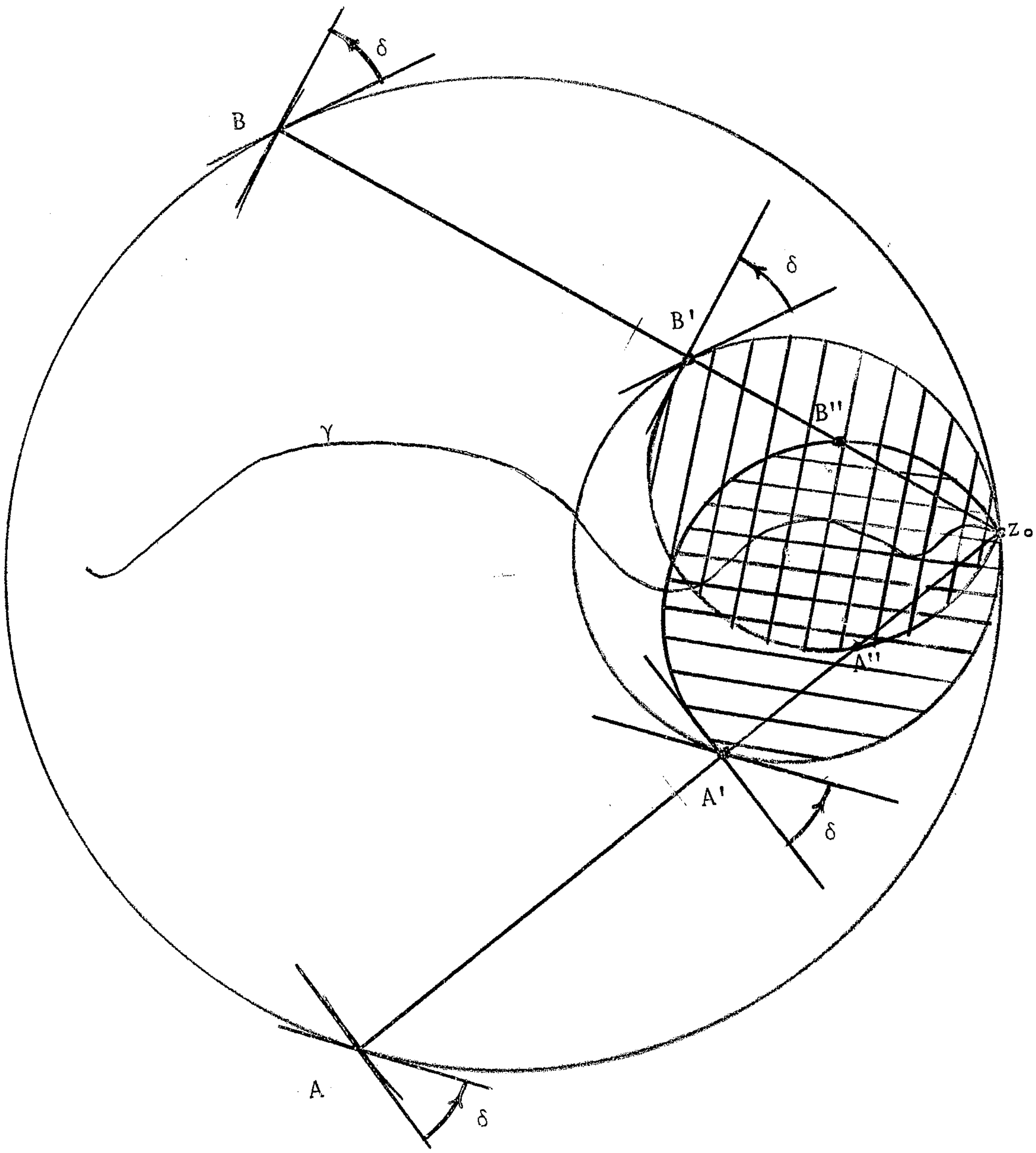

Figure 12

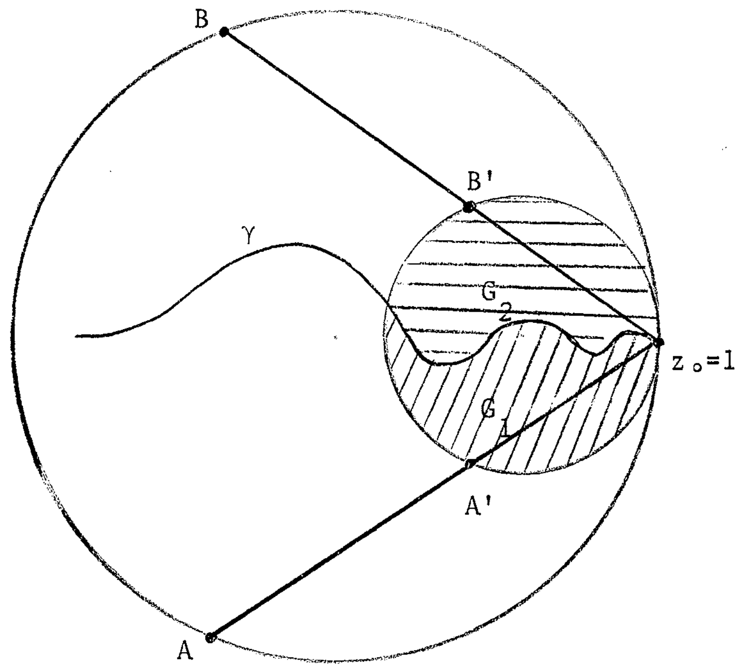

Figure 13

The hypothesis for Lemma 4.16 is met and so

(4.17.11) $|f(z)| \leq |\phi_1(z)|$ for $z \varepsilon G_1$ and similarly

(4.17.12) $|f(z)| \leq |\phi_2(z)|$ for $z \varepsilon G_2$

Consider the lens shaped region which passes through points z_o, A" and B'. Statements 4.17.4 and 4.17.6 imply

$$k < |\phi_1(z)| < k^{\delta/\pi}$$

for z in this region and similarly $k < |\phi_2(z)| < k^{\delta/\pi}$.

Now 4.17.11 and 4.17.12 together with this result gives

$$|f(z)| \leq \max\{|\phi_1(z)|, |\phi_2(z)|\} < k^{\delta/\pi} < \eta$$

for z in this lens shaped region. This lens shaped region's intersection with Δ_ε forms a neighborhood of $z = 1$ relative to Δ_ε; therefore since ε is arbitrary, f approaches zero as z approaches $z = 1$ within Δ_ε.

It should be pointed out that the proof of Theorem 4.17 was dependent on γ being a non-tangential Jordan curve. If γ had been tangent to C at $z = 1$, then either G_1 or G_2 would not be a Jordan domain.

It should be pointed out that although the angular domain Δ_ε was taken to be symmetric, the theorem implies the existence of a unique limit within any angular approach to z_o. This is clear since any non-symmetric angular domain at z_o would always be contained in some Δ_ε. When a function does approach a unique value at z_o within any angular approach, it is said to have that value as an angular limit.

Since a function of H^∞ has a radial limit at almost every point of C, it must also necessarily have an angular limit at almost every point.

The following theorem extends these ideas to show that if a function belonging to H^∞ has an asymptotic value along a circular arc tangent to C at z_o, then the function must have the same value as an angular limit.

<u>Theorem 4.18</u>. Let $f \varepsilon H^\infty$ and Γ be a circle internally tangent to C at z_o. If f approaches the limit w along an arc $\overset{\frown}{B'z_o}$ of Γ then f has the angular limit w at z_o.

<u>Proof</u>: If f is identically zero there is nothing to show. So assume f is not identically zero. Without loss of generality assume

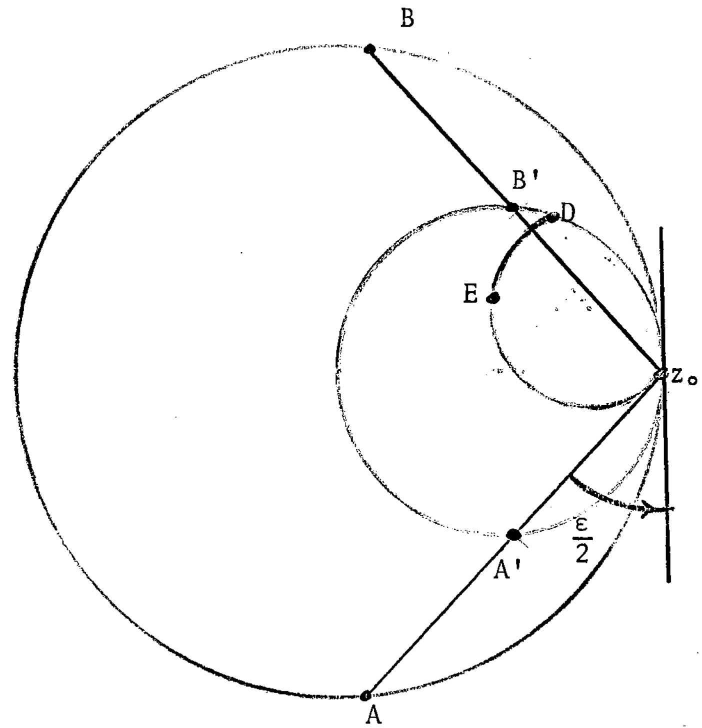

Figure 14

$$w = 0, \; z_o = 1$$

and f is bounded by one. Let Az_o and Bz_o be the chords on the boundary of Δ_ε (see the statement of Theorem 4.17 for the definition of Δ_ε). Let A' and B' be the points of intersection of the chords Az_o and Bz_o with Γ. Let D be any point on $\widehat{z_oB'}$ different from z_o. Now construct a circular arc in U which is tangent to the chord $A'z_o$ at z_o such that it also passes through the point D (see Figure 14). Let E be a point on this arc as shown in the figure. Now define

$$k = \sup_{z \in \widehat{z_o D}} |f(z)| \tag{4.18.1}$$

k is non-zero since $z_o \neq D$ and f is not identically zero. By Lemmas

4.14 and 4.15 there exists a function ϕ analytic in the interior of Γ with

(4.18.2) $|\phi(z)| = 1$ on $\widehat{z_0A'D}$, $|\phi(z)| = k$ on Dz_0 and $|\phi(z)| = k^{\varepsilon/2\pi}$

Notice that the arc $\widehat{z_0ED}$ makes an angle of $\varepsilon/2$ with Γ since it is tangent to the chord Az_0. Now 4.18.2 implies $|\phi(z)| < k^{\varepsilon/2\pi}$ in the lens shaped region bounded by the two arcs $\widehat{z_0ED}$ and $\widehat{Dz_0}$. Recall that f is bounded by one. This fact together with 4.18.1 implies

$$|f(z)| \leq |\phi(z)|$$

everywhere on Γ except z_0 and D. Lemma 4.16 then implies

$$|f(z)| \leq |\phi(z)|$$

in the interior of Γ. In the lens shaped region mentioned above,

(4.18.3) $$|f(z)| < k^{\varepsilon/2\pi}$$

By hypothesis f approaches zero as z approaches z_0 on the arc $\widehat{z_0B'}$. Therefore as D approaches z_0, k approaches zero and so does $k^{\varepsilon/2\pi}$. This together with 4.18.3 gives $C_{\Delta_\varepsilon}(f,z_0) = \{0\}$. Since ε was arbitrary f has the angular limit zero at z_0.

It should be noted that an even stronger result was obtained. It was shown that f approaches zero from within the region bounded by the chord $A'z_0$ and the circular arc $A'B'z_0$.

Theorems 4.17 and 4.18 imply the existence of the following theorem which will not be proven in detail here. This theorem is the more general form of what has become known as Lindelöf's Theorem.

Theorem 4.19 (lindelöf's Theorem). Let f belong to H^∞ and approach limit w along some arc γ lying in U and terminating at a point z_0 of C. Then f has the angular limit w at z_0.

In the notation of Chapter II, if f belongs to H^∞ then $A(f,z_0)$ is at most degenerate. In other words, f can have at most one asymptotic value at a given point of C.

It was seen in detail that $f(z) = e^{\frac{z+1}{z-1}}$ had an angular limit zero at $z = 1$ while $C(f,1) = \overline{U}$. Also an example of a Blaschke product was given which possessed a radial limit of zero at $z = 1$. It will be shown in a later chapter that its total cluster set at $z = 1$ is far from being degenerate, just as was the case with f.

The following existence theorem is in some ways surprising, and coupled with the remarks in the last paragraph, it is even more so. It establishes that for any function defined on U and any point $e^{i\theta}\varepsilon C$, there always exists a simple arc along which the cluster set of f is equal to the complete cluster set of the function at $e^{i\theta}$. For $f(z) = e^{\frac{z+1}{z-1}}$ there exists an arc γ such that $C_\gamma(f,1) = C(f,1)$ but γ must be a tangential arc, for by Lindelöf's Theorem, f must approach zero on any non-tangential arc.

Theorem 4.20. Let f be any function defined on U and $e^{i\theta}\varepsilon C$; then there exists a simple arc γ lying in U and terminating at $e^{i\theta}$ such that

$$C_\gamma(f,e^{i\theta}) = C(f,e^{i\theta}).$$

Proof: Since it is always the case that $C_\gamma(f,e^{i\theta}) \subset C(f,e^{i\theta})$, it only needs to be shown that $C(f,e^{i\theta}) \subset C_\gamma(f,e^{i\theta})$. This will be done by choosing a sequence of points $\{z_n\}$ such that

$$\lim_{n\to\infty} z_n = e^{i\theta}$$

and for $w\varepsilon C(f,e^{i\theta})$, w is an accumulation point of $\{f(z_n)\}$. In addition $\{z_n\}$ is so constructed that $|z_n| < |z_{n+1}|$ for all n; so join the points of $\{z_n\}$ by line segments in a sequential manner and let this be γ.

Consider f as a mapping of U onto the Riemann sphere. Let T_n denote a triangulation of the sphere with each of its triangles having spherical diameter less than 1/3n. Let $T_{n1}, \ldots, T_{nm_n}$ denote the triangles of T_n each of which meet $C(f,e^{i\theta})$ either in its interior or on its boundary. Now consider what is the parallel set of each T_{ni} defined by

$$S_{ni} = \bigcup_{z\varepsilon T_{ni}} \left(N_{1/3n}(z)\right) \text{ (see Figure 15)}$$

Now $\overline{T}_{ni} \subset S_{ni}$ and for each $z_1, z_2 \varepsilon S_{ni}$ there exists $z', z'' \varepsilon T_{ni}$ such that

(4.20.1) $$|z_1 - z'| < 1/3n \text{ and } |z_2 - z''| < 1/3n$$

Now $z', z'' \varepsilon T_{ni}$ implies

$$|z' - z''| < 1/3n$$

therefore

$$|z_1 - z_2| \leq |z_1 - z'| + |z' - z''| + |z'' - z_2| < 1/n$$

Now order the parallel sets as follows

$$S_{11}, S_{12}, \ldots, S_{1m_1}, S_{21}, \ldots, S_{2m_2}, \ldots, S_{n1}, \ldots, S_{nm_n}, \ldots$$

and denote this infinite sequence by re-subscripting as $\{S_j\}$.

