CHARACTERIZATIONS AND PROPERTIES OF SETS OF CONSTANT WIDTH

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OF SETS OF CONSTANT WIDTH

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PREFACE

This dissertation is an exposition of characterizations and properties of sets of constant width. In the plane, a disc is a set of constant width. However, among its properties this one is usually not emphasized. The first paper published on sets of constant width including sets other than discs was by the great Swiss mathematician and astronomer, Leonhard Euler (1707-1783), in the year 1778, a period in which Euler was totally blind. Since then other mathematicians have studied these sets and have written papers concerning them. However, much of their work has been in E_2 and E_3, two and three dimensional Euclidean space, respectively.

There are a number of significant theorems concerning plane sets of constant width but it is not known whether all of the n-dimensional analogues are true or not. As a matter of fact, as late as 1958, Eggleston in his book on convex sets remarks, "Considering the number of papers published on sets of constant width it is surprising how little is known about them."

I want to acknowledge the excellent work of G. D. Chakerian, whose articles on sets of constant width awakened in me an interest in the subject. One of the basic characterizations was first pointed out to me in one of his articles (cf. [8]).

The topic of sets of constant width appeals to a wide range of audiences. Many of the concepts of this paper when restricted to the plane are within easy grasp of the student of high school geometry.

Almost anyone will be fascinated by the constructions of the sets of constant width in E_2 which appear in Chapter I. Some of the simple facts of Chapter I have led to important research and applications in engineering. However, the pure mathematician is interested in the theoretical developments in E_n, $n > 3$, where the intuitive guide of a figure has limitations.

There are at least four engineering applications for sets of constant width. First, a rotor in the shape of a Reuleaux triangle is the basis of a device which transforms constant circular motion to an intermittent linear motion. This is precisely what happens in the gripper of a movie projector. In 1954, Felix Wankel, a German engineer, designed an internal combustion engine where rotors of constant width are employed instead of the conventional pistons. This engine, called the Wankel engine, is very compact, very light in weight in contrast to the power developed, and it is noted for easy starting in cold weather. Two manufacturers are now producing cars equipped with this type of engine. Harry James Watts in 1914 designed a drill which drills square holes. An adaptation of a set of constant width makes this possible. The fourth application is, perhaps, more in the nature of an unsolved problem. The problem is in determining the "roundness" of a roller bearing or ball bearing. It is possible, for instance, for a ball bearing to be of constant width but not round.

I have also investigated a number of properties of sets of constant width in E_2. The first result which is due to Pál asserts the existence of a regular circumscribed hexagon for every plane set of constant width. Very closely related is the fact that every set of constant width in the plane admits a circumscribed rhombus. This follows trivially from the statement of Pál. These two results enable one to prove the following two theorems:

> Blaschke-Lebesque Theorem: For all plane sets of constant width, the circle has the greatest area and the Reuleaux triangle has the least area.
>
> Barbier Theorem: For any plane sets of constant width λ, the perimeter has length $\pi\lambda$.

At the present time, there are available three excellent books on convexity which have sections or chapters devoted to sets of constant width.

The first and best book from the standpoint of sets of constant width is Konvexen Körper by Bonnesen and Fenchel [4]. This book is well written and includes an excellent section on sets of constant width. The book covers practically all of the results on sets of constant width known at the time of publication in 1934. Obviously, this book is not complete in the sense that results discovered since 1934 are not in this book. A major disadvantage to the English speaking student is that it is written in German.

The second book is Convexity by H. G. Eggleston [14]. This entire book by the admission of the author is only a brief introduction to convexity. This is especially true concerning the chapter on sets of constant width. There appears to be an ambiguity in the way Eggleston introduces the idea of a complete set. Also Eggleston's book was published in 1958, and so it, too, is not completely up-to-date.

The third book, an English translation of a Russian book, is Convex Figures by Yaglom and Boltyanskiĭ [37]. In this book there is one chapter on sets of constant width. All of their work is in two and three dimensions. Most of the concepts and proofs are presented from a purely intuitive viewpoint. It was written as a sequal to Euclidean geometry for the gifted Russian high school student, the undergraduate, or the high school teacher. It was hoped that it would awaken geometric intuition, one of the sources of mathematical inspiration. With these objectives in mind, the authors have done an excellent job. The book is ideal as a source of concrete examples and basic intuitive notions concerning sets of constant width. However, this book contains none of the more sophisticated results, and having been printed in 1951, it contains none of the discoveries since that time.

As has been noted before, there have been many papers published in mathematical periodicals on sets of constant width. These articles use a variety of notations and have different objectives. It seems desirable to collect, organize and discuss the subject sets of constant width, using consistent notation throughout.

In the literature on sets of constant width, it was discovered that detailed analytic proofs were noticeably lacking. In fact, almost all the proofs were at best sketchily outlined or omitted entirely. Upon a closer look it was found that the proofs could not always be easily constructed in a straight-forward manner. Proofs of some of the theorems were found to be lengthy and difficult. For these reasons it seems useful to supply proofs in more detail.

Thus the existing literature on sets of constant width, which is excellent in most respects, contained the following situations which are disadvantages to the beginning student in the subject.

1. The best writings of the basic facts are in German.

2. The existing books are not up to date.

3. There are ambiguities in the existing literature, as well as places where there is vagueness.

4. There are no examples of detailed analytic proofs available for the beginner to follow and pattern.

5. Recent discoveries are hidden in scattered periodicals.

In view of these five facts, this dissertation is offered as a remedy for these deficiencies. Chapter I, which is an introduction, consists largely of examples of sets of constant width in E_2 and E_3 and can be read by anyone with a little elementary geometry background. The remaining part of the thesis is on characterizations and properties of sets of constant width which are valid in any Euclidean space E_n, $n > 1$. In Chapters II, III, and IV, a proof is given of the characterization of sets of constant width in terms of the completeness of a set. The first half of the characterization is proved in Chapter II. Chapter III concerns itself with properties of complete sets. These properties are interesting and useful in themselves, but their primary purpose is to enable us to prove the second part of this characterization which is in Chapter IV. Chapters V through VIII include three other characterizations and various properties. Throughout the thesis beyond Chapter I, the emphasis is on detailed analytic proofs. With these objectives in mind, the applications referred to earlier and the results applying only in E_2 are omitted.

Although the discussion is limited to Euclidean space, E_n, it is possible to generalize the theorems and proofs to Hilbert space. The main difficulty would be due to the fact that compactness is not equivalent to closed and boundedness in infinite dimensional Hilbert spaces.

Near the end of Chapter I is a brief review of the most basic fundamental mathematical facts concerning linear spaces, inner products and norms as functions, Euclidean n-space E_n, and hyperplanes. For this part of Chapter I and the remainder of the paper, it is assumed that the reader has a background of functional analysis or of convexity as treated in Parts I and II of Valentine's book [36]. Much of the terminology and notation is from his book and is standard. However, in particular, attention should be called to the usage of H^+ and H^- where H is a hyperplane. Precisely these are defined as follows:

$$H^+ = \{x : f(x) > 0\}$$

and

$$H^- = \{x : f(x) < 0\}.$$

The figures in the text are drawn as two or three dimensional and are supplied to guide the intuition and to provide insight in the nature of the corresponding problem in the plane or in space. However, these figures are not parts of the proofs and can be omitted as far as the logic of proof is concerned.

I wish to thank Loretta Beckham for typing the manuscript. My wife, Eloise, and daughters, Sharon and Pamela, deserve a note of appreciation for the sacrifices they have made that I might attend graduate school. I am especially indebted to Professor E. K. McLachlan, without whose inspiration, encouragement and help this thesis would never have been accomplished. I also want to express appreciation to Professors John Jobe, W. Ware Marsden, and Robert T. Alciatore for serving as members of my advisory committee.

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CHAPTER I

INTRODUCTION

For the simplest example of a set of constant width in the plane, consider a disc of diameter D where the distance between any pair of parallel tangent lines is D (cf. Figure 1-1). Any compact convex set in the plane with this property is a set of constant width. The term "constant width" denotes the property of a set, that the distance between any pair of parallel tangent lines is constant.

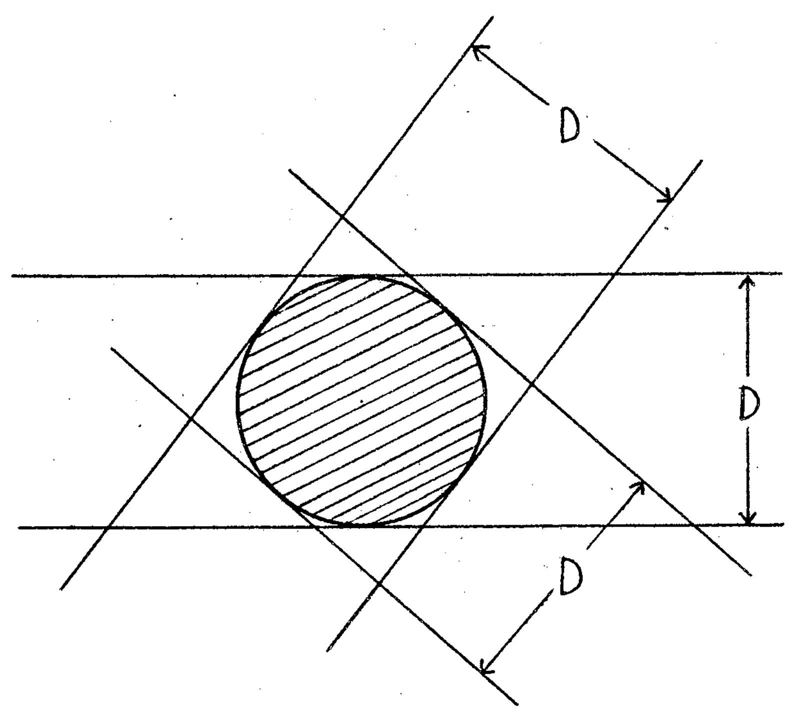

Figure 1-1.

Another possible viewpoint is to consider a pair of fixed parallel lines both of which are tangent to a disc. The disc can be rolled or rotated arbitrarily between these fixed parallel lines. It might seem plausible that the disc is the only such set with this property. However, this is not the case. Actually there are infinitely many plane sets whose width is constant and which therefore can be rolled or rotated between two fixed parallel lines to which they remain tangent throughout.

To illustrate this, consider the following set bounded by three arcs: Let the vertices x_0, x_1, and x_2 of an equilateral triangle be centers of arcs of circles passing through the opposite two vertices and whose radii are s, the length of a side of the equilateral triangle (cf. Figure 1-2). Such a set is called a Reuleaux triangle, after Franz Reuleaux, the nineteenth century German engineer who first noted the constant width property of such a set. The distance between H_0 and H_1 is s for every pair of parallel tangent lines.

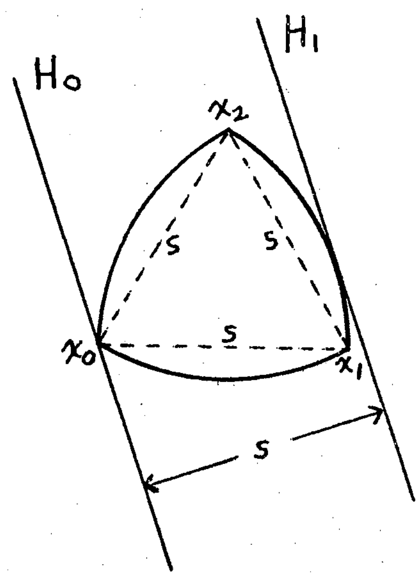

Figure 1-2.

In a similar manner, it is possible to start with any regular plane polygon with an odd number of sides and get a set of constant width. This set is called a Reuleaux polygon. Figure 1-3 illustrates such a set where a regular pentagon is the basis for the construction.

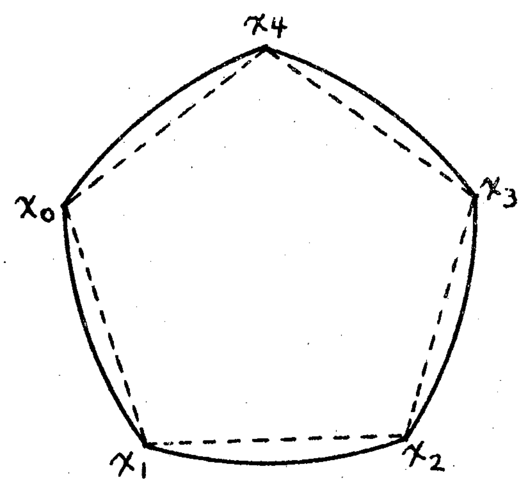

Figure 1-3.

Michael Goldberg prefers the term Reuleaux rotor instead of Reuleaux polygon. He prefers this first of all because a Reuleaux polygon is not a polygon; and secondly, he notes that the other name fails to emphasize the property of freely rolling or rotating between two fixed parallel tangent lines.

An interesting variation of the Reuleaux triangle is the following: Again start with an equilateral triangle with vertices x_0, x_1, and x_2. Denote the length of each side by s. With each vertex as center, draw an arc of radius p where p is greater than s and where the arc is inside the corresponding angle (cf. Figure 1-4). Then with each vertex of the triangle as center draw an arc of radius $p_1 = p - s$ within the angle formed by extending the sides of the triangle. The set bounded by these arcs is of constant width $p + p_1$.

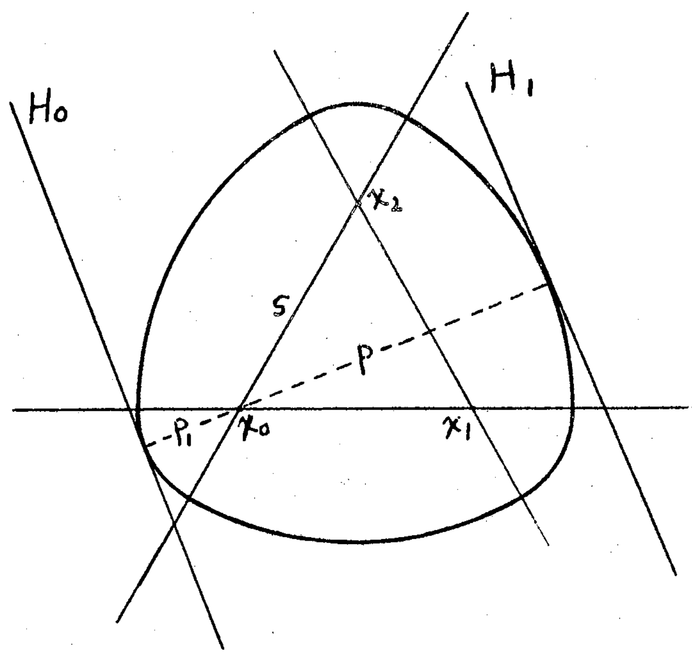

Figure 1-4.

In Figure 1-2 many tangent lines exist for the Reuleaux triangle at each of the boundary points x_0, x_1, and x_2. Any such boundary point with this property is called a corner point. The aforementioned variation of the Reuleaux triangle that is illustrated in Figure 1-4 has no corner point.

Any plane polygon of an odd number of sides can be used as the starting point for the construction of a plane set of constant width. For example in Figure 1-5, let x_0, x_1, and x_2 be the vertices of any triangle where the longest side has endpoints x_0 and x_1. With x_0 as center and $||x_0-x_1||$, the length of the line segment x_0x_1 as radius, construct an arc from x_1 to x_3. Next use x_2 as center and $||x_2-x_3||$ as radius and draw an arc from x_3 to x_4. Next with x_1 as center and $||x_4-x_1||$ as radius, draw an arc from x_4 to x_5. Similarly, using x_0 and x_2 as centers, $||x_5-x_0||$ and $||x_6-x_2||$ as radii respectively,

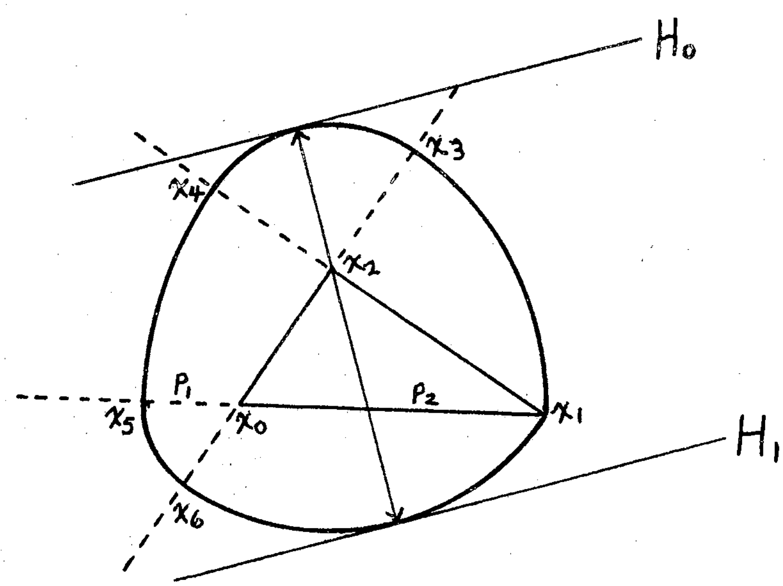

Figure 1-5.

finish the construction as shown. The set bounded by these arcs is a set of constant width and has a corner point at x_1. The distance between H_0 and H_1 is $||x_2-x_3|| + ||x_2-x_6||$. This sum is equivalent to $||x_6-x_0|| + ||x_0-x_3||$ which is equal to $p_1 + p_2$. Notice that the distance between any two parallel tangent lines is $p_1 + p_2$.

It is possible to begin with the same triangle as in Figure 1-5 and construct a set of constant width without a corner point. To eliminate the corner point, the first radius must be bigger than $||x_0 - x_1||$. This is illustrated in Figure 1-6. Note these sets in Figure 1-5 and 1-6 are not symmetric if the original triangle does not have a pair of sides of equal length.

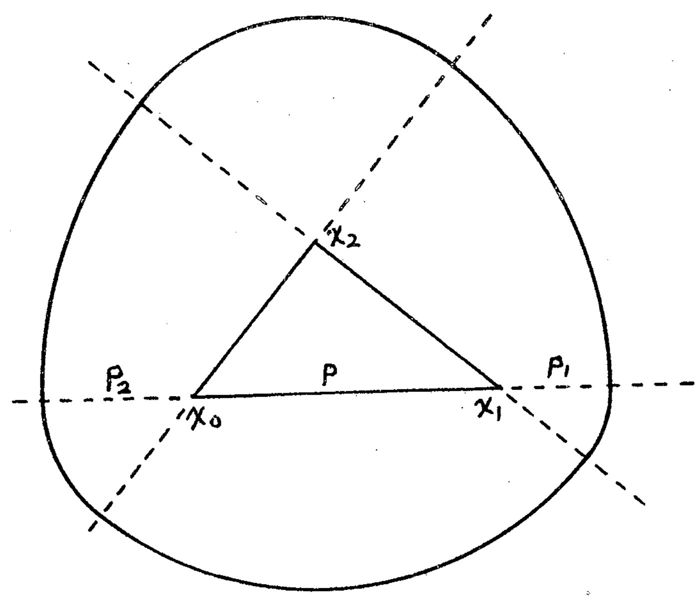

Figure 1-6.

A general method for constructing an unsymmetrical curve of constant width is called the star-polygon method. A convex polygon of an odd number of sides is taken for the basis of this construction. From each vertex construct two adjacent diagonals omitting equally many vertices on the opposite sides of the two diagonals. These diagonals form a star-polygon. Then arcs are drawn using each of the vertices as a center and an appropriate radius. The method is illustrated in Figure 1-7 for the polygon with vertices x_i, $0 \leq i \leq 6$.

From x_0 draw diagonals to x_3 and x_4 and extend them outside the polygon. Similarly draw two diagonals from each vertex. All these diagonals form the star-polygon. With x_0 as center, draw an arc from y_0 to y_1 where the radius of the arc is greater than any of the diagonals of the star-polygon. Use x_4 as center and $\|x_4-y_1\|$ as radius draw an arc from y_1 to y_2. Next use x_1 as center and $\|x_1-y_2\|$ as radius, draw an arc from y_2 to y_3. Continue this same process all around the polygon.

To show that the set bounded by the arcs is a set of constant width, let $\|y_7-y_0\| = \lambda$. Let H_0 and H_1 be a pair of parallel tangent lines as shown in Figure 1-7. The distance between H_0 and H_1 is $\|z_0-x_2\| + \|x_2-z_1\|$. By the equality of the radii of a circle,

$$\begin{aligned}\|z_0-x_2\| + \|x_2-z_1\| &= \|y_5-x_2\| + \|x_2-y_{12}\| \\ &= \|y_6-x_6\| + \|x_6-y_{13}\| \\ &= \|y_7-x_3\| + \|x_3-y_0\| \\ &= \|y_7-y_0\| \\ &= \lambda.\end{aligned}$$

In a similar manner, it can be shown that the distance between any two parallel tangent lines is λ.

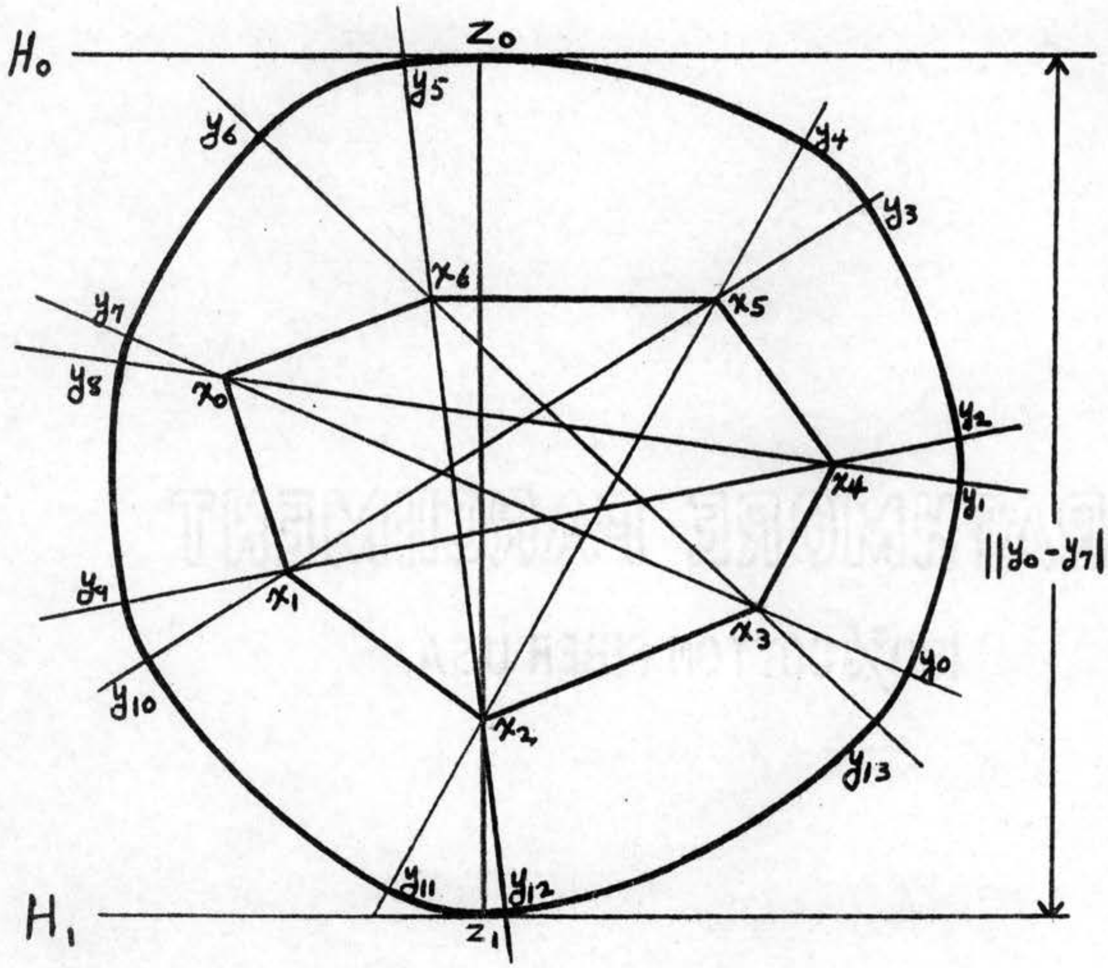
H_0
z_0
y_5
y_6
y_4
y_3
x_6
x_5
y_7
y_8
x_0
y_2
x_4
y_1
$\|y_0 - y_7\|$
y_9
x_1
x_3
y_0
y_{10}
x_2
y_{13}
y_{11}
y_{12}
H_1
z_1

Figure 1-7.

If all the diagonals of the star-polygon have equal length, it is possible to construct a set of constant width which passes through the vertices of the polygon. In this situation each vertex is a corner point. This is shown in Figure 1-8.

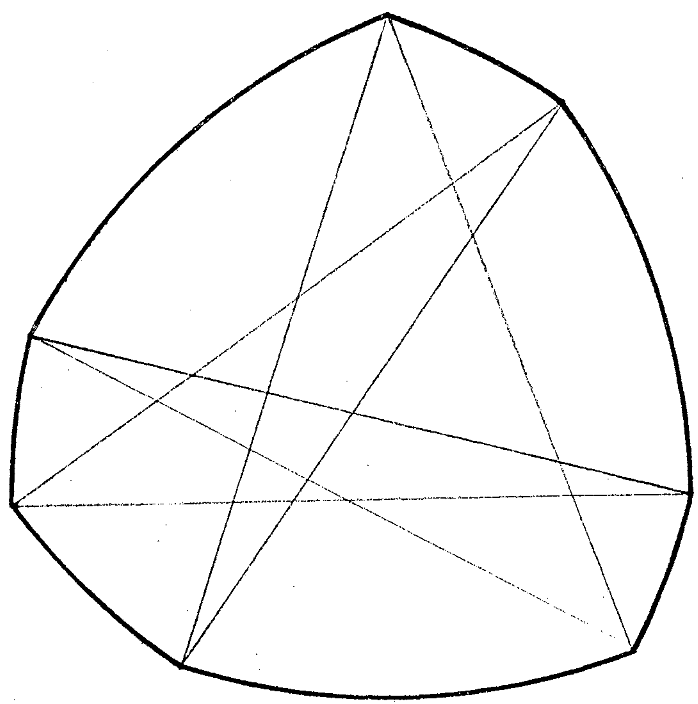

Figure 1-8.

It is known that plane sets of constant width exist such that no parts of their boundaries are arcs of circles. To indicate such a set (cf. Figure 1-9), let C_1, C_2, C_3 be three circles mutually tangent at x_0, x_1, and x_2. Any involute of the figure determined by the three arcs x_0x_1, x_1x_2, and x_2x_0 is a curve of constant width. For proof and further discussion, see [6] and [34].

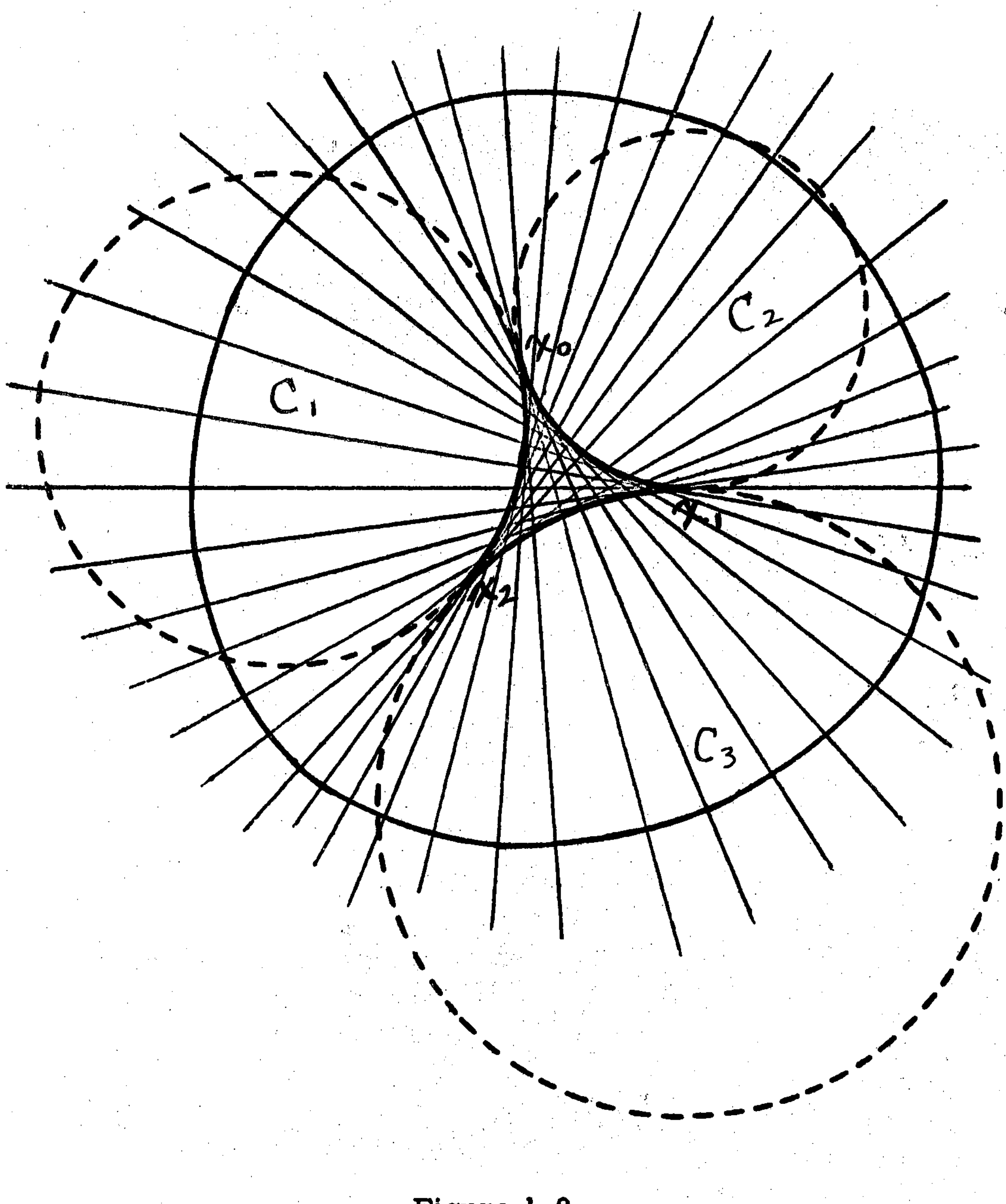

Figure 1-9.

