A STUDY OF UNCERTAINTIES IN PROBABILISTIC MODELS

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in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
May, 1969

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PREFACE

This thesis is concerned with investigating uncertainties associated with probabilistic models and developing a procedure for estimating a measure of that uncertainty. In particular, the response, P_K, of a general probabilistic model is assumed to be a random variable and the variance of this random variable is chosen as a measure of the uncertainty of the probabilistic model. Errors in the forms of the models are not considered in this thesis so that the variance of P_K is evaluated in terms of the uncertainties associated with the basic random variable inputs of the model. These latter uncertainties are assumed to be directly dependent upon empirical data used to estimate the parameters associated with these basic random variable inputs.

The research for this thesis was accomplished under the general Joint Munitions Effectiveness Manual contract of Oklahoma State University from the Department of Defense. The thesis serves as a final report of the work accomplished under that section of the contract covering this research. Quite appropriately, the examples presented are of probabilistic weapons effectiveness models and reflect the applicability of the theory developed here to the evaluation of the uncertainty associated with these models and their random variable inputs.

I would like to take this opportunity to express my appreciation to my major thesis advisor, Dr. A. M. Breipohl, for his able guidance and assistance throughout the development of this thesis. Furthermore, I would like to thank the other members of my thesis committee,

Dr. B. L. Basore, Dr. D. E. Bee, and Professor P. A. McCollum, for their excellent counsel and encouragement.

In addition, I would like to thank the staff of the Office of Engineering Research for their assistance; in particular, Dona Davenport, for her typing excellence and patience.

Also a special word of graditude is in order for the staff of the Oklahoma State University Field Office at Eglin Air Force Base for their generous aid to my research efforts; in particular, the excellent technical advice and personal interest of Jerry Worsham.

Finally, I would like to express appreciation to my wife, Shirley, and our son, Andy, for their ceaseless understanding, encouragement, and patience throughout the preparation of this thesis.

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CHAPTER 1

INTRODUCTION

1.1 Statement of the Problem. The behavior of a physical system is usually described by a series of equations which relate the system response to the parameters of the system and external stimuli. In today's complex society a great number of systems exist that are modeled in probabilistic terms; that is, there is a certain degree of randomness inherent in the system that dictates uncertainty in the response. If the mathematical model chosen to represent the system includes provisions to account for the randomness of the system, one terms the model probabilistic; otherwise, one refers to it as deterministic. The prime purpose of this thesis, then, is to investigate the uncertainty associated with probabilistic models and to develop a probabilistic sensitivity measure.

Generally this measure of uncertainty will describe the accuracy of a predicted response of a probabilistic model. Furthermore, the results and methods employed in such a basic analysis can be useful in locating the principal sources of error in the response and thus indicating where more care should be taken in specifying the parameters which describe the random variables to be used as system inputs. Finally, a measure of uncertainty can aid in the comparison of two or more probabilistic models to determine if the models can be used interchangably.

1.2 Relationship between Probabilistic and Deterministic Models. Since this thesis proposes to expand the present techniques used in

deterministic models to encompass the broader field of probabilistics, it is advantageous to compare these two types of models and indeed to show that a deterministic model can be viewed as a special case of a probabilistic one.

Suppose one considers the following general notation:

η = expected output of a system S (1.2.1)

$\underline{X} = (X_1, X_2, \ldots, X_n)$ = n-dimensional random vector input to system S (1.2.2)

$\underline{x} = (x_1, x_2, \ldots, x_n)$ = n-dimensional vector of expected values of $\underline{X}$ (1.2.3)

$f_{\underline{X}}$ = Joint probability density function of $\underline{X}$

Define Z to be a functional relationship among the components of $\underline{X}$.

$$Z = q(X_1, X_2, \ldots, X_n) \tag{1.2.4}$$

The randomness of $\underline{X}$ thus implies that Z is also a random variable. Z might then be termed a description of the behavior of system S for an input of the random vector $\underline{X}$. The fixed point η then would represent the expected value or most probable response of the system; i.e., E[Z]. In notation one thus writes

$$E[Z] = \int \int \cdots \int q(\underline{\alpha}) f_{\underline{X}}(\underline{\alpha}) d\underline{\alpha} \tag{1.2.5}$$

where $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$$d\underline{\alpha} = d\alpha_1 \, d\alpha_2 \ldots . \, d\alpha_n$$

Equation 1.2.5 then represents the general form of a probabilistic model which is considered in this thesis.

Consider now the change in Equation (1.2.5) when $\underline{X}$ is assumed not to be a random vector. The joint probability function of $\underline{X}$ can now be rewritten (1)

$$f_{\underline{X}}(X_1, X_2, \ldots, X_n) = \delta(X_1 - x_1)\, \delta(X_2 - x_2) \ldots \delta(X_n - x_n) \tag{1.2.6}$$

where $\underline{x}$ is now assumed to be the nominal value of $\underline{X}$ and δ denotes the familiar Dirac delta function. If Equation 1.2.5 is evaluated assuming $f_{\underline{X}}$ is of the form of Equation 1.2.6, we obtain

$$\eta = q(x_1, x_2, \ldots, x_n) \tag{1.2.7}$$

Thus one sees that η and $\underline{x}$ are related deterministically as one would expect. The idea of a deterministic model being a special case of a probabilistic one is not intended to be presented here as an original idea. It is instead presented to in some way justify the extension of deterministic sensitivity analysis to that concerned with probabilistic models.

One has to admit that the representation of any physical system as known numbers is an idealization which may only be taken as a first approximation to reality. The representation of the parameters of a system as random variables is considered a more exact mathematical model. However, in many systems where the variation of the parameters seem negligible or probability distribution functions of the parameters (random variables) are not easily obtained, a deterministic model is often selected to describe the system. Even under these conditions it has become apparent that it is important to evaluate the degree of correspondence between the mathematical model and the real system. No physical

system can presently by envisioned in which its parameters will absolutely coincide with the parameters of its mathematical model. It thus becomes important to determine the influence of the variation of the parameters on the behavior of the system. The ideas of sensitivity analysis were born out of this need.

1.3 Review of Sensitivity in Deterministic Models. Extensive literature exists pertaining to the sensitivity of deterministic systems to variation of parameters. This is especially true in the field of automatic control systems where this sensitivity to the variation of parameters plays an important role in the analysis and synthesis of these systems. Most sensitivity studies presented in the literature depend heavily on perturbation techniques (2), (3). Basic to these developments are the computation of sensitivity coefficients (sometimes referred to as influence coefficients) which are defined as partial derivatives of the particular response variable with respect to the particular parameter variable (4), (5), (6), (7), and (8).

Other less widely known approaches to sensitivity include those associated with tolerance regions, the theory of invariant imbedding, and the game theory technique (9), (2), (10), and (11). Although these three approaches are both interesting and useful, it is felt that the classical perturbation techniques are more readily adaptable to probabilistic models and thus these three ideas will not be discussed further at this point.

Suppose one then returns to the ideas associated with the first approach listed, namely that related to perturbation techniques. As stated previously, the determination of sensitivity coefficients is fundamental to a sensitivity analysis. The basic concept of a sensitivity

coefficient can be traced to Bode (12) although the sensitivity coefficient he presented is the reciprocal of that applied in modern feedback theory. For an extensive list of classical formulas pertinent to the calculation of sensitivity coefficients, the reader is referred to reference (5). In addition, one of the present day leaders in the field of sensitivity analysis, R. Tomovic of Yugoslavia, gives a brief but informative discussion of "The Role of Sensitivity Analysis in Engineering Problems" in which he discusses the state of the art (6). For a more detailed survey of sensitivity analysis in control systems, one is urged to consult Kototovic and Rutman (4). This particular paper presented an excellent bibliography of no less than one-hundred-and-fifty-seven (157) entries.

Passing now from the general to the particular, suppose for convenience of notation one lets

$$\eta = q(x_1, x_2, \ldots, x_n) \tag{1.3.1}$$

$$\Delta x_i = \text{incremental variation from the assumed or desired value of } x_i \tag{1.3.2}$$

where now q represents a deterministic model of a system linking the variables $x_1, x_2, \ldots, x_n$ to the response η. Then define

$$T_i = \left(\frac{\partial q}{\partial x_i}\right)\Bigg|_{\substack{\Delta x_1 = \Delta x_2 = \ldots = \Delta x_n = 0 \\ x_i = \text{desired value for } i = 1, \ldots, n}} \tag{1.3.3}$$

where T_i equals the sensitivity coefficient associated with x_i.

It might be well at this point to mention a slight modification of the defined sensitivity coefficients in Equation 1.3.3 used particularly in feedback control theory. The sensitivity of an overall gain g with

respect to a given parameter k is defined by the equation which follows where d represents the derivative operator

$$S_k^g = \left.\frac{d \ln g}{d \ln k}\right|_{\substack{g = g_N \\ k = k_N}} \tag{1.3.4}$$

which, as mentioned previously, is the reciprocal of that introduced by Bode (12), (5), (13). In the notation 1.3.4, g_N and k_N are considered as the nominal values of g and k respectively. Now Equation 1.3.4 can be written

$$S_k^g = \left.\frac{dg}{g}\frac{k}{dk}\right| = \left(\frac{dg}{dk}\right)\Bigg|_{\substack{g = g_N \\ k = k_N}}^{\substack{k_N \\ g_N \\ g = g_N \\ k = k_N}} \tag{1.3.5}$$

which relates that the sensitivity of g with respect to k is the percentage change in g divided by the percentage change in k which produces the change in g assuming all changes are differentially small. Although this modification is not applied directly in this paper, it appears frequently in the literature and is presented here to point out another form of the standard sensitivity coefficient defined in Equation 1.3.3 (13), (5), and (4).

The reason for introducing sensitivity coefficients as a measure of sensitivity is justified when it is realized that in a great many cases, changes in the system behavior due to parameter variation can be approximated (from a first order Taylor Series Expansion of q about the true value of the x's) as

$$\Delta q = \sum_{i=1}^{n} T_i \, \Delta x_i \tag{1.3.6}$$

If one wishes a more accurate estimate, sensitivity coefficients can be determined for the (k + ℓ)th order (7).

$$T_{ik,j\ell} = \left.\frac{\partial^{k+\ell} q}{\partial x_i^k \partial x_j^\ell}\right| \qquad (1.3.7)$$

$$\Delta x_1 = \Delta x_2 = \ldots = \Delta x_n = 0$$

x_i = desired values for i = 1, . . ., n

x_j = desired values for j = 1, . . ., n

It should be noted that to this point only variation in system behavior in terms of Δx_i has been discussed; where Δx_i is the difference between the actual and the desired values of the system parameters. This is the classical deterministic measure approach. A second, and to this author, more accurate approximation involves a probabilistic measure and is based on finding the expected value of the mean square error of output variation. The idea was first offered by Broome and Young (14) and extended in more general terms to calculating the variance of a system output in terms of the respective variances of the associated parameters (random) by Breipohl and Campbell (15) and Evlanov (16).

Fundamental to these latter two presentations is the assumption that the parameters related to the system are indeed random variables and that Equation 1.3.1 should be written

$$Q = q(X_1, X_2, \ldots, X_n) = q(\underline{X}) \qquad (1.3.8)$$

where Q and all X_i are random variables with

$$E[X_i] = x_i = \text{expected value of } X_i \qquad (1.3.9)$$

and $$\underline{x} = (x_1, x_2, \ldots, x_n) \qquad (1.3.10)$$

If one expands Q about $\underline{x}$ in a first order Taylor series expansion one obtains

$$Q \cong q(\underline{x}) + \sum_{i=1}^{n} \frac{\partial Q}{\partial X_i}\Big|_{\underline{X} = \underline{x}} (X_i - x_i) \tag{1.3.11}$$

Taking the expected value of Q results in

$$E[Q] \cong q(x_1, x_2, \ldots, x_n) \tag{1.3.12}$$

which corresponds to the deterministic model of Equation 1.3.1 if indeed the nominal values of the x's are taken as the most probable or expected value of the X's. Defining the variance of Q as

$$\mathrm{Var}(Q) = E[(Q - E(Q))^2] \tag{1.3.13}$$

and the covariance of X_i and X_j as

$$\mathrm{Cov}(X_i, X_j) = E[(X_i - E(X_i))\ (X_j - E(X_j))] \tag{1.3.14}$$

one has the familiar result:

$$\mathrm{Var}(Q) \cong \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial Q}{\partial X_i}\right)\left(\frac{\partial Q}{\partial X_j}\right)\Big|_{\underline{X} = \underline{x}} \mathrm{cov}(X_i, X_j) \tag{1.3.15}$$

Var (Q) is thus a measure of the sensitivity (or uncertainty) of the system with respect to the variables X_i; i = 1, 2, . . ., n. It should be pointed out that the probabilistic measures which have been applied in this section have been, prior to this time, related only to deterministic models and not to probabilistic models as discussed in the remainder of this thesis.

1.4 Scope of Study. As has been stated previously in this chapter, the basic purpose of this thesis is to investigate uncertainty in

probabilistic models. The general approach employed expands the ideas presented in the previous section on sensitivity analysis of deterministic models to the more general case of probabilistic models and investigates the use of the variance of the system output as a measure of uncertainty. Chapter II presents various system probabilistic models, the first of which is a general model used to introduce notation employed. Then errors associated with assumed density function parameters are investigated for two models. Finally other sources are considered including those related to random conditional loss functions and stochastic tables. Chapter III is devoted to applying the techniques developed in Chapter II to two examples from the field of weapons effectiveness. The probabilistic models involved are essentially those used for computing the probability or, as will be pointed out, the conditional expected probability of accomplishing a certain degree of damage to a target by air delivered weapons. The variance of this conditional probability is taken as the measure of the uncertainty and is related to the variance of the basic random variable inputs of the model. Chapter IV states conclusions drawn from the first three chapters and relates suggestions for further study in the general areas covered by this thesis.

CHAPTER II

DEVELOPMENT OF SYSTEM CONDITIONAL PROBABILISTIC MODELS

2.1 Introduction. The prime purpose of this chapter is to provide a working theory for the investigation of uncertainties in probabilistic models. After the general notation pertaining to conditional probabilistic models is presented, a theoretical approach to obtaining a measure of this uncertainty is examined. Approximations are then made relative to the theoretical models dictated by particular situations. The techniques developed in this chapter are then applied in Chapter III to specific examples in the field of weapons effectiveness.

Although there are numerous types of probabilistic models which could, at this point, be discussed, a representative model has been chosen which is often encountered, especially in the area of weapons effectiveness from which the examples of Chapter III are drawn. The model selected involves the calculation of the expected value of a function of a random vector. Suppose one initially considers the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ with joint probability density function $f_{\underline{X}}$. Now let D_1 be a function of $\underline{X}$; i.e., $D_1(\underline{X})$, so that the conventional form of the probabilistic model specified can be written:

$$E[D_1] = \int_{-\infty}^{\infty}{}_n D_1(\underline{\lambda})\ f_{\underline{X}}(\underline{\lambda})\ d\underline{\lambda} \qquad (2.1.1)$$

where $d\underline{\lambda} = d\lambda_1, d\lambda_2, \ldots, d\lambda_n$ and $\int_{-\infty}^{\infty}{}_n$ represents an nth order multiple integral.

In general, however, the probability density function $f_{\underline{X}}$ depends in a known way on several parameters, say $a_1, a_2, \ldots, a_q$ so that in a broader sense Equation 2.1.1 represents a conditional expected value and should be written

$$E[D_1|a_1, a_2, \ldots, a_q] = \int_{-\infty}^{\infty}{}_n D_1(\underline{\lambda})\, f_{\underline{X}}(\underline{\lambda}|a_1, a_2, \ldots, a_q)\, d\underline{\lambda} \quad (2.1.2)$$

It should be noted that the conditional expected value of Equation 2.1.2 does not represent a random variable in the form shown, and thus to speak of a measure of uncertainty in this value does not seem relevant. However, although the form of Equation 2.1.2 is often assumed to denote a particular situation, a much more general form which is usually a better representation of the actual case, is obtained by considering the parameters of $f_{\underline{X}}$ to be random variables, namely $A_1, A_2, \ldots, A_q$. For notational convenience these A's are termed random variable parameters. One then can write the general form of the probabilistic models investigated in this thesis as follows:

$$E[D_1|\underline{A}] = \int_{-\infty}^{\infty}{}_n D(\underline{\lambda})\, f_{\underline{X}|\underline{A}}(\underline{\lambda}|\underline{A})\, d\underline{\lambda} \quad (2.1.3)$$

where $\underline{A} = (A_1, A_2, \ldots, A_q)$.

One should observe that the notation $f_{\underline{X}|\underline{A}}$ as used in Equation 2.1.3 is slightly different from that usually encountered in texts concerning conditional density functions. Normally the $\underline{A}$ in the argument of $f_{\underline{X}|\underline{A}}$ is written in small type as a particular value; however, in this consideration the identity of $\underline{A}$ as a random vector is retained and the conditional expected value in Equation 2.1.3 is expressed as a function of the random variables $A_1, A_1, \ldots, A_q$. For notational purposes this

conditional expected value is termed a random variable P_A and the function it represents, P_1; that is,

$$P_A = P_1(\underline{A}) = E[D_1|\underline{A}] \tag{2.1.4}$$

The remainder of this thesis deals with the investigation of the uncertainty associated with random variables of the general form of P_A. It should, however, be pointed out that throughout this thesis the forms of the models used are assumed fixed; that is, errors associated with the forms of the models are not considered in this thesis.

Before proceeding to a particular model, suppose one briefly investigates the uncertainty associated with the general probabilistic model of Equation 2.1.3, and in so doing introduces the measure of uncertainty discussed in this thesis. Recalling Equation 2.1.4 it is apparent that one is actually concerned with calculating the uncertainty associated with P_A. Taking the expected value of P_A from Equation 2.1.3 one obtains:

$$E[P_{\underline{A}}] = E[E(D_1|\underline{A})] = \int_{-\infty}^{\infty}{}_q \; P_1(\underline{\alpha}) \; f_{\underline{A}}(\underline{\alpha}) \; d\underline{\alpha} \tag{2.1.5}$$

where $f_{\underline{A}}$ is the joint probability density function of the $\underline{A}$'s and

$$\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_q) \tag{2.1.6a}$$

$$d\underline{\alpha} = d\alpha_1 \, d\alpha_2 \ldots d\alpha_q \tag{2.1.6b}$$

$$P_1(\underline{\alpha}) = \int_{-\infty}^{\infty}{}_n \; D_1(\underline{\lambda}) \; f_{\underline{X}|\underline{A}}(\underline{\lambda}|\underline{\alpha}) \; d\underline{\lambda} \tag{2.1.6c}$$

Now, in general, when speaking of the uncertainty associated with P_A, one is interested in the expected error between P_A and the expected value of P_A; that is, one needs to know how much faith to place in a

particular sample value of P_A, say P_A^*, where the asterisk (*) indicates a particular sample value. Since the variance of P_A gives a measure of the concentration of P_A about the expected value of P_A, Var (P_A) has been chosen as the measure of uncertainty associated with P_A examined in this paper. One calculates this variance for the general model as follows:

$$\text{Var}[P_A] = E[(P_A - E(P_A))^2] \tag{2.1.7a}$$

$$\text{Var}[P_A] = \int_{-\infty}^{\infty}{}_{q} [P_1(\underline{\alpha}) - E(P_{\underline{A}})]^2 \, f_A(\underline{\alpha}) \, d\underline{\alpha} \tag{2.1.7b}$$

or in terms of D_1

$$\text{Var}[P_A] = \int_{-\infty}^{\infty}{}_{q} \{ \int_{-\infty}^{\infty}{}_{n} D_1(\underline{\lambda}) f_{\underline{X}|\underline{A}} (\underline{\lambda}|\underline{\alpha}) \, d\underline{\lambda}$$

$$- \int_{-\infty}^{\infty}{}_{q} [\int_{-\infty}^{\infty}{}_{n} D_1(\underline{\tau}) f_{\underline{X}|\underline{A}} (\underline{\tau}|\underline{\beta}) \, d\underline{\tau}] f_{\underline{A}} (\underline{\beta}) \, d\underline{\beta}\}^2 f_{\underline{A}} (\underline{\alpha}) \, d\underline{\alpha} \tag{2.1.7c}$$

Although Equation 2.1.7c relates the basic form of $\text{Var}(P_A)$ for the general model, the rest of this chapter presents modifications to this measure of uncertainty which, as is shown in the next section, depends directly on the particular model of P_A investigated.

2.2 Model 1 - Theoretical Model. Having introduced in the previous section the basic terminology associated with conditional probabilistic models, the discussion of this section focuses on a particular theoretical model and relates the uncertainty of its conditional random response to the uncertainties associated with certain basic random variable parameters of the system model. An ideal approach to computing uncertainty is first considered and then approximations are introduced which are necessitated by the mathematical aspects of the models themselves.

For this theoretical model, consider again the random vector $\underline{X}$ described in the previous section and the general form of the probabilistic model of Equations 2.1.3 and 2.1.4.

$$P_A = E[D_1|\underline{A}] = \int_{-\infty}^{\infty}{}_n D_1(\underline{\lambda})\ f_{\underline{X}|\underline{A}}\ (\underline{\lambda}|\underline{A})\ d\underline{\lambda} \qquad (2.2.1)$$

where again D_1 is a function of $\underline{X}$. Furthermore, for this particular theoretical model, additional knowledge is assumed, namely that $\underline{X}$ is a function $\underline{G}$ of s random variables $Y_1, Y_2, \ldots, Y_s$:

$$\underline{X} = \underline{G}(Y_1, Y_2, \ldots, Y_s) \qquad (2.2.2)$$

where $Y_1, Y_2, \ldots, Y_s$ have a joint probability density function (JPDF) depending upon the random variable parameters $B_1, B_2, \ldots, B_m$ which in turn have a JPDF of $f_{B_1, B_2, \ldots, B_m}$. Now consider the following definition for notational convenience:

$$\underline{Y} = (Y_1, Y_2, \ldots, Y_s) \qquad (2.2.3a)$$

$$\underline{B} = (B_1, B_2, \ldots, B_m) \qquad (2.2.3b)$$

Since additional knowledge is now assumed concerning P_A, this knowledge needs to be incorporated into the calculation of the $\mathrm{Var}(P_A)$ which was introduced in Section 2.1 as the measure of uncertainty associated with P_A. As a first step, consider the relationship between $\underline{A}$ and $\underline{B}$. Now one reasons that if $\underline{X}$ depends upon $\underline{Y}$, and $\underline{Y}$ in turn depends upon $\underline{B}$, then the knowledge of $\underline{X}$ and its parameters $\underline{A}$ must depend basically upon the random vector parameter $\underline{B}$. In general then the following assumption is made:

$$A_i = g_i(\underline{B}) \qquad i = 1, \ldots, q \qquad (2.2.4)$$

that is; A_i is assumed to be a function of $\underline{B}$. In vector notation, one writes Equation 2.2.4 as follows:

$$\underline{A} = g(\underline{B}) \tag{2.2.5}$$

Now since each A_i of this theoretical model is assumed to be a function of $\underline{B}$, the conditional density of $\underline{X}$ appearing in Equation 2.2.1 could be fundamentally rewritten as a conditional density conditioned on $\underline{B}$; that is,

$$f_{\underline{X}|\underline{A}} = f_{\underline{X}|\underline{g}(B)} \quad \text{which is termed} \quad f_{\underline{X}|\underline{B}} \tag{2.2.6}$$

Thus an alternative way of viewing the conditional expected value of D_1 for this theoretical model is:

$$E[D_1|\underline{B}] = \int_n D_1(\underline{\lambda})\ f_{\underline{X}|\underline{B}}\ (\underline{\lambda}|\underline{B})\ d\underline{\lambda} \tag{2.2.7}$$

Suppose one now denotes the conditional expected value of Equation 2.2.7 as P_B and the function it represents as P_2 so that

$$P_B = P_2(\underline{B}) = E[D_1|\underline{B}] \tag{2.2.8}$$

P_B then represents a random variable which is the conditional expected value of D_1 conditioned upon the basic random vector $\underline{B}$. One now proceeds to calculate $Var(P_B)$ by first computing the expected value of P_B as follows:

$$E[P_B] = \int_{-\infty}^{\infty}{}_m \int_{-\infty}^{\infty}{}_n D_1(\underline{\lambda})\ f_{\underline{X}|\underline{B}}\ (\underline{\lambda}|\underline{\beta})\ f_{\underline{B}}(\underline{\beta})\ d\underline{\lambda}d\underline{\beta} \tag{2.2.9}$$

where again the notation $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ and $d\underline{\beta} = d\beta_1\ d\beta_2 \ldots d\beta_m$ is employed. The variance of P_B can then be calculated directly:

$$\mathrm{Var}(P_B) = E[(P_B - E(P_B))^2] \tag{2.2.10a}$$

$$\mathrm{Var}(P_B) = \int_{-\infty}^{\infty}\{P_2(\beta) - E(P_B)\}^2 f_{\underline{B}}(\underline{\beta})d\underline{\beta} \tag{2.2.10b}$$

Var (P_B) as calculated in Equation 2.2.10a is the measure of uncertainty in P_B of the theoretical model discussed in this section. However, the form of Equation 2.2.10a is rather complex and assumes that the forms of $f_{\underline{X}}$, $f_{\underline{Y}}$, and $f_{\underline{B}}$ are known. It might be noted here that a Bayesian approach to learning these density functions from empirical data is presented in section 2.3. For now though, consider a more simplified approach to approximating $\mathrm{Var}(P_B)$.

Recalling Equations 2.2.7 and 2.2.8 one can write

$$P_B = P_2(\underline{B}) = E[D_1|\underline{B}] \tag{2.2.11}$$

Now make the following definition:

$$E[B_i] = b_i \tag{2.2.12a}$$

$$\mathrm{Var}[B_i] = v_i \tag{2.2.12b}$$

If one expands $P_2(\underline{B})$ about $\underline{b} = (b_1, b_2, \ldots, b_m)$ in a Taylor Series Expansion, one acquires

$$P_B = P_2(\underline{B}) = P_2(\underline{b}) + \sum_{i=1}^{m} \frac{\partial P_2(B)}{\partial B_i}\Big|_{\underline{B}=\underline{b}} (B_i - b_i) + \text{Remainder} \tag{2.2.13}$$

Now for the present assume that the remainder is negligible in comparison to the first order terms, an assumption which is investigated more fully in Appendix A. Suffice it to note at this point that if the

remainder of Equation 2.2.13 cannot be neglected, a similar approach to that which follows can be used but will necessarily be more complex due to the additional terms of Equation 2.2.13.

Taking the expected value of $P_2(\underline{B})$ from Equation 2.2.13 yields

$$E[P_B] = E[P_2(\underline{B})] \approx P_2(\underline{b}) \tag{2.2.14}$$

Defining $Var(P_B)$ as

$$Var(P_B) = E[(P_B - E(P_B))^2] \tag{2.2.15}$$

one can calculate directly

$$Var(P_B) \approx E[\{\sum_{i=1}^{m} \frac{\partial P_2(\underline{B})}{\partial B_i}\Big|_{\underline{B}=\underline{b}} (B_i - b_i)\}^2] \tag{2.2.16}$$

$$Var(P_B) = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial P_2(\underline{B})}{\partial B_i} \frac{\partial P_2(\underline{B})}{\partial B_j}\Big|_{\underline{B}=\underline{b}} Cov(B_i, B_j) \tag{2.2.17}$$

where $Cov(B_i, B_j)$ is the covariance of B_i and B_j.

As a particular example of the calculation of the measure of uncertainty associated with the application of the theoretical model discussed in this section, consider the following form often encountered in physical models, particularly in the area covered by the applications of Chapter III.

Let $\underline{X}$ be a random vector with probability density function depending upon a random vector $\underline{M}$, representing the mean, and a random covariance matrix $\underline{V}$. Further assume that $\underline{X}$ is a function of the random vector $\underline{Y}$ with JPDF $f_{\underline{Y}}$ which is dependent upon the random vector $\underline{B}$. Now $\underline{M}$ and $\underline{V}$ can be evaluated in terms of $\underline{B}$ as

$$\underline{M} = g_M(\underline{B}) = g_M(B_1, B_2, \ldots, B_m) \tag{2.2.18a}$$

$$\underline{V} = g_V(\underline{B}) = g_V(B_1, B_2, \ldots, B_m) \tag{2.2.18b}$$

Equation 2.2.18 then corresponds to Equation 2.2.4 where ($\underline{M}$, $\underline{V}$) are a subset of the A_i's. In particular for $\underline{X} = \underline{G}(\underline{Y})$ as in Equation 2.2.2 $\underline{M}$ and $\underline{V}$ are evaluated as follows:

$$\underline{M} = g_M(\underline{B}) = \int_S \underline{G}(\tau) f_{\underline{Y}|\underline{B}}(\tau|\underline{B})\, d\tau \tag{2.2.19a}$$

$$\underline{V} = g_V(\underline{B}) = \int_S (\underline{G}(\tau) - g_M(\underline{B}))^2 f_{\underline{Y}|\underline{B}}(\tau|\underline{B})\, d\tau \tag{2.2.19b}$$

Again consider the conditional expected value of a function of $\underline{X}$, namely $D(\underline{X})$, conditioned on $\underline{B}$.

$$E[D|\underline{B}] = \int_{-\infty}^{\infty} D(\lambda) f_{\underline{X}|\underline{B}}(\lambda|\underline{B})\, d\lambda \tag{2.2.20}$$

Defining P_D as representing the random variable associated with Equation 2.2.20 and P_3 as the functional relationship between P_D and $\underline{B}$ one can write

$$P_D = E[D|\underline{B}] = P_3(\underline{B}) \tag{2.2.21}$$

Following the form of Equation 2.2.17 one can write the variance of P_D directly:

$$\mathrm{Var}(P_D) \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{\partial P_D}{\partial B_i}\right)\left(\frac{\partial P_D}{\partial B_j}\right)\Big|_{\underline{B}=\underline{b}} \mathrm{cov}(B_i, B_j) \tag{2.2.22}$$

Recall that P_D can be written as a function of $\underline{M}$ and $\underline{V}$, say $P_4(\underline{M},\underline{V})$, similar to the form of Equation 2.2.1 where $\underline{M}$ and $\underline{V}$ correspond to A's.

$$P_D = P_4(\underline{M}, \underline{V}) = \int_{-\infty}^{\infty}{}_n \, D_1(\underline{\lambda}) \; f_{\underline{X}|\underline{M}, \underline{V}}(\underline{\lambda}|\underline{M}, \underline{V}) \; d\underline{\lambda} \tag{2.2.23}$$

Now one can evaluate $\frac{\partial P_D}{\partial B_i}$ by the application of the chain rule for differentiation.

$$\frac{\partial P_D}{\partial B_i} = \left(\frac{\partial P_D}{\partial \underline{M}} \frac{\partial g_{\underline{M}}}{\partial B_i} + \frac{\partial P_D}{\partial \underline{V}} \frac{\partial g_{\underline{V}}}{\partial B_i}\right) \Bigg| \begin{matrix} \underline{M} = E[\underline{M}] \\ \underline{V} = E[\underline{V}] \\ \underline{B} = \underline{b} \end{matrix} \tag{2.2.24}$$

which is a shorthand representation of the following:

$$\frac{\partial P_D}{\partial B_i} = \left(\sum_{k=1}^{n} \frac{\partial P_D}{\partial M_k} \frac{\partial g_{m_k}}{\partial B_i} + \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\partial P_D}{\partial V_{k\ell}} \frac{\partial g v_{k\ell}}{\partial B_i}\right) \Bigg| \begin{matrix} \underline{M} = E[\underline{M}] \\ \underline{V} = E[\underline{V}] \\ \underline{B} = \underline{b} \end{matrix} \tag{2.2.25}$$

If Equation 2.2.24 is substituted into Equation 2.2.22 one obtains

$$\mathrm{Var}(P_D) \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{\partial P_D}{\partial \underline{M}} \frac{\partial g_{\underline{M}}}{\partial B_i} + \frac{\partial P_D}{\partial \underline{V}} \frac{\partial g_{\underline{V}}}{\partial B_i}\right)$$

$$\left(\frac{\partial P_D}{\partial \underline{M}} \frac{\partial g_{\underline{M}}}{\partial B_j} + \frac{\partial P_D}{\partial \underline{V}} \frac{\partial g_{\underline{V}}}{\partial B_j}\right) \Bigg| \mathrm{cov}(B_i, B_j) \begin{matrix} \underline{M}=E[\underline{M}] \\ \underline{V}=E[\underline{V}] \\ \underline{B}=\underline{b} \end{matrix} \tag{2.2.26}$$

Another interesting way of obtaining Equation 2.2.26 is to consider initially P_D as a function of $\underline{M}$ and $\underline{V}$ and expand P_D about $(E(\underline{M}), E(\underline{V}))$ in a Taylor Series expansion similar to that of Equation 2.2.13. The $\mathrm{Var}(P_D)$ obtained corresponding to Equation 2.2.22 is as follows:

$$\mathrm{Var}(P_D) \approx \left(\frac{\partial P_D}{\partial \underline{M}}\right)^2 \Big|_{\substack{\underline{M}=E[\underline{M}] \\ \underline{V}=E[\underline{V}]}} \mathrm{Var}[\underline{M}] + \left(\frac{\partial P_D}{\partial \underline{V}}\right)^2 \Big|_{\substack{\underline{M}=E[\underline{M}] \\ \underline{V}=E[\underline{V}]}} \mathrm{Var}[\underline{V}]$$

$$+ 2\left(\frac{\partial P_D}{\partial \underline{M}} \frac{\partial P_D}{\partial \underline{V}}\right) \Big|_{\substack{\underline{M}=E[\underline{M}] \\ \underline{V}=E[\underline{V}]}} \mathrm{cov}(\underline{M}, \underline{V}) \qquad (2.2.27)$$

Recalling Equations 2.2.18 and 2.2.19 one can expand $\underline{M}$ and $\underline{V}$ about $\underline{b}$ in a first order Taylor Series expansion to acquire

$$\underline{M} \approx g_{\underline{M}}(\underline{b}) + \sum_{i=1}^{m} \frac{\partial g_M(\underline{B})}{\partial B_i} \Big|_{\underline{B}=\underline{b}} (B_i - b_i) \qquad (2.2.28a)$$

$$\underline{V} \approx g_{\underline{V}}(\underline{b}) + \sum_{i=1}^{m} \frac{\partial g_V(\underline{B})}{\partial B_i} \Big|_{\underline{B}=\underline{b}} (B_i - b_i) \qquad (2.2.28b)$$

Again a higher order expansion could be used but is not at this point for notational convenience. A criteria for choosing what order approximation is needed for any particular case is given in Appendix A. From Equation 2.2.28 one can caluclate the mean and variance of $\underline{M}$ and $\underline{V}$ directly as follows:

$$E[\underline{M}] \approx g_M(\underline{b}) \qquad (2.2.29a)$$

$$E[\underline{V}] \approx g_V(\underline{b}) \qquad (2.2.29b)$$

$$\mathrm{Var}[\underline{M}] \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{\partial g_M(\underline{B})}{\partial B_i}\right)\left(\frac{\partial g_M(\underline{B})}{\partial B_j}\right) \Big|_{\underline{B}=\underline{b}} \mathrm{Cov}(B_i, B_j) \qquad (2.2.29c)$$

$$\mathrm{Var}[\underline{V}] \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{\partial g_V(\underline{B})}{\partial B_i}\right)\left(\frac{\partial g_V(\underline{B})}{\partial B_j}\right) \Big|_{\underline{B}=\underline{b}} \mathrm{Cov}(B_i, B_j) \qquad (2.2.29d)$$

$$\mathrm{Cov}[\underline{M}, \underline{V}] \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{\partial g_M(\underline{B})}{\partial B_i}\right)\left(\frac{\partial g_V(\underline{B})}{\partial B_j}\right) \Big|_{\underline{B}=\underline{b}} \mathrm{Cov}(B_i, B_j) \qquad (2.2.29e)$$

Finally, if Equation 2.2.29 is substituted into Equation 2.2.27, one can calculate $\mathrm{Var}(P_D)$ as follows:

$$\mathrm{Var}(P_D) \approx \left(\frac{\partial P_D}{\partial \underline{M}}\right)^2 \Big|_{\substack{\underline{M} = g_M(\underline{b}) \\ \underline{V} = g_V(\underline{b})}} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial g_M}{\partial B_i} \frac{\partial g_M}{\partial B_j} \Big|_{\underline{B}=\underline{b}} \mathrm{Cov}(B_i, B_j)$$

$$+ \left(\frac{\partial P_D}{\partial \underline{V}}\right)^2 \Big|_{\substack{\underline{M} = g_M(\underline{b}) \\ \underline{V} = g_V(\underline{b})}} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial g_V}{\partial B_i} \frac{\partial g_V}{\partial B_j} \Big|_{\underline{B}=\underline{b}} \mathrm{Cov}(B_i, B_j)$$

$$+2\left(\frac{\partial P_D}{\partial \underline{M}}\right)\left(\frac{\partial P_D}{\partial \underline{V}}\right) \Big|_{\substack{\underline{M} = g_M(\underline{b}) \\ \underline{V} = g_V(\underline{b})}} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial g_M}{\partial B_i} \frac{\partial g_V}{\partial B_j} \Big|_{\underline{B}=\underline{b}} \mathrm{Cov}(B_i, B_j) \qquad (2.2.30)$$

which reduces directly to Equation 2.2.26.