$\overline{T}_{11} \subset S_1$ and $\overline{T}_{11} \cap C(f,e^{i\theta}) \neq \phi$ so there exists $z_1 \varepsilon N_1^*(e^{i\theta})$ such that $f(z_1) \varepsilon S_1$. Now let $r_2 < \min(1/2, |e^{i\theta}-z_1|)$; then there exists $z_2 \varepsilon N_{r_2}^*(e^{i\theta})$ such that $f(z_2) \varepsilon S_2$. Similarly for $r_n < \min(1/n, |e^{i\theta}-z_{n-1}|)$ there exists $z_n \varepsilon N_{r_n}^*(e^{i\theta})$ such that $f(z_n) \varepsilon S_n$. Because of the construction of the S_n and the increasing fine-ness of the triangulation for each $w \varepsilon C(f,e^{i\theta})$, each n and for each $\varepsilon > 0$ there exists $z_k \varepsilon N_{r_n}^*(e^{i\theta})$ such that $|f(z_k) - w| < \varepsilon$. This completes the details.

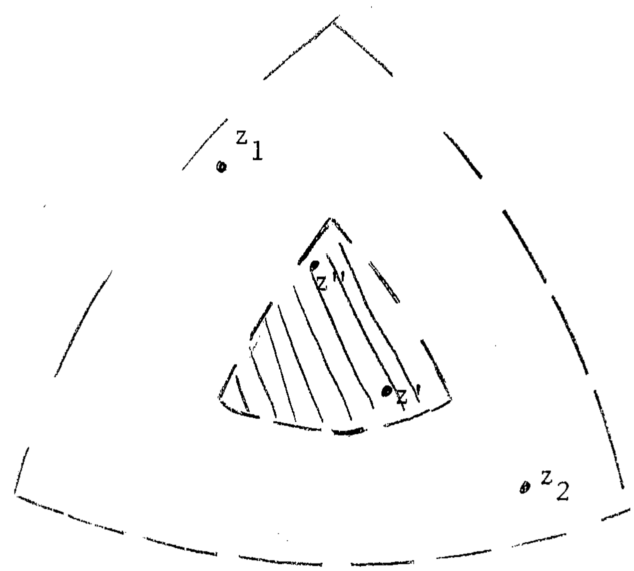

Figure 15

CHAPTER V

FUNCTIONS OF CLASS $\mathcal{A}$

This chapter concerns those functions of H^{∞} which have radial limits of modulus one almost everywhere. The class of functions with this property will be called class $\mathcal{A}$. The Fatou boundary function of a function of class $\mathcal{A}$ must have modulus one almost everywhere on C. Theorems 3.3 and 4.11 imply that each Blaschke product belongs to class $\mathcal{A}$. The function of Example 2.16 also belongs to $\mathcal{A}$. From the maximum modulus theorem it follows that any non-constant function of this class is of modulus less than one everywhere in U. Intuitively a non-constant function of class $\mathcal{A}$ is an analytic function which maps U into U in such a way that the following is true: The image of almost every radius is a Jordan curve lying in U and terminating at a point of C. The class $\mathcal{A}$ is clearly a very restrictive subclass of H^{∞}, but the exciting results which can be obtained for it make it worth studying.

Definition 5.1. A function f belongs to class $\mathcal{A}$ if it belongs to H^{∞} and $|f^*(e^{i\theta})| = 1$ almost everywhere.

The following theorem shows how one function of this class can generate a non-denumerable number of other functions of the class.

Theorem 5.2. If $f \varepsilon \mathcal{A}$ then

$$\exp\left(\frac{f(z)+\alpha}{f(z)-\alpha}\right) \ \varepsilon\ \mathcal{A} \quad \text{for each } \alpha,\ |\alpha| = 1$$

and

$$\frac{f(z)-\alpha}{1-\bar{\alpha}f(z)} \in \mathcal{A} \text{ for each } \alpha,\ |\alpha| < 1$$

Proof: Consider

$$g(z) = \frac{z+\alpha}{z-\alpha},\ |\alpha| = 1$$

Then g is a linear fractional transformation which maps the open unit disk onto the open left half plane. Of course f maps U into U. Therefore

$$g(f(z)) = \frac{f(z)+\alpha}{f(z)-\alpha}$$

maps U into the open left half plane and

$$\text{Re}[g(f(z))] < 0.$$

Let

$$F(z) = \exp\left(\frac{f(z)+\alpha}{f(z)-\alpha}\right)$$

$$|F(z)| = e^{\text{Re}[g(f(z))]} < 1 \text{ for } z \in U.$$

Since g maps C onto the imaginary axis and

$$|f^*(e^{i\theta})| = 1$$

almost everywhere, it follows that

$$\text{Re}[g(f^*(e^{i\theta}))] = 0 \text{ almost everywhere.}$$

Thus

$$|F^*(e^{i\theta})| = \left|\lim_{r\to 1} \exp\left(\frac{f(re^{i\theta})+\alpha}{f(re^{i\theta})-\alpha}\right)\right| = \lim_{r\to 1} e^{\text{Re}[g(f(re^{i\theta}))]} = e^0$$

almost everywhere. So F belongs to H^∞ and $|F^*(e^{i\theta})| = 1$ almost everywhere, which implies F belongs to $\mathcal{A}$.

Let

$$h(z) = \frac{z-\alpha}{1-\bar{\alpha}z}, \qquad |\alpha| < 1,$$

and

$$G(z) = h(f(z)).$$

Now h is a linear fractional transformation which maps U onto U and C onto C. Therefore G maps U into U and

$$G^*(e^{i\theta}) = h(f^*(e^{i\theta})).$$

$f^*(e^{i\theta}) \varepsilon C$ almost everywhere, so $h(f^*(e^{i\theta})) \varepsilon C$ and

$$|G^*(e^{i\theta})| = 1$$

almost everywhere. Thus G belongs to H^∞ and has radial limits of modulus one almost everywhere.

The first theorem of interest states that at a singularity of a function of class $\mathcal{A}$, the cluster set is the closed unit disk. In Chapter III an example was given of a Blaschke product which had every point of C as a limit point of its zeros. This implies zero belongs to the cluster set at each point of C. Since the Blaschke product has radial limits of modulus one almost everywhere, it follows from Theorem 5.3 that each point of C is a singularity and the cluster set at each point is $\bar{U}$.

The following theorem only requires the function to have radial limits of modulus one on an arc which contains the singularity.

Theorem 5.3. Let $f \varepsilon H^\infty$ with $|f(z)|<1$ in U and let

$$|f^*(e^{i\theta})| = 1$$

almost everywhere on

$$A = \{e^{i\theta} | \theta_1 < \theta < \theta_2\}.$$

Then at every singular point $e^{i\theta_0}$ of f on A

$$C(f, e^{i\theta_0}) = \overline{U}.$$

Proof: Without loss of generality let $e^{i\theta_0} = 1$. Since C(f,1) is closed, it is sufficient to show $w \in C(f,1)$ for each w, $|w| < 1$.

Suppose $0 \notin C(f,1)$; then there exists a $\delta > 0$ such that

$$N_\delta(0) \cap C(f,1) = \phi.$$

So there exists $N^*_{\rho_0}(1)$ such that $f(N^*_{\rho_0}(1)) \cap N_\delta(0) = \phi$. In other words

$$|f(z)| > \delta \quad \text{for each } z \in N^*_{\rho_0}(e^{i\theta_0}).$$

Since f has radial limits of modulus one almost everywhere on A, there exists ρ, $0<\rho<\rho_0$, such that f has radial limits of modulus one at $e^{i\rho}$ and $e^{-i\rho}$. (See Figure 16). Now define

$$\Phi(z) = \begin{cases} f(z) & \text{if } z \in F_\rho \cap U \\ \dfrac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}} & \text{if } z \in \overline{F}_\rho \setminus \overline{U} \\ f^*(z) & \text{if } z = e^{i\rho}, e^{-i\rho} \end{cases}$$

where F_ρ is the open disk with center at z = 1 which has its boundary pass through $e^{i\rho}$ and $e^{-i\rho}$. Then $\Phi(z)$ is continuous on the boundary B of F_ρ so

$$F(z) = \frac{1}{2\pi i} \int_B \frac{\Phi(\zeta)}{\zeta - z} d\zeta$$

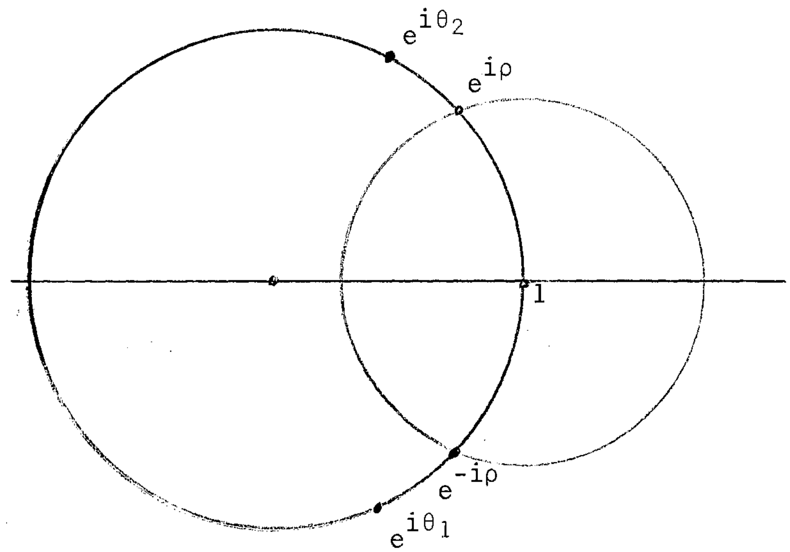

Figure 16. The First Illustration to the Proof of Theorem 5.3

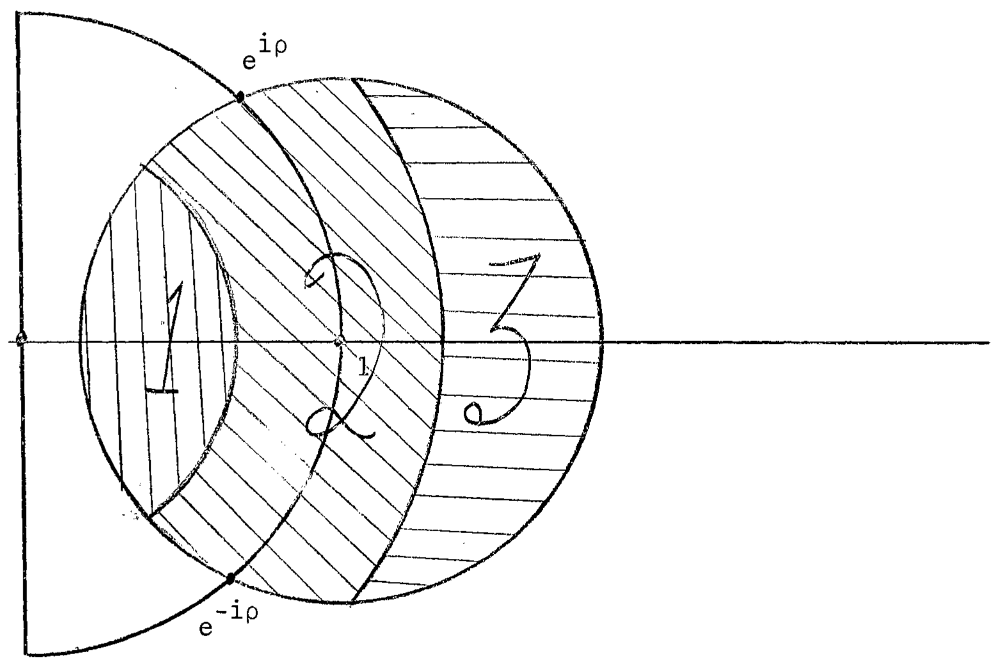

Figure 17. The Second Illustration to the Proof of Theorem 5.3

is analytic in F_ρ.

Consider

$$|z| = 1 + \varepsilon \text{ and } |z| = 1 - \varepsilon$$

which separates F_ρ into three regions as shown in Figure 17 if ε is sufficiently small. Then

$$F(z) = \frac{1}{2\pi i} \int_R \frac{\Phi(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{R'} \frac{\Phi(\zeta)}{\zeta - z}\, d\zeta$$

where R is the boundary of region 1 and R' is the boundary of the union of regions 2 and 3. If $z \varepsilon$Region 1 then

$$\frac{1}{2\pi i} \int_{R'} \frac{\Phi(\zeta)}{\zeta - z}\, d\zeta = 0.$$

So

$$F(z) = \frac{1}{2\pi i} \int_R \frac{\Phi(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_R \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \text{ for } z\varepsilon\text{Region 1.}$$

Since ε is arbitrary, $F(z) = f(z)$ for each $z \varepsilon F_\rho \cap U$. Similarly

$$F(z) = \frac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}}$$

for each $z \varepsilon F_\rho \setminus \overline{U}$.

Hence $F(z)$ is an analytic function in F_ρ which agrees with f in $F_\rho \cap U$, and this contradicts the fact $z = 1$ is a singularity. Let $|w_0| < 1$; then $w_0 \notin C(f,1)$ if and only if

$$0 \notin C\left(\frac{f(z) - w_0}{1 - \overline{w}_0 f(z)},\ 1\right).$$

Therefore the proof is complete.

Fatou's theorem established the existence of radial limits almost everywhere for a function of H^{∞}. The investigation here, for functions of class $\mathcal{A}$, will include a study of the actual values of the radial limits. It was shown in Chapter III that any function f belonging to H^{∞} has a representation of the form

$$f(z) = B(z)e^{-g(z)}.$$

It will be shown that if f belongs to $\mathcal{A}$, then so does $e^{-g(z)}$. Now

$$f^*(e^{i\theta}) = B^*(e^{i\theta}) \exp[\lim_{r\to 1}(-g(re^{i\theta}))] \text{ almost everywhere.}$$

and

$$|f^*(e^{i\theta})| = |B^*(e^{i\theta})| \exp[\lim_{r\to 1} \operatorname{Re}(-g(re^{i\theta}))] \text{ almost everywhere.}$$

So it will be of interest to study the possible behavior of $\lim_{r\to 1} \operatorname{Re}(g(re^{i\theta}))$. It should be noted that $e^{-g(z)}$ belonging to H^{∞} requires only that $e^{-\operatorname{Re}(g(z))}$ be bounded above and therefore $\operatorname{Re}(g(z))$ bounded below. Therefore g need not be a bounded analytic function and it would not be consistent to replace $\lim_{r\to 1} g(re^{i\theta})$ by $g^*(e^{i\theta})$. Once it has been shown that $e^{-g(z)}$ belongs to $\mathcal{A}$, $\lim_{r\to 1} g(re^{i\theta})$ exists and is zero almost everywhere. Of course if f is a Blaschke product, then g is identically zero and of no interest. It is important, then, to distinguish between those functions which are Blaschke products and those which are not.

It will now be shown that $e^{-g(z)}$ belongs to $\mathcal{A}$ whenever

$$f(z) = B(z)e^{-g(z)}$$

does and

$$\lim_{r\to 1} \operatorname{Re}(g(re^{i\theta})) = 0$$

almost everywhere.

Theorem 5.4. If

$$f(z) = B(z)e^{-g(z)}$$

belongs to $\mathcal{A}$ where B(z) is a Blaschke product, then $e^{-g(z)}$ belongs to $\mathcal{A}$, $\operatorname{Re}(g(z)) \geq 0$ and $\lim_{r\to 1} \operatorname{Re}(g(re^{i\theta})) = 0$ almost everywhere.

