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GRADUATE COLLEGE

FORCED VIBRATION OF LAMINATED ANISOTROPIC PLATES INCLUDING THICKNESS-SHEAR AND DAMPING EFFECTS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

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degree of

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By

CHUN CHUNG SIU

Norman, Oklahoma

1972

FORCED VIBRATION OF LAMINATED ANISOTROPIC PLATES

INCLUDING THICKNESS-SHEAR AND DAMPING EFFECTS

APPROVED BY

Charles W. Bert

Davis M. Egle

William N. Huff

Edward F. Blick

L. A. Comp

DISSERTATION COMMITTEE

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SYMBOLS

A_o	area involved in shear-stress integration (fig. 8)
A_i, B_i	constants depending upon the boundary conditions (Appendix D)
A_{ij}	stretching stiffness of the plate
$\bar{A}_{ij}$	complex version of A_{ij}
a	length of plate
B_{ij}	bending-stretching coupling stiffness of the plate
$\bar{B}_{ij}$	complex version of B_{ij}
b	width of plate
C_i	constants depending upon the initial conditions
C_{ij}	Cauchy elastic coefficient
C_m, C_n	characteristic parameters tabulated in ref. 45
c	viscous damping coefficient
c_s	shear wave-propagation velocity
D	total dissipative energy
D_A	energy dissipated per unit of plate area
D_{ij}	bending and twisting stiffnesses of the plate
$\bar{D}_{ij}$	complex version of D_{ij}
E_{11}, E_{22}	Young's moduli in the x,y directions
ΔF	horizontal shear force per unit width
F_{ij}	thickness-shear stiffnesses of the plate
$\bar{F}_{ij}$	complex stiffness coefficients associated with F_{ij}
$\{f\}$	generalized-force column matrix

$g_{ij}^{(\ )}$	loss tangents; for example $g_{11}^{(A)}$ signifies the loss tangent associated with the longitudinal stretching stiffness A_{11}
h	total thickness of the plate
I	integral form defined in Appendix E
i	unit imaginary number $\equiv \sqrt{-1}$
K	thickness shear factor
K_{ij}	composite shear coefficient
$\tilde{L}$	amplitude of Lagrangian energy difference
$[M]$	mass matrix
M_{ij}	stress couples, moment per unit width
m,n	$\cos\theta$, $\sin\theta$
m_o, m_1, m_2	mass per unit of plate area, first moment of mass per unit area, second moment of mass per unit area
N_{ij}	membrane stress resultants, force per unit width
Q_i	thickness-shear stress resultant
$Q_{k\ell}$	reduced stiffness coefficients transformed to plate coordinates (x,y); see eq. (A-16)
$Q_{k\ell}^{*}$	reduced (plane-stress) stiffness coefficients with respect to material symmetry axes (X,Y); defined in eq. (A-6)
q	normal pressure acting on plate
R	density ratio $(\equiv \rho^{(1)}/\rho^{(2)})$
$[S]$	complex stiffness matrix
T	kinetic energy
T_A	kinetic energy per unit of plate area
$[T_r]$	transformation matrix defined in eq. (A-12)
t	time
U	strain energy

U_A strain energy per unit of plate area

U_{mn} undetermined longitudinal-displacement parameters

U_s shear strain energy

U_s' shear strain energy associated with equivalent uniform, longitudinal thickness-shear strain

U_v strain energy per unit of plate volume

u,v,w displacements in the x,y,z directions

u_o,v_o,w_o displacements of middle surface in x,y,z directions

V volume

V_{mn} undetermined transverse-displacement parameter

W total work done

W_{mn} undetermined normal-displacement parameter

X,Y,Z coordinates of the material-symmetry axes

x,y,z rectangular coordinates in the longitudinal, transverse, and thickness directions

Z_m,Z_n characteristic parameters tabulated in ref. 45

z_k,z_{k-1} thickness-direction coordinates of outer and inner faces of the k-th layer

$\{\ \}$ column matrix

$[\ ]$ square matrix

α,β normalized arguments in the x,y directions

β $C_{55}^{(1)}/C_{55}^{(2)}$ in Appendix D only

ϵ_{ij} strain components with respect to x,y axes

ϵ_{IJ} strain components with respect to X,Y axes

ϵ_{xz} strain corresponding to longitudinal thickness-shear action (Section III and Appendix D)

ϵ_{xz}' longitudinal thickness-shear strain weighted according to eq. (55)

ζ_2	z_2/z_1 (Appendix D)
θ	angle of orientation of an individual layer
κ_{ij}	curvature changes
ν	Poisson's ratio
ν_{12}, ν_{21}	major and minor in-plane Poisson's ratios
ξ	generalized coordinate
$\{\xi_{mn}\}$	column matrix representing generalized displacements
π	pi $\approx$ 3.1415962
ρ	density
$\sum$	summation symbol
σ_{ij}	stress components with respect to x,y axes
σ_{IJ}	stress components with respect to X,Y axes
Φ	assumed modal function
Ψ_{xmn}, Ψ_{ymn}	undetermined rotation parameters
Ψ_x, Ψ_y	angles of rotation in the xz and yz planes
Ω_s	ω/c_s
ω	circular frequency of vibration

Superscripts:

(k)	denotes a typical kth layer
$\hat{}$	denotes a damping quantity
$\dot{}$	denotes differentiation with respect to time
$\sim$	denotes that the quantity is an amplitude
o	refers to the middle surface of the plate
I,R	denote the respective imaginary and real parts of a complex quantity

Subscripts:

,	denotes partial differentiation with respect to the variable following the comma, i.e. $u_{o,x} = \partial u_o/\partial x$
k	integers ; $k = 1,2,\ldots K$
ℓ	integers ; $\ell = 1,2,\ldots L$
m	integers ; $m = 1,2,\ldots M$
n	integers ; $n = 1,2,\ldots N$

SECTION I

INTRODUCTION

1.1 Introductory Remarks

The continuing demand for increased structural efficiency in many advanced aerospace vehicles has resulted in development of laminated structures made of advanced fiber-reinforced composite materials, such as boron-epoxy, graphite-epoxy and boron-aluminum (reference 1). One of the most common structural elements for such vehicles is the rectangular plate or panel, for which numerous stable static, buckling, and free-vibrational analyses have been performed (reference 2). These advanced vehicles must maintain their structural integrity not only during high staticly applied mechanical and thermal loadings, but also in a variety of dynamic environments. In order to predict the dynamic stresses to which a structure will be subjected, it is necessary to perform a dynamic response analysis including the effects of damping properties as well as the various stiffnesses. An example of this type of requirement is found in the Space Shuttle now under NASA preliminary development (reference 3)

A <u>composite material</u> is defined as two or more materials joined together to form a nonhomogeneous material, which is used in constructing structures. Although such a material is nonhomogeneous on a micro scale, it behaves macroscopically as if it were a homogeneous, anisotropic material. An <u>anisotropic</u> material is one which exhibits different properties when tested at different directional orientations within the body. For example, a single layer of composite material containing unidirectionally oriented fibers (hereafter referred to as a unidirectional composite) has considerably greater stiffness and strength in the direction of the fibers than it does in a direction transverse to the fibers (see figure 1). This anisotropic aspect is one of several which makes structural analyses for composite-material structures more complicated (reference 4) than those for ordinary isotropic materials, such as aluminum alloys or many unfilled plastics.

A single layer of unidirectional composite has certain orthogonal axes of material symmetry and thus is said to be orthotropic, and behaves in considerably simpler fashion than a general anisotropic material. If the material is thin it is called plane orthotropic. However, for purposes of obtaining a more efficient design, it is often advantageous to place different layers or plies at different orientations. For the case of a thin plate, this means that we must consider it as a plane anisotropic material. This treatment is intermediate between the complicated general anisotropic case and the much simpler plane orthotropic case.

A laminate consists of two or more layers integrally joined together. It is said to be laminated symmetrically if all of the layers above and below the midplane of the laminate have the same dimensions, properties, and orientation (if the layers are orthotropic); see figure 2. A laminate is said to be balanced if all of the layers oriented at $+\theta$ are balanced by an equal number of identical layers at an orientation of $-\theta$; see figure 3. If a laminate is not symmetrically laminated, coupling occurs between in-plane (either normal or shear) stress on one hand and either bending or twisting deformation on the other hand. In such a case as a laminate consisting of a metallic substrate and a few layers of overlaying filamentary composite material, considerable bending-stretching coupling can occur and thus, it is necessary to consider not only the in-plane and bending stiffnesses but also the coupling stiffnesses which couple together the other two effects (reference 4). Obviously, this is considerably more complicated to analyze than a simple single layer.

Unfortunately, it appears that no one has yet succeeded in devising a lamination scheme that is both symmetrical and balanced. However, in the special, yet practical, case of multiple identical layers oriented alternately at $+\theta$ and $-\theta$, as the number of layers is increased, the bending-coupling effect diminishes (reference 5). For more than ten such layers, the bending-stretching coupling may be neglected for most engineering purposes and, thus, the laminate may be treated macroscopically as if it were a single-layer plate.

It has long been recognized that, due to the relatively low shear stiffnesses of composite materials in planes normal to the laminating plane, the thickness-shear* deformations must be included in the analysis, even when the plate is relatively thin (references 6-8), in order to achieve reasonably good predictions of laminate flexural behavior. Although several existing laminated plate theories include thickness-shear flexibility, all of them require an ad hoc assumption of the required shear correction factor. In this report, two different, relatively simple procedures are

---

*Sometimes referred to as "transverse shear", especially in the case of beams. The terminology used here follows that established by Yu for sandwich plates (reference 9). Here the term, transverse, is reserved for the in-plane direction which is normal to the longitudinal direction.

presented to enable rational prediction of the appropriate composite shear correction factor (K). The frequencies calculated using these values for K in shear-flexible laminated plate vibrational analysis give good agreement with the exact three-dimensional elasticity solution, which is computationally much more complicated.

The use of high-damping polymeric materials in the form of a thin layer or tape has come into widespread use as a structural damper to reduce the vibrational response of aircraft panels, especially in high-noise regions such as in the vicinity of jet engines. When the polymeric material is added on either one side or both sides, it is known as an unconstrained damping layer; when the polymeric material is placed between two or more layers, it is called a constrained damping layer (or layers). Appropriate analyses and design procedures have been developed for including either type of damping layer, assuming all layers (metal and polymer) are isotropic (references 10-12). However, so far as known, no detailed solutions have been published on vibration of laminated composite-material plates which included material damping. Such an analysis is presented in this report and the results are compared with experimental data reported recently by Clary (reference 13).

1.2 A Brief Survey of Selected Vibrational Analyses of Laminated Plates

Apparently the first vibrational analysis was carried out by Pister (reference 14) for a thin plate arbitrarily laminated of isotropic layers. In this case, the net effect of the bending-stretching coupling resulted in a reduced flexural stiffness.

Stavsky (reference 15) formulated a coupled bending-stretching dynamic theory for thin plates laminated of composite-material layers, but he did not present any numerical results. Apparently the first published results of the vibrational analysis of such plates is due to Ashton and Waddoups (reference 16), who used the Rayleigh-Ritz method to analyze rectangular plates. These results compared reasonably well with experimental results for the completely free and cantilever cases. A similar analytical and experimental study, but involving simply supported and free boundary conditions, was carried out by Hikami (reference 17). Additional analysis of the free-edge case was carried out by Ashton (reference 18).

For simply supported plates, Whitney and Leissa (references 19,20) presented closed-form solutions for the natural frequencies in the case of cross-ply and angle-ply lamination schemes. As would be expected, their results showed a strong effect of bending-stretching coupling in lowering the natural frequencies.

The case of clamped boundary conditions is more complicated analytically, but more representative of practical aerospace structures. Rayleigh-Ritz and experimental investigations of such structures were carried out independently by Ashton and Anderson (reference 21) and by Bert and Mayberry (reference 22).

The first more accurate vibrational analysis of laminated plates including thickness-shear flexibility was made by Ambartsumyan; see reference 23. He assumed an arbitrary distribution of thickness-shear stresses through the thickness. However, in carrying out his actual calculations, Ambartsumyan assumed a simple parabolic distribution. It can be shown by a simple mechanics-of-materials analysis (Jourawski shear theory; see Section

III) that the thickness-shear stress distribution must be discontinuous from layer to layer; thus, the simple parabolic distribution does not hold. Ambartsumyan did not give any numerical results for vibration of laminated plates; however, Whitney (reference 24) did so, using Ambartsumyan's basic theory.

Using another approach, Yang et al (reference 25) entended the Mindlin homogeneous, isotropic, dynamic plate analysis (reference 26) to the laminated anisotropic case. They assumed a thickness-shear angle which is independent of the thickness coordinate (z) and then integrated the stress equations of motion to obtain the governing partial differential equations. After integration, they introduced a thickness-shear coefficient in an ad hoc fashion to correlate the predicted frequencies with known results.

Also Yang et al introduced the coupling inertial effect (present only in the case of plates laminated unsymmetrically with respect to mass distribution), as well as the familiar translational and rotatory inertia effects present in homogeneous (or mass-symmetrically laminated) plates. The inclusion of these higher-order inertial effects is consistent with the inclusion of thickness-shear flexibility.

The analysis presented in this report is an improvement over both the Ambartsumyan and Yang et al analyses, in that it presents simple rational means for calculating the shear coefficient, rather than assuming it à priori.

An alternative to considering shear deformation per se is to make a microlaminar analysis, such as considered by Biot (reference 27) and Bolotin (reference 28). However, this approach has not been used very extensively so far.

Another approximate approach is to use the Voigt and Reuss models to determine the properties of an equivalent homogeneous, anisotropic, shear-flexible plate. This method was originated independently by Postma (reference 29), White and Angona (reference 30), and Rytov (reference 31) to investigate wave propagation in a continuum consisting of alternating layers of stiff and flexible isotropic materials. This concept was applied recently to plates with experimental verification for the special case of a beam, by Achenbach and Zerbe (reference 32).

All of the analyses mentioned above are approximate formulations in this sense: interlaminar compatibility is impossible unless both of the in-plane Poisson's ratios are identical in all of the layers. The reason for this is that, in a plate, one can have discontinuities in the strain component in a given direction in the plane, due to the Poisson contraction caused by a stress resultant or a stress couple acting in the perpendicular direction in the plane. This deficiency can be removed by using the approach introduced recently by Hsu and Wang (reference 33) and Wang (reference 34) for laminated shells. Another technique is to apply the nonhomogeneous three-dimensional elasticity approach, such as used recently by Srinivas et al (reference 35-37) for a special case of simply supported edges. Still another method is to use finite elements in the thickness direction, as introduced recently by Tso et al (reference 38). Unfortunately, all of the more accurate analyses mentioned in this paragraph are quite complicated computationally and thus are not amendable to engineering analyses of practical structural elements. Furthermore, the work of reference 37 showed that for structurally reasonable values of the following modal parameter, the Mindlin-type theory (such as extended here) is sufficiently accurate for determination of natural frequencies, but not for determination of the associated stress distribution.

There are numerous analyses in the literature on vibration of damped plates with isotropic layers; also there are a few vibrational analyses of single-layer, anisotropic plates and a few quasi-static (creep) analyses of laminated, anisotropic plates. However, vibrational analyses of laminated, anisotropic plates are quite limited. Dong (reference 39) indicated the solution for the dynamic response of a simply-supported rectangular plate arbitrarily laminated of orthotropic, visoelastic plies modeled as a standard linear solid.

SECTION II

FORMULATION OF THE THEORY OF LAMINATED, SHEAR-FLEXIBLE PLATES

In this section is presented a theoretical analysis of sinusoidally forced vibration of laminated, anisotropic plates including bending-stretching coupling, thickness-shear, all three types of inertia effects, and material damping. First the assumptions, on which the analysis is based, are stated explicitly. The general analysis is applicable to plates with any combination of natural boundary conditions at their edges. The analysis begins with the anisotropic stress-strain relations for a single layer and proceeds through the stiffness and damping constitutive relations, formulation of the various types of energy and work terms, and culminates in the formulation of an eigenvalue problem by application of the extended Rayleigh-Ritz method.

2.1 Hypotheses

The following assumptions are made in the analysis presented here:

H1. All displacements are assumed to be sufficiently small so that the linear strain-displacement relations are sufficiently accurate.

H2. The layers which make up the plate are linearly elastic and may be isotropic or orthotropic with any arbitrary orientation in the plane of the plate.

H3. The layers are sufficiently thin that thickness-normal-stress effects may be neglected, i.e. the layers are assumed to have finite stiffnesses which resist membrane, bending, and thickness shearing stresses.

H4. The layers are assumed to be bonded together perfectly.

H5. The plate is assumed to have all components of translational, translational-rotatory coupling, and rotatory inertia.

H6. All material damping effects are assumed to be small. They are incorporated by using stiffnesses which are complex rather than real (App. B,ref. 40); the complex-stiffness array is assumed to be symmetric (App. I,ref. 40). All other thermal effects are neglected.

H7. All initial-stress effects are neglected.

H8. All interactions with a surrounding fluid can be either neglected or considered to be included in the values used for the material-damping coefficients.

H9. The excitation consists of a uniformly distributed normal pressure loading, sinusoidal with respect to time.

2.2 Kinematics

The displacement field is assumed to be (hypothesis H3):

$$\begin{aligned} u(x,y,z,t) &= u_o(x,y,t) + z\psi_x(x,y,t) \\ v(x,y,z,t) &= v_o(x,y,t) + z\psi_y(x,y,t) \\ w(x,y,z,t) &= w_o(x,y,t) \end{aligned} \tag{1}$$

where u,v,w are the displacement components in the x,y,z directions (see figure 4); u_o, v_o, w_o are the displacement components at the middle surface of the multi-layer plate; ψ_x and ψ_y are the weighted middle-surface rotations in the respective xz and yz planes; and t is time.

Within the framework of hypothesis H1, the total engineering strains are given by the following in-plane strain-displacement relations:

$$\epsilon_{ij} = \epsilon^{o}_{ij} + z\kappa_{ij} \qquad \begin{array}{c}\text{(no sum)}\\ (ij = xx,yy,xy)\end{array} \tag{2}$$

where

$$\epsilon^{o}_{xx} = u_{o,x} \quad ; \quad \epsilon^{o}_{yy} = v_{o,y} \quad ; \quad \epsilon^{o}_{xy} = v_{o,x} + u_{o,y} \tag{3}$$

$$\kappa_{xx} = \psi_{x,x} \quad ; \quad \kappa_{yy} = \psi_{y,y} \quad ; \quad \kappa_{xy} = \psi_{y,x} + \psi_{x,y} \tag{4}$$

Also the following thickness-shear strain-displacement relations hold:

$$\epsilon_{xz} = w_{o,x} + \psi_x \quad ; \quad \epsilon_{yz} = w_{o,y} + \psi_y \tag{5}$$

where a subscript comma denotes partial differentiation with respect to the variable following the comma, i.e. $u_{o,x} \equiv \partial u_o/\partial x$.