One sees from Equations 2.2.26 and 2.2.30 that the sensitivity measure for P_D; i.e., $\mathrm{Var}(P_D)$, can be approximated by a sum of the variances and covariances of the B_i's weighted by two types of sensitivity coefficients: one linking P_D to $\underline{M}$ and/or $\underline{V}$ and one, in turn, linking $\underline{M}$ and/or $\underline{V}$ to $\underline{B}$.

In this section, then, a theoretical approach to obtaining a measure of the uncertainty associated with probabilistic models has been presented. In addition, an approximation to this theoretical approach

which was dictated by the complexity of the theoretical model has been investigated. It is not proposed that this approximation is the only method in which the uncertainty related to P_D can be linked to the uncertainties of the basic random variable of the models, nor is it asserted that this must be the "best" way, however one wishes to define "best." On the other hand, though, a practical approach has been presented which is similar to the deterministic measures of sensitivity. Furthermore, this approach proves to be particularly simple if no difficulty is involved in obtaining the sensitivity coefficients (partial derivatives) and if covariance information is easily attained. In most situations, however, covariance information is usually unknown and must be estimated from empirical data. The modifications to $\mathrm{Var}(P_k)$ when estimators are employed are covered in the next sections.

2.3 Estimator Approximations. Often in physical situations either or both of the density functions $f_{\underline{A}}$ and $f_{\underline{B}}$ is unknown and must be learned or estimated from experimental data. This implies that if these estimated densities are used in the theoretical model, then the $\mathrm{Var}(P_B)$ of Equation 2.2.10b so calculated is now only an estimate of $\mathrm{Var}(P_B)$; call it $\hat{\mathrm{Var}}(P_B)$. In this section a Bayesian approach to learning $f_{\underline{A}}$ and $f_{\underline{B}}$ is discussed and the results are applied to calculating $\hat{\mathrm{Var}}(P_B)$.

Furthermore, in the approximation used to evaluate $\mathrm{Var}(P_D)$ in the previous section, the $\underline{B}$ values available usually correspond to estimators of the various parameters of $\underline{Y}$. When an actual set of data is taken, B_i takes on a particular value, say b_i^*, which is an estimate of the ith particular parameter of $f_{\underline{Y}}$ given the data available. The covariance terms of Equation 2.2.26 and 2.2.30 then correspond to covariances of the estimators which must in turn be estimated from the data. If these

terms are used, and indeed one seldom has a choice, an estimated $\mathrm{Var}(P_D)$ results which is termed $\hat{\mathrm{Var}}(P_D)$. The second part of this section discusses the use of estimators to calculate $\hat{\mathrm{Var}}(P_D)$.

Bayesian Learning. As mentioned in the previous section, $f_{\underline{A}}$ and $f_{\underline{B}}$ are seldom, if ever, known in a physical problem and this necessarily complicates the use of the theoretical model of Section 2.2. Suppose as a first simple example, one considers the general theoretical model of Equation 2.1.3; that is, $\underline{X}$ is not assumed at this point to be a function of $\underline{Y}$. In order to calculate $E[P_A]$ in Equation 2.1.5 and $\mathrm{Var}[P_A]$ in Equation 2.1.7c, one is required to know $f_{\underline{A}}$. Assume that the form of $f_{\underline{X}}$ is known but that the only available density function of A is f_{OA} which is based on previous experience with random vectors such as $\underline{A}$. Suppose a sample value of $\underline{X}$ is taken, say $\underline{x}_1$, and Bayes' Theorem is subsequently applied (17).

$$f_{\underline{A}|\underline{X}_1}(\underline{w}|\underline{x}_1) = \frac{f_{\underline{X}|\underline{A}}(\underline{x}_1|\underline{w}) f_{OA}(\underline{w})}{\int_{-\infty}^{\infty} f_{\underline{X}|\underline{A}}(\underline{x}_1|\underline{\tau}) f_{OA}(\underline{\tau}) d\tau} \tag{2.3.1}$$

Now $f_{\underline{A}|\underline{x}_1}$ is the a posteriori density function of $\underline{A}$ after the sample $\underline{x}_1$ is taken and becomes f_{1A} that is the a priori density function of $\underline{A}$ before a second sample $\underline{x}_2$ is taken. This process can be repeated for each independent sample taken, each time using the previously calculated a posteriori density of $\underline{A}$ as the present a priori density function. After many samples, under rather general conditions, the a posteriori density function of $\underline{A}$ can be shown to closely approximate $f_{\underline{A}}$, a fact that is discussed by several authors and will not be investigated in further detail at this point (17). It should, however, be added that

after several samples have been taken, the dependence of the learned or estimated $f_{\underline{A}}$ upon $f_{0\underline{A}}$ is usually not great (18).

Assuming that a density function of $\underline{A}$ was learned from z samples, say $\hat{f}_{\underline{A}}$, where $\hat{f}_{\underline{A}} = f_{\underline{A}|x_1, x_2, \ldots, x_z}$, one could calculate an estimate of $E[P_A]$ and $Var(P_A)$ by substituting $\hat{f}_A$ into Equation 2.1.5 and 2.1.7c for $f_{\underline{A}}$ obtaining

$$\hat{E}[P_A] = \int_{-\infty}^{\infty}{}_q \, P_1(\underline{\alpha}) \, \hat{f}_{\underline{A}}(\underline{\alpha}) \, d\underline{\alpha} \tag{2.3.2a}$$

$$\hat{Var}[P_A] = \int_{-\infty}^{\infty}{}_q \{ \int_{-\infty}^{\infty}{}_n D_1(\underline{\lambda}) f_{\underline{X}|\underline{A}}(\underline{\lambda}|\underline{\alpha}) d\underline{\lambda}$$

$$- \int_{-\infty}^{\infty}{}_q [\int_{-\infty}^{\infty}{}_n D_1(\underline{\tau}) f_{\underline{X}|\underline{A}}(\underline{\tau}|\underline{\beta}) d\underline{\tau}] \hat{f}_{\underline{A}}(\underline{\beta}) d\underline{\beta}\}^2 \hat{f}_{\underline{A}}(\underline{\alpha}) d\underline{\alpha} \tag{2.3.2b}$$

$\hat{Var}[P_A]$ now provides a measure of the uncertainty associated with P_A conditioned on the data used to calculate $\hat{f}_{\underline{A}}$.

Consider now the more complex case of Equation 2.2.10b where it is assumed that f_B must be learned from empirical data. If f_Y is assumed known and samples of $\underline{Y}$ are taken, the same procedure as that used for $\hat{f}_{\underline{A}}$ can be followed to obtain an estimate of $f_{\underline{B}}$, say $\hat{f}_{\underline{B}}$. However, suppose due to the nature of the variables involved, samples of $\underline{X}$ are selected rather than of $\underline{Y}$. If the assumption is made as in Equation 2.2.5 that $\underline{A}$ equals $g(\underline{B})$, one can calculate $\hat{f}_{\underline{B}}$ directly from $\hat{f}_{\underline{A}}$ by observing that

$$\hat{f}_{\underline{B}} = \frac{f(b|x_1, x_2, \ldots, x_z)}{B|x_1, x_2, \ldots, x_z} = \frac{\frac{f(g(b)|x_1, x_2, \ldots, x_z)}{A|x_1, x_2, \ldots, x_z}}{|J|} = \frac{\hat{f}_{\underline{A}}}{|J|} \tag{2.3.3}$$

where J represents the Jacobian of the transformation between $\underline{B}$ and $\underline{A}$.

Another approach to calculating $\hat{f}_{\underline{B}}$ would be to assure initially an a priori density function of $f_{\underline{B}}$, say f_{OB}, and apply Bayes' theorem directly. If one selects a sample, say x_1, one obtains

$$f_{\underline{B}|x_1}(\underline{\theta}|x_1) = \frac{f_{\underline{X}|\underline{B}}(\underline{x}_1|\underline{\theta}) f_{OB}(\underline{\theta})}{\int_{-\infty}^{\infty} f_{\underline{X}|\underline{B}}(x_1|\underline{\tau}) f_{OB}(\underline{\tau}) d\underline{\tau}} \qquad (2.3.4)$$

where $f_{\underline{X}|\underline{B}}(x_1|\underline{\theta}) = f_{\underline{X}|g(\underline{B})}^{(x_1|g(\underline{B}))}$

The function $f_{\underline{B}|\underline{X}_1}$ represents a posteriori density function of B and if one proceeds with the iterative process, as previously with $f_{\underline{A}}$, a density function of B can be learned, say $\hat{f}_{\underline{B}}$. One should note that this $\hat{f}_{\underline{B}}$ should be equivalent to $\hat{f}_{\underline{B}}$ of Equation 2.3.3 if the a priori density function f_{OB} is obtained from f_{OA}' by a transformation similar to Equation 2.3.3.

Substitution now of the learned density of $\underline{B}$ into Equation 2.2.10b can provide an estimate of $\mathrm{Var}(P_B)$ as $\hat{\mathrm{Var}}(P_B)$. It should be pointed out at this point that some degree of difficulty may be encountered in estimating $f_{\underline{B}}$ due to the nature of the transformation involved; for example, J might be equal to zero. However, since the Bayesian learning technique is not used in the examples of this thesis and has been presented here only briefly in introduction, these difficulties will not be pursued further at this point.

However, a method for applying Bayes' learning to facilitate the computation of an estimate of the measure of uncertainty associated with the theoretical model of 2.2 has been briefly introduced. This

method, as has been stated, is not applied in Chapter III due to the complexity of the problems involved there.

Estimators of Unknowns. In most practical situations the $\underline{B}$ values appearing in Equations 2.2.26 and 2.2.30 correspond to estimators of the parameters of $f_{\underline{Y}}$. For the purpose of this thesis, these estimators are assumed to be minimum variance unbiased estimators MVUE. When a set of data has been taken, estimators of $\mathrm{Cov}(B_i, B_j)$ and $E[B_i]$ can be computed (again assume MVUE) and substituted for the parameters in Equations 2.2.26 and 2.2.30. The resulting estimate of $\mathrm{Var}(P_D)$, $\hat{\mathrm{Var}}(P_D)$, is conditioned upon the data used to calculate these particular parameters estimates. If the following definitions are made

$$\mathrm{Cov}^*(B_i, B_j) = \text{estimate of } \mathrm{Cov}[B_i, B_j] \tag{2.3.5a}$$

$$\underline{b}^* = \text{estimate of } E[\underline{B}] \tag{2.3.5b}$$

one can write a modification of Equation 2.2.25 as

$$\hat{\mathrm{Var}}(P_D) = \sum_{i=1}^{m}\sum_{j=1}^{m}\left(\frac{\partial P_D}{\partial \underline{M}}\frac{\partial g_M}{\partial B_i} + \frac{\partial P_D}{\partial \underline{V}}\frac{\partial g_V}{\partial B_i}\right)\left(\frac{\partial P_D}{\partial \underline{M}}\frac{\partial g_M}{\partial B_j} + \frac{\partial P_D}{\partial \underline{V}}\frac{\partial g_V}{\partial B_j}\right)\Big|_{\substack{\underline{M}=g_M(\underline{b}^*)\\ \underline{V}=g_V(\underline{b}^*)\\ \underline{B}=\underline{b}^*}} \mathrm{Cov}^*(B_i, B_j) \tag{2.3.6}$$

The form of Equation 2.3.6 represents the general measure of uncertainty in P_D applicable to the examples of Chapter III and provides sensitivity information linking P_D to the experimental data used to estimate the basic parameter associated with the model.

2.4 Estimator Model. In previous sections the discussion was devoted to an ideal theoretical model and presented both an exact and

an approximation of $Var(P_n)$ where P_n corresponds in general to any probabilistic model discussed. In section 2.2 the number of random variable parameters of $\underline{X}$ considered was reduced to $\underline{M}$ and $\underline{V}$, the mean vector and covariance matrix of $\underline{X}$ respectively, and presented as functions of $\underline{B}$. At that point no restrictions upon the assumed functional relationship between $\underline{M}$ and/or $\underline{V}$ and $\underline{B}$ were made. In some physical models, especially those considered in Chapter III, characteristics of the specific model and the nature of the data available dictate a particular representation for $\underline{M}$ and $\underline{V}$. That is, one is forced to give up some of the generality of the theoretical model of section 2.2 so that specific problems encountered in physical examples can be more realistically investigated. In particular, many situations exist where it is necessary to model $\underline{M}$ and $\underline{V}$ by assuming a set of values for the parameters associated with $f_{\underline{Y}}$; that is, these parameters are assumed fixed and the models are derived empirically with random variable coefficients. The uncertainty associated with these random variable coefficients is assumed to include the uncertainty related to random variable parameters of $f_{\underline{Y}}$. Consider then the following functional representation for $\underline{M}$ and $\underline{V}$:

$$\underline{M} = \underline{H}(b_1^*, b_2^*, \ldots, b_m^*, J_1, J_2, \ldots, J_u) = \underline{H}(\underline{b}^*, \underline{J}) \qquad (2.4.1a)$$

$$\underline{V} = \underline{L}(b_1^*, b_2^*, \ldots, b_m^*, K_1, K_2, \ldots, K_t) = \underline{L}(\underline{b}^*, \underline{K}) \qquad (2.4.1b)$$

where $\underline{b}^*$ corresponds to estimates of the parameters of $f_{\underline{Y}}$; that is, particular values of $\underline{B}$. The J's and K's of Equation 2.4.1 are random variable coefficients obtained empirically with the assumption that the uncertainties associated with $\underline{J}$ and $\underline{K}$ compensate for the use of $\underline{b}^*$. In actual application $\underline{J}$ and $\underline{K}$ have to be estimated from empirical data. If

$\hat{\underline{J}}$ and $\hat{\underline{K}}$ are designated as the respective estimator of the expected values of $\underline{J}$ and $\underline{K}$, estimator models for $\underline{M}$ and $\underline{V}$ can be formed as follows:

$$\hat{\underline{M}} = \underline{H}(b_1^*, b_2^*, \ldots, b_m^*, \hat{J}_1, \hat{J}_2, \ldots \hat{J}_u) = \underline{H}(\underline{b}^*, \hat{\underline{J}}) \qquad (2.4.2a)$$

$$\hat{\underline{V}} = \underline{L}(b_1^*, b_2^*, \ldots, b_m^*, \hat{K}_1, \hat{K}_2, \ldots \hat{K}_t) = \underline{L}(\underline{b}^*, \hat{\underline{K}}) \qquad (2.4.2b)$$

Having introduced the notation concerning $\underline{M}$ and $\underline{V}$ for the estimator models, one can now write directly an expression representing the associated probabilistic model, namely that corresponding to the conditional expected value of $D(\underline{X})$:

$$E[D|\hat{\underline{M}}, \hat{\underline{V}}] = \int_{-\infty}^{\infty}{}_n \; D(\underline{\lambda}) \; f_{\underline{X}|\hat{\underline{M}}, \hat{\underline{V}}}(\underline{\lambda}|\hat{\underline{M}}, \hat{\underline{V}}) d\underline{\lambda} \qquad (2.4.3)$$

If one now defines the conditional expected value of D in Equation 2.4.3 as P_E and the function it represents as P_5, one can then write the following:

$$P_E = E[D|\hat{\underline{M}}, \hat{\underline{V}}] = P_5(\hat{\underline{M}}, \hat{\underline{V}}) \qquad (2.4.4)$$

However, one could also write P_E in terms of $\underline{b}^*$, $\hat{\underline{J}}$, and $\hat{\underline{K}}$ as a function P_6; that is,

$$P_E = E[D|\hat{\underline{M}} = \underline{H}(\underline{b}^*, \hat{\underline{J}}), \hat{\underline{V}} = \underline{L}(\underline{b}^*, \hat{\underline{K}})] = P_6(\underline{b}^*, \hat{\underline{J}}, \hat{\underline{K}}) \qquad (2.4.5)$$

Suppose now one proceeds to calculate $E[P_E]$ and $Var(P_E)$ by applying the approximation techniques introduced in Section 2.2 to the estimator model. As one recalls, there are two approaches to approximating $Var(P_E)$: one method beginning with P_E in the form of Equation 2.4.4; the other, in the form of Equation 2.4.5. Since the examples of Chapter III are more easily adapted to the former method, that method has been chosen here to begin this derivation.

If one expands P_5 about $E[\hat{\underline{M}}]$ and $E[\hat{\underline{V}}]$ in a Taylor Series Expansion, one acquires:

$$P_E = P_5[E(\hat{\underline{M}}),\ E(\hat{\underline{V}})] + \frac{\partial P_5}{\partial \hat{\underline{M}}}\Bigg|_{\substack{\hat{\underline{M}}=E(\hat{\underline{M}}) \\ \hat{\underline{V}}=E(\hat{\underline{V}})}} [\hat{\underline{M}} - E(\hat{\underline{M}})] + \frac{\partial P_5}{\partial \hat{\underline{V}}}\Bigg|_{\substack{\hat{\underline{M}}=E(\hat{\underline{M}}) \\ \hat{\underline{V}}=E(\hat{\underline{V}})}} [\hat{\underline{V}} - E(\hat{\underline{V}})] + \text{Remainder} \tag{2.4.6}$$

Again the assumption is made that the remainder can be neglected. See Appendix A for further discussion concerning the truncation of the series with first order terms. If one now takes the expected value of P_E from Equation 2.4.6, one obtains:

$$E[P_E] = P_5[E(\hat{\underline{M}}),\ E(\hat{\underline{V}})] \tag{2.4.7}$$

and in turn can calculate the following directly as in Equation 2.2.27.

$$\mathrm{Var}(P_E) = \left(\frac{\partial P_5}{\partial \hat{\underline{M}}}\right)^2\Bigg|_{\substack{\hat{\underline{M}}=E[\hat{\underline{M}}] \\ \hat{\underline{V}}=E[\hat{\underline{V}}]}} \mathrm{Var}(\hat{\underline{M}}) + \left(\frac{\partial P_5}{\partial \hat{\underline{V}}}\right)^2\Bigg|_{\substack{\hat{\underline{M}}=E[\hat{\underline{M}}] \\ \hat{\underline{V}}=E[\hat{\underline{V}}]}} \mathrm{Var}(\hat{\underline{V}}) + \left(\frac{\partial P_5}{\partial \hat{\underline{M}}}\right)\left(\frac{\partial P_5}{\partial \hat{\underline{V}}}\right)\Bigg|_{\substack{\hat{\underline{M}}=E[\hat{\underline{M}}] \\ \hat{\underline{V}}=E[\hat{\underline{V}}]}} \mathrm{cov}(\hat{\underline{M}},\ \hat{\underline{V}}) \tag{2.4.8}$$

Now one must evaluate $\mathrm{Var}(\hat{\underline{M}})$, $\mathrm{Var}(\hat{\underline{V}})$, and $\mathrm{cov}(\hat{\underline{M}},\ \hat{\underline{V}})$. As has been implied before in Equation 2.4.2, the following definition is made: $\hat{\underline{J}} = (\hat{J}_1,\ \hat{J}_2,\ \ldots,\ \hat{J}_u)$ and $\hat{\underline{K}} = (\hat{K}_1,\ \hat{K}_2,\ \ldots,\ \hat{K}_t)$. Associated with $\hat{\underline{J}}$ and and $\hat{\underline{K}}$ are the following parameters (19):

$$E[\hat{\underline{J}}] = \underline{j} \tag{2.4.9a}$$

$$E[\hat{\underline{K}}] = \underline{k} \tag{2.4.9b}$$

$$\mathrm{Cov}[\hat{J}_r,\ \hat{J}_s] = \sigma^2_{\hat{J}_r \hat{J}_s} \tag{2.4.9c}$$

$$\mathrm{Cov}[\hat{K}_r, \hat{K}_s] = \sigma^2_{\hat{K}_r \hat{K}_s} \tag{2.4.9d}$$

$$\mathrm{Cov}[\hat{J}_r, \hat{K}_s] = \sigma^2_{\hat{J}_r \hat{K}_s} \tag{2.4.9e}$$

If one expands $\underline{\hat{M}}$ about $\underline{j}$ and $\underline{\hat{V}}$ about $\underline{k}$ in Equation 2.4.2, one acquires for a first order approximation from a Taylor Series expansion (See Appendix A.)

$$\underline{M} = \underline{H}(\underline{b}^*, \underline{j}) + \sum_{i=1}^{u} \frac{\partial \underline{H}}{\partial J_i}\Big|_{\underline{\hat{J}}=\underline{j}} (\hat{J}_i - j_i) \tag{2.4.10a}$$

$$\underline{\hat{V}} = \underline{L}(\underline{b}^*, \underline{k}) + \sum_{i=1}^{t} \frac{\partial \underline{L}}{\partial k_i}\Big|_{\underline{\hat{K}}=\underline{k}} (\hat{K}_i - k_i) \tag{2.4.10b}$$

Again one can calculate the respective expected value and covariance as in Equation 2.2.29.

$$E[\underline{\hat{M}}] = \underline{H}(\underline{b}^*, \underline{j}) \tag{2.4.11a}$$

$$E[\underline{\hat{V}}] = \underline{L}(\underline{b}^*, \underline{k}) \tag{2.4.11b}$$

$$\mathrm{Var}[\underline{\hat{M}}] = \sum_{r=1}^{u} \sum_{s=1}^{u} \frac{\partial \underline{H}}{\partial \hat{J}_r} \frac{\partial \underline{H}}{\partial \hat{J}_s}\Big|_{\underline{\hat{J}}=\underline{j}} \sigma^2_{\hat{J}_r \hat{J}_s} \tag{2.4.11c}$$

$$\mathrm{Var}[\underline{\hat{V}}] = \sum_{r=1}^{t} \sum_{s=1}^{t} \frac{\partial \underline{L}}{\partial \hat{K}_r} \frac{\partial \underline{L}}{\partial \hat{K}_s}\Big|_{\underline{\hat{K}}=\underline{k}} \sigma^2_{\hat{K}_r \hat{K}_s} \tag{2.4.11d}$$

$$\mathrm{Cov}[\underline{\hat{M}}, \underline{\hat{V}}] = \sum_{r=1}^{u} \sum_{s=1}^{t} \frac{\partial \underline{H}}{\partial \hat{J}_r} \frac{\partial \underline{L}}{\partial \hat{K}_s}\Big|_{\substack{\underline{\hat{J}}=\underline{j} \\ \underline{\hat{K}}=\underline{k}}} \sigma^2_{\hat{J}_r \hat{K}_s} \tag{2.4.11e}$$

Substituting Equation 2.4.11 into Equation 2.4.8 yields

$$\mathrm{Var}(P_E) = \left(\frac{\partial P_5}{\partial \hat{\underline{M}}}\right)^2 \Bigg| \sum_{r=1}^{u} \sum_{s=1}^{u} \frac{\partial \underline{H}}{\partial \hat{J}_r} \frac{\partial \underline{H}}{\partial \hat{J}_s} \Bigg|_{\substack{\hat{\underline{M}}=\underline{H}(\underline{b}^*,\underline{j}) \\ \hat{\underline{V}}=\underline{L}(\underline{b}^*,\underline{k})}} \sigma^2_{\hat{J}_r \hat{J}_s} \Bigg|_{\hat{\underline{J}}=\underline{j}}$$

$$+ \left(\frac{\partial P_5}{\partial \hat{\underline{V}}}\right)^2 \Bigg| \sum_{r=1}^{t} \sum_{s=1}^{t} \frac{\partial \underline{L}}{\partial \hat{K}_r} \frac{\partial \underline{L}}{\partial \hat{K}_s} \Bigg|_{\substack{\hat{\underline{M}}=\underline{H}(\underline{b}^*,\underline{j}) \\ \hat{\underline{V}}=\underline{L}(\underline{b}^*,\underline{k})}} \sigma^2_{\hat{K}_r \hat{K}_s} \Bigg|_{\hat{\underline{K}}=\underline{k}}$$

$$+ \left(\frac{\partial P_5}{\partial \hat{\underline{M}}}\right)\left(\frac{\partial P_5}{\partial \hat{\underline{V}}}\right) \Bigg| \sum_{r=1}^{u} \sum_{s=1}^{t} \frac{\partial \underline{H}}{\partial \hat{J}_r} \frac{\partial \underline{L}}{\partial \hat{K}_s} \Bigg|_{\substack{\hat{\underline{M}}=\underline{H}(\underline{b}^*,\underline{j}) \\ \hat{\underline{V}}=\underline{L}(\underline{b}^*,\underline{k})}} \sigma^2_{\hat{J}_r \hat{K}_s} \Bigg|_{\substack{\hat{\underline{J}}=\underline{j} \\ \hat{\underline{K}}=\underline{k}}} \qquad (2.4.12)$$

Now if one could assume that the parameters of $\hat{\underline{J}}$ and $\hat{\underline{K}}$ in Equations 2.4.9 were known, one could calculate $\mathrm{Var}(P_E)$. However, unfortunately these parameters are not usually known and must be estimated from data concerning $\hat{\underline{J}}$ and $\hat{\underline{K}}$ (19). If one denotes the estimates of these parameters by an asterisk (*), the resulting estimate of the $\mathrm{Var}(P_E)$ can be written as follows:

$$\hat{\mathrm{Var}}(P_E) = \left(\frac{\partial P_5}{\partial \hat{\underline{M}}}\right)^2 \Bigg| \sum_{r=1}^{u} \sum_{s=1}^{u} \frac{\partial \underline{H}}{\partial \hat{J}_r} \frac{\partial \underline{H}}{\partial \hat{J}_s} \Bigg|_{\substack{\hat{\underline{M}}=\underline{H}(\underline{b}^*,\underline{j}^*) \\ \hat{\underline{V}}=\underline{L}(\underline{b}^*,\underline{k}^*)}} \sigma^{2*}_{\hat{J}_r \hat{J}_s} \Bigg|_{\hat{\underline{J}}=\underline{j}^*}$$

$$+ \left(\frac{\partial P_5}{\partial \hat{\underline{V}}}\right)^2 \Bigg| \sum_{r=1}^{t} \sum_{s=1}^{t} \frac{\partial \underline{L}}{\partial \hat{K}_r} \frac{\partial \underline{L}}{\partial \hat{K}_s} \Bigg|_{\substack{\hat{M}=\underline{H}(\underline{b}^*, \underline{j}^*) \\ \hat{V}=\underline{L}(\underline{b}^*, \underline{k}^*) \\ \hat{\underline{K}}=\underline{k}^*}} \sigma^{2*}_{\hat{K}_r \hat{K}_s}$$

$$+ \left(\frac{\partial P_5}{\partial \hat{\underline{M}}}\right)\left(\frac{\partial P_5}{\partial \hat{\underline{V}}}\right) \Bigg| \sum_{r=1}^{u} \sum_{s=1}^{t} \frac{\partial \underline{H}}{\partial \hat{J}_r} \frac{\partial \underline{L}}{\partial \hat{K}_s} \Bigg|_{\substack{\hat{\underline{M}}=\underline{H}(\underline{b}^*,\underline{j}^*) \\ \hat{\underline{V}}=\underline{L}(\underline{b}^*,\underline{k}^*) \\ \hat{\underline{J}}=\underline{j}^* \\ \hat{\underline{K}}=\underline{k}^*}} \sigma^{2*}_{\hat{J}_r \hat{K}_s} \qquad (2.4.13)$$

Equation 2.4.13 now presents a measure of the uncertainty of the probabilistic model associated with P_E. Observe that for the estimator model of this section, $\hat{Var}(P_E)$ is dependent upon the estimates of the parameters of $f_{\underline{Y}}$ and the estimates of the expected value and covariances of $\hat{\underline{J}}$ and $\hat{\underline{K}}$. Then $\hat{Var}(P_E)$ relates the uncertainty of P_E to the uncertainty associated with $\hat{\underline{K}}$, $\hat{\underline{J}}$, and in turn to $\hat{\underline{B}}$ primarily via the uncertainty associated with empirical data used for the estimates indicated by asterisks (*) in Equation 2.4.13.

2.5 Other Sources of Error. Throughout the first four sections of this chapter, errors primarily associated with the estimation of parameters used in probability density functions appearing in probabilistic models has been discussed. In this section two other possible sources of errors, conditional loss functions and stochastic tables, are briefly explained.