Proof: Since $f \in \mathcal{A}$,

$$|f^*(e^{i\theta})| = 1$$

almost everywhere. In Chapter III it was shown

$$|B^*(e^{i\theta})| = 1$$

almost everywhere. Therefore the equation

$$|f^*(e^{i\theta})| = |B^*(e^{i\theta})| \exp[-\lim_{r\to 1} \operatorname{Re}(g(re^{i\theta}))]$$

implies

(5.4.1) $$\exp[-\lim_{r\to 1} \operatorname{Re}(g(re^{i\theta}))] = 1 \text{ almost everywhere.}$$

In Theorem 3.8 it was shown that $e^{-g(z)}$ belongs to H^∞. Therefore Statement 5.4.1 implies $e^{-g(z)}$ belongs to $\mathcal{A}$. By the maximum modulus theorem e^{-g} is of modulus less than one in U or is a constant of modulus one. In either case

$$|e^{-g(z)}| = e^{-\operatorname{Re}(g(z))} \leq 1$$

which implies $\operatorname{Re}(g(z)) \geq 0$. Statement 5.6.1 implies

$$\lim_{r\to 1} \operatorname{Re}(g(re^{i\theta})) = 0$$

almost everywhere.

Notice that Theorem 5.2 implies $\exp\left(\frac{z+1}{z-1}\right)$ belongs to $\mathcal{A}$ since $f(z) = z$ belongs. It is obvious that $\exp\left(\frac{z+1}{z-1}\right)$ is not a Blaschke product since it is non-constant and has no zeros. If it were not for the absence of zeros, it would not be immediately obvious that the function fails to be a Blaschke product. The following theorem gives a necessary and sufficient condition for a function of H^{∞} to be a Blaschke product.

Theorem 5.5. Let $f \varepsilon H^{\infty}$; then

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \left| \log|f(re^{it})| \right| dt = 0$$

is a necessary and sufficient condition that f be a Blaschke product.

Proof: Necessity follows from Theorem 4.11. Now suppose

$$\lim_{r\to 0} \left| \log|f(re^{it})| \right| dt = 0. \tag{5.5.1}$$

Define

$$\log^{+}|f(z)| = \begin{cases} \log|f(z)| & \text{if } \log|f(z)| \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then 5.5.1 implies

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log^{+}|f(re^{it})|dt = 0.$$

Let $r_1 < r_2$ and note $\log^{+}|f(re^{it})|$ is continuous on C_{r_2}. Define

$$h(z) = \log^{+}|f(z)|$$

$$H(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(r_2e^{it}) \frac{r_2^2 - r^2}{r_2^2 - rr_2\cos(\theta-t)+r^2}\, dt.$$

Then H is harmonic in D_{r_2} with

$$H(z) = h(z) = \log^+|f(z)|$$

for $z\varepsilon C_{r_2}$. By Lemma 4.16, $\log^+|f(z)|\leq H(z)$ for $z\varepsilon \overline{D}_{r_2}$.

Therefore

$$\int_{-\pi}^{\pi} \log^+|f(r_1e^{it})|dt \leq \int_{-\pi}^{\pi} H(r_1e^{it})dt = 2\pi H(0) = \int_{-\pi}^{\pi} H(r_2e^{it})dt = \int_{-\pi}^{\pi} \log^+|f(re^{it})|dt.$$

Therefore for each r, 0<r<1

$$0 \leq \int_{-\pi}^{\pi} \log^+|f(re^{it})|dt \leq \lim_{r\to 1}\int_{-\pi}^{\pi} \log^+|f(re^{it})|dt = 0.$$

Since $\log^+|f(re^{it})|$ is non-negative, it must be identically zero in U.

Therefore

(5.5.2) $$\log|f(z)| \leq \log^+|f(z)| = 0 \text{ and so}$$

(5.5.3) $$|f(z)| \leq 1 \text{ for } z\varepsilon U.$$

Now f = Bg where g has no zeros, $g\varepsilon H^{\infty}$ and B is a Blaschke product. Therefore

$$\log\left|\frac{1}{g}\right| = \log|B| - \log|f|$$

Statements 5.5.1 and Theorem 4.11 imply

$$\lim_{r\to 1}\int_{-\pi}^{\pi} \log\left|\frac{1}{g(re^{it})}\right|dt = \lim_{r\to 1}\int_{-\pi}^{\pi} \log|B(re^{it})|dt + \lim_{r\to 1}\int_{-\pi}^{\pi} \log|f(re^{it})|dt = 0$$

The same reasoning as was used with the function f to show 5.5.2 and 5.5.3 can now be used to show $\log\left|\frac{1}{g(z)}\right| \leq 0$ and thus $\left|\frac{1}{g(z)}\right| \leq 1$. Since it has already been noted that $|g(z)| \leq 1$, it must be the case that $|g(z)| = 1$ for all $z \varepsilon U$. So

$$f(z) = e^{i\lambda}B(z)$$

and therefore is a Blaschke product.

The following theorems and lemmas will provide the theory necessary to establish a desired integral representation for Re(g). This representation will then be used as an aid in investigating functions of the class $\mathscr{A}$. Lemma 5.6 is really just a special case of Theorem 5.7.

Lemma 5.6. Let

$$v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} dt$$

then

$$\lim_{r\to 1} \frac{\partial v}{\partial \theta} = 1.$$

Proof: Let

$$P_r(t-\theta) = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}$$

and recall that

$$P_r(t-\theta) = \text{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right) = 1 + 2\sum_{1}^{\infty} r^n \cos n(t-\theta).$$

Therefore

$$v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\, dt + \frac{1}{\pi}\sum_{1}^{\infty} r^n \int_{-\pi}^{\pi} t \cos n(t-\theta)dt$$

and

$$v(r,\theta) = \frac{1}{\pi}\sum_{1}^{\infty} r^n[(\cos(n\theta))\int_{-\pi}^{\pi} t\cos(nt)\,dt + (\sin n\theta)\int_{-\pi}^{\pi} t\sin(nt)\,dt].$$

Now t cosnt is an odd function so $\int_{-\pi}^{\pi} t\cos(nt)\,dt = 0$. Thus

$$v(r,\theta) = \sum_{1}^{\infty} a_n r^n \sin(n\theta)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)dt = \frac{2}{n}(-1)^{n-1}.$$

So

$$v(r,\theta) = 2(r\sin\theta - \frac{r^2}{2}\sin 2\theta + \dots)$$

and

$$\frac{\partial v}{\partial\theta} = 2(r\cos\theta - r^2\cos 2\theta + \dots)$$

$$\frac{\partial v}{\partial\theta} = 2\mathrm{Re}(r(\cos\theta+i\sin\theta) - r^2(\cos 2\theta+i\sin 2\theta) + \dots)$$

$$\frac{\partial v}{\partial\theta} = 2\mathrm{Re}(\sum_{1}^{\infty}(-1)^{n-1}r^n e^{in\theta}) = 2\mathrm{Re}\left(\frac{re^{i\theta}}{1+re^{i\theta}}\right) = 2\mathrm{Re}(\frac{z}{z+1})$$

and

$$\lim_{z\to 1} 2\left(\frac{z}{z+1}\right) = 1.$$

<u>Theorem 5.7</u>. Let f be an integrable function and

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,P_r(t-\theta)dt.$$

If $f'(\theta_0)$ exists, then

$$\lim_{r\to 1}\left[\frac{\partial u}{\partial\theta}\right]_{\theta=\theta_0} = f'(\theta_0).$$

Proof: Note first of all that

$$\frac{\partial P_r(t-\theta)}{\partial t} = - \frac{(1-r^2)\sin(t-\theta)}{(1-2r\cos(t-\theta)+r^2)^2} = - \frac{\partial P_r(t-\theta)}{\partial \theta} \tag{5.7.1}$$

Without loss of generality, assume $\theta_0 = 0$ and $f(0) = 0$ and consider the case $f'(0) = 0$. Now there exists a function $\eta(t)$ such that

$$f'(0) + \eta(t) = \frac{f(t)-f(0)}{t}$$

with $\eta(t)\to 0$ as $t\to 0$. Then $t\eta(t) = f(t)$.

$$\left(\frac{\partial u}{\partial \theta}\right)_{\theta=0} = \frac{-1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\partial}{\partial t}\,P_r(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{(1-r^2)2r\sin t}{(1-2r\cos t+r^2)^2}\,dt$$

$$= \frac{1}{2\pi}\int_{|t|<\delta} t\eta(t)\,\frac{(1-r^2)2r\sin t}{(1-2r\cos t+r^2)^2}\,dt + \frac{1}{2\pi}\int_{\delta\le|t|\le\pi} f(t)\,\frac{(1-r^2)2r\sin t}{(1-2r\cos t+r^2)^2}\,dt$$

$$= I_1 + I_2$$

For every $\varepsilon>0$ there exists a δ such that $0<\delta<\pi$ and $|\eta(t)|<\varepsilon$ whenever $|t|\le\delta$. Therefore

$$|I_1| \le \frac{\delta\varepsilon}{2\pi}\int_{-\delta}^{\delta} \frac{(1-r^2)2r|\sin t|}{(1-2r\cos t+r^2)^2}dt \le \frac{2\pi\varepsilon}{2\pi}\int_{0}^{\pi} \frac{(1-r^2)2r\sin t}{(1-2r\cos t+r^2)^2}\,dt.$$

If $|t| \ge \delta$, then $1-2r\cos t+r^2 \ge 1-2r\cos\delta+r^2 \ge \sin^2\delta$ where the last inequality holds because, as a function of r, $1-2r\cos\delta+r^2$ achieves its minimum at $r=\cos\delta$. So

$$|I_2| \le \frac{2M(1-r)}{\sin^2\delta} .$$

Therefore

$$\left|\left(\frac{\partial u}{\partial \theta}\right)_{\theta=0}\right| \leq \varepsilon \int_0^{\pi} \frac{(1-r^2)2r\sin t}{(1-2r\cos t+r^2)^2}\,dt + \frac{2M(1-r)}{\sin^2\delta}$$

and

(5.7.2) $$\lim_{r\to 1}\left|\left(\frac{\partial u}{\partial \theta}\right)_{\theta=0}\right| = 0$$

If $f'(0) \neq 0$, consider

$$F(t) = f(t) - f'(0)t;$$

then $F'(0) = 0$ and

$$u(re^{i\theta})-f'(0)v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)P_r(t-\theta)\,dt.$$

Now F possesses all the conditions which were assumed on f to show 5.7.2. Therefore if the same proof is applied to F,

$$\lim_{r\to 1}\left[\frac{\partial u}{\partial \theta} - f'(0)\frac{\partial v}{\partial \theta}\right]_{\theta=0} = 0.$$

Therefore

$$\lim_{r\to 1}\left[\frac{\partial u}{\partial \theta}\right]_{\theta=0} = f'(0)\lim_{r\to 1}\left[\frac{\partial v}{\partial \theta}\right]_{\theta=0} = f'(0)$$

where the last equality follows from Lemma 5.6.

The following two theorems from measure theory are also needed to help establish the desired representation for $\mathrm{Re}(g(z))$.

Theorem 5.8. Let f be a non-negative measurable function and

$$\eta(E) = \int_E f\,dt \text{ where E is a Borel set.}$$

Then η is a measure on Borel sets and

$$\int_X g\,d\eta = \int_X gf\,dt$$

for every measurable function g on X with range in $[0,\infty)$.

Proof: See Rudin, page 23.

Theorem 5.9. Let $\{\mu_n\}$ be a sequence of countably additive set functions defined for Borel sets such that $0 \leq \mu_n(E) < M$ for each n, each Borel set E, and M a fixed positive number. Then there exists a subsequence $\{\mu_{n_k}\}$ such that

$$\lim_{k\to\infty} \mu_{n_k} = \mu$$

is a non-negative, countably additive set function.

Proof: See Tsuji, page 34.

Theorem 5.10. Let $u(r,\theta)$ be a positive harmonic function in U, then there exists a nondecreasing function $\chi(\theta)$ such that

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\,d\chi(t)$$

and $\lim_{r\to 1} u(r,\theta) = \chi'(\theta)$ for each value $\theta\varepsilon(-\pi,\pi)$ such that $\chi'(\theta)$ exists.

Proof: Let $|z|<\rho<1$; then by Theorem 4.1

$$u(r,\theta) = \int_{-\pi}^{\pi} u(\rho,t)\,\frac{\rho^2 - r^2}{\rho^2-2r\rho\cos(t-\theta)+r^2}\,dt.$$

Define an additive set function η_ρ by

$$\eta_\rho(E) = \int_E u(\rho,t)\,dt \geq 0 \text{ where E is a Borel set.}$$

Then

$$u(r,\theta) = \int_C \frac{\rho^2 - r^2}{\rho^2-2r\rho\cos(t-\theta)+r^2}\,d\eta_\rho \text{ by Theorem 5.8.}$$

Now consider the sequence $\{\eta_\rho\}$ and note

$$\eta_\rho(E) \leq \int_{-\pi}^{\pi} u(\rho,t)dt = 2\pi u(0,0).$$

By Theorem 5.9 there exists a subsequence $\{\eta_{\rho_k}\}$ with $\rho_k \to 1$ such that

$$\lim_{k\to 1} \eta_{\rho_k} = \eta$$

where η is a non-negative, additive set function. Therefore

$$u(r,\theta) = \lim_{k\to 1} \int_C \frac{\rho_k^2 - r^2}{\rho_k^2 - 2r\rho_k \cos(t-\theta) + r^2} d\eta_{\rho_k} = \int_C P_r(t-\theta)d\eta.$$

Now define

$$\chi(\theta) = \int_{-\pi}^{\theta} d\eta = \eta([-\pi,\theta]); \text{ then}$$

χ is non-decreasing and

$$u(r,\theta) = \int_{-\pi}^{\pi} P_r(t-\theta)d\chi(t).$$

Note $\chi(\theta)$ non-decreasing implies $\chi'(\theta)$ exists almost everywhere. Integration by parts gives

$$u(r,\theta) = \frac{1}{2\pi}\left[P_r(t-\theta)\chi(t)\right]_{t=-\pi}^{t=\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi(t) \frac{\partial}{\partial t} P_r(t-\theta)dt$$

$$u(r,\theta) = \frac{1}{2\pi}\left[P_r(t-\theta)\chi(t)\right]_{t=-\pi}^{t=\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi(t) \frac{\partial}{\partial \theta} P_r(t-\theta)dt.$$

From Theorem 5.7, the last integral approaches $\chi'(t)$ and the first term approaches zero as $r\to 1$. Therefore

$$\lim_{r\to 1} u(r,\theta) = \chi'(\theta)$$

at all points where $\chi'(\theta)$ exists.

This theorem completes the ground work for the desired representation of Re(g). The following theorem is the desired representation.