2.3 Stiffness Constitutive Relations

In line with hypothesis H2, the following stress-strain relations are assumed to hold for each individual layer of which the plate is comprised:

$$\begin{Bmatrix} \sigma_{xx}^{(k)} \\ \sigma_{yy}^{(k)} \\ \sigma_{yz}^{(k)} \\ \sigma_{xz}^{(k)} \\ \sigma_{xy}^{(k)} \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 & 0 & Q_{16}^{(k)} \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 & 0 & Q_{26}^{(k)} \\ 0 & 0 & Q_{44}^{(k)} & Q_{45}^{(k)} & 0 \\ 0 & 0 & Q_{45}^{(k)} & Q_{55}^{(k)} & 0 \\ Q_{16}^{(k)} & Q_{26}^{(k)} & 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(k)} \\ \epsilon_{yy}^{(k)} \\ \epsilon_{yz}^{(k)} \\ \epsilon_{xz}^{(k)} \\ \epsilon_{xy}^{(k)} \end{Bmatrix} \qquad (6)$$

where $\sigma_{xx}^{(k)}$, $\sigma_{yy}^{(k)}$ are the in-plane normal stresses; $\sigma_{xz}^{(k)}$, $\sigma_{yz}^{(k)}$ are the thickness-shear stresses; $\sigma_{xy}^{(k)}$ is the in-plane shear stress; the Q_{ij} are the reduced stiffness coefficients which are applicable to a thin layer (see Appendix A); superscript (k) denotes a typical layer k, where $k = 1,2,3...,n$; and n is the total number of layers of which the plate is comprised.

It is noted that it is more consistent mathematically to write the stress-strain relations in either matrix form as

$$\left\{ \sigma_i^{(k)} \right\} = [Q_{ij}^{(k)}] \left\{ \epsilon_j^{(k)} \right\} \quad (i,j=1,2,4,5,6) \qquad (7)$$

or in tensor notation as

$$\sigma_{ij}^{(k)} = Q_{ijk\ell}^{(k)} \ \epsilon_{k\ell}^{(k)} \qquad (i,j=x,y,z) \qquad (8)$$

However, here a mixed notation is used for engineering convenience: double lettered subscripts (xx,yy,yz,xz,xy) following classical elasticity theory notation for stresses and strains, and double numbered subscripts (i,j=1,2,4,5,6) to reduce the number of subscripts used on the stiffness coefficients.

As is conventional in plate and shell theory, it is convenient to introduce generalized forces which are applicable to the whole laminate, rather than only a specific distance from the plate middle surface. The stress components, $\sigma_{ij}^{(k)}$, are functions of x,y,z; therefore, integrating them through the thickness yields the following generalized forces, which are functions of x,y only:

$$\left\{N_{xx},N_{yy},N_{xy},Q_x,Q_y\right\} \equiv \int_{-h/2}^{h/2} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)},\sigma_{xz}^{(k)},\sigma_{yz}^{(k)}\right\} dz$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)},\sigma_{xz}^{(k)},\sigma_{yz}^{(k)}\right\} dz \qquad (9)$$

$$\left\{M_{xx},M_{yy},M_{xy}\right\} \equiv \int_{-h/2}^{h/2} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)}\right\} z\,dz$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)}\right\} z\,dz \qquad (10)$$

where h ≡ total plate thickness, the quantity (z_k-z_{k-1}) is the thickness of an individual layer (see figure 5), N_{ij} ≡ membrane stress resultants (force per unit length), Q_i ≡ thickness-shear stress resultants (force per unit length), and M_{ij} are the stress couples (moment per unit length).

Inserting the stress-strain relations, given by equations (6) into equations (9) and (10), one obtains the following constitutive relations for the composite:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hdashline M_{xx} \\ M_{yy} \\ M_{xy} \\ \hdashline Q_y \\ Q_x \end{Bmatrix} = \left[\begin{array}{ccc:ccc:cc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ \hdashline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & F_{44} & F_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & F_{55} \end{array}\right] \begin{Bmatrix} \epsilon^o_{xx} \\ \epsilon^o_{yy} \\ \epsilon^o_{xy} \\ \hdashline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \\ \hdashline \epsilon_{yz} \\ \epsilon_{xz} \end{Bmatrix} \quad (11)$$

where the respective stretching, bending-stretching coupling, and bending stiffnesses of the plate are defined as follows:

$$\left\{A_{ij}, B_{ij}, D_{ij}\right\} \equiv \int_{-h/2}^{h/2} Q_{ij}^{(k)} \left\{1, z, z^2\right\} dz \quad (i,j=1,2,6) \qquad (12)$$

and the thickness-shear stiffnesses of the plate are as follows:

$$F_{ij} \equiv K_{ij} \int_{-h/2}^{h/2} Q_{ij}^{(k)} \; dz \qquad (i,j=4,5) \qquad (13)$$

where K_{ij} ≡ composite shear coefficient, introduced to account for the thickness-shear strain variation through the thickness. Section III presents several approximate analyses to predict K_{ij} analytically.

The A_{ij}, B_{ij}, and D_{ij} submatrices appearing in equation (11) are present in the classical Kirchhoff-type theory of laminated plates (see refs. 4,5,15-22). In the case of an orthotropic plate all of the terms with subscripts 16 and 26, called the cross-elasticity or shear-coupling terms, vanish. Furthermore, when the layers are all isotropic materials, only two of each set of the terms with subscripts 11,12,22, and 66 are independent.

If the plate is either homogeneous (i.e. single layer) or laminated symmetrically about the plate middle surface, the bending-stretching coupling stiffness submatrix B_{ij} vanishes. Then the only remaining terms are the A_{ij} and D_{ij} submatrices, which are present in classical, homogeneous thin-plate theory.

The thickness-shear stiffness coefficients F_{ij} account for the presence of thickness-shear strains, in a manner which can be considered to be a generalization of that used by Mindlin (ref. 26) for homogeneous, isotropic plates. To reduce the present theory to the classical Kirchhoff-type plate theory, the thickness-shear strains would be omitted; thus, from equations (5), one would obtain:

$$\psi_x = - w_{o,x} \; ; \quad \psi_y = - w_{o,y} \tag{14}$$

Furthermore, one would delete the thickness-shear strains, ε_{xz} and ε_{yz}, from equations (11) and the thickness-shear stress resultants, Q_x and Q_y, would be computed from equilibrium considerations only.

2.4 Strain Energy

The differential of the strain-energy density (strain energy per unit volume) is given by:

$$dU_v^{(k)} = \sigma_{xx}^{(k)} \, d\varepsilon_{xx}^{(k)} + \sigma_{yy}^{(k)} \, d\varepsilon_{yy}^{(k)} + \sigma_{xy}^{(k)} \, d\varepsilon_{xy}^{(k)}$$

$$+\sigma_{xz}^{(k)}\, d\epsilon_{xz}^{(k)} + \sigma_{yz}^{(k)} d\epsilon_{yz}^{(k)} \tag{15}$$

Integration of equation (15) yields:

$$U_v^{(k)} = (1/2)\ [\sigma_{xx}^{(k)}\ \epsilon_{xx}^{(k)} + \sigma_{yy}^{(k)}\ \epsilon_{yy}^{(k)} + \sigma_{xy}^{(k)}\ \epsilon_{xy}^{(k)}$$

$$+ \sigma_{yz}^{(k)}\ \epsilon_{yz}^{(k)} + \sigma_{xz}^{(k)}\ \epsilon_{xz}^{(k)}\ ] \tag{16}$$

The strain energy per unit of plate area (U_A) is the integral of $U_v^{(k)}$ over the total thickness of the plate:

$$U_A = \int_{-h/2}^{h/2} U_v^{(k)} dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} U_v^{(k)} dz \tag{17}$$

Substituting equations (16), (6), and (2) into equation (17), performing the integration, and using equations (12) and (13), one obtains the following expression for the strain energy per unit area:

$$U_A = (1/2)\ [A_{11}(\epsilon_{xx}^{o})^2 + 2A_{12}\ \epsilon_{xx}^{o}\ \epsilon_{yy}^{o} + A_{22}(\epsilon_{yy}^{o})^2 + 2A_{16}\ \epsilon_{xx}^{o}\ \epsilon_{xy}^{o}$$

$$+ 2A_{26}\ \epsilon_{yy}^{o}\ \epsilon_{xy}^{o} + A_{66}\ (\epsilon_{xy}^{o})^2 + 2B_{11}\ \epsilon_{xx}^{o}\ \kappa_{xx}$$

$$+ 2\ B_{12}(\epsilon_{xx}^{o}\kappa_{yy} + \epsilon_{yy}^{o}\ \kappa_{xx}) + 2\ B_{22}\ \epsilon_{yy}^{o}\ \kappa_{yy}$$

$$+ 2\ B_{16}(\epsilon_{xx}^{o}\ \kappa_{xy} + \epsilon_{xy}^{o}\ \kappa_{xx}) + 2B_{26}(\ \epsilon_{yy}^{o}\ \kappa_{xy} + \epsilon_{xy}^{o}\ \kappa_{yy})$$

$$+ 2\ B_{66}\ \epsilon_{xy}^{o}\ \kappa_{xy} + D_{11}\ \kappa_{xx}^{2} + 2\ D_{12}\ \kappa_{xx}\kappa_{yy} + D_{22}\ \kappa_{yy}^{2}$$

$$+ 2\ D_{16}\ \kappa_{xx}\ \kappa_{xy} + 2\ D_{26}\ \kappa_{yy}\ \kappa_{xy} + D_{66}\ \kappa_{yy}^{2} + F_{44}\ \epsilon_{yz}^{2}$$

$$+ 2\ F_{45}\ \epsilon_{yz}\ \epsilon_{xz} + F_{55}\ \epsilon_{xz}^{2}\ ] \tag{18}$$

2.5 Damping Coefficients and Dissipative Energy

For a material with Kimball-Lovell structural damping (Appen. B, ref. 40), the stress-strain rate relation for sinusoidal motion at a circular frequency ω can be expressed as follows:

$$\hat{\sigma} = (b/\omega)\ \dot{\varepsilon} \tag{19}$$

where b is a material constant and $\dot{\varepsilon}$ is the strain rate.

The strain is given by

$$\varepsilon = |\varepsilon|\ \epsilon^{i\omega t} \tag{20}$$

where $i \equiv \sqrt{-1}$.

Thus,

$$\dot{\varepsilon} = i\omega\varepsilon \tag{21}$$

Substituting equation (21) into equation (19), one obtains:

$$\hat{\sigma} = \hat{Q}\varepsilon \tag{22}$$

where

$$\hat{Q} \equiv ib \tag{23}$$

Generalizing equation (23) to the entire array of stresses and strains in layer "k", in contracted notation analogous to equation (7), we obtain the following expression, where $\hat{Q}_{ij}^{(k)}$ is symmetric (hypothesis H6):

$$\left\{ \hat{\sigma}_i^{(k)} \right\} = [\hat{Q}_{ij}^{(k)}] \left\{ \varepsilon_j^{(k)} \right\} \tag{24}$$

Equation (24) is the same as equation (7), except that here the presence of the hat symbols ($\hat{\ }$) denotes damping quantities rather than elastic quantities.

The energy dissipated per unit of plate area, due to damping, is denoted by the symbol $\hat{D}$ The expression for it in terms of the midplane strains (ε^o_{ij}) and the curvatures (κ_{ij}) is the same as equation (18), except for the presence of the hat symbols over all of the A_{ij}, B_{ij}, D_{ij}, and F_{ij} quantities, where

$$\hat{A}_{ij} \equiv ig^{(A)}_{ij} A_{ij} \quad , \quad \hat{B}_{ij} \equiv ig^{(B)}_{ij} B_{ij} \ , \quad \hat{D}_{ij} \equiv ig^{(D)}_{ij} D_{ij} \ ,$$

$$\hat{F}_{ij} \equiv ig^{(F)}_{ij} F_{ij} \tag{25}$$

where $g^{(\ )}_{ij}$ are loss tangents (see Appendix B, ref. 40)

2.6 Kinetic Energy

The differential kinetic energy of an elemental volume (dx by dy by dz) is given by:

$$(\rho^{(k)}/2)\ (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\ dx\ dy\ dz$$

where $\rho^{(k)}$ is the density at the point (x,y,z) and $\dot{u},\dot{v},\dot{w}$ are the velocity components in the x,y,z directions respectively.

Now the kinetic energy per unit of plate area is obtained from the differential quantity given above by dividing by dx dy and integrating over the entire thickness of the plate, as follows:

$$T_A = \int_{-h/2}^{h/2} (\rho^{(k)}/2)(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, dz \tag{26}$$

Substituting the kinematic relations, equations (1), into equation (26) and performing the integration, assuming that the density is uniform through the thickness of each individual layer, one arrives at the following expression:

$$T_A = (m_o/2)(\dot{u}_o^2 + \dot{v}_o^2 + \dot{w}_o^2) + m_1\ (\dot{u}_o\dot{\psi}_x + \dot{v}_o\ \dot{\psi}_y)$$

$$+ (m_2/2)\ (\dot{\psi}_x^2 + \dot{\psi}_y^2) \tag{27}$$

where m_o, m_1, and m_2 are respectively the mass per unit of plate area, first moment of mass per unit area (coupling inertia), and second moment of mass per unit area (mass moment of inertia), defined as follows:

$$\left\{m_o, m_1, m_2\right\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \rho^{(k)} \left\{1, z, z^2\right\} dz \tag{28}$$

Only the coefficient m_o appears in classical Kirchhoff theory for the dynamics of thin, homogeneous plates. The Mindlin dynamic plate theory of homogeneous plates (ref. 26) contains both m_o and m_2, while the dynamic theory originated by Yang et al (ref. 25) for laminated plates contains m_o, m_1, and m_2.

2.7 Work Done by External Forces

The only external force considered here is a uniformly distributed normal pressure which has a sinusoidal wave form in time. Thus, it can be written in complex variable notation as follows:

$$q(x,y,t) = \tilde{q}\, e^{i\omega t} \tag{29}$$

where $\tilde{q} \equiv$ amplitude of the normal pressure.

The following expression for the total work done may be derived easily from the principle of virtual work:

$$W = \int_0^a \int_0^b q\,(x,y,t)\; w_o\,(x,y)\; dx\, dy \tag{30}$$

2.8 Application of the Extended Rayleigh-Ritz Method

Since the present problem is one of steady-state harmonic excitation denoted in complex-variable exponential forms, the displacements and rotations are proportional to $e^{i\omega t}$ also. It is noted that the phase angle between the response and the excitation is taken care of by the imaginary components of these quantities. Thus, the time dependence cancels out in all of the energy terms which appear in Lagrange's equation. Therefore, it is necessary to consider only the amplitudes of the respective energy and work terms. Then, as shown in detail in Appendix B, the amplitude of the Lagrangian energy difference can be expressed as follows:

$$\tilde{L} = (\tilde{T} + \tilde{W}) - (\tilde{U} + \tilde{D}) \tag{31}$$

where $\tilde{D},\tilde{T},\tilde{U},\tilde{W}$ are the amplitudes of the dissipative, kinetic, and strain energies and of the work done by the external forces, respectively.

The strain and damping energy terms can be combined by introducing complex stiffnessses, as discussed in Ap.B,ref.40. Making this substitution and integrating over the plate area, we obtain the following:

$$\begin{aligned}\tilde{U}+\tilde{D} = \tfrac{1}{2}\int_0^a\int_0^b \Big\{&\bar{A}_{11}\tilde{u}^2_{o,x} + 2\bar{A}_{12}\tilde{u}_{o,x}\tilde{v}_{o,y} + \bar{A}_{22}\tilde{v}^2_{o,y} + 2(\bar{A}_{16}\tilde{u}_{o,x}+\bar{A}_{26}\tilde{v}_{o,y})(\tilde{v}_{o,x}+\tilde{u}_{o,y}) \\
&+\bar{A}_{66}(\tilde{v}_{o,x}+\tilde{u}_{o,y})^2 + 2\bar{B}_{11}\tilde{u}_{o,x}\tilde{\psi}_{x,x} + 2\bar{B}_{12}(\tilde{u}_{o,x}\tilde{\psi}_{y,y}+\tilde{v}_{o,y}\tilde{\psi}_{x,x})+2\bar{B}_{22}\tilde{v}_{o,y}\tilde{\psi}_{y,y} \\
&+2\bar{B}_{16}[\tilde{u}_{o,x}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y})+(\tilde{v}_{o,x}+\tilde{u}_{o,y})\tilde{\psi}_{x,x}] + 2\bar{B}_{26}[\tilde{v}_{o,y}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y}) \\
&+ (\tilde{v}_{o,x}+\tilde{u}_{o,y})\tilde{\psi}_{y,y}] + 2\bar{B}_{66}(\tilde{v}_{o,x}+\tilde{u}_{o,y})(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y}) + \bar{D}_{11}\tilde{\psi}^2_{x,x} \\
&+2\bar{D}_{12}\tilde{\psi}_{x,x}\tilde{\psi}_{y,y}+\bar{D}_{22}\tilde{\psi}^2_{y,y}+ 2\bar{D}_{16}\tilde{\psi}_{x,x}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y}) +2\bar{D}_{26}\tilde{\psi}_{y,y}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y}) \\
&+\bar{D}_{66}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y})^2 + \bar{F}_{44}(\tilde{w}_{o,y}+\tilde{\psi}_y)^2 + 2\bar{F}_{45}(\tilde{w}_{,y}+\tilde{\psi}_y)(\tilde{w}_{o,x}+\tilde{\psi}_x) \\
&+\bar{F}_{55}(\tilde{w}_{o,x}+\tilde{\psi}_x)^2\Big\}\,dx\,dy \qquad (32)\end{aligned}$$

where the superscript (~) denotes that the quantity is an amplitude, and

$$\bar{A}_{ij},\ \bar{B}_{ij},\ \bar{D}_{ij} = (A_{ij}+\hat{A}_{ij})\ ,\ (B_{ij}+\hat{B}_{ij}),\ (D_{ij}+\hat{D}_{ij});\ i,j=1,2,6$$

$$\bar{F}_{ij} = F_{ij}+\hat{F}_{ij}\quad ;\ i,j=4,5 \qquad (33)$$

The sum of the amplitudes of the kinetic energy and the work done by external forces can be expressed as follows:

$$\tilde{T}+\tilde{W} = (\omega^2/2)\int_0^a\int_0^b[m_o(\tilde{u}_o^2+\tilde{v}_o^2+\tilde{w}_o^2)+2m_1(\tilde{u}_o\tilde{\psi}_x+\tilde{v}_o\tilde{\psi}_y)$$

$$+ m_2(\tilde{\psi}_x^2+\tilde{\psi}_y^2)]\,dx\,dy + \int_0^a\int_0^b \tilde{q}\,\tilde{w}_o\,dx\,dy \qquad (34)$$

To apply the extended Rayleigh-Ritz method, the assumed functions for the amplitudes of the displacements and rotations are given the following general form:

$$\tilde{u}_o = \sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\Phi_{um}(\alpha)\Phi_{un}(\beta)$$

$$\tilde{v}_o = \sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\Phi_{vm}(\alpha)\Phi_{vn}(\beta)$$

$$\tilde{w}_o = \sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn}\Phi_{wm}(\alpha)\Phi_{wn}(\beta) \qquad (35)$$

$$\tilde{\psi}_y = \sum_{m=1}^{M}\sum_{n=1}^{N} \Psi_{ymn}\Phi_{\psi ym}(\alpha)\Phi_{\psi yn}(\beta)$$

$$\tilde{\psi}_x = \sum_{m=1}^{M}\sum_{n=1}^{N} \Psi_{xmn}\Phi_{\psi xm}(\alpha)\Phi_{\psi xn}(\beta)$$

The modal coefficients $U_{mn}, V_{mn}, W_{mn}, \Psi_{ymn}, \Psi_{xmn}$ are the undetermined parameters, the Φ's are assumed modal functions, $\alpha \equiv x/a$, $\beta \equiv y/b$.

Substituting equations (32), (34), and (35) into equation (31) for the Lagrangian energy-difference amplitude $(\tilde{L})$, one obtains the result presented in Appendix C1.