Conditional Loss Functions. Thus far, not much has been said concerning the function $D(\underline{X})$ which has appeared in each of the models presented. In almost every case of these types of probabilistic models, D can be considered as a loss function; that is, P_n, as it has previously

been defined, is the integral of a conditional density function weighted by $D(\underline{X})$. If $D(\underline{X})$ were defined to be $+\underline{1}$ over the domain of $\underline{X}$, then P would necessarily equal $+\underline{1}$. In general, if D is some other function over the domain of $\underline{X}$, P_n would be different from 1. In particular, if $D(\underline{X})$ is defined

$$D(\underline{X}) \leq 1 \text{ over the domain of } \underline{X} \tag{2.5.1}$$

then $P_n \leq 1$

and the reduction or loss of P_n from the norm can be directly related to the form of $D(\underline{X})$. This dependence of P_n upon $D(\underline{X})$ necessitates investigating the uncertainties associated with the loss function.

Often in practice D, or at least the parameters associated with D, is found to be derived directly from experimental data. Suppose that D, besides being a function of $\underline{X}$ is also a function of another set of random variables, $FZ_1, FZ_2, \ldots, FZ_r$. Define $\underline{FZ} = (FZ_1, FZ_2, \ldots, FZ_r)$.

Upon examination, one observes that D can be written directly as a function of $\underline{X}$ and $\underline{FZ}$, say $D_6(\underline{X}, \underline{FZ})$. Following techniques of the previous sections, one can expand D_6 about the expected values of $\underline{X}$ and $\underline{FZ}$; where

$$E[\underline{X}] = E[E(\underline{X}|\underline{M}, \underline{V})] = E[\underline{M}] \tag{2.5.2}$$

$$\underline{m}_x \equiv E[\underline{M}] \tag{2.5.3}$$

$$D_6[\underline{X}, \underline{FZ}] \approx D_6(\underline{m}_x, E[\underline{FZ}]) + \sum_{i=1}^{r} \frac{\partial D_6}{\partial \hat{FZ}_i}\Big|_{\underline{FZ}=E[\underline{FZ}]} (FZ_i - E[FZ_i])$$

$$+ \frac{\partial D_6}{\partial \underline{X}}\Big|_{\underline{X}=\underline{m}_x} (\underline{X} - \underline{m}_x) \tag{2.5.4}$$

Again for notational convenience the Taylor Series expansion is terminated with first order terms. (See Appendix A) The expected value of D_6 can now be calculated from Equation 2.5.4 as follows:

$$E[D_6] \approx D_6[\underline{m}_x, E[\underline{FZ}]] \tag{2.5.5}$$

Likewise one can compute the variance of D_6 using Equation 2.5.4 with Equation 2.5.5 and assuming $\underline{X}$ and $\underline{\hat{FZ}}$ to be independent, obtain the following:

$$\text{Var}[D_6] \approx \sum_{i=1}^{r}\sum_{j=1}^{r} \frac{\partial D_6}{\partial FZ_i}\frac{\partial D_6}{\partial FZ_j}\Big|_{\underline{FZ}=E[\underline{FZ}]} \text{Cov}[FZ_i, FZ_j] + \left(\frac{\partial D_6}{\partial \underline{X}}\right)^2\Big|_{X=\underline{m}_x} \text{Var } \underline{X} \tag{2.5.6}$$

where Var X is calculated as

$$\text{Var } X = E[\text{Var}(\underline{X}|\underline{M}, \underline{V})] + \text{Var}[E(\underline{X}|\underline{M}, \underline{V})] \tag{2.5.7a}$$

$$\text{Var } \underline{X} = E[\underline{V}] + \text{Var}[\underline{M}] \tag{2.5.7b}$$

Now if one assumes as in Equation 2.2.18 that $\underline{M}$ and $\underline{V}$ are related to the basic random vector parameter $\underline{B}$, one can recall from Equation 2.2.29 the following:

$$E[\underline{M}] = \underline{m} \approx g_{\underline{M}}(\underline{b}) \tag{2.5.8a}$$

$$E[\underline{V}] \approx g_{\underline{V}}(\underline{b}) \tag{2.5.8b}$$

$$\text{Var}[\underline{M}] \approx \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\frac{\partial g_{\underline{M}}(\underline{B})}{\partial B_i}\right)\left(\frac{\partial g_{\underline{M}}(\underline{B})}{\partial B_j}\right)\Big|_{\underline{B}=\underline{b}} \text{Cov}(B_i, B_j) \tag{2.5.8c}$$

Similarly referring to the notation of Section 2.3, an estimate of variance of D can be calculated from Equations 2.5.6, 2.5.7, and 2.5.8 as follows:

$$\hat{\text{Var}}(D) = \sum_{i=1}^{r}\sum_{j=1}^{r} \frac{\partial D_6}{\partial \hat{FZ}_i}\frac{\partial D_6}{\partial \hat{FZ}_j}\Big|_{\underline{FZ}=\underline{fz}^*} \text{Cov}^*[FZ_i, FZ_j] + \left(\frac{\partial D_6}{\partial \underline{X}}\right)^2\Big|_{\underline{X}=g_M(\underline{b}^*)} (g_v(\underline{b}^*)$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m} \left(\frac{\partial g_M(B)}{\partial B_i}\right)\left(\frac{\partial g_M(\underline{B})}{\partial B_j}\right)\Big|_{\underline{B}=\underline{b}^*} \text{Cov}^*(B_i, B_j)) \qquad (2.5.9)$$

where

$\text{Cov}^*[FZ_i, FZ_j]$ = estimate of $\text{Cov}[FZ_i, FZ_j]$

$\text{Cov}^*[B_i, B_j]$ = estimate of $\text{Cov}[B_i, B_j]$

$\underline{fz}^*$ = estimate of $E[FZ]$

$\underline{b}^*$ = estimate of $E[\underline{B}]$

Thus another component that contributes to the variance of the general random variable P_n has been introduced. Before this component can be added to any of the models given, it must be multiplied by the sensitivity coefficient linking D to P_n, namely $(\frac{\partial P_n}{\partial D})^2$. An interesting example of the use of these additional sensitivity terms is presented in the second example of Chapter III.

In this section then, the uncertainty of a probabilistic model has been extended to include errors inherent to the conditional loss

function D and associated with experimental data used to estimate the parameters of D.

Stochastic Tables. Often in the process of obtaining some desired result by means of a particular probabilistic model, one is required to use some parameter from an experimentally determined table of values. To be more explicit, suppose that in the model being used, one was required to determine the drag coefficient C of a particular object as a function of its velocity v. The drag coefficient is usually determined from measurements made in a wind tunnel. Since each measurement is subject to experimental errors and uncertainties, many measurements of C are made at each of a number of preselected values of v. Then at any value of v, say v_o, an average C, say $\overline{C}_o$ is used as an entry into the drag coefficient table. Thus for any particular v_o, the $\overline{C}_o$ entered into the tables is an estimate of the expected value of C_o given the data available.

The values of the table then represent a stochastic process, not as a function of time, but as a function of v; that is, from one set of data one particular table would result; from another, a slightly different table.

In the terminology of the previous sections then, one necessarily assumes that each entry into the table corresponds to a random variable with uncertainty which affects the uncertainty of P_E. The sensitivity terms then associated with $\mathrm{Var}(P_E)$ would be of the form

$$\sum_{i=1}^{a} \sum_{j=1}^{a} \left(\frac{\partial P_E}{\partial C_i}\right)\left(\frac{\partial P_E}{\partial C_j}\right) \mathrm{Cov}(C_i, C_j) \qquad (2.5.10)$$

where a is the number of entries in the table. The form of Equation 2.5.10 assumes that the C_i's are independent of any other random

variables, say L_i's. If this cannot be assumed in a particular case, Equation 2.5.10 must be supplemented with additional terms expressing this dependence, such as

$$\sum_{j=1}^{w} \sum_{i=1}^{a} \left(\frac{\partial P_E}{\partial C_i}\right)\left(\frac{\partial P_E}{\partial L_j}\right) \mathrm{Cov}(C_i, L_j) \tag{2.5.11}$$

where w equals the number of random variable dependent on the C_i's.

It might be interesting to examine the $\mathrm{Cov}(C_i, C_j)$ of Equation 2.5.10.

$$\mathrm{Cov}(C_i, C_j) = E[(C_i - E(C_i))(C_j - E(C_j))] \tag{2.5.12}$$

$$\mathrm{Cov}(C_i, C_j) = E[(C(v_i) - E[C(v_i)])(C(v_j) - E[C(v_j)])] \tag{2.5.13}$$

In stochastic processes normally encountered, the variable t_i (time) replaces v_i (velocity). With this observation, one sees that $\mathrm{Cov}(C_i, C_j)$ corresponds to the autocovariance of $C(\underline{v})$.

The purpose of this section has been to briefly touch on two sources of error not covered directly by the previous sections. Conditional loss functions were introduced and the uncertainty associated with them was used to expand $\hat{\mathrm{Var}}(P_E)$ to include additional sensitivity terms. Tables used in probabilistic models were considered as stochastic tables; that is, each entry in any particular table was assumed to be only an estimate of the expected value of that entry. The uncertainties related to these tables were in turn related to the uncertainty in P_E.

CHAPTER III

PROBABILISTIC MODELS IN WEAPONS EFFECTIVENESS PROBLEMS

3.1 Introduction. This chapter is primarily devoted to investigation of particular applications of the techniques developed in Chapter II to the field of weapons effectiveness. In particular, two specific weapon-target probabilistic models are presented and a measure of the uncertainty associated with each is calculated. This chapter begins with a brief discussion of a general weapons effectiveness problem, then the example models previously mentioned are examined in detail, including the computer programs employed and the assumptions made.

Consider initially the following general weapons effectiveness problem. One is concerned with finding the probability P_K that a desired level of damage is accomplished against a particular target by using a specified weapon. Now one can in general define P_K as follows:

$$P_K = \int_{-\infty}^{\infty}{}_2 \; D_a(\underline{\tau}) f_{\underline{X}}(\underline{\tau}) d\underline{\tau} \tag{3.1.1}$$

where the random vector $\underline{X} = (X_R, X_D)$ and represents the actual impact point of the weapon with the subscripts R and D denoting range and deflection components of $\underline{X}$.

The function D_a which appears under the integral in Equation 3.1.1 is the damage function associated with a weapon target combination; that is, for a target positioned at $\underline{\tau} = 0$, $D_a(\underline{\tau}_0)$ is the level of damage

(desired damage = 1.0) to the target caused by a weapon impacting at $\underline{\tau}_0$. In this thesis, two types of damage functions are considered depending upon the specific weapon-target damage mechanism; in particular, (1) a blast sensitive target damage function and (2) a fragment sensitive damage function. A blast sensitive target is defined to be one which has associated with it a definite geometric figure within which the weapon (or weapons) must impact in order to achieve a measure of damage. On the other hand, a fragment sensitive target relates to one in which the major damage mechanism is due to fragmentation effects rather than to a direct impact of the weapon. The former type of damage function is assumed in the example presented in Section 3.2, and the latter in Section 3.3, where respectively each damage function is described in more detail.

The value p_K as described by Equation 3.1.1 is often referred to as the expected damage to the target involved, a notation which is apparent from the form of the equation. One can thus write p_K in a manner corresponding to Equation 2.1.1 as follows:

$$p_K = E[D_a] = \int_{-\infty}^{\infty}{}_2 \, D_a(\underline{\tau}) \, f_{\underline{X}}(\underline{\tau}) \, d\underline{\tau} \tag{3.1.2}$$

In general there are numerous ways of modeling Equation 3.1.2 depending upon weapon characteristics, target vulnerability, and assumptions made concerning the damage function D_a, and the impact point joint density $f_{\underline{X}}$. Due to the complexity of the problem, most of the simulation and computations are accomplished via the digital computer. In particular, this thesis considers models (programs) used in connection with the Joint Munitions Effectiveness Manual, JMEM. These models are designed to compute a single value of P_K for a given set of system input conditions

which are, for the most part, obtained from data derived from experimental tests.

Since the system input conditions are at best only estimates of the mean value of the input parameters based on random experiments, the value of p_K computed by the system models represents a random variable conditional probability. This implies then, that Equation 3.1.2 should be rewritten as follows:

$$P_K = P_I(\underline{I}) = E[D_a|\underline{I}] = \int_{-\infty}^{\infty} D_a(\underline{x}|\underline{I})\ f_{\underline{X}|\underline{I}}\ (\underline{x}|\underline{I})\ d\underline{x} \qquad (3.1.3)$$

The $\underline{I}$ which appears in Equation 3.1.3 denotes information derived from experimental data; that is, P_K, which is calculated by a series of simulation programs is actually a conditional expected damage, conditioned upon data used to compute the estimates of the mean value inputs to the programs. In general terms, $\underline{I}$ corresponds to the information needed to estimate such random variable parameters as $\underline{A}$, $\underline{B}$, and $\underline{FZ}$ presented in Chapter II.

The assumption is now made that any uncertainty in P_K is directly associated with the uncertainties in the models themselves and/or with the uncertainties in the values of the random variable parameter used as inputs into the models. For the purposes of this thesis, it has been assumed, as stated in Chapter II, that adequate models are available so that the discussion presented here is concerned with the uncertainties associated with the model input variables. In general, all variables for which input values to the system models have been estimated from experimental data are considered to be random variables. Likewise, it follows directly that P_K, as a function to these input variables, is a random variable whose uncertainty should be related to the uncertainty

associated with the input random variables. In general terms, for the weapons effectiveness examples of this chapter, these system input random variables range from weapon release conditions to weapon fragmentation characteristics and to target vulnerability parameters.

The measure of uncertainty of P_K derived from the analysis in this chapter can be useful in dealing with weapon effectiveness models by providing a measure of the confidence one should have in the particular value of P_K which results as output of the system models. Furthermore, the methods developed here can aid in locating the principal sources of error in the models of P_K, thus indicating where more data should be taken in obtaining estimates of the mean values of the random variables used as system inputs.

In addition, this measure of uncertainty may well assist in the comparison of "open end" (short-hand, manually calculated) solutions and "closed end" (computer) solutions, indicating whether these two types of solutions should be used interchangably.

For example, consider two models for P_K corresponding to P_{K_1} and P_{K2}. Let

$$E[P_{K1}] = \mu_1 \tag{3.1.4a}$$

$$E[P_{K2}] = \mu_2 \tag{3.1.4b}$$

$$Var[P_{K1}] = \sigma_1^2 \tag{3.1.4c}$$

$$Var[P_{K2}] = \sigma_2^2 \tag{3.1.4d}$$

$$Cov[P_{K1},P_{K2}] = \sigma_{12}^2 \tag{3.1.4e}$$

Define a new random variable

$$P_{\Delta} = P_{K1} - P_{K2} \tag{3.1.5}$$

where

$$E[P_{\Delta}] = \mu_1 - \mu_2$$

$$Var[P_{\Delta}] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}^2$$

$$SD(P_{\Delta}) = \sqrt{Var(P_{\Delta})} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_{12}^2}$$

In particular, consider the case where P_{K1} and P_{K2} are two independent models; i.e., $\sigma_{12}^2 = 0$. One might make the decision to use the two models interchangably then if $|E[P_{\Delta}]| \leq SD(P_{\Delta})$; i.e., $|\mu_1 - \mu_2| \leq \sqrt{\sigma_1^2 + \sigma_2^2}$.

3.2 Example 1: Blast Sensitive Target Model. Consider again the general weapons effectiveness model of Equation 3.1.3.

$$P_K = P_K(\underline{I}) = E[D_a|\underline{I}] = \int_{-\infty}^{\infty}{}_2 \; D_a(\underline{\tau}) \; f_{\underline{X}|\underline{I}}(\underline{\tau}|\underline{I}) \; d\underline{\tau} \tag{3.2.1}$$

Under certain target-weapon combination situations, one is concerned with only the blast effects of a weapon and correspondingly the associated damage function D_a is defined as a function of only the distance of a target point from the impact point of the weapon. This damage function corresponds to that associated with the blast sensitive target introduced in the previous section. For the purpose of the example of this section, a blast sensitive target with a rectangular vulnerable area R_v is assumed and D_a is defined as follows:

$$D_a(\underline{\tau}) = p_{D|H} \; P_H(\underline{\tau}) \tag{3.2.2}$$

where

$$P_{D|H} = \text{probability of damage given that } R_v \text{ is hit} \qquad (3.2.2b)$$

$$P_H(\underline{\tau}) = 1 \text{ if } \underline{\tau} \in R_v \qquad (3.2.2c)$$

$$= 0 \text{ otherwise}$$

In other words, $D_a(\tau)$ has a constant value of $p_{D|H}$ over the area described by R_v and is zero elsewhere. It should be noted that damage functions of the form of D_a are often referred to as cookie cutter damage functions Under the assumptions of Equation 3.2.2, one can now write the model of P_K as presented by this example as

$$P_K = P_K(\underline{I}) = E[D_a|\underline{I}] = \int_{R_v} p_{D|H} \, f_{\underline{X}|\underline{I}} \, (\underline{\tau}|\underline{I}) \, d\underline{\tau} \qquad (3.2.3)$$

The computer programs used in the JMEM effort which correspond to the model of Equation 3.2.3 are the Stick Bomb Program and the Multiple Round Kill Probability Program (MRKP). These two programs are discussed in detail in the following subsections.

Weapon Delivery. The Stick Bomb Program is a basic "initial" program for many weapons effectiveness situations employing air delivered weapons. This program utilizes release conditions of aircraft and weapon characteristics to predict the intended impact points of a stick of weapons, where the term "stick" refers to a rapid sequential release of several weapons which form a characteristic pattern on the ground. In general, the program actually simulates the trajectories of the falling weapons and predicts where they would impact if only the conditions specified as inputs to the program affected the trajectories and all of these conditions were exactly at their desired values.

Consider again the random vector $\underline{X}$ as representing the actual impact points of the weapons. For simplicity of notation, however, consider for this example the release of only one weapon so that $\underline{X}$ can be expressed as (X_R, X_D) where the subscripts R and D denote range and deflection respectively. It should be pointed out that if λ bombs were released from the aircraft, both X_R and X_D would be λ dimensional vectors. The multiple bomb case is a single extension of the single bomb case and is discussed further at the end of Section 3.2.

Consider now the following notation used in the example of this section:

$$\underline{M}_x = \text{mean vector of } \underline{X} = (M_R, M_D) \tag{3.2.4}$$

$$\underline{V}_x = \text{covariance matrix of } \underline{X} = \begin{vmatrix} V_{RR} & V_{RD} \\ V_{DR} & V_{DD} \end{vmatrix} \tag{3.2.5}$$

where again the subscripts R and D indicate range and deflection components respectively. P_K can now be written as a conditional expected damage conditioned on $\underline{M}_x$ and $\underline{V}_x$ in the manner of Equation 2.2.23

$$P_K = P_{x1}(\underline{M}_x, \underline{V}_x) = E[D_a|\underline{M}_x, \underline{V}_x] = \int_{R_v} P_{D|H}\, f_{\underline{X}|\underline{M}_x, \underline{V}_x}(\underline{\tau}|\underline{M}_x, \underline{V}_x)\, d\underline{\tau} \tag{3.2.6}$$

Furthermore, the impact point $\underline{X}$ is now assumed to be a function of the release conditions of the weapon from the aircraft as well as atmospheric conditions and ballistic characteristics so that $\underline{X}$ can be written as follows:

$$\underline{X} = \underline{G}_x(Y_1, Y_2, \ldots, Y_\phi) \tag{3.2.7}$$

where the Y's are random variables, each Y_i associated with a different but not necessarily independent factor influencing $\underline{X}$.

If the pilot (and/or delivery system) had absolute control over all factors influencing $\underline{X}$; that is, over all Y's and if he had perfect judgment, he could, for a specified set of Y's compute an exact impact point $\underline{X}$. Unfortunately, however, the pilot does not possess this perfect control due to the fact that he is involved with a physical "real world" system; that is, although a particular Y_i, say y_i, is desired for a certain weapon delivery condition, there will exist some random error δ_i associated with Y_i such that the actual value for a desired Y_i is $y_i \pm \delta_i$. Thus, one justifies terming the Y's random variables and defines $f_{\underline{Y}}$ as the joint density function of the Y's. Furthermore, in the Stick Bomb Program, which is in general terms a simulation of $\underline{G}_X$ in Equation 3.2.7, $\underline{M}_x$ and $\underline{V}_x$ are modeled as functions of the knowledge or information available concerning $(Y_1, Y_2, \ldots, Y_\phi)$. Both the theoretical models of Section 2.2 and 2.3 and the estimator model of Section 2.4 need to be examined in order to see if either of these models corresponds to that used in the Stick Bomb Program. However, before either of these models can be examined further, the Stick Bomb Program model is described.

Stick Bomb Delivery Model. Suppose a stick of λ bombs is projected at a target from an aircraft. Associated with the specific release conditions and the types of weapons involved, there will be a characteristic pattern of impact points. Now, if one chooses one point in the pattern as a pattern reference point, say the aimpoint of the first weapon, the position of each weapon in the pattern with respect to that reference point can be specified by the designation of two numbers, ΔR_i and ΔD_i. The notation ΔR_i indicates the range separation between the reference point and the ith weapon; ΔD_i, the separation in deflection. If, as in the example of this section, only one weapon is released, the pattern

aimpoint would necessarily correspond to the aimpoint of that one weapon and ΔR_i and ΔD_i would both equal zero.

Now it is assumed that the pattern as a whole is aimed by attempting to impact the pattern reference point at a designated point in a range-deflection (R_e - D_e) plane associated with the target, say at (r_0, d_0). The R_e - D_e plane is a horizontal plane passing through the center of mass of the target. The coordinate system in which (r_0, d_0) is measured has as its origin the point in the R_e - D_e plane below the release point of the first weapon in the stick. Correspondingly, the range axis associated with the pattern is parallel to the horizontal velocity of the aircraft and the deflection axis is perpendicular to the range axis and lies in the R_e - D_e plane. If the pattern were correctly positioned, the pattern aimpoint would correspond to (r_0, d_0) upon impact. Unfortunately, aiming errors are usually present and the pattern reference point is usually aimed at another point, say (R, D). The aiming errors involved, i.e., (R - r_0) and (D - d_0) are assumed to be independent and normally distributed with means zero and standard deviations S_{ar} and S_{ad} respectively. Primarily these aiming errors are assumed to be due to sight misalignment, wind miscorrection, improper release conditions, and pilot inexperience.

A similar situation exists with regard to the ballistic dispersion. If ballistic errors were not present, the impact points of the ith weapon could be predicted given R and D as (R + ΔR_i, D + ΔD_i). However, the actual impact points are assumed to be normally distributed about (R + ΔR_i, D + ΔD_i) with ballistic standard deviations of S_{br} and S_{bd}. Basically, these ballistic errors are due to unforeseen natural phenomena, air currents, and drag coefficients.

One sees then that S_{ar}^2, S_{ad}^2, S_{br}^2, and S_{bd}^2 provide essentially the same information concerning the deviation of $\underline{X}$ about the expected or intended impact point (r_0, d_0) as does $\underline{V}_x$. Indeed, these four random variables are but a particular case of $\underline{V}_x$ which applies to the weapon delivery model of this section. Since the Stick Bomb Program specifically models S_{ar}, S_{ad}, S_{br}, and S_{bd} and these particular values are subsequently used directly by the Multiple Round Kill Probability Program to calculate a particular value of P_K, a slight modification in the form of the conditional expected damage of Equation 3.2.6 has been made as follows:

$$P_K = P_x(\underline{M}_x, \underline{S}_x) = E[D_a|\underline{M}_x, \underline{S}_x] = \int_{R_v} P_{D|H} \, f_{\underline{X}|\underline{M}_x, \underline{S}_x}(\underline{r}|\underline{M}_x, \underline{S}_x) \, d\underline{r} \tag{3.2.8}$$

where

$$\underline{S}_x = (S_{ar}, S_{ad}, S_{br}, S_{bd})$$

This modification basically is merely a change of notation in variables as $\underline{S}_x$ simply represents a particular form for the general random vector (matrix) $\underline{V}_x$. Thus the techniques of the previous chapter can be applied directly to provide a measure of the uncertainty of P_K by means of examining the uncertainty associated with $\underline{M}_x$ and $\underline{S}_x$ provided that the Stick Bomb Delivery Model can be shown to correspond to one of the models discussed in that chapter.

Now from previous discussion in this section concerning the Stick Bomb Delivery model, one can observe that the expected impact point of $\underline{X}$ should be related to the intended aimpoint of the pattern reference

point. In particular, for the example of the releasing of only one weapon, the mean impact point of $\underline{X}$ should be related to the intended aimpoint of the pattern reference point. In particular, for the example of the releasing of only one weapon, the mean impact point, call it $\underline{m}_0$, is equal to the aimpoint of that one weapon, namely (r_0, d_0). Associated with r_0 and d_0, the uprange release distance and deflection, are a set of intended release conditions $y_1, y_2, \ldots, y_\phi$ that must be met in order to release the weapon aimed at r_0. Many factors influence the values of $y_1, y_2, \ldots, y_\phi$ such that on any particular run, errors occur between the actual valves obtained, call them $y_1^*, y_2^*, \ldots, y_\phi^*$, and the intended value of the y's. Assuming that these errors are neither positively nor negatively biased, the intended y values $y_1, y_2, \ldots, y_\phi$ can be treated as merely estimates of the expected values of the basic underlying random variables $Y_1, Y_2, \ldots, Y_\phi$. Thus, one can write $\underline{m}_0$ in functional notation as

$$\underline{m}_0 = (r_0, d_0) = \underline{H}_x(y_1, y_2, \ldots, y_\phi) = \underline{H}_x(\underline{y}) \tag{3.2.9}$$

Furthermore, consider the Stick Bomb Delivery models for S_{ar}, S_{ad}, S_{br}, and S_{bd} which also depend upon the intended release conditions

$$S_{ar} = S_1(y_1, y_2, \ldots, y_\phi, SA_1, SA_2) \tag{3.2.10a}$$

$$S_{ad} = S_2(y_1, y_2, \ldots, y_\phi, SA_1, SA_3) \tag{3.2.10b}$$

$$S_{br} = S_3(y_1, y_2, \ldots, y_\phi, SA_1) \tag{3.2.10c}$$

$$S_{bd} = S_4(y_1, y_2, \ldots, y_\phi, SB_1) \tag{3.2.10d}$$

or in general notation $\underline{S}_x = \underline{S}(y_1, y_2, \ldots, y_\phi, SA_1, SA_2, SA_3, SB_1)$ where SA_1, SA_2, and SA_3 are empirically determined aiming error

coefficients relating the intended release condition to S_{ar} and S_{ad} and SA_4 is a corresponding ballistic error coefficient relating the release conditions to S_{br} and S_{bd}. Since these coefficients are obtained for experimental data, they have associated with them a certain degree of uncertainty related to the random experiments involved. This uncertainty is assumed to compensate in the model for the use of the estimates of the expected value of the Y's as well as for the uncertainty associated with unknown factors that influence $\underline{X}$.

It is obvious from Equations 3.2.9 and 3.2.10 that the Stick Bomb Delivery Model corresponds to special cases of the estimator models of Section 2.4 since random variable coefficients are assumed. However, before considering the form of the estimator model involved, it is interesting to observe how the theoretical model of Section 2.3 could be applied to the Stick Bomb Program if the aiming and ballistic error coefficients were not available.

Theoretical Model. Consider modeling $\underline{M}_x$ and $\underline{S}_x$ in the modified form of the approximate theoretical models of Equation 2.2.18 for $\underline{X} = G(\underline{Y})$

$$\underline{M}_x = E[\underline{X} \mid \hat{\underline{B}}] = E[G(\underline{Y}) \mid \hat{\underline{B}}] \equiv \underline{H}_B(\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_m) \equiv \underline{H}_B(\hat{\underline{B}}) \quad (3.2.11a)$$

$$\underline{S}_x = \sqrt{\mathrm{Var}(\underline{X} \mid \hat{\underline{B}})} = \sqrt{\mathrm{Var}(G(\underline{Y}) \mid \hat{\underline{B}})} \equiv \underline{L}_B(\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_m) \equiv \underline{L}_B(\hat{\underline{B}}) \quad (3.2.11b)$$

where $\hat{B}_i$ is defined as the MVUE of the ith parameter of $f_{\underline{Y}}$. Following the general techniques of Section 2.2 modified by the use of $\hat{\underline{B}}$ instead of $\underline{B}$, one can expand $\underline{M}_x$ and $\underline{S}_x$ about the expected value of $\hat{\underline{B}}$ in a first order Taylor Series Expansion. See Appendix A for series termination criterion.

$$\underline{M}_x \approx \underline{H}_B[E(\hat{\underline{B}})] + \sum_{i=1}^{m} \frac{\partial \underline{H}_B}{\partial \hat{B}_i} \Big|_{\hat{\underline{B}}=E(\underline{B})} [\hat{B}_i - E(\hat{B}_i)] \tag{3.2.12a}$$

$$\underline{S}_x \approx \underline{L}_B[E(\hat{\underline{B}})] + \sum_{i=1}^{m} \frac{\partial \underline{L}_B}{\partial \hat{B}_i} \Big|_{\underline{B}=E[\underline{B}]} [\hat{B}_i - E(\hat{B}_i)] \tag{3.2.12b}$$

It should be noted that since the $\hat{B}_i$'s have been assumed to be unbiased estimators of the parameters of $f_{\underline{Y}}$ then $E[\hat{B}_i]$ represents the true value of the ith parameter of $f_{\underline{Y}}$.

Continuing in the manner of Section 2.2, the expected value and covariances associated with $\underline{M}_x$ and $\underline{S}_x$ can be obtained as follows:

$$E[\underline{M}_x] \approx \underline{H}_B[E(\hat{\underline{B}})] \tag{3.2.13a}$$

$$E[\underline{S}_x] \approx \underline{L}_B[E(\hat{\underline{B}})] \tag{3.2.13b}$$

$$\mathrm{Var}[\underline{M}_x] \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial \underline{H}_B}{\partial \hat{B}_i} \frac{\partial \underline{H}_B}{\partial \hat{B}_j} \Big|_{\hat{\underline{B}}=E[\hat{\underline{B}}]} \mathrm{Cov}[\hat{B}_i, \hat{B}_j] \tag{3.2.13c}$$

$$\mathrm{Var}[\underline{S}_x] \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial \underline{L}_B}{\partial \hat{B}_i} \frac{\partial \underline{L}_B}{\partial \hat{B}_j} \Big|_{\hat{\underline{B}}=E[\hat{\underline{B}}]} \mathrm{Cov}[\hat{B}_i, \hat{B}_j] \tag{3.2.13d}$$

$$\mathrm{Cov}[\underline{M}_x, \underline{S}_x] \approx \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial \underline{L}_B}{\partial \hat{B}_i} \frac{\partial \underline{L}_B}{\partial \hat{B}_j} \Big|_{\hat{\underline{B}}=E[\hat{\underline{B}}]} \mathrm{Cov}[\hat{B}_i, \hat{B}_j] \tag{3.2.13e}$$

Equations 3.2.13 thus present a measure of the uncertainty of $\underline{M}_x$ and $\underline{S}_x$ as functions of the MVUE of the parameters of $f_{\underline{Y}}$ and indeed could provide the needed information corresponding to Equation 2.2.27 for the calculation of $\mathrm{Var}(P_K)$. Unfortunately, data available for air delivered

munitions does not provide sufficient information at this time to make estimates of the covariance of the $\hat{B}$'s. Therefore, the theoretical model cannot be employed in this example to obtain a measure of the uncertainty associated with P_K.

Estimator Models. Returning now to the Stick Bomb Delivery models of Equations 3.2.9 and 3.2.10, one can observe the correspondence between these models and those estimator models of Section 2.4. In particular, Equation 3.2.9 expresses $\underline{m}_0$ as a specific case of Equation 2.4.2a where the randomness of the J coefficients is neglected and the y's correspond to a subset of the b_i*'s. This implies then that one can write the mean of $\underline{X}$, call it $\hat{\underline{m}}_x$, as the following estimator model.

$$\hat{\underline{m}}_x = \underline{m}_0 = H_x(y_1, y_2, \ldots, y_\phi) \tag{3.2.14}$$

Furthermore, if the following notation is considered:

$$\begin{aligned} SB_1 &= K_4 \quad \text{and} \quad \hat{SB}_1 = \hat{K}_4 \\ SA_i &= K_i \quad \text{and} \quad \hat{SA}_i = \hat{K}_i \end{aligned} \tag{3.2.15a}$$

for i = 1, 2, 3, then the Stick Bomb Delivery model for the standard deviation of $\underline{X}$, call it $\hat{\underline{S}}_x$, can be written as a special case of the estimator model of Equation 2.4.2b as

$$\hat{\underline{S}}_x = \underline{S}(y_1, y_2, \ldots, y_\phi, \hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4) \tag{3.2.15b}$$

where again the y's are a subset of the b_i*'s and the $\hat{K}$'s are estimators of the corresponding error coefficients.