Theorem 5.11. If $e^{-g(z)}$ belongs to $\mathscr{A}$, there exists a non-decreasing function $\chi(\theta)$ with $\chi'(\theta) = 0$ almost everywhere such that

$$\mathrm{Re}(g(re^{i\theta})) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\,d\chi(t)$$

Proof: Since $e^{-g(z)}$ belongs to $\mathscr{A}$, it follows that $\mathrm{Re}(g(re^{i\theta}))$ is a non-negative harmonic function. Theorem 5.10 implies there exists a non-decreasing function $\chi(\theta)$ such that

$$\mathrm{Re}(g(re^{i\theta})) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\,d\chi(t)$$

and

(5.11.1) $$\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta})) = \chi'(\theta)$$

for values of θ where $\chi'(\theta)$ exists. From Theorem 5.4 it is seen that

(5.11.2) $$\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta})) = 0$$

almost everywhere. Statements 5.11.1 and 5.11.2 imply

$$\chi'(\theta) = \lim_{r\to 1} \mathrm{Re}(g(re^{i\theta})) = 0 \text{ almost everywhere.}$$

The next theorem is not directly related to the theory of cluster sets; however, it is needed to establish an auxiliary theorem. Consider a rectifiable curve with parametric equations X(t), Y(t), S(t) where S(t) represents the length. The following theorem is really a generalization of the Pythagorean theorem. It is shown that at almost all points

$$(X'(t))^2 + (Y'(t))^2 = (S'(t))^2.$$

In other words the sum of the squares of the instantaneous rate of change in the x and y coordinates is equal to the square of the instantaneous rate of change of the length of the curve. It should be noted that the derivatives or instantaneous rate of change is taken with respect to the parameter t.

Theorem 5.12. Let Γ be a rectifiable curve given by equations $x=X(t)$, $y=Y(t)$, $t\varepsilon[a,b]$ and let $S(t)$ represent its length. Then

(5.12.1) $\quad (X'(t))^2 + (Y'(t))^2 = (S'(t))^2$ for almost all t, $t\varepsilon(a,b)$

Proof: Since Γ is rectifiable $X(t)$, $Y(t)$ and $S(t)$ are of bounded variation. Therefore $X(t)$, $Y(t)$, and $S(t)$ are differentiable almost everywhere on (a,b).

Now

$$S(t+h)-S(t) \geq [(X(t+h)-X(t))^2+(Y(t+h)-Y(t))^2]^{1/2}$$

for each t, $t+h\varepsilon(a,b)$ because the arc distance is never less than the distance between two points on Γ. Therefore

$$\lim_{h\to 0} \frac{S(t+h)-S(t)}{h} \geq [(\lim_{h\to 0} \frac{x(t+h)-X(t)}{h})^2 + (\lim_{h\to 0} \frac{Y(t+h)-Y(t)}{h})^2]^{1/2}$$

wherever all three limits exist. So

(5.12.2) $\quad S'(t) \geq [(X'(t))^2+(Y'(t))^2]^{1/2}$ for almost all $t\varepsilon(a,b)$.

It will be shown next that

$$(S'(t))^2 \leq (X'(t))^2+(Y'(t))^2 \text{ for almost all } t\varepsilon(a,b)$$

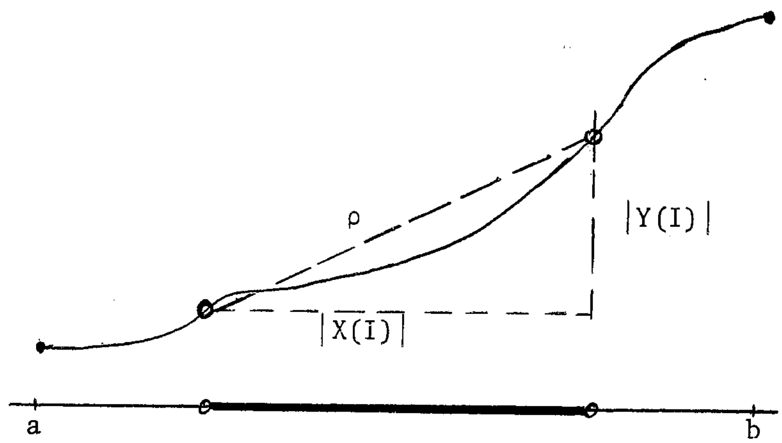

Figure 18

Let A be the set of all points of (a,b) such that

$$(S'(t))^2 > (X'(t))^2 + (Y'(t))^2.$$

For convenience, |I| will be used to denote the length of any finite interval I, and S(I) will denote the arc length of the segment corresponding to an interval I, $I \subset [a,b]$. Also ρ will denote the distance between the endpoints of the segment. (See Figure 18).

Now define

$$(5.12.3) \quad A_n = \left\{ t : \frac{S(I)}{|I|} \geq \left[\left(\frac{|X(I)|}{|I|}\right)^2 + \left(\frac{|Y(I)|}{|I|}\right)^2 \right]^{1/2} + \frac{1}{n}, \text{ if } t \in I \text{ and } |I| < \frac{1}{n} \right\}$$

Then $A = \bigcup_n A_n$.

Let n be fixed and $\varepsilon > 0$ be arbitrary. Since Γ is rectifiable there exists a partition

$$a = t_0 < t_1 < \dots < t_m = b$$

such that $|I_K| < \frac{1}{n}$ and

$$(5.12.4) \qquad S([a,b]) = \sum_{K=1}^{m} S(I_K) < \sum_{K=1}^{m} \rho_K + \varepsilon$$

where

$$I_K = (t_{K-1}, t_K).$$

Let $\sum_{K=1}^{m}{}' S(I_K)$ represent a deleted sum where the sum is taken only over Those I_K such that $I_K \cap A_n \neq \phi$. From 5.12.3

$$(5.12.5) \qquad \sum_{K=1}^{m}{}' S(I_K) \geq \sum_{K=1}^{m}{}' [|X(I_K)|^2 + |Y(I_K)|^2]^{1/2} + \sum_{K=1}^{m}{}' \frac{|I_K|}{n}.$$

It should be noted that

$$[|X(I_K)|^2 + |Y(I_K)|^2]^{1/2} = \rho_K$$

and $A_n \subset \bigcup_{K=1}^{m}{}' \overline{I}_K$. Let $\mu(A_n)$ denote the Lebesque measure of A_n. Then 5.12.4 and 5.12.5 imply

$$(5.12.6) \quad \mu(A_n) \leq \sum_{K=1}^{m}{}' |I_K| \leq n \sum_{K=1}^{m}{}' (S(I_K)-\rho_K) \leq n \sum_{1}^{m} (S(I_K)-\rho_K) \leq n\varepsilon$$

for n fixed, $\varepsilon > 0$ arbitrary. Therefore $\mu(A_n) = 0$ for each n and $\mu(A) = 0$. So for almost all $t \varepsilon (a,b)$

$$(S'(t))^2 \leq (X'(t))^2 + (Y'(t))^2.$$

Consider a non-decreasing, non-constant continuous function F defined on a closed interval [a,b]. Suppose $F'(x) = 0$ for almost all x in (a,b). The Cantor ternary function is an example of such a function. Since F is non-constant F(b) - F(a) is a positive number. It is intuitively clear that the vertical change from F(a) to F(b) does not occur on the set of points of (a,b) where the derivative is zero. The remaining points of (a,b) constitute a set of measure zero which may be further divided into three subsets of measure zero. The first subset is the set of points where the derivative exists and is a non-zero finite number, the second is the set of points where the derivative fails to exist and the third is the set on which the derivative is infinite. It will be shown that all of the vertical change, F(b) - F(a), occurs on the set of measure zero where the derivative is infinite. Therefore there must exist points where the derivative is infinite.

The following theorem proves in detail the facts mentioned in the preceding paragraph. However, it should be mentioned that a more general form of the derivative is considered which is self-explanatory in the proof of the theorem.

Theorem 5.13. Let F(x) be a non-decreasing, non-constant continuous function on [a,b] with $F'(x) = 0$ almost everywhere. Then there exists an $x_0 \varepsilon (a,b)$ such that $F'(x_0) = \infty$.

Proof: Define the following subsets of [a,b].

$A = \{x | x\varepsilon(a,b) \text{ and } F'(x) = 0\}$

$B = \{x | x\varepsilon(a,b) \text{ and } 0<F'(x)<\infty\}$

$C = \{x | x\varepsilon(a,b) \text{ and } F'(x) \text{ does not exist}\}$

$D = \{x | x\varepsilon(a,b) \text{ and } F'(x) = \infty\}$

Let μ represent Lebesque measure. By hypothesis $F'(x) = 0$ almost everywhere on [a,b] so

$$\mu(B) = \mu(C) = \mu(D) = 0$$

and

$$\mu(A) = b - a.$$

It will be shown that

$$\mu(F(A)) = \mu(F(B)) = \mu(F(C)) = 0$$

and therefore

$$\mu(F(D)) = F(b) - F(a) > 0.$$

Thus F(D) is a non-empty set and so is D. In fact D must be a non-denumerable set of measure zero because the image of a denumerable set is denumerable and therefore of measure zero.

As in the proof of the last theorem $|I|$ and $|F(I)|$ will be used to represent the lengths of the intervals $I \subset [a,b]$ and $F(I) \subset [F(a),F(b)]$. Let E be any subset of [a,b] on which F is differentiable and $F'(x) \leq m$ where m is a positive real number.

Let $\varepsilon > 0$ be arbitrary and define

(5.13.1) $E_n = \{x | x \varepsilon E \text{ and } \frac{|F(I)|}{|I|} \leq m+\varepsilon \text{ whenever } |I| < \frac{1}{n} \text{ and } x \varepsilon I\}$

Then $E = \bigcap_1^\infty E_n$ and $E_{n+1} \subset E_n$ so $\lim_{n\to\infty} E_n = E$. There exists an open set G_n such that $E_n \subset G_n \subset (a,b)$,

(5.13.2) $$\mu(G_n) - \mu(E_n) < \varepsilon$$

Since G_n is open there exists a sequence $\{I_{nK}\}$ of disjoint open intervals such that $|I_{nK}| < \frac{1}{n}$ for each K and $G_n \setminus \bigcup_{K=1}^\infty I_{nK}$ is at most countable. G_n may not be equal to the countable union because of the requirement $|I_{nK}| < \frac{1}{n}$. Therefore $F(G_n) \setminus \bigcup_{K=1}^\infty F(I_{nK})$ is at most countable,

(5.13.3) $$\mu(G_n) = \sum_1^\infty |I_{nK}| \text{ and } \mu(F(G_n)) = \sum_{K=1}^\infty |F(I_{nk})|.$$

It may also be assumed without loss of generality that $I_{nK} \cap E_n \neq \phi$ for each K. Then 5.13.1 implies

$$|F(I_{nK})| \leq (m+\varepsilon)|I_{nK}|$$

and

(5.13.4) $$\sum_{K=1}^{\infty} |F(I_{nK})| \leq (m+\varepsilon) \sum_{K=1}^{\infty} |I_{nK}|.$$

Statements 5.13.2, 5.13.3, and 5.13.4 give

$$\mu(F(G_n)) = \sum_{K=1}^{\infty} |F(I_{nK})| \leq (m+\varepsilon) \sum_{K=1}^{\infty} |I_{nK}| = (m+\varepsilon)\mu(G_n)$$

and

(5.13.5) $$\mu(F(E_n)) \leq \mu(F(G_n)) \leq (m+\varepsilon)\mu(G_n) < (m+\varepsilon)(\mu(E_n)+\varepsilon)$$

Since ε is arbitrary

$$\mu(F(E_n)) \leq m \ \ \mu(E_n)$$

Therefore

(5.13.6) $$\mu(F(E)) = \lim_{n\to\infty} \mu(F(E_n)) \leq m \lim_{n\to\infty} \mu(E_n) = m \ \ \mu(E)$$

where E is any subset of [a,b] such that $F'(x) \leq m$ for each $x \varepsilon E$.

Now define

$$A_n = \{x | x\varepsilon(a,b) \text{ and } F'(x) \leq \frac{1}{n}\}$$

Then $A = \bigcap_n A_n$. Statement 5.13.6 implies

$$\mu(F(A_n) \leq \frac{1}{n} \mu(A_n)$$

and since $A \subset A_n$ for each n

$$\mu(F(A)) \leq \mu(F(A_n)) \leq \frac{1}{n} \mu(A_n) \leq \frac{1}{n}(b-a)$$

Therefore

(5.13.7) $$\mu(F(A)) = 0$$

Define

$$B_n = \{x \mid x\varepsilon(a,b) \text{ and } 0<F'(x)\leq n\}$$

and note that $B = \bigcup B_n$, $B_{n+1} \subseteq B_n \subseteq B$. Now $\mu(B) = 0$ implies $\mu(B_n) = 0$ for each n. By Statement 5.13.6

$$\mu(F(B_n)) \leq n \quad \mu(B_n) = 0 \text{ for each } n$$

Therefore

(5.13.8) $$\mu(F(B)) = \lim_{n\to\infty} \mu(F(B_n)) = 0$$

It will be shown next that $\mu(F(C)) = 0$. Consider the curve

$$\Gamma : \quad x = x,\ y = F(x),\ x\varepsilon[a,b]$$

and let $s(x)$ represent its arc length. Consider another parametric representation where the parameter t is the arc length. Define $X(t) = x$ if and only if $s(x) = t$. Then

$$\Gamma : \quad x = X(t),\ y = Y(t) = F(X(t))$$

where $t\varepsilon[0,s(b)]$ and $S(t) = t$ is the arc length. By Theorem 5.12

(5.13.9) $$(X'(t))^2 + (Y'(t))^2 = (S'(t))^2 = 1$$

for almost all $t\varepsilon[0,s(b)]$. Now

$$\frac{d\ F(X(t))}{dt} = F'(X(t))X'(t) \text{ so since } X(t) = x$$

$$F'(x) = F'(x(t)) = \frac{F'(X(t))X'(t)}{X'(t)} = \frac{Y'(t)}{X'(t)}$$

Recall that C is the set of x-values where $F'(x)$ fails to exist. The corresponding t-values are those where $Y'(t)$ or $X'(t)$ does not exist and those where $X'(t) = Y'(t) = 0$. Statement 5.13.9 shows the set of t values is of measure zero. Therefore the set of points on Γ corresponding to the set C has measure zero and

(5.13.10) $$\mu(F(C)) = 0$$

It should be mentioned that $F'(x) = \infty$ where $Y'(t) \neq 0$ and $X'(t) = 0$. It is clearly possible that

$$(Y'(t))^2 + (X'(t))^2 = 1$$

could hold for such a value of t. Therefore Statement 5.13.9 cannot be used with reference to the set D as it was for the set C.

Now F is non-constant and non-decreasing so

$$0<F(b)-F(a) = \mu(F(A))+\mu(F(B))+\mu(F(C))+\mu(F(D))$$

and 5.13.7, 5.13.8 and 5.13.10 give

$$\mu(F(A)) = \mu(F(B)) = \mu(F(C)) = 0.$$

Therefore $\mu(F(D))>0$ and D is non-empty.

In the preceding theorem a general form, frequently used in measure theory, was used for the difference quotient. The form of the derivative was

$$F'(x) = \lim_{|I|\to 0} \frac{|F(I)|}{|I|} = \lim_{\mu(I)\to 0} \frac{\mu(F(I))}{\mu(I)}$$

where I is an open interval containing x. This is equivalent to a symmetric derivative

$$F'(x) = \lim_{\delta\to 0} \frac{F(x+\delta)-F(x-\delta)}{2\delta}$$

Therefore under the hypothesis of the previous theorem there exists a point where the symmetric derivative is infinite.

The following theorem is the first of many exciting ones. It reveals the startling fact that any function of class $\mathcal{A}$ which is not a Blaschke product must admit zero as a radial limit. Since such a function has radial limits of modulus one almost everywhere, it follows that the point where the radial limit is zero, is a singularity.