As might be expected, there are three-dimensional objects of constant width. The sphere is one example. Any two parallel planes tangent to the sphere are a constant distance apart. If a pair of parallel tangent planes is fixed, the sphere can be rolled or rotated arbitrarily and always remains tangent to these planes.

The simplest example of a nonspherical solid of constant width is obtained by rotating a Reuleaux triangle about one of its axes of symmetry (cf. Figure 1-10). This object when viewed from an axis of symmetry appears like a circle, and when observed perpendicular to an axis, has the characteristic shape of a Reuleaux triangle.

A regular tetrahedron can be used to form a set of constant width. A spherical cap is placed on each face of the tetrahedron. Each spherical cap has the opposite vertex as center and a radius

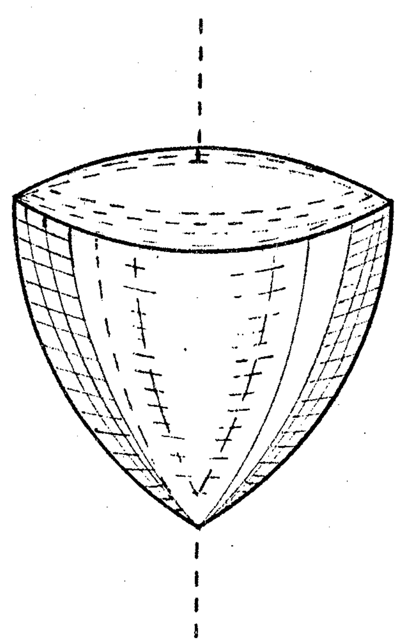

Figure 1-10.

equal to an edge of length s of the tetrahedron. Consider the trihedral angle determined by x_0 (cf. Figure 1-11) and the three edges emanating from x_0. Remove from the spherical cap on $x_1x_2x_3$ any part not contained in this trihedral angle. This is repeated using each vertex x_i and the corresponding trihedral angle determined by x_i, $1 \leq i \leq 3$ and three edges emanating from x_i.

Figure 1-11 shows a cross section T of the spherical cap on $x_1x_2x_3$ formed by arc c and edge x_1x_2. This cross section was formed when a part of the spherical cap was removed by the trihedral angle with vertex x_0. Another cross section T' on edge x_1x_2 and congruent to T is formed when the spherical cap on $x_2x_1x_0$ is cut by the trihedral angle with vertex x_3. Rotate c about edge x_1x_2 through an appropriate angle so that at the end of the rotation c matches the corresponding curve on T'.

When this is done at each edge, the set bounded by the parts of the spherical caps remaining and these rotations of an arc at each edge

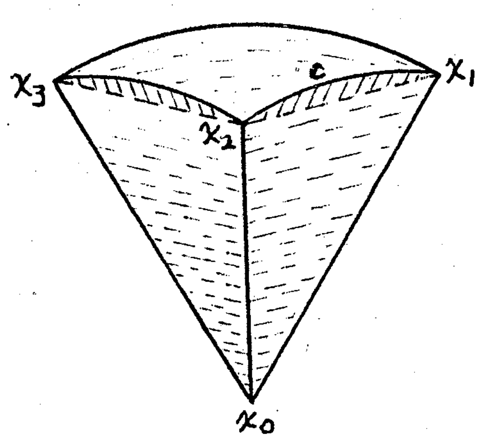

Figure 1-11.

is a set of constant width. This process is the three-dimensional analogue of the process used in the plane as illustrated in Figure 1-4.

All the examples of sets of constant width so far are in E_2 or E_3. But the discussion of sets of constant width fits naturally in E_n.

Euclidean space, E_n, is a real linear space where the vectors $x = (x_1, \ldots, x_n)$ are the real n-tuples. Addition and scalar multiplication are componentwise. There is an inner product

$$(x, y) = \sum_{i=1}^{n} x_i y_i$$

defined for each pair of vectors x and y of E_n, and there is a norm

$$||x|| = \sqrt{(x, x)} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

defined for each vector x in E_n.

It is easy to note that the inner product has the following properties:

1. Linearity:

   $(x+y, z) = (x, z) + (y, z), \quad$ for $x, y, z \in E_n$,

   $(\alpha x, y) = \alpha(x, y), \quad$ for $x, y \in E_n$, $\alpha \in R$.

2. Symmetry:

   $(x, y) = (y, x), \quad$ for $x, y \in E_n$.

3. Positivity:

   $(x, x) > 0, \quad$ if $x \neq \emptyset$ where $\emptyset$ represents the origin.

Using the linearity property of inner product and the property of the additive zero in a linear space, one can show $(\emptyset, x) = 0$ for any $x \in E_n$.

Furthermore, it is also easy to observe that the norm has the following properties:

1. Triangle inequality:

$$||x+y|| \leq ||x|| + ||y||,$$

2. Absolute homogeneity:

$$||\alpha x|| = |\alpha| \; ||x|| \; ,$$

3. $\qquad ||x|| \geq 0$,

4. $\qquad ||x|| \neq 0 \quad \text{if } x \neq \emptyset.$

The distance between x and y is

$$||x-y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \; .$$

Thus $||x||$ represents the distance between the origin, $\emptyset$, and x.

A topology is defined for E_n in terms of the norm. An open set in E_n is a set S such that if $a \in S$, then there exists

$$N(a, r) = \{x : ||x-a|| < r\}$$

contained in S. With this topology, E_n is complete in the sense that each Cauchy sequence in E_n has a limit in E_n. Thus, in summary, E_n is a complete, normed, finite dimensional linear topological inner product space.

In the plane E_2, the tangent lines played a role in observing that a set had constant width. Similarly, in E_3, tangent planes were used. The appropriate generalization of these ideas to E_n is the supporting hyperplane. That is, in any linear space, a hyperplane H is a translate of a maximal proper subspace. Thus in E_2 and E_3 hyperplanes are lines and planes, respectively.

For any hyperplane H, there exists a non-trivial real linear functional f and a real scalar α such that $H = \{x : f(x) = \alpha\}$. Conversely, any such linear functional f defines a hyperplane. The inner product (x, u) for some fixed u which is not the origin is a linear function of x, and $(u, u) > 0$ implies that the function is not identically zero. In fact, if f is linear in E_n, then there is a fixed u such that $f(x) = (x, u)$ for all $x \in E_n$ (cf. [35], Thm 4.81c, page 245).

Therefore in E_n, hyperplanes can always be represented using the inner product. The set $\{x : f(x) = \alpha\}$ is sometimes abbreviated to $[f : \alpha]$.

Two hyperplanes H_0 and H_1 are parallel if one is a non-trivial translate of the other. This means either $H_0 = x_0 + H_1$ or $H_1 = y_0 + H_0$ where neither x_0 nor y_0 is the origin. The distance between parallel hyperplanes H_0 and H_1 is

$$\rho(H_0, H_1) = \inf \{||x-y|| : x \in H_0,\ y \in H_1\}.$$

A hyperplane $H = [f : \alpha]$ bounds a set A if $f(A) \geq \alpha$ or $f(A) \leq \alpha$, where $f(A) = \{f(x) : x \in A\}$. The hyperplane H is said to support the set A if A contains at least one point of A and H bounds A.

In the preceding examples, the "tangent lines" each contained only one point of the set, and the entire set was situated on one side of each "tangent line." Strictly speaking these "tangent lines" should be called lines of support. As a matter of fact, most of the lines of support at a corner point of a set of constant width are not tangent lines.

In E_2, $\rho(H_1, H_2)$ is simply the perpendicular distance between parallel lines of support.

In the examples the sets that were of constant width were all convex and compact. Thus, it is natural to make the following definition:

Definition 1-1. A compact convex set S in E_n is of constant width if every pair of parallel support hyperplanes are the same distance apart.

The first published account of sets of constant width that were not discs was in 1778 by Euler. Since then many mathematicians have studied and contributed to the theory of sets of constant width. Among these mathematicians in the earlier developments we find Reuleaux, Minkowski, Meissner, Blaschke, Lebesque, and Schilling. In more recent times Chakerian, Besicovitch, Eggleston, Cooke, Bonnesen, Fenchel, Hammer and Melzak have written and published articles on sets of constant width.

CHAPTER II

COMPLETE SETS

A bounded set is <u>complete</u> if the diameter of the set is increased whenever any point is adjoined to the set.

Consider the disc K in E_2 (cf. Figure 2-1). The diameter of $K \cup \{x_0\}$ is greater than the diameter of K for any point $x_0 \notin K$. Therefore K is complete as well as a set of constant width.

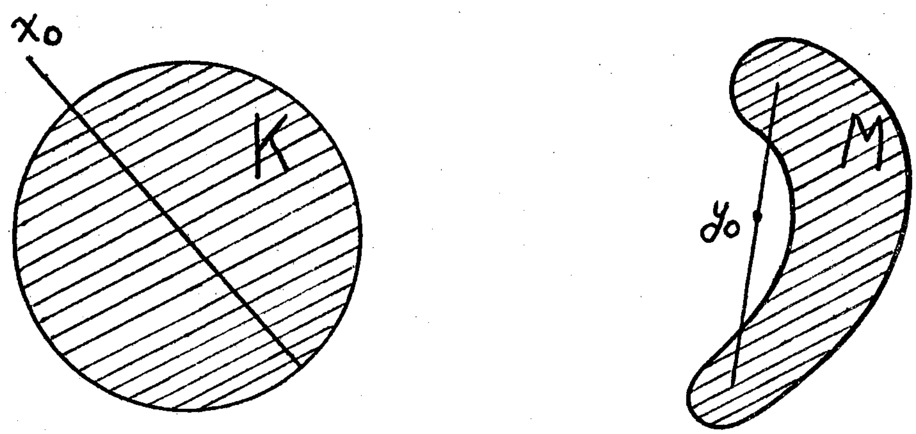

Figure 2-1.

For the set M, y_0 can be adjoined to M, and it is true that there are two points in M whose distance apart is greater than any distance from y_0 to any point in M. So the set M is not complete, and neither is it a set of constant width.

Ernst Meissner, a Swiss mathematician, first introduced the idea of a set being complete. A close relationship exists between complete sets and sets of constant width. A part of this relationship is stated in the following theorem:

Theorem 2-1: A set X in E_n is of constant width λ only if X is complete and of diameter λ.

Since the proof is long it is divided into lemmas. One of these is Lemma 2-3 which is very useful in itself and is used repeatedly elsewhere.

Lemma 2-1: If the set X is compact, there are two points $\bar{x}, \bar{y}$ in X so that $||\bar{x} - \bar{y}|| = D(X)$, the diameter of X.

Proof: The diameter of X is defined as follows:

$$D(X) = \sup \{ ||x - y|| : x, y \in X\}$$

Since X is a bounded set, D(X) is finite. Using the properties of the supremum, there are two sequences $\{x_n\}$ and $\{y_n\}$ of points in X so that

$$\lim_{n \to \infty} ||x_n - y_n|| = D(X).$$

If the sequence $\{x_n\}$ is an infinite set, the boundedness of X implies $\{x_n\}$ has an accumulation point $\bar{x}$ which belongs to X since X is closed. If, however, $\{x_n\}$ is a finite set, then some element must be repeated infinitely often. For simplicity, this element can also be denoted by $\bar{x}$. Notice again in this case that $\bar{x}$ is an element of X. In both cases, a subsequence, $\{x'_n\}$, of $\{x_n\}$ can be selected for which

$$\lim_{n \to \infty} x'_n = \bar{x}.$$

Similarly, there exists a subsequence, $\{y'_n\}$, of $\{y_n\}$ so that

$$\lim_{n \to \infty} y'_n = \bar{y} \in X.$$

It follows that

$$\lim_{n \to \infty} ||x'_n - y'_n|| = D(X).$$

It remains to show that $||\bar{x} - \bar{y}|| = D(X)$. For $\epsilon > 0$ there exist N_1 and N_2 such that $||x'_n - \bar{x}|| \leq \epsilon/3$ for $n > N_1$ and $||y'_n - \bar{y}|| < \epsilon/3$ for $n > N_2$. The statement

$$\lim_{n \to \infty} ||x'_n - y'_n|| = D(X)$$

assures the existence of N_3 such that for $n > N_3$,

$$D(X) - \epsilon/3 < ||x'_n - y'_n|| < D(X) + \epsilon/3.$$

Select $N = \max\{N_1, N_2, N_3\}$, then for $n > N$, one gets

$$\begin{aligned} D(X) - \epsilon/3 &< ||x'_n - y'_n|| \\ &\leq ||x'_n - \bar{x}|| + ||\bar{x} - \bar{y}|| + ||\bar{y} - y'_n|| \\ &< \epsilon/3 + ||\bar{x} - \bar{y}|| + \epsilon/3. \end{aligned}$$

Thus,

$$D(X) - \epsilon < ||\bar{x} - \bar{y}|| \leq D(X)$$

for any $\epsilon > 0$, and hence $D(X) = ||\bar{x} - \bar{y}||$.

<u>Lemma 2-2</u>: There are two parallel support hyperplanes $H_{\bar{x}}$, $H_{\bar{y}}$ of X at $\bar{x}$ and $\bar{y}$, respectively, so that the distance between $H_{\bar{x}}$ and $H_{\bar{y}}$ is

$$||\bar{x} - \bar{y}|| = D(X) = \lambda.$$

Proof: Let $H_{\bar{x}}$ and $H_{\bar{y}}$ be hyperplanes defined as follows:

$$H_{\bar{x}} = \{x : (x-\bar{x}, \bar{y}-\bar{x}) = 0\},$$

$$H_{\bar{y}} = \{x : (x-\bar{y}, \bar{x}-\bar{y}) = 0\}.$$

Clearly,

$$\bar{x} \in H_{\bar{x}} \text{ and } \bar{y} \in H_{\bar{y}}.$$

The positivity property of the inner product,

$$(\bar{y}-\bar{x}, \bar{y}-\bar{x}) > 0$$

shows that

$$\bar{y} \in (H_{\bar{x}})^{+} = \{x : (x-\bar{x}, \bar{y}-\bar{x}) > 0\}.$$

If we let $x_0 \in H_{\bar{x}}$, then

$$(x_0-\bar{x}, \bar{y}-\bar{x}) = 0.$$

Thus

$$(x_0, \bar{y}) - (x_0, \bar{x}) - (\bar{x}, \bar{y}) + (\bar{x}, \bar{x}) = 0.$$

Multiplying by two and adding two terms leads to

$$(x_0, x_0) - 2(x_0, \bar{x}) + 2(\bar{x}, \bar{x}) - 2(\bar{x}, \bar{y}) + (\bar{y}, \bar{y}) = (x_0, x_0) - 2(x_0, \bar{y}) + (\bar{y}, \bar{y}).$$

Therefore,

$$||x_0 - \bar{x}||^2 + ||\bar{x} - \bar{y}||^2 = ||x_0 - \bar{y}||^2.$$

This can be interpreted as the Pythagorean relationship for the triangle whose vertices are $\bar{x}, \bar{y}$ and x_0, where $\bar{x}$ is the vertex of the right angle and x_0 is an arbitrary point in $H_{\bar{x}}$ (cf. Figure 2-2).

It has already been shown that $\bar{y} \in (H_{\bar{x}})^{+}$ and $\bar{x} \in H_{\bar{x}}$. In order to show that $H_{\bar{x}}$ is a support hyperplane, it must now be shown that $(y-\bar{x}, \bar{y}-\bar{x}) \geq 0$ for every $y \in X$. An indirect method of proof will be

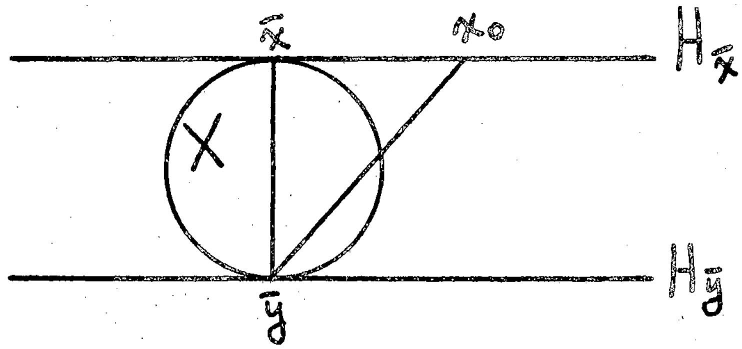

Figure 2-2.

used here, that is, suppose

$$(y-\bar{x},\ \bar{y}-\bar{x}) \ < \ 0. \tag{2-1}$$

From the properties of the inner product we know

$$(\bar{x}-\bar{y},\ \bar{y}-\bar{x}) \ < \ 0.$$

Therefore,

$$(y,\ \bar{y}-\bar{x}) \ < \ (\bar{x},\ \bar{y}-\bar{x}) \ < \ (\bar{y},\ \bar{y}-\bar{x}). \tag{2-2}$$

Let

$$\delta = \frac{(\bar{x},\ \bar{y}-\bar{x}) - (y,\ \bar{y}-\bar{x})}{(\bar{y},\ \bar{y}-\bar{x}) - (y,\ \bar{y}-\bar{x})}$$

$$= \frac{(\bar{x}-y,\ \bar{y}-\bar{x})}{(\bar{y}-y,\ \bar{y}-\bar{x})}\ .$$

From (2-2), $0 < \delta < 1$. Since $\bar{y}$ and y are points in X, it follows from the convexity of X that

$$z \ = \ \delta\bar{y} + (1 - \delta)\, y$$

is a point of X. Then

$$(z, \bar{y} - x) = (\delta \bar{y} + (1 - \delta) y, \bar{y} - \bar{x})$$

$$= \delta (\bar{y} - y, \bar{y} - \bar{x}) + (y, \bar{y} - \bar{x})$$

$$= \frac{(\bar{x} - y, \bar{y} - \bar{x})}{(\bar{y} - y, \bar{y} - \bar{x})} (\bar{y} - y, \bar{y} - \bar{x}) + (y, \bar{y} - \bar{x})$$

$$= (\bar{x} - y, \bar{y} - \bar{x}) + (y, \bar{y} - \bar{x})$$

$$= (\bar{x}, \bar{y} - \bar{x})$$

or

$$(z - \bar{x}, \bar{y} - \bar{x}) = 0.$$

Hence, $z \in H_{\bar{x}}$ (cf. Figure 2-3).

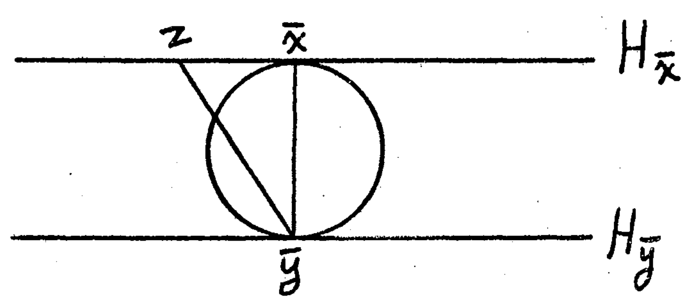

Figure 2-3.

By the Pythagorean relationship

$$||\bar{y} - \bar{x}||^2 + ||\bar{x} - z||^2 = ||\bar{y} - z||^2.$$

Since $||\bar{x} - z||^2 \geq 0$, then

$$||\bar{y} - \bar{x}|| \leq ||\bar{y} - z||.$$

An evaluation of $||\bar{y} - z||$ leads to the following:

$$\begin{aligned} ||\bar{y} - z|| &= ||\bar{y} - (\delta\bar{y} + (1 - \delta)y)|| \\ &= ||(1 - \delta)\bar{y} - (1 - \delta)y)|| \\ &= ||(1 - \delta)(\bar{y} - y)|| \\ &= (1 - \delta)\,||\bar{y} - y|| \\ &< ||\bar{y} - y|| . \end{aligned}$$

Therefore,

$$||\bar{y} - x|| \; < \; ||\bar{y} - y||. \tag{2-3}$$

However, since $y, \bar{y} \in X$, it must be true that

$$D(X) = ||\bar{y} - \bar{x}|| \geq ||\bar{y} - y||$$

which contradicts (2-3). So (2-1) is false. Therefore, $H_{\bar{x}}$ is a support hyperplane of X at $\bar{x}$, and by a similar argument $H_{\bar{y}}$ is a support hyperplane of X at $\bar{y}$.

It will now be shown that

$$H_{\bar{x}} = H_{\bar{y}} + (\bar{x} - \bar{y}),$$

that is, $H_{\bar{x}}$ and $H_{\bar{y}}$ are parallel hyperplanes. For any $y \in H_{\bar{y}}$,

$$(y - \bar{y},\; \bar{x} - \bar{y}) = 0.$$

Then

$$\begin{aligned} (y + \bar{x} - \bar{y},\; \bar{y} - \bar{x}) &= (y - \bar{y},\; \bar{y} - \bar{x}) + (\bar{x},\; \bar{y} - \bar{x}) \\ &= (\bar{x},\; \bar{y} - \bar{x}), \end{aligned}$$

or

$$((y - \bar{x} - \bar{y}) - \bar{x}, \bar{y} - \bar{x}) = 0.$$

Hence

$$y - \bar{x} - \bar{y} \in H_{\bar{x}},$$

and thus,

$$H_{\bar{x}} \supset H_{\bar{y}} + (\bar{x} - \bar{y}).$$

Similarly, for any $x \in H_{\bar{x}}$,

$$(x - \bar{x}, \bar{y} - \bar{x}) = 0.$$

Then

$$(x - \bar{x} + \bar{y}, \bar{x} - \bar{y}) = (x - \bar{x}, \bar{y} - \bar{x}) + (\bar{y}, \bar{x} - \bar{y})$$

$$= (\bar{y}, \bar{x} - \bar{y}),$$

or

$$((x - \bar{x} + \bar{y}) - \bar{y}, \bar{x} - \bar{y}) = 0.$$

This means $x - \bar{x} + \bar{y} \in H_{\bar{y}}$ or $x \in H_{\bar{y}} + (\bar{x} - \bar{y})$ and hence

$$H_{\bar{x}} \subset H_{\bar{y}} + (\bar{x} - \bar{y}).$$

So $H_{\bar{x}}$ is a translate of $H_{\bar{y}}$ and these two hyperplanes are parallel.

The distance between the hyperplanes $H_{\bar{x}}$ and $H_{\bar{y}}$,

$$\rho(H_{\bar{x}}, H_{\bar{y}}) = \inf \{ ||x - y|| : x \in H_{\bar{x}}, y \in H_{\bar{y}} \},$$

is less than or equal to $||\bar{x} - \bar{y}||$. Suppose there exist x_0, $y_0 \in H_{\bar{x}}$, $H_{\bar{y}}$, respectively, such that

$$||x_0 - y_0|| < ||\bar{x} - \bar{y}||. \tag{2-4}$$

First, since $x_0 \in H_{\bar{x}}$ and $y_0 \in H_{\bar{y}}$, it follows that

$$(x_0, \bar{y} - \bar{x}) = (\bar{x}, \bar{y} - \bar{x})$$

and

$$(y_0, \bar{x} - \bar{y}) = (\bar{y}, \bar{x} - \bar{y}).$$

Therefore,

$$(x_0 - y_0, \bar{y} - \bar{x}) = (\bar{x} - \bar{y}, \bar{y} - \bar{x}) \tag{2-5}$$

which is equivalent to

$$(x_0 - y_0 - \bar{x} + \bar{y}, \bar{y} - \bar{x}) = 0. \tag{2-6}$$

From (2-4) we have

$$||x_0 - y_0||^2 < ||\bar{x} - \bar{y}||^2$$

which implies

$$(x_0 - y_0, x_0 - y_0) < (\bar{x} - \bar{y}, \bar{x} - \bar{y}). \tag{2-7}$$

By (2-5) the right half of (2-7) is equal to $(x_0 - y_0, \bar{x} - \bar{y})$. Therefore,

$$(x_0 - y_0 - \bar{x} + \bar{y}, x_0 - y_0) < 0. \tag{2-8}$$

Combining (2-6) with (2-8) leads to

$$(x_0 - y_0 - \bar{x} + \bar{y}, x_0 - y_0 - \bar{x} + \bar{y}) < 0,$$

which is a contradiction of the positivity property of the inner product.

So it must be true that

$$\rho(H_{\bar{x}}, H_{\bar{y}}) = ||\bar{x} - \bar{y}||.$$

But, $||\bar{x} - \bar{y}|| = D(X)$. Since X is a set of constant width λ, $\rho(H_{\bar{x}}, H_{\bar{y}}) = \lambda$. Finally, $D(X) = \lambda$.

<u>Lemma 2-3</u>: For any closed convex set K in E_n and $y_0 \notin K$ there is a unique orthogonal projection $x_0 \in K$ of y_0 onto K and

$$H_{x_0} = \{x : (x - x_0, y_0 - x_0) = 0\}$$

as a support hyperplane of K.

Proof: The existence of x_0 is described in the lemma comes from the following important theorem in functional analysis (cf. [11], Thm. 1.12.3, p. 94):

Theorem 2-2: **If L is a Hilbert space, $K \subset L$, K is closed and** convex and $y_0 \in L$, then there exists a unique point $x_0 \in K$ such that $||y_0 - x_0|| \leq ||y_0 - x||$ for every $x \in K$.

The point x_0, with the property stated in the theorem, is called the orthogonal projection of y_0 onto K. Since Euclidean space is one example of a Hilbert space, in E_n the theorem asserts the existence of the orthogonal projection x_0 of y_0 onto K.

It remains to show

$$H_{x_0} = \{x : (x - x_0, y_0 - x_0) = 0\}$$

is a support hyperplane of K. Since $(y_0 - x_0, y_0 - x_0) > 0$, then $y_0 \in (H_{x_0})^+$. Furthermore, $(x_0 - x_0, y_0 - x_0) = 0$ shows that $x_0 \in H_{x_0}$ (cf. Figure 2-4). We wish to show H_{x_0} bounds K, that is $(y - x_0, y_0 - x_0) \leq 0$ for every $y \in K$.

Suppose

$$(y - x_0, y_0 - x_0) > 0 \tag{2-9}$$

for some $y \in K$. This means $y \in (H_{x_0})^+$ as shown in Figure 2-4.

Let

$$z = \delta(x_0 - y) + y = \delta x_0 + (1 - \delta) y$$

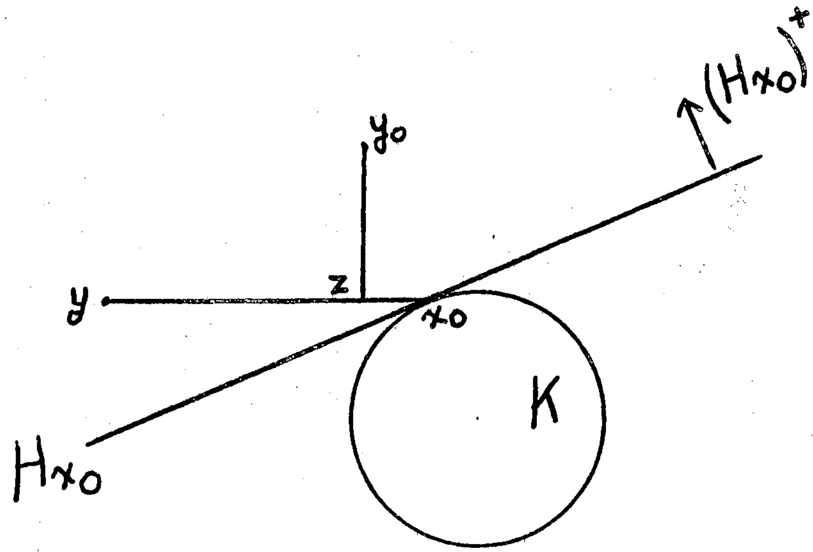

Figure 2-4.

where

$$\delta = \frac{(x_0 - y,\ y_0 - y)}{(x_0 - y,\ x_0 - y)} .$$

It shall be shown that $0 < \delta < 1$, and thus z is between y and x_0.