Now that the models used in the Stick Bomb Program have been introduced; i.e., $\hat{m}_x$ and $\hat{S}_x$, one needs to relate these models to P_K so that a measure of the uncertainty of P_K can be investigated corresponding to

that of Equation 2.4.12 or 2.4.13. In particular, the Multiple Round Kill Probability Program has to be examined.

Multiple Round Kill Probability Program. The MRKP Program combines the delivery information obtained from the Stick Bomb Program with particular target vulnerability information and calculates P_K. For the purpose of this example, the target information is assumed fixed and the only variable inputs into the program that are considered random are S_{ar}, S_{ad}, S_{br}, and S_{bd}. As the example of Section 3.3 illustrates, the addition of other input random variables is handled in an analogous manner as for S_{ar}, S_{ad}, S_{br}, and S_{bd}.

Consider then the following functional relationship representing the Multiple Round Kill Probability model of P_K.

$$P_K(\underline{S}_x) = \int_{L_R} p_{D|H} \, f_{\underline{X_R|S_x}} (\tau_R|\underline{S}_x) \, d\tau_R \int_{L_D} f_{\underline{X_D|S_x}} (\tau_D|\underline{S}_x) \, d\tau_D \quad (3.2.17)$$

where L_R and L_D denote range length and deflection width respectively of the rectangular vulnerable area R_v. Employing Φ_R and Φ_D as dummy variables for R and D respectively one can write

$$f_{\underline{X_R|S_x}} (\tau_R|\underline{S}_x) = \int_{-\infty}^{\infty} f_{\underline{X_R|R,\, S_x}} (\tau_R|\Phi_R, \underline{S}_x) \, f_R(\Phi_R|\underline{S}_x) \, d\Phi_R \quad (3.2.18a)$$

and

$$f_{\underline{X_D\, S_x}} (\tau_D|\underline{S}_x) = \int_{-\infty}^{\infty} f_{\underline{X_D|D,\, S_x}} (\tau_D|\Phi_D, \underline{S}_x) \, f_D(\Phi_D|\underline{S}_x) \, d\Phi_D \quad (3.2.18b)$$

Substituting Equation 3.2.18 into Equation 3.2.17 yields the resulting model for P_K.

$$P_K(\underline{S}_x) = \int_{L_R} \int_{-\infty}^{\infty} p_{D|H} \, f_{\underline{X}_R|R,\,\underline{S}_x}(\tau_R|\Phi_R,\,\underline{S}_x)\, f_R(\Phi_R|\underline{S}_x)\, d\Phi_R \cdot$$

$$\int_{L_D} \int_{-\infty}^{\infty} f_{\underline{X}_D|D,\underline{S}_x}(\tau_D|\Phi_D,\,\underline{S}_x)\, f_D(\Phi_D|\underline{S}_x)\, d\Phi_D \qquad (3.2.19)$$

where under the assumptions of the MRKP model (N denoting normal density function)

$$f_{x_R|R,\,\underline{S}_x} \sim N(R,\, S_{br}^2) \qquad (3.2.20a)$$

$$f_{x_D|D,\,\underline{S}_x} \sim N(D,\, S_{bd}^2) \qquad (3.2.20b)$$

$$f_{R|\underline{S}_x} \sim N(r_0,\, S_{ar}^2) \qquad (3.2.20c)$$

$$f_{D|\underline{S}_x} \sim N(d_0,\, S_{ad}^2) \qquad (3.2.20d)$$

so that

$$f_{\underline{x}_R|\underline{S}_x} \sim N(r_0,\, S_{ar}^2 + S_{br}^2) \qquad (3.2.20e)$$

$$f_{\underline{x}_D|\underline{S}_x} \sim N(d_0,\, S_{ad}^2 + S_{bd}^2) \qquad (3.2.21f)$$

To avoid the rather complex notation associated with P_K in Equation 3.2.19, only the functional notation

$$P_K = P_K(\underline{S}_x) = P_K(S_{ar},\, S_{ad},\, S_{br},\, S_{bd}) \qquad (3.2.21)$$

is used in the main body of this thesis and corresponds to Equation 2.4.4 for a general estimator model.

Variance of P_K. Now that the appropriate Stick Bomb Delivery and MRKP estimator models have been introduced for the example of this section, a measure of the uncertainty associated with P_K must be examined. In particular, the uncertainty associated with P_K; i.e. $Var(P_K)$, must be related to the uncertainties associated with $\underline{S}_x$ and in turn to the uncertainties related to the error coefficients estimators $\hat{K}_1$, $\hat{K}_2$, $\hat{K}_3$, and $\hat{K}_4$ in an analogous manner to that exhibited for the general estimator-model of Equation 2.4.4. Therefore, following the general techniques of Section 2.4, P_K of Equation 3.2.18 is expanded about $E[\underline{S}_x] = (E(S_{ar}), E(S_{ad}), E(S_{br}), E(S_{bd}))$ in a Taylor Series Expansion and yields

$$P_K = P_K[E(\underline{S}_x)] + \sum_{i=1}^{4} \frac{\partial P_K}{\partial S_i}\Big|_{\underline{S}_x=E(S_x)} [S_i - E(S_i)] + \text{Remainder} \qquad (3.2.22)$$

where $\underline{S}_x \equiv (S_{ar}, S_{ad}, S_{br}, S_{bd}) \equiv (S_1, S_2, S_3, S_4)$. Only first order terms are assumed significant in the derivation that follows; however, a complete parallel derivation appears in Appendix C for a second order approximation to P_K in Equation 3.2.22.

Taking the expected value of P_K in Equation 3.2.22 as

$$E[P_K] \approx P_K[E(\underline{S}_x)] \qquad (3.2.23)$$

the variance of P_K can be calculated directly,

$$Var[P_K] \approx E[(P_K - E(P_K))^2]$$

$$Var[P_K] \approx E\left[\left(\sum_{i=1}^{4} \frac{\partial P_K}{\partial S_i}\Big|_{\underline{S}_x=E(\underline{S}_x)} [S_i - E(S_i)]\right)^2\right]$$

$$\mathrm{Var}[P_K] \approx \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial P_K}{\partial S_i} \frac{\partial P_K}{\partial S_j} \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_j) \qquad (3.2.24)$$

The $\mathrm{Var}[P_K]$ in Equation 3.2.24 thus relates a measure of the uncertainty associated with P_K to that related to $\underline{S}_x$. Now in order to estimate the uncertainties associated with $\underline{S}_x$ and relate these uncertainties to the basic random variables involved, one should recall the estimator model of Equation 3.2.15

$$\hat{\underline{S}}_x = \underline{S}(y_1, y_2, \ldots, y_\phi, \hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4) = \underline{S}(\underline{y}, \hat{\underline{K}}) \qquad (3.2.25)$$

There exists for each component of $\hat{\underline{S}}_x$; i.e., $\hat{S}_{ar}$, $\hat{S}_{ad}$, $\hat{S}_{br}$, and $\hat{S}_{bd}$, a specific functional relationship between the error coefficient estimators and the particular error component corresponding to Equation 3.2.10. However, for notational convenience, the general form of Equation 3.2.25 is used at this point for the derivation of the measure of uncertainties related to $\hat{\underline{S}}_x$. A parallel derivation appears in Appendix B which employs the exact form of the functional models representing $\hat{S}_{ar}$, $\hat{S}_{ad}$, $\hat{S}_{br}$, and $\hat{S}_{bd}$.

In general, the uncertainties associated with $\hat{\underline{S}}_x$ are estimated by variances and covariances related to $\hat{\underline{S}}_x$. Since the MRKP model assumes the components of $\hat{\underline{S}}_x$ are independent, it is necessary only to investigate the variances associated with $\hat{S}_{ar}$, $\hat{S}_{ad}$, $\hat{S}_{br}$, and $\hat{S}_{bd}$ for the purposes of this example.

The estimator coefficients of Equation 3.2.25 are assumed to possess the following parameters which correspond to Equation 2.4.9 in the general estimator model discussion (19).

$$E[\hat{K}_i] = k_i \tag{3.2.26a}$$

$$Cov[\hat{K}_i, \hat{K}_j] = \sigma^2_{\hat{K}_i\hat{K}_j} \tag{3.2.26b}$$

Following the general procedure employed in Section 2.4, one can expand $\underline{\hat{S}}_x$ about $\underline{k} = (k_1, k_2, k_3, k_4)$ in a first order Taylor Series Expansion. A second order approximation derivation of the variance related to $\underline{\hat{S}}_x$ appears in Appendix C, while a criteria for termination of the series is related in Appendix A.

$$\underline{\hat{S}}_x = \underline{S}(\underline{y}, \underline{k}) + \sum_{i=1}^{4} \frac{\partial S_i}{\partial K_i} \Big|_{\underline{\hat{K}}=\underline{k}} (\hat{K}_i - k_i) \tag{3.2.27}$$

Again taking the expected value of $\underline{\hat{S}}_x$ as

$$E[\underline{\hat{S}}_x] = \underline{S}(\underline{y}, \underline{k}) \tag{3.2.28a}$$

one can calculate the variance of $\underline{\hat{S}}_x$:

$$Var[\hat{S}_x] = E[(\hat{S}_x - E(\hat{S}_x))^2]$$

$$Var[\hat{S}_x] = \sum_{i=1}^{4} \sum_{j=1}^{4} \left(\frac{\partial S}{\partial \hat{K}_j}\right)^2 \Big|_{\underline{\hat{K}}=\underline{k}} \sigma^2_{\hat{K}_i\hat{K}_j} \tag{3.2.28b}$$

where Equation 3.2.28b corresponds to Equation 2.4.11d in the general estimator model derivation.

Assuming $Var(\underline{\hat{S}}_x)$ to be a good estimate of $Var(\underline{\hat{S}}_x)$ and recalling again that the MRKP model assumes independence of the components of $\underline{S}_x$, one can substitute Equation 3.2.28 into Equation 3.2.24 to obtain

$$\text{Var } P_K \approx \sum_{i=1}^{4} \left(\frac{\partial P_K}{\partial S_i}\right)^2 \Bigg| \sum_{j=1}^{4} \sum_{\ell=1}^{4} \frac{\partial \underline{S}_i}{\partial \hat{K}_j} \frac{\partial \underline{S}_i}{\partial \hat{K}_\ell} \Bigg| \sigma^2_{\hat{K}_j \hat{K}_\ell} \qquad (3.2.29)$$

$$\hat{S}_x = \underline{S}(\underline{y}, \underline{k}) \qquad \hat{\underline{K}} = \underline{k}$$

In general, the parameters of Equations 3.2.26 are not known and must be estimated from the empirical data pertaining to $\hat{\underline{K}}$. Denoting the estimate of these parameters by asterisks (*), one can finally write the resulting estimate of $\text{Var}(P_K)$ corresponding to Equation 2.4.13.

$$\hat{\text{Var}}[P_K] \approx \sum_{i=1}^{4} \left(\frac{\partial P_K}{\partial S_i}\right)^2 \Bigg| \sum_{j=1}^{4} \sum_{\ell=1}^{4} \frac{\partial \underline{S}_i}{\partial \hat{K}_j} \frac{\partial \underline{S}_i}{\partial \hat{K}_\ell} \Bigg| \sigma^{*2}_{\hat{K}_j \hat{K}_\ell} \qquad (3.2.30)$$

$$\hat{\underline{S}}_x = S(\underline{y}, \underline{k}^*) \qquad \hat{\underline{K}} = \underline{k}^*$$

Equation 3.2.30 now represents a measure of the uncertainty associated with P_K in this example. One observes that $\hat{\text{Var}}(P_K)$ is dependent upon the estimates of the parameters related to the error coefficients and the sensitivity coefficients relating $\hat{\underline{K}}$ to $\hat{\underline{S}}_x$ and $\hat{\underline{S}}_x$ to P_K. A quantitative discussion of the results of implementing Equation 3.2.30 via Program VPK(see Appendix E) follows in the next subsection.

Quantitative Results for Example 1. As has been stated previously in this section, this example utilizes models associated with the Joint Munitions Effectiveness Manual; in particular, the Stick Bomb Delivery Model and the Multiple Round Kill Probability Model. Figure 1 illustrates the relationship between the two models.

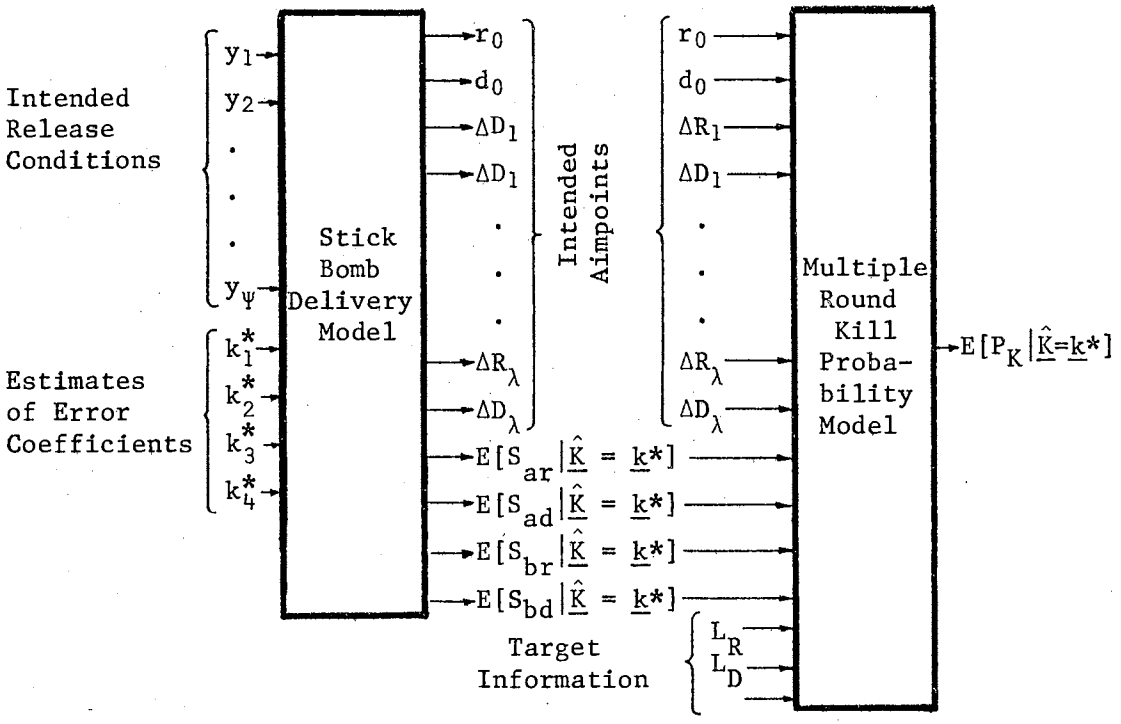

Figure 1

P_K Model for Example 1

The sensitivity coefficients relating $\hat{K}$ to $\hat{S}_x$ were obtained directly from the equation relating $\hat{K}$ to $\hat{S}_x$ by computation of the required partial derivatives. The sensitivity coefficients relating $\hat{S}_x$ to P_K were obtained by varying the values of S_{ar}, S_{ad}, S_{br}, and S_{bd} used as inputs

into the MRKP Program, obtaining corresponding resultant values of P_K, and subsequently employing finite difference techniques to approximate the required partial derivatives. It might be of interest to note that for the calculation of each partial derivative associated with a specific function, five values of the particular variable involved were chosen, including the intended value of the variable. In turn, a central difference table was formed for the five resulting functional values centered at the function evaluated at the intended value of the variable. The partial derivatives were then calculated from the appropriate entries in the central difference table.

The sensitivity coefficients along with estimates of the parameters associated with $\underline{\hat{K}}$ serve as inputs into a general purpose computer program developed especially for the research connected with this thesis. The program is termed Program VPK and is used to evaluate $\hat{Var}(P_K)$ as discussed in this thesis. A complete documentation of Program VPK appears in Appendix E. Figure 2 illustrates the use of VPK for this example.

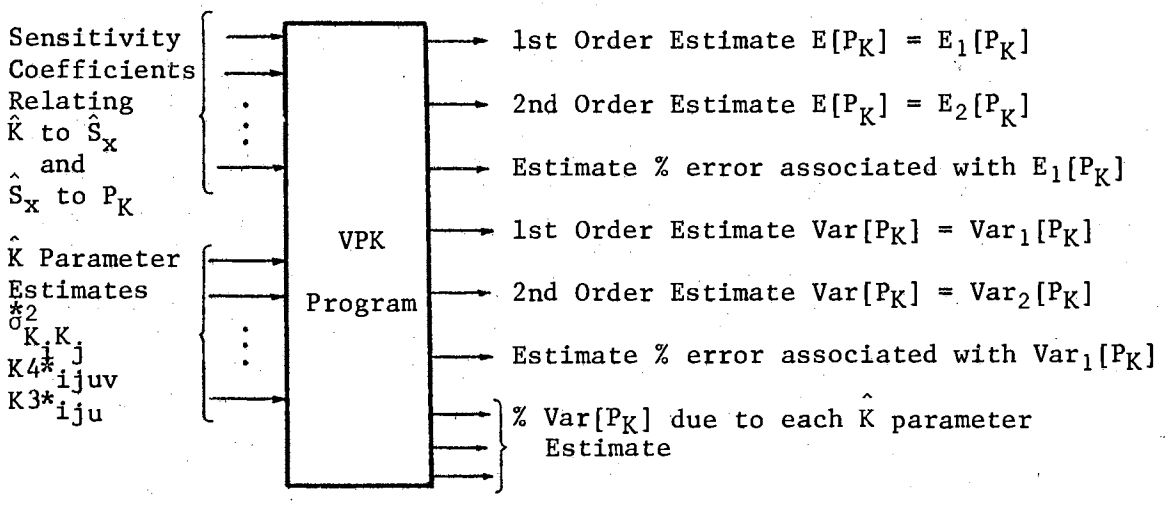

Figure 2. $Var(P_K)$ Model for Example 1

In Figure 2 $\sigma^{*2}_{K_iK_j}$ = estimate of $E[(\hat{K}_i - k_i)(\hat{K}_j - k_j)]$

$K3^*_{iju}$ = estimate of $E[(\hat{K}_i - k_i)(\hat{K}_j - k_j)(\hat{K}_u - k_u)]$

$K4^*_{ijuv}$ = estimate of $E[(\hat{K}_i - k_i)(\hat{K}_j - k_j)(\hat{K}_u - k_u)(\hat{K}_v - k_v)]$

The quantitative results pertaining to Example 1 are presented in Tables I through VI. For the most part, the uncertainty parameters are related in terms of standard deviation SD rather than variances so that the numbers are more readily interpreted. It should be pointed out that due to the nature of this example, the specific names of the weapons and targets assumed are not given so that this thesis can remain unclassfied. Furthermore, the values of several of the parameters used are not revealed since they may also be of a classified nature. The omission of these few specifics should not devalue the worth of this example since it is not the numerical results of this thesis that should be emphasized, but rather the application of a straight-forward technique for estimating the uncertainty associated with complex probabilistic models.

Table I presents the assumed estimated inputs to the Stick Bomb Delivery and MRKP Models which include the intended release conditions $\underline{y}$, the dimensions of the vulnerable target area, and the estimated target height. Also included are estimates of the parameters of the error coefficients; i.e., $\underline{k}^*$ and $\sigma^*_{\hat{K}_i\hat{K}_j}$. The values of $\sigma^{*2}_{K_iK_j}$ have been strictly assumed for this example and bear no relation to actual data measurement. However, for purposes of comparison, $\sigma^*_{\hat{K}_i\hat{K}_i}$ has been chosen so that the ratio $\sigma^*_{\hat{K}_i\hat{K}_i}/k^*_i$ is approximately the same for each i. Furthermore, for this example, the $\hat{K}_i$ have been assumed independent, an assumption that is removed for Example 2. For an actual physical case, the parameter $\sigma^{*2}_{\hat{K}_i\hat{K}_j}$ could be obtained from available empirical data.

TABLE I

Model Inputs for Example 1

Inputs	Symbol	Model	Estimated Value
Release Altitude of Weapon	y_1	Stick	3000 feet
Velocity of A/C at Release	y_2	Stick	450 knots
Dive Angle of A/C at Release	y_3	Stick	30 degrees
Ejection Angle of Weapon	y_4	Stick	90 degrees
Ejection Velocity of Weapon	y_5	Stick	CL*
Target Height	h	Stick/MRKP	6.7 feet
Vulnerable Target Length	L_R	MRKP	155.6 feet
Vulnerable Target Width	L_D	MRKP	143.9 feet
E[Aiming Error Coefficient AAA]	k_1*	Stick	CL*
E[Aiming Error Coefficient BBB]	k_2*	Stick	CL*
E[Aiming Error Coefficient DDD]	k_3*	Stick	CL*
E[Ballistic Error Coefficient DIS]	k_4*	Stick	CL*
$Cov(\hat{K}_1, \hat{K}_1)$	$\sigma^{*2}_{\hat{K}_1\hat{K}_1}$	VPK	1.0 feet^2
$Cov(\hat{K}_2, \hat{K}_2)$	$\sigma^{*2}_{\hat{K}_2\hat{K}_2}$	VPK	4.0 feet^2
$Cov(\hat{K}_3, \hat{K}_3)$	$\sigma^{*2}_{\hat{K}_3\hat{K}_3}$	VPK	4.0 feet^2
$Cov(\hat{K}_4, \hat{K}_4)$	$\sigma^{*2}_{\hat{K}_4\hat{K}_4}$	VPK	0.25 feet^2
$Cov(\hat{K}_i, \hat{K}_j)$ $i \neq j$	$\sigma^{*2}_{\hat{K}_i\hat{K}_j}$	VPK	0.0 feet^2
$E(S_{ar})$	REP	MRKP	272.7 feet
$E(S_{ad})$	DEP	MRKP	155.2 feet
$E(S_{br})$	SIGY	MRKP	37.2 feet
$E(S_{bd})$	SIGX	MRKP	25.0 feet

*CL = Classified Data

Tables II and III relate the sensitivity coefficients relating $\underline{\hat{K}}$ to $\underline{\hat{S}}_X$ and $\underline{\hat{S}}_X$ to P_K respectively. In turn, Table IV lists some of the outputs of Program VPK for Example 1. Of special interest in Table IV is the column labeled "Estimate of Percent Error in First Order Approximation." One sees that under the assumption of Example 1, the first order approximation employed is assumed sufficiently accurate under the criterion expressed in Appendix A.

TABLE II

Sensitivity Coefficients
Relating $\underline{\hat{K}}$ to $\underline{\hat{S}}_x$

Variable K	Sensitivity Coefficient for S_I			
	I = 1	I = 2	I = 3	I = 4
PSK (I, 1)	2.51	4.42	0.0	0.0
PSK (I, 2)	11.91	0.00	0.0	0.0
PSK (I, 3)	0.00	6.51	0.0	0.0
PSK (I, 4)	0.00	0.00	7.5	5.0
CPSK (I, 1, 1)	0.29	0.43	0.0	0.0
CPSK (I, 1, 2)	-0.11	0.00	0.0	0.0
CPSK (I, 1, 3)	0.00	-1.26	0.0	0.0
CPSK (I, 1, 4)	0.00	0.00	0.0	0.0
CPSK (I, 2, 2)	0.04	0.00	0.0	0.0
CPSK (I, 2, 3)	0.00	0.00	0.0	0.0
CPSK (I, 2, 4)	0.00	0.00	0.0	0.0
CPSK (I, 3, 3)	0.00	0.00	0.0	0.0
CPSK (I, 3, 4)	0.00	0.00	0.0	0.0
CPSK (I, 4, 4)	0.00	0.00	0.0	0.0

$$\text{PSK (I, J)} = \frac{\partial S_I}{\partial \hat{K}_J}$$

$$\text{CPSK (I, J, L)} = \frac{\partial^2 S_I}{\partial \hat{K}_J \partial \hat{K}_L}$$

TABLE III

Sensitivity Coefficients for Example 1
Relating $\underline{S}$ to $\underline{P_K}$

Variable S	Sensitivity Coefficients for P_K
PFS (1)	-0.00025293
PFS (2)	-0.00043256
PFS (3)	-0.00007654
PFS (4)	-0.00006970
CPFS (1, 1)	0.00000167
CPFS (1, 2)	0.00000151
CPFS (1, 3)	0.00000081
CPFS (1, 4)	0.00000024
CPFS (2, 2)	0.00000479
CPFS (2, 3)	0.00000045
CPFS (2, 4)	0.00000128
CPFS (3, 3)	-0.00000182
CPFS (3, 4)	0.00000007
CPFS (4, 4)	-0.00000261

$$\text{PFS (I)} = \frac{\partial P_K}{\partial S_I}$$

$$\text{CPFS (I, J)} = \frac{\partial^2 P_K}{\partial S_I \partial S_J}$$

TABLE IV

Uncertainty in P_K for Example 1

Item	First Order Approximation	Second Order Approximation	Estimate of Percent Error in First Order Approximation
$Var(S_{ar})$	574.054	574.111	0.01
$SD(S_{ar})$	23.959	23.960	<0.001
$Var(S_{ad})$	189.215	189.358	<0.001
$SD(S_{ad})$	13.756	13.760	<0.001
$Var(S_{br})$	13.850	13.850	0.00
$SD(S_{br})$	3.721	3.721	0.00
$Var(S_{bd})$	6.248	6.248	0.00
$SD(S_{bd})$	2.499	2.499	0.00
$E(P_K)$	0.07382	0.07475	1.26
$Var(P_K)$	0.00007224	0.00007311	1.2
$SD(P_K)$	0.008499	0.008550	0.6

Now as has been alluded to previously, one of the main purposes of this thesis is to relate the uncertainty associated with P_K to the uncertainties associated with basic random variable inputs to the model; that is, $\hat{Var}(P_K)$ must be related directly to the parameter estimates associated with $\hat{K}$. Table V lists the percent of the first order approximation of $Var(P_K)$ which can be attributed directly to the estimates of the parameters associated with $\underline{\hat{K}}$. One observes from Table V that $Var(P_K)$ is more sensitive for this example to the uncertainties associated with $\hat{K}_2$ and $\hat{K}_3$ which indicates that more care should be employed in estimating the parameters associated with these two random variables.

As an added observation, consider the ratio of the $SD(P_K)$ to $E(P_K)$ which is now termed the uncertainty ratio SDR; that is

$$SDR = \frac{SD(P_K)}{E(P_K)}$$

Now for Example 1

$$SDR = \frac{0.008550}{0.07475} = .1142$$

Table VI shows the behavior of SDR as $E(P_K)$ is varied for the same parameter estimates for $\underline{\hat{K}}$. One observes that for a change in $E(P_K)$ of 66.8 percent, SDR changes less than 8 percent. Thus, one might choose SDR to be yet another, and more general, measure of the uncertainty associated with P_K.

TABLE V

Variance of P_K Components for Example 1

Variance Components	$Var(P_K)$ = 0.00007224
Percent $Var(P_K)$ due to $\sigma^{*2}_{\hat{K}_1\hat{K}_1}$	5.57
Percent $Var(P_K)$ due to $\sigma^{*2}_{\hat{K}_2\hat{K}_2}$	50.39
Percent $Var(P_K)$ due to $\sigma^{*2}_{\hat{K}_3\hat{K}_3}$	43.89
Percent $Var(P_K)$ due to $\sigma^{*2}_{\hat{K}_4\hat{K}_4}$	0.15

TABLE VI

Uncertainty Ratio

$E(P_K)$	$SD(P_K)$	SDR
0.1399	0.01540	.1100
0.1162	0.01302	.1133
0.0985	0.01120	.1137
0.0847	0.00972	.1149
0.0738	0.00855	.1159
0.0651	0.00759	.1165
0.0579	0.00678	.1171
0.0519	0.00611	.1179
0.0469	0.00554	.1181

In summary, Example 1 has presented a simple probabilistic model for P_K as a function of four basic independent random variables: $\hat{K}_1$, $\hat{K}_2$, $\hat{K}_3$, and $\hat{K}_4$. Both a first and a second order approximation of the uncertainty related to P_K; i.e., $\hat{Var}(P_K)$, has been presented in terms of the assumed parameters associated with $\underline{\hat{K}}$; $\sigma^{*2}_{\hat{K}_i\hat{K}_i}$. Furthermore, estimates of the errors involved in approximating $E(P_K)$ and $Var(P_K)$ by first order approximations have been shown to be less than 5 percent in both cases. In addition, it has been observed that, under the assumptions of this example, P_K is more sensitive to uncertainties in $\hat{K}_2$ and $\hat{K}_3$ than in $\hat{K}_1$ and $\hat{K}_4$ which suggest that more care should be employed in evaluating the parameter estimates associated with $\hat{K}_2$ and $\hat{K}_3$. Finally, the uncertainty ratio SDR has been introduced as a further, and more general, measure of the uncertainty of P_K.

Multiple Weapon Delivery. Before proceeding with the second and more complex example, consider briefly the multiple weapon delivery case in light of Example 1. As stated previously, the computation of a measure of the uncertainty of P_K for the delivery of a stick of λ weapons is a simple extension of the techniques employed for the single weapon delivery case. From the discussion concerning the Stick Bomb Delivery Model, the mean impact point for the ith weapon, say m_i, is given by $(r_0 + \Delta r_i, d_0 + \Delta d_i)$ where (r_0, d_0) represents the intended pattern aimpoint and Δr_i and Δd_i denote range and deflection separation respectively between the pattern reference point and the position of the ith weapon. One could thus write the following

$$m_i = (r_0 + \Delta r_i, d_0 + \Delta d_i) \tag{3.2.35}$$

Again as in the single bomb case, r_0 and d_0 can be calculated from the intended release conditions. Furthermore, Δr_i and Δd_i can also be calculated from these intended release conditions. This is accomplished by means of the Stick Bomb Delivery Model.

Now the variance associated with aiming in the ith impact point should be the same as that related to the jth impact point since the aiming error is assumed to be applied to the pattern as a unit. On the hand, however, the ballistic error for the ith weapon would not necessarily be the same error as that for the jth weapon since the ballistic error is assumed to affect each bomb independently. Thus one should write the estimater model for the standard deviation associated with the ith impact point as

$$\hat{S}_{xi} = (\hat{S}_{ar}, \hat{S}_{ad}, \hat{S}_{bri}, \hat{S}_{bdi}) \tag{3.2.36}$$

where $\hat{S}_{ar}$ and $\hat{S}_{ad}$ are calculated using the release conditions for the middle (or theoretical middle) weapon in the pattern. Although the MRKP Model does not consider multiple values for ballistic errors, consider for now the general case where $\hat{S}_{bri}$ and $\hat{S}_{bdi}$ are calculated for the release conditions associated with the ith impact point. The specific application of the MRKP Model is discussed at the end of this section.