Theorem 5.14. Let $f\varepsilon\mathcal{A}$, then if f is not a Blaschke product there exists an $e^{i\theta_0}\varepsilon C$ such that $C_\rho(f,e^{i\theta_0}) = \{0\}$.

Proof: $f\varepsilon\mathcal{A}$ so $f = Be^{-g}$, $\mathrm{Re}(g(re^{i\theta})) \geq 0$ and

$$\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta})) = 0$$

almost everywhere. Now

$$\overline{\lim_{r\to 1}} |f(re^{i\theta})| \leq \overline{\lim_{r\to 1}} |B(re^{i\theta})| \exp\{-\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta}))\}$$

at points where $\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta}))$ exists. Since $|B(re^{i\theta})| \leq 1$ it follows that

$$\overline{\lim_{r\to 1}} |f(re^{i\theta})| \leq \exp\{-\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta}))\} \tag{5.14.1}$$

The major part of the proof is to demonstrate that there exists a θ_0 such that

$$\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta_0})) = \infty.$$

Then Statement 5.14.1 implies

$$\overline{\lim_{r\to 1}}\ |f(re^{i\theta_0}))| \leq e^{-\infty} = 0$$

and thus

$$C_\rho(f,e^{i\theta_0}) = \{0\}.$$

By Theorem 5.11 there exists a non-decreasing function $\chi(\theta)$ with $\chi'(\theta) = 0$ almost everywhere such that

$$\text{(5.14.2)} \qquad \text{Re}(g(re^{i\theta})) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)d\chi(t)$$

Let

$$|t-\theta| = 1-r = \delta<1.$$

Since $\cos x>1 - \frac{x^2}{2}$,

$$\cos(t-\theta)>1 - \frac{(1-r)^2}{2} = \frac{1}{2} + r - \frac{r^2}{2}.$$

Therefore

$$2r\cos(t-\theta)>2r^2+r-r^3 \text{ and } 1-r^2-r+r^3>1-2r\cos(t-\theta)+r^2.$$

This implies

$$P_r(t-\theta) = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} > \frac{1}{1-r} = \frac{1}{\delta} > 0$$

Thus

$$\text{Re}(g(re^{i\theta})) \geq \frac{1}{2\pi}\int_{\theta-\delta}^{\theta+\delta} P_r(t-\theta)d\chi(t) > \frac{1}{2\pi\delta}\int_{\theta-\delta}^{\theta+\delta} d\chi(t) = \frac{1}{2\pi\delta}[\chi(\theta+\delta)^- - \chi(\theta-\delta)^+]$$

Therefore

$$\text{(5.14.3)} \qquad \text{Re}(g(re^{i\theta})) > \frac{1}{2\pi\delta}[\chi(\theta+\delta)^- - \chi(\theta-\delta)^+]$$

for r such that $1-r = \delta < 1$.

If $\chi(\theta)$ fails to be continuous on $(-\pi,\pi)$ there exists a θ_0, a real number $m>0$ and $\delta_0>0$ such that $\chi(\theta_0+\delta)^- - \chi(\theta_0-\delta)^+ > m$ for each δ, $0<\delta<\delta_0$.

Therefore

$$\mathrm{Re}(g(re^{i\theta_\circ})) > \frac{1}{2\pi\delta} m \text{ for } 0<\delta<\delta_\circ$$

and

$$\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta_\circ})) > \lim_{\delta\to 0} \frac{m}{2\pi\delta} = \infty \tag{5.14.4}$$

Now consider the case where $\chi(\theta)$ is continuous and recall that $\chi'(\theta) = 0$ almost everywhere. $\chi(\theta)$ is non-constant because if it were constant Statement 5.14.2 would imply

$$\mathrm{Re}(g(re^{i\theta})) = 0$$

and

$$g(re^{i\theta}) = i\alpha.$$

Then

$$f(z) = B(z)e^{i\alpha}$$

which contradicts the hypothesis that f is not a Blaschke product. Now $\chi(\theta)$ satisfies the hypothesis of Theorem 5.13 so there exists a $\theta_\circ$ such that

$$\lim_{\delta\to 0} \frac{\chi(\theta_\circ+\delta)-\chi(\theta_\circ-\delta)}{2\delta} = \infty \tag{5.14.5}$$

Statements 5.14.3 and 5.14.4 imply

$$\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta_\circ})) > \frac{m}{\pi} \lim_{\delta\to 0} \frac{\chi(\theta_\circ+\delta)-\chi(\theta_\circ-\delta)}{2\delta} = \infty \tag{5.14.6}$$

Then 5.14.4 and 5.14.6 imply there exists a $\theta_\circ$ such that

$$\lim_{r\to 1} \mathrm{Re}(g(re^{i\theta_\circ})) = \infty$$

which was what was to be shown.

Examples were given in Chapter III which illustrate that a Blaschke product may or may not admit zero as a radial limit. However, any infinite Blaschke product must necessarily have at least one point on C which is a limit point of zeros and therefore zero belongs to the cluster set at that point. Since a Blaschke product has radial limits of modulus one almost everywhere, any point of C which has zero in its cluster set is a singular point. Therefore, with Theorem 5.14 it is clear that any function of class $\mathcal{A}$ which is not a finite Blaschke product, has a least one singularity on C.

The following theorem is no less surprising than the previous one and it deals with functions of class $\mathcal{A}$ which are not finite Blaschke products. If such a function assumes the value w, $|w| < 1$, at most a finite number of times, then w is a radial limit of the function.

Theorem 5.15. Let $f \in \mathcal{A}$ and let f not be a finite Blaschke product. Then if f assumes the value w, $|w| < 1$ at most a finite number of times in U, there exists $e^{i\theta_0}$ such that

$$C_\rho(f, e^{i\theta_0}) = \{w\}.$$

Proof: Consider $f \in \mathcal{A}$ satisfying the hypothesis. By Theorem 5.2 the function

$$F(z) = \frac{f(z)-w}{1-\overline{w}f(z)}$$

belongs to $\mathcal{A}$. Since f assumes the value w, at most a finite number of times, it follows that F can have at most a finite number of zero's (counting multiplicities). Therefore F is not an infinite Blaschke product. Since f is not a finite Blaschke product, it follows from previous remarks that there exists $e^{i\theta_0}$ which is a singularity of f.

Theorem 5.3 implies $0 \varepsilon C(f, e^{i\theta_0})$. This implies there exists a sequence $\{z_n\} \subset U$ such that

$$\lim_{n\to\infty} z_n = e^{i\theta_0}$$

and

$$\lim_{n\to\infty} f(z_n) = 0.$$

Therefore

$$\lim_{n\to\infty} F(z_n) = -w$$

and so $-w \varepsilon C(F, e^{i\theta_0})$. This indicates that F is not a finite Blaschke product since any such product is analytic on C and of modulus one there.

Since F is not a Blaschke product, it follows from Theorem 5.14 that there exists $e^{i\theta_0}$ such that

$$\lim_{r\to 1} F(re^{i\theta_0}) = 0$$

and thus

$$\lim_{r\to 1} f(re^{i\theta_0}) = w$$

Therefore

$$C_\rho(f, e^{i\theta_0}) = \{w\}.$$

There are examples of functions of class $\mathcal{A}$ which omit one value of the open unit disk. Such a function is

$$f(z) = e^{\frac{z+1}{z-1}}$$

which omits the value zero. This possibility is not restricted to functions which fail to be Blaschke products, as an example will illustrate. Another example will be given to show that a function of class $\mathcal{A}$ may admit two values of modulus less than one as radial limits.

Example 5.16. Let

$$f(z) = e^{\frac{z+1}{z-1}}$$

and

$$F(z) = \frac{f(z)+e^{-1}}{1+e^{-1}f(z)},$$

then both functions belong to $\mathcal{A}$. Also

$$F^*(z) = \frac{f^*(z)+e^{-1}}{1+e^{-1}f^*(z)}$$

Now f has radial limits of modulus one everywhere except at z = 1 and where $f^*(1) = 0$. This implies $F^*(e^{i\theta}) \neq 0$ for each $e^{i\theta} \varepsilon C$. By the contrapositive of Theorem 5.14, F(z) is a Blaschke product. Now f omits the value zero so F omits the value e^{-1}. Also note that F has a radial limit of e^{-1} at z = 1.

Example 5.17. Now consider the function

$$G(z) = \exp\left(\frac{F(z)+1}{F(z)-1}\right)$$

where F is the Blaschke product of the previous example. Note that

$$G^*(1) = \exp\frac{e^{-1}+1}{e^{-1}-1}$$

and G omits the value zero. Now $G^*(e^{i\theta}) = 0$ if and only if $f^*(e^{i\theta}) = 1$ and this occurs at infinitely many points of C (see Example 2.16). Note also that z = 1 is a limit point of the set of points where $G^*(e^{i\theta}) = 0$.

The next theorems deal with functions such as those of the previous examples which actually omit values. If a function of $\mathcal{A}$ omits one value, every singularity on C must either have that value as a radial limit or be a limit point of singularities, each of which has the value as its radial limit. Notice how strong this statement is. If the function has any isolated singularities, the radial limit at each must exist and be precisely the value omitted. This leads to the fact that a function of $\mathcal{A}$ which omits two values cannot have any isolated singularities.

Theorem 5.18. Let f be a non-constant function of class $\mathcal{A}$. Let $e^{i\theta_0}$ be a singularity of f and

$$A_\varepsilon = \{e^{i\theta} \mid -\pi \leq \theta_0-\varepsilon<\theta_0+\varepsilon \leq \pi\}.$$

Also let $w \in U$ and

$$E = \{e^{i\theta} \mid f^*(e^{i\theta}) = w\}.$$

If $f(z) \neq w$ for each z in U, then $e^{i\theta_0}$ belongs to the closure of E.

Proof: Without loss of generality assume $e^{i\theta} = 1$. The proof will be given for $w = 0$. If $w \neq 0$, the proof can be applied to

$$F(z) = \frac{f(z)-w}{1-\overline{w}f(z)}$$

So assume f is non-zero in U and that $z = 1$ is a singularity of f. Then

$$f(z) = e^{i\lambda}\cdot e^{-g(z)}$$

since f has no zeros in U. Now

$$\mathrm{Re}(g(re^{i\theta})) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\,d\chi(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right) d\chi(t)$$

(See Statements 4.1.2 and 4.1.3.)

Also

$$h(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\, d\chi(t) \text{ is analytic in } U.$$

(See Rudin's <u>Real and Complex Analysis</u>, page 225.)

Now

$$Re(g(z)) = Re(h(z))$$

so g and h differ by only an imaginary constant. Since h(0) is real,

$$g(z) = h(z) + iv(0) \text{ where } v(z) = Im(g(z))$$

It will be shown later that $\chi(\theta)$ is non-constant on the interval $\{\theta:|\theta|<\varepsilon\}$. Then by the same technique as that used in Theorem 5.14, there exists a $\theta_\circ$, $|\theta_\circ| < \varepsilon$, such that

$$\lim_{r\to 1} f(re^{i\theta_\circ}) = 0.$$

Since ε is arbitrary either f has the radial limit zero at $z = 1$ or $z = 1$ is the limit point of a sequence $\{e^{i\theta_n}\}$ with

$$\lim_{r\to 1} f(re^{i\theta_n}) = 0.$$

Therefore $z = 1$ belongs to the closure of E.

It will now be shown that $\chi(\theta)$ is non-constant on $\{\theta:|\theta|<\varepsilon\}$. Suppose $\chi(\theta)$ is constant on this interval. Then

$$\int_{-\varepsilon}^{\varepsilon} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\, d\chi\ (t) = 0$$

and

$$g(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{-\varepsilon} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\, d\chi(t) + \frac{1}{2\pi} \int_{\varepsilon}^{\pi} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\, d\chi(t)+iv(0)$$

Consider θ, $|\theta|<\frac{\varepsilon}{2}$ and t, $\varepsilon \le |t| \le \pi$. Then $|e^{it}-re^{i\theta}|>\frac{\varepsilon}{4}$ for $0<r\le 1$. Let

$$F = \{t: \varepsilon \leq |t| \leq \pi\}$$

Then

$$(5.18.1) \qquad \left| \frac{1}{2\pi} \int_F \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\, d\chi(t) - \frac{1}{2\pi}\int_F \frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}}\, d\chi(t)\right| =$$

$$\left| \frac{1}{2\pi} \int_F \frac{2(r-1)e^{i(\theta-t)}}{(e^{it}-re^{i\theta})(e^{it}-e^{i\theta})}\, d\chi(t)\right| \leq \frac{1}{2\pi} \cdot \frac{2(1-r)}{\varepsilon/4\cdot\varepsilon/4} = \frac{16(1-r)}{\varepsilon^2}$$

for $|\theta|<\frac{\varepsilon}{2}$. Let ε be fixed. Since Statement 5.18.1 is true for all θ, $|\theta|<\frac{\varepsilon}{2}$, $g(re^{i\theta})$ approaches a unique limit along any approach to $z = 1$. This implies $C(f,1)$ is degenerate which contradicts Theorem 5.3 since $z = 1$ is a singularity. Therefore $\chi(\theta)$ is non-constant.

Theorem 5.19. If $f\varepsilon\mathcal{A}$ and omits two values of modulus less than one, f has no isolated singularities.

Proof: Suppose $e^{i\theta_0}$ is an isolated singularity and f omits w_1 and w_2 in U. Since $e^{i\theta_0}$ is an isolated singularity there exists a δ such that

$$|f^*(e^{i\theta})| = 1$$

for all θ, $0<|\theta_0-\theta|<\delta$. This fact together with Theorem 5.18 implies $f^*(e^{i\theta})$ must equal both values w_1 and w_2 which is a contradiction.

The following theorem is a localization of Theorem 5.19 to those isolated singularities which are limit points of zeros. Not only does a function of $\mathcal{A}$ with an isolated singularity omit at most one value in U, but it omits at most one value in the neighborhood of the singularity providing the singularity is a limit point of zeros.

<u>Theorem 5.20</u>. If $e^{i\theta_0}$ is an isolated singularity of f, $f\varepsilon\mathcal{A}$ such that $e^{i\theta_0}$ is the limit point of zeros of f, then f omits at most one value w, $|w|<1$ in each $N_\varepsilon^*(e^{i\theta_0})$. (i.e. $U\setminus R(f,e^{i\theta_0})$ is at most degenerate.)

<u>Proof</u>: Without loss of generality, let $e^{i\theta_0} = 1$ and

$$A = \{e^{i\theta} \mid \ |\theta|<\varepsilon\}$$

be an arc on which the only singularity is $z = 1$. Now define

$$F(z) = \begin{cases} f(z) \text{ if } z\varepsilon N^*(1) \\ \dfrac{1}{f\left(\frac{1}{\bar{z}}\right)} \text{ if } z\varepsilon N(1)\setminus\overline{U} \\ f^*(z) \text{ if } z\varepsilon C\cap N_\varepsilon(1) \end{cases}$$

By Schwarz's reflection principle, F is analytic everywhere in $N_\varepsilon(1)$ except at $z = 1$, which is an essential singularity, and at points z_0 for which $f\left(\frac{1}{\bar{z}_0}\right) = 0$ where poles occur. Let $\{z_k\}\subset N_\varepsilon^*(1)$ such that $f(z_k) = 0$ and $\lim_{k\to\infty} z_k = 1$. Then $\left\{\frac{1}{\bar{z}_k}\right\}\subset N_\varepsilon(1)$ and f has poles at each of these points.