Inequality (2-9) leads to the following:

$$(y, y_0) - (x_0, y) > (x_0, y_0) - (x_0, x_0)$$

$$(x_0, x_0) - (x_0, y) > (x_0, y_0) - (y, y_0) . \qquad (2\text{-}10)$$

By adding $-(x_0, y) + (y, y)$ to (2-10),

$$(x_0, x_0) - (x_0, y) - (x_0, y) + (y, y) > (x_0, y_0) - (y, y_0) - (x_0, y) + (y, y)$$

or

$$(x_0 - y,\ x_0 - y) > (x_0 - y,\ y_0 - y).$$

This means that $\delta < 1$. Since $y \in K$ and x_0 is the orthogonal projection of y_0, then

$$(y_0 - y, y_0 - y) \geq (y_0 - x_0, y_0 - x_0). \tag{2-11}$$

Adding

$$(y, y_0 - x_0) > (x_0, y_0 - x_0)$$

to (2-11) gives:

$$(y, y_0 - x_0) + (y_0 - y, y_0 - y) > (y_0, y_0 - x_0),$$

or

$$(y - y_0, y_0 - x_0) + (y_0 - y, y_0 - y) > 0. \tag{2-12}$$

Since the real inner product is symmetric and since an interchange in signs in each argument leaves the outside sign positive, (2-12) can be rewritten as

$$(x_0 - y_0, y_0 - y) + (y_0 - y, y_0 - y) > 0$$

or

$$(x_0 - y, y_0 - y) > 0.$$

Therefore, the numerator of δ, as well as the denominator, is also positive and $0 < \delta < 1$. So by the convexity of K, $z = \delta x_0 + (1 - \delta) y$ is a point of K. By routine manipulation and substitution

$$\begin{aligned} ||y_0 - z||^2 + ||x_0 - z||^2 &= (y_0 - y - \delta(x_0 - y), y_0 - y - \delta(x_0 - y)) \\ &\quad + (x_0 - y - \delta(x_0 - y), x_0 - y - \delta(x_0 - y)) \\ &= (y_0 - y, y_0 - y) - 2\delta(x_0 - y, y_0 - y) \\ &\quad + \delta^2(x_0 - y, x_0 - y) + (1 - \delta)^2(x_0 - y, x_0 - y). \end{aligned} \tag{2-13}$$

Let

$$\alpha = (x_0 - y, y_0 - y)$$

and

$$\beta = (x_0 - y, x_0 - y),$$

then $\delta = \alpha\beta^{-1}$. Then (2-13) becomes

$$(y_0 - y, y_0 - y) - 2\,\delta\alpha + \delta^2\beta + (1 - \delta)^2\beta$$

or

$$(y_0 - y, y_0 - y) - 2\alpha^2\beta^{-1} + \beta(1 - 2\alpha\beta^{-1}) + 2\alpha^2\beta^{-1}$$

which is equivalent to

$$(y_0 - y, y_0 - y) + \beta - 2\alpha .$$

Using the values for α and β, yields

$$(y_0 - y, y_0 - y) + (x_0 - y, x_0 - y) - 2(x_0 - y, y_0 - y).$$

Rewriting this in the form

$$(y_0 - y, y_0 - y) - (x_0 - y, y_0 - y) + (x_0 - y, x_0 - y) - (x_0 - y, y_0 - y)$$

results in

$$(y_0 - y, y_0 - x_0) + (x_0 - y, x_0 - y_0)$$

which is equal to

$$(y_0 - x_0, y_0 - x_0) \text{ or } ||y_0 - x_0||^2.$$

Therefore

$$||y_0 - z||^2 + ||x_0 - z||^2 = ||y_0 - x_0||^2. \qquad (2\text{-}14)$$

Since $0 < \delta < 1$, then $z \neq x_0$ and $||x_0 - z||^2 > 0$. Hence, from (2-14),

$$||y_0 - z|| < ||y_0 - x_0||. \qquad (2\text{-}15)$$

However, since $z \in K$, we see that

$$||y_0 - z|| \geq ||y_0 - z_0||$$

which contradicts (2-15). Hence (2-9) is false and H_{x_0} bounds K.

Proof of Theorem 2-1: Lemmas 2-1 and 2-2 have shown that $D(X) = \lambda$. It remains only to show that X is complete. Suppose the contrary. Let y_0 be any point not in X and show

$$D(\{y_0\} \cup X) > D(X).$$

By Lemma 2-3, there is $x_0 \in X$ such that

$$||y_0 - x_0|| \leq ||y_0 - x||$$

for every $x \in X$. Furthermore,

$$H_{x_0} = \{x : (x - x_0, y_0 - x_0) = 0\}$$

is a support hyperplane of X.

Let

$$H = H_{x_0} + \lambda\mu \, (x_0 - y_0),$$

where

$$\mu = ||x_0 - y_0||^{-1} \tag{2-16}$$

and let

$$x' = x_0 + \lambda\mu \, (x_0 - y_0). \tag{2-17}$$

Then

$$||x_0 - x'||^2 = ||x_0 - x_0 - \lambda\mu \, (x_0 - y_0||^2$$

$$= \lambda^2 \mu^2 (x_0 - y_0, x_0 - y_0)$$

$$= \lambda^2,$$

and hence $||x_0 - x'|| = \lambda$ (cf. Figure 2-5).

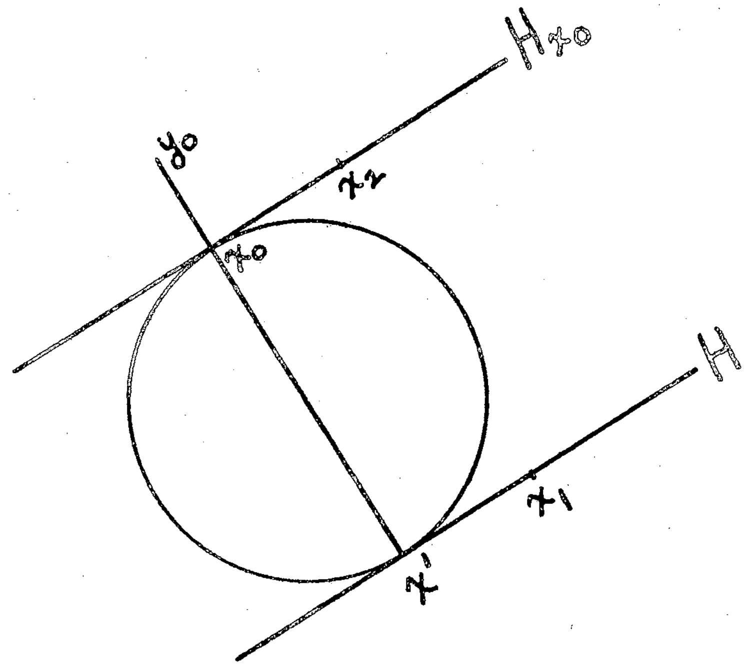

Figure 2-5.

It will now be shown that

$$H = \{x : (x, x_0 - y_0) = (x', x_0 - y_0)\}.$$

Let $x_1 \in H$ which means

$$x_1 = x_2 + \lambda\mu (x_0 - y_0)$$

for some $x_2 \in H_{x_0}$. Then

$$\begin{aligned}(x_1, x_0 - y_0) &= (x_2 + \lambda\mu (x_0 - y_0), x_0 - y_0) \\ &= (x_2, x_0 - y_0) + \lambda\mu (x_0 - y_0, x_0 - y_0) \\ &= (x_0, x_0 - y_0) + \lambda\mu (x_0 - y_0, x_0 - y_0) \\ &= (x_0 + \lambda\mu (x_0 - y_0), x_0 - y_0)\end{aligned}$$

or using (2-17)

$$= (x', x_0 - y_0).$$

Therefore,

$$(x_1, x_0 - y_0) = (x', x_0 - y_0),$$

which implies that

$$H \subset \{x : (x, x_0 - y_0) = (x', x_0 - y_0)\}.$$

To show inclusion the other way select x_3 so that

$$(x_3, x_0 - y_0) = (x', x_0 - y_0).$$

Then

$$(x_3 - \lambda\mu (x_0 - y_0), y_0 - x_0) = (x_3, y_0 - x_0) - \lambda\mu (x_0 - y_0, y_0 - x_0)$$

$$= -(x', x_0 - y_0) + \lambda\mu (x_0 - y_0, x_0 - y_0)$$

or using (2-17)

$$= (x_0, y_0 - x_0),$$

which means that

$$x_3 - \lambda\mu (x_0 - y_0) \in H_{x_0}.$$

Since

$$x_3 = x_3 - \lambda\mu (x_0 - y_0) + \lambda\mu (x_0 - y_0),$$

it follows that

$$x_3 \in H_{x_0} + \lambda\mu (x_0 - y_0) = H.$$

Therefore,

$$H \supset \{x : (x, x_0 - y_0 = (x', x_0 - y_0)\}.$$

The distance between H_{x_0} and H is λ. Since $x_0 \in H_{x_0}$, $x' \in H$, it follows that

$$\rho(H_{x_0}, H) \leq ||x_0 - x'|| = \lambda.$$

To show that $\rho(H_{x_0}, H) = \lambda$ suppose there exist y_1, y_2 where $y_1 \in H_{x_0}$, $y_2 \in H$ and

$$||y_1 - y_2|| < ||x_0 - x'|| = \lambda. \quad (2\text{-}18)$$

If $y_2 \in H$, then

$$y_2 = x_3 + \lambda\mu (x_0 - y_0)$$

where $x_3 \in H_{x_0}$. Then

$$||y_1 - y_2||^2 = ||y_1 - x_3 - \lambda\mu (x_0 - y_0)||^2$$

$$= (y_1 - x_3, y_1 - x_3) - 2\lambda\mu (x_0 - y_0, y_1 - x_3) + \lambda^2\mu^2 (x_0 - y_0, x_0 - y_0).$$

Since y_1 and x_3 are in H_{x_0},

$$(y_1 - x_0, y_0 - x_0) = 0,$$

$$(x_3 - x_0, y_0 - x_0) = 0,$$

and therefore, the term $2\lambda\mu (x_0 - y_0, y_1 - x_3)$ is zero. Therefore,

$$||y_1 - y_2||^2 = (y_1 - x_3, y_1 - x_3) + \lambda^2.$$

Since

$$(y_1 - x_3, y_1 - x_3) \geq 0,$$

thus

$$||y_1 - y_2|| \geq \lambda$$

which is contrary to (2-18). So the distance between H_{x_0} and H is λ.

The hyperplane H bounds the set X. Since, by definition of x',

$$(x' - x_0, x_0 - y_0) = (\lambda\mu (x_0 - y_0), x_0 - y_0) > 0,$$

and hence it follows that

$$(x', x_0 - y_0) > (x_0, x_0 - y_0)$$

which implies that $x_0 \in H^-$. So it is necessary to show

$$(y, x_0 - y_0) \leq (x', x_0 - y_0)$$

for all $y \in X$.

Suppose, however, that

$$(y, x_0 - y_0) > (x', x_0 - y_0) \tag{2-19}$$

for some $y \in X$ where $y \neq x'$. Using the inequalities

$$(y, x_0 - y_0) > (x', x_0 - y_0) > (x_0, x_0 - y_0),$$

it is possible to see that if

$$\alpha = \frac{(y - x', x_0 - y_0)}{(y - x_0, x_0 - y_0)},$$

$0 < \alpha < 1$. Let

$$z = \alpha x_0 + (1 - \alpha) y$$

$$= \alpha(x_0 - y) + y$$

which is in X by the convexity of X (cf. Figure 2-6). Using the definition of α, the following calculation:

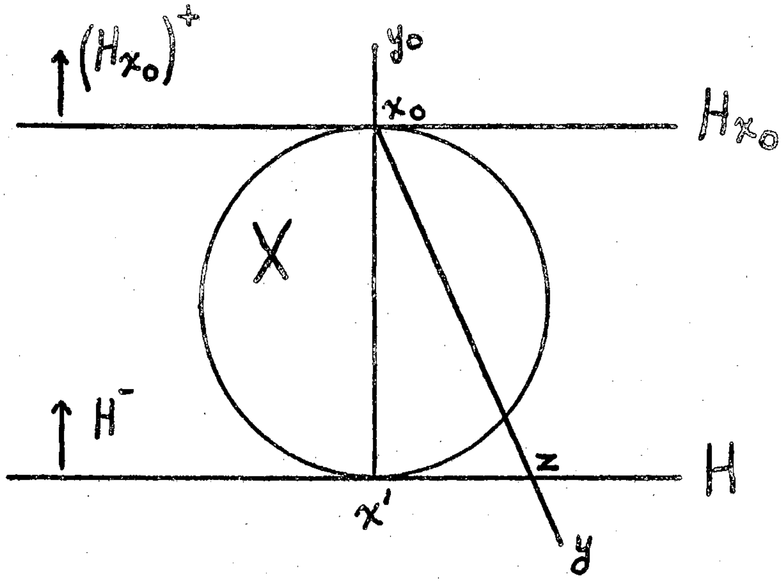

Figure 2-6.

$$(z,\ x_0 - y_0) = \alpha\,(x_0 - y,\ x_0 - y_0) + (y,\ x_0 - y_0)$$

$$= (x' - y,\ x_0 - y_0) + (y,\ x_0 - y_0)$$

$$= (x',\ x_0 - y_0),$$

shows that $z \in H$.

By using the properties of the inner product and substituting for x' and α,

$$(x', x'-x_0) - (z, x'-x_0) = (x_0 + \lambda\mu\,(x_0-y_0),\ \lambda\mu(x_0-y_0)) - (\alpha(x_0-y) + y,\ \lambda\mu(x_0-y_0))$$

$$= \lambda\mu\,(x_0,\ x_0-y_0) + \lambda^2 - \alpha\lambda\mu\,(x_0-y, x_0-y_0) - \lambda\mu(y, x_0-y_0)$$

$$= \lambda\mu(x_0, x_0-y_0 + \lambda^2 - \lambda\mu\,(x'-y, x_0-y_0) - \lambda\mu\,(y, x_0-y_0)$$

$$= \lambda\mu\,(x_0, x_0 - y_0) + \lambda^2 - \lambda\mu\,(x', x_0 - y_0)$$

$$= \lambda\mu\,(x_0, x_0 - y_0) + \lambda^2 - \lambda\mu\,(x_0 + \lambda\mu(x_0 - y_0), x_0 - y_0)$$

$$= \lambda\mu\,(x_0, x_0 - y_0) + \lambda^2 - \lambda\mu\,(x_0, x_0 - y_0) - \lambda^2$$

$$= 0\,.$$

Therefore,

$$(x', x' - x_0) = (z, x' - x_0). \qquad (2\text{-}20)$$

Expanding (2-20) and multiplying by 2 leads to

$$-2(x_0, x') + 2(x', x') - 2(x', z) = -2(x_0, z). \qquad (2\text{-}21)$$

Adding $(x_0, x_0) + (z, z)$ to both sides of (2-21) results in

$$(x_0, x_0) - 2(x_0, x') + 2(x', x') - 2(x', z) + (z, z) = (x_0, x_0) - 2(x_0, z) + (z, z)$$

which is equivalent to

$$||x_0 - x'||^2 + ||x' - z||^2 = ||x_0 - z||^2.$$

Since $x' \neq z$,

$$||x_0 - x'|| < ||x_0 - z||$$

which implies

$$\lambda < ||x_0 - z||.$$

But x_0 and z are both in X and $D(X) = \lambda$ means $||x_0 - z|| \leq \lambda$. This contradiction implies (2-19) is false. Therefore,

$$(y, x_0 - y_0) \leq (x', x_0 - y_0)$$

for every $y \in X$, and H is a hyperplane that bounds X.

If $x' \in X$, then H would be a supporting hyperplane for X. Suppose $x' \notin X$. It is impossible for any z, not x', to be in $H \cap X$ since $||x_0 - z|| > \lambda$. Therefore, $H \cap X = 0$, and X is not a set of constant width λ. So x' must be in X, and H is a supporting hyperplane of X at x'.

Since y_0, x_0, and x' are on the same line with x_0 in between y_0 and x',

$$||y_0 - x'|| = ||y_0 - x_0|| + ||x_0 - x'||.$$

Since $y_0 \neq x_0$, $||y_0 - x_0|| > 0$, and consequently,

$$||y_0 - x'|| > ||x_0 - x'|| = \lambda .$$

The preceding paragraphs show

$$D(X) < D(\{y_0\} \cup X)$$

for any $y_0 \notin X$. Hence X is complete.

CHAPTER III

PROPERTIES OF COMPLETE SETS

Completeness of a set as defined in Chapter II is a very simple notion. However, it turns out that a complete set has many significant properties, making completeness a powerful idea.

It will be convenient to use the following notation. For any $a \in E_n$, $r \in R$, let

$$N(a, r) = \{x : \|x - a\| < r\},$$

$$D(a, r) = \{x : \|x - a\| \leq r\},$$

$$C(a, r) = \{x : \|x - a\| = r\}.$$

In proving properties of complete sets, we will need the following lemma:

Lemma 3-1: For any bounded set S, D(S) = D(conv S). The symbol conv S represents the convex hull of the set S.

The proof is omitted since this is a well known result in general convexity (cf. [14], Thm. 12, p. 23).

A complete set has many remarkable properties. Lemmas 3-2 and 3-3 express two such properties. These properties give some insight into the structure of a complete set.

Lemma 3-2: If S is a complete set, then S is closed.

Proof: If S is not closed there must be an accumulation point x_0 for the set S but $x_0 \notin S$. The diameter $D(S \cup \{x_0\}) > \delta$ where $\delta = D(S)$. Thus, there is $x_1 \in S$ where $||x_1 - x_0|| > \delta$. Let

$$||x_1 - x_0|| - \delta = \epsilon > 0.$$

Consider $N(x_0, \epsilon/2)$. Since x_0 is an accumulation point of S,

$$N(x_0, \epsilon/2) \cap S \neq 0.$$

So let

$$x_2 \in N(x_0, \epsilon/2) \cap S.$$

By the triangle inequality

$$||x_1 - x_0|| \leq ||x_1 - x_2|| + ||x_2 - x_0||.$$

This inequality implies $\epsilon + \delta \leq \delta + \epsilon/2$ which says $\epsilon/2 \leq 0$, a contradiction.

Lemma 3-3: If S is complete set, then S is convex.

Proof: For any set S, $S \subset \text{conv } S$ and by completeness $S = \text{conv } S$ or $D(S) < D(\text{conv } S)$. By Lemma 3-1, $D(S) = D(\text{conv } S)$, so the only possibility is for $S = \text{conv } S$, Since conv S is a convex set, it follows that S is convex.

We will have need for the following notation:

$$S(X) = \cap \{D(x,\lambda) : x \in X\},$$

The next lemma further describes a complete set in terms of n-dimensional spheres whose centers are points of the set.

Lemma 3-4: For any complete set X of diameter λ, $S(X) = X$.

Proof: If we select any $x_0 \in X$, then $||x_0 - x|| \leq \lambda$ for every $x \in X$. This means $x_0 \in D(x, \lambda)$ for every $x \in X$. Therefore,

$$x_0 \in \bigcap \{D(x, \lambda) : x \in X\}$$

which proves $X \subset S(X)$.

Suppose $S(X) \not\subset X$. This means there is at least one $y \in S(X)$ where $y \notin X$. Thus, $y \in D(x, \lambda)$ for every $x \in X$. This implies

$$D(X \cup \{y\}) = D(X)$$

which contradicts the hypothesis of completeness.

In a complete set X, we find a generalized concept of convexity. In any convex set, the line segment determined by two points in the set is contained in the set. In a complete set of diameter λ, every arc of radius λ and joining any two points in the set is contained in the set, and by the convexity of the set, all points between the arcs are in the set (cf. Figure 3-1). A set having this property is called λ-arc convex. In a 3-dimensional set X, this condition implies that between any two points of X lies a football shaped region that is completely contained in X.

Lemma 3-5: If X is complete and $D(X) = \lambda$, then X is λ-arc convex.

Proof: Let x_1 and x_2 be any points of X. Then since $D(X) = \lambda$, $||x_1 - x_2|| \leq \lambda$. For simplicity let $\lambda = 1$. Consider any arc of radius one and with endpoints x_1 and x_2. Without loss of generality, let the

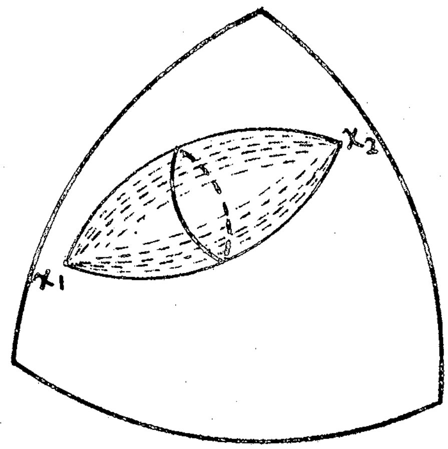

Figure 3-1.

center of the circle be the origin ϕ (cf. Figure 3-2).

Notice that

$$X = \bigcap \{D(x,\lambda) : x \in X\} \subset D(x_1, \lambda) \cap D(x_2, \lambda) = A.$$

That is, $x \in A$ if and only if $||x - x_1|| \leq 1$ and $||x - x_2|| \leq 1$.

Let x_4 be in arc x_1x_2 and show $x_4 \in X$. This result seems intuitively obvious when looking at the figure. However, the problem is an n-dimensional problem, and therefore requires an analytic proof.

Suppose $x_4 \notin X$. Then by completeness of X there is a point $y \in X$ so that $||x_4 - y|| > 1$. Since $y \in X \subset A$, $||y - x_1|| \leq 1$ and $||y - x_2|| \leq 1$. Also, if $x_4 \notin X$ then $x_4 \neq x_1$ or x_2. Hence it follows that $0 < \alpha < \beta \leq \pi/3$ where α and β are as indicated in Figure 3-2.

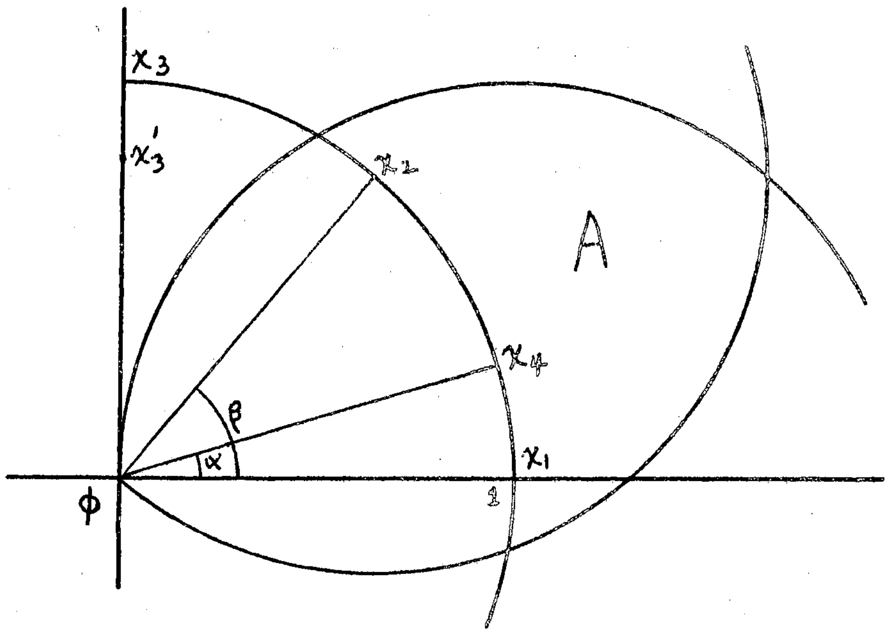

Figure 3-2.

The method of proof here is indirect and under the supposition $x_4 \notin X$, $||x_4 - y|| \leq 1$ will be shown to hold, which contradicts $||x_4 - y|| > 1$.

First, $||y - x_1|| \leq 1$ leads to the following:

$$||y - x_1||^2 = (y, y) - 2(x_1, y) + (x_1, x_1) \leq 1 .$$

Since $(x_1, x_1) = 1$, then

$$(y, y) - 2(x_1, y) \leq 0.$$

Therefore,

$$0 < (y, y) \leq 2(x_1, y). \tag{3-1}$$

Similarly,

$$0 < (y, y) \leq 2(x_2, y) \qquad (3\text{-}2)$$

follows from $||y - x_2|| \leq 1$.

Using the Gram-Schmidt orthogonalization process, let

$$x'_3 = x_2 - \frac{(x_1, x_2)}{||x_1|| \; ||x_1||} x_1 .$$

Recall that

$$||x_1|| = ||x_2|| = 1$$

and

$$(x_1, x_2) = ||x_1|| \; ||x_2|| \cos \beta .$$

Therefore,

$$(x_1, x_2) = \cos \beta ,$$

and

$$x'_3 = x_2 - (\cos \beta) x_1 .$$

Furthermore,

$$\begin{aligned} ||x'_3||^2 &= (x_2, x_2) - 2 \cos \beta (x_1, x_2) + \cos^2 \beta (x_1, x_1) \\ &= 1 - 2 \cos^2 \beta + \cos^2 \beta \\ &= 1 - \cos^2 \beta . \end{aligned}$$

Let

$$x_3 = \frac{x'_3}{||x'_3||} .$$

Therefore,

$$x_3 = \frac{1}{\sqrt{1 - \cos^2\beta}} x_2 - \frac{\cos\beta}{\sqrt{1 - \cos^2\beta}} x_1$$

$$= \frac{1}{\sin\beta} x_2 - \frac{\cos\beta}{\sin\beta} x_1 .$$

Then x_4 is equal to $(\cos\alpha)x_1 + (\sin\alpha)x_3$ or

$$x_4 = (\cos\alpha)x_1 + \frac{\sin\alpha}{\sin\beta} x_2 - \frac{(\sin\alpha)(\cos\beta)}{\sin\beta} x_1 . \tag{3-3}$$

Next the needed inequality

$$\sin\beta - \sin\alpha + \sin(\alpha - \beta) \le 0 \tag{3-4}$$

is established. This inequality is shown to hold for the special range of values $0 < \alpha < \beta \le \pi/3$ by the following argument:

First $\cos(\beta - \alpha) \le 1$ for any angles α and β and $\cos\alpha > \cos\beta$ or $\cos\alpha - \cos\beta > 0$ for $0 < \alpha < \beta \le \pi/3$. Thus,

$$\cos(\beta - \alpha) \le 1 + \cos\alpha - \cos\beta,$$

$$\cos\beta\cos\alpha + \sin\beta\sin\alpha \le 1 + \cos\alpha - \cos\beta,$$

or

$$\sin\beta\sin\alpha \le 1 + \cos\alpha - \cos\beta - \cos\alpha\cos\beta.$$

Factoring

$$\sin\beta\sin\alpha \le (1 + \cos\alpha)(1 - \cos\beta)$$

and multiplying by $\sin\alpha > 0$ gives

$$\sin\beta\sin^2\alpha \le \sin\alpha(1 + \cos\alpha)(1 - \cos\beta).$$

Substituting for $\sin^2\alpha$ gives

$$\sin\beta\ (1 - \cos^2\alpha) \leq \sin\alpha\ (1 + \cos\alpha)(1 - \cos\beta),$$

or

$$\sin\beta\ (1 + \cos\alpha)(1 - \cos\alpha) \leq (1 + \cos\alpha)(1 - \cos\beta)(\sin\alpha)$$

or

$$\sin\beta\ (1 - \cos\alpha) \leq \sin\alpha\ (1 - \cos\beta)$$

after dividing by $1 + \cos\alpha$. Then

$$\sin\beta - \sin\beta\cos\alpha \leq \sin\alpha - \sin\alpha\cos\beta$$

or

$$\sin\beta - \sin\alpha \leq \sin\beta\cos\alpha - \sin\alpha\cos\beta$$

which is

$$\sin\beta - \sin\alpha \leq \sin(\beta - \alpha),$$

$$\sin\beta - \sin\alpha - \sin(\beta - \alpha) \leq 0,$$

or

$$\sin\beta - \sin\alpha + \sin(\alpha - \beta) \leq 0.$$

For a specific y, $(x_2, y) \leq (x_1, y)$ or $(x_1, y) \leq (x_2, y)$. If $(x_2, y) \leq (x_1, y)$, the following will show $||x_4 - y|| \leq 1$. Observe

$$||x_4 - y||^2 = (y, y) - 2(x_4, y) + (x_4, x_4). \tag{3-5}$$

Using (3-2), (3-3), and $(x_4, x_4) = 1$, equation (3-5) leads to

$$||x_4 - y||^2 \leq 1 + 2(x_2, y) - 2\cos\alpha(x_1, y) - 2\frac{\sin\alpha}{\sin\beta}(x_2, y) + 2\frac{\sin\alpha\cos\beta}{\sin\beta}(x_1, y)$$

$$\leq 1 + 2\left[\frac{\sin\beta - \sin\alpha}{\sin\beta}\right](x_2, y) + 2\left[\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\beta}\right](x_1, y).$$

(3-6)

Using $0 < (x_2, y) \leq (x_1, y)$, the quantity of (3-6) is not greater than

$$1 + \frac{2}{\sin \beta} [\sin \beta - \sin \alpha + \sin \alpha \cos \beta - \cos \alpha \sin \beta] (x_1, y)$$

which is not greater than

$$1 + \frac{2}{\sin \beta} [\sin \beta - \sin \alpha + \sin (\alpha - \beta)] (x_1, y).$$

Therefore, using the inequalities $\sin \beta > 0$, $(x_1, y) > 0$, and (3-4), the inequality $||x_4 - y|| \leq 1$ follows.

If $(x_1, y) \leq (x_2, y)$, then it follows that

$$||x_4 - y||^2 = (y, y) - 2(x_4, y) + (x_4, x_4)$$

$$\leq 1 + 2(x_1, y) - 2 \cos \alpha (x_1, y) - 2 \frac{\sin \alpha}{\sin \beta} (x_2, y) + \frac{2 \sin \alpha \cos \beta}{\sin \beta} (x_1, y)$$

$$\leq 1 + 2 \left[\frac{\sin \beta - \sin \beta \cos \alpha + \sin \alpha \cos \beta}{\sin \beta} \right] (x_1, y) - 2 \frac{\sin \alpha}{\sin \beta} (x_2, y)$$

$$\leq 1 + \frac{2}{\sin \beta} [\sin \beta - \sin \beta \cos \alpha + \sin \alpha \cos \beta - \sin \alpha] (x_1, y)$$

$$\leq 1 + \frac{2}{\sin \beta} [\sin \beta - \sin \alpha + \sin (\alpha - \beta)] (x_1, y).$$

Again $||x_4 - y|| \leq 1$. So in either case, $||x_4 - y|| \leq 1$ which contradicts $||x_4 - y|| > 1$.