The general functional relationship representing the probability of kill model for the multiple weapon delivery case can then be written as

$$P_{KM} = P_{KM}(m_1, m_2, \ldots, m_\lambda, \hat{S}_{ar}, \hat{S}_{ad}, \hat{S}_{br1}, \hat{S}_{br2}, \ldots, \hat{S}_{br\lambda}, \hat{S}_{bd1}, \hat{S}_{bd2}, \ldots, \hat{S}_{bd\lambda}) \tag{3.2.37}$$

which corresponds to Equation 3.2.16 for the single weapon delivery case and, in general, to Equation 3.2.3. Following the techniques of Section 2.4, and assuming again that aiming and ballistic error are independent, one can write a first order approximation of the measure of uncertainty corresponding to that for P_E in Equation 2.4.13.

$$\begin{aligned}\hat{Var}(P_{KM}) = &\left(\frac{\partial P_{KM}}{\partial \hat{S}_{ar}}\right)^2 Var(\hat{S}_{ar}) + \left(\frac{\partial P_{KM}}{\partial \hat{S}_{ad}}\right)^2 Var(\hat{S}_{ad}) \\ &+ \sum_{i=1}^{\lambda}\sum_{j=1}^{\lambda} \frac{\partial P_{KM}}{\partial \hat{S}_{bri}} \frac{\partial P_{KM}}{\partial \hat{S}_{brj}} Cov(\hat{S}_{bri}, \hat{S}_{brj}) \\ &+ \sum_{i=1}^{\lambda}\sum_{j=1}^{\lambda} \frac{\partial P_{KM}}{\partial \hat{S}_{bdi}} \frac{\partial P_{KM}}{\partial \hat{S}_{bdj}} Cov(\hat{S}_{bdi}, \hat{S}_{bdj})\end{aligned} \tag{3.2.38}$$

where the sensitivity coefficients associated with $\hat{S}_{bri}$ and $\hat{S}_{bdi}$ are calculated for $\hat{S}_{bri}$ and $\hat{S}_{bdi}$ evaluated for the intended release conditions related to the ith weapon. In turn, the sensitivity coefficients

associated with $\hat{S}_{ar}$ and $\hat{S}_{ad}$ are calculated for $\hat{S}_{ar}$ and $\hat{S}_{ad}$ evaluated at the intended release conditions related to the middle weapon of the pattern. The covariance terms of Equation 3.2.38 indicate some measure of dependence among the ballistic errors. However, if one examines the model more closely, one observes that for a given pattern aimpoint, the ballistic errors are conditionally independent given the aiming error. In turn, the uncertainty associated with the pattern aimpoint is included in the aiming error terms so that the covariance terms of Equation 3.2.38 can be omitted. One can then write the following form for the measure of uncertainty involved in P_{KM}:

$$\begin{aligned} \mathrm{Var}(P_{KM}) = {} & \left(\frac{\partial P_{KM}}{\partial \hat{S}_{ar}}\right)^2 \mathrm{Var}(S_{ar}) + \left(\frac{\partial P_{KM}}{\partial \hat{S}_{ad}}\right)^2 \mathrm{Var}(\hat{S}_{ad}) \\ & + \sum_{i=1}^{\lambda} \left(\frac{\partial P_{KM}}{\partial \hat{S}_{bri}}\right)^2 \mathrm{Var}(\hat{S}_{bri}) \\ & + \sum_{j=1}^{\lambda} \left(\frac{\partial P_{KM}}{\partial \hat{S}_{bdj}}\right)^2 \mathrm{Var}(\hat{S}_{bdj}) \end{aligned} \tag{3.2.39}$$

Now as was alluded to previously, the MRKP Model does not consider a separate ballistic error for each impact point. Instead, an average $\hat{S}_{br}$ and $\hat{S}_{bd}$ are calculated using the release conditions for the middle weapon in the pattern. In turn $\hat{S}_{br}$ and $\hat{S}_{bd}$ are used as the ballistic errors associated with the ith impact point. In relation to the MRKP Model then, Equation 3.2.39 should be written as follows

$$\mathrm{Var}(P_{KM}) = \left(\frac{\partial P_{KM}}{\partial \hat{S}_{ar}}\right)^2 \mathrm{Var}(\hat{S}_{ar}) + \left(\frac{\partial P_{KM}}{\partial \hat{S}_{ad}}\right)^2 \mathrm{Var}(\hat{S}_{ad})$$

$$+ \left(\frac{\partial P_{KM}}{\partial \hat{S}_{br}}\right)^2 \mathrm{Var}(\hat{S}_{br}) + \left(\frac{\partial P_{KM}}{\partial \hat{S}_{bd}}\right)^2 \mathrm{Var}(\hat{S}_{bd}) \qquad (3.2.40)$$

where the sensitivity coefficients are evaluated for the intended release conditions associated with the middle weapon of the pattern. It should be observed that if $\hat{S}_{br}$ and $\hat{S}_{bd}$ are literally substituted into Equation 3.2.39 for each $\hat{S}_{bri}$ and $\hat{S}_{bdi}$ then, the last two terms of Equation 3.2.40 should be multiplied by λ. However, the sensitivity coefficients in Equation 3.2.40 which relate $\hat{S}_{br}$ and $\hat{S}_{bd}$ to P_{KM} are obtained by varying the single value of $\hat{S}_{br}$ and $\hat{S}_{bd}$ used as inputs for the model which in turn varies the respective standard deviations associated with <u>all</u> the impact points. The sensitivity coefficients calculated in this manner relate the sensitivities associated with all the impact points and thus justify the omission of the λ factors. One might observe that the error in P_{KM} associated with using average ballistic errors in the MRKP should be negligible due to the relative insensitivity of P_{KM} to S_{br} and S_{bd} which has been illustrated previously in Example 1.

3.3 Example 2: Fragment Sensitive Target. A fragment sensitive target is considered in this section as a second example of the uncertainty associated with a particular weapons effectiveness probabilistic model. Recall again the general weapons effectiveness model of Equation 3.1.3

$$P_K = P_K(\underline{I}) = E[D_F|\underline{I}] = \int_{-\infty}^{\infty}{}_2\, D_F(\underline{\tau})\, f_{\underline{X}|\underline{I}}(\underline{\tau}|\underline{I})\, d\underline{\tau} \qquad (3.3.1)$$

For the fragment sensitive target model, the damage function D_F is considerably more complex than that used in the blast sensitive model and corresponds to the damage function discussed in Section 2.5; that is, D_F is, in general, a function of the impact point and fragmentation characteristics of the particular weapon deployed. For the purpose of this example, the following form is assumed for the damage function and corresponds to that used by Snow (20) in Rand's Simplified Target Coverage Model.

$$D_F(\underline{X}, D_0, RM_R, RM_D) = D_0 \exp\{-D_0[(\frac{X_R}{RM_R})^2 + (\frac{X_D}{RM_D})^2]\} \qquad (3.3.2)$$

D_F relates a symmetric damage function with center value D_0 and elliptic constant damage level contours. Equation 3.2.2 corresponds to an analytical function which has been fit to an empirical fragmentation damage function obtained from fragmentation data. The random variable parameters D_0, RM_R and RM_D are acquired from the empirical damage function and thus relate the fragmentation data to the analytical damage function of Equation 3.2.2. The purpose of this example then, is to relate the uncertainty associated with the empirical fragmentation data, as well as delivery accuracy uncertainty, to the uncertainty in P_K. In order to do

this, each individual model employed must be examined. The empirical fragmentation function is obtained by means of a Lethal Area Program which also contains a subroutine that in turn calculates RM_R and RM_D directly from the empirical fragmentation function. For this example D_0 is assumed to have the value 1.0 and will not be considered a random variable. This assumption corresponds to assuming that the probability of damage to the target at the impact point is 1.0 which is reasonable for the weapon and target assumed for this example. Another program, termed Quickie, combines the damage function information obtained in the Lethal Area Program, with target and weapon delivery specifications to obtain P_K, the conditional expected damage to the target. As in the blast sensitive example, the target is assumed fixed and the weapon delivery information is derived from the Stick Bomb Delivery Model.

In order to better understand the conditional aspects of P_K, consider the following notation used in this example:

$$P_K = E[D_F | RM_R, RM_D, \hat{S}_x] = \int_{-\infty}^{\infty}{}_2 \, D_F(\underline{\tau}, \underline{RM}_R, \underline{RM}_D) \, f_{\underline{X}|\underline{S}_x}(\underline{\tau}|\hat{\underline{S}}_x) \, d\underline{\tau} \tag{3.3.3a}$$

where

$$RM_R = FR_1(Z_1, Z_2, \ldots, Z_w) \equiv FR_1(\underline{Z}) \tag{3.3.3b}$$

$$RM_D = FR_2(Z_1, Z_2, \ldots, Z_w) \equiv FR_2(\underline{Z}) \tag{3.3.3c}$$

$$Z_i = \text{ith component associated with fragmentation data} \tag{3.3.3d}$$

$$\hat{\underline{S}}_x = (\hat{S}_{ar}, \hat{S}_{ad}, \hat{S}_{br}, \hat{S}_{bd}) = \underline{S}(\underline{y}, \hat{\underline{K}}) \tag{3.3.3e}$$

and $\hat{\underline{S}}_x$ is equivalent to Equation 3.2.25. The Z values are explained in detail at a later point. From Equation 3.3.3 one can see that P_K is a conditional expected damage, conditioned directly on RM_R, RM_D and $\hat{\underline{S}}_x$ and in turn on $\underline{Z}$ and $\hat{\underline{K}}$. Thus the uncertainty associated with P_K must be linked directly to the uncertainty related to $\underline{Z}$ and $\hat{\underline{K}}$ by the techniques of Chapter II. Before considering any particular model for the computation of a measure of uncertainty for P_K, suppose each individual program model is examined along with the assumptions made for that model; i.e., Stick Bomb Weapon Delivery, Lethal Area, and Quickie Programs associated with the JMEM effort.

Stick Bomb Weapon Delivery. As for the example of the previous section, the weapon delivery is assumed to be modeled by the Stick Bomb Weapon Delivery Program. Thus one can write immediately the estimator models for the mean and standard deviation of $\underline{X}$(the actual impact point) corresponding to Equations 3.2.14 and 3.2.15 respectively

$$\hat{\underline{m}}_x = \underline{M}_0 = H_x(y_1, y_2, \ldots, y_\phi) \equiv H_x(\underline{y}) \tag{3.3.4a}$$

$$\hat{\underline{S}}_x = \underline{S}(y_1, y_2, \ldots, y_\phi, \hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4) \equiv \underline{S}(\underline{y}, \hat{\underline{K}}) \tag{3.3.4b}$$

where

$$\hat{\underline{S}}_x = (\hat{S}_{ar}, \hat{S}_{ad}, \hat{S}_{br}, \hat{S}_{bd}) \equiv (\hat{S}_1, \hat{S}_2, \hat{S}_3, \hat{S}_4) \tag{3.3.4c}$$

Again the release of only one weapon is considered to reduce the complexity of the problem.

Since the model is identical to that used in Example 1, one can also write the expected value of $\hat{\underline{S}}_x$ and a measure of the uncertainty associated with $\hat{\underline{S}}_x$ directly from Equation 3.2.28

$$E[\hat{S}_x] = S(\underline{y}, \underline{k}) \tag{3.3.5a}$$

$$\mathrm{Var}(\hat{S}_x) = \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial \underline{S}}{\partial \hat{K}_i} \frac{\partial \underline{S}}{\partial \hat{K}_j} \Big|_{\hat{\underline{K}}=\underline{k}} \sigma^2_{\hat{K}_i \hat{K}_j} \tag{3.3.5b}$$

where

$\hat{K}_i$ = estimator of the ith error coefficient

k_i = expected value of the ith error coefficient

$\sigma^2_{\hat{K}_i \hat{K}_j}$ = covariance of estimators of the ith and jth error coefficients

Lethal Area Model. The Lethal Area Model provides detailed information about the destructive capability of a weapon on a specified target. This information is output in the form of an empirical damage function that gives probability of full damage P_{FD} as a function of target position relative to the point on the target directly under the burst point. An analytic damage function is then fit to this empirical damage function and the pertinent parameters associated with the analytical damage functions serve as additional outputs of this Lethal Area Model. Another measure of the weapons effectiveness is computed as lethal area which corresponds to the integration of the probability of full damage over the area of effects. In general, lethal area is a damage index which when multiplied by the number of uniformly distributed targets per unit area, gives the expected number of targets completely damaged.

The Lethal Area Model recognizes two distinct and independent damage mechanisms: fragmentation and blast. The probability of full

damage at a point T_p measured with reference to the impact point of the weapon, is computed from the formula given in reference (21).

$$P_{FD}(T_p) = 1 - [1 - P_{Kb}(T_p)][1 - P_{Kf}(T_p)] \qquad (3.3.6)$$

where P_{Kb} is the probability of full damage from blast effects alone and P_{Kf} is the probability of full damage from fragmentation effects alone. It should be noted that the assumed independence between fragmentation and blast effects might prove to be questionable under certain situations; however, since the Lethal Area model used assumes this independence, the same assumption is made for the example of this section. In the model, P_{Kb} is assumed to depend only on the distance R_b which is the distance from the impact (burst) point to the target point T_p. From the impact point to a range R_{b1}, P_{Kb} is assumed to be unity. Beyond R_{b1}, P_{Kb} is assumed to decrease linearly to zero at R_{b2}. The model thus requires the entry of particular values of R_{b1} and R_{b2} which are termed the blast radii.

On the other hand, the computation of P_{Kf} is considerably more complex than that of P_{Kb}. In particular, P_{Kf} is calculated directly from extensive empirical fragmentation data. The lethal capability of a given fragment depends on its mass, its impact velocity, and its shape. It is not the purpose of this thesis to examine in depth the computation of P_{Kf} but to point out its functional dependence on empirically derived fragmentation characteristics. These characteristics, which are now called components of fragment data, are obtained from bomb arena tests and include such items as

(1) fragment shape

(2) spatial distribution of fragments

(3) fragment mass densities

(4) initial fragment velocities

(5) fragment drag coefficients

(6) weight of unexploded bomb case

A more complete description of the internal structure of the Lethal Area model is given in reference (21).

Since the values used by the Lethal Area model for the components of fragment data are obtained via a few experiments, these values are at best only estimates of the means of the basic underlying random variables involved. Likewise the blast radii as well as other model inputs could also be treated as estimates of the expected values of other random variables. In general one could write

$$P_{FD}(T_p) = FR_3(T_p, Z_1, Z_2, \ldots, Z_{ww}) \tag{3.3.7}$$

where

Z_i = estimator of the expected value of the ith factor affecting P_{FD}

The uncertainty associated with the Z components; i.e., estimates of covariances of Z_i and Z_j, can now be related to the uncertainty of P_K in a manner similar to that of the theoretical models of Sections 2.2 and 2.3. For the purposes of this example, only four Z components were chosen to illustrate the theory involved. This was primarily due to lack of sufficient data concerning the other components but is also convenient, as the complexity of the computations involved is reduced accordingly. It should be pointed out that the general program written to evaluate $\hat{Var}(P_x)$; i.e., Program VPK, which is described in Appendix E,

considers the uncertainty associated with NZ of the $\underline{Z}$ components.

The four Z components considered are all components of fragment data and are listed as:

(1) number of fragments counted N = Z_1

(2) weight of unexploded bomb case CASWGT = Z_2

(3) total fragment weight recovered FW = Z_3

(4) fragment drag coefficients, CD = Z_4

Now Z_1 and Z_3, number of fragments counted and fragment weight, reflect uncertainties due to errors made in methods and equipment used in the weapons test arena. Quantitative measure of these uncertainties can be made by relatively simple methods such as equipment calibration; that is, estimates of the variances and covariances associated with Z_1 and Z_3 can be estimated directly from the test arena data.

On the other hand, Z_2, CASWGT represents a random variable whose uncertainty cannot be traced to the arena tests. Z_2 is the weight of the unexploded weapon case and the uncertainty associated with Z_2 is related to the fact that weapon cases are mass produced; i.e., production errors are inherent. The uncertainty related to Z_2 is important in the Lethal Area Model since the value assumed for Z_2 is used to compute the total number of fragments considered by the model; that is, Z_2 is used to extrapolate Z_1 for the lethal area model. A measure of the uncertainty associated with Z_2 can be obtained by examining the actual production records of several munition plants and estimating a value for $\text{Var}(Z_2)$ from this data.

The fourth component Z_4, the fragment drag coefficient is determined from measurements made in a wind tunnel. Recalling the discussion concerning stochastic drag coefficients tables in Section 2.5, one can

express the uncertainty associated with each "average" value of the drag coefficient table as a function of the data used to calculate each entry into the table; i.e., $Cov(C_i, C_j)$ where C_i and C_j are the ith and jth entry into the table respectively.

The uncertainties associated with the four Z's considered in this example have been expressed and now must in turn be related to the uncertainty of P_K. Now as one recalls, the Lethal Area model computes an empirical damage function where each point in this damage function corresponds to a particular P_{FD}; i.e., a particular value of T_P. It has been shown that P_{FD} is a function of the Z components which implies that the empirical damage function then in a larger sense is a function of the Z components also. Furthermore, the random variable parameters linking the empirical damage function to the analytical damage function used in the Quickie Target Coverage Model, namely RM_R and RM_D, must also in turn be functions of the Z components. One thus justifies the previously written Equations 3.3.3b and 3.3.3c.

$$RM_R = FR_1(Z_1, Z_2, Z_3, Z_4) \equiv FR_1(\underline{Z}) \tag{3.3.8a}$$

$$RM_D = FR_2(Z_1, Z_2, Z_3, Z_4) \equiv FR_2(\underline{Z}) \tag{3.3.8b}$$

Equations 3.3.8 simply relates that the analytical damage function used is indeed a function of the factors affecting the empirical damage function as one would expect.

Consider now the uncertainties associated with RM_R and RM_D which are related to the uncertainties of the Z's. Denoting $\underline{RM} = (RM_R, RM_D)$ and $\underline{FR} = (FR_1, FR_2)$, expand $\underline{RM}$ about $\underline{z} = E[Z]$ in a Taylor Series Expansion

$$\underline{RM} = \underline{FR}(\underline{z}) + \sum_{i=1}^{4} \frac{\partial \underline{FR}}{\partial Z_i} \Big|_{\underline{Z}=\underline{z}} [Z_i - z_i] \tag{3.3.9}$$

Again a first order expansion is employed at this point to reduce the notational complexity; however, a second order derivation for this example is analogous to that which appears in Appendix C for Example 1. One is also referred to Appendix A for a discussion concerning the Taylor Series termination criteria. Taking the expected value of $\underline{RM}$ as

$$E[\underline{RM}] \approx \underline{FR}(\underline{z}) \tag{3.3.10}$$

one can calculate the variance of $\underline{RM}$ directly corresponding for this example to the variance of $\underline{V}$ or $\underline{M}$ calculated in Equation 2.2.29

$$\mathrm{Var}[\underline{RM}] \approx \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial FR}{\partial Z_i} \frac{\partial FR}{\partial Z_j} \Big|_{\underline{Z}=\underline{z}} \mathrm{Cov}[Z_i, Z_j] \tag{3.3.11}$$

In particular one can write the following components of $\mathrm{Var}[\underline{RM}]$

$$\mathrm{Var}[RM_R] \approx \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial FR_1}{\partial Z_i} \frac{\partial FR_1}{\partial Z_j} \Big|_{\underline{Z}=\underline{z}} \mathrm{Cov}[Z_i, Z_j] \tag{3.3.12a}$$

$$\mathrm{Var}[RM_D] \approx \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial FR_2}{\partial Z_i} \frac{\partial FR_2}{\partial Z_j} \Big|_{\underline{Z}=\underline{z}} \mathrm{Cov}[Z_i, Z_j] \tag{3.3.12b}$$

$$\mathrm{Cov}[RM_R, RM_D] \approx \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial FR_1}{\partial Z_i} \frac{\partial FR_2}{\partial Z_j} \Big|_{\underline{Z}=\underline{z}} \mathrm{Cov}[Z_i, Z_j] \tag{3.3.12c}$$

where it is understood that $\mathrm{Cov}[Z_i, Z_j]$ and $\underline{z}$ are estimated from empirical data.

It should be noted at this point that an additional program was used in order to obtain the required sensitivity coefficients; i.e., the partial derivatives for Equation 3.3.12. The program called FRAME-1 was

developed by the Defense Technology Laboratory in Santa Clara, California to be used in connection with the Lethal Area program which is used in this example. Originally the program was designed to implement an error analysis of the random variable parameter Lethal Area, LA, and provide a level of confidence associated with LA. Although the program was not initially intended to be used to calculate sensitivity coefficients, it did provide a method of varying the particular Z values used by the Lethal Area Program and thus was easily modified to provide data suitable for calculating the sensitivity coefficients by standard finite difference techniques.

Quickie Target Coverage Model. Assuming the delivery of one weapon aimed at the center of the target and that the input probability densities of Equations 3.2.20 again apply, one can write a functional representation of the P_K model used in this example; namely, the Quickie Target Coverage Model:

$$P_K = P_{FC}(RM_R, RM_D, \underline{\hat{S}}_X) = E[D_F | RM_R, RM_D, \underline{\hat{S}}_X] = \int_{-\infty}^{\infty}{}_2 \exp[(\frac{\tau_R}{RM_R})^2 + (\frac{\tau_D}{RM_D})^2] f_{\underline{X}|\underline{\hat{S}}_x}(\tau_R, \tau_D|\underline{\hat{S}}_x) d\tau_R d\tau_D \quad (3.3.13)$$

which will be referred to as simply $P_{FC}(\underline{RM}, \underline{\hat{S}}_x)$. The multiple bomb drop case can again be attained by a simple but rather lengthy extension similar to that associated with Example 1. A function representation of the multiple bomb P_K, say P_{KM}, is written which corresponds to Equation 3.2.37.

$$P_{KM} = P_{FCM}(\underline{m}, RM_R, RM_D, \underline{\hat{S}}_x) \quad (3.3.14)$$

where P_{FCM} represents the function involved. Due to the notational

complexity associated with the multiple bomb case, only the delivery of a single weapon is considered in this example.

Now suppose one makes the following change of notation

$$RM_R = W_1 \quad \text{and} \quad E[RM_R] = \gamma_1 \tag{3.3.15a}$$

$$RM_D = W_2 \quad \text{and} \quad E[RM_D] = \gamma_2 \tag{3.3.15b}$$

$$\hat{S}_{ar} = W_3 \quad \text{and} \quad E[\hat{S}_{ar}] = \gamma_3 \tag{3.3.15c}$$

$$\hat{S}_{ad} = W_4 \quad \text{and} \quad E[\hat{S}_{ad}] = \gamma_4 \tag{3.3.15d}$$

$$\hat{S}_{br} = W_5 \quad \text{and} \quad E[\hat{S}_{br}] = \gamma_5 \tag{3.3.15e}$$

$$\hat{S}_{bd} = W_6 \quad \text{and} \quad E[\hat{S}_{bd}] = \gamma_6 \tag{3.3.15f}$$

so that

$$\underline{W} = (W_1, W_2, \ldots, W_6) \equiv (RM_R, RM_D, \underline{\hat{S}}_x)$$

where γ_1 and γ_2 are obtained from Equation 3.3.10 and γ_3, γ_4, γ_5, and γ_6 are obtained from Equation 3.3.5a. Expanding P_K in Equation 3.3.13 about $E[\underline{W}] = (\gamma_1, \gamma_2, \ldots, \gamma_6) \equiv (\underline{\gamma})$ in a first order Taylor Series Expansion yields

$$P_K = P_{FC}(\underline{\gamma}) + \sum_{i=1}^{6} \frac{\partial P_{FC}}{\partial W_i} \Big|_{\underline{W}=\underline{\gamma}} [W_i - \gamma_i] \tag{3.3.17}$$

See Appendix A for Taylor Series approximation criteria. Take the expected value of P_K from Equation 3.3.17 as

$$E(P_K) = P_{FC}(\underline{\gamma}) \tag{3.3.18}$$

then one can directly calculate a measure of the uncertainty associated with P_K corresponding to Equation 3.2.24.

$$\mathrm{Var}(P_K) \simeq \sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial P_{FC}}{\partial W_i} \frac{\partial P_{FC}}{\partial W_j} \Big|_{\underline{W}=\underline{\gamma}} \mathrm{Cov}(W_i, W_j) \tag{3.3.19}$$

where from Equation 3.3.12c for i = 1, 2; j = 1, 2 one obtains

$$\mathrm{Cov}(W_i, W_j) \simeq \sum_{u=1}^{4} \sum_{v=1}^{4} \frac{\partial FR_i}{\partial Z_u} \frac{\partial FR_j}{\partial Z_v} \Big|_{Z=z} \mathrm{Cov}(Z_u, Z_v) \tag{3.3.20}$$

If i = 3, 4, 5, 6 and j = 3, 4, 5, 6, one has from Equation 3.3.5b

$$\mathrm{Cov}(W_i, W_j) \simeq \sum_{u=1}^{4} \sum_{v=1}^{4} \frac{\partial \underline{S}_i}{\partial \hat{K}_u} \frac{\partial \underline{S}_j}{\partial \hat{K}_v} \Big|_{\underline{\hat{K}}=\underline{k}} \sigma^2_{\hat{K}_u \hat{K}_v} \tag{3.3.21}$$

Recalling that W_1 and W_2 denote the analytic damage function radii and W_3, W_4, W_5, and W_6 correspond to aiming and ballistic errors, it seems reasonable to assume that W_1 and W_2 are independent of W_3, W_4, W_5, and W_6 so that

$$\mathrm{Cov}(W_i, W_j) = 0 \quad \text{for} \quad i = 1, 2;\ j = 3, 4, 5, 6 \tag{3.3.22}$$

Equation (3.2.22) merely relates the assumption that the fragmentation spray from a weapon is not dependent upon the impact point of the weapon. This is not to imply that fragmentation is not a function of impact angle and impact velocity as surely it is. Impact angle and impact velocity are two of the $\underline{Z}$ components affecting $\underline{RM}$ which were not considered in this sensitivity study.

Now in general the covariance and expected value terms appearing in Equations 3.3.20 and 3.3.21 are unknown as has been alluded to before, and thus must be estimated from appropriate empirical data. Denoting

these estimated values with asterisks (*), Equations 3.3.19, 3.3.20, and 3.3.21 can be modified to obtain an estimate of the $Var(P_K)$ corresponding to Equation 2.4.13.

$$\hat{Var}(P_K) \approx \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial P_{FC}}{\partial W_i}\frac{\partial P_{FC}}{\partial W_j}\Big|_{\underline{W}=\underline{\gamma}} \sum_{u=1}^{4}\sum_{v=1}^{4} \frac{\partial FR_i}{\partial Z_u}\frac{\partial FR_j}{\partial Z_v}\Big|_{\underline{Z}=\underline{z}^*} Cov^*[Z_i, Z_j]$$

$$+ \sum_{i=3}^{6}\sum_{j=3}^{6} \frac{\partial P_{FC}}{\partial W_i}\frac{\partial P_{FC}}{\partial W_j}\Big|_{\underline{W}=\underline{\gamma}} \sum_{u=1}^{4}\sum_{v=1}^{4} \frac{\partial \underline{S}_i}{\partial \hat{K}_u}\frac{\partial \underline{S}_j}{\partial \hat{K}_v}\Big|_{\hat{\underline{K}}=\underline{k}^*} \hat{\sigma}^{*2}_{K_i K_j} \qquad (3.3.23)$$

Equation 3.3.23 now represents a measure of the uncertainty associated with P_K for this example. One can observe that $\hat{Var}(P_K)$ is dependent upon estimates of parameters related to the fragmentation data components and to the error coefficients. Furthermore, $Var(P_K)$ is a function of the sensitivity coefficients relating $\hat{K}$ to $\underline{\hat{S}}_x$, $\underline{Z}$ to $\underline{RM}$, $\underline{\hat{S}}_x$ to P_K, and $\underline{RM}$ to P_K. A quantitative discussion of the results pertaining to this example follow.

Quantitative Results for Example 2. The following JMEM programs are employed in Example 2:

(1) Stick Bomb Delivery Model

(2) Lethal Area Model

(3) Frame Model - (sets up inputs for Lethal Area Program)

(4) Quickie Target Coverage Model

Figure 3 illustrates the relationship among these models.

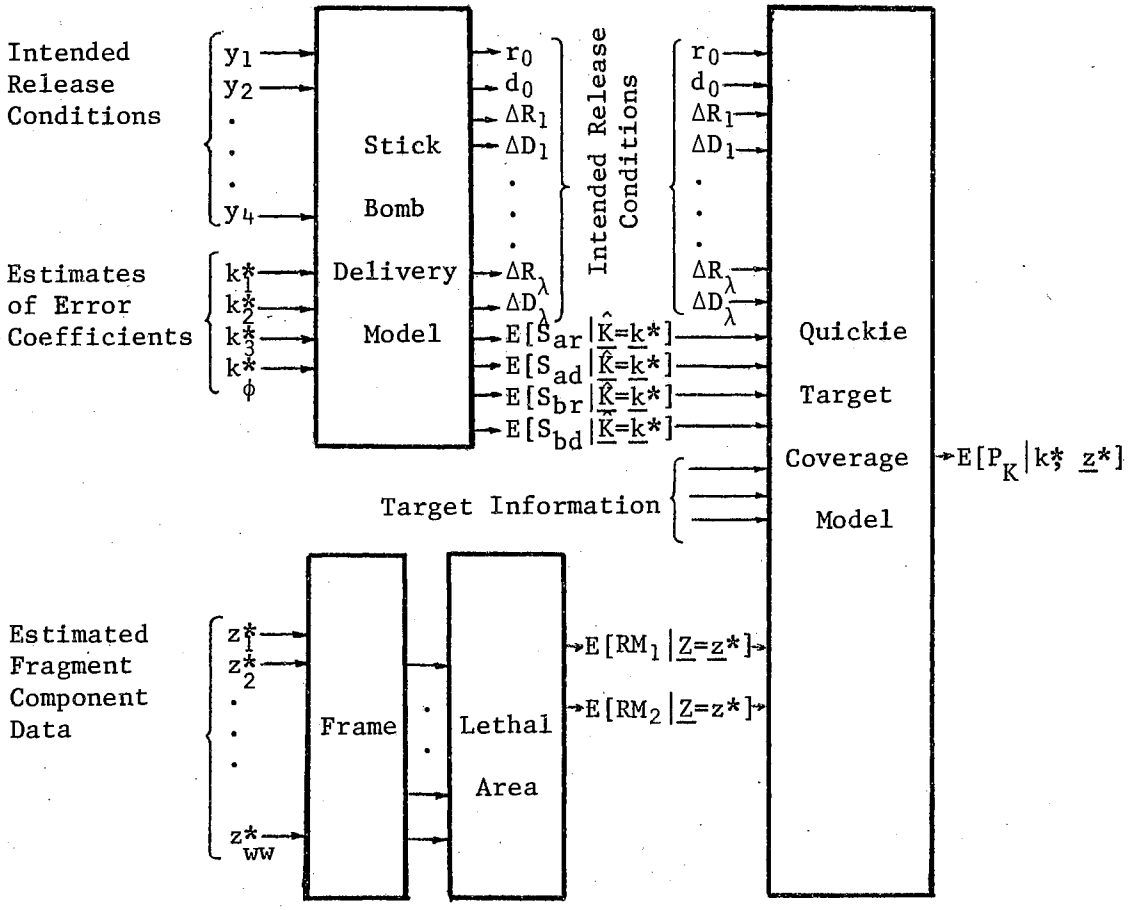

Figure 3

P_K Model for Example 2

Now the sensitivity coefficients relating $\underline{\hat{K}}$ to $\underline{\hat{S}}_x$ are the same as those used in Example 1 which are calculated directly from the functional relationships between $\underline{\hat{S}}_x$ and $\underline{\hat{K}}$. The sensitivity coefficients relating $\underline{Z}$ to $\underline{RM}$ were calculated using finite difference techniques as has been related previously in this section. In particular, it should be pointed out again that the Program FRAME aided greatly in the calculation of the partial derivatives required as it could be set to automatically vary the respective Z values in a sequential manner. Finally the sensitivity coefficients relating $\underline{\hat{S}}_x$ and $\underline{RM}$ to P_K were obtained by computing P_K for various values of $\underline{\hat{S}}_x$ and $\underline{RM}$ and calculating the required partial derivative again by means of finite difference techniques.

The sensitivity coefficients as well as estimates of the parameters associated with $\underline{\hat{K}}$ and $\underline{Z}$ serve as inputs into the VPK Program which, in turn, calculates first and second order approximations for $\text{Var}(P_K)$ and gives an estimate of the percent error associated with the first order approximation of $\text{Var}(P_K)$.

The quantitative results associated with Example 2 are summarized in Tables VII through XI in much the same manner as Tables I through VI related the quantitative results of Example 1. Table VII lists the estimated values which are used as inputs into the various models employed. Tables VIII and IX list the sensitivity coefficients relating $\underline{Z}$ to $\underline{RM}$ and $\underline{\hat{S}}_x$ and $\underline{RM}$ to P_K. The sensitivity coefficients pertaining to $\underline{\hat{K}}$ and $\underline{\hat{S}}_x$ were given previously in Table II.