Hamilton's principle can be stated mathematically as follows:

$$\delta\int_{t_1}^{t_2} \tilde{L}\, dt = 0 \tag{36}$$

To achieve as close of an approximation as possible to equation (36), the Lagrangian function $\tilde{L}$ is minimized by setting its partial derivatives with respect to the modal coefficients respectively equal to zero, i.e.

$$\partial\tilde{L}/\partial U_{k\ell} = 0 \;;\quad \partial\tilde{L}/\partial V_{k\ell} = 0 \;;\quad \partial\tilde{L}/\partial W_{k\ell} = 0 \;;\quad \partial\tilde{L}/\partial \Psi_{xk\ell} = 0$$
$$\partial\tilde{L}/\partial \Psi_{yk\ell} = 0 \quad;\quad k = 1,2,\ldots, M \quad;\ \ell = 1,2,\ldots, N \tag{37}$$

Equations (37) represent a set of M x N nonhomogeneous, linear algebraic equations.

2.9 Reduction to Matrix Form

For computational convenience, it is desirable to express the set of equations (37) in matrix form as follows:

$$[S]\left\{\xi_{mn}\right\} - \omega^2[M]\left\{\xi_{mn}\right\} = \left\{f\right\} \tag{38}$$

where $[S]$ ≡ complex stiffness matrix, $[M]$ ≡ mass matrix, and $\left\{\xi_{mn}\right\}$ and $\left\{f\right\}$ are column matrices representing the generalized displacements and generalized forces, respectively, i.e.

$$\left\{\xi_{mn}\right\} \equiv \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \Psi_{ymn} \\ \Psi_{xmn} \end{Bmatrix} \quad ; \quad \left\{f\right\} \equiv \begin{Bmatrix} 0 \\ 0 \\ \tilde{q} \\ 0 \\ 0 \end{Bmatrix} \tag{39}$$

Equation (38) can be written in explicit complex form as follows:

$$[S^{(R)} + iS^{(I)}] \left\{\xi_{mn}^{(R)} + i\xi_{mn}^{(I)}\right\} - \omega^2 \, [M^{(R)}] \left\{\xi_{mn}^{(R)} + i\xi_{mn}^{(I)}\right\} = \left\{f^{(R)}\right\} \tag{40}$$

where superscripts (R) and (I) denote the real and imaginary parts of the complex quantities appearing in equation (38).

Equation (40) can be solved for the response matrix, partitioned into real and imaginary parts, as follows:

$$\begin{Bmatrix} \xi_{mn}^{(R)} \\ \hline \xi_{mn}^{(I)} \end{Bmatrix} = \left[\begin{array}{c|c} S^{(R)} - \omega^2 M^{(R)} & -S^{(I)} \\ \hline S^{(I)} & S^{(R)} - \omega^2 M^{(R)} \end{array}\right]^{-1} \begin{Bmatrix} f^{(R)} \\ \hline 0 \end{Bmatrix} \tag{41}$$

The modal coefficients are placed in the inverse matrix in equation (41). For the specific numerical case treated M=N=2, i.e. m,n=1,2. Thus, the complete matrix is of order 40 x 40 with each submatrix being of order 2 x 2. Thus, the problem has been reduced to a standard algebraic eigenvalue problem which can be solved by an available computer suboutline for the IBM 360 Series 50 digital computer. Complete computer program documentation is presented in Appendix F.

SECTION III

DERIVATION OF THICKNESS-SHEAR FACTORS FOR LAMINATES

In this section are presented two entirely different approaches for calculating the thickness-shear factors for a laminate. The first approach is to extend the Jourawski static theory of shear-flexible beams (reference 41), as presented in elementary mechanics-of-materials textbooks, to a laminated beam. The other approach is to extend, to the laminate case, Mindlin's method (ref. 26) of matching the pure thickness-shear-mode frequencies predicted by two-dimensional dynamic elasticity and by Timoshenko one-dimensional, shear-flexible beam theory (references 42, 43).

In the case of isotropic plates, there is only one independent thickness-shear factor (K), since $K_{45}=0$ and $K_{55}=K_{44}=K$. The static approach mentioned above yields a value of K = 0.833 (reference 44), while Mindlin's dynamic approach results in a value of 0.822 for K. Srinivas et al. (ref. 35) showed that use of either of these values for K in Mindlin's plate theory results in a very close approximation to the lower natural frequencies computed by exact, dynamic, three-dimensional theory of elasticity for rather thick, homogeneous isotropic rectangular plates simply supported on all edges.

In the case of orthotropic plates, there are two independent thickness-shear factors (K_{44} and K_{55}), since $K_{45}=0$ and $K_{55}\neq K_{44}$. In the case of a plane anisotropic plate, such as one consisting of a unidirectional composite material with its major material-symmetry axis oriented at an acute angle (θ) with its edges, there are three independent, non-zero K_{ij}. However, for this case, Appendix A presents transformations from which the three F_{ij}, proportional to K_{ij} and defined in equation (13), can be calculated from the following data: F_{44} and F_{55} for the orthotropic case ($\theta=0^o$) and the value of θ in the anisotropic case. Thus, the general problem of determining K_{ij} is reduced to one of calculating K_{44} and K_{55} for the orthotropic case.

To calculate K_{55} for an orthotropic laminate by either of the two approaches mentioned above, one considers a beam oriented in the x direction and laminated of layers having properties $E_{11}^{(k)}$ * and $C_{55}^{(k)}$. To calculate K_{44}, $x \rightarrow y$, $E_{11}^{(k)} \rightarrow E_{22}^{(k)}$, and $C_{55}^{(k)} \rightarrow C_{44}^{(k)}$.

To provide a check for the results, the fundamental eigenvalues for free vibration calculated by laminated, shear-flexible plate theory (Section II), using these values of K_{ij}, are compared with the exact laminated elasticity theory values given by Srinivas et al. (ref. 35).

3.1 Static Approach

To calculate the horizontal shear force per unit width, ΔF, acting on a cross section, figures 6 and 7 are used. For consistency plate notation is used, even though the theory is only one dimensional.

The bending stress in typical layer "k", at a distance z from the midplane of the laminate, is calculated as follows:

$$\sigma_{xx}^{(k)} = E_{11}^{(k)} \kappa_{xx}\, z \tag{42}$$

*For thin plates as treated in this dissertation, $E_{11}^{(k)} \rightarrow Q_{11}^{(k)}$, where $Q_{11}^{(k)} = E_{11}^{(k)} / (1 - \nu_{12}^{(k)} \nu_{21}^{(k)})$

where κ_{xx} ≡ longitudinal bending curvature and $E_{11}^{(k)}$ ≡ longitudinal Young's modulus of layer "k".

The longitudinal stress couple (bending moment per unit width) is defined as follows:

$$M_{xx} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} z \, dz \qquad (43)$$

The longitudinal curvature can be expressed in terms of M_{xx} as follows:

$$\kappa_{xx} = M_{xx}/D_{11} \qquad (44)$$

where D_{11} ≡ longitudinal flexural rigidity.

From equations (42) and (44), the bending stresses acting on the left $(x=x_1)$ and right $(x = x_1+\Delta x)$ sides of the elements are:

$$\begin{aligned} \sigma_{xx}^{(k)}(x_1) &= (E_{11}^{(k)} z/D_{11}) M_{xx} \quad ; \\ \sigma_{xx}^{(k)}(x_1+\Delta x) &= (E_{11}^{(k)} z/D_{11})(M_{xx}+\Delta M_{xx}) \end{aligned} \qquad (45)$$

Since both of these bending stress distributions act on identical cross-sectional areas, the horizontal shear force per unit width, which acts on the element, is given by:

$$\Delta F = \int_{A_o} [\sigma_{xx}^{(k)}(x_1+\Delta x) - \sigma_{xx}^{(k)}(x_1)] \, dA \qquad (46)$$

where A_o ≡ area shown in figure 8.

Combining equations (45) and (46), one obtains:

$$\Delta F = (\Delta M_{xx}/D_{11})Y(z) \tag{47}$$

where

$$Y(z) \equiv \int_{A_o} E_{11}^{(k)} z \, dA \ .$$

The force per unit width, ΔF, must equilibrate the horizontal shear stress, $\sigma_{xz}^{(k)}$, acting on the bottom face of the element shown in figure 7. Thus, we have:

$$\Delta F = \sigma_{xz}^{(k)} \Delta x \tag{48}$$

Hooke's law in shear for layer "k" is

$$\sigma_{xz}^{(k)} = C_{55}^{(k)} \epsilon_{xz}^{(k)} \tag{49}$$

where $C_{55}^{(k)}$ and $\epsilon_{xz}^{(k)}$ are the modulus and strain corresponding to longitudinal thickness-shear action in layer "k".

Substituting equation (49) into equation (48), one obtains the following result:

$$\Delta F = C_{55}^{(k)} \epsilon_{xz}^{(k)} \Delta x \tag{50}$$

The following expression for the longitudinal shear strain at a distance z is obtained by equating the right-hand sides of equation (47) and (50):

$$\epsilon_{xz}^{(k)}(x,z) = [Y(z)/(C_{55}^{(k)}\ D_{11})]\ Q_x(x) \tag{51}$$

where Q_x is the thickness-shear stress resultant (force per unit width), which is related to M_{xx} by static equilibrium as follows:

$$Q_x = \Delta M_{xx}/\Delta x \tag{52}$$

The shear strain energy for a laminated, rectangular-cross-section beam, shown in figure 9, is:

$$U_s = \tfrac{1}{2} \int_A C_{55}^{(k)} [\epsilon_{xz}^{(k)}]^2\ dA \tag{53}$$

The shear strain energy associated with an equivalent uniform, longitudinal thickness-shear strain in a laminated, rectangular-cross-section beam of cross-sectional area A is:

$$U_s' = (K_{55}/2)(\epsilon_{xz}')^2 \int_A C_{55}^{(k)}\ dA \tag{54}$$

where K_{55} is the longitudinal thickness-shear factor and ϵ_{xz}' is weighted according to:

$$\epsilon_{xz}' = [\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xz}^{(k)}\ \epsilon_{xz}^{(k)}\ dz][\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xz}^{(k)}\ dz]^{-1} \tag{55}$$

Equating the right-hand sides of equations (53) and (54) yields an equation; then substituting equations (49) and (51) into that equation, one arrives at the following explicit expression for the longitudinal thickness-shear factor

$$K_{55} = \frac{\left[\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Y(z)\, dz \right]^2}{\left[\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} C_{55}^{(k)}\, dz \right] \left\{ \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} (C_{55}^{(k)})^{-1} [Y(z)]^2\, dz \right\}} \tag{56}$$

Equation (56) is a general expression for the static, longitudinal thickness-shear factor for an arbitrarily laminated beam. It is a relatively simple algebraic expression which depends upon only the longitudinal Young's and thickness-shear moduli of the individual layers ($E_{11}^{(k)}$ and $C_{55}^{(k)}$) and the lamination geometry. Numerical results are presented in later examples.

3.2 Dynamic Approach

Due to the inherent mathematical form of the equations of dynamic elasticity theory for a multi-material (multi-layer) medium, it is not feasible to present a general solution for the thickness-shear factor obtained by using the dynamic approach. Even in the case of a specific class of laminate, such as the three-layer, symmetrically laminated one, it is not possible to obtain an explicit expression for the thickness-shear factor. In fact, the complexity of the dynamic case, which involves the dynamic elasticity-theory analysis of a shear-flexible laminated beam, approaches that of the shear-flexible laminated plate and thus negates much of the advantage of using the thickness-shear factor approach to laminated, shear-flexible plate dynamics. However, in order to illustrate the dynamic approach and to provide a numerical comparison with the static approach and with the exact results of Srinivas et al (ref. 35), an analysis for the symmetrical three-layer case is presented in Appendix D.

SECTION IV

MODAL FUNCTIONS USED FOR VARIOUS PLATE BOUNDARY CONDITIONS

In applying the extended Rayleigh-Ritz method, the approximate modal shapes assumed must satisfy the kinematic boundary conditions. However, to improve convergence, it is desirable to satisfy the force-type boundary conditions also. For a rectangular plate, the three cases of boundary conditions most commonly encountered in practice are discussed in Sections 4.1-4.3.

It should be noted tha C_m, C_n, Z_m, Z_n are characteristic parameters tabulated in reference 45. Also, five boundary conditions per edge need to be prescribed rather than four as in classical theory.

4.1 Simply Supported on All Edges

The boundary conditions considered here are simply-supported edges in the sense that

$$\begin{aligned} N_{xx} = M_{xx} = \tilde{v}_o = \tilde{w}_o = \tilde{\psi}_y = 0 \quad & \text{at } x = 0,a \ (\alpha=0,1) \\ N_{yy} = M_{yy} = \tilde{u}_o = \tilde{w}_o = \tilde{\psi}_x = 0 \quad & \text{at } y = 0,b \ (\beta=0,1) \end{aligned} \tag{57}$$

As was pointed out by Wang (reference 46), it is impossible for a separable-form deflection function to satisfy the above boundary conditions for the case of a plate of anisotropic material (Q_{16} and $Q_{26} \neq 0$). However, the following modal functions permit eqs. (35) to satisfy eq. (57) exactly in the orthotropic case, but only approximately in the anisotropic case:

$$\Phi_{um} = \Phi_{\psi xm} = \cos m\pi\alpha \; ; \quad \Phi_{vm} = \Phi_{wm} = \Phi_{\psi ym} = \sin m\pi\alpha$$
$$\Phi_{vn} = \Phi_{\psi yn} = \cos n\pi\beta \; ; \quad \Phi_{un} = \Phi_{wn} = \Phi_{\psi xn} = \sin n\pi\beta \tag{58}$$

For the simply supported case, we substitute equations (58) into equations (C-1 through C-6). The evaluations of the integral forms are presented in Appendix E.

4.2 Fully Clamped on All Edges

The boundary conditions for a fully clamped laminated plate are as follows:

$$\tilde{u}_o = \tilde{v}_o = 0 \quad \text{at } x = 0,a \text{ and } y = 0,b$$
$$\tilde{w}_o = \tilde{w}_{o,x} = \tilde{\psi}_y = 0 \quad \text{at } x = 0,a \tag{59}$$
$$\tilde{w}_o = \tilde{w}_{o,y} = \tilde{\psi}_x = 0 \quad \text{at } y = 0,b$$

The assumed modal functions selected to permit equations (35) to satisfy equations (59) are as follows:

$$\Phi_{um} = \Phi_{vm} = \Phi_{\psi xm} = \Phi_{\psi ym} = \sin 2m\pi\alpha$$
$$\Phi_{un} = \Phi_{vn} = \Phi_{\psi xn} = \Phi_{\psi yn} = \sin 2n\pi\beta$$
$$\Phi_{wm} = \cosh Z_m\alpha - \cos Z_m\alpha - C_m\,(\sinh Z_m\alpha - \sin Z_m\alpha) \tag{60}$$
$$\Phi_{wn} = \cosh Z_n\beta - \cos Z_n\beta - C_n\,(\sinh Z_n\beta - \sin Z_n\beta)$$

For the fully clamped case, we substitute equations (60) into equations (C1 through C6). The evaluations of the integral forms are presented in Appendix E.

4.3 Free on All Edges

The boundary conditions appropriate for a laminated plate free on all edges can be expressed mathematically as follows:

$$
\begin{aligned}
&N_{xy} = M_{xy} = 0 && \text{at } x = 0,a \text{ and } y = 0,b \\
&N_{xx} = Q_x = M_{xx} = 0 && \text{at } x = 0,a \\
&N_{yy} = Q_y = M_{yy} = 0 && \text{at } y = 0,b
\end{aligned}
\tag{61}
$$

The assumed modal functions selected to permit equations (35) to satisfy equations (61) are as follows:

$$
\begin{aligned}
&\Phi_{um} = \Phi_{vm} = \Phi_{\psi xm} = \Phi_{\psi ym} = 1 - \cos 2m\pi\alpha \\
&\Phi_{un} = \Phi_{vn} = \Phi_{\psi xn} = \Phi_{\psi yn} = 1 - \cos 2n\pi\alpha \\
&\Phi_{wm} = \cos m\pi\alpha \ ; \qquad \Phi_{wn} = \cos n\pi\beta
\end{aligned}
\tag{62}
$$

For the free edge case, we substitute equations (62) into equations (C1 through C6). The evaluation of the resulting integral forms are presented in Appendix E.

SECTION V

NUMERICAL RESULTS AND COMPARISON WITH RESULTS OF OTHER INVESTIGATORS

In this section a comparison is made between the numerical results obtained in this investigation and those obtained by other investigators.

5.1 Thickness-Shear Factors for Laminates

Two example problems are analyzed and, where possible, results of Example 1 are compared with solutions obtained from other sources. The results of Example 2 are presented for use in design.

Example 1: Three-layer, symmetric, isotropic laminate - The coordinate system is shown in figure 10. All geometrical and material property ratios correspond to the data of ref. 35 given as follows:

$$z_2/z_1 = h^{(2)}/[h^{(2)}+2h^{(1)}] = 0.8 \ , \quad E_{11}^{(1)}/E_{11}^{(2)} = 15,$$

$$C_{55}^{(1)}/C_{55}^{(2)} = 15, \quad \rho^{(1)}/\rho^{(2)} = 1 \ , \text{ and } \nu^{(1)} = \nu^{(2)} = 0.3$$

Assuming only the x-direction is considered in the static and dynamic approaches which were developed in Section III, then the following results are obtained: (1) static case - the result obtained from equation (56) can be expressed as $K_{55}=0.651$, (II) dynamic case - the result obtained by equating eqs. (D-13) and (D-27) is $K_{55}=0.350$.

When the three-layer, laminate thickness parameter is chosen to be $z_2/z_1 = 1$, the analysis given in Section III can be reduced to a single-layer homogeneous, isotropic material. Thus, using the data $E_{11}^{(1)}/E_{11}^{(2)} = C_{55}^{(1)}/C_{55}^{(2)} = \rho^{(1)}/\rho^{(2)} = 1$ and $\nu^{(1)} = \nu^{(2)} = 0.3$, the static case gives $K_{55} = 0.833$ and dynamic case results in $K_{55} = 0.822$. These results are identical with the values of the static and the dynamic shear factors obtained for single-layer plates by previous investigators (refs. 44 and 16).

Example 2: Multiple alternating layers of two materials. - The geometrical and material property ratios are given as

$$h^{(2)}/h^{(1)} = 1, \quad E_{11}^{(1)}/E_{11}^{(2)} = C_{55}^{(1)}/C_{55}^{(2)} = \text{from } 0.01 \text{ to } 100$$

The laminates vary from two layers to nine layers. Since the static case is a relatively simple algebraic expression which considers only longitudinal Young's and thickness-shear moduli of the individual layers ($E_{11}^{(k)}$ and $C_{55}^{(k)}$)and the laminate geometry, for design purposes, eq. (56) will be used. The results are shown in table 1 and figure 11.

5.2 Plate Simply Supported on All Edges

Here we consider only free vibration of plates made of composite material having material-symmetry axes coinciding with the geometric coordinates (plate edges) as shown in figure 12. Neglecting energy dissipation, the time integral of the difference between the potential and kinetic energies attains a stationary value. The displacement and rotations are proportional to $e^{i\omega t}$. The time dependence in the strain and kinetic energies cancel each other, i.e. only the amplitudes of the strain energy and kinetic energy need be considered. Thus, we use only the real part of homogeneous eqs. (C1-C6) with proper boundary conditions, see eq. (57), to apply Example 3.