Therefore the supposition $x_4 \notin X$ is false and $x_4 \in X$. Since x_4 was an arbitrary point in the arc, the lemma is proved.

A <u>convex body</u> is a convex set which has at least one interior point. Every complete set, except a trivial set, turns out to be a convex body.

<u>Lemma 3-6</u>: If X is complete of diameter λ and has at least two points, then X is a convex body.

Proof: Let x_1, x_2 be any two points of X. By Lemma 3-3, X is convex and hence $x_3 = (1/2)x_1 + (1/2)x_2$ is a point in X. Let

$$r = ||x_1 - x_3|| \quad \text{and} \quad c = \lambda - \sqrt{\lambda^2 - r^2}\,.$$

It will be shown that $N(x_3, c) \subset X$, and therefore X has an interior point.

Select an arbitrary point $x_4 \in N(x_3, c)$. If x_4 is on line segment with endpoints x_1, x_2, then by convexity, $x_4 \in X$. If x_4 is not on the line segment, consider the plane determined by x_1, x_2, and x_4. In this plane let x_5, x_6 be centers of circles with radii λ and passing through x_1, x_2 as shown in Figure 3-3. By Lemma 3-5 and convexity, the set bounded by these arcs, which includes x_4, is in X. Therefore, $N(x_3, c) \subset X$.

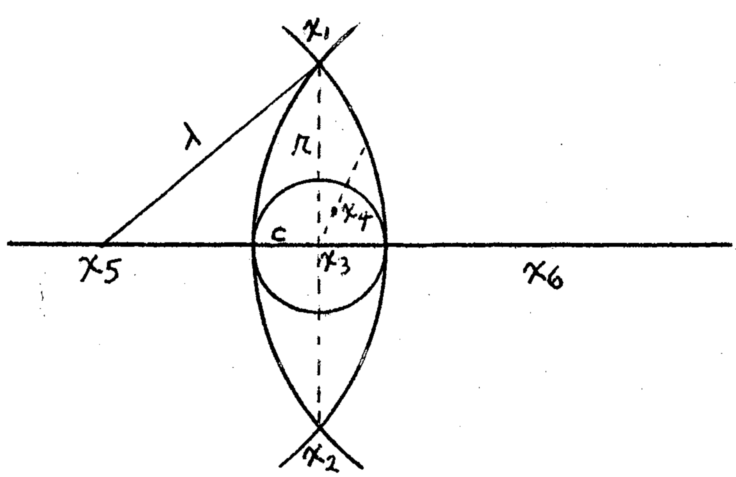

Figure 3-3.

The proof of Lemma 3-6 is somewhat intuitive since it can be reduced to a problem in the plane. However, an analytic proof can be supplied.

Lemma 3-7 states properties of a translate of a complete set. This may seem rather artificial, but its real significance will appear in the proof of Lemma 3-9.

Lemma 3-7: If X is complete and has diameter λ, then $X_1 = X + x - y$ is convex, complete and $D(X_1) = \lambda$.

Proof:

i) X_1 is convex. Let x_1, y_1 be in X_1 which means $x_1 = x_0 + x - y$, $x_2 = y_0 + x - y$ for some x_0 and $y_0 \in X$. For $0 \leq \alpha \leq 1$,

$$\begin{aligned} \alpha x_1 + (1-\alpha) y_1 &= \alpha x_0 + \alpha(x-y) + (1-\alpha) y_0 + (1-\alpha)(x-y) \\ &= \alpha x_0 + (1-\alpha) y_0 + (\alpha + 1 - \alpha)(x-y) \\ &= \alpha x_0 + (1-\alpha) y_0 + x - y \in X + x - y, \end{aligned}$$

which shows X_1 is convex.

ii) X_1 is complete. If X_1 is not complete, then there is some $x_0 \notin X_1$ for which

$$D(\{x_0\} \cup X_1) = \lambda .$$

Then $x_0 + y - x \notin X$ and by completeness of X,

$$D(\{x_0 + y - x\} \cup X) > \lambda .$$

This means there is $y_0 \in X$ so that

$$||(x_0 + y - x) - y_0|| > \lambda$$

or

$$||x_0 - (y_0 + x - y)|| > \lambda .$$

The point $y_0 + x - y \in X_1$ and the statement

$$||x_0 - (y_0 + x - y)|| > \lambda$$

contradict the hypothesis,

$$D(\{x_0\} \cup X_1) = \lambda .$$

iii) $D(X_1) = \lambda$. For every $x, y \in X$, $||x - y|| \leq \lambda$, and there are two points $\overline{x}, \overline{y} \in X$ so that $||\overline{x} - \overline{y}|| = \lambda$. Let x_2, y_2 be any points in X_1. Then $x_2 = x_1 + x - y$ and $y_2 = y_1 + x - y$ for some x_1, y_1 in X and

$$||x_2 - y_2|| = ||x_1 - y_1|| \leq \lambda.$$

This means $D(X_1) \leq \lambda$. Since $\overline{x}, \overline{y} \in X$, $\overline{x} + x - y$ and $\overline{y} + (x - y)$ are in X_1. Consequently,

$$||\overline{x} + x - y - (\overline{y} + x - y)|| = ||\overline{x} - \overline{y}|| = \lambda,$$

Combining this result with $D(X_1) \leq \lambda$ leads to $D(X_1) = \lambda$.

The following definitions are introduced to be used in the next set of lemmas:

$B_u = \rho(L_1, L_2)$ where L_1, L_2 are the parallel support hyperplanes perpendicular to u and where $||u|| = 1$,

$W = \inf \{ B_u : u \in E_n, \ ||u|| = 1\}$,

$W_u = \sup \{ ||x - y|| \ ; x, y \in X, \ x - y = \gamma u, \ \gamma \in R\}$,

$$d = \inf \{W_u : u \in E_n, \ ||u|| = 1\} .$$

Lemma 3-8: If X is complete then for any arbitrary direction u there must be $\bar{x}, \bar{y} \in X$ so that $||\bar{x} - \bar{y}|| = W_u$.

Since X is a bounded set, the W_u is finite. Using the meaning of supremum, there are two sequences $\{x_n\}$ and $\{y_n\}$ consisting of points in X where for each n, $x_n - y_n$ has direction u and

$$\lim_{n \to \infty} ||x_n - y_n|| = W_u.$$

The sequence $\{x_n\}$, if it is an infinite set, must have an accumulation point $\bar{x}$ by the boundedness of X. If it is a finite set, then some point is repeated infinitely often which can also be denoted by $\bar{x}$. In either case, it is possible to select a subsequence $\{x'_n\}$ of $\{x_n\}$ so that

$$\lim_{n \to \infty} x'_n = \bar{x}.$$

The vector $\bar{x} \in X$ since X is closed.

Similarly, there is a sequence $\{y'_n\}$ so that

$$\lim_{n \to \infty} y'_n = \bar{y} \in X.$$

Furthermore,

$$\lim_{n \to \infty} ||x'_n - y'_n|| = W_u.$$

Using these three limit statements, it will be shown that $||\bar{x} - \bar{y}|| = W_u$. For $\epsilon > 0$ there are three numbers N_1, N_2, N_3 so that

$$||x'_n - \bar{x}|| \leq \epsilon/3$$

whenever $n > N_1$,

$$||y'_n - \bar{y}|| \leq \epsilon/3$$

whenever $n > N_2$, and

$$\Big| \, ||x'_n - y'_n|| - W_u \Big| \leq \epsilon/3$$

whenever $n > N_3$. Let

$$N = \max\{N_1, N_2, N_3\}.$$

Then

$$\begin{aligned} W_u - \epsilon/3 &\leq ||x'_n - y'_n|| \\ &\leq ||x'_n - \bar{x}|| + ||\bar{x} - \bar{y}|| + ||\bar{y} - y'_n|| \\ &\leq 2\epsilon/3 + ||\bar{x} - \bar{y}|| \end{aligned}$$

whenever $n > N$. Therefore,

$$W_u - \epsilon \leq ||\bar{x} - \bar{y}||,$$

but by definition

$$||\bar{x} - \bar{y}|| \leq W_u.$$

Therefore,

$$W_u - \epsilon \leq ||\bar{x} - \bar{y}|| \leq W_u$$

for every positive ϵ and hence

$$||\bar{x} - \bar{y}|| = W_u.$$

Lemma 3-9: For a complete set X and x, y two points in X for which $||x - y|| = W_u$, there are parallel support hyperplanes H_1, H_2 of X at x and y, respectively.

Proof: Let $X_1 = X + x - y$. We shall first show that $X \cap \text{int } X_1 = 0$.

On the contrary suppose $p_1 \in X \cap \text{int } X_1$. From functional analysis

$$\text{int } X_1 = \text{int } (X + x - y)$$

$$= \text{int } X + x - y.$$

Since $p_1 \in \text{int } X_1$,

$$p_1 = p + x - y$$

for some $p \in \text{int } X$, and

$$p_1 - p = x - y. \tag{3-7}$$

If $p \in \text{int } X$, there must be $N(p, \epsilon) \subset X$ for some $\epsilon > 0$. Define a real number

$$\delta = \frac{\epsilon/2 + ||p - p_1||}{||p - p_1||}.$$

Let

$$x_0 = p_1 + \delta(p - p_1) \tag{3-8}$$

and show $x_0 \in N(p, \epsilon) \subset X$ (cf. Figure 3-4).

Subtracting p from both sides of (3-8) leads to

$$x_0 - p = p_1 - p + \delta(p - p_1)$$

$$= (\delta - 1)(p - p_1)$$

or

$$x_0 - p = \frac{\epsilon/2}{||p - p_1||} (p - p_1).$$

Thus,

$$||x_0 - p|| = \epsilon/2 < \epsilon,$$

and hence

$$x_0 \in N(p, \epsilon) \subset X.$$

The statement

$$||p - p_1|| = \frac{||x_0 - p_1||}{\delta} \qquad (3\text{-}9)$$

follows from (3-8).

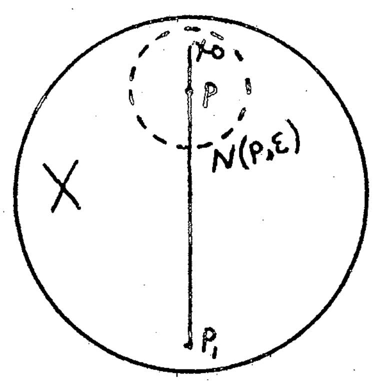

Figure 3-4.

Since $\delta > 1$ it follows that

$$\frac{||x_0 - p||}{\delta} < ||x_0 - p_1||. \qquad (3\text{-}10)$$

Then from (3-7) and (3-8),

$$\begin{aligned} x_0 - p_1 &= \delta\,(p - p_1) \\ &= \delta\,(y - x) \end{aligned}$$

means $x_0 - p_1$ has the same direction as $y - x$ and

$$||x_0 - p_1|| \leq ||x - y|| = W_u$$

since $x_0, p_1 \in X$. Therefore, from (3-9) and (3-10) $||p - p_1|| < W_u$ but

$$||p - p_1|| = ||x - y|| = W_u.$$

This contradiction shows $X \cap \text{int } X_1 = 0$.

By Lemma 3-7 X_1 is convex, complete, $D(X_1) = \lambda$, and X_1 is a convex body by Lemma 3-6. Then core X_1 = int X_1 since X_1 is a convex body (cf. Theorem 1.16 in [36]). Since $X \neq 0$, core $X_1 \neq 0$, $X \cap \text{core } X_1 = 0$, there is a hyperplane H_1 separating X and X_1 (cf. Figure 3-5 and Theorem 2.7 in [36]).

The hyperplane H_1 is a support hyperplane for X and X_1 since $x \in X \cap X_1$. A linear functional f, $f \neq 0$, and a real number α exists so that

$$H_1 = \{x : f(x) = \alpha\}.$$

Assign the value β to $f(y - x)$. With loss of generality assume $f(X_1) \geq \alpha$.

Let $H_2 = H_1 + y - x$ and show that H_2 is a support hyperplane for X. The point $x \in H_1$, and so $x + y - x = y$ is a point in H_2. In fact, $y \in X \cap H_2$. Let $x_0 \in X$ and $X = X_1 + y - x$. Then $x_0 = x_1 + y - x$ for some $x_1 \in X$. The value

$$f(x_0) = f(x_1) + \beta \geq \alpha + \beta.$$

If $H_2 = [f : \alpha + \beta]$, then H_2 would be a support hyperplane of X.

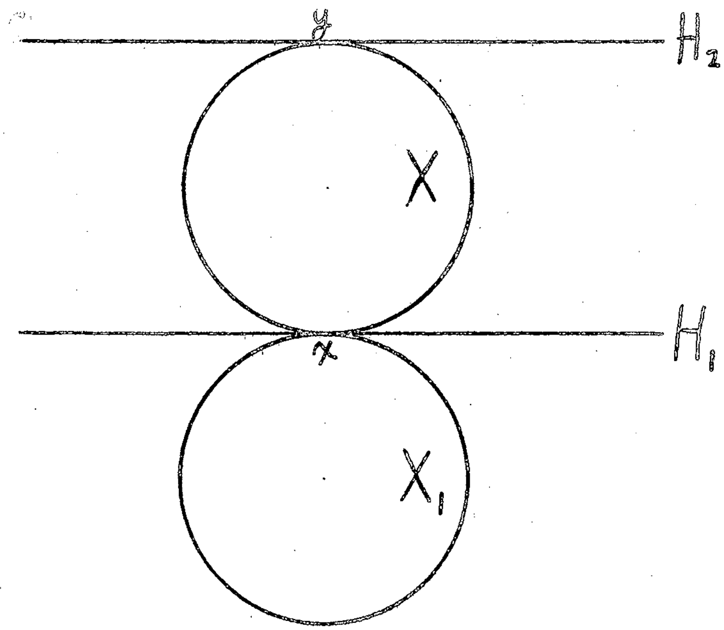

Figure 3-5.

If $h_2 \in H_2$ then $h_2 = h_1 + y - x$ for some $h_1 \in H_1$. Thus,

$$f(h_2) = f(h_1) + f(y - x)$$
$$= \alpha + \beta .$$

So the hyperplane $H_2 \subset [f : \alpha + \beta]$. If $x_0 \in [f : \alpha + \beta]$, then

$$f(x_0) = \alpha + \beta$$
$$= \alpha + f(y - x)$$

or

$$f(x_0 - y + x) = \alpha .$$

Thus,

$$x_0 - y + x \in H_1$$

which implies

$$x_0 \in H_1 + y - x = H_2.$$

Therefore, $[f: \alpha + \beta] \subset H_2$. So $H_2 = [f:\alpha+\beta]$ and H_2 is a support hyperplane for X. Since H_2 was defined as a translate of H_1, the two are parallel and satisfy the conclusion of the lemma.

Lemma 3-10: For the complete set X, d = W.

Proof: From the definitions of d and W, $d \leq W$. By Lemma 3-8 for any direction u, there is a chord xy in X so that $||x - y|| = W_u$. From Lemma 3-9 there are two parallel support hyperplanes H_1, H_2 of X at x, y, respectively, So

$$W \leq \rho(H_1, H_2) \leq ||x - y|| = W_u,$$

and this statement holds for every u. Therefore $\inf W_u \geq W$, but $\inf W_u = d$. The statement of the lemma follows from $d \geq W$ and $d \leq W$

Lemma 3-11: If X is complete, there are two points $x_1, x_2 \in X$ such that $||x_1 - x_2|| = d$.

Proof: By Lemma 3-8 for each u_n, there are two points $\overline{x}_n, \overline{y}_n \in X$ such that

$$||\overline{x}_n - \overline{y}_n|| = W_{u_n}.$$

The set

$$\{W_u : u \in E_n, \ ||u|| = 1\}$$

is bounded below by zero, and so $\inf \{W_u\} = d$ exists. Hence there are sequences $\{\overline{x}_n\}$, $\{\overline{y}_n\}$ so that

$$\lim_{n \to \infty} ||\overline{x}_n - \overline{y}_n|| = d.$$

The sequence $\{\bar{x}_n\}$ either has an accumulation point or some element repeated infinitely often. In either case, denote the element by x_1. Similarly $\{\bar{y}_n\}$ has such a point y_1. Select subsequences $\{x'_n\}$, $\{y'_n\}$ for which

$$\lim_{n \to \infty} x'_n = x_1 ,$$

$$\lim_{n \to \infty} y'_n = x_2 ,$$

and

$$\lim_{n \to \infty} ||x'_n - y'_n|| = d.$$

Both points x_1, x_2 are in X by the closure property of X.

The following standard limit procedure shows $||x_1 - x_2|| = d$. For $\epsilon > 0$ there exist three numbers N_1, N_2, N_3 so that

$$||x'_n - x_1|| < \epsilon/3$$

whenever $n > N_1$,

$$||y'_n - x_2|| < \epsilon/3$$

whenever $n > N_2$, and

$$\Big| \, ||x'_n - y'_n|| - d \, \Big| < \epsilon/3$$

whenever $n > N_3$.

For $n > N = \max \{ N_1, N_2, N_3 \}$,

$$||x_1 - x_2|| \leq ||x_1 - x'_n|| + ||x'_n - y'_n|| + ||y'_n - x_2||$$

$$< \epsilon/3 + (d + \epsilon/3) + \epsilon/3$$

$$= d + \epsilon.$$

Also,

$$\begin{aligned} d - \epsilon/3 &< ||x'_n - y'_n|| \\ &\leq ||x'_n - x_1|| + ||x_1 - x_2|| + ||x_2 - y'_n|| \\ &< \epsilon/3 + ||x_1 - x_2|| + \epsilon/3, \end{aligned}$$

which implies

$$d - \epsilon < ||x_1 - x_2||.$$

Combining

$$||x_1 - x_2|| < d + \epsilon$$

with

$$d - \epsilon < ||x_1 - x_2||$$

gives

$$\Big| \, ||x_1 - x_2|| - d \, \Big| < \epsilon .$$

Therefore, $||x_1 - x_2|| = d$.

Lemma 3-12: If X is a complete set and $x_1, x_2 \in X$ for which $||x_1 - x_2|| = d$, then there are parallel support hyperplanes H_1, H_2 of X at x_1, x_2, respectively, such that $x_1 - x_2$ is perpendicular to H_1 and H_2.

Proof: By Lemma 3-9 there exist two parallel support hyperplanes H_1, H_2 of X at x_1, x_2 respectively. Suppose $x_1 - x_2$ is not perpendicular to H_1 or H_2. Then

$$\rho(H_1, H_2) < ||x_1 - x_2|| = d = W \leq \rho(H_1, H_2).$$

But this last statement is impossible.

Lemma 3-13: If X is a complete set and has diameter λ, then for any x in the boundary of X, there is a point y in X for which $||x - y|| = \lambda$.

Proof: Suppose for every set $x \in X$, $||x - x_1|| < \lambda$. Since X is bounded, let

$$\lambda' = \sup \{ ||x - x_1|| : x \in X \} \leq \lambda.$$

If $\lambda' = \lambda$, there is a sequence $\{x_n\}$ of points in X such that

$$\lim_{n \to \infty} ||x_n - x_1|| = \lambda.$$

The sequence $\{x_n\}$ either has an accumulation point or some element is repeated infinitely often. In either case, denote the element by $\bar{x}$. By the closure of X, $\bar{x}$ is an element of X. One can select a subsequence $\{x'_n\}$ of $\{x_n\}$ so that

$$\lim_{n \to \infty} x'_n = \bar{x}$$

and

$$\lim_{n \to \infty} ||x'_n - x_1|| = \lambda.$$

For $\epsilon > 0$, there is N so that

$$\begin{aligned} \lambda - \epsilon/2 &\leq ||x'_n - x_1|| \\ &\leq ||x'_n - \bar{x}|| + ||\bar{x} - x_1|| \\ &\leq \epsilon/2 + ||\bar{x} - x_1|| \end{aligned}$$

whenever $n > N$. Thus,

$$\lambda - \epsilon \leq ||\bar{x} - x_1|| \leq \lambda$$

for every $\epsilon > 0$ which means $||\bar{x} - x_1|| = \lambda$. However, it was assumed $||x - x_1|| < \lambda$ for all $x \in X$. Therefore, it follows that for all $x \in X$,

$$||x - x_1|| \leq \lambda' < \lambda.$$

Now let $\epsilon = (1/2)(\lambda - \lambda')$ and consider a point $x_3 \in N(x_1, \epsilon)\setminus X$ (cf. Figure 3-6). For every $x \in X$,

$$\begin{aligned} ||x - x_3|| &\leq ||x - x_1|| + ||x_1 - x_3|| \\ &\leq \lambda' + \frac{\lambda - \lambda'}{2} \\ &= \frac{\lambda + \lambda'}{2} \\ &< \lambda \end{aligned}$$

which contradicts completeness of X. Therefore, there is $y \in X$ such that $||y - x_1|| = \lambda$.

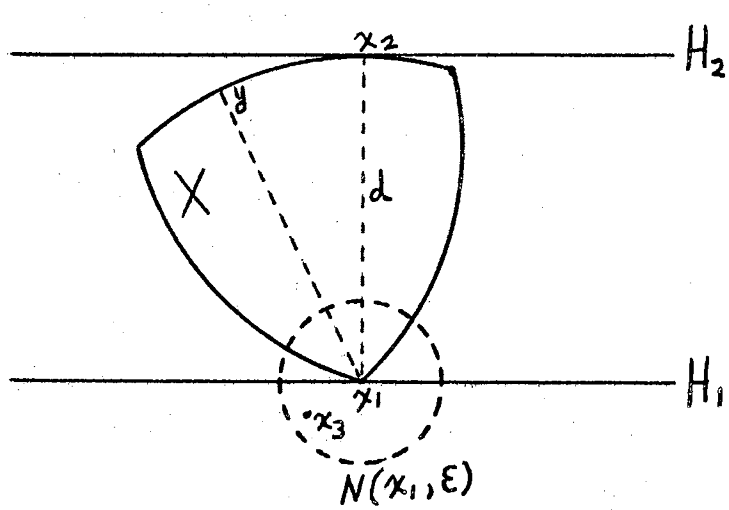

Figure 3-6.

CHAPTER IV

A COMPLETE SET IS A SET OF CONSTANT WIDTH

We now further investigate the relationship of complete sets and sets of constant width. It should be noted that Theorem 4-1 is the converse of Theorem 2-1. Thus together these theorems form a useful characterization of sets of constant width.

Theorem 4-1: If X is complete and has diameter λ, then X is a set of constant width λ.

Proof: An indirect proof will be given, and it will be divided into two parts.

(1) There are parallel support hyperplanes H_1, H_2 of X at x_1, x_2 of X, respectively, so that $||x_1 - x_2|| = d$, and $x_1 - x_2$ is perpendicular to both H_1 and H_2. (This is the same constant d as defined in Chapter III.)

(2) By Lemma 3-13, there is $y \in X$ such that $||y - x_1|| = \lambda$ since x_1 is a boundary point of X. The arc yx_2 of a circle through y and x_2 with radius λ and in the plane determined by y, x_1, and x_2 has a point x_4 which is not in X. This will give us a contradiction to Lemma 3-5.

Proof of (1): If X is not of constant width, there are two parallel support hyperplanes P_1, P_2 of X so that

$$\rho(P_1, P_2) = d \neq \lambda.$$

The value of d must be less than λ since $D(X) = \lambda$. Therefore $d < \lambda$ under the supposition that X is not of constant width.

Lemma 3-11 asserts the existence of two points x_1, x_2 in X such that $||x_1 - x_2|| = d$, and Lemma 3-12 shows that there are two parallel support hyperplanes H_1, H_2 of X at x_1, x_2 so that $x_1 - x_2$ is perpendicular to H_1 and H_2.

Proof of (2): Consider the plane determined by x_1, x_2 and y. Let $\emptyset$ be the center of a circle through x_2 and y with radius λ as shown in Figure 4-1.

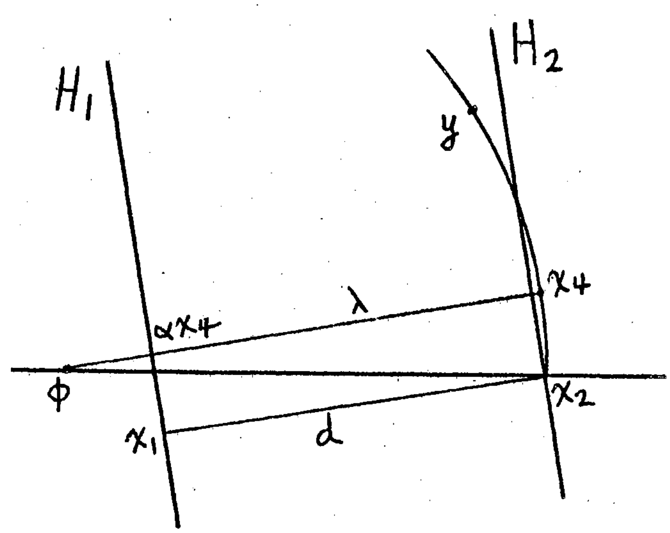

Figure 4-1.

Select the vector x_4 in arc yx_2 and such that the line determined by $\emptyset$ and x_4 is parallel to $x_2 - x_1$. It will be shown that $x_4 \notin X$ which will contradict Lemma 3-5.

First observe

$$(x_4, x_2 - x_1) = ||x_4|| \; ||x_2 - x_1|| \cos 0^\circ = \lambda d.$$

Select α so that

$$(\alpha x_4 - x_1, x_2 - x_1) = 0$$

which means $\alpha x_4 \in H_1$. From this, α must equal

$$\frac{(x_1, x_2 - x_1)}{(x_4, x_2 - x_1)} = \frac{(x_1, x_2 - x_1)}{\lambda d} .$$

If $\alpha \leq 0$,

$$\lambda - ||\alpha x_4|| > d$$

which means $x_4 \notin X$. If $\alpha > 0$,

$$||\alpha x_4|| = \sqrt{(\alpha x_4, \alpha x_4)}$$

$$= \alpha \sqrt{(x_4, x_4)}$$

or

$$\alpha \lambda = (x_1, x_2 - x_1) \lambda \lambda^{-1} d^{-1}$$

$$= (x_1, x_2 - x_1) d^{-1}.$$

It will be shown that

$$\lambda - ||\alpha x_4|| > d$$

which means $x_4 \notin X$. First,

$$\lambda - ||\alpha x_4|| = \lambda - (x_1, x_2 - x_1)\, d^{-1}$$

so if

$$(x_1, x_2 - x_1) \leq 0,$$

then

$$\lambda - ||\alpha x_4|| > d.$$

So assume

$$(x_1, x_2 - x_1) > 0. \qquad (4\text{-}1)$$

By the triangle inequality,

$$||\emptyset - x_2|| \leq ||\emptyset - x_1|| + ||x_1 - x_2||. \qquad (4\text{-}2)$$

If equality holds in (4-2), then the following relationships hold:

$$||x_2|| = ||x_1|| + ||x_1 - x_2||,$$

$$||x_2|| - ||x_1|| = ||x_1 - x_2||,$$

$$||x_2||^2 - 2||x_1||\ ||x_2|| + ||x_1||^2 = ||x_1 - x_2||^2$$

$$= ||x_1||^2 - 2(x_1, x_2) + ||x_2||^2$$

$$||x_1||\ ||x_2|| = (x_1, x_2).$$

From linear algebra, for $||x_1|| \neq 0$ and $||x_2|| \neq 0$, then

$$(x_1, x_2) = ||x_1||\ ||x_2||$$

if and only if x_1, x_2 are linearly dependent. So in this situation

$x_2 = \delta x_1$ for some $\delta \in \mathbb{R}$. Also, since

$$||x_1 - y||^2 = ||y||^2,$$

then

$$(x_1, x_1) - 2(x_1, y) + (y, y) = (y, y)$$

or

$$(x_1, x_1) = 2(x_1, y). \tag{4-3}$$

In particular,

$$(x_1, y) > 0. \tag{4-4}$$

Since $||x_1 - y|| = \lambda$ and $||x_2 - y|| \leq \lambda$, angle β is less than or equal to $\pi/3$ (cf. Figure 4-2). Consequently,

$$(y - x_1, x_2 - x_1) > 0. \tag{4-5}$$

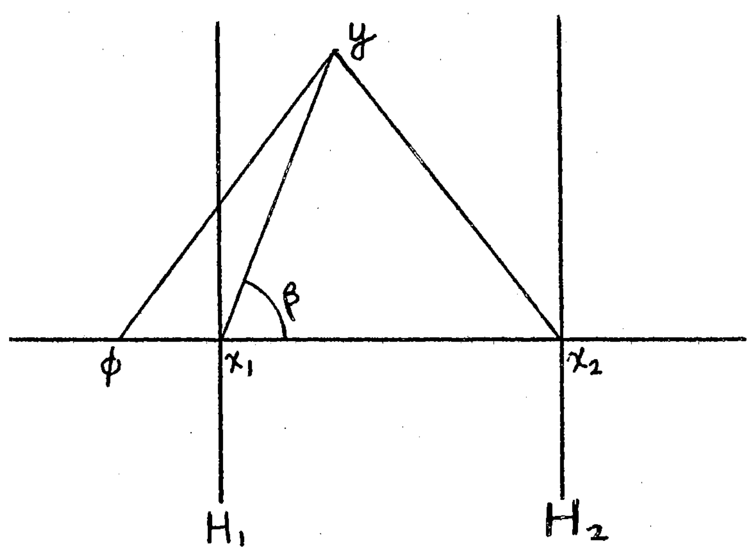

Figure 4-2.