Table X relates the uncertainty parameters obtained from the VPK Program. One should note that the components of $\underline{\hat{S}}_x$ are not assumed independent for this example. Table XI then lists the percent of the first

TABLE VII

Model Inputs for Example 2

Inputs	Symbol	Model	Estimated Value
Release Altitude of Weapon	y_1	Stick	3,000 feet
Velocity of A/C at Release	y_2	Stick	450 knots
Dive Angle of A/C at Release	y_3	Stick	30 degrees
Ejection Angle of Weapon	y_4	Stick	90 degrees
Ejection Velocity of Weapon	y_5	Stick	CL*
Target Height	h	Stick/Quickie	0.125 feet
Vulnerable Target Length	L_R	Quickie	100 feet
Vulnerable Target Width	L_D	Quickie	50 feet
E[Aiming Error Coefficient AAA]	k_1*	Stick	CL*
E[Aiming Error Coefficient BBB]	k_2*	Stick	CL*
E[Aiming Error Coefficient DDD]	k_3*	Stick	CL*
E[Ballistic Error Coefficient DIS]	k_4*	Stick	CL*
Number of Fragments Recovered From Arena Test	$N=Z_1$	Lethal Area	388
Weight of Bomb Case	$CA=Z_2$	Lethal Area	124,953.5 grams
Total Fragment Weight Recovered	$FW=Z_3$	Lethal Area	1,230.97 grams
Average Drag Coefficient	$CD=Z_4$	Lethal Area	0.55
$Cov(\hat{K}_1, \hat{K}_1)$	$\sigma^{*2}_{\hat{K}_1\hat{K}_1}$	VPK	1.0 feet^2
$Cov(\hat{K}_2, \hat{K}_2)$	$\sigma^{*2}_{\hat{K}_2\hat{K}_2}$	VPK	4.0 feet^2
$Cov(\hat{K}_3, \hat{K}_3)$	$\sigma^{*2}_{\hat{K}_3\hat{K}_3}$	VPK	4.0 feet^2
$Cov(\hat{K}_4, \hat{K}_4)$	$\sigma^{*2}_{\hat{K}_4\hat{K}_4}$	VPK	0.25 feet^2
$Cov(\hat{K}_1, \hat{K}_2)$	$\sigma^{*2}_{\hat{K}_1\hat{K}_2}$	VPK	1.0 feet^2

*CL = Classified Data

TABLE VII (Continued)

Inputs	Symbol	Model	Estimated Value
$Cov(\hat{K}_1, \hat{K}_3)$	$\sigma^{*2}_{\hat{K}_1\hat{K}_3}$	VPK	1.0 feet^2
$Cov(\hat{K}_2, \hat{K}_3)$	$\sigma^{*2}_{\hat{K}_2\hat{K}_3}$	VPK	2.0 feet^2
$E[(K_1 - k_1)^4]$	$K4_{1111}$	VPK	3.0 feet^4
$E[(K_2 - k_2)^4]$	$K4_{2222}$	VPK	48 feet^4
$E[(K_3 - k_3)^4]$	$K4_{3333}$	VPK	48 feet^4
$E[(K_4 - k_4)^4]$	$K4_{4444}$	VPK	.187 feet^4
$Cov(Z_1, Z_1)$	$\sigma^{*2}_{Z_1Z_1}$	VPK	400
$Cov(Z_2, Z_2)$	$\sigma^{*2}_{Z_2Z_2}$	VPK	5,020,000 grams^2
$Cov(Z_3, Z_3)$	$\sigma^{*2}_{Z_3Z_3}$	VPK	576 grams^2
$Cov(Z_4, Z_4)$	$\sigma^{*2}_{Z_4Z_4}$	VPK	0.0001
$Cov(Z_1, Z_3)$	$\sigma^{*2}_{Z_1Z_3}$	VPK	200 grams
$Cov(Z_2, Z_3)$	$\sigma^{*2}_{Z_2Z_3}$	VPK	40,000 grams^2
$Cov(Z_3, Z_4)$	$\sigma^{*2}_{Z_3Z_4}$	VPK	0.10 grams
$E[(Z_1 - z_1)^4]$	$Z4_{1111}$	VPK	480,000
$E[(Z_2 - z_2)^4]$	$Z4_{2222}$	VPK	0.785×10^{14} grams^4
$E[(Z_3 - z_3)^4]$	$Z4_{3333}$	VPK	31,200 grams^4
$E[(Z_4 - z_4)^4]$	$Z4_{4444}$	VPK	0.300×10^{-8}
All other estimates of parameters of $\hat{K}$ and Z	-	VPK	0.0

TABLE VIII

Sensitivity Coefficients for Example 2
Relating Z to RM

Item	Sensitivity Coefficients for RM_I	
	RM_1	RM_2
PRZ (I, 1)	0.112×10^{-1}	0.372×10^{-1}
PRZ (I, 2)	0.405×10^{-4}	0.173×10^{-3}
PRZ (I, 3)	-0.487×10^{-2}	-0.100×10^{-1}
PRZ (I, 4)	-0.775	-0.243×10^{2}
CPRZ (I, 1, 1)	-0.188×10^{-3}	$0.521 \times 10^{-}$
CPRZ (I, 1, 2)	0.365×10^{-7}	0.116×10^{-6}
CPRZ (I, 1, 3)	0.305×10^{-4}	-0.721×10^{-4}
CPRZ (I, 1, 4)	-0.296×10^{-1}	-0.904×10^{-1}
CPRZ (I, 2, 2)	-0.147×10^{-7}	0.379×10^{-7}
CPRZ (I, 2, 3)	0.233×10^{-6}	-0.690×10^{-6}
CPRZ (I, 2, 4)	-0.378×10^{-5}	-0.753×10^{-4}
CPRZ (I, 3, 3)	-0.177×10^{-3}	0.602×10^{-3}
CPRZ (I, 3, 4)	-0.219×10^{-2}	-0.333×10^{-2}
CPRZ (I, 4, 4)	-0.695×10^{3}	0.114×10^{4}

$$\text{PRZ (I, J)} = \frac{\partial RM_I}{\partial Z_J}$$

$$\text{CPRZ (I, J, L)} = \frac{\partial^2 RM_I}{\partial Z_J \partial Z_L}$$

TABLE IX

Sensitivity Coefficients for Example 2
Relating $\hat{\underline{S}}_x$ and $\underline{RM}$ to P_K

Variable S	Sensitivity Coefficient of P_K	Variable RM	Sensitivity Coefficient of P_K
PRS (1)	-0.426×10^{-4}	PFR (1)	0.574×10^{-3}
PFS (2)	-0.710×10^{-4}	PFR (2)	0.207×10^{-3}
PFS (3)	-0.575×10^{-5}	CPFR (1, 1)	-0.749×10^{-6}
PFS (4)	-0.118×10^{-4}	CPFR (1, 2)	0.988×10^{-5}
CPFS (1, 1)	0.293×10^{-6}	CPFR (2, 2)	-0.698×10^{-6}
CPFS (1, 2)	0.251×10^{-6}	Variable RM & S	Sensitivity Coefficients of P_K
CPFS (1, 3)	-0.460×10^{-6}		
CPFS (1, 4)	0.400×10^{-6}	CPFRS (1, 1)	-0.202×10^{-5}
CPFS (2, 2)	0.801×10^{-7}	CPFRS (1, 2)	-0.340×10^{-5}
CPFS (2, 3)	0.333×10^{-6}	CPFRS (L, 3)	-0.250×10^{-6}
CPFS (2, 4)	0.222×10^{-6}	CPFRS (1, 4)	-0.584×10^{-6}
CPFS (3, 3)	-0.291×10^{-6}	CPFRS (2, 1)	-0.703×10^{-6}
CPFS (3, 4)	0.00	CPFRS (2, 2)	-0.108×10^{-5}
CPFS (4, 4)	-0.441×10^{-6}	CPFRS (2, 3)	-0.938×10^{-7}
		CPFRS (2, 4)	-0.988×10^{-6}

$$\text{PFS (I)} = \frac{\partial P_K}{\partial S_I} \qquad \text{PFR (I)} = \frac{\partial P_K}{\partial RM_I}$$

$$\text{CPFS (I, J)} = \frac{\partial^2 P_K}{\partial S_I \partial S_J} \qquad \text{CPFR (I, J)} = \frac{\partial^2 P_K}{\partial RM_I \partial RM_J}$$

$$\text{CPFRS (I, J)} = \frac{\partial^2 P_K}{\partial RM_I \partial S_J}$$

TABLE X

Uncertainty in P_K for Example 2

Item	First Order Approximation	Second Order Approximation	Estimate of Percent Error in First Order Approximation
$Cov(\hat{S}_1, \hat{S}_1)$	642.66	642.72	<.01
$Cov(\hat{S}_1, \hat{S}_2)$	237.86	238.05	<.01
$Cov(\hat{S}_1, \hat{S}_3)$	0.00	0.00	0.00
$Cov(\hat{S}_1, \hat{S}_4)$	0.00	0.00	0.00
$Cov(\hat{S}_2, \hat{S}_2)$	249.04	248.47	0.23
$Cov(\hat{S}_2, \hat{S}_3)$	0.00	0.00	0.00
$Cov(\hat{S}_2, \hat{S}_4)$	0.00	0.00	0.00
$Cov(S_3, S_3)$	14.04	14.04	0.00
$Cov(S_3, S_4)$	9.41	9.41	0.00
$Cov(S_4, S_4)$	6.31	6.31	0.00
$Cov(RM_R, RM_R)$	0.35	0.23	20.00
$Cov(RM_D, RM_D)$	0.14	0.17	21.20
$Cov(RM_R, RM_D)$	0.58	0.50	13.80
$E(P_K)$	0.012588	0.012084	4.16
$Var(P_K)$	0.3879×10^{-5}	0.3854×10^{-5}	0.64
$SD(P_K)$	0.001969	0.001963	0.305
SDR	0.1565	0.1625	3.85

TABLE XI

Variance of P_K for Example 2

Variance Component Source	Percent Contribution
$Cov(\hat{K}_1, \hat{K}_1)$	4.44
$Cov(\hat{K}_1, \hat{K}_2)$	10.86
$Cov(\hat{K}_1, \hat{K}_3)$	9.76
$Cov(\hat{K}_2, \hat{K}_2)$	26.58
$Cov(\hat{K}_2, \hat{K}_3)$	23.89
$Cov(\hat{K}_3, \hat{K}_3)$	21.48
$Cov(\hat{K}_4, \hat{K}_4)$	0.07
E1(1, 1, 1, 1)	0.04
E1(2, 2, 2, 2)	0.55
E1(3, 3, 3, 3)	0.37
E1(4, 4, 4, 4)	<0.01
$Cov(Z_1, Z_1)$	2.03
$Cov(Z_1, Z_3)$	-.70
$Cov(Z_2, Z_2)$	1.42
$Cov(Z_2, Z_3)$	-.61
$Cov(Z_3, Z_3)$	.96
$Cov(Z_3, Z_4)$	.27
$Cov(Z_4, Z_4)$	.07
E2(1, 1, 1, 1)	<.01
E2(2, 2, 2, 2)	<.01
E2(3, 3, 3, 3)	<.01
E2(4, 4, 4, 4)	<.01

order approximation of $Var(P_K)$ that can be attributed directly to the particular parameter estimates associated with $\hat{\underline{K}}$ and Z.

Consider the results of Example 2 as presented in Tables VII through XI. One observes the increase in complexity of Example 2 in comparison with Example 1 due to the consideration of addition variables. Furthermore, again as in Example 1, the values given for the parameters are strictly assumed and bear no relation to actual estimates derived from actual empirical data.

One should note that most of the sensitivity coefficients presented in Tables VIII through IX follow a simple pattern that each sensitivity coefficient decreases in magnitude as the order increases. This is not the case for the sensitivity coefficient related to Z_4; i.e., CD, the drag coefficient, which due to its small nominal value, has a large sensitivity coefficient. On the other hand, fortunately, the large sensitivity is balanced by a very small variance of Z_4. However, the higher order sensitivity coefficient of Z_4 affects the $Cov(RM_i, RM_j)$ as seen in Table X, so that the second order approximation of $Cov(RM_i, RM_j)$ is used in both the first and second order approximations of $Var(P_K)$ (see Equation 3.2.24).

Table X relates the first order approximation of $E(P_K)$ and $Var(P_K)$ as meeting the criterion developed in Appendix A. One notes that the estimated error is even smaller for this example than that of Example 1. Of particular interest is the uncertainty ratio listed as 0.1565 which indicates that to some extent this model has associated with it a higher degree of uncertainty than that of the model of Example 1. This, of course, is a rather broad general statement based partially on intuition. A much better comparison could be made if both models were considered for

the same target-weapon combination. One would, however, expect a higher degree of uncertainty associated with Example 2 than with Example 1 due to the increased number of sources to provide uncertainty for Example 2.

As a final observation, consider the percent contribution to $Var(P_K)$ as presented in Table XI. Again as in Example 1, the primary source of uncertainty lies in parameters associated with $\hat{K}_2$ and $\hat{K}_3$. Although the percent contribution of the $\underline{Z}$ parameter estimates are small in comparison with those of $\hat{\underline{K}}$, the relative contribution within the Z component groups is informative. In particular, $Var(P_K)$ is most sensitive to Z_1, the number of fragments counted, for the parameter estimates used in this example. This would in turn indicate a need to provide for more care in the recovery of fragments from the arena tests.

The minus signs of two of the contributions indicate that the associated sensitivity coefficients were such that the sums of the terms associated with these parameters estimates were negative. This indicates that if the two random variables are positively correlated, the net contribution due to the associated variances is not as great as would be expected if the covariance terms were neglected.

In summary, the uncertainty associated with a complex probabilistic model has been investigated and first and second order approximations of a measure of that uncertainty have been evaluated. Furthermore, the uncertainty measure has been linked directly to parameter estimates of the basic model random variable inputs and in turn partitioned according to uncertainty contributions, thus indicating the major sources of uncertainty associated with $Var(P_K)$.

CHAPTER IV

SUMMARY AND CONCLUSIONS

4.1 Summary. The objective of this thesis has been to investigate the uncertainty associated with probabilistic models and to develop a procedure for estimating a measure of that uncertainty. Denoting the response of a general probabilistic model as P_K, the variance of P_K [that is, $Var(P_K)$] was chosen as this measure of uncertainty. In particular, P_K has been assumed to be a function of several basic random variables corresponding specifically in this thesis to model inputs. $Var(P_K)$ is in turn estimated in terms of the estimated parameters obtained from empirical data associated with these basic random variables.

Basically this thesis has been developed as an extension of sensitivity analysis of deterministic models to encompass probabilistic models. Chapter I thus reviews deterministic model sensitivity theory and lays the foundation from which the rest of the thesis is developed.

Chapter II presents various system probabilistic models and investigates errors associated with assumed density functions of parameters as well as errors related to conditional loss functions. Initially a theoretical model is introduced and an exact representation of $Var(P_K)$ is developed. Several difficulties associated with the theoretical model are discussed and subsequently an approximation of $Var(P_K)$ is developed through the use of Taylor's Series Approximations. The estimator model is then introduced as an alternative to the theoretical model for use in more specialized situations. An estimate of $Var(P_K)$ for

the estimator model is developed with aid of Taylor's Series Approximation.

Chapter III presents two examples in which the techniques developed in Chapter II are used. Both examples are taken from the field of weapons effectiveness where P_K represents the conditional expected damage to a target attacked by a group of specified weapons. Example 1 is concerned with an area target vulnerable to blast effects only, and is modeled by the Stick Bomb Delivery Model and the Multiple Round Kill Probability Model. Example 2 considers a much more complex problem: an area target vulnerable to both blast and fragmentation effects. Model used from the JMEM include the Stick Bomb Delivery Model, the Quickie Target Coverage Model, the Frame Model, and the Lethal Area Model. In both examples, first and second order approximations of $E(P_K)$ and $Var(P_K)$ are calculated along with estimates of the percent error associated with the first order approximation. Furthermore, the estimates of $Var(P_K)$ are partitioned into percent contributions due to parameter estimates of the basic random variables associated with the models.

The Taylor Series Approximation and the assumptions associated with the series termination criterion are discussed in Appendix A. Also included in Appendix A is the development of an estimate of the error associated with using the Nth order approximation of P_K to estimate $E(P_K)$ and $Var(P_K)$. In addition, a brief documentation appears in Appendix E of the VPK Program which was developed to calculate estimates of $E(P_K)$ and $Var(P_K)$.

4.2 Observations and Conclusions. An approximation has been developed that estimates the uncertainty associated with probabilistic models; i.e., Var (P_K). As was alluded to previously, it is not proposed

that this estimate must be the best estimate, however one wishes to define "best," but it does present a practical approach to estimating the probabilistic model uncertainty similar to that used with deterministic models.

Examples 1 and 2 are assumed to present models that are fairly representative of general probabilistic models. One observes that lower order approximations of $Var(P_K)$ are quite acceptable in the examples if one accepts the estimated error criterion presented in Appendix A. These assumptions are based primarily upon the fact that the errors associated with the Nth order approximation of $E(P_K)$ and $Var(P_K)$ are greater than the errors associated with any higher order approximations. In general, this is intuitively the case for most probabilistic models encountered; however, models may exist in which the criterion of Appendix A is not valid. For this reason, higher order approximations of P_K than that chosen to estimate $Var(P_K)$ should be investigated before the error criterion of Appendix A is applied.

One further observes that the uncertainty ratio introduced in Example 1 can provide even a more general measure of the uncertainty associated with a given probabilistic model than $\hat{Var}(P_K)$. In particular, it has been shown that SDR provides a relatively constant uncertainty measure over a wide spectrum of intended model inputs values for a given set of parameter estimates associated with the basic random variables of the model.

As was pointed out in Chapters I and III, the uncertainty measures developed in this thesis should aid in the comparison of two or more models. In particular, as presented in Section 3.2, the respective uncertainty measures of the various models under investigation can be

used to determine if particular models should be used interchangeably; that is, one can set up decision criterion based upon the uncertainty measures associated with each model.

Finally, one should observe that the partitioning of the estimate of $Var(P_K)$ into percent components related to the parameter estimates should indicate where major sources of uncertainty lie and thus specify where more data should be taken and where more money should be spent on research and testing.

4.3 Recommendations for Further Study. As indicated at a previous point in this thesis, errors associated with the form of the models themselves are not considered in this thesis. It is recommended that these errors be investigated in light of the discussion of this thesis. For example, one might consider each approximation made in a particular model to correspond to a random variable with uncertainty associated with the order of the error involved in that particular approximation. Information of this type could be combined to produce a measure of the uncertainty associated with the model form. This measure of uncertainty could then be used to supplement the measure of uncertainty developed in this thesis related to the model inputs. A measure of the model form uncertainty would be useful in determining where better approximations should be made in the model.

As a further recommendation, it is suggested that the sensitivity coefficients be investigated in more detail than was provided by this thesis. Since many of the sensitivity coefficients were computed using finite difference techniques, they are essentially dependent upon the particular values chosen from which the partial derivatives were calculated. There is, therefore, some degree of uncertainty that can

be linked to the sensitivity coefficients. This suggests the treating of the sensitivity coefficients as random variables and in turn relating their uncertainty to $Var(P_K)$.

Furthermore, it would be advantageous to investigate further the estimated error associated with the Nth order approximation of the functional representation of P_K. Since the error is directly related to the point α (see Appendix A), a better estimate of α would be desirable. In addition, one might investigate the possibility of evaluating an upper bound on the error associated with a given approximation, in particular, the evaluation of R_N for α equal to β as mentioned in Appendix A.

Finally, the Bayesian learning techniques presented briefly in Section 2.3 should be investigated in more detail as a possible alternative procedure in the calculation of $Var(P_K)$. Furthermore, in turn, the effects of the addition of more data concerning a particular random variable for the Bayesian learning technique should be compared with the corresponding effects for the Taylor's Series Approximation method as presented in this thesis.

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APPENDIX A

TAYLOR'S SERIES EXPANSION

A.1 Introduction. The purpose of this Appendix is to discuss in general the Taylor's Series approximations employed throughout this thesis. Taylor's Formula with Remainder is discussed for functions of several non-random variables and then is extended for functions of random vectors. The remainder term is examined in light of the models involved in this thesis and a criterion is given for estimating the error associated with a given Nth order approximation.

A.2 Taylor's Formula with Remainder. Consider initially a non-random vector $\underline{x} \equiv (x_1, x_2, \ldots, x_n)$ and let $g(\underline{x})$ be defined and continuous and have continuous differentials up to the $(N + 1)$st order for $\underline{a} - \underline{r}_a < \underline{x} < \underline{a} + \underline{r}_a$, where $\underline{a} = (a_1, a_2, \ldots, a_n)$ and $\underline{r}_a = (r_1, r_2, \ldots, r_n)$. For a function of n variables the kth order differential at $\underline{a} = (a_1, a_2, \ldots, a_n)$ is defined by Williamson, Crowell, and Trotter (22) to be the following polynomial in $\underline{x} = (x_1, x_2, \ldots, x_n)$:

$$d^k_{\underline{a}}\, g(\underline{x}) = \left(x_1 \frac{\partial}{\partial x_1} + \ldots + x_n \frac{\partial}{\partial x_n}\right)^k_a g$$

or

$$d^k_{\underline{a}}\, g(\underline{x}) = \sum_{k_1 + \ldots + k_n = k} \binom{k}{k_1 \ldots k_n} x_1^{k_1} \ldots x_n^{k_n} \frac{\partial^k g\,(\underline{a})}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \quad \text{(A.2.1)}$$

where $\binom{k}{k_1 \ldots k_n} = \dfrac{k!}{k_1! \ldots k_n!}$

In terms of differentials, Taylor's Formula with Remainder can be expressed for each x in the region about $\underline{a}$ defined by $\underline{r}_a$ as

$$g(\underline{x}) = g(\underline{a}) + \frac{1}{1!}(d_a^1 g)(\underline{x} - \underline{a}) + \frac{1}{2!}(d_a^2 g)(\underline{x} - \underline{a})$$

$$+ \ldots + \frac{1}{N!} d_{\underline{a}}^N g(\underline{x} - \underline{a}) + R_N \qquad (A.2.2)$$

where

$$R_N = \frac{1}{(N+1)!} d_{\underline{c}}^{N+1} g(\underline{x} - \underline{a})$$

and $\underline{c} = (c_1, c_2, \ldots, c_n)$ is some point in the region defined by $\underline{r}_a$ about $\underline{a}$. The error associated with approximating $g(\underline{x})$ by the first N + 1 terms of Equation A.2.2; that is, terms up to and including $\frac{1}{N!} d_a^N g(\underline{x}-\underline{a})$, is precisely the remainder R_n. An upper bound on this error would be calculated if the point $\underline{b}$ where the absolute value of the (N + 1)th differential is maximum

$$|R_N| \leq \left| \frac{1}{(N+1)!} d_{\underline{b}}^{N+1} g(\underline{x} - \underline{a}) \right| \qquad (A.2.3)$$

It should be pointed out that except for simple functions with n small the calculation of an upper bound on R_N is a tedious job.

A.3 Extension of Taylor's Formula with Remainder to Random Vectors. Now suppose one considers a random vector $\underline{X} \equiv (X_1, X_2, \ldots X_n)$ and a function $G(\underline{X})$ which is defined and continuous and has continuous differentials up to the (N + 1)th order over the range of $\underline{X}$ denoted by S_0. Define $\underline{\mu} \equiv E[\underline{X}] \equiv [E(X_1), E(X_2), \ldots E(X_n)]$. Then Taylor's Formula with Remainder can be written for the random function $G(\underline{X})$ as

$$G(\underline{X}) = G(\underline{\mu}) + \frac{1}{1!} d^1_{\underline{\mu}} G(\underline{X} - \underline{\mu}) + \frac{1}{2!} (d^2_{\underline{\mu}} G)(\underline{X} - \underline{\mu})$$

$$+ \cdots + \frac{1}{N!} d^N_{\underline{\mu}} G(\underline{X} - \underline{\mu}) + [R_N]_\alpha \qquad \text{(A.3.1)}$$

where

$$[R_N]_\alpha = \frac{1}{(N+1)!} d^{N+1}_\alpha G(\underline{X} - \underline{\mu})$$

where $\underline{\alpha}$ is a particular value of $\underline{X}$ belonging to S_0. Now the expected error associated with approximating $G(\underline{X})$ by the first $(N + 1)$ terms of Equation A.3.1 is given by

$$E[R_N]_\alpha = \frac{1}{(N+1)!} E[d^{N+1}_\alpha G(\underline{X} - \underline{\mu})] \qquad \text{(A.3.2)}$$

or letting $k = N + 1$

$$E[R_N]_\alpha = \frac{1}{k!} E\Big[\sum_{k_1+\ldots+k_n=k} \binom{k}{k_1 \ldots k_n} (X_1 - \mu_1)^{k_1} \ldots (X_n - \mu_n)^{k_n}$$

$$\frac{\partial^k G(\alpha)}{\partial X_1^{k_1} \ldots \partial X_n^{k_n}}\Big] \qquad \text{(A.3.3)}$$

or

$$E[R_N]_\alpha = \frac{1}{k!} \sum_{k_1+\ldots+k_n=k} \binom{k}{k_1 \ldots k_n} E[(X_1 - \mu_1)^{k_1} \ldots (X_n - \mu_n)^{k_n}] \cdot$$

$$\frac{\partial^k G(\underline{\alpha})}{\partial X_1^{k_1} \ldots \partial X_n^{k_n}} \qquad \text{(A.3.4)}$$

Now $E[R_N]$ could be found if the value of $\underline{\alpha}$ were known or an upper bound could be estimated for $|E(R_N)|$ if the point β belonging to S_0 could be

evaluated such that

$$|E[R_N]_\alpha| \leq |E[R_N]_\beta| \tag{A.3.5}$$

As would be expected, evaluating β is a very difficult process except for very simple functions and small n. However, one needs to estimate $E[R_N]$ so that some simple measure of the approximate error involved in a given Taylor Series Approximation can be given. Now $\underline{\alpha}$ is a particular value of $\underline{X}$, and so for the estimation of $E[R_N]_\alpha$ suppose one evaluates $E[R_N]_\alpha$ at $\underline{\alpha}$ equal to the expected value of $\underline{X}$; i.e., $\underline{\alpha} = \underline{\mu}$. Therefore, one obtains $\overset{*}{E}[R_N]$ as an estimate of the error associated with approximating G(x) by the first (N + 1) terms of its Taylor's Series expansion where

$$\overset{*}{E}[R_N] = E[R_N]_\alpha \text{ for } \underline{\alpha} = \underline{\mu} \tag{A.3.6}$$

Consider for example expanding $G(\underline{X})$ about $\underline{\mu}$ in a first order Taylor series expansion as is used predominately throughout this thesis

$$G(\underline{X}) = G(\underline{\mu}) + \sum_{i=1}^{n} \frac{\partial G}{\partial X_i} \Big|_{\underline{X}=\underline{\mu}} (X_i - \mu_i) \tag{A.3.7}$$

Now the estimate of the expected error involved in approximating $E[G(\underline{X})]$ by taking the expected value of Equation A.3.7 can be evaluated as

$$\overset{*}{E}[R_1] = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 G}{\partial X_i \partial X_j} \Big| \text{Cov}(X_i, X_j) \tag{A.3.8}$$

which is simply the expected value of the next term in the series.

Now suppose one considers the error associated with approximating Var(G(X)) using the Nth order approximation of $G(\underline{X})$. For notational

convenience, let G_N denote the first (N + 1) terms of G(X) in Equation A.3.1; that is, G_N equals the Nth order approximation of $G(\underline{X})$. Then one can write

$$\text{Var } G(X) = E\{[G_N + [R_N]_\alpha - E(G_N + [R_N]_\alpha)]^2\} \tag{A.3.9a}$$

or

$$\text{Var } G(X) = E\{[(G_N - E(G_N)) + ([R_N]_\alpha - E[R_N]_\alpha)]^2\} \tag{A.3.9b}$$

which in turn can be evaluated as

$$\text{Var } G(X) = \text{Var}(G_N) + \text{Var}[R_N]_\alpha + 2\text{Cov}[G_N[R_N]_\alpha] \tag{A.3.10}$$

which is denoted now as

$$\text{Var } G(\underline{X}) = \text{Var } (G_N) + V_{N_\alpha} \tag{A.3.11}$$

Now $\text{Var}(G_N)$ denotes the approximation of Var G(X) by assuming the Nth order approximation of $G(\underline{X})$. The remaining term on the right hand side of Equation A.3.11, namely V_{N_α}, represents the error associated with approximating $\text{Var}(G_X)$ by $\text{Var}(G_N)$, where the sensitivity coefficients associated with R_N and thus with V_{N_α} are evaluated at the point $\underline{\alpha}$ belonging to S_0. If a point β_2 belonging to S_0 could be found where

$$|V_{N_\alpha}| \leq |V_{N\beta_2}| \tag{A.3.12}$$

an upper bound on $|V_{N_\alpha}|$ could be calculated. However, the evaluation of β_2 is a very difficult process except for simple functions and small n. However, one does need to estimate V_{N_α} so that some simple measure of the approximate error associated with employing $\text{Var}(G_N)$ for Var G(X) can be given. As with $E[R_N]_\alpha$ suppose one evaluates V_{N_α} at $\underline{\alpha}$ equal to the

expected value of $\underline{X}$; i.e., $\underline{\alpha} = \underline{\mu}$. Thus one obtains V_N^* as an estimate of V_{N_α} where

$$V_N^* = V_{N_\alpha} \qquad \text{for } \underline{\alpha} = \underline{\mu} \tag{A.3.13}$$

One should note that for $\underline{\alpha} = \underline{\mu}$, R_N is simply the next term in the series approximation of $G(\underline{X})$ and V_N^* is the sum of the terms added to $\text{Var}(G_N)$ by assuming the next higher order approximation of $G(\underline{X})$; i.e.,

$$|V_N^*| = |\text{Var}(G_{N+1}) - \text{Var}(G_N)| \tag{A.3.14}$$

In order to illustrate the validity of Equation A.3.14, consider the following: Let the (N+1)th order approximation of $G(\underline{X})$ be denoted by

$$G_{N+1} = G_N + T_{N+1} \tag{A.3.15}$$

where from Equation A.3.1

$$T_{N+1} = \frac{1}{(N+1)!} \, d_\mu^{N+1} \, G(\underline{X} - \underline{\mu}) \tag{A.3.16}$$

Now T_{N+1} is precisely $[R_N]_{\underline{\alpha}=\underline{\mu}}$ (A.3.17)

Therefore the (N+1)th order approximation of $\text{Var}(G(\underline{X}))$ can be written as

$$\text{Var}(G_{N+1}) = E[(G_{N+1} - E(G_{N+1}))^2] = E[(G_N + [R_N]_{\underline{\alpha}=\underline{\mu}} - E(G_N) - E[R_N]_{\underline{\alpha}=\underline{\mu}})^2] \tag{A.3.18}$$

or

$$\text{Var}(G_{N+1}) = E\{[(G_N - E(G_N)) + ([R_N]_{\underline{\alpha}=\underline{\mu}} - E[R_N]_{\underline{\alpha}=\underline{\mu}})]^2\} \tag{A.3.19}$$

$$\mathrm{Var}(G_{N+1}) = E[(G_N - E(G_N))^2] + E[([R_N]_{\underline{\alpha=\mu}} - E[R_N]_{\underline{\alpha=\mu}})^2]$$

$$+ 2E[(G_N - E(G_N))([R_N]_{\underline{\alpha=\mu}} - E[R_N]_{\underline{\alpha=\mu}})] \qquad \text{(A.3.20)}$$

or

$$\mathrm{Var}(G_{N+1}) = \mathrm{Var}\ G_N + \mathrm{Var}[R_N]_{\underline{\alpha=\mu}} + 2\ \mathrm{Cov}[G_N,\ [R_N]_{\underline{\alpha=\mu}}] \qquad \text{(A.3.21)}$$

However, from Equations A.3.10 and A.3.13

$$V_N^* = \mathrm{Var}[R_N]_{\underline{\alpha=\mu}} + 2\ \mathrm{Cov}[G_N,\ [R_N]_{\underline{\alpha=\mu}}] \qquad \text{(A.3.22)}$$

so that Equation A.3.21 can be written as follows

$$\mathrm{Var}(G_{N+1}) = \mathrm{Var}(G_N) + V_N^* \qquad \text{(A.3.23)}$$

Equation A.3.14 now follows directly from Equation A.3.23. Thus V_N^* is an estimate of the error associated with $\mathrm{Var}(G(\underline{X}))$ due to using the Nth order approximation of $G(\underline{X})$. For the purpose of this thesis, the following criterion on N was chosen

$$\text{Choose N such that } |V_N^*| \leq .05 \qquad \text{(A.3.24)}$$

For the examples of Section 3.2 and 3.3, first order approximations were employed as suitable approximations with reference to the above criterion on N.

APPENDIX B

VARIANCE OF $\underline{S}_x$

B.1 Introduction. In this Appendix a detailed derivation of the covariance associated with $\underline{S}_x$ is presented, where $\underline{S}_x$ is defined in Chapter III to be the standard deviation vector associated with $\underline{X}$, the weapon impact point. Since the variance related to the range components of $\underline{S}_x$; namely, S_{ar} and S_{br} are derived in an analogous manner as those related to deflection components S_{ad} and S_{bd}, a description of the calculations of the estimates of $Var(S_{ar})$ and $Var(S_{br})$ is made and then the results are appropriately modified to provide $Var(S_{ad})$ and $Var(S_{bd})$. It should be pointed out that the equations modeling S_{ar}, S_{ad}, S_{br}, and S_{bd} which are used in this Appendix have been empirically derived from actual weapon delivery tests and correspond to the models assumed in the Stick Bomb Delivery Model.