Example 3: Three-layer, symmetric, isotropic plates - The elastic coefficients are those of isotropic material:

$$Q_{11}^{(k)} = Q_{22}^{(k)} = E^{(k)}/\left\{1 - [\nu^{(k)}]^2\right\}, Q_{12}^{(k)} = \nu^{(k)}E^{(k)}/\left\{1-[\nu^{(k)}]^2\right\}$$

$$Q_{44}^{(k)} = Q_{55}^{(k)} = Q_{66}^{(k)} = E^{(k)}/\left\{2[1+\nu^{(k)}]\right\}, \quad Q_{45}^{(k)}=0 \quad (k=1,2)$$

Geometrical and material property ratios correspond to the data of ref. 35, given as follows

$$z_2/z_1 = h^{(2)}/[h^{(2)}+2h^{(1)}] = 0.8, \quad E_{11}^{(1)}/E_{11}^{(2)} = C_{55}^{(1)}/C_{55}^{(2)} = 15,$$

$$\rho^{(1)}/\rho^{(2)} = 1, \quad \nu^{(1)} = \nu^{(2)} = 0.3$$

and the following dimensionless geometric parameter defined in ref. 35:

$$g_k^2 \equiv \left\{[m\pi h^{(k)}/a]^2 + [n\pi h^{(k)}/b]^2\right\}^{\frac{1}{2}} = \begin{cases} 0.0002\pi^2 & \text{for } k=1 \\ 0.0128\pi^2 & \text{for } k=2 \end{cases}$$

When the thickness of three-ply laminates is chossen to be $z_2/z_1=1$, it can be reduced to a single-layer material as a special case. Then the material properties are as follows:

$$Q_{ij}^{(1)} = Q_{ij}^{(2)} \qquad i,j = 1, 2, 4, 5, 6$$

$$E_{11}^{(1)} / E_{11}^{(2)} = C_{55}^{(1)}/C_{55}^{(2)} = \rho^{(1)}/\rho^{(2)} = 1 \; ; \; \nu^{(1)} = \nu^{(2)} = 0.3$$

$$g_1 = 0, \quad g_2 = 12.$$

A FORTRAN IV program is employed to compute the lowest eigenvalues of plates with simply supported edges. The two cases considered are: (1) single-layer homogeneous, isotropic, and (2) three-ply symmetrical construction. The results are shown as a solid line in figures 13 and 14. The static and dynamic shear factors calculated in examples 1 and 2 are also shown for comparison.

The static Jourawski shear theory and dynamic Timoskenko type theory appear to give good results for K as evaluated by inserting in laminated, shear flexible plate theory and comparing the lowest eigenvalues with those given in refs. 35 and 14.

5.3 Plate Free on All Edges

The problem considered here is forced vibration of a rectangular plate made of an arbitrary number of composite-material layers each having its major material-symmetry axis oriented at an arbitrary angle with geometrical coordinates as shown in figure A-2.

Since the present problem is one of steady-state harmonic excitation including material damping effects, the complete set of eqs. (C1-C6) with proper boundary conditions, eqs. (61), must be used.

Example 4: Boron/epoxy plate with twenty-four parallel plies oriented at an arbitrary angle. - The input-data material properties are Young's and shear moduli (E_{11}, E_{22}, C_{44}, C_{55}, C_{66}), major Poisson's ratio (ν_{12}), bending and twisting stiffnesses (D_{11}, D_{22}, D_{12}, D_{66}), and their corresponding loss tangents, as generated in ref. 47 from constituent-material experimental properties. It is noticed that all of the above input data are functions of frequency. Geometrical and material properties correspond to the data of ref. 13, as follows:

length, $a = 18.19$ in ; width, $b = 2.75$ in.

thickness, $h = 0.034$ in. ; density, $\rho = 0.000194$ lb-sec^2/in^4

angle of orientation, $\theta = 0^\circ, 10^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$

A FORTRAN IV program is employed to compute the eigenvectors of a plate with all edges free. To obtain the resonant frequencies of the first five modes, two different resonance criteria are applied. In the peak-amplitude method, the system is excited harmonically and the amplitude at a particular point ($\alpha = 0$, $\beta = 1$) is measured over a range of frequency. The amplitude of the response depends not only on the dynamic characteristics of the system, but also on the amplitude of the force applied to it. For a linear system the peak amplitude is taken as maximum displacement per unit amplitude of force. Throughout this report, this ratio is referred to as the response. A typical result of response with varying frequency is shown in figure 15 for an angle of orientation, $\theta = 10^\circ$. The complete results are as tabulated in Table II.

In the modified Kennedy-Pancu method, the modal shapes, phase relations between motions at various points, and coupling between the various degrees of freedom are assumed to be unaffected by the presence of a small amount of damping. Vectors are used for the analysis of the results of forced vibration by making a polar plot (called the Argand plane) of the displacement vector (amplitude and phase angle). In the vicinity of resonance, the resulting curve would be an approximately circular arc. Each point on the circle locus corresponds to a value of frequency.

In the modified Kennedy-Pancu method, the resonant frequency is defined to correspond to the point of the Argand-plot curve having maximum change in arc length (Δs) per fixed change in frequency ($\Delta\omega$). The change in arc length Δs can be determined approximately in analytical fashion as the response amplitude $(W/\tilde{q})$ multiplied by the difference in the phase angles corresponding to the two distinct frequencies (separated by $\Delta\omega$), i.e.

$$\Delta s = (W/\tilde{q})\ \Delta\varphi$$

The parameter $\Delta s/\Delta\omega$ is plotted versus excitation frequency in figure 16 for a specific angle of orientation, $\theta = 10^{\circ}$.

As can be seen in the figure, the curves exhibits sharp spikes and thus the Kennedy-Pancu resonant frequencies, i.e. the frequencies corresponding to the maximum points, can readily be found. The results obtained in this fashion for all of the plates used by Clary are tabulated in Table II.

A discussion of experimental methods used to determine material damping in composite materials is presented in Ap. H, ref. 40. Here the damping ratios are calculated by using the modified Kennedy-Pancu method as expressed in eq. (B-69), Ap. B, ref. 40. It is noted that the damping ratio is the ratio of material damping coefficient to critical material damping coefficient. The results associated with each mode are tabulated completely in Table II.

The nodal patterns of the first five modes are calculated from the response at selected points on the surface of the plate. Twenty-five points in the x direction and five points in the y direction are used. The nodal patterns are shown in figures 17-22. The results appear to be in good agreement with the data of ref. 13.

VI. CONCLUSIONS

The analyses in Sections II and IV were developed for the vibrational problem of composite-material plates with thickness-shear flexibility and material damping. The theory used in the analyses is that of Yang, Norris, and Stavsky (ref. 25), which can be considered to be the laminated, anisotropic version of Mindlin's dynamic plate theory.

To account for the effects of thickness-shear deformation, a shear correction factor K was introduced. A variety of methods to arrive at an appropriate shear factor have been proposed for homogeneous plates; the most popular ones resulted in K = 0.833 for a static distribution (ref. 44) and 0.822 for the dynamic case (ref. 26). Srinivas et al. (ref. 35) showed that use of either of these values for K in Mindlin's plate theory gives a very close approximation to the lower natural frequencies computed by exact, dynamic, three-dimensional elasticity theory for rather thick, homogeneous, isotropic rectangular plates simply supported on all edges.

Unfortunately to date no one has proposed a means of rational calculation of the shear factor for a laminated plate, Srinivas et al. made an exact, dynamic three-dimensional elasticity analysis of simply supported laminated plates. However, their analysis is quite tedious computationally and thus they have presented numerical results for only a very limited number of cases.

In the present investigation, both static (Jourauski shear theory) and dynamic (Mindlin type analysis for pure thickness-shear motion) approaches were used to derive a shear factor K for laminates.

An assessment of the accuracy of these theories was made by comparing the lowest eigenvalues calculated by the laminated, shear-flexible plate theory with the laminated, three-dimensional exact values given by Srivinvas et al. (ref. 35).

It has been found that the shear factor K for a laminate is not the same value as that for a homogeneous, isotropic member. The numberical examples indicated that the value of K depends on the layer properties and the stacking sequence of the laminate.

Using peak-amplitude and modified Kennedy-Pancu methods, the first five resonant frequencies were calculated for various laminated boron/epoxy plates with all edges free. The resonant frequencies obtained by the two techniques differed by only a very small amount, and were in good agreement with the results obtained both experimentally and analytically by Clary (ref. 13). Furthermore, the nodal patterns have been found to be satisfactory or as good as the data given by Clary. Finally, the damping-ratio results were in good agreement with the experimental ones obtained by Clary. No comparison with analytical results for lmainated-plate damping could be made, since they have not been available previously.

APPENDIX A

NOTATION AND TRANSFORMATION FOR ELASTIC COEFFICIENTS

The generalized Hooke's law (Cauchy equations) for a general anisotropic material in terms of rectangular coordinates x,y,z are as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{21} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \quad \text{(A-1)}$$

where σ_{ij} and ϵ_{ij} are stress and strain components, and $C_{k\ell}$ are the Cauchy stiffness coefficients for a three-dimensional body. It can be shown that the matrix of Cauchy coefficients is symmetric, provided that the material is conservative (elastic), so that

$$C_{\ell k} = C_{k\ell} \quad \text{(A-2)}$$

Usually a single layer of fiber-reinforced composite material has the fibers oriented more or less parallel to the top and bottom surfaces of the layer, i.e. material-symmetry plane XY coincides with surface plane xy. Thus, it is orthotropic in the yz and xz planes, and the shear-normal or cross-elasticity coefficients with subscripts i4 and i5 (i=1,2,3,6) vanish:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \tag{A-3}$$

The only exception to the above, i.e. the only case when the general form (A-1), rather than (A-3), must be used is the case of a shingle-laminated composite, as shown in figure A-1 (reference 48).

If, in addition to being parallel to the top and bottom faces of the layer, the fibers are also oriented in a direction parallel to one of the coordinate directions in the plane of the layer, the material-symmetry plates XY,YZ,ZX all coincide with the coordinate planes xy, yz,zx and the material is said to be completely <u>orthotropic</u>. In this case the Cauchy relations are:

$$\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{ZZ} \end{Bmatrix} = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{13} \\ C^*_{12} & C^*_{22} & C^*_{23} \\ C^*_{13} & C^*_{23} & C^*_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \epsilon_{ZZ} \end{Bmatrix} ; \begin{Bmatrix} \sigma_{YZ} \\ \sigma_{XZ} \\ \sigma_{XY} \end{Bmatrix} = \begin{Bmatrix} C^*_{44}\,\epsilon_{YZ} \\ C^*_{55}\,\epsilon_{XZ} \\ C^*_{66}\,\epsilon_{XY} \end{Bmatrix} \tag{A-4}$$

where the asterisks denote the orthotropic case.

In the case of a thin layer of fiber-reinforced composite material in which the thickness-normal stress (σ_{ZZ}) can be neglected (see Hypothesis H3, Section 2.1), the third equation in set (A-4) can be solved for ϵ_{ZZ} in terms of ϵ_{XX} and ϵ_{YY}. Then equations (A-4) can be simplified as follows:

$$\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \end{Bmatrix} = \begin{bmatrix} Q^*_{11} & Q^*_{12} \\ Q^*_{12} & Q^*_{22} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \end{Bmatrix} ; \begin{Bmatrix} \sigma_{YZ} \\ \sigma_{XZ} \\ \sigma_{XY} \end{Bmatrix} = \begin{Bmatrix} Q^*_{44} \ \epsilon_{YZ} \\ Q^*_{55} \ \epsilon_{XZ} \\ Q^*_{66} \ \epsilon_{XY} \end{Bmatrix} \tag{A-5}$$

where the reduced orthotropic stiffness coefficients ($Q^*_{k\ell}$) are related to the Cauchy three-dimensional orthotropic stiffness coefficients ($C^*_{k\ell}$) as follows:

$$\begin{aligned} Q^*_{11} &\equiv C^*_{11} - (C^*_{13})^2/C^*_{33} \quad ; \quad Q^*_{12} \equiv C^*_{12} - (C^*_{13}\, C^*_{23}/C^*_{33}) \ ; \\ Q^*_{22} &\equiv C^*_{22} - (C^*_{23})^2/C^*_{33} \quad ; \quad Q^*_{ij} \equiv C^*_{ij} \ (ij = 44,55,66) \end{aligned} \tag{A-6}$$

Equation (A-5) can be written in the form of a single matrix equation as follows:

$$\{\sigma_{IJ}\} = [Q^*_{k\ell}]\,\{\epsilon_{IJ}\} \tag{A-7}$$

where

$$\left\{\sigma_{IJ}\right\} \equiv \left\{\begin{array}{c} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{YZ} \\ \sigma_{XZ} \\ \sigma_{XY} \end{array}\right\} , \quad \left\{\epsilon_{IJ}\right\} \equiv \left\{\begin{array}{c} \epsilon_{XX} \\ \epsilon_{YY} \\ \frac{1}{2}\epsilon_{YZ} \\ \frac{1}{2}\epsilon_{XZ} \\ \frac{1}{2}\epsilon_{XY} \end{array}\right\} \qquad \text{(A-8)}$$

$$[Q^*_{k\ell}] \equiv \begin{bmatrix} Q^*_{11} & Q^*_{12} & 0 & 0 & 0 \\ Q^*_{12} & Q^*_{22} & 0 & 0 & 0 \\ 0 & 0 & 2Q^*_{44} & 0 & 0 \\ 0 & 0 & 0 & 2Q^*_{55} & 0 \\ 0 & 0 & 0 & 0 & 2Q^*_{66} \end{bmatrix} \qquad \text{(A-9)}$$

where the presence of the factor 1/2 in the ϵ_{IJ} matrix is used to make it a second-rank tensor and this necessitates the presence of the factor 2 appearing in equation (A-9).

In structural-panel applications of composite materials, usually there are design requirements for multiple orientations of fiber-reinforced composite-material layers. Therefore, it is essential that a set of transformation relations be used to calculate the stiffness coefficients for any desired orientation from that associated with the major material-symmetry direction (fiber direction) as shown in figure A-2. Such relations are developed in the ensuing paragraphs.

Arbitrary orthogonal axes in the plane of the laminate and making an angle θ with the material-symmetry axes (X,Y) are designated as the x,y axes. Then the components of stress and strain can be transformed from the x,y axes to the X,Y axes as follows:

$$\left\{\sigma_{IJ}\right\} = [T_r]\left\{\sigma_{ij}\right\} \tag{A-10}$$

$$\left\{\epsilon_{IJ}\right\} = [T_r]\left\{\epsilon_{ij}\right\} \tag{A-11}$$

where $\left\{\sigma_{IJ}\right\}$ and $\left\{\epsilon_{IJ}\right\}$ are as defined in equations (A-8); $\left\{\sigma_{ij}\right\}$ and $\left\{\epsilon_{ij}\right\}$ are similar except that the subscripts X,Y,Z are replaced by the subscripts x,y,z; and $[T_r]$ is the transformation, defined as follows:

$$[T_r] \equiv \begin{bmatrix} m^2 & n^2 & 0 & 0 & -2mn \\ n^2 & m^2 & 0 & 0 & 2mn \\ 0 & 0 & m & -n & 0 \\ 0 & 0 & n & m & 0 \\ mn & -mn & 0 & 0 & m^2-n^2 \end{bmatrix} \tag{A-12}$$

where

$$m \equiv \cos\theta \quad , \quad n \equiv \sin\theta. \tag{A-13}$$

For the case when $\theta \neq 0$, we have

$$\left\{\sigma_{ij}\right\} = [Q_{k\ell}]\left\{\epsilon_{ij}\right\} \tag{A-14}$$

In view of equations (A-7,A-10,A-11,A-14),

$$[Q_{k\ell}] = [T_r]^{-1}[Q^*_{k\ell}][T_r] \tag{A-15}$$

By substituting equations (A-9) and (A-12) and then performing the matrix operations indicated in equation (A-15) the elements of the $[Q_{k\ell}]$ matrix are related to those of the $[Q^*_{k\ell}]$ matrix as follows:

$$\begin{aligned}
Q_{11} &= Q^*_{11}m^4 + 2\ (Q^*_{12} + 2Q^*_{66})\ m^2n^2 + Q^*_{22}n^4 \\
Q_{12} &= Q^*_{12}m^4 + (Q^*_{11} + Q^*_{22} - 4Q^*_{66})m^2n^2 + Q^*_{12}n^4 \\
Q_{22} &= Q^*_{22}m^4 + 2(Q^*_{12} + 2Q^*_{66})m^2n^2 + Q^*_{11}n^4 \\
Q_{66} &= Q^*_{66}m^4 + (Q^*_{11} + Q^*_{22} - 2Q^*_{12} - 2Q^*_{66})m^2n^2 + Q^*_{66}n^4 \\
Q_{16} &= (2Q^*_{66} + Q^*_{12} - Q^*_{11})m^3n - (2Q^*_{66} + Q^*_{12} - Q^*_{22})mn^3 \qquad \text{(A-16)} \\
Q_{26} &= -(2Q^*_{66} + Q^*_{12} - Q^*_{22})m^3n + (2Q^*_{66} + Q^*_{12} - Q^*_{11})mn^3 \\
Q_{44} &= Q^*_{44}m^2 + Q^*_{55}n^2 \\
Q_{45} &= Q^*_{55}mn - Q^*_{44}mn \\
Q_{55} &= Q^*_{55}m^2 + Q^*_{44}n^2
\end{aligned}$$

APPENDIX B

DERIVATION OF ENERGY DIFFERENCE

It is assumed that the generalized coordinate ξ and generalized force f have the following forms:

$$\xi = \tilde{\xi}\, e^{i\omega t}$$
$$f = \tilde{q}\, e^{i\omega t} \qquad \text{(B-1)}$$

where $\tilde{\xi}$ is a complex form representing $\tilde{u}_o, \tilde{v}_o, \tilde{w}_o, \tilde{\psi}_x$ and $\tilde{\psi}_y$.