(5.20.1) Let f omit w_0, $|w_0|<1$ in $N_\varepsilon^*(1)$.

Note $|f^*(e^{i\theta})| = 1$ for $0<|\theta|<\varepsilon$. So if F is to assume the value $\frac{1}{\bar{w}_0}$, $\left|\frac{1}{\bar{w}_0}\right| > 1$, it must do so outside the closed unit disk. Suppose there exists $z'\varepsilon N_\varepsilon(1)\setminus\overline{U}$ such that $F(z') = \frac{1}{\bar{w}_0}$. This implies

(5.20.2) $$\frac{1}{f\left(\frac{1}{\bar{z}'}\right)} = \frac{1}{\bar{w}_0} \text{ and } f\left(\frac{1}{\bar{z}'}\right) = w_0$$

If $z' \varepsilon N_\varepsilon(1)\setminus\overline{U}$, $\frac{1}{z'} \varepsilon N^*(1)$ which implies Statements 5.20.1 and 5.20.2 are contradictory. Therefore if f were to omit two values in $N_\varepsilon{}^*(1)$, F would omit four values in $N_\varepsilon(1)$. This contradicts Picard's theorem which states that F can omit at most two values in $N_\varepsilon(1)$.

Much has been said about the existence of points where the radial limit is of modulus less than one. The amazing fact is that any function of class $\mathcal{A}$ must admit every value $e^{i\alpha}$ as a radial limit and do so in each neighborhood of a singularity.

Theorem 5.21. Let $f \varepsilon \mathcal{A}$ and $e^{i\theta_0}$ be a singular point of f. Also let

$$A_\varepsilon = \{e^{i\theta}: |\theta-\theta_0|<\varepsilon\}.$$

Then $C \subset f^*(A_\varepsilon)$

Proof: Let $w_0 \varepsilon C$; then

$$F(z) = \exp\{\frac{f(z)+w_0}{f(z)-w_0}\} \quad \varepsilon \mathcal{A}.$$

Since $e^{i\theta_0}$ is a singular point of f, it is also a singular point of F. So F cannot be a finite Blaschke product and it omits the value zero. By Theorem 5.18, either $F^*(e^{i\theta_0}) = 0$ or $e^{i\theta_0}$ is the limit point of a sequence where F^* is zero. Therefore there exists $e^{i\theta_1} \varepsilon A$ such that $F^*(e^{i\theta_1}) = 0$; and thus $f^*(e^{i\theta_1}) = w_0$.

Several of the previous theorems deal with values which are assumed by the function a finite number of times. A first guess might be that little can be said of a value which a function assumes infinitely often. However, the following theorem characterizes those functions of class $\mathcal{A}$ which assume a value infinitely often and fail to have the value as a radial limit.

Theorem 5.22. Let $f \varepsilon \mathcal{A}$ and suppose f assumes the value w_0, $|w_0|<1$ infinitely often in U. If f fails to have w_0 as a radial limit, then f is of form

$$f(z) = \frac{B(z)+w_0}{1+\overline{w}_0 B(z)} \tag{5.22.1}$$

Proof: Let

$$g(z) = \frac{f(z)-w_0}{1-\overline{w}_0 f(z)} .$$

Since w_0 is not a radial limit of f, zero is not a radial limit of g. By the contrapositive of Theorem 5.14, g is a Blaschke product. So let B(z) = g(z) and

$$B(z)-\overline{w}_0 B(z)f(z) = f(z)-w_0$$

$$f(z)(1+\overline{w}_0 B(z)) = B(z)+w_0$$

$$f(z) = \frac{B(z)+w_0}{1+\overline{w}_0 B(z)}$$

The following theorem establishes an amazing possibility. If a function belonging to $\mathcal{A}$ is not of form 5.22.1 then it must admit every value of the unit disk as a radial limit. This is not an empty subclass for Ohtsuka[1] has constructed such a function.

Theorem 5.23. Let $f \varepsilon \mathcal{A}$. If f is not of form 5.22.1 then f admits every value w, $w \varepsilon \overline{U}$ as a radial limit.

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[1] M. Ohtsuka, "Note on Functions Bounded and Analytic in the Unit Circle". Procedures American Mathematical Society, Vol. 5 (1954), pp. 533-535.

Proof: If $|w_o|<1$ and f assumes w_o a finite number of times, then f must admit w_o as a radial limit by Theorem 5.15. If f assumes the value w_o an infinite number of times, w_o is a radial limit by Theorem 5.22. If $|w_o| = 1$ it follows from Theorem 5.21.

CHAPTER VI

SELECTED RESULTS IN A MORE GENERAL SETTING

The theory presented in this chapter, although somewhat varied in its context, includes two very important theorems from the theory of cluster sets. It is because of their importance that they are included here. A problem will be given at the close of this chapter to illustrate the application of several of the major theorems presented in this paper. This problem, it is hoped, will illustrate the application of a theorem, in a local sense, to a function which does not satisfy the hypothesis over the entire region.

The first theorem is auxiliary in nature and is needed to establish Theorem 6.2. This theorem deals with an arbitrary subset S of U. It states that there are at most a countable number of points on C where it is possible to find two arcs, one lying in S and the other lying in $U \setminus S$.

<u>Theorem 6.1</u>. Let S be a subset of U and

$$K = \left\{ e^{i\theta} : \text{ there exist arcs } \Gamma, \Gamma' \text{ at } e^{i\theta} \text{ such that } \Gamma \subset S \text{ and } \Gamma' \subset U \setminus S \right\}$$

Then K is countable.

<u>Proof</u>: For each pair of rational numbers α and ρ, $0<\alpha<\pi/2$, $0<\rho<1$ define the following set.

(6.1.1)

$$E_{\alpha\rho} = \left\{ \begin{array}{l} e^{i\theta} \mid \text{there exist arcs } \Gamma, \Gamma' \text{ at } e^{i\theta} \text{ with } \Gamma \subset S \text{ and} \\ \Gamma' \subset U \setminus S \text{ such that if } \zeta \text{ is the endpoint of } \Gamma \text{ and } \zeta' \\ \text{is the endpoint of } \Gamma', \text{ then } |\zeta| = |\zeta'| = \rho \\ \theta - \pi/4 < \arg z < \theta + \frac{\pi}{4} \text{ and} \\ \rho < |z| < 1 \text{ for } z \in \Gamma \cup \Gamma', \ z \neq \zeta, \zeta' \\ \theta - \frac{\pi}{4} < \arg \zeta' < \theta + \frac{\pi}{4} - \alpha < \arg \zeta < \theta + \pi/4 \end{array} \right\}$$

Figure 19 illustrates the definition of $E_{\alpha\rho}$. It shows that $e^{i\theta}$ belongs to $E_{\alpha\rho}$ for the given set S in the figure and for the fixed values of α and ρ. The point $e^{i\theta'}$ does not belong to $E_{\alpha\rho}$ because it is impossible to construct an arc Γ' satisfying the definition of $E_{\alpha\rho}$. Define a set $E'_{\alpha\rho}$ in the same way as $E_{\alpha\rho}$ except interchange the roles of Γ and Γ'. Figure 20 illustrates that $e^{i\theta'} \varepsilon E'_{\alpha\rho}$. It is possible, of course, that $E_{\alpha\rho} \cap E'_{\alpha\rho} \neq \phi$. Since α and ρ are rational, there are a countable number of sets of the types $E_{\alpha\rho}$ and $E'_{\alpha\rho}$. It will be shown that each of these sets contains a countable number of elements. Then define

$$K = \{e^{i\theta} | e^{i\theta} \varepsilon E_{\alpha\rho}, E'_{\alpha\rho} \text{ for each rational } \alpha \text{ and } \rho; \ 0<\alpha<\pi/2, \ 0<\rho<1\}$$

and note that K is countable. It will also be shown that for each $e^{i\theta}$ which belongs to $C \setminus K$ and each pair of arcs at $e^{i\theta}$, both arcs intersect S or both intersect $U \setminus S$.

Consider a set $E \subset C$ such that each point of E is isolated. Then for each point of E there exists an arc on C containing that point of E and no others. Each of these arcs has a positive arc length and the sum of the lengths of all such arcs is less than 2π. Hence there can be at most a countable number of such arcs, and therefore

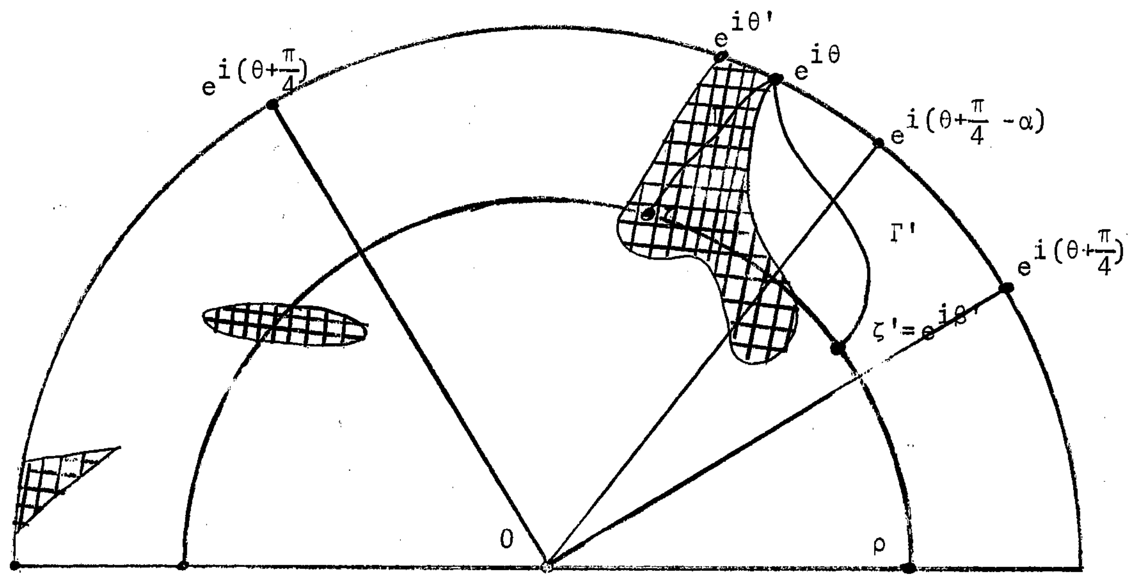

Figure 19. The First Illustration for the Proof of Theorem 6.1. The Shaded Area Represents the Set S.

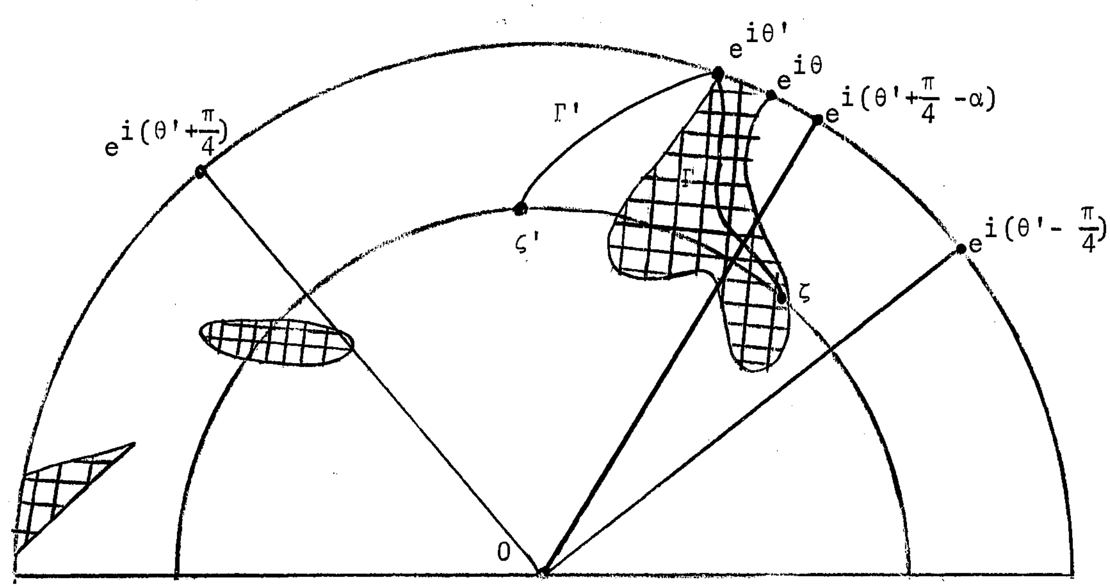

Figure 20. The Second Illustration for the Proof of Theorem 6.1.

countably many points in E. It will be shown now that each set $E_{\alpha\rho}$ is a set of isolated points and therefore countable. Let $e^{i\theta} \varepsilon E_{\alpha\rho}$ and Γ, Γ' be two arcs at $e^{i\theta}$ which satisfy the conditions in 6.1.1. Let $\zeta' = \rho e^{i\beta'}$ and note that 6.1.1 implies $\arg \zeta' = \beta' < \theta + \frac{\pi}{4} - \alpha$. Now suppose there exists an $e^{i\theta_1}$ belonging to $E_{\alpha\rho}$ such that

(6.1.2) $$0 < \theta - \theta_1 < \theta + \frac{\pi}{4} - \alpha - \beta'.$$

This implies

(6.1.3) $$\theta_1 > \beta' + \alpha - \frac{\pi}{4}.$$

Since $e^{i\theta_1}$ belongs to $E_{\alpha\rho}$, there exists arcs Γ_1 and Γ_1' at $e^{i\theta_1}$ which satisfy 6.1.1. Let ζ_1 be the endpoint of Γ_1 with $|\zeta_1| = \rho$ and $\theta_1 + \frac{\pi}{4} - \alpha < \arg \zeta_1$ as given in 6.1.1. This result together with 6.1.3 yields

$$\arg\zeta' = \beta' < \theta - \alpha + \frac{\pi}{4} < \arg\zeta_1$$

Now note that $\theta_1 < \theta$ and Γ' is an arc connecting ζ' to $e^{i\theta}$ whilc Γ_1 is an arc connecting ζ_1 to $e^{i\theta_1}$. (See Figure 21.) These conditions and the fact the two arcs must lie in an annular section imply the arcs intersect. This contradicts the condition that $\Gamma' \subset U \setminus S$ and $\Gamma_1 \subset S$. Since θ_1 could be any value such that 6.1.2 holds, it follows that $e^{i\theta}$ is isolated from the right. A similar argument can be used to show $e^{i\theta}$ is isolated in $E_{\alpha\rho}$ from the left. Therefore $E_{\alpha\rho}$ is a set of isolated points and therefore countable. The same type argument can be used to show $E'_{\alpha\rho}$ is countable.