B.2 Range Impact Point Model. Consider for now the range component of impact point. Assume: (Capital Letters = Random Variables, small letters = parameters).

$$X_r = \text{Range Impact Point} = \text{R.V.} \tag{B.2.1}$$

$$X_r \sim N(R, S_{br}^2) \tag{B.2.2}$$

$$R = \text{Actual Aimpoint} = \text{R.V.} \tag{B.2.3}$$

$$R \sim N(r_0, S_{ar}^2) \tag{B.2.4}$$

where

r_0 = intended aimpoint distance from release (in range). At present no specific distribution will be assumed for S^2_{ar} and S^2_{br}; but, we will assume the following:

$$E[S^2_{br}] = \sigma^2_{br} = \text{unknown parameter; ballistic variance} \qquad \text{(B.2.5a)}$$

$$V[S^2_{br}] = \alpha^2_{br} = \text{unknown parameter; variance of } S^2_{br} \qquad \text{(B.2.5b)}$$

$$E[S^2_{ar}] = \sigma^2_{ar} = \text{unknown parameter; aiming variance} \qquad \text{(B.2.6a)}$$

$$V[S^2_{ar}] = \alpha^2_{ar} = \text{unknown parameter; variance of } S^2_{ar} \qquad \text{(B.2.6b)}$$

thus

$$E[X_r] = E[E[X_r|R, S^2_{br}]] = E[R] = r_0 \qquad \text{(B.2.7a)}$$

$$Var(X_r) = Var[E[X_r|R, S^2_{br}]] + E[Var[X_r|R, S^2_{br}]] \qquad \text{(B.2.7b)}$$

$$Var(X_r) = Var[R] + E[S^2_{br}] \qquad \text{(B.2.9c)}$$

$$Var(X_r) = Var[E[R|S^2_{ar}]] + E[Var[R|S^2_{ar}]] + E[S^2_{br}] \qquad \text{(B.2.7d)}$$

$$Var(X_r) = 0 + \sigma^2_{ar} + \alpha^2_{br} \qquad \text{(B.2.7e)}$$

In summary then

$$E[X_r] = r_0 \qquad \text{(B.2.8)}$$

$$Var[X_r] = \sigma^2_{ar} + \sigma^2_{br} \qquad \text{(B.2.9)}$$

$$SD[X_r] = \sqrt{\sigma^2_{ar} + \sigma^2_{br}} = \text{standard deviation of } X_r$$

Assume a model for S_{ar} of the form which corresponds to that used in the Stick Bomb Program and denoted by the function S_1 of Equation 3.2.10a.

$$S_{ar} = \sqrt{K_1^2 t_0^2 + K_2^2 \frac{sr_0^2}{ha_0^2}} \qquad \text{(B.2.10)}$$

where

t_0 = calculated time of fall corresponding to intended release conditions associated with hitting r_0.

sr_0 = calculated slant range for intended release conditions.

ha_0 = calculated harp angle for intended release conditions.

k_1 = random variable whose uncertainty is assumed to partially compensate for error between actual release conditions and intended release conditions; in particular, to compensate for errors in velocity due to miscalculation of wind velocity.

k_2 = random variable whose uncertainty is assumed to compensate (along with K_1) for errors between actual release conditions and intended release conditions.

NOTE: A model for S_{br} will be discussed later in this appendix.

Consider the model of Equation B.2.10 further. Suppose a pilot desires to drop a weapon or a stick of weapons on a particular target. From the physics of the trajectory problem, his uprange release distance can be calculated and corresponds to his intended range aimpoint distance r_0. Now, associated with r_0 are a set of intended release conditions $y_1, y_2, \ldots, y_\phi$ that must be met in order to release the weapon actually aimed at r_0. Many factors influence the values of $y_1, y_2, \ldots, y_\phi$ such that on any particular run errors occur between the actual values obtained, call them $y_1^*, y_2^*, \ldots, y_\phi^*$, and the intended values

of the y's. If we assume (which we do at this point) that these errors are neither positively nor negatively biased, we can treat the intended y values $y_1, y_2, \ldots, y_\phi$ as merely the expected values of the basic underlying random variables $Y_1, Y_2, \ldots, Y_\phi$. Thus we see that $y_1^*, y_2^*, \ldots, y_\phi^*$) is simply a sample point or an outcome in the space defined by $Y_1, Y_2, \ldots, Y_\phi$.

Now suppose one defines the following random variables as functions of $Y_1, Y_2, \ldots, Y_\phi$.

$$T = g_1(Y_1, Y_2, \ldots, Y_\phi) = \text{time of fall of weapon} \qquad \text{(B.2.11a)}$$

$$SR = g_2(Y_1, Y_2, \ldots, Y_\phi) = \text{slant range at weapon release} \qquad \text{(B.2.11b)}$$

$$Ha = g_3(Y_1, Y_2, \ldots, Y_\phi) = \text{sine of angle between slant range at release and horizontal (Harp angle)} \qquad \text{(B.2.11c)}$$

For a set of intended release conditions $y_1, y_2, \ldots, y_\phi$, the following is obtained

$$E[T|Y_1 = y_1, Y_2 = y_2, \ldots, Y_\phi = y_\phi] = t_0 \tilde{=} g_1(y_1, y_2, \ldots, y_\phi) \qquad \text{(B.2.12a)}$$

$$E[SR|Y_1 = y_1, Y_2 = y_2, \ldots, Y_\phi = y_\phi] = sr_0 \tilde{=} g_2(y_1, y_2, \ldots, y_\phi) \qquad \text{(B.2.12b)}$$

$$E[H_a|Y_1 = y_1, Y_2 = y_2, \ldots, Y_\phi = y_\phi] = ha_0 \tilde{=} g_3(y_1, y_2, \ldots, y_\phi) \qquad \text{(B.2.12c)}$$

One sees then that t_0, sr_0, and ha_0 are related directly to the intended release conditions. Again referring to the model of Equation B.2.10

$$S_{ar} = \sqrt{K_1^2\, t_0^2 + K_2^2 \frac{sr_0^2}{ha_0^2}} \qquad (B.2.13)$$

one recalls from Chapter II that the random variables K_1 and K_2 are assumed to account for errors in the release conditions due to errors in aiming.

If one assumes for a series of test runs for set values of t_0, sr_0, ha_0, that is, for a series of sets of intended release conditions y's, that particular values of S_{ar} can be observed and K_1 and K_2 can be estimated from this data, say as $\hat{K}_1$ and $\hat{K}_2$ respectively, then the following estimator model is obtained

$$\hat{S}_{ar} = \sqrt{\hat{K}_1^2 t_0^2 + \hat{K}_2^2 \frac{sr_0^2}{ha_0^2}} \qquad (B.2.14)$$

Another way of looking at our estimator model is to let I_{ar} designate the information obtained from the data in order to compute $\hat{K}_1$ and $\hat{K}_2$. The designation I_{ar} would necessarily represent a random variable whose uncertainty is tied to that of the uncertainty of the data involved. Thus, one can rewrite $\hat{S}_{ar}$ as

$$\hat{S}_{ar} = E[S_{ar}|I_{ar}] = \sqrt{E[K_1|I_{ar}]^2\, t_0^2 + E[K_2|I_{ar}]^2 \frac{sr_0^2}{ha_0^2}} \qquad (B.2.15)$$

Where $E[S_{ar}|I_{ar}]$, $E[K_1|I_{ar}]$, and $E[K_2|I_{ar}]$ are random variables; i.e., functions of the random variable I_{ar}. For example, if $I_{ar} = i_{ar1}$ represents information from one particular set of data, then

$$E[S_{ar}|I_{ar} = i_{ar1}] = \sqrt{E[K_1|I_{ar} = i_{ar1}]^2 \, t_0^2 + E[K_2|I_{ar} = i_{ar1}]^2 \frac{sr_0^2}{ha_0^2}} \tag{B.2.16}$$

where $E[S_{ar}|I_{ar} = i_{ar1}]$, $E[K_1|I_{ar} = i_{ar1}]$, and $E[K_2|I_{ar} = i_{ar1}]$ are numbers computed with $I_{ar} = i_{ar1}$.

Now assuming that $E[K_1|I_{ar}] = \hat{K}_1$ and $E[K_2|I_{ar}] = \hat{K}_2$ are estimators for K_1 and K_2 derived from the data, the following unknown parameters can be associated with these estimators

$$k_1 = E[\hat{K}_1] = E[E[K_1|I_{ar}]] \tag{B.2.17a}$$

$$k_2 = E[\hat{K}_2] = E[E[K_2|I_{ar}]] \tag{B.2.17b}$$

$$\sigma^2_{\hat{K}_1\hat{K}_1} = Var[\hat{K}_1] \tag{B.2.17c}$$

$$\sigma^2_{\hat{K}_2\hat{K}_2} = Var[\hat{K}_2] \tag{B.2.17d}$$

$$\sigma^2_{\hat{K}_1\hat{K}_2} = cov[\hat{K}_1, \hat{K}_2] \tag{B.2.17e}$$

Recall the estimator model

$$\hat{S}_{ar} = E[S_{ar}|I_{ar}] = \sqrt{\hat{K}_1^2 \, t_0^2 + \hat{K}_2^2 \frac{sr_0^2}{ha_0^2}} \tag{B.2.18}$$

Expanding $E[S_{ar}|I_{ar}]$ about k_1 and k_2 in a first order Taylor series expansion, we have

$$\hat{S}_{ar} = SQAR + \frac{k_1 t_0^2}{SQAR}(\hat{K}_1 - k_1) + \frac{k_2 sr_0^2(\hat{K}_2 - k_2)}{ha_0^2 SQAR} \qquad (B.2.19)$$

where $SQAR = \sqrt{k_1^2 t_0^2 + \frac{k_2^2 sr_0^2}{ha_0^2}}$

so that if we take the expected value of Equation B.2.15 we have

$$E[E(S_{ar}|I_{ar})] = E\{\hat{S}_{ar}\} \cong \sqrt{k_1^2 t_0^2 + k_2^2 \frac{sr_0^2}{ha_0^2}} \qquad (B.2.20)$$

Furthermore defining

$$Var\{E(S_{ar}|I_{ar})\} = E[\{S_{ar}|I_{ar}) - E(E(S_{ar}|I_{ar}))\}^2] \qquad (B.2.21)$$

we have

$$Var\{E(S_{ar}|I_{ar})\} = Var(\hat{S}_{ar}) \cong \frac{k_1^2 t_0^2}{SQAR^2} Var(\hat{K}_1) + \frac{k_2^2 sr_0^4}{ha_0^4 SQAR^2} Var(\hat{K}_2)$$

$$+ \frac{2t_0^2 sr_0^2 k_1 k_2}{ha_0^2 SQAR^2} Cov(\hat{K}_1, \hat{K}_2) \qquad (B.2.22)$$

where

$$SQAR = \sqrt{k_1^2 t_0^2 + \frac{k_2^2 sr_0^2}{ha_0^2}}$$

or

$$Var\{\hat{S}_{ar}\} = \frac{k_1^2 t_0^4}{SQAR^2} \sigma^2_{\hat{K}_1\hat{K}_1} + \frac{k_2^2 sr_0^4}{ha_0^4 SQAR^2} \sigma^2_{\hat{K}_2\hat{K}_2} + \frac{2t_0^2 sr_0^2 k_1 k_2}{ha_0^2 SQAR^2} \sigma^2_{\hat{K}_1\hat{K}_2} \qquad (B.2.23)$$

As has been pointed out before, the parameters k_1, k_2, $\sigma^2_{\hat{K}_1\hat{K}_1}$, $\sigma^2_{\hat{K}_2\hat{K}_2}$, and $\sigma^2_{\hat{K}_1\hat{K}_2}$ are not known and cannot be calculated exactly. However, for any given set of data corresponding, for example, to $I_{ar} = i_{arj}$, one could calculate a particular value of k_1 and k_2 for that set of data; say,

$$k^*_{1_j} = E[\hat{K}_1 | I_{ar} = i_{arj}] \tag{B.2.24a}$$

$$k^*_{2_j} = E[\hat{K}_2 | I_{ar} = i_{arj}] \tag{B.2.24b}$$

One could also calculate particular values for $\sigma^2_{\hat{K}_1\hat{K}_1}$ and $\sigma^2_{\hat{K}_1\hat{K}_2}$ respectively as,

$$\sigma^{2*}_{\hat{K}_1\hat{K}_{1j}} = \text{var}(\hat{K}_1 | I_{ar} = i_{arj}) \tag{B.2.25a}$$

$$\sigma^{2*}_{\hat{K}_2\hat{K}_{2j}} = \text{var}(\hat{K}_2 | I_{ar} = i_{arj}) \tag{B.2.25b}$$

$$\sigma^{2*}_{\hat{K}_1\hat{K}_{2j}} = \text{cov}(\hat{K}_1, \hat{K}_2 | I_{ar} = i_{arj}) \tag{B.2.25c}$$

For example, one might employ standard regression techniques to calculate $k^*_{1_j}$ and $k^*_{2_j}$ as estimates of regression coefficients associated with the $\hat{S}_{ar}$ model of Equation B.2.10. Weapons could be dropped for varying intended values of t_0, sr_0, and ha_0 and sample values of S_{ar} could be obtained for the calculation of $k^*_{1_j}$ and $k^*_{2_j}$. Furthermore, from the regression analysis of the data obtained, covariances of the estimator $\hat{K}$'s could be estimated corresponding to $\sigma^{*2}_{\hat{K}_i\hat{K}_j}$.

Now, if one had n sets of data corresponding to $I_{ar} = i_{ar1}$, $I_{ar} = i_{ar2}$, . . ., $I_{ar} = i_{arn}$ one could form average estimates by averaging over j, namely,

$$k_1^* = E[\hat{K}_1 | I_{ar} = i_{ar1} \text{ U } i_{ar2} \text{ U } \ldots \text{ U} i_{arn}]$$

$$= E[\hat{K}_1 | I_{ar} = i_{ar}] \qquad \text{(B.2.26a)}$$

$$k_2^* = E[\hat{K}_2 | I_{ar} = i_{ar1} \text{ U } i_{ar2} \text{ U } \ldots \text{ U} i_{arn}]$$

$$= E[\hat{K}_2 | I_{ar} = i_{ar}] \qquad \text{(B.2.26b)}$$

$$\sigma^{*2}_{\hat{K}_1\hat{K}_1} = \text{Var}[\hat{K}_1 | I_{ar} = i_{ar1} \text{ U } i_{ar2} \text{ U } \ldots \text{ U} i_{arn}]$$

$$= \text{Var}[\hat{K}_1 | I_{ar} = i_{ar}] \qquad \text{(B.2.26c)}$$

$$\sigma^{*2}_{\hat{K}_2\hat{K}_2} = \text{Var}[\hat{K}_2 | I_{ar} = i_{ar} \text{ U } i_{ar} \text{ U } \ldots \text{ U} i_{arn}]$$

$$= \text{Var}[\hat{K}_2 | I_{ar} = i_{ar}] \qquad \text{(B.2.26d)}$$

$$\sigma^{*2}_{\hat{K}_1\hat{K}_2} = \text{cov}[\hat{K}_1, \hat{K}_2 | I_{ar} = i_{ar1} \text{ U } i_{ar2} \text{ U } \ldots \text{ U} i_{arn}]$$

$$= \text{cov}[\hat{K}_1, \hat{K}_2 | I_{ar} = i_{ar}] \qquad \text{(B.2.26e)}$$

where i_{ar} represents the total information obtained from the n sets of data. Thus, k_1^*, k_2^*, $\sigma^{*2}_{\hat{K}_1\hat{K}_1}$, $\sigma^{*2}_{\hat{K}_2\hat{K}_2}$, and $\sigma^{*2}_{\hat{K}_1\hat{K}_2}$ are the estimates of the parameters k_1, k_2, $\sigma^{2}_{\hat{K}_1\hat{K}_1}$, $\sigma^{2}_{\hat{K}_2\hat{K}_2}$, and $\sigma^{2}_{\hat{K}_1\hat{K}_2}$ given all the available data. Suffice it to say that if one had one set of data or twenty sets of data, the final resulting estimates are termed k_1, k_2, $\sigma^{2}_{\hat{K}_1\hat{K}_1}$, $\sigma^{2}_{\hat{K}_1\hat{K}_2}$ so that the notation will be consistent. The estimate of $E[\hat{S}_{ar}]$ can now be calculated as

$$E[\hat{S}_{ar} | I_{ar} = i_{ar}] = \sqrt{k_1^{*2} t_0^2 + \frac{k_2^{*2} sr_0^2}{ha_0^2}} \qquad \text{(B.2.27)}$$

This value, Equation B.2.27, is assumed to be the actual output of the Stick Bomb Program for $(\frac{REP}{.6745})$ with k_1^* and k_2^* as normally given. k_1^* and k_2^* are assumed to have been calculated from a set or sets of data corresponding to $I_{ar} = i_{ar}$. Since we assume that the Stick Bomb Output corresponding to Equation 2.2.7 is a conditional expected value of $\hat{S}_{ar}$, conditioned on all available data, a measure of the uncertainty associated with $\hat{S}_{ar}$ is needed which is related to the data; that is, one needs to calculate an estimate of $\hat{S}_{ar}$ based on the data that is available. If $\sigma^2_{\hat{K}_1\hat{K}_1}$, $\sigma^2_{\hat{K}_2\hat{K}_2}$, and $\sigma^2_{\hat{K}_1\hat{K}_2}$ were known, they could be substituted into Equation B.2.23 and $var(\hat{S}^2_{ar})$ calculated directly. Unfortunately, these parameters are not known; but, as has been pointed out previously, these parameters can be estimated from the available data. If these estimates are substituted into Equation B.2.23 for the parameters which they estimate, an estimate of the $var(\hat{S}_{ar})$ results

$$\hat{Var}\{\hat{S}_{ar}\} = Var(\hat{S}_{ar} | I_{ar} = i_{ar}) = \frac{k_1^{*2} t_0^4}{SQAT^2} \hat{\sigma}^{*2}_{\hat{K}_1\hat{K}_1} + \frac{k_2^{*2} sr_0^4}{ha_0^2 SQAT^2}$$

$$+ \frac{2t_0^2 sr_0^2 k_1^* k_2^*}{ha_0 SQAT^2} \hat{\sigma}^{*2}_{\hat{K}_1\hat{K}_2} \qquad \text{(B.2.28)}$$

where

$$SQAT = \sqrt{k_1^{*2} t_0^2 + \frac{k_2^{*2} sr_0^2}{ha_0^2}}$$

Equation B.2.28 is chosen to be the measure of the uncertainty associated with $\hat{S}_{ar}$ and, as one can observe, it is directly related to the estimates of the variances and covariance associated with $\hat{K}_1$ and $\hat{K}_2$. To be more specific, suppose one relates the results of this derivation to the Stick Bomb Program.

$\hat{K}_1 = E[K_1 | I_{ar}]$ = estimator of aiming error coefficient (AAA) given set (or sets) of data corresponding to I_{ar}.

$\hat{K}_2 = E[K_2 | I_{ar}]$ = estimator of aiming error coefficient (BBB) given set (or sets) of data corresponding to I_{ar}.

$k_1^* = E[K_1 | I_{ar} = i_{ar}]$ = particular value of $\hat{K}_1$ for set of data corresponding to $I_{ar} = i_{ar}$. This is particular value of AAA, say a.

$k_2^* = E[\hat{K}_2 | I_{ar} = i_{ar}]$ = particular value of $\hat{K}_2$ for set of data corresponding to $I_{ar} = i_{ar}$. This is particular value of BBB, say b.

$E[\hat{S}_{ar} | I_{ar} = i_{ar}]$ = particular value of $\hat{S}_{ar}$ using k_1^* and k_2^* as particular values of $\hat{K}_1$ and $\hat{K}_2$ respectively. As a computer output, this corresponds to the variable $(\frac{REP}{.6745})$.

$Var[\hat{S}_{ar} | I_{ar} = i_{ar}]$ = measure of uncertainty of $\hat{S}_{ar}$ given data corresponding to $I_{ar} = i_{ar}$. This obtained as output of VPK program.

Thus far only errors due to aiming have been considered. However, even if one could aim without error; i.e., R ≠ R.V., one would still encounter error in the range impact point X_r due to ballistic dispersion. Assume a model of the form which corresponds to that used by the Stick Bomb Delivery Model and is denoted by Equation 3.2.10c as S_3.

$$S_{br} = K_4 \frac{sr_0}{ha_0} \tag{B.2.29}$$

where K_4 = random variable whose uncertainty is assumed to account for errors between actual aimpoint and actual impact point.

Since the exact slant range and harp angle are unknown for any particular weapon release, the model for the ballistic variance is assumed to be a function of sr_0 and ha_0 which, as desired values of SR and Ha, are assumed to represent average slant range and harp angle involved.

Again one assumes that particular values of S_{br} can be observed for a series of test runs for set values of sr_0 and ha_0 and K_4 can be estimated, say $\hat{K}_4$. Thus the following estimator ballistic model is obtained

$$\hat{S}_{br} = \hat{K}_4 \frac{sr_0}{ha_0} \tag{B.2.30}$$

or letting I_{br} represent the information from the data now involved,

$$\hat{S}_{br} = E[S_{br} | I_{br}] = E[K_4 | I_{br}] \frac{sr_0}{ha_0} \tag{B.2.31}$$

Following the same procedure as was followed for the aiming error model, one acquires

$$E[\hat{S}_{br} | I_{br} = i_{br}] = E[\hat{K}_4 | I_{br} = i_{br}] \frac{sr_0}{ha_0} = k_4^* \frac{sr_0}{ha_0} \tag{B.2.32}$$

and

$$\mathrm{Var}[\hat{S}_{br} | I_{br} = i_{br}] = \frac{sr_0^2}{ha_0^2} \; \overset{*2}{\sigma}_{\hat{K}_4\hat{K}_4} \tag{B.2.33}$$

where

k_4^* = $E[\hat{K}_4 | I_{br} = i_{br}]$ = particular value of $\hat{K}_4$ for set of data corresponding to $I_{br} = i_{br}$. This is particular value of (DIS).

$\overset{*2}{\sigma}_{\hat{K}_4\hat{K}_4}$ = $\mathrm{Var}[\hat{K}_4 | I_{br} = i_{br}]$ = estimate of the variance of $\hat{K}_4$ given $I_{br} = i_{br}$.

$E[\hat{S}_{br} | I_{br} = i_{br}]$ = particular value of $\hat{S}_{br}$ using k_4^* as particular value of K_4. As a computer output, this corresponds to the variable (SIGY).

$\mathrm{Var}[\hat{S}_{br} | I_{br} = i_{br}]$ = measure of uncertainty of $\hat{S}_{br}$ given data corresponding to $I_{br} = i_{br}$. This obtained as output of VPK Program.

B.3 Deflection Impact Point Model. Now that estimates have been derived for the variance associated with S_{ar} and S_{br} in Equation B.2.28 and B.2.33 respectively, the associated derivation must be extended to provide estimates of the variances associated with S_{ad} and S_{bd}. The estimator model related to S_{ad} corresponding to that of Equation B.2.14 for S_{ar} is as follows:

$$\hat{S}_{ad} = \sqrt{\hat{K}_1^2 t_0^2 + \hat{K}_3^2 sr_0^2} \tag{B.3.1}$$

Equation B.3.1 corresponds to the functional notation of Equation 3.2.10b. If the form of $\hat{S}_{ad}$ is compared with that of $\hat{S}_{ar}$, one can observe that $\hat{K}_3$ replaces $\hat{K}_2$ and the term ha_0^2 is absent in the estimator model $\hat{S}_{ad}$. The ha_0^2 term was necessary in the range computations since the error components are originally calculated in a plane perpendicular to slant range and must be divided by the sine of the harp angle (ha_0) in order to express the associated error in the R_e - D_e plane. Following parallel notation to that used in Section B.2 with only subscripts changed to correspond to deflection components, a representation for $\hat{Var}(\hat{S}_{ad})$ can be written corresponding to Equation B.2.28 as

$$\hat{Var}(\hat{S}_{ad}) \tilde{=} \frac{k_1^{*2} t_0^4}{SQAD^2} \hat{\sigma}^{*2}_{K_1K_1} + \frac{k_3^{*2} sr_0^4}{SQAD^2} \hat{\sigma}^{*2}_{K_3K_3} + \frac{2t_0^2 sr_0^2 k_1^* k_3^*}{SQAD^2} \hat{\sigma}^{*2}_{K_1K_3} \qquad (B.3.2)$$

where

$$SQAD = \sqrt{k_1^{*2} t_0^2 + k_3^{*2}\, sr_0^2}$$

Finally the estimator model related to S_{bd} corresponding to Equation B.2.30 for $\hat{S}_{br}$ is given as

$$\hat{S}_{bd} = \hat{K}_4\, sr_0 \qquad (B.3.3)$$

where again the absence of the ha_0^2 term is noted. Equation B.3.3 represents the functional form of Equation 3.2.10d. Following the derivation leading to Equation B.2.33, one can write a representation for $\hat{Var}(S_{bd})$ directly as

$$\hat{Var}(S_{bd}) \tilde{=} sr_0^2 \hat{\sigma}^{*2}_{K_4K_4} \qquad (B.3.4)$$

Now relating the models of Equation B.3.2 and B.3.4 to the Stick Bomb Delivery Model directly as was done for the range components of Section B.2, one obtains

$\hat{K}_3 = E[K_3|I_{ad}]$ = estimator of aiming error coefficient (DDD) given set (or sets) of data corresponding to I_{ad}.

$\hat{K}_4 = E[K_4|I_{bd}]$ = estimator of ballistic error coefficient (DIS) given set (or sets) of data corresponding to I_{bd}.

$k_3^* = E[\hat{K}_3|I_{ad} = i_{ad}]$ = particular value of $\hat{K}_3$ for set of data corresponding to $I_{ad} = i_{ad}$. This is particular value of DDD, say d.

$k_4^* = E[\hat{K}_4|I_{bd} = i_{bd}]$ = particular value of $\hat{K}_4$ for set of data corresponding to $I_{bd} = i_{bd}$. This is particular value of (DIS). It should be noted that $I_{bd} = i_{br}$ since only one coefficient needs to be determined.

$\sigma^{*2}_{\hat{K}_3\hat{K}_3} = Var[\hat{K}_3|I_{ad} = i_{ad}]$ = estimate of the variance of $\hat{K}_3$ given $I_{ad} = i_{ad}$.

$E[\hat{S}_{ad}|I_{ad} = i_{ad}]$ = particular value of $\hat{S}_{ad}$ using k_1^* and k_3^* as particular values of K_1^* and K_3^* respectively. As a computer output, this corresponds to the variable $(\frac{DEP}{.6745})$.

$E[\hat{S}_{bd} | I_{bd} = i_{bd}]$ = particular value of $\hat{S}_{bd}$ using k_4^* as particular value of $\hat{K}_4$. As a computer output, this corresponds to the variable (SIGX).

$Var[\hat{S}_{ad} | I_{ad} = i_{ad}]$ = measure of uncertainty of $\hat{S}_{ad}$ given data corresponding to $I_{ad} = i_{ad}$. This is obtained as output of VPK Program.

$Var[\hat{S}_{bd} | I_{bd} = i_{bd}]$ = measure of uncertainty of $\hat{S}_{bd}$ given data corresponding to $I_{bd} = i_{bd}$. This obtained as output of VPK Program.

Since aiming and ballistic errors are assumed to be independent in both range and deflection for the model associated with the examples of this thesis, it is not necessary to calculate the associated covariance terms. If this assumption had not been made concerning independence, straight forward calculations of covariance terms could have been made by following the techniques leading to Equation 2.4.11 with $\underline{\hat{V}}$ replaced by $\underline{\hat{S}}_x$.

Equations B.2.28, B.2.33, B.3.2, and B.3.4 have thus presented estimates of $Var(\hat{S}_{ar})$, $Var(\hat{S}_{ad})$, $Var(\hat{S}_{br})$, and $Var(\hat{S}_{bd})$ respectively and linked then to the uncertainties associated with the estimation of the aiming and ballistic error coefficients obtained from empirical data.

APPENDIX C

SECOND ORDER DERIVATION FOR EXAMPLE 1

C.1 Introduction. The purpose of this appendix is to present a parallel derivation of the $\hat{\text{Var}}\ (P_K)$ for the example of Section 3.2 assuming that second order terms are required for the Taylor Series Expansion used. In order that the first order and second order approximations can be compared equation for equation, the equation numbers in Section 3.2 are listed directly below their corresponding equations of this appendix.

C.2 Variance of P_K. Suppose one begins with the MRKP model of P_K as given in Equation 3.2.21.

$$P_K = P_K(\underline{S}_x) = P_K(S_{ar}, S_{ad}, S_{br}, S_{bd}) \qquad \begin{matrix}(C.2.1)\\(3.2.21)\end{matrix}$$

Expanding P_K about $E[\underline{S}_x]$ in a second order Taylor Series expansion

$$P_K = P_K[E(\underline{S}_x)] + \sum_{i=1}^{4} \frac{\partial P_K}{\partial S_i}\Big|_{\underline{S}_x = E(\underline{S}_x)} [S_i - E(S_i)]$$

$$+ \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j}\Big|_{\underline{S}_x = E(\underline{S}_x)} [S_i - E(S_i)][S_j - E(S_j)] \qquad \begin{matrix}(C.2.2)\\(3.2.22)\end{matrix}$$

Taking the expected value of P_K as

$$E[P_K] \cong P_K[E(\underline{S}_x)] + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j}\Big|_{\underline{S}_x = E(\underline{S}_x)} \text{Cov}(S_i, S_j) \qquad \begin{matrix}(C.2.3)\\(3.2.23)\end{matrix}$$

$$\text{Var}[P_K] \approx E\{[P_K(E(\underline{S}_x)) + \sum_{i=1}^{4} \frac{\partial P_K}{\partial S_i} \Big|_{\underline{S}_x = E(\underline{S}_x)} (S_i - E(S_i))$$

$$+ \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j} \Big|_{\underline{S}_x = E(\underline{S}_x)} (S_i - E(S_i))(S_j - E(S_j))$$

$$- P_K(E(S_x)) - \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j} \Big|_{\underline{S}_x = E(\underline{S}_x)} \text{Cov}(S_i, S_j)]^2\} \qquad \text{(C.2.4)}$$

$$\text{Var}(P_k) \approx \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial P_K}{\partial S_i} \frac{\partial P_K}{\partial S_j} \Big|_{\underline{S}_x = E(\underline{S}_x)} \text{Cov}(S_i, S_j)$$

$$+ \frac{1}{4} \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{u=1}^{4} \sum_{v=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j} \frac{\partial^2 P_K}{\partial S_u \partial S_v} \Big|_{\underline{S}_x = E(\underline{S}_x)} E[(S_i - E(S_i))(S_j - E(S_j))$$

$$(S_u - E(S_u))(S_v - E(S_v))]$$

$$+ \frac{1}{4} \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{u=1}^{4} \sum_{v=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j} \frac{\partial^2 P_K}{\partial S_u \partial S_v} \Big|_{\underline{S}_x = E(\underline{S}_x)} \text{Cov}(S_i, S_j)\ \text{Cov}(S_u, S_v)$$

$$+ \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{u=1}^{4} \frac{\partial P_K}{\partial S_i} \frac{\partial^2 P_K}{\partial S_j \partial S_u} \Big|_{\underline{S}_x = E(\underline{S}_x)} E[(S_i - E(S_i))(S_j - E(S_j))(S_u - E(S_u))]$$

$$- \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{u=1}^{4} \frac{\partial P_K}{\partial S_i} \frac{\partial^2 P_K}{\partial S_j \partial S_u} \Big|_{\underline{S}_x = E(\underline{S}_x)} E(S_i - E(S_i))\ \text{Cov}(S_j, S_u)$$

$$- \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{u=1}^{4} \sum_{v=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j} \frac{\partial^2 P_K}{\partial S_u \partial S_v} \Big|_{\underline{S}_x = E(\underline{S}_x)} \text{Cov}(S_i, S_j)\ \text{Cov}(S_u, S_v) \qquad \text{(C.2.5)}$$

Combining and eliminating the fifth sum since $E(S_i - E(S_i)) = 0$, one obtains the following results:

$$\mathrm{Var}(P_K) = \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial P_K}{\partial S_i}\frac{\partial P_K}{\partial S_j}\Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_j)$$

$$+ \frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{u=1}^{4}\sum_{v=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j}\frac{\partial^2 P_K}{\partial S_u \partial S_v}\Big|_{\underline{S}_x = E(\underline{S}_x)} E[(S_i - E(S_i))$$

$$(S_j - E(S_j))(S_u - E(S_u))(S_v - E(S_v))]$$

$$- \frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{u=1}^{4}\sum_{v=1}^{4} \frac{\partial^2 P_K}{\partial S_i \partial S_j}\frac{\partial^2 P_K}{\partial S_u \partial S_v}\Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_j)\ \mathrm{Cov}(S_u, S_v)$$

$$+ \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{u=1}^{4} \frac{\partial P_k}{\partial S_i}\frac{\partial^2 P_k}{\partial S_j \partial S_u}\Big|_{\underline{S}_x = E(\underline{S}_x)} E[(S_i - E(S_i))$$

$$(S_j - E(S_j))(S_u - E(S_u))] \qquad \begin{matrix}(C.2.6)\\(3.2.24)\end{matrix}$$

It should be noted that Equation C.2.6 has been derived with no assumption made concerning the form or the distribution of any of the random variable components of $\underline{S}_x$. Furthermore, Equation C.2.6 represents a general model in the sense that the $\underline{S}_x$ components have not been assumed to be independent in this derivation.