By assumption (4-1), $(x_1, x_2 - x_1) > 0$ and so $(y, x_2 - x_1) > 0$. Then

$$(y, x_2 - x_1) = (y, \delta x_1 - x_1)$$

$$= (\delta - 1)(y, x_1)$$

$$> 0.$$

By (4-4) δ must be greater than 1. However, (4-5) implies

$$(y - x_1, \delta x_1 - x_1) = (\delta - 1)(y - x_1, x_1) > 0$$

and so

$$(\delta - 1)(x_1 - y, x_1) < 0$$

which implies

$$(x_1, x_1 - y) < 0 \qquad (4\text{-}6)$$

since $\delta > 1$. However, by (4-3) and (4-4),

$$(x_1, x_1) = 2(x_1, y)$$

and $(x_1, y) > 0$ proves

$$(x_1, x_1) > (x_1, y)$$

which means $(x_1, x_1 - y) > 0$, contradicting (4-6). Therefore,

$$||x_2|| < ||x_1|| + ||x_1 - x_2||,$$

and this is equivalent to

$$\lambda < ||x_1|| + d.$$

Transposing d and squaring gives

$$(\lambda - d)^2 < ||x_1||^2$$

or

$$-(x_1, x_1) < -(\lambda - d)^2. \qquad (4\text{-}7)$$

It is now possible to evaluate $(x_1, x_2 - x_1)$ in terms of λ and d as follows: Write

$$d^2 = ||x_1 - x_2||^2$$

in the form

$$d^2 = (x_1, x_1) - 2(x_1, x_2) + (x_2, x_2)$$

which is equivalent to

$$(x_1, x_1) - 2(x_1, x_2) = d^2 - (x_2, x_2)$$

$$= d^2 - \lambda^2.$$

Therefore,

$$2(x_1, x_2) - (x_1, x_1) = \lambda^2 - d^2 \qquad (4\text{-}8)$$

or

$$(x_1, x_2) = [\lambda^2 - d^2 + (x_1, x_2)]2^{-1}.$$

Rewriting equation (4-8) leads to

$$(x_1, x_2) - (x_1, x_1) = \lambda^2 - d^2 - (x_1, x_2).$$

Therefore,

$$0 < (x_1, x_2 - x_1) = \lambda^2 - d^2 - [\lambda^2 - d^2 + (x_1, x_1)]\, 2^{-1}$$

$$= [\lambda^2 - d^2 - (x_1, x_1)]\, 2^{-1},$$

Equation (4-7) implies

$$[\lambda^2 - d^2 - (x_1, x_1)]\, 2^{-1} < [\lambda^2 - d^2 - (\lambda - d)^2]\, 2^{-1}$$

$$= (2\, d\lambda - 2d^2)\, 2^{-1}$$

$$= d(\lambda - d).$$

Therefore,

$$0 < (x_1, x_2 - x_1) < d\,(\lambda - d)$$

and

$$\lambda - (x_1, x_2 - x_1)\, d^{-1} > d.$$

From this it follows that $\lambda - ||\alpha x_4|| > d$. Thus $x_4 \notin X$, a contradiction. Recall that the supposition X is not of constant width, thus leading to this contradiction. Therefore X being complete and of diameter λ implies X has constant width λ.

Meissner was the first one to recognize the close relationship of complete sets and sets of constant width as stated in Theorems 2-1 and 4-1. This relationship, which is a characterization of sets of constant width, can now be summarized in the following theorem:

Theorem 4-2: A set X in E_n is of constant width λ if and only if X is complete and of diameter λ.

Meissner proved the theorem for n = 2, 3 in 1911, but it was not until 1928 that Börge Jessen of Copenhagen proved the theorem for any integer n. The proof given here follows the outline of a proof sketched by Eggleston [14].

CHAPTER V

ENCLOSING SETS IN COMPLETE SETS

In the plane it seems intuitively obvious that a set of diameter λ could be contained in some set of constant width λ. In fact Pál was the first to prove this result in a paper published in 1920 [30]. Lebesque proved the general result for arbitrary n in 1921 [24]. In 1922, for the case of the plane, Reinhardt gave a proof different from the one given by Pál [32].

Two proofs of the general theorem will be given here. The first but longer proof uses the Blaschke Convergence Theorem (Blaschke Selection Theorem). The second proof uses Zorn's Lemma.

Let

$$A_\rho = \bigcup_{a \in A} D(a, r), \qquad 0 \leq \rho ,$$

be called the <u>parallel set</u> of A. If the distance between two non-empty bounded sets A and B in a Minkowski space L_n,

$$d(A, B) = \inf \{ \rho : A \subset B_\rho, \ B \subset A_\rho \},$$

then d defines a metric on the bounded sets in L_n. A sequence of convex sets A_i in a Minkowski space L_n is said to converge to a convex set A if

$$\lim_{i \to \infty} d(A_i, A) = 0.$$

The Blaschke Convergence Theorem states: A uniformly bounded infinite collection of closed convex sets in a Minkowski space contains a sequence which converges to a non-empty compact convex set (cf. [36] for a proof).

Theorem 5-1: A set M in E_n of diameter λ is a subset of a complete set of diameter λ.

Proof: The steps of the proof are as follows:

(1) The diameter of Cl (M), the closure of M, is D(M).

(2) The set $A(M) = \{x : D(\{x\} \cup M) = D(M)\}$ is closed.

(3) If $\rho(x, M) = \inf \{ ||x - m|| : m \in M\}$, then there is $\bar{x} \in A(M)$ for which $\rho(\bar{x}, M) = \rho(M) = \sup \{\rho(x, M) : x \in A(M)\}$.

(4) There exists a complete set M* such that $M \subset M^*$, $D(M^*) = \lambda$.

Proof of (1): Suppose $\lambda' = D(\text{Cl } M) > D(M) = \lambda$ and let $\epsilon = \lambda' - \lambda > 0$.

There must be two points $\bar{x}, \bar{y}$ in Cl M for which

$$||\bar{x} - \bar{y}|| > \lambda' - \epsilon/4 = \lambda + \epsilon - \epsilon/4$$

$$= \lambda + 3\epsilon/4.$$

Both $\bar{x}$ and $\bar{y}$ cannot be elements of M. So let $\bar{x}$ be in Cl $M \setminus M$. Since $\bar{x}$ is a limit point of M, there is a sequence $\{x_n\}$ of points in M and

$$\lim_{n \to \infty} x_n = \bar{x}. \tag{5-1}$$

Since $\overline{y}$ is in Cl M, there exists m* in $N(\overline{y}, \epsilon/4) \cap M$. Then using triangle inequality

$$||\overline{x} - m^*|| + ||m^* - \overline{y}|| \geq ||\overline{x} - \overline{y}||.$$

But

$$||\overline{x} - \overline{y}|| > \lambda' - \epsilon/4$$

results in

$$||\overline{x} - m^*|| \geq \lambda' - \epsilon/4 - ||m^* - \overline{y}||$$

$$> \lambda' - \epsilon/4 - \epsilon/4$$

or

$$||\overline{x} - m^*|| > \lambda' - \epsilon/2 = \epsilon + \lambda - \epsilon/2$$

$$= \lambda + \epsilon/2. \qquad (5\text{-}2)$$

From (5-1) for $\epsilon/2 > 0$, there must be an N so that $||x_n - \overline{x}|| < \epsilon/2$ whenever $n > N$.

Then for n_0 greater than N,

$$||\overline{x} - m^*|| \leq ||\overline{x} - x_{n_0}|| + ||x_{n_0} - m^*||$$

$$< \epsilon/2 + \lambda.$$

But this contradicts (5-2).

Hence for the theorem itself, let M be a closed set which does not affect its diameter. Since the diameter of M, λ implies M is bounded, then it can be assumed that M is a compact set.

Proof of (2): Let $A(M) = \{x : D(\{x\} \cup M) = D(M)\}$. Suppose A(M) has an accumulation point $\overline{x}$ which is not in A(M). Let $\lambda' = D(\{\overline{x}\} \cup M) > D(M) = \lambda$ and $\epsilon = \lambda' - \lambda$. Using the properties of an accumulation point, there is a sequence $\{x_n\}$ of points in A(M) for which

$$\lim_{n \to \infty} x_n = \bar{x}$$

(cf. Figure 5-1)

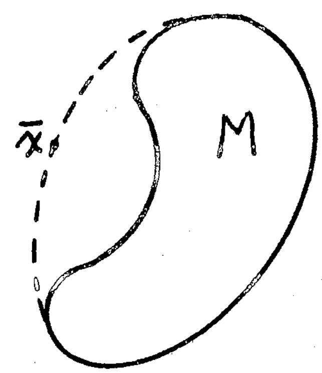

Figure 5-1.

Using the compactness of M and continuity of the norm function, there is an m* in M so that

$$||\bar{x} - m^*|| > \lambda' - \epsilon/2 = \lambda + \epsilon/2. \qquad (5\text{-}3)$$

For $\epsilon/2$, an N exists so that $||x_n - \bar{x}|| < \epsilon/2$, whenever $n > N$. Select x_{n_0} so that $n_0 > N$. Then

$$||\bar{x} - m^*|| \leq ||\bar{x} - x_{n_0}|| + ||x_{n_0} - m^*|| < \epsilon/2 + \lambda$$

which contradicts (5-3).

Proof of (3): Define $\rho(M) = \sup\{\rho(x, M) : x \in A(M)\}$. Since $\rho(x, M)$ is bounded above by λ for every x in A(M), $\rho(M)$ exists. Therefore, for each positive integer n, there is an x_n so that

$$\rho(x_n, M) > \rho(M) - 1/n.$$

The sequence $\{x_n\}$ is a bounded infinite set of points in A(M) or some point is repeated infinitely often. In either case a subsequence $\{x'_n\}$ can be selected which for simplicity can be called the subsequence $\{x_n\}$ and such that

$$\lim_{n \to \infty} x_n = \bar{x}.$$

Now $\bar{x}$ belongs to A(M) since A(M) is closed (cf. Figure 5-2).

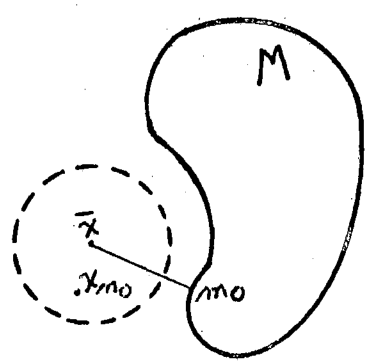

Figure 5-2.

To show the equality of $\rho(\bar{x}, M)$ and $\rho(M)$, suppose $\rho(\bar{x}, M) < \rho(M)$ and let $\rho(M) - \rho(\bar{x}, M) = \epsilon$. For $\epsilon/2$, there is N_1 so that $||x_n - \bar{x}|| < \epsilon/2$ whenever $n > N_1$. There also must be N_2 so that

$$\rho(x_n, M) > \rho(M) - 1/n > \rho(M) - \epsilon/2$$

whenever $n > N_2$. Let n_0 be greater than the maximum of N_1 and N_2. Therefore,

$$||x_{n_0} - \overline{x}|| < \epsilon/2$$

and

$$\rho(x_{n_0}, M) > \rho(M) - \epsilon/2. \tag{5-4}$$

The compactness of M assures the existence of a point m_0 in M for which

$$||\overline{x} - m_0|| = \rho(\overline{x}, M).$$

Then

$$||x_{n_0} - m_0|| \leq ||x_{n_0} - \overline{x}|| + ||\overline{x} - m_0|| < \epsilon/2 + \rho(\overline{x}, M).$$

But $\rho(\overline{x}, M) = \rho(M) - \epsilon$ and therefore

$$||x_{n_0} - m_0|| < \rho(M) - \epsilon/2.$$

From the definition

$$\rho(x_{n_0}, M) = \inf \{||x_{n_0} - m|| : m \in M\},$$

$$\rho(x_{n_0}, M) < \rho(M) - \epsilon/2,$$

contradicting (5-4).

Proof of (4): If M is not complete, select x_1 such that

$$D(\{x_1\} \cup M) = D(M)$$

and

$$\rho(x_1, M) = \sup \{\rho(x, M) : D(\{x\} \cup M) = D(M)\}.$$

By (3) such an x_1 exists.

If M is not complete, there is at least one $x_0 \in A(M)$ which is not in M. By the compactness of M, $\rho(x_0, M)$ is positive. Therefore, $\sup\{\rho(x, M) : x \in A(M)\}$ is positive which implies that x_1 is not in M (cf. Figure 5-3).

Let

$$M_1 = \text{Cl conv}(\{x_1\} \cup M).$$

By (1)

$$D(M_1) = D(\text{Cl } M_1) = D(\text{conv}(\{x_1\} \cup M)).$$

By Lemma 3-1

$$D(\text{conv}(\{x_1\} \cup M)) = D(\{x_1\} \cup M)).$$

Since $D(\{x_1\} \cup M) = \lambda$, it follows that $D(M_1) = \lambda$.

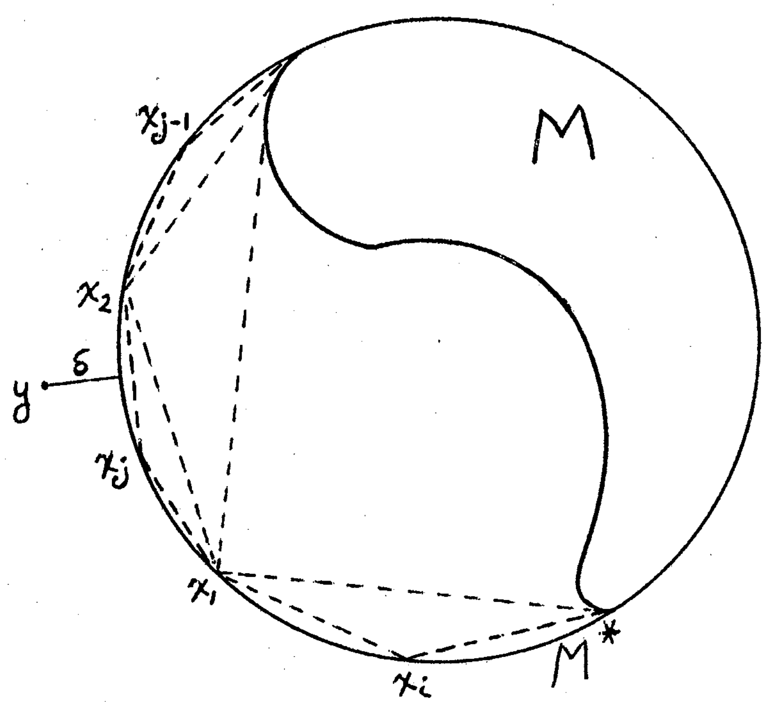

Figure 5-3.

Observe that M is a proper subset of M_1, and that M_1 is closed and convex. In a similar manner if M_1 is not complete, select x_2 not in M such that $D(\{x_2\} \cup M) = D(M)$, and

$$\rho(x_2, M_1) = \sup \{\rho(x, M_1) : D(\{x\} \cup M_1) = D(M_1) = \lambda\}.$$

Then let

$$M_2 = \text{Cl conv } (\{x_2\} \cup M_1).$$

The diameter if M_2 is λ and M_1 is a proper subset of M_2. The process is continued in this same manner.

If this process does not end after finitely many steps, there exists an infinite sequence $\{M_i\}$ of closed convex sets which is a properly increasing sequence and uniformly bounded. Therefore, by the Blaschke Selection Theorem, there is an infinite subsequence $\{M_i'\}$ from the sets in the sequence $\{M_i\}$ which converges to a compact convex set M^*.

A new sequence $\{M_i''\}$ is now selected from the sequence $\{M_i'\}$. The first element M_1'' is selected to be M_1'. If M_1'' is contained in M_2', then let M_2'' equal M_2'. If M_1'' is not contained in M_2', there is an integer $n_0 > 1$ so that M_1'' is contained in M'_{n_0}. This must be true for if we suppose M_1'' is not contained in M_n' for every $n > 1$, then M_n' is contained in M_1'' for infinitely many values of n, which is not true. In this case let

$$M_2'' = M'_{n_0}.$$

Continuing this process produces a subsequence $\{M_i''\}$ which is a subsequence of $\{M_i'\}$ and hence also converges to M^*. For simplicity denote the new subsequence, $\{M_i''\}$ by $\{M_i\}$. Observe that it is a properly increasing sequence of compact convex sets each of which has diameter λ.

Suppose there is a set M_{n_0} which is not contained in M*. This means there would be an x_0 in M_{n_0} but not in M* (cf. Figure (5-4). By the compactness of M*, there is m* in M* so that

$$\rho(x_0, M^*) = ||x_0 - m^*|| = \epsilon > 0.$$

Then from

$$\lim_{n \to \infty} d(M_n, M^*) = 0,$$

there is an integer N so that $d(M_n, M^*) < \epsilon/2$ whenever $n > N$. This means $M_n \subset (M^*)_{\epsilon/2}$ for $n > N$. If $n_0 > N$, then $M_{n_0} \subset (M^*)_{\epsilon/2}$. But this is impossible since $\rho(x_0, M^*) = \epsilon$. If $n_0 \leq N$, then

$$M_{n_0} \subset M_{n+1} \subset (M^*)_{\epsilon/2},$$

and there is the same difficulty. Therefore, the entire sequence $\{M_i\}$ is contained in M*.

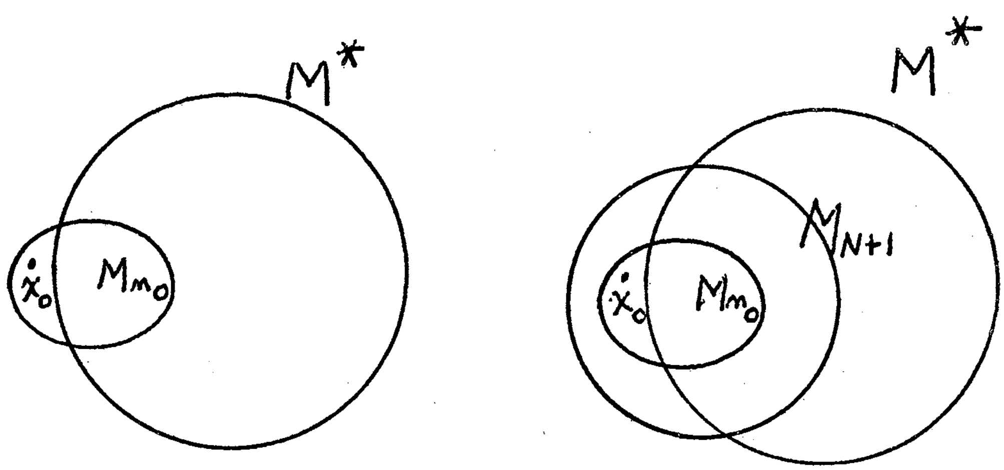

Figure 5-4.

To prove that $D(M^*) = \lambda$, observe that since $M_i \subset M^*$ and $D(M_i) = \lambda$ for every i, the diameter of M^* cannot be less than λ. So suppose $D(M^*) = \lambda' > \lambda$ and let $\lambda' - \lambda = \epsilon$. Then

$$d(M_i, M^*) = \inf \{\rho : M_i \subset (M^*)_\rho, \ (M^*) \subset (M_i)_\rho\}$$

$$= \inf \{\rho : M^* \subset (M_i)_\rho\}$$

$$\geq \epsilon/3 ,$$

which contradicts

$$\lim_{i \to \infty} d(M_i, M^*) = 0,$$

(cf. Figure 5-5). Therefore, $D(M^*) = \lambda$.

Suppose M^* is not complete. Thus there must be y not in M^* such that $D(\{y\} \cup M^*) = \lambda$ (cf. Figure 5-3). Let

$$\delta = \rho(y, M^*) = \inf \{||y - m|| : m \in M^*\}.$$

Since M^* is compact, there is m^* in M^* for which $||y - m^*|| = \delta$. The point y cannot be equal to m^*, and therefore δ is positive.

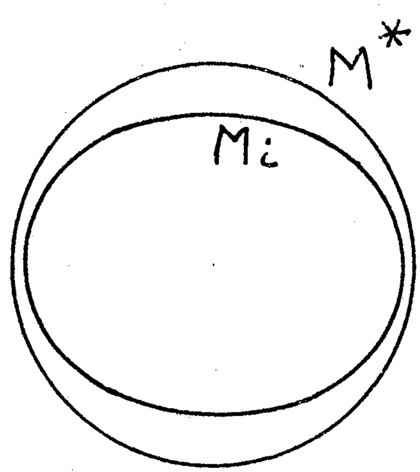

Figure 5-5.

Consider the two points x_i, x_j with $j > i$ where x_i and x_j are the special points used to define M_i and M_j. Let

$$\rho(M_{j-1}) = \rho(x_j, M_{j-1}) = \sup \{\rho(x, M_{j-1}) : D(\{x\} \cup M_{j-1}) = \lambda\}.$$

Since

$$D(\{y\} \cup M_{j-1}) = \lambda, \quad M_{j-1} \subset M_j, \quad y \notin M_j,$$

then

$$||y - x|| \leq \rho(M_{j-1})$$

for some x in M_j (cf. Figure 5-6).

Because $\delta = \rho(y, M^*)$ and all M_i are in M^*, $\delta \leq ||y - x||$. From $x_i \in M_{j-1}$, it follows that

$$\rho(M_{j-1}) \leq ||x_j - x_i||.$$

Therefore $0 < \delta \leq ||x_j - x_i||$. But it is impossible to have infinitely many distinct points in a compact region of E_n, every two of which are at least δ distance apart where δ is fixed and positive. Every infinite

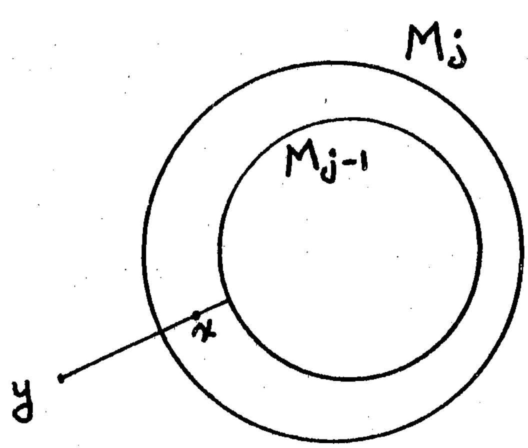

Figure 5-6.

subset of a compact subset of E_n must have an accumulation point. This contradiction shows that M* is complete. In fact, M* is a complete set of diameter λ of which M is a subset.

The second proof of Theorem 5-1 makes use of Zorn's Lemma which states that every non-empty partially ordered family $\mathfrak{B}$ in which every linearly ordered subset has an upper bound in $\mathfrak{B}$ must have a maximal element.

Let

$$\mathfrak{B} = \{F_i : M \subset F_i, \; D(F_i) = \lambda\}.$$

This family is non-empty since M is a member. Set inclusion partially orders $\mathfrak{B}$.

Let the set $\{F_\alpha : \alpha \epsilon A\}$ represent a linearly ordered subset $\mathfrak{B}'$ of $\mathfrak{B}$. Set

$$F = \bigcup_{\alpha \epsilon A} F_\alpha .$$

The set M is contained in F since M is in F_α for every α. Let x_i and x_j be points in F where $i \leq j$. Then x_i and x_j are in F_j and $D(F_j) = \lambda$ implies $||x_i - x_j|| \leq \lambda$. This means that $D(F) = \lambda$ and that F is an upper bound of $\mathfrak{B}'$ and F is in $\mathfrak{B}$.

Therefore, by Zorn's Lemma, there is a maximal element M* in $\mathfrak{B}$.

Suppose M* is not complete. Thus there must be x_0 not in M* such that $D(\{x_0\} \cup M^*) = \lambda$. Since M* is in $\mathfrak{B}$, M is in M* and so M is in $\{x_0\} \cup M^*$. Therefore the set $\{x_0\} \cup M^*$ is an element in the family $\mathfrak{B}$. But M* is properly contained in $\{x_0\} \cup M^*$ which contradicts the maximality property of M*. Therefore, M* is complete and satisfies the requirements of the theorem.

CHAPTER VI

OTHER PROPERTIES OF SETS OF CONSTANT WIDTH

There are other ways of characterizing sets of constant width besides noting that they are complete sets. A second characterization will be given in terms of boundary points. The following theorem summarizes this characterization:

<u>Theorem 6-1</u>: A set K is of constant width λ if and only if K is closed, convex with diameter λ and such that for each boundary point x_b of K there is a y in K whose distance from x_b is λ.

Proof: The "only if" portion follows from Lemma 3-13, hence it remains only to prove the "if" portion.

To prove this, by Theorem 4-2 it is sufficient to show that K is a complete set. Let p be an arbitrary point not in K. Let x_1 be the orthogonal projection of p onto K. This means x_1 is in K, and

$$||p - x_1|| \leq ||p - x|| \qquad (6\text{-}1)$$

for every x in K (cf. Figure 6-1).

If K is equal to the boundary of K, then x_1 is a boundary point. Suppose that K is not equal to the boundary of K. Then it will be shown that x_1 is a boundary point. Thus suppose that x_1 is an interior point of K. This means there is an open set $N(x_1,\epsilon)$ in K for some value of ϵ. Then there is y in the relative interior of the line segment x_1p

such that y is in $N(x_1, \epsilon)$ for which $||p - y|| < ||p - x_1||$, contradicting (6-1). Therefore, x_1 is a boundary point of K. By the hypothesis there is $x_2 \epsilon K$ so that $||x_1 - x_2|| = \lambda$. From Lemma 2-3,

$$H = \{x : (x, p - x_1) = (x_1, p - x_1)\}$$

is a support hyperplane for K and separates K from point p.

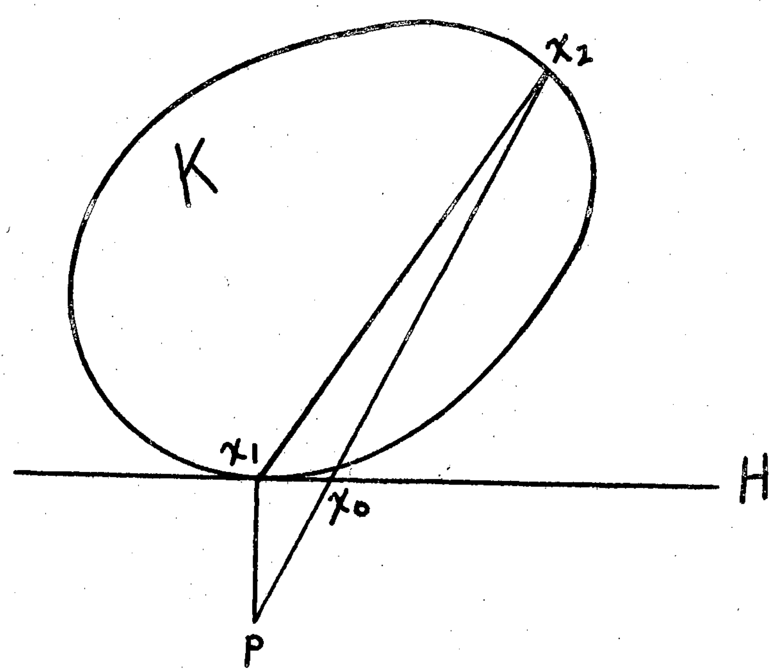

Figure 6-1.

Two cases are considered here. First, if x_2 is in H, then using the Pythagorean relationship, $||x_1 - x_2|| = \lambda$ and $||p - x_1|| \neq 0$ results in $||p - x_2|| > \lambda$ which means that K is complete

The second case is for x_2 not in H. Let

$$\alpha = \frac{(p - x_1, p - x_1)}{(x_2 - p, x_1 - p)} .$$

From the definition of H,

$$(x_2 - x_1, x_1 - p) > 0.$$

By adding

$$(x_1 - p, x_1 - p) > 0,$$

the inequality

$$(x_2 - p, x_1 - p) > 0$$

results, and thus α is positive. Rewrite

$$(x_2 - x_1, p - x_1) < 0$$

in the form

$$(x_2, p - x_1) < (x_1, p - x_1).$$

To this inequality add

$$(-p, p - x_1) = (-p, p - x_1)$$

and get

$$(x_2 - p, p - x_1) < (x_1 - p, p - x_1).$$

Therefore

$$(p - x_2, p - x_1) > (p - x_1, p - x_1),$$

and $\alpha < 1$. Let

$$x_0 = p + \alpha(x_2 - p).$$

To see that $x_0 \in H$, observe that

$$(p + \alpha(x_2 - p) - x_1,\ p - x_1) = (p - x_1,\ p - x_1) + (\alpha(x_2 - p),\ p - x_1)$$

$$= (p - x_1,\ p - x_1) + \frac{(p - x_1,\ p - x_1)}{(x_2 - p,\ x_1 - p)} (x_2 - p,\ p - x_1)$$

$$= 0.$$

This shows that x_0 is in H. Therefore,

$$||x_2 - p|| = ||x_2 - x_0|| + ||x_0 - p||.$$

But

$$||x_2 - x_0|| + ||x_0 - x_1|| \geq ||x_2 - x_1||$$

or

$$||x_2 - x_0|| \geq ||x_2 - x_1|| - ||x_0 - x_1||$$

and

$$||x_0 - p|| > ||x_0 - x_1||.$$

Therefore,

$$||x_2 - p|| > ||x_2 - x_1|| - ||x_0 - x_1|| + ||x_0 - x_1||$$

$$= ||x_2 - x_1||$$

$$= \lambda.$$

So for any p not in K, the diameter of $(\{p\} \cup K)$ is greater than D(K) which means that K is complete and therefore of constant width. This finishes the proof of Theorem 6-1.