Let $e^{i\theta}$ belong to $C \setminus K$ and Γ_1, Γ_2 be arcs at $e^{i\theta}$. Now suppose $\Gamma_1 \subset U \setminus S$ and $\Gamma_2 \subset S$. Now consider the sector

$$R = \{z \mid z \varepsilon U \text{ and } \theta - \frac{\pi}{4} < \arg z < \theta + \frac{\pi}{4}\}$$

and note that each of the arcs Γ_1, Γ_2 must have subarcs which lie entirely in R. Therefore there exists a circle C_ρ, $0<\rho<1$, which intersects each of these subarcs. (See Figure 22.) Let Γ' and Γ be subarcs of Γ_1 and Γ_2 respectively such that each connects a point of C_ρ to $e^{i\theta}$ and

$$\rho < |z| < 1 \text{ for } z\varepsilon\Gamma'\cup\Gamma, \text{ z not an endpoint of } \Gamma', \text{ or } \Gamma.$$

(See Figure 22.) Let ζ and ζ' be the points on C_ρ which are endpoints of Γ and Γ' respectively. Without loss of generality, assume $\arg\zeta' < \arg\zeta$. Choose γ such that $\gamma\varepsilon C_\rho$, $\arg\zeta' < \arg\gamma < \arg\zeta$ and $\theta + \frac{\pi}{4} - \arg\gamma = \alpha$ is rational. Now $\gamma\varepsilon R$ which implies

$$\theta - \frac{\pi}{4} < \arg\gamma < \theta + \frac{\pi}{4}$$

and

$$0 < \theta + \frac{\pi}{4} - \arg\gamma = \alpha < \frac{\pi}{2}.$$

Therefore all conditions of 6.1.1 are satisfied for Γ and Γ' at $e^{i\theta}$ which implies $e^{i\theta}\varepsilon E_{\alpha\rho}$. This contradicts $e^{i\theta}\varepsilon C\setminus K$. Therefore for each point $e^{i\theta}$ which belongs to $C\setminus K$ and every pair of arcs at $e^{i\theta}$ either both arcs intersect S or both intersect $U\setminus S$.

It has been seen in the example

$$f(z) = \exp\left(\frac{z+1}{z-1}\right)$$

that it is possible to find two arcs terminating at $z = 1$ such that the cluster sets along these arcs are disjoint. It will be recalled in this Example 2.16 that $C_{G_r}(f,1)\cap C_{G_{r'}}(f,1) = \phi$ whenever $r \neq r'$. The next theorem by Bagemihl is quite surprising.

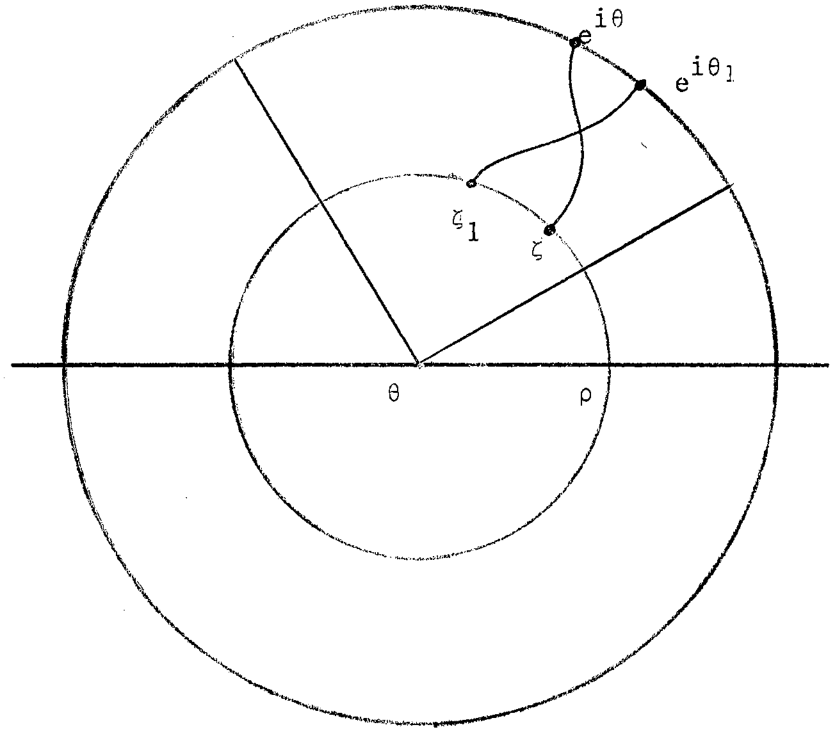

Figure 21. The Third Illustration for the Proof of Theorem 6.1

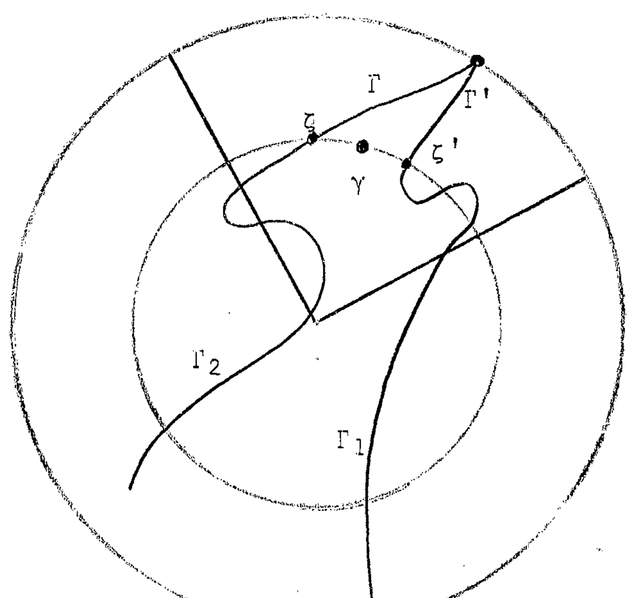

Figure 22. The Fourth Illustration for the Proof of Theorem 6.1

Theorem 6.2. Let f be an arbitrary function defined on U. Then there exists a subset K of C such that K is countable and for each $e^{i\theta}$ belonging to $C\setminus K$ with Γ_1 and Γ_2 any two arcs at $e^{i\theta}$,

$$C_{\Gamma_1}(f,e^{i\theta}) \cap C_{\Gamma_2}(f,e^{i\theta}) \neq \phi.$$

Proof: A spherical neighborhood on the Riemann sphere will be called rational if it has a rational radius and if the stereographic projection of its center is either infinity or a complex number whose real and imaginary parts are rational. There are only countably many rational neighborhoods on the sphere and therefore only countably many open subsets of the sphere each of which is the union of rational neighborhoods. Let the sequence $\{G_n\}$ denote this collection of open sets. Define

$$S_n = \{z \mid z\varepsilon U \text{ and } f(z)\notin G_n\}$$

By Theorem 6.1 there exists a subset K_n of C such that K_n is countable and for each $e^{i\theta}$ which belongs to $C\setminus K_n$ and each pair of arcs at $e^{i\theta}$ either both arcs intersect S_n or both intersect $U\setminus S_n$. Define $K = \bigcup_1^{\infty} K_n$ and note that K is at most countable. It will be shown that

$$C_{\Gamma_1}(f,e^{i\theta}) \cap C_{\Gamma_2}(f,e^{i\theta}) \neq \phi$$

for each $e^{i\theta}$ in $C\setminus K$ and each pair of arcs Γ_1 and Γ_2 at $e^{i\theta}$.

Let $e^{i\theta}\varepsilon C\setminus K$ and Γ_1, Γ_2 be arcs at $e^{i\theta}$. Suppose

$$C_{\Gamma_1}(f,e^{i\theta}) \cap C_{\Gamma_2}(f,e^{i\theta}) = \phi.$$

Then for each $w\varepsilon C_{\Gamma_1}(f,e^{i\theta})$ there exists a rational neighborhood N(w) on the sphere such that $\overline{N(w)} \cap C_{\Gamma_2}(f,e^{i\theta}) = \phi$. Since this is true for each such w, there exists a G_m such that

(6.2.1) $$C_{\Gamma_1}(f,e^{i\theta}) \subset G_m$$

and

(6.2.2) $$C_{\Gamma_2}(f,e^{i\theta}) \subset \text{Riemann sphere} \setminus \overline{G}_m$$

Let $\{\Gamma_1^k\}$ be a nested sequence of subarcs of Γ_1 whose endpoints converge to $e^{i\theta}$, and let $\{\Gamma_2^k\}$ be a similar sequence of subarcs of Γ_2. Now $e^{i\theta} \varepsilon C \setminus K$ implies $e^{i\theta} \notin K_n$ for each n. Therefore for a fixed k, the pair of arcs, Γ_1^k, Γ_2^k either both intersect S_k or both intersect $U \setminus S_k$. Therefore there exists an infinite sequence

$$\{(\Gamma_1^{k_j}, \Gamma_2^{k_j})\}$$

such that exactly one of the following holds:

(6.2.3) $\Gamma_1^{k_j}$ and $\Gamma_2^{k_j}$ intersect S_n for each j and for each n.

(6.2.4) $\Gamma_1^{k_j}$ and $\Gamma_2^{k_j}$ intersect $U \setminus S_n$ for each j and for each n.

If 6.2.3 holds, then in particular $\Gamma_1^{k_j}$ and $\Gamma_2^{k_j}$ each intersect S_m. Consider the sequence $\{z_j\}$ such that $z_j \varepsilon \Gamma_1 \cap S_m$. By the way these arcs were constructed, the sequence $\{z_j\}$ converges to $e^{i\theta}$ and by the definition of S_m, $f(z_j) \notin G_m$ for each j. Therefore

$$C_{\Gamma_1}(f,e^{i\theta}) \cap (U \setminus G_m) \neq \phi$$

and this contradicts 6.2.1.

If 6.2.4 holds, then there exists a sequence $\{z_j\} \subset \Gamma_2$ such that the sequence approaches $e^{i\theta}$ and $f(z_j) \varepsilon G_n$ for each j. This implies

$$C_{\Gamma_2}(f,e^{i\theta}) \cap \overline{G}_m \neq \phi \text{ which contradicts 6.2.2.}$$

Therefore the supposition is false and

$$C_{\Gamma_1}(f,e^{i\theta}) \cap C_{\Gamma_2}(f,e^{i\theta}) \neq \phi.$$

It should be recalled that a value w is an asymptotic value of f at $e^{i\theta}$ if there exists an arc Γ at $e^{i\theta}$ along which f approaches w. In other words, the cluster set along Γ is precisely $\{w\}$. It was shown in Lindelöf's Theorem that a function belonging to H^∞ has at most one asymptotic value at any given point of C. It is, of course, possible to find a function f outside the class H^∞ such that f has two asymptotic values at some point of C. If a function has two or more asymptotic values at a point, the point is called an ambiguous point.

<u>Definition 6.3</u>. Let f be defined on U. A point $e^{i\theta}$ is an ambiguous point of f if f has two distinct asymptotic values at $e^{i\theta}$.

If $e^{i\theta}$ is an ambiguous point of a function f, there exist arcs Γ and Γ' such that

$$C_\Gamma(f,e^{i\theta}) = \{w_1\} \text{ and } C_{\Gamma'}(f,e^{i\theta}) = \{w_2\}$$

where w_1 and w_2 are distinct asymptotic values of f at $e^{i\theta}$. This implies that at any ambiguous point $e^{i\theta}$

$$C_\Gamma(f,e^{i\theta}) \cap C_{\Gamma'}(f,e^{i\theta}) = \phi.$$

However, by Theorem 6.2, this can occur at only a countable number of points. Therefore any function defined on U can have at most a countable number of ambiguous points. The following theorem states this fact and is known as Bagemihl's Ambiguous Point Theorem.

Theorem 6.4. Let f be defined on U. Then f has at most a countable number of ambiguous points.

Proof: Immediate from Theorem 6.2.

It has been pointed out that a function belonging to H^{∞} has no ambiguous points. Theorem 6.6 shows that any meromorphic function which omits three values of the extended plane also has no ambiguous points.

Lemma 6.5. There exists an analytic map of

$$\{z \mid z \neq 0, 1, \infty\}$$

onto the open half plane.[1]

Theorem 6.6. Let f be meromorphic and nonconstant in U. Let f omit three values in U. Then if w is an asymptotic value of f at $e^{i\theta}$, f has the angular limit of w at $e^{i\theta}$.

Proof: It can be assumed without loss of generality that the three values omitted are 0, 1, ∞. Let v(z) be the function of Lemma 6.5 which maps $\{z \mid z \neq 0, 1, \infty\}$ onto the open upper half plane. Now consider

$$F(z) = \frac{v(f(z))-v(i)}{v(f(z))-\overline{v(i)}} .$$

[1]Rudin, pp. 322-323.

Since v(i) lies in the upper half plane, $\overline{v(i)}$ lies in the lower half plane. This implies $|v(f(z))-v(i)|<|v(f(z))-\overline{v(i)}|$ and

$$|F(z)| < 1 \text{ for } z\varepsilon U.$$

Therefore F is a bounded analytic function. Furthermore, since f approaches the limit w along some arc Γ at $e^{i\theta}$, F approaches the limit $\frac{v(w)-v(i)}{v(w)-\overline{v(i)}}$ along the same arc. Since F(z) belongs to H^{∞}, Lindelöf's Theorem implies F has the angular limit $\frac{v(w)-v(i)}{v(w)-\overline{v(i)}}$ at $e^{i\theta}$. As a result, f must have the angular limit w at $e^{i\theta}$.

Corollary 6.7. If f is meromorphic in U and omits three values, f has no ambiguous points.

Proof: Immediate from Theorem 6.6.

The next theorem, called the Gross-Iverson Theorem, relates the function values to the values in the set $C(f,e^{i\theta})\setminus C_B(f,e^{i\theta})$. It shows that a meromorphic function must assume all values of the set $C(f,e^{i\theta})\setminus C_B(f,e^{i\theta})$ in each partial neighborhood of $e^{i\theta}$ with at most two exceptions. It can also be shown that each exceptional value is an asymptotic value of f at $e^{i\theta}$. In the familiar example

$$f(z) = \exp \frac{z+1}{z-1}$$

zero is such an exceptional value. It should be pointed out that $C(f,e^{i\theta})\setminus C_B(f,e^{i\theta})$ may be empty, and in such cases the Gross-Iverson Theorem is of no interest. If f is a Blaschke product, or any function of class $\mathcal{A}$, with a non-isolated singularity then

$$C(f,e^{i\theta}) = C_B(f,e^{i\theta}) = \overline{U}.$$

The Gross-Iverson Theorem now follows.

Theorem 6.8. (Gross-Iverson). If f is a non-constant meromorphic in U, then every value of

$$C(f,e^{i\theta})\setminus C_B(f,e^{i\theta})$$

is assumed infinitely often in each partial neighborhood $N^*_\varepsilon(e^{i\theta})$ with at most two exceptions. Each exception is an asymptotic value of f at $e^{i\theta}$.

Proof: Suppose there are three exceptions. Without loss of generality let

$$0,\ 1,\ \infty \in [C(f,e^{i\theta})\setminus C_B(f,e^{i\theta})] \text{ and} \tag{6.8.1}$$

$N^*_{\rho_0}(e^{i\theta})$ be a partial neighborhood where f omits these three values. That is

$$0,\ 1,\ \infty \notin f(N^*_{\rho_0}(e^{i\theta})) \tag{6.8.2}$$

It will be shown that this supposition leads to the conclusion that f has zero and infinity as asymptotic values at $e^{i\theta}$. This contradicts Corollary 6.7. Therefore the supposition is false and f can omit at most two values of $C(f,e^{i\theta})\setminus C_B(f,e^{i\theta})$ in any partial neighborhood of $e^{i\theta}$.