Under the assumption pertaining to the MRKP model; namely, (1) the independence of the components of $\underline{S}_x$ and (2) the normality relationships

of Equation 3.2.20, one can reduce Equation C.2.6 for use in the Examples of Section 3.2. Consider

$$E[(S_i - E(S_i))(S_j - E(S_j))(S_u - E(S_u))(S_v - E(S_v))] \qquad (C.2.7)$$

For $i = j = u = v$

$$E[(S_i - E(S_i))^4] = 3\ Cov^2\ (S_i, S_i) \qquad (C.2.8)$$

For $i = j = u \neq v$

$$E[(S_i - E(S_i))^3(S_v - E(S_v))] = 0 \qquad (C.2.9)$$

For $i = j \neq u = v$

$$E[(S_i - E(S_i))^2(S_u - E(S_u))^2] = Cov(S_i, S_i)\ Cov(S_u, S_u) \qquad (C.2.10)$$

For $i = j \neq u \neq v$

$$E[(S_i - E(S_i))^2(S_u - E(S_u))(S_v - E(S_v))] = 0 \qquad (C.2.11)$$

For $i \neq j \neq u \neq v$

$$E[(S_i - E(S_i)(S_j - E(S_j))(S_u - E(S_u)(S_v - E(S_v))] = 0 \qquad (C.2.12)$$

Next consider

$$E[(S_i - E(S_i))(S_j - E(S_j))(S_u - E(S_u))] \qquad (C.2.13)$$

For $i = j = u$

$$E[(S_i - E(S_i))^3] = 0 \qquad (C.2.14)$$

For $i = j \neq u$

$$E[(S_i - E(S_i))^2(S_u - E(S_u))] = 0 \qquad (C.2.15)$$

For $i \neq j \neq u$

$$E[(Si - E(Si))(Sj - E(Sj))(Su - E(Su))]= 0 \qquad (C.2.16)$$

Thus Equation C.2.6 can be reduced under the assumption of the MRKP model to obtain

$$\mathrm{Var}(P_K) = \sum_{i=1}^{4} \left(\frac{\partial P_K}{\partial S_i}\right)^2 \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_j)$$

$$+ \frac{3}{4} \sum_{i=1}^{4} \left(\frac{\partial^2 P_K}{\partial S_i^2}\right) \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}^2(S_i, S_i)$$

$$+ \frac{1}{4} \sum_{i=1}^{4} \sum_{\substack{j=1 \\ i \neq j}}^{4} \frac{\partial^2 P_K}{\partial S_i^2} \frac{\partial^2 P_K}{\partial S_j^2} \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_i)\ \mathrm{Cov}(S_j, S_j)$$

$$+ \frac{1}{2} \sum_{i=1}^{4} \sum_{\substack{j=1 \\ i \neq j}}^{4} \left(\frac{\partial^2 P_K}{\partial S_i \partial S_j}\right)^2 \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_i)\ \mathrm{Cov}(S_j, S_j)$$

$$- \frac{1}{4} \sum_{i=1}^{4} \sum_{\substack{j=1 \\ i \neq j}}^{4} \left(\frac{\partial^2 P_K}{\partial S_i \partial S_j}\right)^2 \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_i)\ \mathrm{Cov}(S_j, S_j) \qquad \text{(C.2.17)}$$

which reduces to

$$\mathrm{Var}(P_K) = \sum_{i=1}^{4} \left(\frac{\partial P_K}{\partial S_i}\right)^2 \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_i)$$

$$+ \frac{1}{4} \sum_{i=1}^{4} \sum_{j=1}^{4} \left(\frac{\partial^2 P_K}{\partial S_i^2}\right)\left(\frac{\partial^2 P_K}{\partial S_j^2}\right) \Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_i)\ \mathrm{Cov}(S_j, S_j)$$

$$+\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\frac{\partial^2 P_K}{\partial S_i \partial S_j}\right)^2\Big|_{\underline{S}_x = E(\underline{S}_x)} \mathrm{Cov}(S_i, S_i)\ \mathrm{Cov}(S_j, S_j) \qquad (C.2.18)$$

Equation C.2.18 now corresponds to the second order approximation of $\mathrm{Var}(P_K)$ under the assumptions of the example of Section 3.2.

C.3 Variance of S_x. The $\mathrm{Var}(P_K)$ in Equation C.2.18 relates a measure of the uncertainty associated with P_K to that related to $\underline{S}_x$; i.e., $\mathrm{Cov}(S_i, S_i)$. Now the uncertainties associated with the error coefficients need to be related to $\mathrm{Cov}(S_i, S_i)$. Recalling the estimator model of Equation 3.2.25

$$\hat{\underline{S}}_x = \underline{S}(\underline{y}, \hat{\underline{K}}) \qquad (C.3.1)\ (3.2.25)$$

where

$$\hat{\underline{S}}_x = (\hat{S}_{ar}, \hat{S}_{ad}, \hat{S}_{br}, \hat{S}_{bd}) \equiv (S_1, S_2, S_3, S_4)$$

one can obtain a second order estimate of $\mathrm{Var}(\hat{S}_t)$, for t = 1, 2, 3, 4, in a manner analogous to that used to obtain $\mathrm{Var}(P_K)$ in Equation C.2.18. Defining

$$\overset{*}{\sigma}{}^2_{\hat{K}_i \hat{K}_j} = \text{estimate of } \mathrm{Cov}(\hat{K}_i, \hat{K}_j) \qquad (C.3.1a)$$

$$K4^*_{ijuv} = \text{estimate of } E[(\hat{K}_i - k_i)(\hat{K}_j - k_j)(\hat{K}_u - k_u)(\hat{K}_v - k_v)] \qquad (C.3.1b)$$

$$K3^*_{iju} = \text{estimate of } E[(\hat{K}_i - k_i)(\hat{K}_j - k_j)(\hat{K}_u - k_u)] \qquad (C.3.1c)$$

one can write

$$\hat{\mathrm{Var}}(S_t) \approx \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial S_t}{\partial \hat{K}_i}\frac{\partial S_t}{\partial \hat{K}_j}\Big|_{\underline{\hat{K}}=\underline{k}} \hat{\sigma}^{*2}_{\hat{K}_i\hat{K}_j}$$

$$+\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{u=1}^{4}\sum_{v=1}^{4} \frac{\partial^2 S_t}{\partial \hat{K}_i \partial \hat{K}_j}\frac{\partial^2 S_t}{\partial \hat{K}_u \partial \hat{K}_v}\Big|_{\underline{\hat{K}}=\underline{k}} K4^{*}_{ijuv}$$

$$-\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{u=1}^{4}\sum_{v=1}^{4} \frac{\partial^2 S_t}{\partial \hat{K}_i \partial \hat{K}_j}\frac{\partial^2 S_t}{\partial \hat{K}_u \partial \hat{K}_v}\Big|_{\underline{\hat{K}}=\underline{k}} \hat{\sigma}^{*2}_{\hat{K}_i\hat{K}_j}\,\hat{\sigma}^{*2}_{\hat{K}_u\hat{K}_v}$$

$$+\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{u=1}^{4} \frac{\partial S_t}{\partial \hat{K}_i}\frac{\partial^2 S_t}{\partial \hat{K}_j \partial \hat{K}_u}\Big|_{\underline{\hat{K}}=\underline{k}} K3^{*}_{iju} \qquad \text{(C.3.2)}$$

Now $\hat{\mathrm{Var}}(S_t)$ can be substituted for $\mathrm{Cov}(S_t, S_t)$ in Equation C.2.18 to provide an estimate of Var (P_k), say $\hat{\mathrm{Var}}(P_k)$. Thus a second order estimate of $\mathrm{Var}(P_1)$ is obtained which depends on estimates of the third and fourth multiple moments of $\underline{\hat{K}}$ as well as the covariance terms.

Equations C.2.18 and C.3.2 have been implemented by Program VPK which is discussed further in Appendix E. Suffice it to point out here that the inputs to VPK include all sensitivity coefficients related to the model as well as $\hat{\sigma}^{*2}_{\hat{K}_i\hat{K}_j}$, $K4^{*}_{ijuv}$, and $K3^{*}_{iju}$ for i, j, u, v equal to 1, 2, 3, and 4.

APPENDIX D

DERIVATION OF MULTIPLE ROUND KILL PROBABILITY MODEL

D.1 Introduction. The purpose of this Appendix is to present the Multiple Round Kill Probability Model as used in Section 3.2 for Example 1 corresponding to the functional representation $P_{KM}(\underline{S}_x)$ of Equation 3.2.25. The single weapon delivery model, i.e., $P_K(\underline{S}_x)$ of Equation 3.2.16 represents a special case of P_{KM} for which ΔR_i and ΔD_i are zero.

D.2 Basic Assumptions. Basic to the MRKP model are several assumptions. In particular, a pattern of λ weapons delivered is described by specifying a pattern reference point and a set of values $(\Delta R_i, \Delta D_i)$ for $i = 1, \ldots, \lambda$ which specify the relative positions of the λ weapons to the reference point. It is assumed that the pattern reference point of the released weapons is actually aimed at a point (R, D) although the intended aimpoint is (r_0, d_0). Further it is assumed that the aiming errors $(R - r_0)$ and $(D - d_0)$ are independent normal random variables with means zero and variances S^2_{ar} and S^2_{ad} respectively. The actual impact point of the ith weapon is specified by (X_{R_i}, X_{D_i}) and the associated ballistic errors $[X_{R_i} - (R + \Delta R_i)]$ and $[X_{D_i} - (D + D_i)]$ are assumed to be conditionally independent (given R and D) normal random variables with means zero and variances S^2_{br} and S^2_{bd} respectively. Thus one can summarize the assumption concerning R, D, X_{R_i}, and X_{D_i} as follows:

$$R \sim N(r_0, S_{ar}^2) \tag{D.2.1a}$$

$$D \sim N(d_0, S_{ad}^2) \tag{D.2.1b}$$

$$X_{R_i} \sim N(R + \Delta R_i, S_{br}^2) \tag{D.2.1c}$$

$$X_{D_i} \sim N(D + \Delta D_i, S_{bd}^2) \tag{D.2.1d}$$

where X_{R_i} and X_{D_i} are assumed conditionally independent.

Since the MRKP model evaluates P_{KM} as a function of particular values of $\underline{S}_x$, ΔR_i and ΔD_i, for $i = 1, \ldots, \lambda$, the following derivation is presented for

$$S_{ar} = \sigma_{ar} \tag{D.2.2a}$$

$$S_{ad} = \sigma_{ad} \tag{D.2.2b}$$

$$S_{br} = \sigma_{br} \tag{D.2.2c}$$

$$S_{bd} = \sigma_{bd} \tag{D.2.2d}$$

$$\Delta R_i = \alpha_{r_i} \qquad i = 1, \ldots, \lambda \tag{D.2.2e}$$

$$\Delta D_i = \alpha_{d_i} \qquad i = 1, \ldots, \lambda \tag{D.2.2f}$$

In addition, the MRKP model makes one strong assumption about the manner in which the weapons are delivered. In particular, the model assumes that the weapons are delivered along the range axis, which in turn lies parallel to one side of the rectangular vulnerable area R_v. Furthermore, the weapons are assumed to be traveling in approximately the same direction when they reach the target. When traveling in the direction of the flight of the weapons, one denotes the leading edge of

R_v as A_{R1}; the far edge, A_{R2}; the left side A_{D1}, and the right side, A_{D2}; that is A_{R1} and A_{R2} are measured in the range direction and A_{D1} and A_{D2}, the deflection direction.

D.3 Model Derivation. Basically the MRKP model is designed to compute the probability of accomplishing a desired degree of damage to a target by the attack of λ weapons. The probability is usually referred to as the probability of kill and corresponds to P_{KM} of Equation 3.2.25. For the MRKP model the following basic equation is employed.

$$P_{KM} = \text{Prob(at least 1 killing hit}|\lambda \text{ weapons released) Prob}(\lambda \text{ weapons delivered}) \quad \text{(D.3.1)}$$

where Prob(λ weapons delivered) corresponds to the system reliability RLBTY, which is the probability that λ weapons are delivered. It is assumed that either λ or zero weapons are delivered. One can thus write Equation D.3.1 as

$$P_{KM} = \text{Prob(at least 1 killing hit}|\lambda \text{ weapons released)*RLBTY} \quad \text{(D.3.2)}$$

Now denote the following

$$PKW = \text{Prob(at least 1 killing hit}|\lambda \text{ weapons released)} \quad \text{(D.3.3)}$$

which can be evaluated recalling the pattern dependence on (R, D) as

$$PKW = \int_{-\infty}^{\infty}{}_2 \text{ Prob(at least 1 killing hit}|(R, D) = (w_r, w_d)) \; f_{R,D}(w_r, w_d)\, dw_r\, dw_d \quad \text{(D.3.4)}$$

Now denoting

$$\text{PKRD} = \text{Prob(at least 1 killing hit}|\text{R, D)} \tag{D.3.5}$$

one can write

$$\text{PKRD} = 1 - \text{Prob(no killing hit}|\text{R, D)} \tag{D.3.6}$$

or

$$\text{PKRD} = 1 - \prod_{i=1}^{\lambda} \text{Prob(ith weapon not killing hit}|\text{R, D)} \tag{D.3.7}$$

and

$$\text{PKRD} = 1 - \prod_{i=1}^{\lambda} (1 - \text{Prob(ith weapon is killing hit}|\text{R, D)}) \tag{D.3.8}$$

But

$$\text{Prob(ith weapon kills}|\text{R, D)} = \text{PC} * \text{Prob(ith weapon hit } R_V|\text{R, D)} \tag{D.3.9}$$

where PC = Prob(ith weapon is killing hit|R, D and ith weapon hits R_V)

Let

$$PX_i(R, D) = \text{Prob(ith weapon hits } R_V|\text{R, D)} \tag{D.3.10}$$

so that recalling the assumption of independence for X_{R_i} and X_{D_i} one can write

$$PX_i(\text{RM D}) = \int_{A_{R_1}}^{A_{R_2}} \int_{A_{D_1}}^{A_{D_2}} f_{X_{R_i}|R}(\tau_{r_i}|R) f_{X_{D_i}|D}(\tau_{d_i}|D)\, d\tau_{r_i}\, d\tau_{d_i} \tag{D.3.11}$$

From Equations D.3.6 through D.3.11 one can write Equation D.3.4 as

$$PKW = \int_{-\infty}^{\infty} \left(1 - \prod_{i=1}^{\lambda} \left(1 - PC \int_{A_{R_1}}^{A_{R_2}} \int_{A_{D_1}}^{A_{D_2}} f_{X_{R_i}|R}(\tau_{r_i}|w_r) f_{X_{D_i}|D}(\tau_{d_i}|w_d)\right.\right.$$

$$\left.\left. d\tau_{r_i}\, d\tau_{d_i}\right)\right) f_{R,D}(w_r, w_d)\, dw_r dw_d \qquad \text{(D.3.12)}$$

Recall from Equations D.2.1 and D.2.2

$$f_{X_{R_i}|R}(\tau_{r_i}|w_r) = \frac{1}{\sqrt{2\pi}\,\sigma_{br}} \exp - \left[\frac{\tau_{r_i} - (w_r + \alpha_{r_i})}{\sqrt{2}\,\sigma_{br}}\right]^2 \qquad \text{(D.3.13a)}$$

$$f_R(w_r) = \frac{1}{\sqrt{2\pi}\,\sigma_{ar}} \exp - \left[\frac{w_r - r_0}{\sqrt{2}\,\sigma_{ar}}\right]^2 \qquad \text{(D.3.13b)}$$

$$f_{X_{D_i}|D}(\tau_d|w_d) = \frac{1}{\sqrt{2\pi}\,\sigma_{bd}} \exp - \left[\frac{\tau_{d_i} - (w_d - \alpha_{d_i})}{\sqrt{2}\,\sigma_{bd}}\right]^2 \qquad \text{(D.3.13c)}$$

$$f_D(w_d) = \frac{1}{\sqrt{2\pi}\,\sigma_{ad}} \exp - \left[\frac{w_d - d_0}{\sqrt{2}\,\sigma_{ad}}\right]^2 \qquad \text{(D.3.13d)}$$

Consider the following change of variable

$$\text{Let } t = \frac{w_r - r_0}{\sigma_{ar}} \qquad \text{(D.3.14a)}$$

$$s = \frac{w_d - d_0}{\sigma_{ad}} \qquad \text{(D.3.14b)}$$

$$u_i = \left[\frac{\tau_{r_i} - (r_0 + \sigma_{ar} t) - \alpha_{r_i}}{\sqrt{2}\,\sigma_{br}}\right] \qquad \text{(D.3.14c)}$$

$$v_i = \left[\frac{\tau_{di} - (d_0 + \sigma_{ad}t) - \sigma_{d_i}}{\sqrt{2}\ \sigma_{bd}}\right] \qquad \text{(D.3.14d)}$$

so that

$$dt = \frac{dw_r}{\sigma_{ar}} \qquad \text{(D.3.14e)}$$

$$ds = \frac{dw_d}{\sigma_{ad}} \qquad \text{(D.3.14f)}$$

$$du_i = \frac{d\tau_{ri}}{\sqrt{2}\ \sigma_{br}} \qquad \text{(D.3.14g)}$$

$$dv_i = \frac{d\tau_{di}}{\sqrt{2}\ \sigma_{bd}} \qquad \text{(D.3.14h)}$$

and define

$$Fl(i, 1, t) = \frac{2}{\sqrt{\pi}} \int_{A_1(t)}^{A_2(t)} \exp - (u^2)\ du \qquad \text{(D.3.15)}$$

$$Fl(i, 2, s) = \frac{2}{\sqrt{\pi}} \int_{A_3(s)}^{A_4(s)} \exp - (v^2)\ dv \qquad \text{(D.3.16)}$$

where

$$A_1(t) = \frac{A_{R_1} - (r_0 + \alpha_{r_i}) - \sigma_{ar}\ t}{\sqrt{2}\ \sigma_{br}}$$

$$A_2(t) = \frac{A_{R_2} - (r_0 + \alpha_{r_i}) - \sigma_{ar}\ t}{\sqrt{2}\ \sigma_{br}}$$

$$A_3(s) = \frac{A_{D_1} - (d_0 + \alpha_{di}) - \sigma_{ad}\, s}{\sqrt{2}\ \sigma_{bd}}$$

$$A_4(s) = \frac{A_{D_2} - (d_0 + \alpha_{di}) - \sigma_{ad}\, s}{\sqrt{2}\ \sigma_{bd}}$$

One can now write Equation D.3.12 simply as

$$PKW = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 - \prod_{i=1}^{\lambda} \left(1 - PC * \frac{1}{8\pi}\, F1(i, 1, t)\ F2(i, 2, 5)\right)\right)$$

$$e^{-t^2/2}\ e^{-s^2/2}\ dtds \qquad \text{(D.3.17)}$$

Finally, using Equation D.3.2, D.3.3, and D.3.17, one can write the MRKP model representation for P_{KM} as

$$P_{KM} = RLBTY * \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 - \prod_{i=1}^{\lambda} \left(1 - PC * \frac{1}{8\pi}\, F1(i, 1, t)\ F2(i, 2, 5)\right)\right)$$

$$e^{-t^2/2} e^{-s^2/2} dtds \qquad \text{(D.3.18)}$$

APPENDIX E

VPK PROGRAM DOCUMENTATION

E.1 Introduction. The VPK Program was developed to evaluate estimates of the uncertainty associated with a probabilistic model. In particular, it calculates both a first and a second order approximation of $Var(P_G)$, where P_G refers to a general probabilistic model, and computes an estimate of the error associated with the first order approximation. The program was initially written for use with weapons effectiveness models where P_G represented a conditional expected damage to a target, conditioned on several random variable parameters which serve as inputs to the particular model under study. The VPK Program inputs include sensitivity coefficients and estimates of expected moments of the random variable parameters. As an added output, the program lists the percentage variance associated with each random variable parameter.

E.2 Model. In particular, VPK was developed to investigate the uncertainty associated with a probabilistic model with response P_G depending upon two random vectors $\underline{S}$ and $\underline{R}$ as

$$P_G = G_P(\underline{S}, \underline{R}) \tag{E.2.1}$$

where

$$\underline{S} = (S_1, S_2, \ldots, S_{NS}) \qquad NS \leq 8 \tag{E.2.2a}$$

$$\underline{R} = (R_1, R_2, \ldots, R_{NR}) \qquad NR \leq 8 \tag{E.2.2b}$$

In Example 1 and 2 of Chapter III, $\underline{S}$ corresponds to $\hat{\underline{S}}_x$ and $\underline{R}$ to $\underline{RM}$.

Now each component of $\underline{S}$ is considered to be a function of another random vector $\underline{K}$. Likewise, each component of $\underline{R}$ is assumed to be a function of a random vector $\underline{Z}$; that is,

$$S_I = G_{SI}(\underline{K}) \qquad \text{(E.2.3a)}$$

$$R_I = G_{RI}(Z) \qquad \text{(E.2.3b)}$$

where

$$\underline{K} = (K_1, K_2, \ldots, K_{NK}) \qquad NK \leq 8$$

$$\underline{Z} = (Z_1, Z_2, \ldots, Z_{NZ}) \qquad NZ \leq 8$$

$\underline{K}$ corresponds to $\hat{\underline{K}}$ in Examples 1 and 2 of Chapter III and $\underline{Z}$ is represented by $\underline{Z}$ in Example 2 for NZ equal to four. Figure 4 illustrates the general model assumed for P_G.

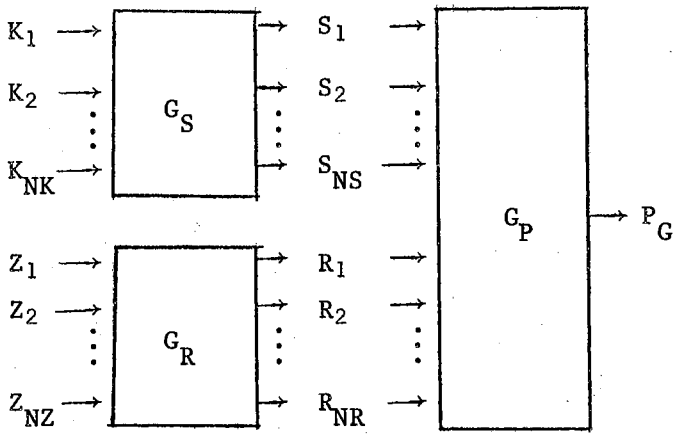

Figure 4

General Probabilistic Model P_G

The VPK Program investigates uncertainty associated with P_G in terms of uncertainties related to $\underline{K}$ and $\underline{Z}$. Letting

$$k_I = E[K_I] \tag{E.2.5a}$$

$$z_I = E[Z_I] \tag{E.2.5b}$$

one can write the uncertainty parameters associated with $\underline{K}$ and $\underline{Z}$ as

$$E1[I, J, U, V] = E[(K_I - k_I)(K_J - k_J)(K_U - k_U)(K_V - k_V)] \tag{E.2.6a}$$

$$E2[I, J, U, V] = E[(Z_I - z_I)(Z_J - z_J)(Z_U - z_U)(Z_V - z_V)] \tag{E.2.6b}$$

where a zero subscript indicates ommission of that factor. For example

$$Cov(K_1, K_2) = E1(0, 0, 1, 2) = E[(K_1 - k_1)(K_2 - k_2)] \tag{E.2.7}$$

Now define the following sensitivity coefficients needed for a second order approximation of $Var(P_G)$

$$PFS(I) = \frac{\partial P_G}{\partial S_I} \tag{E.2.8a}$$

$$PFR(I) = \frac{\partial P_G}{\partial R_I} \tag{E.2.8b}$$

$$CPFS(I, J) = \frac{\partial^2 P_G}{\partial S_I \partial S_J} \tag{E.2.8c}$$

$$CPFR(I, J) = \frac{\partial^2 P_G}{\partial R_I \partial R_J} \tag{E.2.8d}$$

$$CPFRS = \frac{\partial^2 P_G}{\partial R_I \partial S_J} \tag{E.2.8e}$$

$$PSK(I, J) = \frac{\partial S_I}{\partial K_J} \tag{E.2.8f}$$

$$PRZ(I, J) = \frac{\partial R_I}{\partial Z_J} \qquad \text{(E.2.8g)}$$

$$CPSK(I, J, L) = \frac{\partial^2 S_I}{\partial K_J \partial K_L} \qquad \text{(E.2.8h)}$$

$$CPRZ(I, J, L) = \frac{\partial^2 R_I}{\partial Z_J \partial Z_L} \qquad \text{(E.2.8i)}$$

Having defined the basic probabilistic model and the random variables and parameters associated with it, suppose one now proceeds to describe Program VPK in terms of input-output information.

E.3 Program Inputs. All inputs to VPK are punched onto cards. The format for each card and a brief description of the inputs are given in Table XII. The input data cards for a single probabilistic model analysis are of five types and are assembled in the following order:

(1) Type 1 Card: Title Card: used to identify particular model

(2) Type 2 Card: Control Card: specifies option codes

(3) Type 3 Cards: Data Information Cards: specifies data pertaining to model. This information is not used by program but is transferred directly to output list (optional).

(4) Type 4 Cards: Sensitivity Coefficients: input list of sensitivity coefficients corresponding to Equation E.2.8.

(5) Type 5 Cards: Parameter Estimates: input list of parameters associated with $\underline{K}$ and $\underline{Z}$ corresponding to Equation E.2.8.

One should note that the number of inputs has been limited: eight for each variable. This is due to the storage requirements of the program which as presented here requires approximately 200K bytes. Should more inputs be needed and sufficient core storage is available, the program can be extended by merely increasing the dimensions of the arrays employed by the program.

TABLE XII

VPK INPUT DATA CARDS

Card	Card Column	Fortran Format	Parameter Symbol	Description
1. Title Card (1 card)	1-80	20A4	Title	Alphanumeric information to identify model
2. Control Card (1 card)	1	I1	NK	No. of K's considered; NK $\leq$ 8
	4	I1	NZ	No. of Z's considered; NZ $\leq$ 8
	7	I1	NS	No. of S's considered; NS $\leq$ 8
	10	I1	NR	No. of R's considered; NR $\leq$ 8
	11-20	F10.0	EPK	1st order approximation of $E(P_G)$ (Response of model for $\underline{K}$ and $\underline{Z}$ at intended values)
	23	I1	NDATA	No. of data information cards
	26	I1	INDS	S Dependency Code = 0, All S's dependent = 1, All S's independent
	29	I1	INDR	R Dependency Code = 0, All R's dependent = 1, All R's independent
3. Data information cards (no limit)	1-80	20A4	DAT1	Information desired in output list
4. Sensitivity Coefficients				
A. 1 card (omit if NS = 0)	1-10	E10.4	PFS(1)	1st order Sens. Coef. relating $\underline{S}$ to P_G; $PFS(I) = \frac{\partial P_G}{\partial S_I}$
	⋮	⋮	⋮	
	71-80	E10.4	FFS(8)	
B. 1 card (omit if NR = 0)	1-10	E10.4	PFR(1)	1st order Sens. Coef. relating $\underline{R}$ to P_G; $PFR = \frac{\partial P_G}{\partial R_I}$
	⋮	⋮	⋮	
	71-80	E10.4	PFR(8)	

TABLE XII (continued)

Card	Card Column	Fortran Format	Parameter Symbol	Description
C. NS Cards (omit if NS = 0) I = 1, 2, ..., NS	1-10	E10.4	CPFS(I, 1)	2nd Order Sens Coef. relating $\underline{S}$ to P_c;
	71-80	E10.4	CPFS(I, 8)	$CPFS(I,J)=\frac{\partial^2 P_G}{\partial S_I \partial S_J}$
D. NR Cards (omit if NR = 0) I = 1, 2, ..., NR	1-10	E10.4	CPFR (I, 1)	2nd Order Sens. Coef. relating $\underline{R}$ to P_G;
	71-80	E10.4	CPFR(I, 8)	$CPFR(I,J)=\frac{\partial^2 P_G}{\partial R_I \partial R_J}$
E. NS Cards (omit if NR or NS = 0) I = 1, 2, ..., NS	1-10	E10.4	CPFRS(I, 1)	2nd Order Sens. Coef. relating both $\underline{R}$ and $\underline{S}$ to P_G;
	71-80	E10.4	CPFRS(I, 8)	$CPFRS(I,J)=\frac{\partial^2 P_G}{\partial R_I \partial S_J}$
F. NS Cards (omit if NS = 0) I = 1, 2, ..., NS	1-10	E10.4	PSK (I, 1)	1st Order Sens. Coef. relating $\underline{K}$ to S_I;
	71-80	E10.4	PSK (I, 8)	$PSK(I,J) = \frac{\partial S_I}{\partial K_J}$
G. NR Cards (omit if NR = 0) I = 1, 2, ..., NR	1-10	E10.4	PRZ (I, 1)	1st Order Sens. Coef. relating $\underline{Z}$ to R_I;
	71-80	E10.4	PRZ (I, 8)	$PRZ(I,J) = \frac{\partial R_I}{\partial Z_J}$
H. NS sets of NK cards each (omit if NS = 0) I=1, 2,...NS J=1, 2,...NK	1-10	E10.4	CPSK(I,J,1)	2nd Order Sens. Coef. relating $\underline{K}$ to S_I;
	71-80	E10.4	CPSK(I,J,8)	$CPSK(I,J,L)=\frac{\partial S_I}{\partial K_J \partial K_L}$

TABLE XII (continued)

Card	Card Column	Fortran Format	Parameter Symbol	Description
I. NR sets of NZ cards each (omit if NR = 0) I=1,2,...,NR J=1,2,...,NZ	1-10	E10.4	CPRZ(I,J,1)	2nd Order Sens Coef. relating $\underline{Z}$ to R_I; $CPRZ(I,J,L) = \frac{\partial R_I}{\partial Z_J \partial Z_L}$
	71-80	E10.4	CPRZ(I,J,8)	
5. Parameter Estimate Cards				
A. For K (use as many cards as needed; 6 estimates per card) (omit if NS = 0)	1	I1	K_I	Designates particular parameter estimate (see Equation E.2.5a)
	2	I1	K_J	
	3	I1	K_U	
	4	I1	K_V	
	5-13	E9.4	E1[I,J,U,V]	Parameter Estimate
	.	.	.	
	.	.	.	
	.	.	.	
	65	I1	K_I	Designates particular parameter estimate (see Equation E.2.5a)
	66	I1	K_J	
	67	I1	K_U	
	68	I1	K_V	
	69-78	E9.4	E1[I,J,U,V]	Parameter Estimate
	80	I1	IS	= 1 if last K parameter card = 0 otherwise
B. For Z (use as many cards as needed; 6 estimates per card) (omit if NR = 0)	1	I1	Z_I	Designates particular parameter estimate (see Equation E.2.5b)
	2	I1	Z_J	
	3	I1	Z_U	
	4	I1	Z_V	
	5-13	E9.4	E2[I,J,U,V]	Parameter Estimate
	.	.	.	
	.	.	.	
	.	.	.	
	65	I1	Z_I	Designates particular parameter estimate (see Equation E.2.5b)
	66	I1	Z_J	
	67	I1	Z_U	
	68	I1	Z_V	
	69-78	E9.4	E2[I,J,U,V]	Parameter Estimate
	80	I1	IS	= 1 if last Z parameter card = 0 otherwise

E.4 Program Output. Each output of the VPK Program is clearly labeled so that it can easily be interpreted. In general, the output list consists of the following parts.

(1) Title: taken directly from Title card

(2) Data Information: card by card listing of Data Information cards

(3) Sensitivity Coefficients: complete list of sensitivity coefficients

(4) Uncertainty of P_G

A. First Order estimate of $E[P_G]$

B. Second Order estimate of $E[P_G]$

C. Estimate of percent error in first order estimate of $E[P_K]$

D. First order estimate of $Var(P_G)$

E. Second order estimate of $Var(P_G)$

F. Estimate of percent error in first order estimate of $Var[P_G]$

G. Ratio of $SD(P_G)$ to $E(P_G)$

(5) Variance Components

A. List estimates assumed for each parameter associated with $\underline{K}$ and $\underline{Z}$

B. List variance components contributed to each parameter estimate

C. List percentage of $Var(P_K)$ due to each assumed parameter estimate

VITA

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