If X is a convex set and H is a support hyperplane of X at some point x_0, then a vector $u \neq \emptyset$ orthogonal to H at x_0 is called a normal of the set X at x_0. The line containing x_0 and determined by the vector u is called the <u>normal line</u> of the set X at x_0. If the normal line of the set X determined by a normal u contains two boundary points of X at which parallel support hyperplanes of X exist and such that u is orthogonal to these hyperplanes, then u is called a <u>double normal</u> of X.

If the set X is a set of constant width λ, then every normal is a double normal. For by Lemmas 3-8 and 3-9, if u is an arbitrary direction, then there are two points $\bar{x}$ and $\bar{y}$ in X for which

$$||\bar{x}-\bar{y}|| = W_u = \sup \{ ||x-y|| : x, y \in X, \ x-y = \alpha u, \ \alpha \neq 0\},$$

and there are two parallel support hyperplanes H_1, H_2 of X at $\bar{x}, \bar{y}$, respectively. If $\bar{x} - \bar{y}$ is not orthogonal to H_1 and H_2, then $||\bar{x} - \bar{y}|| > \lambda$, a contradiction of $D(X) = \lambda$. Therefore, $\bar{x} - \bar{y}$ is a double normal of X for direction u.

Suppose there is another normal line of X at x_0 in the direction of u. If $x_0 \in X \cap H_1$, then $||x_0 - \bar{y}|| > \lambda$, which is contrary to $D(X) = \lambda$. A similar statement can be made if $x_0 \in X \cap H_2$. Therefore, the vector $\bar{x} - \bar{y}$ determines the only normal line for direction u.

It is also immediate that each hyperplane of support of a set X of constant width contains exactly one point of X. If two points of X were in a support hyperplane, then two normal lines of X would be determined for the same direction, contradicting the preceding result.

The family of sets of constant width is closed under Minkowski addition; that is, if X and Y are two sets of constant width λ and ρ, respectively, the Minkowski sum,

$$X + Y = \{x + y : x \in X, \ y \in Y\},$$

has the constant width $\lambda + \rho$. This theorem follows easily from the lemmas derived in Chapter III.

Theorem 6-2: If X and Y have constant width λ and ρ, respectively, then X + Y has constant width $\lambda + \rho$.

Proof: Let u be any direction in E_n such that $||u|| = 1$. Then there are two points x_0, x_1 in X, two points y_0, y_1 in Y so that $x_0 - x_1 = \lambda u$, and $y_0 - y_1 = \rho u$. In addition,

$$H_0 = \{x : (x-x_0, \lambda u) = 0\},$$

$$H_1 = \{x : (x-x_1, \lambda u) = 0\}$$

are parallel support hyperplanes of X, and

$$L_0 = \{x : (x-y_0, \rho u) = 0\},$$

$$L_1 = \{x : (x-y_1, \rho u) = 0\}$$

are parallel support hyperplanes of Y (cf. Figure 6-2).

Let

$$M_0 = \{x : (x - (x_0 + y_0), u) = 0\}.$$

First of all,

$$x_0 + y_0 \in M_0 \cap (X + Y).$$

Next, let x + y be any point in X + Y where $x \in X$ and $y \in Y$. Since $x \in X$ then $(x - x_0, u) \leq 0$ and $y \in Y$ implies $(y - y_0, u) \leq 0$. Adding these two inequalities results in

$$(x + y - (x_0 + y_0), u) \leq 0$$

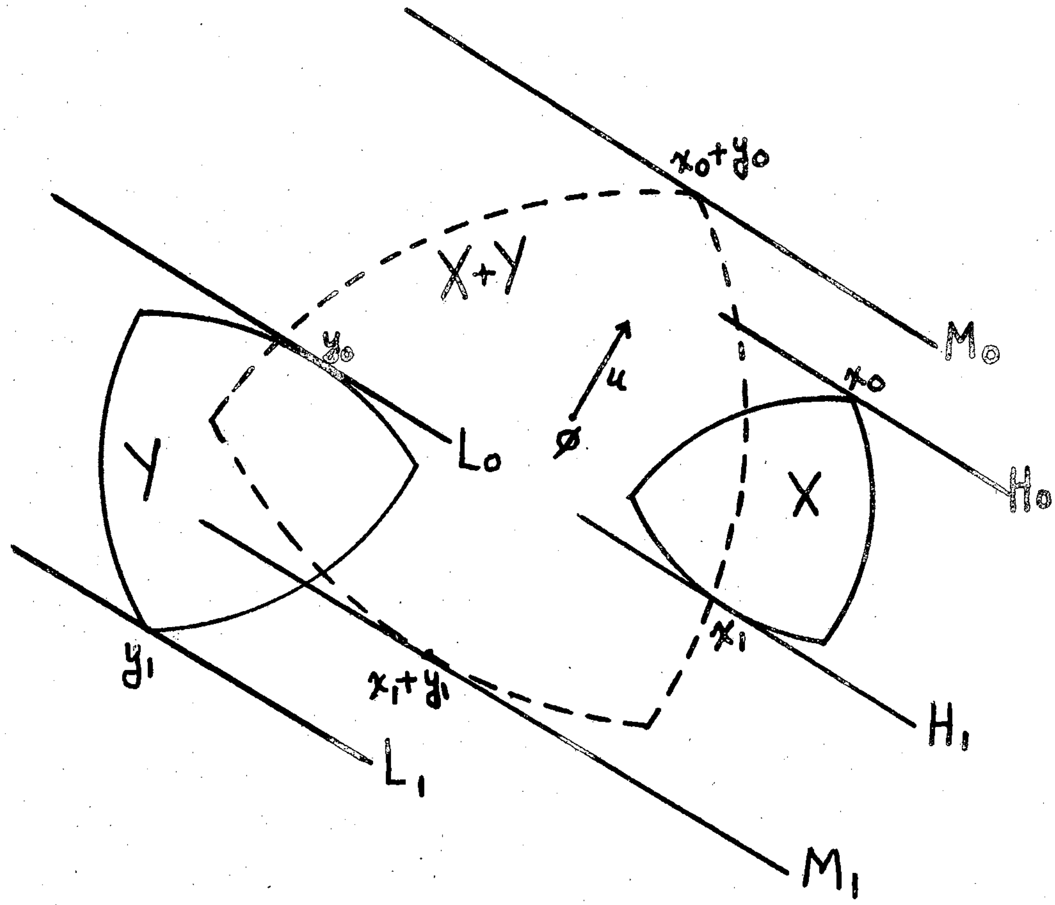

Figure 6-2.

which implies that M_0 bounds the set X + Y. This combined with $x_0 + y_0 \in M_0$ implies that M_0 is a support hyperplane of X + Y. Similarly, one can show that

$$M_1 = \{x : (x - (x_1 + y_1), u) = 0\}$$

is a support hyperplane of X + Y and that M_0 and M_1 are parallel by showing one is a translate of the other.

To find the distance between M_0 and M_1, recall that

$$x_0 + y_0 \in M_0 \cap X + Y,$$

$$x_1 + y_1 \in M_1 \cap X + Y,$$

and that

$$x_0 + y_0 - (x_1 + y_1)$$

is a double normal of X + Y with respect to M_0 and M_1. Then

$$||x_0 + y_0 - (x_1 + y_1)|| = ||(\lambda + \rho) u|| = \lambda + \rho.$$

Therefore,

$$\rho(M_0, M_1) = \lambda + \rho.$$

Since u was arbitrarily chosen, X + Y is a set of constant width $\lambda + \rho$.

Using the same techniques one can show that if X is a set of constant width λ, then δX has constant width $|\delta|\lambda$ where δ is a real number different from zero.

Before proceeding to the next characterization, a lemma needed in its proof is inserted here.

Lemma 6-1: If a is in $C(\phi, r) = \{x : ||x|| = r\}$ then $H_0 = \{x : (x - a, a) = 0\}$ is a support hyperplane for $D(\phi, r)$.

Proof: Since a is in H_0 and ϕ is in H_0^-, H_0 will be a support hyperplane of $D(\phi, r)$ if it can be shown that $(x_1 - a, a) \leq 0$ for all x_1 in $D(\phi, r)$.

Let x_1 be in $D(\phi, r)$ and so $||x_1|| \leq r$ (cf. Figure 6-3). Suppose $(x_1 - a, a) > 0$. Then $(x_1, a) > (a, a) = r^2$. From this it follows that $-2(x_1, a) < -2r^2$. Adding $||x_1||^2 \leq r^2$ and $||a||^2 = r^2$ leads to

$$||x_1||^2 - 2(x_1, a) + ||a||^2 < 0$$

or

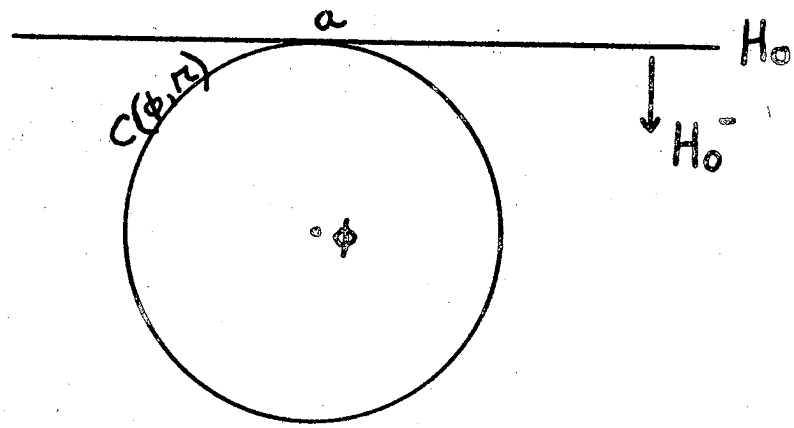

Figure 6-3.

$$||x_1 - a||^2 < 0,$$

a contradiction of the positivity property of the norm function. Therefore, H_0 is a support hyperplane of $D(\phi, r)$.

The following theorem is another characterization of sets of constant width.

Theorem 6-3: A necessary and sufficient condition for a set K to be a body of constant width λ is for K - K to be a spherical ball of radius λ.

Proof: First let K be a body of constant width λ. By the two previous theorems K - K is a set of constant width 2λ. For each u, there are two points x_0, x_1 in K such that $||x_0 - x_1|| = \lambda$. Observe that $x_0 - x_1$ is a point in K - K. Therefore, for each direction u, there exists a point in K - K whose distance from the origin is exactly λ.

Let x_2 be in K and y_2 in $(-1)K$. For y_2 to be in $(-1)K$ means there is y_3 in K so that $y_2 = -y_3$ and $x_2 + y_2$ is in K - K. Then

$$||x_2 + y_2|| = ||x_2 - y_3||$$

and x_2, y_3 both in K imply $||x_2 - y_3|| \leq \lambda$. Therefore, K - K is spherical ball of radius λ.

Suppose that K - K is a spherical ball of radius λ. Let ϕ be the center of K - K and select $u \in E_n$ so that $||u|| = 1$. Since $||\lambda u|| = \lambda$, λu is in the boundary of K - K (cf. Figure 6-4). But λu in K - K means

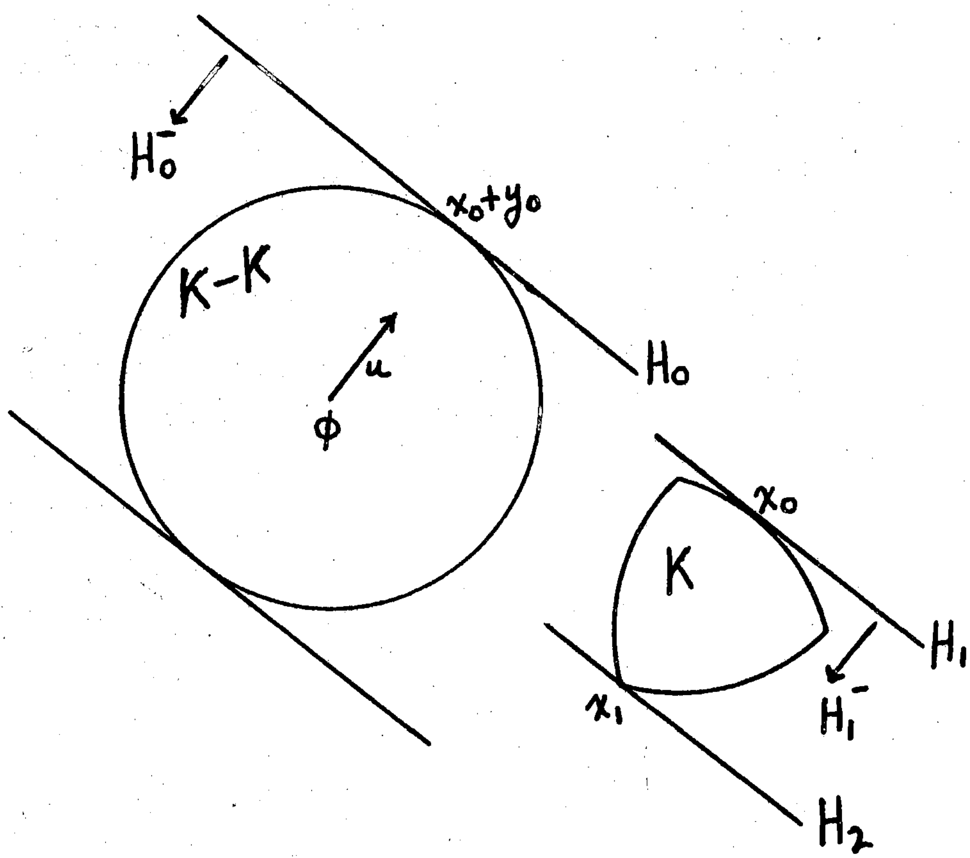

Figure 6-4.

that $\lambda u = x_0 + y_0$ for some x_0 in K and y_0 in (-K) or $y_0 = -x_1$ for some x_1 in K. So $\lambda u = x_0 - x_1$ and $||\lambda u|| = \lambda$ implies $||x_0 - x_1|| = \lambda$.

By Lemma 6-1,

$$H_0 = \{x : (x - (x_0 + y_0), u) = 0\}$$

is a support hyperplane for K - K. Let

$$H_1 = \{x : (x - x_0, u) = 0\}.$$

Select an arbitrary y in K. Then $y + y_0$ is in K - K and

$$(y + y_0 - (x_0 + y_0), u) = (y - x_0, u) \leq 0.$$

Thus, H_1 is a support hyperplane of K. Similarly,

$$H_2 = \{x : (x - x_1, u) = 0\}$$

is parallel to H_1 and is a support hyperplane of K. The vector $x_0 - x_1$ is perpendicular to H_1 and H_2, $||x_0 - x_1|| = \lambda$, and therefore $\rho(H_1, H_2) = \lambda$. Since u was an arbitrary direction, the set K is of constant width λ.

CHAPTER VII

ORTHOGONAL PROJECTIONS

For any set X, let H be an arbitrary hyperplane in E_n and select a direction v perpendicular to H. Now consider the collection of points $X(H) = \{x + \lambda v : x \in X, \lambda \in R$ such that $x + \lambda v \in H\}$. The set X(H) will be called the orthogonal projection of X on the hyperplane H.

Consider the following question: What can be said about the orthogonal projections of a set of constant width? Hermann Minkowski, a Polish mathematician, was the first to observe the following property:

Theorem 7-1: If X is a set of constant width λ, then any orthogonal projection of X is a set of constant width λ.

Proof: Let H_0 be any hyperplane in E_n and such that the origin is in H_0. In this case H_0 is a subspace. Select u in H_0 such that $||u|| = 1$ (cf. Figure 7-1). Take u' such that $||u'|| = 1$, $(u, u') = 0$, and $(x, u') = 0$ for every x in H_0. In fact, u' is perpendicular to H_0 and

$$H_0 = \{x : (x, u') = 0\}.$$

For the direction u, there are two points $\bar{x}, \bar{y}$ in X for which $\bar{x} - \bar{y} = \lambda u$, two parallel support hyperplanes H_1, H_2 of X at $\bar{x}, \bar{y}$ respectively, $\bar{x} - \bar{y}$ is perpendicular to H_1 and H_2, and $\rho(H_1, H_2) = \lambda$. It is also known that

$$H_1 = \{x : (x - \bar{x}, u) = 0\},$$

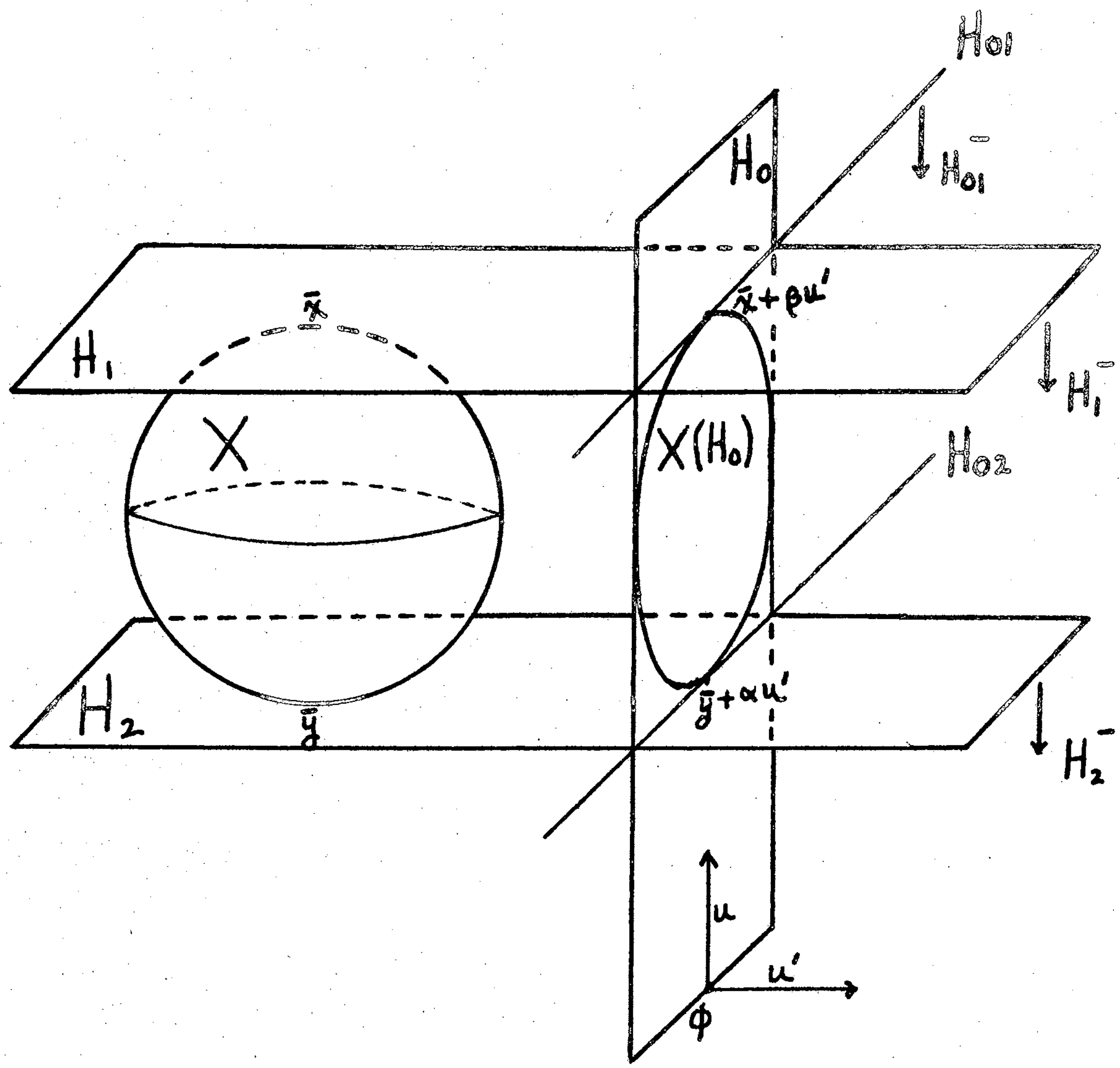

Figure 7-1.

and

$$H_2 = \{x : (x - \overline{y},\ u) = 0\}.$$

Select $\beta = -(\overline{x}, u')$. Then

$$(\overline{x} + \beta u',\ u') = (\overline{x} - (\overline{x}, u')\, u', u')$$

$$= (\overline{x}, u') - (\overline{x}, u')(u', u') = 0.$$

Therefore, $\bar{x} + \beta u'$ is in H_0. Also observe that

$$(\bar{x} + \beta u', u) = \beta (u', u) = 0$$

which implies $\bar{x} + \beta u'$ is in H_1. Similarly, for $\alpha = -(\bar{y}, u')$, the point $\bar{y} + \alpha u' \in H_0 \cap H_2$.

The hyperplane H_0 can be written as $\bar{x} + \beta u' + L_0$ where L_0 is some subspace. Also $H_1 = \bar{x} + \beta u' + L_1$ where L_1 is some subspace. At this point define $H_{01} = H_0 \cap H_1$. It follows that

$$H_0 \cap H_1 = \bar{x} + \beta u' + L_0 \cap L_1.$$

Since $L_0 \cap L_1$ is a subspace, H_{01} is a translate of a subspace.

From the equation

$$\dim(L_0 \cap L_1) = \dim L_0 + \dim L_1 - \dim(L_0 + L_1)$$

it follows that $\dim (L_0 \cap L_1) = n - 2$ since L_0 and L_1 are (n-1)-dimensional and $L_0 + L_1$ is n-dimensional. Therefore, H_{01} is a hyperplane in H_0 and we can write

$$H_{01} = \{x \in H_0 : (x - \bar{x}, u) = 0\}.$$

From

$$\begin{aligned}(\bar{y} + \alpha u' - \bar{x}, u) &= (y - \bar{x}, u) + \alpha(u', u) \\ &= (\bar{y} - \bar{x}, u) \\ &= -\lambda(u, u) \\ &< 0,\end{aligned}$$

observe that $\bar{y} + \alpha u' \in H_{01}^-$. Let x_1 be in X then $x_1 + \delta u' \in H_0$ for

$\delta = -(x_1, u')$. Then

$$(x_1 + \delta u' - \bar{x},\ u) = (x_1 - \bar{x},\ u) + \delta(u', u)$$

$$= (x_1 - \bar{x}, u)$$

$$\leq 0$$

since $x_1 \in X$ and $X \subset H_1^- \cup H_1$. Therefore,

$$(x_1 + \delta u' - \bar{x},\ u) \leq 0$$

and $\bar{x} + \beta u' \in H_{01}$ means that H_{01} is a support hyperplane for $X(H_0)$.

Similarly, define $H_{02} = H_0 \cap H_2$ and show that

$$H_{02} = \{x \in H_0 : (x - \bar{y},\ u) = 0\}$$

and is a support hyperplane for $X(H_0)$.

It can be shown that H_{01} is parallel to H_{02} by showing $H_{02} = \bar{y} - \bar{x} + H_{01}$.

Let x_{02} be any element in H_{02}. Consider

$$(\bar{x} - \bar{y},\ x_{02} - \bar{y} + \alpha u') = (\lambda u,\ x_{02} - \bar{y}) + (\lambda u, \alpha u') = 0.$$

This implies $\bar{x} - \bar{y}$ is perpendicular to H_{02} and similarly, $\bar{x} - \bar{y}$ is perpendicular to H_{01}. But

$$\bar{x} + \beta u' - \bar{y} - \alpha u' = \bar{x} - \bar{y}$$

and

$$||\bar{x} + \beta u' - \bar{y} - \alpha u'|| = \lambda.$$

So H_{01}, H_{02} are parallel support hyperplanes of $X(H_0)$. Furthermore, $\rho(H_{01}, H_{02}) = \lambda$.

Since u was an arbitrary unit vector in H_0, $X(H_0)$ is a set of constant width λ.

Lemma 7-1. If x, y are nontrivial vectors in E_n, there is a vector $z \epsilon E_n$ such that $(x, z) = (y, z) = 0$ if and only if $n \geq 3$ or $x = \lambda y$ when $n = 2$.

Proof: The conditions $(x, z) = 0$ and $(y, z) = 0$ imply that

$$x_1 z_1 + x_2 z_2 + \ldots + x_n z_n = 0,$$

$$y_1 z_1 + y_2 z_2 + \ldots + y_n z_n = 0.$$

If

$$A = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix},$$

then there is a nontrivial solution if and only if $n - \rho(A) > 0$ where $\rho(A)$ is the rank of the matrix A. If $n \geq 3$, there is a nontrivial solution z. If $n = 2$, $n - \rho(A) > 0$ if and only if $\rho(A) = 1$. From $\rho(A) = 1$ it follows that $x = \lambda y$.

This lemma will be now used in proving the following theorem which is the converse of the preceding theorem.

Theorem 7-2: If X is a closed bounded convex body such that each orthogonal projection of X is a set of constant width, then X is a set of constant width.

Proof: In any closed bounded set X in E_n there are two points x_0, x_1 in X so that

$$||x_0 - x_1|| = \max \{ ||x - y|| : x, y \epsilon X \},$$

By translation of X, let $x_0 = \phi$. Take H_0 to be any hyperplane in E_n such that ϕ and x_1 are in H_0 (cf. Figure 7-2).

Let $X(H_0)$ be the projection of X on H_0. Select any u in E_n such that $||\mathbf{u}|| = 1$. By Lemma 7-1, there exists a vector u' in E_n so that

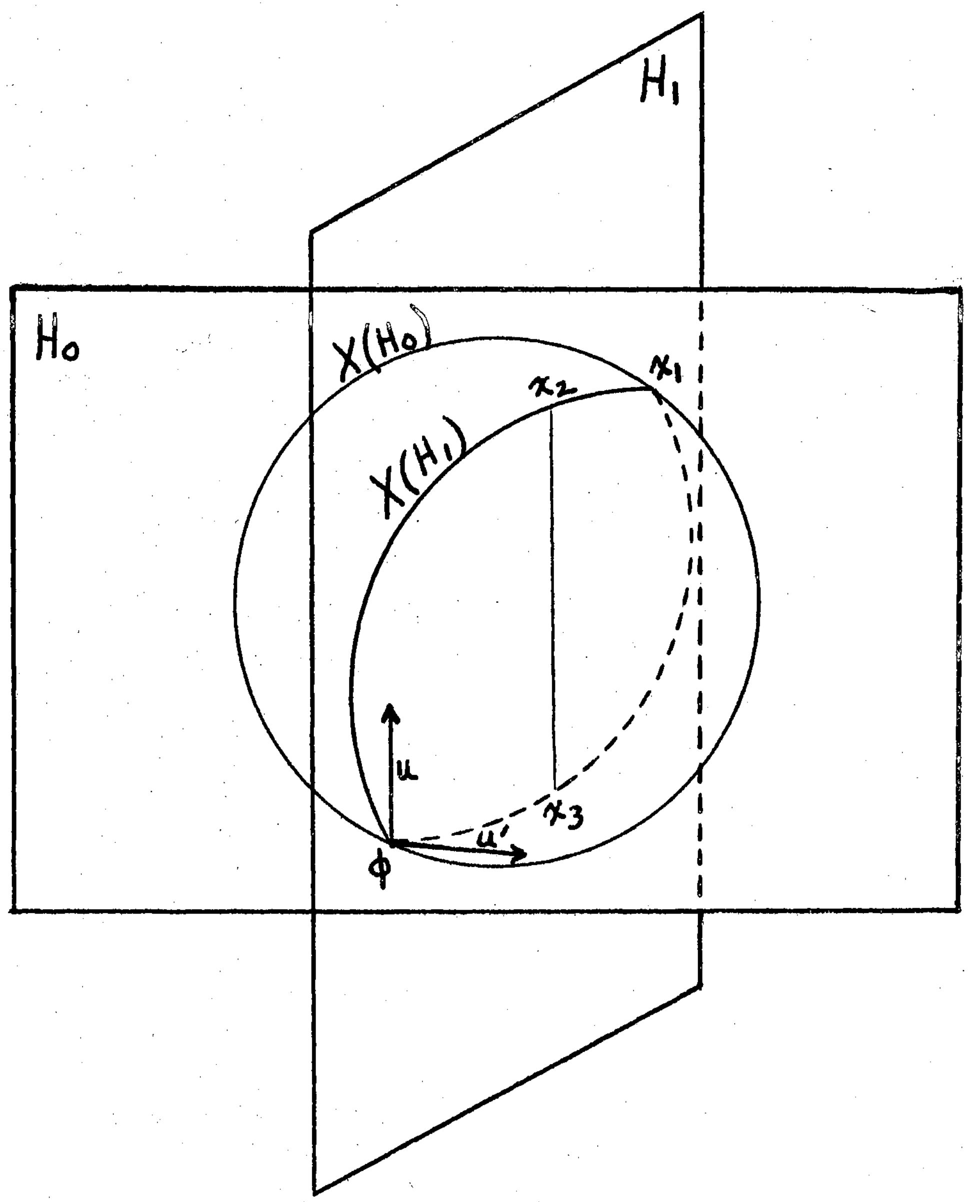

Figure 7-2.