The strain energy can be expressed as

$$U = (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}\, \xi_i \xi_j = \tilde{U}\, e^{2i\omega t} \qquad \text{(B-2)}$$

where Q_{ij} are stiffness coefficients and $\tilde{U}$ is the amplitude of the strain energy given by:

$$\tilde{U} = (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}\, \tilde{\xi}_i\, \tilde{\xi}_j \qquad \text{(B-3)}$$

For the damping energy, a dissipation function suitable for material damping (reference 49) is developed. Hence, we introduce a dissipation energy function for material damping as follows:

$$D = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (b_{ij}/\omega)\ \dot{\xi}_i\, \dot{\xi}_j = (i\omega/2) \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{Q}_{ij}\, \tilde{\xi}_i \tilde{\xi}_j\, e^{2i\omega t} = (i\omega)\tilde{D}\, e^{2i\omega t}$$

(B-4)

where b_{ij} are structural damping coefficients, $\hat{Q}_{ij} = ib_{ij}$ and $\tilde{D}$ is the amplitude of dissipation energy given by:

$$\tilde{D} = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{Q}_{ij}\, \tilde{\xi}_i \tilde{\xi}_j \qquad \text{(B-5)}$$

The kinetic energy of the system is

$$T = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}\, \dot{\xi}_i \dot{\xi}_j = -(\omega^2/2) \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \tilde{\xi}_i \tilde{\xi}_j\, e^{2i\omega t} = -\tilde{T}\, e^{2i\omega t}$$

$$\text{(B-6)}$$

where m_{ij} are inertia coefficients and $\tilde{T}$ is the amplitude of kinetic energy given by

$$\tilde{T} = (\omega^2/2) \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}\, \tilde{\xi}_i\, \tilde{\xi}_j \qquad \text{(B-7)}$$

The work done by the uniformly distributed normal force is

$$W = f \sum_{i=1}^{n} \xi_i = \tilde{q}\, e^{i\omega t} \sum_{i=1}^{n} \xi_i = \tilde{W}\, e^{2i\omega t} \qquad \text{(B-8)}$$

where $\tilde{q}$ is the amplitude of the normal pressure and $\tilde{W}$ is the amplitude of the work done

$$\tilde{W} = \tilde{q} \sum_{i=1}^{n} \tilde{\xi}_i \qquad \text{(B-9)}$$

The Lagrangian equation takes the following form (reference 50)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) + \frac{\partial D}{\partial \dot{\xi}} + \frac{\partial U}{\partial \xi} = f \qquad \text{(B-10)}$$

Each term of equation (B-10) can be expressed in terms of its amplitude as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) = - \frac{\partial \widetilde{T}}{\partial \widetilde{\xi}} e^{i\omega t} , \quad \frac{\partial D}{\partial \dot{\xi}} = \frac{\partial \widetilde{D}}{\partial \widetilde{\xi}} e^{i\omega t}$$

$$\text{(B-11)}$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial \widetilde{U}}{\partial \widetilde{\xi}} e^{i\omega t} , \quad f = \frac{\partial W}{\partial \xi} = \frac{\partial \widetilde{W}}{\partial \widetilde{\xi}} e^{i\omega t}$$

With equations (B-11) introduced, one can rewrite equation (B-10) as follows:

$$\frac{\partial}{\partial \widetilde{\xi}} (\widetilde{T} + \widetilde{W} - \widetilde{U} - \widetilde{D}) \quad e^{i\omega t} = 0 \qquad \text{(B-12)}$$

or

$$\frac{\partial}{\partial \widetilde{\xi}} (\widetilde{T} + \widetilde{W} - \widetilde{U} - \widetilde{D}) = 0 \qquad \text{(B-13)}$$

Equation (B-13) is analogous to one presented by Volterra (reference 51).

Thus, the amplitude of the Lagrangian energy difference can be expressed as follows

$$\widetilde{L} = (\widetilde{T} + \widetilde{W}) - (\widetilde{U} + \widetilde{D}) \qquad \text{(B-14)}$$

when used in conjunction with Hamilton's principle, expressed in equation (36).

APPENDIX C

COMPLETE ENERGY EXPRESSIONS

C1. The Energy Difference

$$\tilde{L} = \frac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{N}\int_0^1\int_0^1 \Big\{(\bar{A}_{11}/a^2)(U_{mn}\Phi'_{um}\Phi_{un})^2+(2\bar{A}_{12}/ab)(U_{mn}\Phi'_{um}\Phi_{un})(V_{mn}\Phi_{vm}\Phi'_{vn})$$

$$+2\bar{A}_{16}(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})$$

$$+2\bar{A}_{26}(b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})+\bar{A}_{66}(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})^2$$

$$+2\bar{B}_{11}(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})+2\bar{B}_{12}[(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn}$$

$$+b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})]+2\bar{B}_{22}(b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})$$

$$+2\bar{B}_{16}[(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})+a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}$$

$$+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})]+2\bar{B}_{26}[b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}$$

$$+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})+(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})]$$

$$+2\bar{B}_{66}(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})(a^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})$$

$$+(\bar{D}_{11}/a^2)(\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})^2+(2\bar{D}_{12}/ab)(\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})(\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})$$

$$+(\bar{D}_{22}/b^2)(\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})^2+2\bar{D}_{16}(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}$$

$$+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})+2\bar{D}_{26}(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})$$

$$+\bar{D}_{66}(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})^2+K_{55}\bar{A}_{55}(a^{-1}W_{mn}\Phi'_{wm}\Phi_{wn}+\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn})^2$$

$$+2K_{45}\bar{A}_{45}(a^{-1}W_{mn}\Phi'_{wm}\Phi_{wn}+\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn})(b^{-1}W_{mn}\Phi_{wm}\Phi'_{wn}+\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn})$$

$$+K_{44}\bar{A}_{44}(b^{-1}W_{mn}\Phi_{wn}\Phi'_{wn}+\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn})^2\Big\}\; ab\; d\alpha\; d\beta$$

$$-(\omega^2/2)\sum_{m=1}^{M}\sum_{n=1}^{N}\int_0^1\int_0^1 \Big\{m_o[(U_{mn}\Phi_{um}\Phi_{un})^2+(V_{mn}\Phi_{vm}\Phi_{vn})^2+(W_{mn}\Phi_{wm}\Phi_{wn})^2]$$

$$+2m_1[(U_{mn}\Phi_{um}\Phi_{un})(\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn})+(V_{mn}\Phi_{vm}\Phi_{vn})(\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn})]$$

$$+m_2\left[(\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn})^2+(\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn})^2\right]\Big\}\ ab\ d\alpha\ d\beta$$

$$-\sum_{m=1}^{M}\sum_{n=1}^{N}\int_o^1\int_o^1 \tilde{q}\ W_{mn}\Phi_{wm}\Phi_{wn}\ ab\ d\alpha\ d\beta \qquad (C-1)$$

where

$$\Phi_m' = \partial\Phi_m/\partial\alpha\ ,\Phi_n' = \partial\Phi_n/\partial\beta$$

C2. Equations for Minimizing the Energy Difference

$$\partial\tilde{L}/\partial U_{k\ell} = (b/a)\bar{A}_{11}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_o^1 \Phi_{um}'\Phi_{uk}'\ d\alpha \int_o^1 \Phi_{un}\Phi_{u\ell}d\beta$$

$$+\bar{A}_{12}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_o^1 \Phi_{vm}\Phi_{uk}'d\alpha\int_o^1 \Phi_{vn}'\Phi_{u\ell}d\beta$$

$$+(b/a)\bar{A}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_o^1 \Phi_{vm}'\Phi_{uk}'d\alpha\int_o^1 \Phi_{vn}\Phi_{u\ell}d\beta$$

$$+\bar{A}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_o^1 \Phi_{um}\Phi_{uk}'d\alpha\int_o^1 \Phi_{un}'\Phi_{u\ell}d\beta$$

$$+\bar{A}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_o^1 \Phi_{um}'\Phi_{uk}d\alpha\int_o^1 \Phi_{un}\Phi_{u\ell}'d\beta$$

$$+(a/b)\bar{A}_{26}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_o^1 \Phi_{vm}\Phi_{uk}d\alpha\int_o^1 \Phi_{vn}'\Phi_{u\ell}'d\beta$$

$$+\bar{A}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_o^1 \Phi_{vm}'\Phi_{uk}d\alpha\int_o^1 \Phi_{vn}\Phi_{u\ell}'d\beta$$

$$+(a/b)\bar{A}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_o^1 \Phi_{um}\Phi_{uk}d\alpha\int_o^1 \Phi_{un}'\Phi_{u\ell}'d\beta$$

$$+(b/a)\bar{B}_{11}\sum_{m=1}^{M}\sum_{n=1}^{N} \Psi_{xmn}\int_o^1 \Phi_{\psi xm}'\Phi_{uk}'d\alpha\int_o^1 \Phi_{\psi xn}\Phi_{u\ell}d\beta$$

$$+\bar{B}_{12}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi'_{uk}d\alpha\int_0^1\Phi'_{\psi yn}\Phi_{u\ell}d\beta$$

$$+(b/a)\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi'_{\psi ym}\Phi'_{uk}d\alpha\int_0^1\Phi_{\psi yn}\Phi_{u\ell}d\beta$$

$$+\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi'_{uk}d\alpha\int_0^1\Phi'_{\psi xn}\Phi_{u\ell}d\beta$$

$$+\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi'_{\psi xm}\Phi_{uk}d\alpha\int_0^1\Phi_{\psi xn}\Phi'_{u\ell}d\beta$$

$$+(a/b)\bar{B}_{26}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi_{uk}d\alpha\int_0^1\Phi'_{\psi yn}\Phi'_{u\ell}d\beta$$

$$+\bar{B}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi'_{\psi ym}\Phi_{uk}d\alpha\int_0^1\Phi_{\psi yn}\Phi'_{u\ell}d\beta$$

$$+(a/b)\bar{B}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{uk}d\alpha\int_0^1\Phi'_{\psi xn}\Phi'_{u\ell}d\beta$$

$$-ab\,\omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_oU_{mn}\int_0^1\Phi_{um}\Phi_{uk}d\alpha\int_0^1\Phi_{un}\Phi_{u\ell}d\beta$$

$$-ab\,\omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_1\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{uk}d\alpha\int_0^1\Phi_{\psi xn}\Phi_{u\ell}d\beta = 0 \qquad \text{(C-2)}$$

There is another equation, for $\partial\tilde{L}/\partial V_{k\ell}=0$, which is analogous to eq. (C-2) with the following substitutions:

$$u \leftrightarrow v\ , \quad U \leftrightarrow V,\ x \leftrightarrow y,\quad 1 \leftrightarrow 2,\quad \Phi \leftrightarrow \Phi',\quad a \leftrightarrow b. \qquad \text{(C-3)}$$

$$\partial\tilde{L}/\partial W_{k\ell} = (b/a)K_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn}\int_0^1 \Phi'_{wm}\Phi'_{wk}d\alpha\int_0^1 \Phi_{wn}\Phi_{w\ell}d\beta$$

$$+bK_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1 \Phi_{\psi xm}\Phi'_{wk}d\alpha\int_0^1 \Phi_{\psi xn}\Phi_{wxn}\Phi_{w\ell}d\beta$$

$$+K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn}\int_0^1 \Phi'_{wm}\Phi_{wk}d\alpha\int_0^1 \Phi_{wn}\Phi'_{w\ell}d\beta$$

$$+K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn}\int_0^1 \Phi_{wm}\Phi'_{wk}d\alpha\int_0^1 \Phi'_{wn}\Phi_{w\ell}d\beta$$

$$+aK_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1 \Phi_{\psi xm}\Phi_{wk}d\alpha\int_0^1 \Phi_{\psi xn}\Phi'_{w\ell}d\beta$$

$$+b\ K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1 \Phi_{\psi ym}\Phi'_{wk}d\alpha\int_0^1 \Phi_{\psi yn}\Phi_{w\ell}d\beta$$

$$+(a/b)K_{44}\bar{A}_{44}\sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn}\int_0^1 \Phi_{wm}\Phi_{wk}d\alpha\int_0^1 \Phi'_{wn}\Phi'_{w\ell}d\beta$$

$$+aK_{44}\bar{A}_{44}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1 \Phi_{\psi ym}\Phi_{wk}d\alpha\int_0^1 \Phi_{\psi yn}\Phi'_{w\ell}d\beta$$

$$-ab\ \omega^2\sum_{m=1}^{M}\sum_{n=1}^{N} m_o\ W_{mn}\int_0^1 \Phi_{wm}\Phi_{wk}d\alpha\int_0^1 \Phi_{wn}\Phi_{w\ell}d\beta$$

$$-ab\tilde{q}\ \sum_{m=1}^{M}\sum_{n=1}^{N}\int_0^1 \Phi_{wk}d\alpha\int_0^1 \Phi_{w\ell}d\beta = 0 \qquad \text{(C-4)}$$

$$\partial\tilde{L}/\partial\Psi_{xk\ell} = (b/a)\bar{B}_{11}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_0^1 \Phi'_{um}\Phi'_{\psi xk}d\alpha\int_0^1 \Phi_{un}\Phi_{\psi x\ell}d\beta$$

$$+\bar{B}_{12} \sum_{m=1}^{M} \sum_{n=1}^{N} V_{mn} \int_0^1 \Phi_{vm} \Phi'_{\psi xk} d\alpha \int_0^1 \Phi'_{vn} \Phi_{\psi x\ell} d\beta$$

$$+\bar{B}_{16} \sum_{m=1}^{M} \sum_{n=1}^{N} U_{mn} \int_0^1 \Phi'_{um} \Phi_{\psi xk} d\alpha \int_0^1 \Phi_{un} \Phi'_{\psi x\ell} d\beta$$

$$+(b/a)\bar{B}_{16} \sum_{m=1}^{M} \sum_{n=1}^{N} V_{mn} \int_0^1 \Phi'_{vm} \Phi'_{\psi xk} d\alpha \int_0^1 \Phi_{vn} \Phi_{\psi x\ell} d\beta$$

$$+\bar{B}_{16} \sum_{m=1}^{M} \sum_{n=1}^{N} U_{mn} \int_0^1 \Phi_{um} \Phi'_{\psi xk} d\alpha \int_0^1 \Phi'_{un} \Phi_{\psi x\ell} d\beta$$

$$+(a/b)\bar{B}_{26} \sum_{m=1}^{M} \sum_{n=1}^{N} V_{mn} \int_0^1 \Phi_{vm} \Phi_{\psi xk} d\alpha \int_0^1 \Phi'_{vn} \Phi'_{\psi x\ell} d\beta$$

$$+\bar{B}_{66} \sum_{m=1}^{M} \sum_{n=1}^{N} V_{mn} \int_0^1 \Phi'_{vm} \Phi_{\psi xk} d\alpha \int_0^1 \Phi_{vn} \Phi'_{\psi x\ell} d\beta$$

$$+(a/b)\bar{B}_{66} \sum_{m=1}^{M} \sum_{n=1}^{N} U_{mn} \int_0^1 \Phi_{um} \Phi_{\psi xk} d\alpha \int_0^1 \Phi'_{un} \Phi'_{\psi x\ell} d\beta$$

$$+(b/a)\bar{D}_{11} \sum_{m=1}^{M} \sum_{n=1}^{N} \Psi_{xmn} \int_0^1 \Phi'_{\psi xm} \Phi'_{\psi xk} d\alpha \int_0^1 \Phi_{\psi xn} \Phi_{\psi x\ell} d\beta$$

$$+\bar{D}_{12} \sum_{m=1}^{M} \sum_{n=1}^{N} \Psi_{ymn} \int_0^1 \Phi_{\psi ym} \Phi'_{\psi xk} d\alpha \int_0^1 \Phi'_{\psi yn} \Phi_{\psi x\ell} d\beta$$

$$+(b/a)\bar{D}_{16} \sum_{m=1}^{M} \sum_{n=1}^{N} \Psi_{ymn} \int_0^1 \Phi'_{\psi ym} \Phi'_{\psi xk} d\alpha \int_0^1 \Phi_{\psi yn} \Phi_{\psi x\ell} d\beta$$

$$+\bar{D}_{16} \sum_{m=1}^{M} \sum_{n=1}^{N} \Psi_{xmn} \int_0^1 \Phi'_{\psi xm} \Phi_{\psi xk} d\alpha \int_0^1 \Phi_{\psi xn} \Phi'_{\psi x\ell} d\beta$$

$$+\bar{D}_{16} \sum_{m=1}^{M} \sum_{n=1}^{N} \Psi_{xmn} \int_0^1 \Phi_{\psi xm} \Phi'_{\psi xk} d\alpha \int_0^1 \Phi'_{\psi xn} \Phi_{\psi x\ell} d\beta$$

$$+(a/b)\bar{D}_{26}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi_{\psi xk}d\alpha\int_0^1\Phi'_{\psi yn}\Phi'_{\psi x\ell}d\beta$$

$$+\bar{D}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi'_{\psi ym}\Phi_{\psi x\ell}d\alpha\int_0^1\Phi_{\psi yn}\Phi'_{\psi x\ell}d\beta$$

$$+(a/b)\bar{D}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{\psi xk}d\alpha\int_0^1\Phi'_{\psi xn}\Phi'_{\psi x\ell}d\beta$$

$$+bK_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi'_{wm}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{wn}\Phi_{\psi x\ell}d\beta$$

$$+ab\ K_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{\psi xn}\Phi_{\psi x\ell}d\beta$$

$$+a\ K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi_{wm}\Phi_{\psi xk}d\alpha\int_0^1\Phi'_{wn}\Phi_{\psi x\ell}d\beta$$

$$+ab\ K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{\psi yn}\Phi_{\psi x\ell}d\beta$$

$$-ab\ \omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_1U_{mn}\int_0^1\Phi_{um}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{un}\Phi_{\psi x\ell}d\beta$$

$$-ab\ \omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_2\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{\psi xn}\Phi_{\psi x\ell}d\beta = 0 \qquad \text{(C-5)}$$

There is an analogous expression for $\partial\tilde{L}/\partial\Psi_{yk\ell}=0$, using the transformations (C-3), plus the following additional one:

$$45 \leftrightarrow 55 \qquad \text{(C-6)}$$

APPENDIX D

DERIVATION OF THICKNESS-SHEAR FACTOR FOR THE THREE-LAYER, SYMMETRICALLY LAMINATED CASE USING THE DYNAMIC APPROACH

D1. Dynamic Elasticity Analysis of an Individual Layer Undergoing Pure Thickness-Shear Motion

For an individual layer undergoing pure thickness-shear motion in the xz plane, the only non-zero strain component is ϵ_{xz}. Now it is assumed that the layer is orthotropic, so that the only non-zero stress component is σ_{xz}, given by equation (49). Then the general stress equations of motion for three-dimensional, dynamic elastic theory reduce to the following two equations:

$$\sigma_{xz,z} = \rho u,_{tt} \quad ; \quad \sigma_{xz,x} = \rho w,_{tt} \tag{D-1}$$

The longitudinal thickness-shear strain can be calculated from the displacements (u,w) in the x,z directions, respectively, as follows:

$$\epsilon_{xz} = u,_z + w,_x \tag{D-2}$$

The following displacement equations of motion are obtained by substituting equations (49) and (E-2) into equations (E-1):

$$c_s^2(u,_{zz} + w,_{xz}) = u,_{tt} \; ; \; c_s^2(u,_{xz} + w,_{xx}) = w,_{tt} \tag{D-3}$$

where c_s is the shear wave-propagation velocity defined by:

$$c_s^2 = C_{55}/\rho \tag{D-4}$$

However, for pure longitudinal thickness-shear motion, the displacements are independent of axial position x. Thus, all derivatives of displacements with respect to x vanish and equations (D-3) reduce to the following single expression:

$$c_s^2 \, u_{,zz} = u_{,tt} \tag{D-5}$$

Equation (D-5) is the familiar, one-dimensional wave equation, which is solved easily by the separation-of-variables method, with the following solution for simple harmonic motion:

$$u(z,t) = (A \cos \Omega_s z + B \sin \Omega_s z)(C_1 \cos \omega t + C_2 \sin \omega t) \tag{D-6}$$

where $\Omega_s \equiv \omega/c_s$, A and B are constants depending upon the boundary conditions, and C_1 and C_2 are constants which depend upon the initial conditions.

D2. Dynamic Elasticity Analysis for Three-Layer, Symmetrically Laminated Case

Here we consider the special case of a three-layer, symmetrically laminated member, in which the two identical outer layers are designated by superscript 1 and the middle layer is denoted by superscript 2, as shown in figure 10. The proper boundary conditions for the anti-symmetric modes in pure thickness-shear motion are

$$u^{(1)}(z_2,t) = u^{(2)}(z_2,t) \; ; \; u,_z^{(1)}(z_1,t) = 0 \; ;$$
$$u,_z^{(1)}(z_2,t) = u,_z^{(2)}(z_2,t) \; ; \; u^{(2)}(0,t) = 0 \tag{D-7}$$

where z_1 and z_2 are dimensions shown in figure 9.