It will now be shown with much effort that f has 0 and ∞ as asymptotic values at $e^{i\theta}$. Recall

$$C_B(f,e^{i\theta}) = \bigcap_{\eta} \overline{\bigcup_{0<|\theta-\alpha|<\eta} C(f,e^{i\alpha})}.$$

Since $0 \notin C_B(f,e^{i\theta})$ there exists positive numbers ε_1 and δ_1 such that

$$\varliminf_{z\to e^{i\alpha}} |f(z)| > \varepsilon_1 \tag{6.8.3}$$

for each α, $0<|\theta-\alpha|<\delta_1$. Also $\infty \notin C_B(f,e^{i\theta})$ implies there exists positive numbers ε_2 and δ_2 such that

(6.8.4) $$\underline{\lim}_{z\to e^{i\alpha}} \left[\frac{1}{|f(z)|}\right] > \varepsilon_2 \text{ for each } \alpha,\ 0<|\theta-\alpha|<\delta_2$$

So for $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$, $\delta = \min\{\delta_1,\delta_2\}$ and $0<|\theta-\alpha|<\delta$.

(6.8.5) $$\underline{\lim}_{z\to e^{i\alpha}} |f(z)| > \varepsilon \text{ and } \underline{\lim}_{z\to e^{i\alpha}} \left[\frac{1}{|f(z)|}\right] > \varepsilon$$

by Statements 6.8.3 and 6.8.4.

Now let $\rho = \min \{\rho_o, \delta/2\}$. Then $N^*_\rho(e^{i\theta}) \subset N^*_{\rho_o}(e^{i\theta})$ and 6.8.5 holds for $e^{i\alpha}\varepsilon N_\rho(e^{i\theta})$, $\alpha \neq \theta$. By Statement 6.8.2 $\infty \notin f(N^*_\rho(e^{i\theta}))$. This implies f is analytic in $N^*_\rho(e^{i\theta})$.

Now consider the open sets

(6.8.6) $$H_n = \{z|z\varepsilon U \text{ and } |f(z)| < 1/n\}$$

and let N be the smallest positive integer such that

(6.8.7) $$1/N < \varepsilon$$

where ε is the positive number in Statement 6.8.5. Note that $H_{n+1} \subset H_n$ and $1/n < \varepsilon$ for n>N. Now H_n can be decomposed into pairwise disjoint open sets each of which is connected. Intuitively an open connected set is simply connected if it contains no "holes". If a component of H_n had such a hole, it would be possible to construct a simply connected domain which contains the hole and has its boundary in H_n. The maximum modulus theorem applies to this simply connected domain and requires $|f(z)| \leq \frac{1}{n}$ everywhere. This contradicts the existence of such a hole, (6.8.8) So every component of H_n is an open simply connected set.

It will now be shown that there exists an n>N such that component of H_n is contained entirely in $N^*_\rho(e^{i\theta})$. Suppose that the contrary is true. Since $0\varepsilon C(f,e^{i\theta})$, there exists a point $z'_p \varepsilon N^*_{\frac{\rho}{2p}}(e^{i\theta})$ such that

$$|f(z'_p)| < \frac{1}{N+p} .$$

Thus $z'_p \varepsilon H_{N+p}$. Let O_p be the component of H_{N+p} which contains z'_p. By supposition O_p must also contain a point of U outside $N^*_\rho(e^{i\theta})$. Since O_p is connected (see 6.8.8), the frontier of O_p is a closed connected set. Therefore there exists a continuum K_p lying in the frontier of O_p which connects a point of $|z-e^{i\theta}| = \rho$ to a point of $|z-e^{i\theta}| = \frac{\rho}{2^p}$. Now define

(6.8.9) $K = \bigcap_m \overline{\bigcup_{p=m}^{\infty} K_p}$ where K_p is a continuum lying in FrO_p, O_p a component of H_{N+p}, K_p connects a point of $|z-e^{i\theta}| = \rho$ to a point of

$$|z - e^{i\theta}| = \frac{\rho}{2^p} ,$$

and

$$|f(z)| = \frac{1}{N+p} ,$$

if $z\varepsilon FRO_p$. It will be shown in Lemma 6.9 that K is a continuum connecting a point of

$$|z - e^{i\theta}| = \rho \text{ to } e^{i\theta}.$$

Let $z\varepsilon K$; this implies $z\varepsilon \overline{\bigcup_{p=m}^{\infty} K_p}$ for each m. Thus there exists a sequence of points from a subsequence of $\{K_p\}$ which converge to z. From 6.8.9 the function values of this sequence converge to zero and by Statement 6.8.2 $0 \notin N^*_\rho(e^{i\theta})$. Therefore $z\varepsilon C$. In other words, although K is the limit of a sequence of continua each of which lies in U, K lies on C. Thus K is an arc on C connecting a point of

$$|z - e^{i\theta}| = \rho \text{ to } e^{i\theta}.$$

The statements above imply $0 \varepsilon C(f,e^{i\alpha})$ for each $e^{i\alpha} \varepsilon K$, $\alpha \neq \theta$ which gives $0 \varepsilon C_B(f,e^{i\theta})$. This contradicts 6.8.1.

Therefore there exists a n>N such that

(6.8.10) A component of H_n lies entirely in $N^*_\rho(e^{i\theta})$.

Let G be such a component of H_n. Statement 6.8.3 and the fact

$$f(z) < 1/n < 1/N < \varepsilon$$

imply FrG does not contain any points of C except possibly $e^{i\theta}$. If $e^{i\theta}$ does not belong to the frontier of G, f is analytic at each point of FrG as well as each point of G. Note that

$$|f(z)| = 1/n$$

for each $z \varepsilon$FrG and that $f(z) \neq 0$ for $z \varepsilon G \subset N^*_\rho(e^{i\theta})$. These facts imply by the maximum modulus theorem that f(z) is constant of modulus one in G. This is a contradiction to the hypothesis that f is non-constant. Thus

(6.8.11) $e^{i\theta} \varepsilon$FrG and FrG contains no other points of C.

It is claimed that G contains a component of H_{n+1}. If $G \cap H_{n+1} = \phi$,

(6.8.12) $$\frac{1}{n+1} \leq |f(z)| < \frac{1}{n} \text{ for } z \varepsilon G.$$

Now G is simply connected and therefore conformally equivalent to the open unit disk U. Let g: U→G represent this equivalence. Then $h = f \circ g$ belongs to H^∞ and is also analytic at all points of C except $g^{-1}(e^{i\theta})$. In addition

$$\frac{1}{n+1} \le |h(z)| < \frac{1}{n}$$

in U with

$$|h^*(e^{i\beta})| = \frac{1}{n}$$

except at $g^{-1}(e^{i\theta})$. By the Maximum Modulus Theorem 4.8 $|h(z)| < \frac{1}{n}$ or h is constant of modulus $\frac{1}{n}$ in U. Now $\frac{1}{h(z)}$ also belongs to H^∞, which implies $\left|\frac{1}{h(z)}\right| < n$ or constant of modulus n. This gives h constant of modulus $\frac{1}{n}$ in U and thus f constant of modulus $\frac{1}{n}$ in G. This is a contradiction, so H_{n+1} has a component G_1 contained in G.

Consequently, there exists a nested sequence

$$G \supset G_1 \supset G_2 \supset \ldots \supset G_p \supset \ldots \qquad (6.8.13)$$

of simply connected open sets with

$$|f(z)| < \frac{1}{p} < \frac{1}{N} < \varepsilon$$

for $z \varepsilon G_p$. Under the same technique as that applied to G, $e^{i\theta} \varepsilon FrG_p$ for each p.

Consider a point z belonging to $\bigcap_{p=1}^{\infty} G_p$. Then if z lies in $N^*_{\rho_o}(e^{i\theta})$,

$$|f(z)| < \frac{1}{p}$$

for each p = 1, 2, ... Thus f(z) = 0 for each point of $\bigcap_{p=1}^{\infty} G_p$ which also lies in $N^*_{\rho_o}(e^{i\theta})$. However, Statement 6.8.2 implies this set of points is void. Therefore the sequence $\{G_p\}$ converges to a set on the boundary C. The fact that this sequence is nested in G and G only contains one point of C (see 6.8.11) implies

$$\bigcap_{p=1}^{\infty} G_p = \{e^{i\theta}\}$$

Therefore the diameters of the sequence $\{G_p\}$ tend to zero as $p\to\infty$.

Now select a sequence $\{z_p\}$ such that $z_p \varepsilon G_p$ and $|z_p| < |z_{p+1}|$ for each p. Then because $G_{p+1} \subseteq G_p$, both z_p and z_{p+1} belong to G_p. Since G_p is an open connected set it is also arcwise connected. So there exists an arc lying entirely in G_p which connects z_p and z_{p+1}. Note that $|f(z)| < \frac{1}{p}$ for each z on this arc. The sequence of such arcs forms an arc γ_1 at $e^{i\theta}$ and f approaches zero as z approaches $e^{i\theta}$ along γ_1.

The same argument may be applied to $\frac{1}{f(z)}$ to construct an arc γ_2 terminating at $e^{i\theta}$ along which $\frac{1}{f(z)}$ approaches zero. Thus f(z) approaches ∞ along γ_2.

Lemma 6.9. In the context of Theorem 6.8

$$K = \bigcap_m \overline{\bigcup_m^{\infty} K_p} \text{ is a continuum}$$

(See Statement 6.8.9)

Proof: K is obviously closed. Assume K is not connected. Let $z_p \varepsilon K_p$ with

$$|z_p - e^{i\theta}| = \frac{\rho}{2p} .$$

Now K not connected implies there exists closed disjoint sets F_1 and F_2 such that $K = F_1 \cup F_2$ with $e^{i\theta} \varepsilon F_2$. There must exist disjoint open sets 0_1 and 0_2 with $F_1 \subseteq 0_1$ and $F_2 \subseteq 0_2$. Let $w \varepsilon F_1 \subset K$. Then

$$w \varepsilon \overline{\bigcup_m^{\infty} K_p} \text{ for each } m,$$

which implies there exists a subsequence $\{K_{p_j}\}$ such that

(6.9.1) $$K_{p_j} \cap 0_1 \neq \phi \text{ for each } j.$$

Now $e^{i\theta} \varepsilon F_2 \subset 0_2$ and $z_p \varepsilon K_p$ with $\lim_{p\to\infty} z_p = e^{i\theta}$ implies there exists a positive integer P such that $z_p \varepsilon K_p \cap 0_2$ for p>P.

Therefore K_{p_j} for p_j>P contains a point of 0_1 and a point of 0_2 and must also contain a point of the frontier of 0_2. Let $v_j \varepsilon K_{p_j} \cap Fr0_2$ and let v be a limit point of $\{v_j\}$. Then v must also belong to K and to $Fr0_2$. This contradicts $F = F_1 \cup F_2$. Therefore K is connected and thus a continuum.

The following result will conclude this chapter and this paper. Although the result is not in itself of basic interest, the method of proof gives a good illustration of the application of the theory presented in this paper to a particular problem. The following discussion will set the scene for the problem.

It has been shown in Fatou's Theorem that a bounded analytic function defined on U will have a finite radial limit at almost every point of C. The problem mentioned above will deal with analytic functions defined on U which need not be bounded. Consider the example

$$g(z) = \exp\left(-\frac{z+1}{z-1}\right) .$$

This function is just the multiplicative inverse of the familiar example

$$f(z) = \exp\left(\frac{z+1}{z-1}\right) .$$

Therefore it is easy to see that g has radial limits of modulus one everywhere except at z = 1 where the radial limit is infinite. So g is an example of an analytic function on U which has a finite radial limit everywhere except at one point, yet g does not belong to H^∞ .

An arc which lies in U, terminates at $e^{i\theta}$, and lies entirely to the right of the diameter to $e^{i\theta}$, is called a right arc. A left arc is defined similarly. If f has an asymptotic value w along a right arc, w is called a right asymptotic value. A left asymptotic value is defined similarly.

The following problem utilizes the idea of left and right asymptotic values to establish a necessary and sufficient condition for a function, analytic in U, to have finite radial limits at all except a countable number of points of C.

Problem 6.10. Let f be analytic in U. A necessary and sufficient condition that f have a finite radial limit except on a countable set, is that f have a right and left asymptotic value, at least one of which is finite everywhere on C except on a countable set.

Proof: Necessity of the condition results from f being continuous in U. The proof of sufficiency will provide the illustration desired.

Let E_1 be the points of C where f has a left and right asymptotic values, at least one of which is finite. Then by hypothesis $C\setminus E_1$ is at most countable.

If $e^{i\theta}\varepsilon E_1$, there exists a right arc Γ_θ^r and left arc Γ_θ^ℓ at $e^{i\theta}$ such that at least one of the following is finite.

$$\lim_{\substack{z\to e^{i\theta}\\ z\varepsilon\Gamma_\theta^r}} f(z) = w_\theta^r, \qquad \lim_{\substack{z\to e^{i\theta}\\ z\varepsilon\Gamma_\theta^\ell}} f(z) = w_\theta^\ell.$$

Let

$$E_2 = \{e^{i\theta} | e^{i\theta} \varepsilon E_1 \text{ and } w_\theta^r = w_\theta^\ell\}.$$

Then by Bagemihl's ambiguous point theorem, $E_1\setminus E_2$ is at most countable. If $e^{i\theta} \varepsilon E_2$, $w_\theta^r = w_\theta^\ell$ and this is a finite value. Join the initial points of Γ_θ^r and Γ_θ^ℓ by a Jordan arc J in U to form a closed Jordan curve

$$\Gamma = \Gamma_\theta^r \cup \Gamma_\theta^\ell \cup J \cup \{e^{i\theta}\}$$

Let G_θ be the domain enclosed by Γ. Note that G_θ is conformally equivalent to the open disk U. If $g: U \to G_\theta$ represents this equivalence, $f \circ g$ is analytic in U. In addition, since f is analytic at all points of Γ except $e^{i\theta}$, $f \circ g$ is analytic at all points of $\overline{U}$ except $g^{-1}(e^{i\theta})$. Now let

$$E_3 = \{e^{i\theta} | e^{i\theta} \varepsilon E_2 \text{ and } f \text{ is bounded in } G_\theta\}.$$

If $z \varepsilon E_3$, $f \circ g$ is bounded and thus belongs to H^∞. Also note that $f \circ g$ has the asymptotic value w_θ^r along an arc on C. By Lindelöf's Theorem, 4.19, $f \circ g$ approaches w_θ^r along any path in U which terminates at $g^{-1}(e^{i\theta})$. Therefore f has $w_\theta^r = w_\theta^\ell$ as a finite radial limit at $e^{i\theta}$. If $e^{i\theta} \varepsilon E_2 \setminus E_3$, $w_\theta^r = w_\theta^\ell$ and f is unbounded in G_θ. Then $f \circ g$ is unbounded. Therefore

$$C_B(g, g^{-1}(e^{i\theta})) = \{w_\theta^r\} = \{w_\theta^1\}$$

and $\infty \ \varepsilon \ C(g, g^{-1}(e^{i\theta}))$. Since $\infty \ \varepsilon \ [C(g, g^{-1}(e^{i\theta})) \setminus C_B(g, g^{-1}(e^{i\theta}))]$ and g does not assume the value ∞ in U, by Theorem 6.8 g has ∞ as an asymptotic value at $g^{-1}(e^{i\theta})$. This implies f has ∞ as an asymptotic value at $e^{i\theta}$ as well as $w_\theta^r = w_\theta^\ell$. Therefore $e^{i\theta}$ is an ambiguous point, so by the ambiguous point theorem $E_2 \setminus E_3$ is at most countable.

So it has been shown that f has a finite radial limits on E_3 and $C \setminus E_3$ is countable. This completes the proof.

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APPENDIX

SYMBOLS USED

VITA

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