$(u, u') = 0$ and $(u', x_1) = 0$. Let $H_1 = \{x : (x, u') = 0\}$ and consider $X(H_1)$ the projection of X on H_1. By hypothesis $X(H_1)$ is a set of constant width, and it is here asserted that this constant width is equal to $||x_1||$. To see this, let x_p, y_p be the orthogonal projections of x, y, respectively. Observe that $||x - y|| \geq ||x_p - y_p||$. Since

$$||\phi - x_1|| = \max \{ ||x - y|| : x, y \in X\},$$

it follows that $||x_1|| \geq ||x_p - y_p||$ for all x_p, y_p in $X(H_1)$. Notice that ϕ and x_1 are in $X(H_1)$. Therefore, $X(H_1)$ has constant width $||x_1||$.

Since $X(H_1)$ is a set of constant width, for direction u, there must be two points x_2, x_3 in $X(H_1)$ so that $||x_2 - x_3|| = ||x_1||$ and there are two parallel support hyperplanes H_{21}, H_{31} of $X(H_1)$ for which $\rho(H_{21}, H_{31}) = ||x_1||$.

Then

$$H_{21} = \{x \in H_1 : (x - x_2,\ x_3 - x_2) = 0\},$$

and

$$H_{31} = \{x \in H_1 : (x - x_3,\ x_2 - x_3) = 0\}.$$

Let

$$H_2 = \{x : (x - x_2,\ x_3 - x_2) = 0\},$$

and

$$H_3 = \{x : (x - x_3,\ x_2 - x_3) = 0\},$$

and note that H_2 and H_3 are parallel.

Since H_{21} is a support hyperplane of $X(H_1)$, there must be $\overline{x}_2$ in X such that

$$0 = (\overline{x}_2 + \alpha u' - x_2,\ x_3 - x_2)$$

$$= (\overline{x}_2 - x_2,\ x_3 - x_2) + (\alpha u',\ x_3 - x_2)$$

$$= (\bar{x} - x_2,\ x_3 - x_2).$$

This $\bar{x}_2$ is in H_2 and similarly, there is $\bar{x}_3$ in $X \cap H_3$.

Now consider any $x \in X$ and let $\bar{x}$ be the projection of x on H_1. So $\bar{x}$ is in $X(H_1)$, and $(\bar{x} - x_2, x_3 - x_2) \geq 0$ implies that

$$(-x_2,\ x_3 - x_2) \geq (-\bar{x},\ x_3 - x_2). \tag{7-1}$$

Then

$$(x - x_2,\ x_3 - x_2) = (x,\ x_3 - x_2) - (x_2,\ x_3 - x_2).$$

Using inequality (7-1),

$$(x, x_3 - x_2) - (x_2, x_3 - x_2) \geq (x, x_3 - x_2) - (\bar{x}, x_3 - x_2)$$

$$= (x - \bar{x},\ x_3 - x_2)$$

$$= 0.$$

Therefore, $(x - x_2, x_3 - x_2) \geq 0$, and H_2 is a support hyperplane of X. In a similar manner H_3 is a support hyperplane of X, and

$$\rho(H_2, H_3) = ||x_1||.$$

The direction u was arbitrarily selected, and so for every direction there are two parallel support hyperplanes of X, and the distance between them is always $||x_1||$. Therefore, X is a set of constant width.

Note that in the hypothesis of Theorem 7-2, the width of each orthogonal projection is not designated. Observe also that Theorems 7-1 and 7-2 form another characterization of sets of constant width.

An interesting question presents itself. What is the least number of directions for this converse to be true? It seems reasonable that a countable dense set of directions would be sufficient. A simple example can be constructed in E_3 showing that three directions are not sufficient.

CHAPTER VIII

INSPHERES AND CIRCUMSPHERES

Let K be a convex and compact set in E_n. An insphere of K is a sphere of largest radius contained in K, and a circumsphere is a sphere of smallest radius containing K. A circumsphere of K is unique, but in general an insphere is not unique except where K is a set of constant width (cf. [27]). The minimal spherical shell of K exists uniquely and consists of the closed set of points between two concentric spheres such that K is contained in the closed set of points and such that the difference of the radii of these spheres is a minimum. For the content of this chapter, the existence of insphere, circumsphere, and minimal spherical shell for K is assumed. Recall that conv (X) is the convex hull of X and bdy K is the boundary of K.

A proof will be given of the theorem stating that the insphere and circumsphere of a set of constant width λ are concentric and that the sum of their radii is λ. The preliminary results are important in themselves, but their main purpose is the proof of the theorem.

The first result, Theorem 8-1, presents a property of the circumsphere of an arbitrary convex compact set. In E_2, a circumsphere is simply a circle, commonly called a circumcircle. The property described in this theorem, interpreted in E_2, intuitively means that for a compact set K and its circumcircle C, there must be at least one diameter of C where each of its endpoints lies in the boundary of K and on C.

Theorem 8-1: Let K be a compact convex set in E_n with circumsphere $D(\phi, r)$, then $\phi \in \text{conv}(C(\phi, r) \cap \text{bdy } K)$.

Proof: Suppose $\phi \notin \text{conv}(C(\phi, r) \cap \text{bdy } K)$. The set $\text{conv}(C(\phi, r) \cap \text{bdy } K)$ is a closed convex set in E_n. Hence, by Lemma 2-3, there is x_0, the orthogonal projection of ϕ on $\text{conv}(C(\phi, r) \cap \text{bdy } K)$, and $H_0 = \{x : (x - x_0, x_0) = 0\}$ is a support hyperplane of $\text{conv}(C(\phi, r) \cap \text{bdy } K)$. Let $y_0 = (1/2) x_0$ and $H_1 = \{x : (x - y_0, y_0) = 0\}$. The set $\text{conv}(C(\phi, r) \cap \text{bdy } K)$ is contained in H_1^+ (cf. Figure 8-1).

Let $K' = K \cap \text{complement } H_1^+$. The set K' is closed and bounded and therefore compact. Let $D = \{ ||x - \phi|| : x \in K'\}$. The

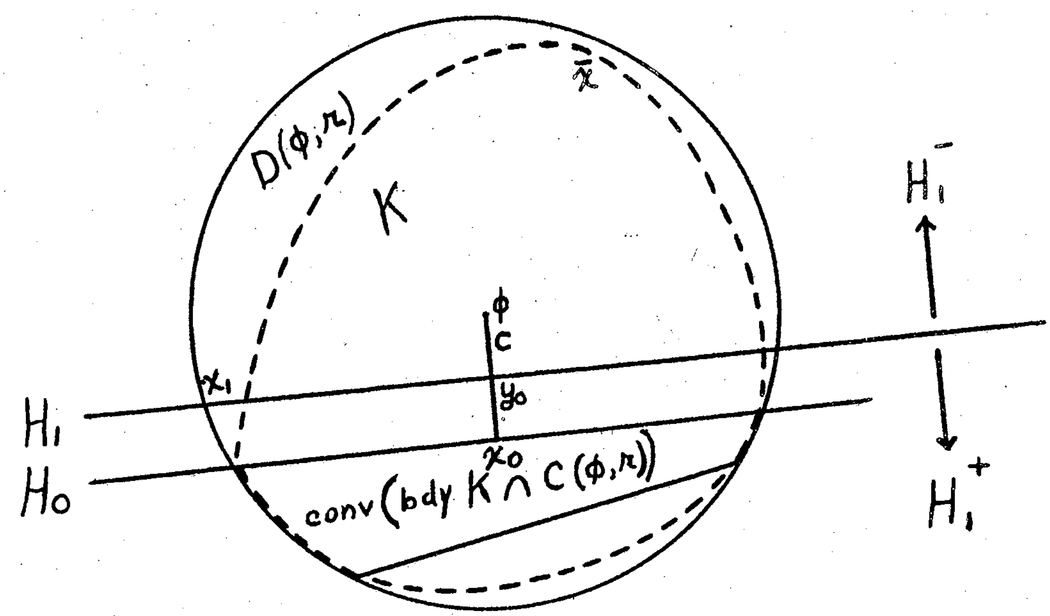

Figure 8-1.

norm, a continuous function, attains its maximum value on a compact set. Therefore, there is $\bar{x}$ in K' such that $||\bar{x}|| \geq ||x||$ for every $x \in K'$. Let $||\bar{x}|| = r'$ where r' is less than or equal to r.

To prove that $\bar{x}$ is not in $C(\phi, r)$ suppose $\bar{x} \in C(\phi, r)$. First, $\bar{x}$ is in K', which means that $\bar{x}$ is in bdy $K \cap C(\phi, r)$, and thus $\bar{x}$ is in H_1^+. But $\bar{x} \in K'$ implies $\bar{x}$ is in complement of H_1^+. This contradiction shows $\bar{x} \notin C(\phi, r)$ and therefore $||\bar{x}|| = r' < r$.

Let $r - r' = \delta > 0$. Select c such that $c \in$ intv $\phi\, y_0$ and $||c|| \leq \delta/2$. Since $c \in$ intv $\phi\, y_0$, $c = \alpha y_0$ for some α such that $0 < \alpha < 1$. Select x_1 in $H_1 \cap C(\phi, r)$.

Let

$$S = \{x : ||x - c|| \leq \max[r - \delta/2, ||x_1 - c||]\}$$

It will be shown that S is a sphere with center c and radius less than r and such that $K \subset S$, thus contradicting the assumption that $D(\phi, r)$ is a circumsphere.

Obviously, $r - \delta/2 < r$. Then,

$$\begin{aligned} ||x_1 - c||^2 &= ||x_1 - y_0||^2 + ||y_0 - c||^2 \\ &= ||x_1 - y_0||^2 + ||y_0 - \alpha y_0||^2 \\ &< ||x_1 - y_0||^2 + ||y_0||^2 \\ &= ||x_1||^2 \\ &= r^2, \end{aligned}$$

shows $||x_1 - c|| < r$.

It yet remains to demonstrate $K \subset S$. First, let $x \in K'$ and then

$$\begin{aligned} ||x - c|| &\leq ||x|| + ||c|| \\ &\leq ||\overline{x}|| + ||c|| \\ &\leq r - \delta + \delta/2 \\ &= r - \delta/2 \end{aligned}$$

shows that $x \epsilon S$. Now select any $x \epsilon K$ where x is in $H_1 \cup H_1^+$. The point x in $H_1 \cup H_1^+$ implies that $(x - y_0, y_0) \geq 0$ or $(x, y_0) \geq (y_0, y_0)$. Since $x_1 \epsilon H_1$, $(x_1 - y_0, y_0) = 0$ or $(x_1, y_0) = (y_0, y_0)$. Therefore, $(x, y_0) \geq (x_1, y_0)$. The real number α is positive, so multiplying this last inequality by -2α leads to

$$-2(\alpha y_0, x) \leq -2(\alpha y_0, x_1).$$

Substituting c for αy_0 yields

$$-2(c, x) \leq -2(c, x_1).$$

To this inequality add $||x||^2 \leq ||x_1||^2$ and $||c||^2 = ||c||^2$ resulting in

$$||x||^2 - 2(c, x) + ||c||^2 \leq ||x_1||^2 - 2(c, x_1) + ||c||^2$$

which is equivalent to

$$||x - c||^2 \leq ||x_1 - c||^2$$

or

$$||x - c|| \leq ||x_1 - c||.$$

Thus, again $x \epsilon S$, and hence $K \subset S$ where S has a smaller radius than the radius of the circumsphere. Thus we have reached a contradiction to the hypothesis of the theorem.

The following theorem, which is the converse of Theorem 8-1, together with Theorem 8-1, forms a characterization of the circumsphere.

Theorem 8-2: If K is compact convex contained in $D(\phi, r)$ such that $\phi \in$ conv (bdy K $\cap$ $C(\phi, r)$), then $D(\phi, r)$ is the circumsphere of K.

Proof: Suppose $D(\phi, r)$ is not a circumsphere. This means that there is S, a circumsphere with center x_0 and radius r', where $r' < r$ and $K \subset S$ (cf. Figure 8-2).

Since K is closed, bdy $K \subset K$. Then

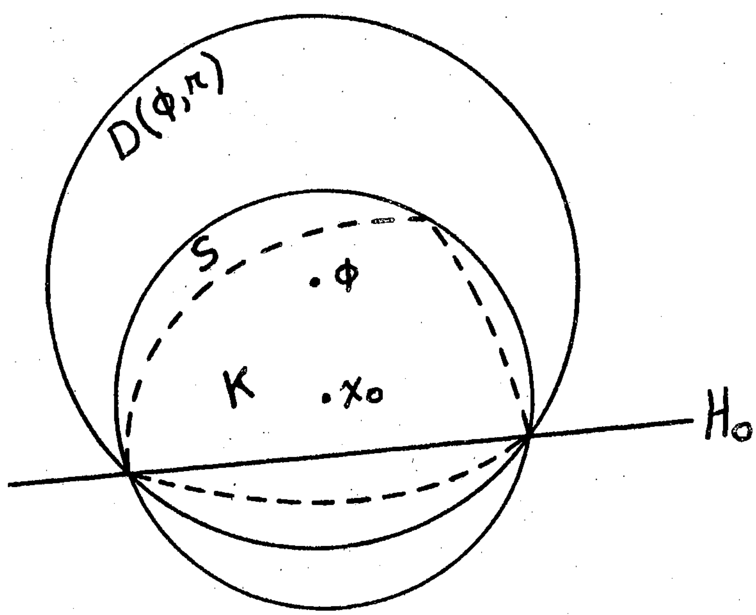

Figure 8-2.

$$\text{bdy } K \cap C(\phi, r) \subset S \cap C(\phi, r).$$

If $S \cap C(\phi, r) = 0$, then bdy $K \cap C(\phi, r) = 0$, but by hypothesis $\phi \in \text{conv}(\text{bdy } K \cap C(\phi, r))$. This contradiction shows that $S \cap C(\phi, r) \neq 0$ and $\phi \neq x_0$. The set bdy K is contained in S, and so

$$\text{bdy } K \cap C(\phi, r) \subset S \cap C(\phi, r).$$

This leads to

$$\text{conv}(\text{bdy } K \cap C(\phi, r)) \subset \text{conv}(S \cap C(\phi, r)).$$

The assumption $\phi \in \text{conv}(\text{bdy } K \cap C(\phi, r))$ implies that $\phi \in \text{conv}(S \cap C(\phi, r))$.

Let

$$H_0 = \{x : (x, x_0) = (1/2)(r^2 + ||x_0||^2 - r'^2)\}.$$

If

$$\alpha = (1/2)(r^2 + ||x_0||^2 - r'^2),$$

then α is positive. The inner product $(x_0, \phi) = 0 < \alpha$ implies that $\phi \in H_0^-$.

Let $a_i \in S \cap C(\phi, r)$. This implies $||a_i - x_0|| \leq r'$, and leads to

$$||a_i||^2 - 2(x_0, a_i) + ||x_0||^2 \leq r'^2.$$

This last inequality is equivalent to

$$-2(x_0, a_i) \leq r'^2 - ||x_0||^2 - ||a_i||^2.$$

Using the fact $||a_i|| = r$ and the last inequality results in the statement

$$(x_0, a_i) \geq (1/2)(r^2 + ||x_0||^2 - r'^2) = \alpha.$$

Since the origin ϕ is in conv $(S \cap C(\phi, r))$,

$$\phi = \sum_{i=1}^{m} \beta_i a_i$$

where

$$\sum_{i=1}^{m} \beta_i = 1, \quad \beta_i \geq 0, \quad a_i \in S \cap C(\phi, r).$$

Then

$$\begin{aligned}
(x_0, \phi) &= (x_0, \sum_{i=1}^{m} \beta_i a_i) \\
&= \sum_{i=1}^{m} \beta_i (x_0, a_i) \\
&\geq \sum_{i=1}^{m} \beta_i \alpha \\
&= \alpha \sum_{i=1}^{m} \beta_i \\
&= \alpha.
\end{aligned}$$

Therefore, $(x_0, \phi) \geq \alpha$ implies that $\phi \in H_0 \cup H_0^+$, a contradiction of $\phi \in H_0^-$.

Before proceeding with a characterization of an insphere, a lemma needed in the characterization is inserted.

Lemma 8-1: Let M be any convex compact body in E_n containing the origin, and let

$$t = \min \{ ||x|| : x \in \text{bdy } M \}.$$

If $||x_1|| \leq t$, then $x_1 \in M$.

Proof: Suppose $x_1 \notin M$ and let x_2 be the orthogonal projection of x_1 onto M (cf. Figure 8-3). The point x_2 is a boundary point of M.

Let

$$H = \{x : (x - x_2, x_1 - x_2) = 0\}.$$

By Lemma 2-3, H is a support hyperplane of M separating x_1 and ϕ. In fact, x_1 is in H^+ and ϕ is in H^-.

Let

$$\beta = \frac{(x_2, x_1 - x_2)}{(x_1, x_1 - x_2)} .$$

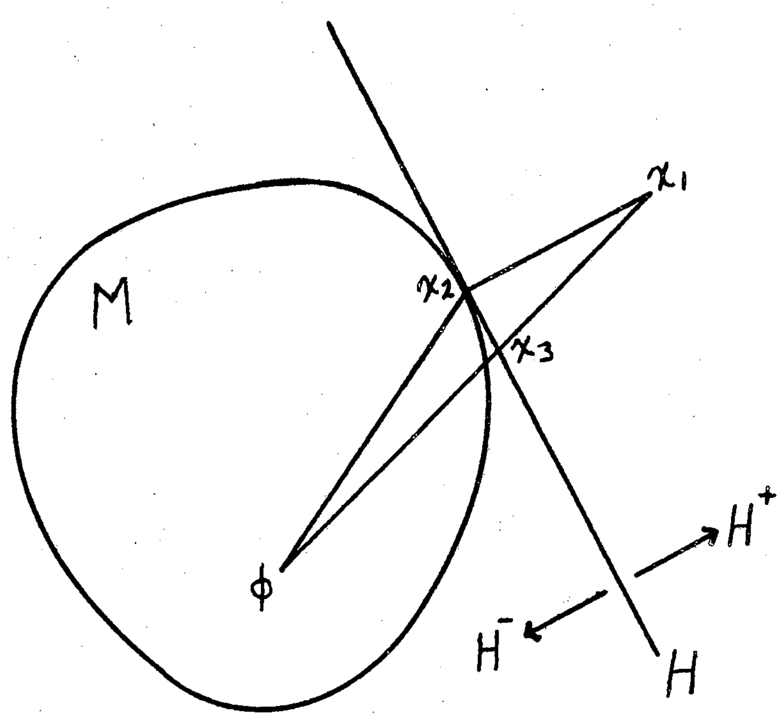

Figure 8-3.

First, $(x_2, x_1 - x_2)$ is positive since $\phi \in H^-$, and as clearly $(x_1 - x_2, x_1 - x_2)$ is positive. Adding these two inequalities results in $(x_1, x_1 - x_2) > 0$, which implies that β is positive. Also, from $(x_1 - x_2, x_1 - x_2) > 0$ it follows that

$$(x_1, x_1 - x_2) > (x_2, x_1 - x_2),$$

and hence $\beta < 1$.

Let $x_3 = \beta x_1$. Then

$$(\beta x_1 - x_2, x_1 - x_2) = \left(\frac{(x_2, x_1 - x_2)}{(x_1, x_1 - x_2)} x_1 - x_2, x_1 - x_2 \right)$$

$$= \frac{(x_2, x_1 - x_2)}{(x_1, x_1 - x_2)} (x_1, x_1 - x_2) - (x_2, x_1 - x_2)$$

$$= (x_2, x_1 - x_2) - (x_2, x_1 - x_2)$$

$$= 0$$

shows that x_3 is in H. Therefore,

$$||x_1|| = ||x_3|| + ||x_1 - x_3||.$$

By the Pythagorean relationship,

$$||x_1 - x_3||^2 = ||x_1 - x_2||^2 + ||x_2 - x_3||^2.$$

Since $||x_1 - x_2||$ is not zero,

$$||x_1 - x_3 > ||x_2 - x_3||.$$

Using the triangle inequality,

$$||\phi - x_3|| + ||x_3 - x_2|| \geq ||\phi - x_2||,$$

it follows that

$$||x_3|| \geq ||x_2|| - ||x_2 - x_3||.$$

Therefore,

$$||x_1|| = ||x_3|| + ||x_1 - x_3||$$

$$> ||x_2|| - ||x_2 - x_3|| + ||x_2 - x_3||$$

$$= ||x_2||,$$

and hence $||x_1|| > ||x_2|| \geq t$ since x_2 is a boundary point of M and

$$t = \min \{||x|| : x \in \text{bdy } M\}.$$

But by hypothesis $||x_1|| \leq t$. This contradiction shows that $x_1 \in M$.

Theorem 8-3: If K is a compact convex body in E_n and $D(\phi, r)$ is an insphere, then $\phi \in \text{conv}(\text{bdy } K \cap C(\phi, r))$.

Proof: Suppose $\phi \notin \text{conv}(\text{bdy } K \cap C(\phi, r))$ and let x_0 be the orthogonal projection of ϕ onto $\text{conv}(\text{bdy } K \cap C(\phi, r))$. Then $H_0 = \{x : (x - x_0, x_0) = 0\}$ is a support hyperplane for $\text{conv}(\text{bdy } K \cap C(\phi, r))$. Let $y_0 = (1/2)x_0$, $H_1 = \{x : (x - y_0, y_0) = 0\}$, and $K' = \text{bdy } K \cap \text{complement } H_1^+$ (cf. Figure 8-4). The set K' is compact and non-empty.

To see that K' is non-empty, assume it is empty which means $\text{bdy } K \subset H_1 \cup H_1^+$. So for any element $b \in \text{bdy } K$, $(b, y_0) \geq (y_0, y_0)$. Since ϕ is in interior of K, there are two points b and d in the boundary

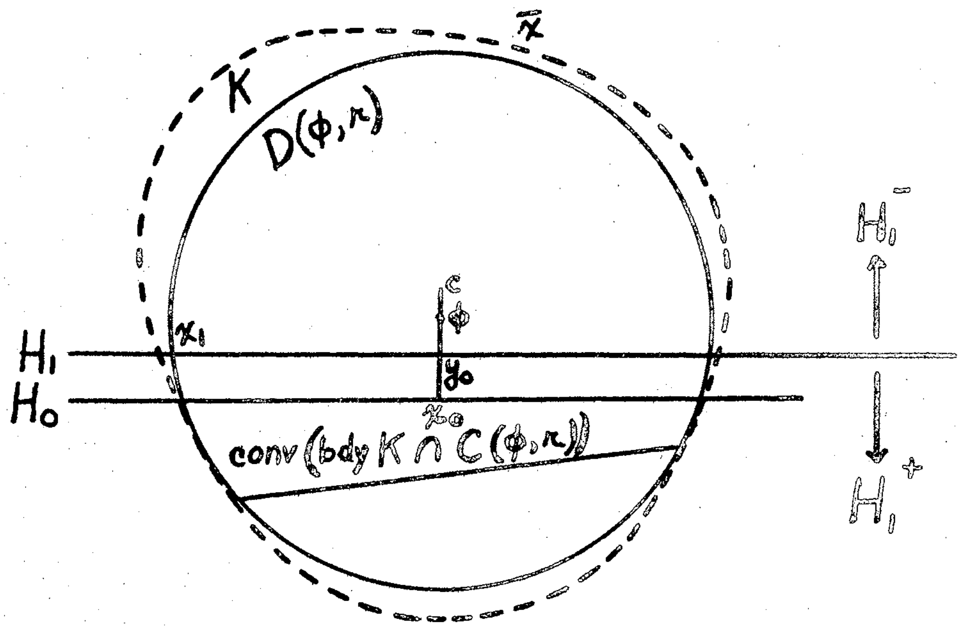

Figure 8-4.

of K such that $\phi = \alpha b + (1 - \alpha)d$ for some $\alpha > 0$. Then

$$(\alpha b + (1 - \alpha)d - y_0, y_0) = \alpha(b, y_0) + (1 - \alpha)(d, y_0) - (y_0, y_0).$$

But

$$\alpha(b, y_0) \geq \alpha(y_0, y_0) \quad \text{and} \quad (1 - \alpha)(d, y_0) \geq (1 - \alpha)(d, y_0)$$

imply that

$$\alpha(b, y_0) + (1 - \alpha)(d, y_0) - (y_0, y_0) \geq 0,$$

and hence $\phi \in H_1^+$. But $(\phi - y, y_0) < 0$ implies that $\phi \in H_1^-$, a contradiction.

Let $D = \{||x - \phi|| : x \in K'\}$. Since the norm is a continuous function and K' is compact, there is a point $\bar{x}$ in K' such that

$||\bar{x}|| \leq ||x||$ for all x in K'. Let $||\bar{x}|| = r' \geq r$.

To show that $\bar{x}$ is not in $C(\phi, r)$, assume the contrary. Since $\bar{x} \in K'$ then $\bar{x}$ is in bdy $K \cap C(\phi, r)$. The hyperplane H_0 supports

$$\text{conv}\,(\text{bdy } K \cap C(\phi, r))$$

and $\bar{x}$ in $H_0 \cup H_0^+$ implies that

$$(\bar{x} - x_0, x_0) = (\bar{x} - 2y_0, 2y_0) \geq 0.$$

This is equivalent to

$$2(\bar{x}, y_0) - 4(y_0, y_0) \geq 0$$

or

$$(\bar{x}, y_0) \geq 2(y_0, y_0) > (y_0, y_0).$$

This leads to $(\bar{x} - y_0, y_0) > 0$ which implies that $\bar{x} \in H_1^+$. But $\bar{x} \in K' \subset$ (complement (H_1^+)). Therefore, $\bar{x}$ is not in $C(\phi, r)$ and $||\bar{x}|| = r' > r$. Let $\delta = r' - r > 0$.

Select c so that $\phi \in$ intv cy_0 and $||c|| \leq \delta/2$. Thus

$$\phi = \alpha c + (1 - \alpha) y_0$$

for $0 < \alpha < 1$ and

$$c = \frac{\alpha - 1}{\alpha} y_0 \quad \text{where} \quad \frac{\alpha - 1}{\alpha} < 0.$$

Take any point x_1 in $H_1 \cap C(\phi, r)$ and let

$$S = \left\{ x : ||x - c|| \leq \min \left[r + \delta/2,\ ||x_1 - c|| \right] \right\}.$$

To finish the theorem, it is now proved that the radius of S is greater than r and that $S \subset K$. This will be a contradiction of the hypothesis that $D(\phi, r)$ is an insphere of K.

First observe that $r + \delta/2 > r$. Then

$$\begin{aligned} ||x_1 - c||^2 &= ||x_1 - y_0||^2 + ||y_0 - c||^2 \\ &> ||x_1 - y_0||^2 + ||y_0||^2 \\ &= ||x_1||^2 \\ &= r^2 \end{aligned}$$

implies that $||x_1 - c|| > r$. Therefore the radius of S is greater than r. Let $s \in S \cap H_1^+$; that is, since $s \in S$, $||s - c||^2 \leq ||x_1 - c||^2$ or equivalently

$$||s||^2 - 2(s, c) + ||c||^2 \leq ||x_1||^2 - 2(x_1, c) + ||c||^2.$$

Therefore,

$$||s||^2 \leq ||x_1||^2 - 2(x_1 - s, c).$$

Since $s \in H_1^+$, $(s - y_0, y_0) > 0$ which implies that $(s, y_0) > (y_0, y_0)$. The point $x_1 \in H_1$ leads to $(x_1, y_0) = (y_0, y_0)$. Therefore, from these two statements

$$(s, y_0) > (x_1, y_0). \tag{8-1}$$

But rewriting (8-1) and substituting for y_0 results in

$$\begin{aligned} 0 < (s - x_1, y_0) &= (s - x_1, \frac{\alpha}{\alpha - 1} c) \\ &= \frac{\alpha}{\alpha - 1} (s - x_1, c) \\ &= \frac{\alpha}{1 - \alpha} (x_1 - s, c). \end{aligned}$$

From this it can be concluded that $0 < (x_1 - s, c)$ since $\alpha(1-\alpha)^{-1}$ is positive. From

$$||s||^2 \leq ||x_1||^2 - 2(x_1 - s, c)$$

it follows that $||s|| \leq ||x_1|| = r$, which proves that $s \in D(\phi, r) \subset K$ or $S \cap H_1^+ \subset K$.

Now select any $s \in S \cap$ complement H_1^+. Suppose there is such an s which is not in K. Since s is in S then $||s - c|| \leq r + \delta/2$. If s is not in K, $||s|| > ||\bar{x}||$ by Lemma 8-1. Knowing $||\bar{x}|| = r' = \delta + r$, it follows that $\delta + r < ||s||$. Therefore,

$$\begin{aligned} ||s|| &= ||s - c + c|| \\ &\leq ||s - c|| + ||c|| \\ &\leq r + \delta/2 + \delta/2 \\ &= r + \delta \\ &< ||s||, \end{aligned}$$

a contradiction. This demonstrates that S is a sphere with radius greater than r and $S \subset K$, a contradiction of the hypothesis that $D(\phi, r)$ is an insphere.

<u>Lemma 8-2</u>: Let H be a hyperplane such that $a \in H \cap C(\phi, r)$ and a is not orthogonal to H, then H is not a support hyperplane for $D(\phi, r)$.

Proof: Let x_2 be the orthogonal projection of ϕ onto H (cf. Figure 8-5). Therefore $||x_2|| = r' < r$.