Substituting equation (D-6) into equations (D-7) yields the following set of expressions:

$$A_1 \cos \Omega_s^{(1)} z_2 + B_1 \sin \Omega_s^{(1)} z_2 = A_1 \cos \Omega_s^{(2)} z_2$$
$$+ B_2 \sin \Omega_s^{(2)} z_2 \; ; \; - \Omega_s^{(1)} A_1 \sin \Omega_s^{(1)} z_2 + \Omega_s^{(1)} B_1 \cos \Omega_s^{(1)} z_2 =$$
$$- \Omega_s^{(2)} A_2 \sin \Omega_s^{(2)} z_2 + \Omega_s^{(2)} B_2 \cos \Omega_s^{(2)} z_2 \; ; \; -\Omega_s^{(1)} A_1 \sin \Omega_s^{(1)} z_1$$
$$+ \Omega_s^{(1)} B_1 \cos \Omega_s^{(1)} z_1 = 0 \; ; \; A_2 = 0 \tag{D-8}$$

or

$$\begin{bmatrix} \cos \Omega_s^{(1)} z_2 & \sin \Omega_s^{(1)} z_2 & - \sin \Omega_s^{(2)} z_2 \\ -\Omega_s^{(1)} \sin \Omega_s^{(1)} z_2 & \Omega_s^{(1)} \cos \Omega_s^{(1)} z_2 & -\Omega_s^{(2)} \cos \Omega_s^{(2)} z_2 \\ -\Omega_s^{(1)} \sin \Omega_s^{(1)} z_1 & \Omega_s^{(1)} \cos \Omega_s^{(1)} z_1 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{D-9}$$

The homogeneous system of linear algebraic equations (D-9) has a nontrivial solution if, and only if, the determinant of its coefficient matrix is equal to zero. The resulting determinantal equation has as its solutions the roots of the following transcendental equation:

$$\Omega_s^{(1)} \tan \Omega_s^{(2)} z_2 = - \Omega_s^{(2)} \cot (\Omega_s^{(1)} z_2 - \Omega_s^{(1)} z_1) \qquad \text{(D-10)}$$

From the definitions of $\Omega^{(k)}$ and $c_s^{(k)}$, we have

$$\Omega_s^{(1)}/\Omega_s^{(2)} = c_s^{(2)}/c_s^{(1)} = (C_{55}^{(2)}/C_{55}^{(1)})^{\frac{1}{2}}(\rho^{(1)}/\rho^{(2)})^{\frac{1}{2}} = (R/\beta)^{\frac{1}{2}} \qquad \text{(D-11)}$$

where

$$\beta \equiv C_{55}^{(1)}/C_{55}^{(2)} \quad ; \quad R \equiv \rho^{(1)}/\rho^{(2)} \qquad \text{(D-12)}$$

Then equation (D-10) can be expressed as follows:

$$(R/\beta)^{\frac{1}{2}} \tan [(\zeta_2 \Omega_s^{(1)} z_1 (\beta/R)^{\frac{1}{2}}] = \cot [(1-\zeta_2)\Omega_s^{(1)} z_1] \qquad \text{(D-13)}$$

where

$$\zeta_2 \equiv z_2/z_1 \qquad \text{(D-14)}$$

D3. Dynamic Analysis of a Symmetrically Laminated Timoshenko Beam Undergoing Pure Thickness-Shear Motion

Here we consider a symmetrically laminated Timoshenko beam*. The

*A beam exhibiting both thickness-shear flexibility and rotatory inertia is generally referred to as a Timoshenko beam(refs. 42,43).

axial and thickness (or depth) directions are designated as the x and z axes, respectively. Such a beam could be analyzed as a special case of the laminated plate theory presented in Section II by merely deleting all derivatives with respect to y. However, the beam case is so much simpler and pure thickness-shear motion is such a simple type of motion; therefore, it was decided to make an exact analysis for the present case.

The following kinematic relations hold throughout the entire thickness of the laminate:

$$\kappa_{xx} = \psi_{x,x} \quad ; \quad \epsilon_{xz} = w_{o,x} + \psi_x \tag{D-15}$$

The following stress-strain relations are applicable to a typical layer "k":

$$\sigma_{xx}^{(k)} = E_{11}^{(k)} \epsilon_{xx} = E_{11}^{(k)} \kappa_{xx} z \; ; \; \sigma_{xz}^{(k)} = C_{55}^{(k)} \epsilon_{xz} \tag{D-16}$$

The bending moment and shear force, expressed on the basis of a unit width as in plate theory (Section 2.3) are:

$$M_{xx} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} z \, dz \quad ; \quad Q_x = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xz}^{(k)} \, dz \tag{D-17}$$

Substituting equations (D-15) and (D-16) into equation (D-17) and introducing the shear factor K_{55} as a correction factor to be determined later, one obtains:

$$M_{xx} = D_{11} \psi_{x,x} \quad ; \quad Q_x = K_{55} A_{55} (w_{o,x} + \psi_x) \tag{D-18}$$

where

$$\left\{D_{11}\ ,\ A_{55}\right\} \equiv \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{E_{11}^{(k)} z^2\ ,\ C_{55}^{(k)}\right\} dz \tag{D-19}$$

The equations of motion for a symmetrically laminated Timoshenko beam are identical in form to those governing a homogeneous Timoshenko beam (refs. 41,42), namely:

$$Q_{x,x} = m_o\, w_{o,tt} \quad ; \quad M_{xx,x} - Q_x = m_2 \psi_{x,tt} \tag{D-20}$$

where m_o and m_2 are defined in equations (28).

Inserting equations (D-18) into equation (D-20), one obtains the following set of two coupled equations of motion in terms of the generalized displacements w_o and ψ_x:

$$K_{55}\, A_{55}\, (w_{o,xx} + \psi_{x,x}) = m_o\, w_{o,tt} \tag{D-21}$$

$$D_{11}\, \psi_{x,xx} - K_{55}\, A_{55}\, (w_{o,x} + \psi_x) = m_2\, \psi_{x,tt} \tag{D-22}$$

For pure thickness-shear motion, w_o and ψ_x are independent of axial position x, so that equations (D-21) and (D-22) uncouple and become:

$$w_{o,tt} = 0 \tag{D-23}$$

$$m_2\, \psi_{x,tt} + K_{55}\, A_{55}\, \psi_x = 0 \tag{D-24}$$

Since equation (D-23) does not contain K_{55} and since w_o is not in equation (D-24), we have no further need for equation (D-23).

For steady-state harmonic motion, the solution of equation (D-24) can be expressed as follows:

$$\psi_x = \tilde{\psi}_x e^{i\omega t} \tag{D-25}$$

where $\tilde{\psi}_x$ is a constant.

Substituting equation (D-25) into equation (D-24), we are led to the following relationship:

$$\omega^2 = K_{55} A_{55}/m_2 \tag{D-26}$$

This equation is applicable to any symmetrical laminate. For the special case of a three-layer one, using the notation depicted in figure 10 and the definitions of A_{55} and m_2 from equations (D-19) and (28), one obtains:

$$\omega^2 = 3K_{55} z_1^{-2} (C_{55}^{(1)}/\rho^{(1)})[(\zeta_2/\beta) + 1-\zeta_2] \cdot [(\zeta_2^3/R) + 1 - \zeta_2^3]^{-1} \tag{D-27}$$

where β,R, and ζ_2 are as defined previously.

D4. Determinations of the Thickness-Shear Factor

The longitudinal thickness-shear factor, K_{55}, is determined implicitly by equating ω^2 associated with the lowest non-trivial solution of equation (D-13) to that given by equation (D-27).

APPENDIX E

IDENTIFICATION OF INTEGRAL FORMS

E1. Trigonometric Integrals

The integral forms for three boundary conditions are tabulated in the following form, where $m,k,n,\ell = 1$ to 2:

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{1mk} \equiv \int_0^1 \Phi_{um}\Phi_{uk}d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	$3/2$; $m=k\neq 0$ 1 ; $m\neq k$
$I_{1n\ell} \equiv \int_0^1 \Phi_{un}\Phi_{u\ell}d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$3/2$; $n=\ell\neq 0$ 1 ; $n\neq\ell$
$I_{2mk} \equiv \int_0^1 \Phi_{vm}\Phi_{vk}d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{2n\ell} \equiv \int_0^1 \Phi_{vn}\Phi_{v\ell}d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{3mk} \equiv \int_0^1 \Phi_{um}\Phi_{vk}d\alpha$	0	I_{1mk}	I_{1mk}
$I_{3n\ell} \equiv \int_0^1 \Phi_{un}\Phi_{v\ell}d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{4mk} \equiv \int_0^1 \Phi_{um}\Phi'_{uk}d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	0 ; $m=k\neq 0$ 0 ; $m\neq k$	0 ; $m=k\neq 0$ 0 ; $m\neq k$
$I_{4n\ell} \equiv \int_0^1 \Phi_{un}\Phi'_{v\ell}d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	0 ; $n=\ell\neq 0$ 0 ; $n\neq\ell$	0 ; $n=\ell\neq 0$ 0 ; $n\neq\ell$

Integral Form	Evaluation for Boundary Condition Listed: Simply Supported	Clamped Edges	Free Edges
$I_{5mk} \equiv \int_0^1 \Phi_{vm}\Phi'_{uk}d\alpha$	$k\pi/2$; $m=k\neq0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{5n\ell} \equiv \int_0^1 \Phi_{vn}\Phi'_{u\ell}d\beta$	$n\pi/2$; $n=\ell\neq0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{6mk} \equiv \int_0^1 \Phi_{um}\Phi'_{uk}d\alpha$	0	I_{4mk}	I_{4mk}
$I_{6n\ell} \equiv \int_0^1 \Phi_{un}\Phi'_{u\ell}d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{7mk} \equiv \int_0^1 \Phi_{vm}\Phi'_{vk}d\alpha$	0	I_{4mk}	I_{4mk}
$I_{7n\ell} \equiv \int_0^1 \Phi_{vn}\Phi'_{v\ell}d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{8mk} \equiv \int_0^1 \Phi'_{um}\Phi'_{uk}d\alpha$	$mk\pi^2/2$; $m=k\neq0$ 0 ; $m\neq k$	$2mk\pi^2$;$m=k\neq0$ 0 ; $m\neq k$	$2mk\pi^2$; $m=k\neq0$ 0 ; $m\neq k$
$I_{8n\ell} \equiv \int_0^1 \Phi'_{un}\Phi'_{u\ell}d\beta$	$n\ell\pi^2/2$; $n=\ell\neq0$ 0 ; $n\neq\ell$	$2n\ell\pi^2$;$n=\ell\neq0$ 0 ; $n\neq\ell$	$2n\ell\pi^2$; $n=\ell\neq0$ 0 ; $n\neq\ell$
$I_{9mk} \equiv \int_0^1 \Phi'_{vm}\Phi'_{vk}d\alpha$	$mk\pi^2/2$; $m=k\neq0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{9n\ell} \equiv \int_0^1 \Phi'_{vn}\Phi'_{v\ell}d\beta$	$n\ell\pi^2/2$; $n=\ell\neq0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{10mk} \equiv \int_0^1 \Phi'_{um}\Phi'_{vk}d\alpha$	0	I_{8mk}	I_{8mk}
$I_{10n\ell} \equiv \int_0^1 \Phi'_{un}\Phi'_{v\ell}d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{11mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi_{wk}d\alpha$	0	See G2 for clamped case	0
$I_{11n\ell} \equiv \int_0^1 \Phi_{\psi xn}\Phi_{w\ell}d\beta$	½ ; $n=\ell\neq0$ 0 ; $n\neq\ell$	See G2	0

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{12mk} \equiv \int_0^1 \Phi_{\psi ym}\Phi_{wk}d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{11mk}	I_{11mk}
$I_{12n\ell} \equiv \int_0^1 \Phi_{\psi yn}\Phi_{w\ell}d\beta$	0	$I_{11n\ell}$	$I_{11n\ell}$
$I_{13mk} \equiv \int_0^1 \Phi_{\psi xm}\Phi'_{wk}d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	See G2	0
$I_{13n\ell} \equiv \int_0^1 \Phi_{\psi xn}\Phi'_{w\ell}d\beta$	0	See G2	0
$I_{14mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi_{wk}d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{13mk}	I_{13mk}
$I_{14n\ell} \equiv \int_0^1 \Phi'_{\psi xn}\Phi_{w\ell}d\beta$	0	$I_{13n\ell}$	$I_{13n\ell}$
$I_{15mk} \equiv \int_0^1 \Phi_{\psi ym}\Phi'_{wk}d\alpha$	0	I_{13mk}	I_{13mk}
$I_{15n\ell} \equiv \int_0^1 \Phi_{\psi yn}\Phi'_{w\ell}d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{13n\ell}$	$I_{13n\ell}$
$I_{16mk} \equiv \int_0^1 \Phi'_{\psi ym}\Phi_{wk}d\alpha$	0	I_{13mk}	I_{13mk}
$I_{16n\ell} \equiv \int_0^1 \Phi'_{\psi yn}\Phi_{w\ell}d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{13n\ell}$	$I_{13n\ell}$
$I_{17mk} \equiv \int_0^1 \Phi_{wm}\Phi_{wk}d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	1	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$
$I_{17n\ell} \equiv \int_0^1 \Phi_{wn}\Phi_{w\ell}d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	1	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$
$I_{18mk} \equiv \int_0^1 \Phi'_{wm}\Phi_{wk}d\alpha$	0	0	0
$I_{18n\ell} \equiv \int_0^1 \Phi'_{wn}\Phi_{w\ell}d\beta$	0	0	0

Integral Form	Simply Supported	Clamped Edges	Free Edges
$I_{19mk} \equiv \int_o^1 \Phi'_{wm}\Phi_{wk}d\alpha$	0	I_{18mk}	I_{18mk}
$I_{19n\ell} \equiv \int_o^1 \Phi'_{wn}\Phi_{w\ell}d\beta$	0	$I_{18n\ell}$	$I_{18n\ell}$
$I_{20mk} \equiv \int_o^1 \Phi'_{wm}\Phi'_{wk}d\alpha$	$mk\pi^2/2$; $m=k\neq0$ 0 ; $m\neq k$	$C_m Z_m (C_m Z_m - 2)$	$mk\pi^2/2$; $m=k\neq0$ 0 ; $m\neq k$
$I_{20n\ell} \equiv \int_o^1 \Phi'_{wn}\Phi'_{w\ell}d\beta$	$n\ell\pi^2/2$; $n=\ell\neq0$ 0 ; $n\neq\ell$	$C_n Z_n (C_n Z_n - 2)$	$n\ell\pi^2/2$; $n=\ell\neq0$ 0 ; $n\neq\ell$
$I_{21mk} \equiv \int_o^1 \Phi_{\psi xm}\Phi_{\psi xk}d\alpha$	½ ; $m=k\neq0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{21n\ell} \equiv \int_o^1 \Phi_{\psi xn}\Phi_{\psi x\ell}d\beta$	½ ; $n=\ell\neq0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{22mk} \equiv \int_o^1 \Phi_{\psi ym}\Phi_{\psi yk}d\alpha$	½ ; $m=k\neq0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{22n\ell} \equiv \int_o^1 \Phi_{\psi yn}\Phi_{\psi y\ell}d\beta$	½ ; $n=\ell\neq0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{23mk} \equiv \int_o^1 \Phi_{\psi xm}\Phi_{\psi yk}d\alpha$	0	I_{1mk}	I_{1mk}
$I_{23n\ell} \equiv \int_o^1 \Phi_{\psi xn}\Phi_{\psi y\ell}d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{24mk} \equiv \int_o^1 \Phi_{\psi xm}\Phi'_{\psi yk}d\alpha$	$k\pi/2$; $m=k\neq0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{24n\ell} \equiv \int_o^1 \Phi_{\psi xn}\Phi'_{\psi y\ell}d\beta$	$n\pi/2$; $n=\ell\neq0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{25mk} \equiv \int_o^1 \Phi_{\psi ym}\Phi'_{\psi xk}d\alpha$	$k\pi/2$; $m=k\neq0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{25n\ell} \equiv \int_o^1 \Phi_{\psi yn}\Phi'_{\psi x\ell}d\beta$	$n\pi/2$; $n=\ell\neq0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$

(Column heading over the last three columns: Evaluation for Boundary Condition Listed)

Integral Form	Simply Supported	Clamped Edges	Free Edges
	Evaluation for Boundary Condition Listed		
$I_{26mk} \equiv \int_0^1 \Phi_{\psi xm}\Phi'_{\psi xk}d\alpha$	0	I_{4mk}	I_{4mk}
$I_{26n\ell} \equiv \int_0^1 \Phi_{\psi xn}\Phi'_{\psi x\ell}d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{27mk} \equiv \int_0^1 \Phi_{\psi ym}\Phi'_{\psi yk}d\alpha$	0	I_{4mk}	I_{4mk}
$I_{27n\ell} \equiv \int_0^1 \Phi_{\psi yn}\Phi'_{\psi y\ell}d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{28mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi'_{\psi xk}d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{28n\ell} \equiv \int_0^1 \Phi'_{\psi xn}\Phi'_{\psi x\ell}d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{29mk} \equiv \int_0^1 \Phi'_{\psi ym}\Phi'_{\psi yk}d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{29n\ell} \equiv \int_0^1 \Phi'_{\psi yn}\Phi'_{\psi y\ell}d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{30mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi'_{\psi yk}d\alpha$	0	I_{8mk}	I_{8mk}
$I_{30n\ell} \equiv \int_0^1 \Phi'_{\psi xn}\Phi'_{\psi y\ell}d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{31mk} \equiv \int_0^1 \Phi_{\psi xm}\Phi_{uk}d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{31n\ell} \equiv \int_0^1 \Phi_{\psi xn}\Phi_{u\ell}d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{32mk} \equiv \int_0^1 \Phi_{um}\Phi_{\psi xk}d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{32n\ell} \equiv \int_0^1 \Phi_{un}\Phi_{\psi x\ell}d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$

Integral Form	Evaluation for Boundary Condition Listed: Simply Supported	Clamped Edges	Free Edges
$I_{33mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi_{uk}d\alpha$	0	I_{4mk}	I_{4mk}
$I_{33n\ell} \equiv \int_0^1 \Phi'_{\psi xn}\Phi_{u\ell}d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{34mk} \equiv \int_0^1 \Phi_{\psi xm}\Phi'_{uk}d\alpha$	0	I_{4mk}	I_{4mk}
$I_{34n\ell} \equiv \int_0^1 \Phi_{\psi xn}\Phi'_{u\ell}d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{35mk} \equiv \int_0^1 \Phi_{\psi ym}\Phi_{uk}d\alpha$	0	I_{1mk}	I_{1mk}
$I_{35n\ell} \equiv \int_0^1 \Phi_{\psi yn}\Phi_{u\ell}d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{36mk} \equiv \int_0^1 \Phi'_{\psi ym}\Phi_{uk}d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{36n\ell} \equiv \int_0^1 \Phi'_{\psi yn}\Phi_{u\ell}d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{37mk} \equiv \int_0^1 \Phi_{\psi ym}\Phi'_{uk}d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{37n\ell} \equiv \int_0^1 \Phi_{\psi yn}\Phi'_{u\ell}d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{38mk} \equiv \int_0^1 \Phi_{\psi xm}\Phi_{vk}d\alpha$	0	I_{1mk}	I_{1mk}
$I_{38n\ell} \equiv \int_0^1 \Phi_{\psi xn}\Phi_{v\ell}d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{39mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi_{vk}d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}

Integral Form	Simply Supported	Clamped Edges	Free Edges
$I_{39n\ell} \equiv \int_0^1 \Phi'_{\psi xn}\Phi_{v\ell}\,d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{40mk} \equiv \int_0^1 \Phi_{\psi xm}\Phi'_{vk}\,d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{40n\ell} \equiv \int_0^1 \Phi_{\psi xn}\Phi'_{v\ell}\,d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{41mk} \equiv \int_0^1 \Phi_{\psi ym}\Phi_{vk}\,d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{41n\ell} \equiv \int_0^1 \Phi_{\psi yn}\Phi_{v\ell}\,d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{42mk} \equiv \int_0^1 \Phi'_{\psi ym}\Phi_{vk}\,d\alpha$	0	I_{4mk}	I_{4mk}
$I_{42n\ell} \equiv \int_0^1 \Phi'_{\psi yn}\Phi_{v\ell}\,d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{43mk} \equiv \int_0^1 \Phi_{\psi ym}\Phi'_{vk}\,d\alpha$	0	I_{4mk}	I_{4mk}
$I_{43n\ell} \equiv \int_0^1 \Phi_{\psi yn}\Phi'_{v\ell}\,d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{44mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi'_{uk}\,d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{44n\ell} \equiv \int_0^1 \Phi'_{\psi xn}\Phi'_{u\ell}\,d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{55mk} \equiv \int_0^1 \Phi'_{\psi ym}\Phi'_{uk}\,d\alpha$	0	I_{8mk}	I_{8mk}
$I_{55n\ell} \equiv \int_0^1 \Phi'_{\psi yn}\Phi'_{u\ell}\,d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{56mk} \equiv \int_0^1 \Phi'_{\psi xm}\Phi'_{vk}\,d\alpha$	0	I_{8mk}	I_{8mk}

Integral Form	Evaluation for Boundary Condition Listed: Simply Supported	Clamped Edges	Free Edges
$I_{56n\ell} \equiv \int_0^1 \Phi'_{\psi xn}\Phi'_{v\ell}d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{57mk} \equiv \int_0^1 \Phi'_{\psi ym}\Phi'_{vk}d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{57n\ell} \equiv \int_0^1 \Phi'_{\psi yn}\Phi'_{v\ell}d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$

E2. Combination Trigonometric-Beam Type Integrals

These integrals are related to I_{11mk}, $I_{11n\ell}$, I_{13mk}, and $I_{13n\ell}$ for clamped and free edges of the plate mentioned in Appendix E1. Therefore, these integrals were evaluated and are listed below:

<u>All Edges Clamped</u>.

$$I_{11mk} \equiv \int_0^1 \Phi_{\psi xm}\Phi_{wk}d\alpha$$

$$= \frac{1}{(Z_k)^2 + (2m\pi)^2}\left\{Z_k \sinh Z_k \sin 2m\pi - 2m\pi \cosh Z_k \cos 2m\pi + 2m\pi\right\}$$

$$+ \left\{\frac{\cos[2m\pi + Z_k] - 1}{2[2m\pi + Z_k]} + \frac{\cos[2m\pi - Z_k] - 1}{2[2m\pi - Z_k]}\right.$$

$$- \frac{C_k}{(Z_k)^2 + (2m\pi)^2}\left\{Z_k \cosh Z_k \sin 2m\pi - 2m\pi \sinh Z_k \cos 2m\pi\right\}$$

$$+ C_k\left\{\frac{\sin[2m\pi - Z_k]}{2[2m\pi - Z_k]} - \frac{\sin[2m\pi + Z_k]}{2[2m\pi + Z_k]}\right\}$$

$I_{11n\ell}$ is analogous to I_{11mk} with the following substitutions:

$$m \leftrightarrow n\ , \qquad k \leftrightarrow \ell\ , \qquad \alpha \leftrightarrow \beta$$

$$I_{13mk} \equiv \int_0^1 \Phi_{\psi xm} \Phi'_{wk} d\alpha = \frac{Z_k}{(Z_k)^2 + (2m\pi)^2} \Big\{ Z_k \cosh Z_k \sin 2m\pi$$

$$- 2m\pi \sinh Z_k \cos 2m\pi \Big\} + Z_k \left\{ \frac{\sin[2m\pi - Z_k]}{2[2m\pi - Z_k]} - \frac{\sin[2m\pi + Z_k]}{2[2m\pi + Z_k]} \right\}$$

$$- \frac{Z_k \ C_k}{(Z_k)^2 + (2m\pi)^2} \Big\{ Z_k \sinh Z_k \sin 2m\pi - 2m\pi \cosh Z_k \cos 2m\pi + 2m\pi \Big\}$$

$$- Z_k \ C_k \left\{ \frac{\cos[2m\pi + Z_k]-1}{2[2m\pi + Z_k]} + \frac{\cos[2m\pi - Z_k]-1}{2[2m\pi - Z_k]} \right\}$$

$I_{13n\ell}$ is analogous to I_{13mk} with the following substitutions:

$$m \leftrightarrow n\ , \qquad k \leftrightarrow \ell\ , \qquad \alpha \leftrightarrow \beta$$

APPENDIX F

COMPUTER PROGRAM DOCUMENTATION AND LISTING

The program described in this appendix was programmed to accomplish the following three computations:

1. Calculate the shear factor K for laminates, using Jourawski static shear theory.
2. Calculate the lowest eigenvalue for a simply supported laminated plate without damping.
3. Calculate the amplitude frequency response and modified Kennedy-Pancu frequency response and damping data for a free-edge anisotropic plate with material damping.

Computation 1 was accomplished by using an explicit algebraic expression. Computation 2 was performed by using IBM System/360 Scientific Subroutine Packages NROOT and EIGEN. Computation 3 was performed by Package SMIQ. A complete description of the variables, operations, etc. may be found in IBM Manual 360A-CM-03X, version III, for the subroutines NROOT, EIGEN and SMIQ.

The program was written in FORTRAN IV language as prescribed in IBM System Reference Library Form C-28-6274-3.

The input-data deck was set up as follows:

Computation 1 -

(a) Thickness of each layer (lower and upper limit)

(b) Elastic and shear moduli

Computation 2 -

(a) Plate geometry

(b) Lamination geometry (specially orthotropic)

(c) Moduli and Poisson's ratio data for each ply

(d) Density for each ply

(e) Shear factor (as calculated in Computation 1)

Computation 3 -

(a) Young's and shear moduli for each layer

(b) Poisson's ratios for each layer

(c) Bending and twisting stiffnesses for each layer

(d) Loss tangents corresponding to moduli, Poisson's ratios, and stiffnesses for each layer

(e) Plate geometry

(f) Lamination geometry, including angle of orientation for each layer

(g) Density of each layer

(h) Shear factor (as calculated in Computation 1)

(i) Mode numbers of the assumed modes.

A complete listing of the computer program is presented at the end of this dissertation.

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TABLE I. SHEAR FACTOR FOR LAMINATES WITH MULTIPLE ALTERNATING LAYERS OF TWO MATERIALS

Modulus ratio, $E_{11}^{(1)} / E_{11}^{(2)} = C_{55}^{(1)} / C_{55}^{(2)}$	Number of Layers, n							
	2	3	4	5	6	7	8	9
1	0.833	0.833	0.833	0.833	0.833	0.833	0.833	0.833
10	0.173	0.190	0.203	0.201	0.195	0.193	0.188	0.186
20	0.091	0.087	0.087	0.082	0.072	0.069	0.063	0.062
30	0.062	0.056	0.050	0.047	0.037	0.036	0.032	0.031
40	0.047	0.041	0.033	0.031	0.023	0.022	0.019	0.018
50	0.038	0.032	0.022	0.016	0.015	0.011	0.013	0.012
60	0.032	0.027	0.017	0.016	0.011	0.011	0.009	0.009
70	0.027	0.023	0.014	0.013	0.009	0.008	0.007	0.006
80	0.024	0.020	0.011	0.010	0.008	0.007	0.005	0.005
90	0.021	0.018	0.009	0.008	0.005	0.005	0.004	0.004
100	0.020	0.016	0.007	0.007	0.004	0.004	0.003	0.003

Table II. Frequencies and Damping for a Laminated, Composite-Material Plates

Angle, deg.	Mode No.	Reference 13 Exp. freq., Hz.	Reference 13 Anal. freq., Hz.	Reference 13 Exp. damping ratio, %	Present Results Peak-ampl. freq., Hz.	Present Results Kennedy-Pancu freq., Hz.	Present Results Calc. damping ratio, %
	1	141.8	138.2	0.1	138.9	140.0	0.22
	2	182.2	164.0	1.9	170.0	170.4	1.83
0	3	386.2	365.4	0.9	359.4	360.1	0.97
	4	401.1	381.1	0.2	389.7	390.2	0.24
	5	649.5	639.4	1.4	639.0	639.6	0.95
	1	100.6	93.4	0.6	93.0	92.0	1.19
	2	234.0	226.9	3.4	226.1	225.2	1.28
10	3	275.2	257.9	0.9	264.0	265.5	1.06
	4	485.3	497.5	1.6	495.2	495.0	0.09
	5	523.4	512.8	0.9	512.3	511.0	0.66
	1	57.2	56.1	1.0	57.7	58.7	1.37
	2	163.4	164.2	1.0	166.1	165.0	0.82
30	3	314.9	325.9	1.1	320.0	320.0	1.03
	4	378.4	385.9	1.6	388.0	387.2	1.64
	5	534.2	572.1	1.4	564.3	565.0	0.97

TABLE II. Continued

Angle, deg.	Mode No.	Reference 13: Exp. freq., Hz.	Reference 13: Anal. freq., Hz.	Reference 13: Exp. damping ratio, %	Present Results: Peak-ampl. freq., Hz.	Present Results: Kennedy-Pancu freq., Hz.	Present Results: Calc. damping ratio, %
45	1	46.2	42.4	0.9	45.2	45.1	0.98
	2	131.4	122.0	1.2	124.0	124.0	1.25
	3	257.6	250.0	1.1	249.6	251.0	0.95
	4	278.4	291.5	2.2	293.0	293.4	1.59
	5	433.3	432.0	1.5	429.9	430.0	1.18
60	1	44.1	42.4	0.9	43.0	42.3	1.00
	2	127.0	119.0	1.5	121.5	122.0	1.60
	3	260.8	226.3	3.9	228.0	228.1	2.05
	4	239.9	239.3	1.1	237.0	237.6	1.06
	5	405.1	404.0	1.5	402.3	401.5	1.17
90	1	48.0	49.3	0.8	48.1	48.0	0.87
	2	137.6	136.0	1.2	136.0	136.1	1.23
	3	188.9	161.6	5.3	165.6	164.4	3.31
	4	250.6	267.1	1.0	259.8	259.0	1.10
	5	331.7	331.7	2.1	332.3	331.0	1.98

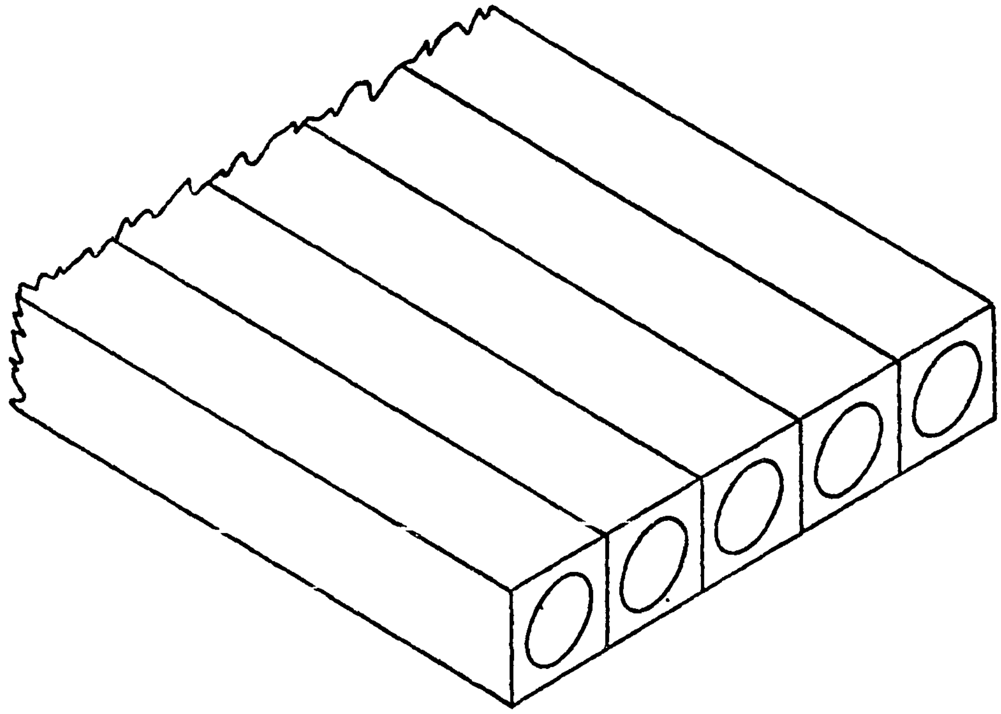

Figure 1. Composite containing undirectional fibers.

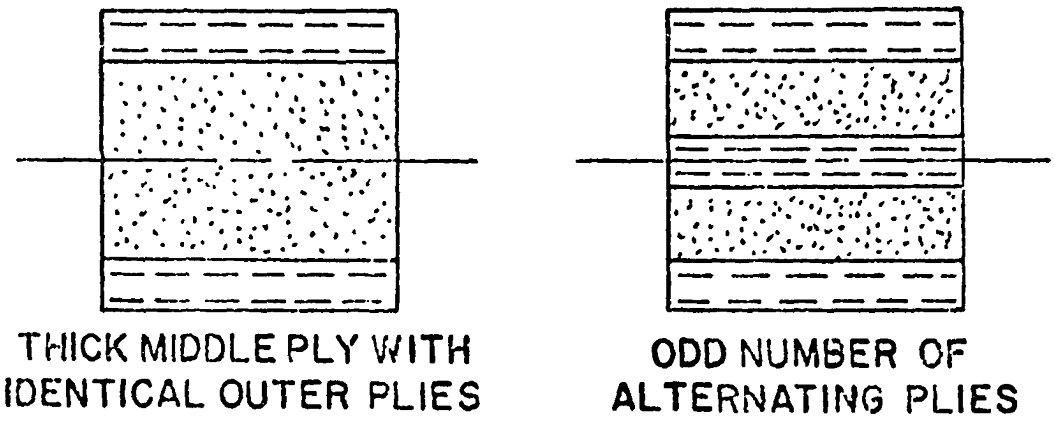

Figure 2. Examples of symmetrical laminates.

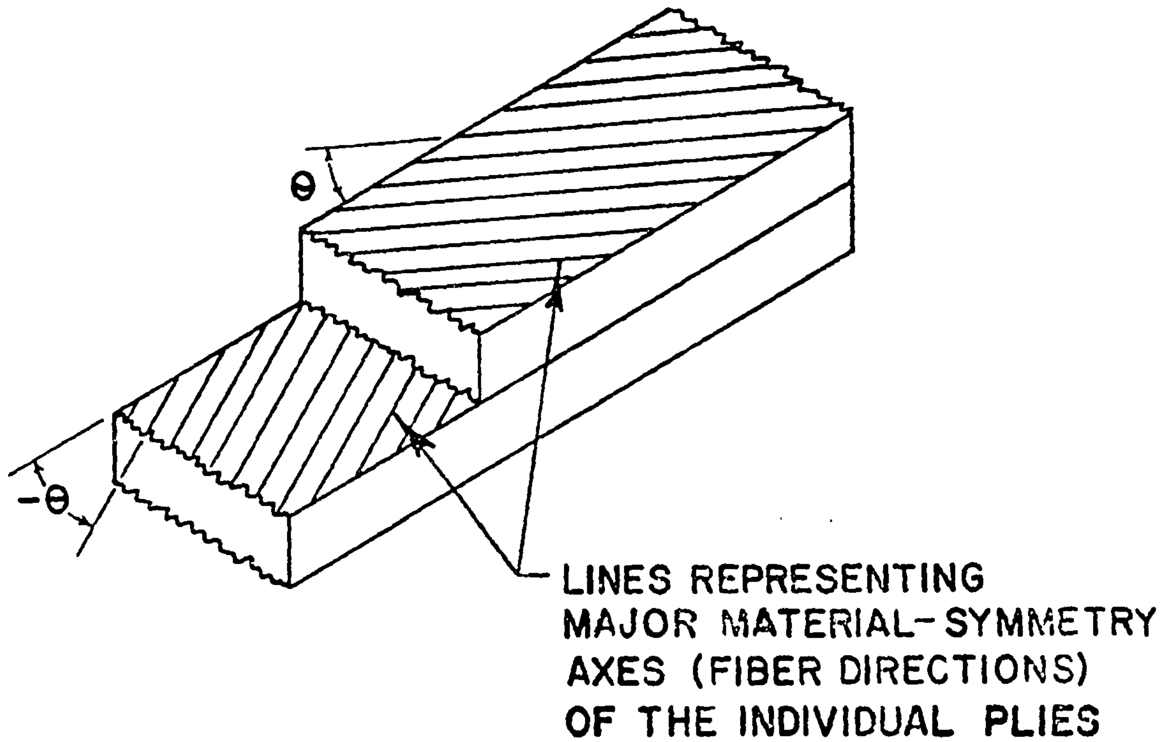

Figure 3. An example of a balanced laminate.

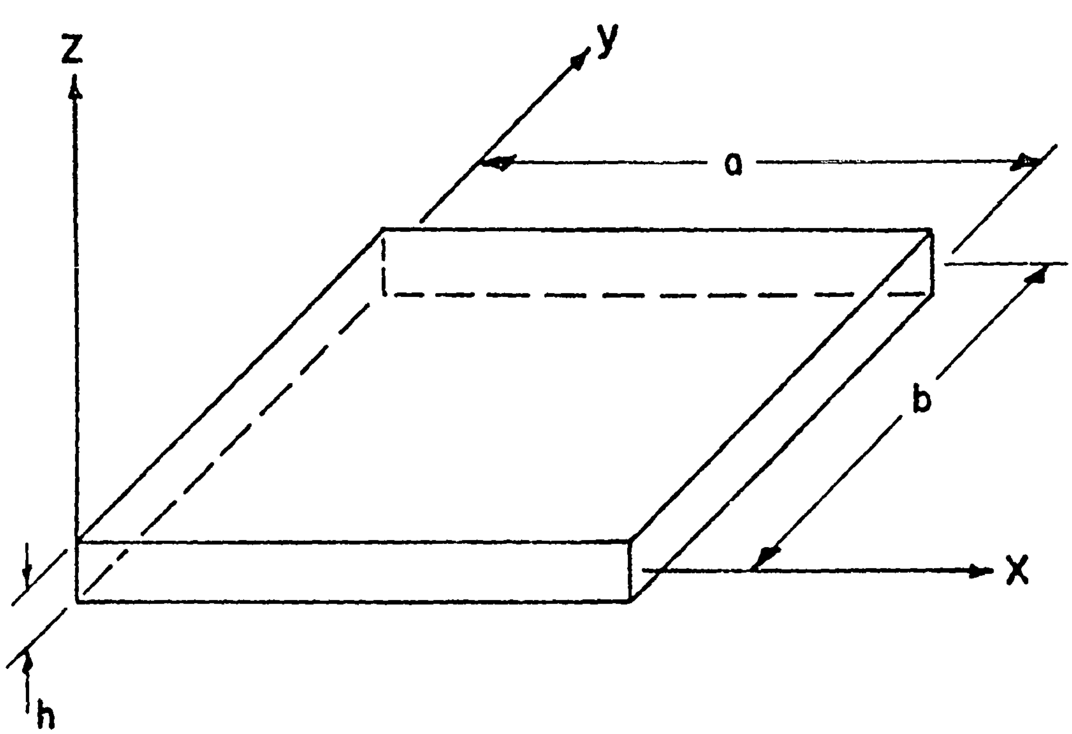

Figure 4. Plate coordinate system.