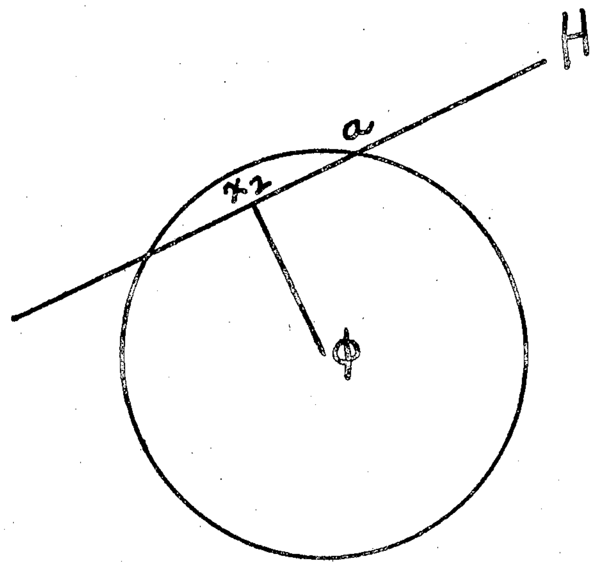

Figure 8-5.

Then the hyperplane $H = \{x : (x - x_2, x_2) = 0\}$ and $\phi \in H^-$. Consider the point $(r/r')\, x_2$ and notice that

$$\|\frac{r}{r'} x_2\| = \frac{r}{r'} \|x_2\| = r.$$

Furthermore,

$$(\frac{r}{r'} x_2 - x_2, x_2) = (\frac{r}{r'} - 1)(x_2, x_2)$$

$$= \frac{r - r'}{r'} (x_2, x_2) > 0.$$

Thus, $(r/r')\, x_2 \in C(\phi, r) \cap H^+$. But $\phi \in D(\phi, r) \cap H^-$ implies that H is not a support hyperplane for $D(\phi, r)$.

This lemma, along with Lemma 6-1, shows that a hyperplane of support of a sphere $D(\phi, r)$ at point a must have the form

$\{x : (x - a,\ a) = 0\}$.

Theorem 8-4: Let K be compact convex body in E_n. If $D(\phi, r)$ is contained in K and $\phi \in$ conv $(C(\phi, r) \cap$ bdy K), then $D(\phi, r)$ is an insphere.

Proof: Suppose that $D(\phi, r)$ is not an insphere. Thus there must be a sphere S with radius $r' > r$ and a center x_0 (cf. Figure 8-6).

Let $H_0 = \{x : (x, x_0) = 0\}$. Since the point ϕ is in conv (bdy K $\cap$ $C(\phi, r)$), then

$$\phi = \sum_{i=1}^{m} \lambda_i y_i ,$$

where

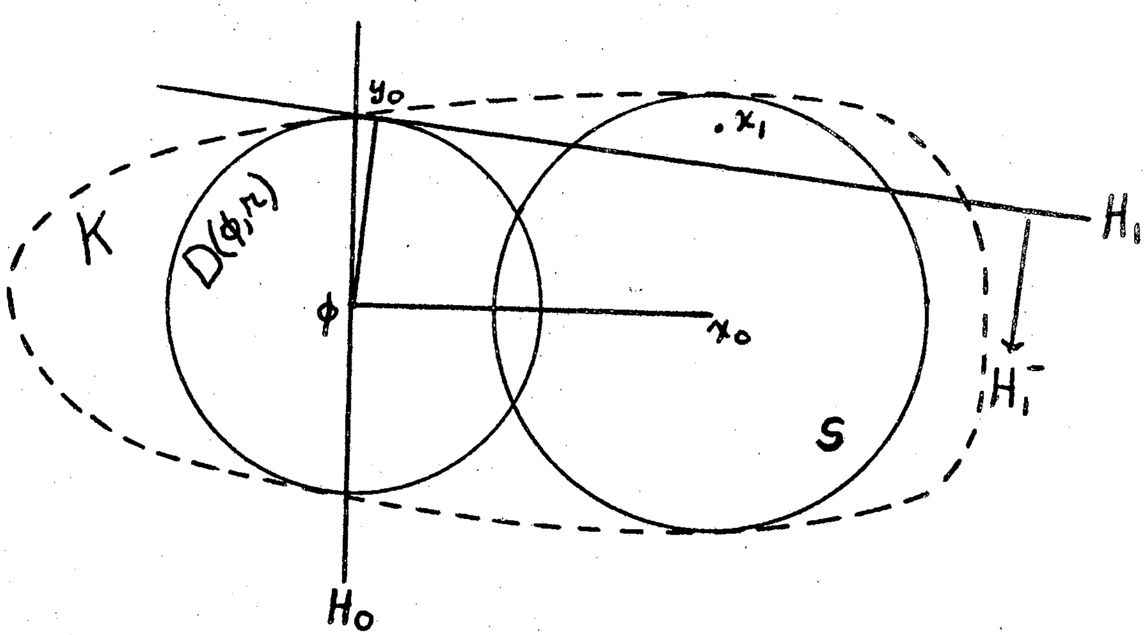

Figure 8-6.

$$\sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0,$$

and $y_i \in$ bdy $K \cap C(\phi, r)$. Then,

$$0 = (\phi, x_0) = \left(\sum_{i=1}^{m} \lambda_i y_i, x_0 \right) = \sum_{i=1}^{m} \lambda_i (y_i, x_0)$$

shows that $(y_i, x_0) \geq 0$ for at least one i. Without loss of generality, let $(y_0, x_0) \geq 0$, and hence $y_0 \in H_0 \cup H_0^+$. Since y_0 is in bdy $K \cap C(\phi, r)$, there is a support hyperplane H_1 for K through y_0 (cf. Theorem 2-15 in [36]). The point y_0 in $C(\phi, r)$ and $C(\phi, r) \subset K$, implies that H_1 is also a supporting hyperplane for $D(\phi, r)$, and H_1 bounds the sphere S.

By Lemmas 6-1 and 8-2, H_1 must have the form $\{x : (x - y_0, y_0) = 0\}$. Let

$$x_1 = x_0 + \left(\frac{r+r'}{2r}\right) y_0.$$

Then

$$\begin{aligned} ||x_1 - x_0|| &= ||x_0 + \left(\frac{r+r'}{2r}\right) y_0 - x_0|| \\ &= \frac{r+r'}{2r} ||y_0|| \\ &= \frac{r+r'}{2r} r \\ &= \frac{r+r'}{2} < r' \end{aligned}$$

shows that $x_1 \in S$. Now

$$(x_1 - y_0, y_0) = (x_0 + \left(\frac{r+r'}{2r}\right) y_0 - y_0, y_0)$$

$$= (x_0 + \left(\frac{r'-r}{2r}\right) y_0, y_0)$$

$$= (x_0, y_0) + \frac{r'-r}{2r} (y_0, y_0).$$

Using the fact $(x_0, y_0) \geq 0$ and

$$\frac{r'-r}{2r} (y_0, y_0) > 0,$$

it follows that $(x_1 - y_0, y_0)$ is positive. Therefore H_1 is not a support hyperplane for K, a contradiction. Therefore $D(\phi, r)$ is an insphere.

The previous two theorems give a characterization of an insphere. In proving the main theorem of this chapter, it turns out to be convenient to have a second characterization of an insphere in terms of an open half sphere.

If $D(a, r)$ is a sphere and a is in a hyperplane H, then $H^+ \cap D(a,r)$ is an <u>open half sphere</u> of $D(a, r)$ determined by H.

<u>Theorem 8-5</u>: Let K be a compact convex body in E_n. If $D(\phi, r)$ is a sphere such that $\phi \in$ conv $(C(\phi, r) \cap$ bdy K), then $C(\phi, r) \cap$ bdy K cannot be contained in any open half sphere of $D(\phi, r)$.

Proof: Suppose $C(\phi, r) \cap$ bdy K is contained in some open half sphere of $D(\phi, r)$. Thus, there is a hyperplane H such that $\phi \in H$ and $C(\phi, r) \cap \text{bdy } K \subset H^+ \cap D(\phi, r)$.

Let $x_0 \in E_n$ where x_0 is perpendicular to H. The form of H must be $\{x : (x, x_0) = 0\}$. If $a_i \in C(\phi, r) \cap$ bdy K, then $(a_i, x_0) > 0$ since $a_i \in H^+ \cap D(\phi, r)$. Since $\phi \in$ conv $(C(\phi, r) \cap$ bdy K),

$$\phi = \sum_{i=1}^{m} \lambda_i a_i ,$$

where

$$\sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0, \quad a_i \in C(\phi, r) \cap \text{bdy } K$$

and hence,

$$0 = (x_0, \phi) = \sum_{i=1}^{m} \lambda_i (a_i, x_0).$$

But $(a_i, x_0) > 0$ for all i implies that

$$\sum_{i=1}^{m} \lambda_i (a_i, x_0) > 0,$$

a contradiction.

Theorem 8-6: Let K be compact convex body in E_n. If $C(\phi, r) \cap$ bdy K is not empty and cannot be contained in any open half sphere of $D(\phi, r)$, then $\phi \in$ conv $(C(\phi, r) \cap$ bdy K).

Proof: Suppose $\phi \notin$ conv $(C(\phi, r) \cap$ bdy K) and let x_0 be the orthogonal projection of ϕ onto conv $(C(\phi, r) \cap$ bdy K), (cf. Figure 8-7).

Let $H_1 = \{x : (x - x_0, x_0) = 0\}$. By Lemma 2-3, H_1 is a support hyperplane of conv $(C(\phi, r) \cap$ bdy K). The origin ϕ is in H_1^-, and therefore, conv $(C(\phi, r) \cap \text{bdy } K) \subset H_1 \cup H_1^+$. Let $y \in C(\phi, r) \cap$ bdy K, and by the support property of H_1, $(y - x_0, x_0) \geq 0$, or

$$(y, x_0) \geq (x_0, x_0) > 0.$$

That is, $(y, x_0) > 0$.

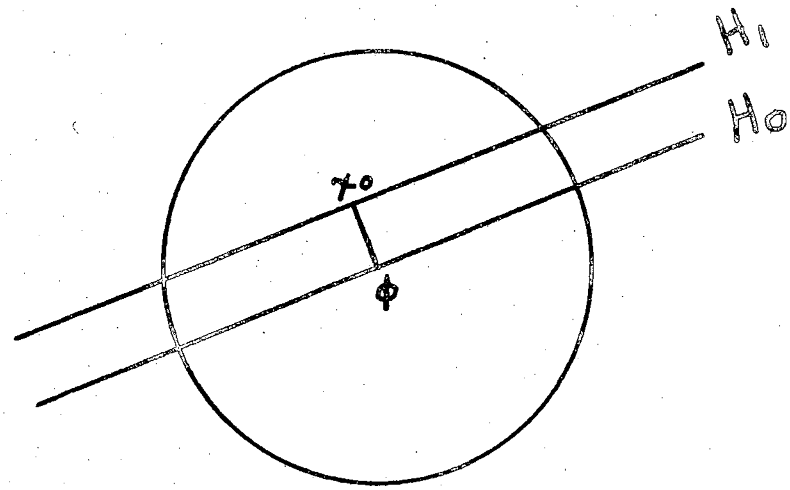

Figure 8-7.

The set $H = \{x : (x, x_0) = 0\}$ is a hyperplane through ϕ. Thus $(y, x_0) > 0$ implies that y is in H_0^+. Therefore, $C(\phi, r) \cap \text{bdy } K$ is contained in H_0^+. Since $C(\phi, r) \cap \text{bdy } K$ is contained in $D(\phi, r)$, it follows that $C(\phi, r) \cap \text{bdy } K \subset H_0^+ \cap D(\phi, r)$. However, this contradicts the hypothesis of the theorem since $H_0^+ \cap D(\phi, r)$ is an open half space of $D(\phi, r)$.

Lemma 8-3: Let X be a compact convex body in E_n whose minimal spherical shell is formed by $C_1 = D(\phi, r_1)$ and $C_2 = D(\phi, r_2)$. If $A = C'_1 \cap \text{bdy } X$ and $B = C'_2 \cap \text{bdy } X$ where $C'_1 = C(\phi, r_1)$ and $C'_2 = C(\phi, r_2)$, then it is impossible for any hyperplane through ϕ to strictly separate A and B.

Proof: Let $r_1 > r_2$ and suppose there is a hyperplane H containing ϕ and strictly separating A and B. Assume $A \in H^+$ and $B \in H^-$ (cf. Figure 8-8).

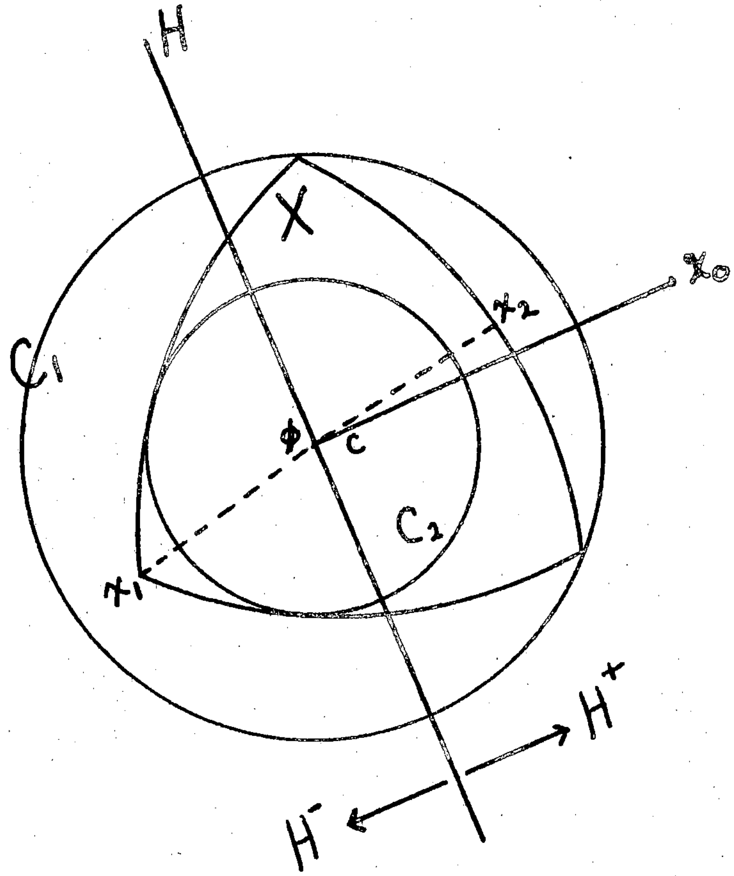

Figure 8-8.

Select an $x_0 \epsilon E_n$ such that x_0 is perpendicular to H at ϕ. Therefore, $H = \{x : (x, x_0) = 0\}$ and $x_0 \epsilon H^+$.

Let $K_1 = X \cap$ complement H^+. The set K_1 is compact and therefore the set $\{||x|| : x \epsilon K_1\}$ has a maximum value at $x_1 \epsilon K_1$. Let $||x_1|| = r'_1 < r_1$ and define $\delta_1 = r_1 - r'_1 > 0$.

Similarly, let $K_2 = \text{bdy } X \cap$ complement H^-. The set $\{||x|| : x \epsilon K_2\}$ has a minimum value at $x_2 \epsilon K_2$. Let $||x_2|| = r'_2 > r_2$ and $r'_2 - r_2 = \delta_2 > 0$.

Let δ be the smaller of δ_1 and δ_2. Select $c = \lambda x_0$ for some positive λ so that $||c|| = \delta/2$.

Let

$$S_1 = D\left(c, \sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2}\right)$$

and

$$S_2 = D\left(c, \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2}\right).$$

If it can be shown that

1) $$\sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2} - \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2} < r_1 - r_2,$$

and

2) $$S_2 \subset X \subset S_1,$$

then since S_1 and S_2 are concentric, it follows that C_1 and C_2 do not form a minimal spherical shell for X.

Proof of 1): For any r_1 and r_2, $(r_1 - r_2)^2 > 0$, which is equivalent to

$$r_1^2 + r_2^2 - 2r_1r_2 > 0.$$

Next, multiplying by the positive number $(\delta/2)^2$ and then adding

$$r_1^2\, r_2^2 + \left(\frac{\delta}{2}\right)^4$$

leads to

$$r_1^2\, r_2^2 + r_1^2 \left(\frac{\delta}{2}\right)^2 + r_2^2 \left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^4 > r_1^2\, r_2^2 + \left(\frac{\delta}{2}\right)^4 + 2r_1r_2 \left(\frac{\delta}{2}\right)^2.$$

By factoring each side, one obtains the relationship

$$\left[r_1^2 + \left(\frac{\delta}{2}\right)^2\right]\left[r_2^2 + \left(\frac{\delta}{2}\right)^2\right] > \left[r_1 r_2 + \left(\frac{\delta}{2}\right)^2\right]^2 .$$

All of the quantities involved are positive. Thus first taking square roots and then multiplying by (-2) leads to

$$-2\sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2}\ \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2} < -2\left[r_1 r_2 + \left(\frac{\delta}{2}\right)^2\right].$$

By adding

$$r_1^2 + r_2^2 + 2\left(\frac{\delta}{2}\right)^2 ,$$

results in

$$r_1^2 + r_2^2 - 2\sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2}\ \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2} + 2\left(\frac{\delta}{2}\right)^2 < -2r_1 r_2 + r_1^2 + r_2^2 . \tag{8-2}$$

By rearranging terms, write (8-2) in the form

$$r_1^2 + \left(\frac{\delta}{2}\right)^2 - 2\sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2}\ \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2} + r_2^2 + \left(\frac{\delta}{2}\right)^2 < (r_1 - r_2)^2$$

which is equivalent to

$$\left[\sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2} - \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2}\right]^2 < (r_1 - r_2)^2 .$$

Again, since all the quantities which are squared are positive, the square roots of both sides result in

$$\sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2} - \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2} < r_1 - r_2 ,$$

which was to be proved.

Proof of 2): First it will be shown that $X \subset S_1$. Let $x \in X \cap$ complement $H^+ = K_1$. By the triangle inequality,

$$||x - c|| \leq ||x|| + ||c||.$$

The value of $||x||$ is less than r_1' since r_1' is the maximum value for the norm function of any point in K_1. Therefore,

$$||x|| + ||c|| \leq r_1' + \delta/2.$$

Then

$$r_1' + \frac{\delta}{2} = r_1 - \delta_1 + \frac{\delta}{2}$$

$$\leq r_1 - \delta_1 + \frac{\delta_1}{2}$$

$$= r_1 - \frac{2\delta_1}{2} + \frac{\delta_1}{2}$$

$$< r_1 < \sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2}.$$

This proves that

$$||x - c|| < \sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2}$$

which shows that $x \in S_1$.

Next, let $x \in X \cap$ complement H^-. Here observe that

$$||x - c||^2 = ||x||^2 - 2(c, x) + ||c||^2.$$

Since $x \epsilon X \subset C_1$, $||x|| \leq r_1$. Hence,

$$||x - c||^2 \leq r_1^2 - 2(c, x) + \left(\frac{\delta}{2}\right)^2.$$

The inner product $(c, x) = \lambda(x_0, x) \geq 0$ since $x \epsilon$ complement H^-. Therefore,

$$r_1^2 - 2(c, x) + \left(\frac{\delta}{2}\right)^2 \leq r_1^2 + \left(\frac{\delta}{2}\right)^2,$$

which implies that

$$||x - c|| \leq \sqrt{r_1^2 + \left(\frac{\delta}{2}\right)^2}.$$

Therefore, $x \epsilon S_1$ and $X \subset S_1$.

It will now be shown that $S_2 \subset X$. Let $x \epsilon S_2 \cap$ complement H^+. Therefore, $x \epsilon S_2$ implies that

$$||x - c||^2 \leq r_2^2 + \left(\frac{\delta}{2}\right)^2$$

which is equivalent to

$$||x||^2 - 2(c, x) + ||c||^2 \leq r_2^2 + \left(\frac{\delta}{2}\right)^2.$$

From this

$$||x||^2 \leq r_2^2 + 2(c, x).$$

The point $x \epsilon$ complement H^+ implies that $(c, x) = \lambda(x_0, x) \leq 0$ or $0 \leq -2(c, x)$. Therefore $||x||^2 \leq r_2^2$, which says that $x \epsilon C_2 \subset X$.

If $x \epsilon S_2 \cap$ complement H^-,

$$||x|| - ||c|| \leq ||x - c|| \leq \sqrt{r_2^2 + \left(\frac{\delta}{2}\right)^2} \leq r_2 + \frac{\delta}{2}.$$

Therefore,

$$||x|| \leq r_2 + \frac{\delta}{2} + \frac{\delta}{2} = r_2 + \delta \leq r_2 + \delta_2 = r_2'.$$

By Lemma 8-1, $x \in X$. Therefore, $S_2 \subset X$, and the proof of the lemma has been completed.

The main objective of this chapter is to present and prove the following theorem:

Theorem 8-7: If X is a set of constant width λ, then the insphere and circumsphere are concentric, and the sum of their radii is λ.

Proof: Let $C_1 = D(\phi, r_1)$ and $C_2 = D(\phi, r_2)$ form the minimal spherical shell for X. Let C_1' and C_2' represent the boundaries of C_1 and C_2, respectively. As in the preceding lemma, let

$$A = \{x : x \in C_1' \cap \text{bdy } X\},$$

and

$$B = \{ x : x \in C_2' \cap \text{bdy } X\}.$$

Let $x_0 \in A$ and let

$$H_0 = \{x : (x - x_0, x_0) = 0\}.$$

By Lemma 6-1, H_0 is a support hyperplane of C_1. The set X is also supported by H_0 at x_0 (cf. Figure 8-9)

Let H_1 be the parallel support hyperplane of X at $y_0 \in X$. Since the point y_0 cannot be an interior point of C_2, $||y_0 - x_0|| \geq r_1 + r_2$. But $||y_0 - x_0|| = \lambda$ and hence $\lambda \geq r_1 + r_2$.

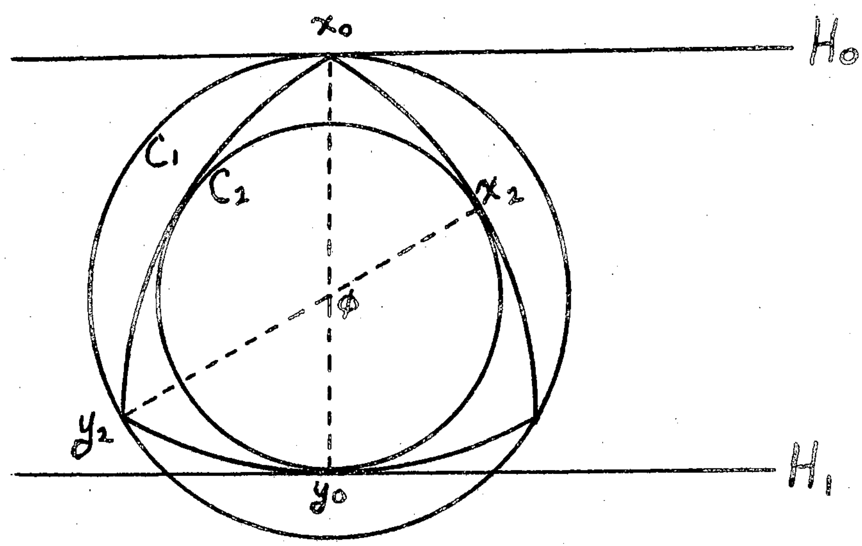

Figure 8-9.

Let $x_2 \in B$ and let H_2 be a support hyperplane of X at x_2. The set C_2 is also supported by H_2 at x_2. By Lemmas 6-1 and 8-2,

$$H_2 = \{x : (x - x_2, x_2) = 0\}.$$

Let H_3 be the parallel support hyperplane of X at y_2. The point y_2 cannot be an exterior point of C_1 and hence $||y_2 - x_2|| \leq r_1 + r_2$. Since $||y_2 - x_2|| = \lambda$, $\lambda \leq r_1 + r_2$. This, along with the inequality $\lambda \geq r_1 + r_2$ implies that $\lambda = r_1 + r_2$.

By Lemma 8-3, there is no hyperplane through ϕ strictly separating A and B. Therefore, A is not contained in any open half sphere of C_1, and B is not contained in any open half sphere of C_2.

Using Theorem 8-6, $\phi \in$ conv (A) and $\phi \in$ conv (B) where $A = C_1' \cap$ bdy X and $B = C_2' \cap$ bdy X.

By Theorem 8-2, $X \subset C_1$ and $\phi \in \text{conv } C_1' \cap \text{bdy } X)$ implies that C_1 is the circumsphere of X. Similarly, using Theorem 8-4, $C_2 \subset X$ and $\phi \in \text{conv } (C_2' \cap \text{bdy } X)$ implies that C_2 is the insphere.

As a corollary to the theorem, consider the following statement:

Corollary 8-1: The radius r_1 of the circumsphere $D(\phi, r_1)$ of a set X of constant width λ in E_n lies between

$$\frac{1}{2}\lambda \quad \text{and} \quad \lambda\sqrt{\frac{n}{2n+2}} \ .$$

Proof: If r_2 is the radius of the insphere, then $r_1 + r_2 = \lambda$. Since $r_1 \geq r_2$, $2r_1 \geq r_1 + r_2 = \lambda$ from which $r_1 \geq (1/2)\lambda$.

Since ϕ is the center of the circumsphere, by Theorem 8-1, $\phi \in \text{conv } (C(\phi, r) \cap \text{bdy } X)$. Then by Caratheodory's theorem,

$$\phi = \sum_{i=1}^{m} \lambda_i x_i$$

where

$$1 < m \leq n + 1, \qquad \sum_{i=1}^{m} \lambda_i = 1, \quad \text{all } \lambda_i \geq 0$$

and $x_i \in C(\phi, r) \cap \text{bdy } X$. Observe that $m - 1 \neq 0$ for if $m - 1 = 0$, then $\phi = x$ where $x \in C(\phi, r)$.

Let $\delta = \max\{||x_i - x_j|| : 1 \leq i \leq m, \ 1 \leq j \leq m\}$. For any fixed j, $1 \leq j \leq m$,

$$(1 - \lambda_j)\,\delta^2 = 1\delta^2 - \lambda_j\,\delta^2 = \sum_{i=1}^{m} \lambda_i\,\delta^2 - \lambda_j\,\delta^2$$

$$\geq \sum_{i=1}^{m} \lambda_i ||x_i - x_j||^2 = \sum_{i=1}^{m} \lambda_i \left[||x_i||^2 - 2(x_i, x_j) + ||x_j||^2 \right]$$

$$= \sum_{i=1}^{m} \lambda_i \left[2r_1^2 - 2(x_i, x_j) \right] = 2r_1^2 \sum_{i=1}^{m} \lambda_i - 2 \sum_{i=1}^{m} \lambda_i (x_i, x_j)$$

$$= 2r_1^2 - 2(\lambda_1 x_1 + \ldots + \lambda_m x_m, x_j) = 2r_1^2 - 2\left(\sum_{i=1}^{m} \lambda_i x_i, x_j \right)$$

$$= 2r_1^2 - 2(\phi, x_j) = 2r_1^2.$$

Therefore, $(1 - \lambda_j)\,\delta^2 \geq 2r_1^2$ and the sum of these inequalities over all j results in

$$\sum_{j=1}^{m} (1 - \lambda_j)\delta^2 \geq \sum_{j=1}^{m} 2r_1^2,$$

which is equivalent to $m\delta^2 - \delta^2 \geq 2r_1^2\, m$. From this

$$m \geq \frac{\delta^2}{\delta^2 - 2r_1^2},$$

and since $n + 1 \geq m$,

$$n + 1 \geq \frac{\delta^2}{\delta^2 - 2r_1^2},$$

which can be changed to

$$(n + 1)\delta^2 - 2r_1^2\,(n + 1) \geq \delta^2$$

or

$$\delta^2 (n + 1 - 1) \geq 2r_1^2 (n + 1)$$

which is equal to

$$\delta^2 \geq 2r_1^2 \left(\frac{n+1}{n}\right) .$$

Since

$$\lambda \geq \delta \geq r_1 \sqrt{\frac{2n+2}{n}} ,$$

the final result

$$r_1 \leq \lambda \sqrt{\frac{n}{2n+2}}$$

follows. Therefore,

$$\frac{1}{2}\lambda \leq r_1 \leq \lambda \sqrt{\frac{n}{2n+2}} .$$

To show that the left limit is attained, consider any sphere. In such a case $r_1 = r_2$, and $2r_1 = \lambda$ or $r_1 = (1/2)\lambda$.

In E_2 consider the Reuleaux triangle (cf. Figure 8-10). In this situation

$$\lambda = \sqrt{\frac{3}{4} r_1^2 + \frac{9}{4} r_1^2} = r_1 \sqrt{3} ,$$

and so,

$$\lambda \sqrt{\frac{n}{2n+2}} = r_1 \sqrt{3} \left(\frac{2}{6}\right)^{1/2} = r_1 .$$

Thus, the right limit is attained.

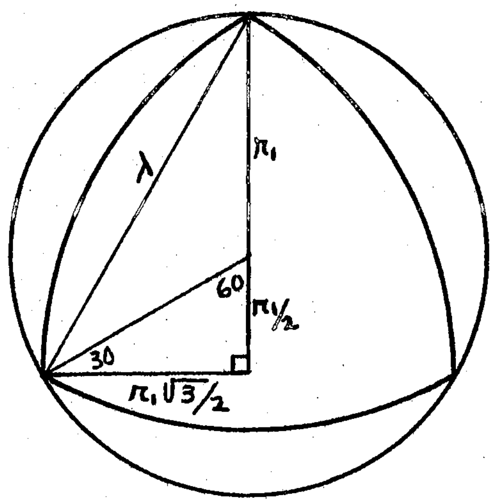

Figure 8-10.

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