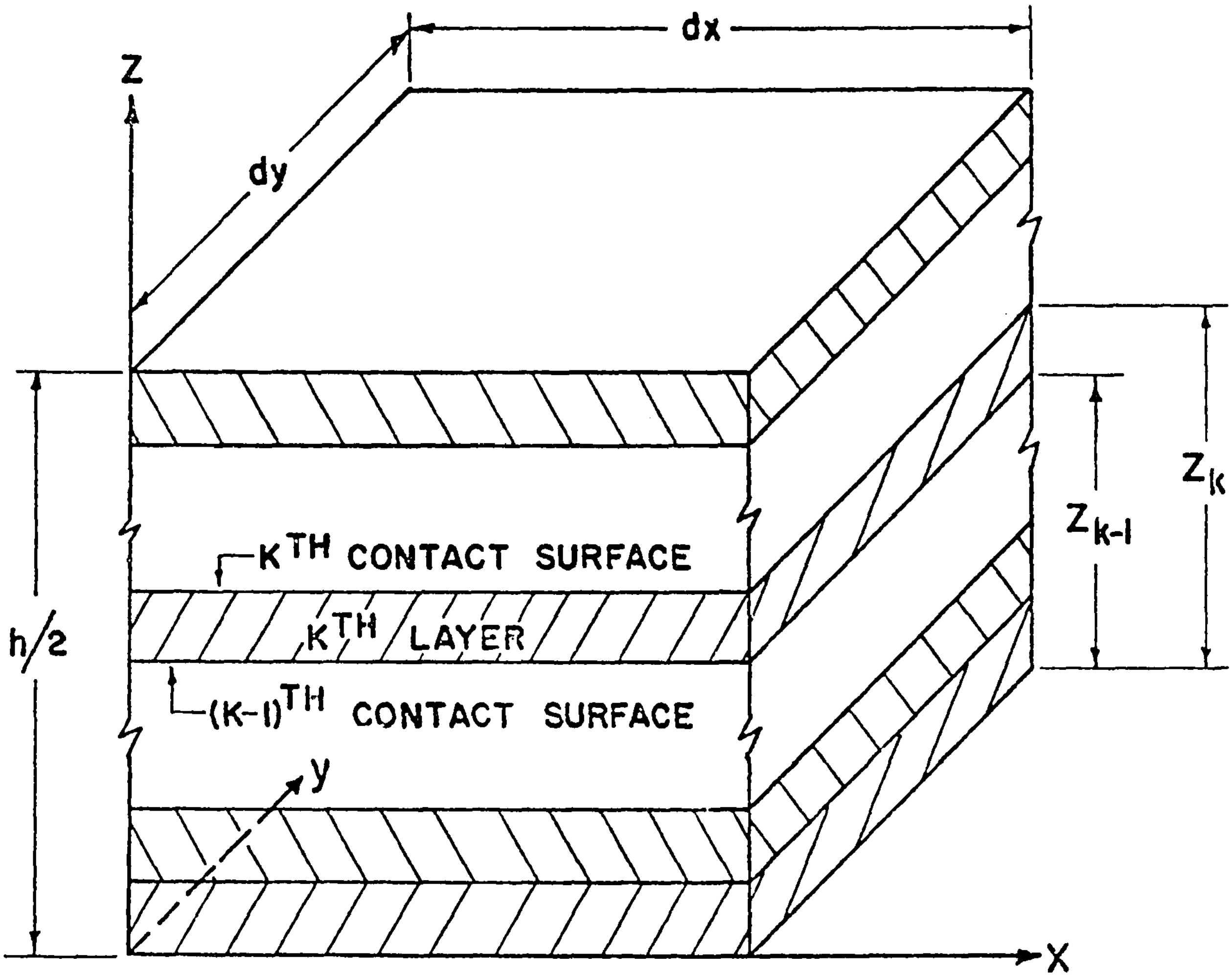

Figure 5. Lamination geometry, showing upper half of plate. (Plane xy is the midplane of the plate).

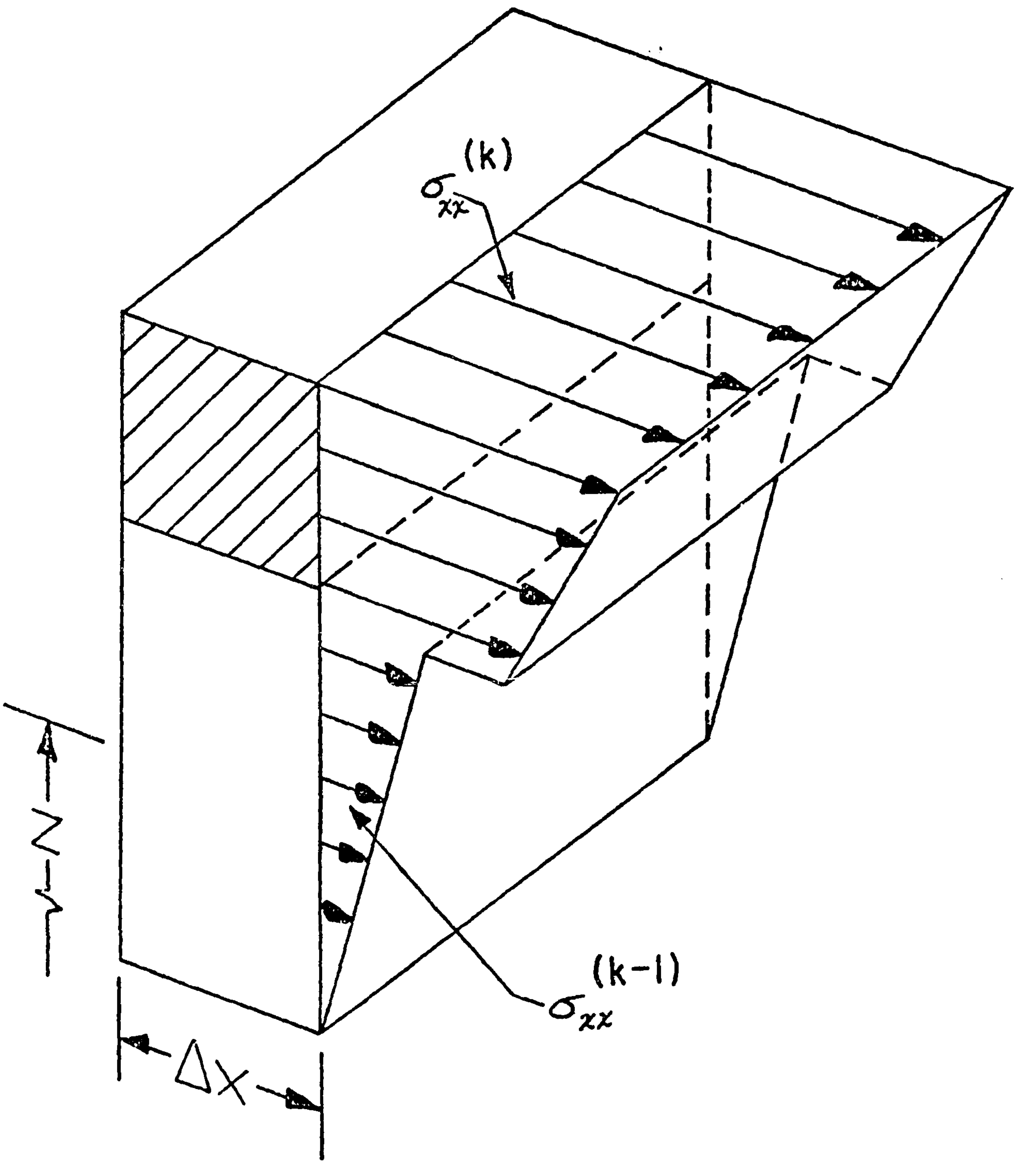

Figure 6. Bending-stress distribution.

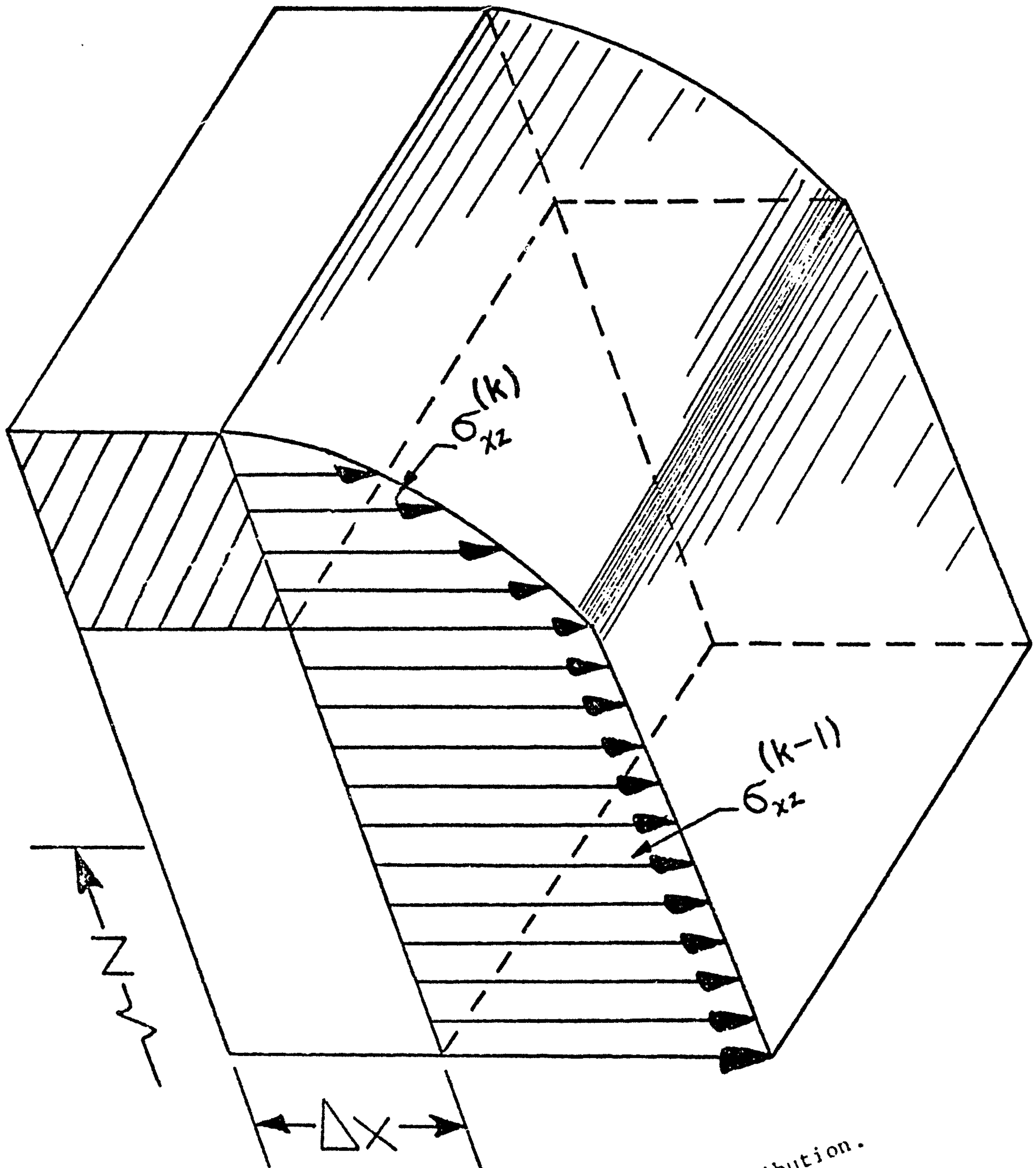

Figure 7. Shear-stress distribution.

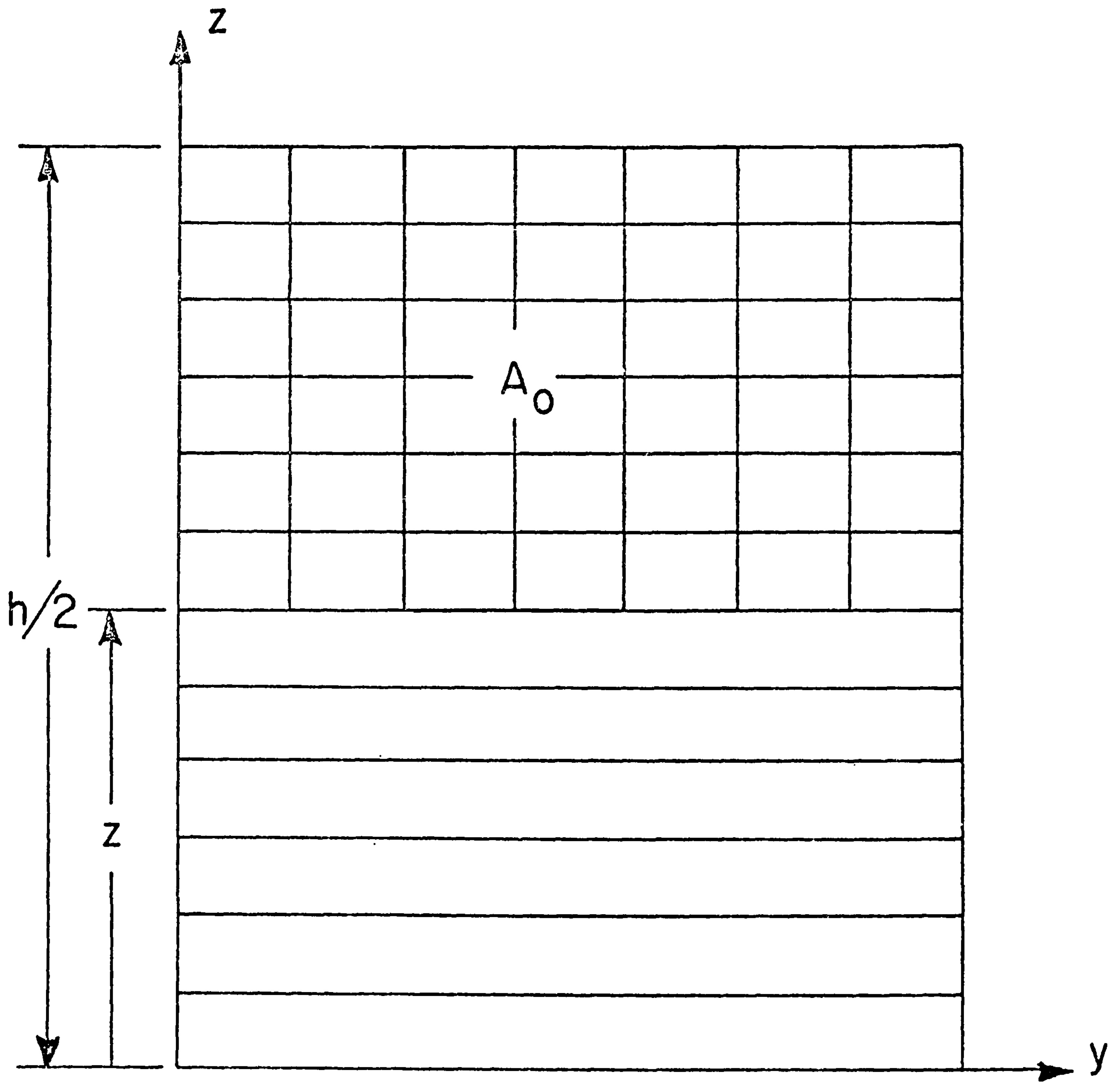

Figure 8. Area of integration A_o.

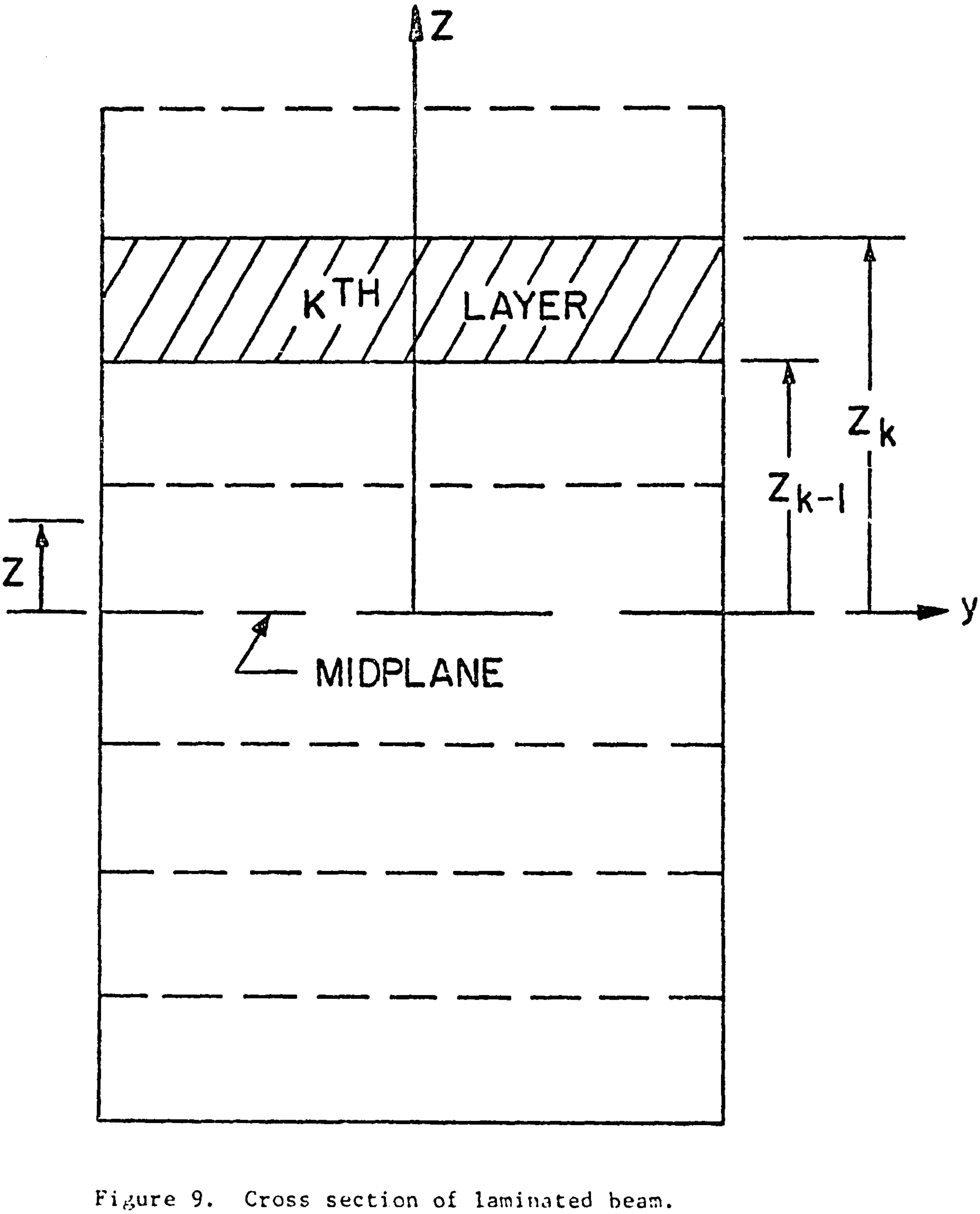

Figure 9. Cross section of laminated beam.

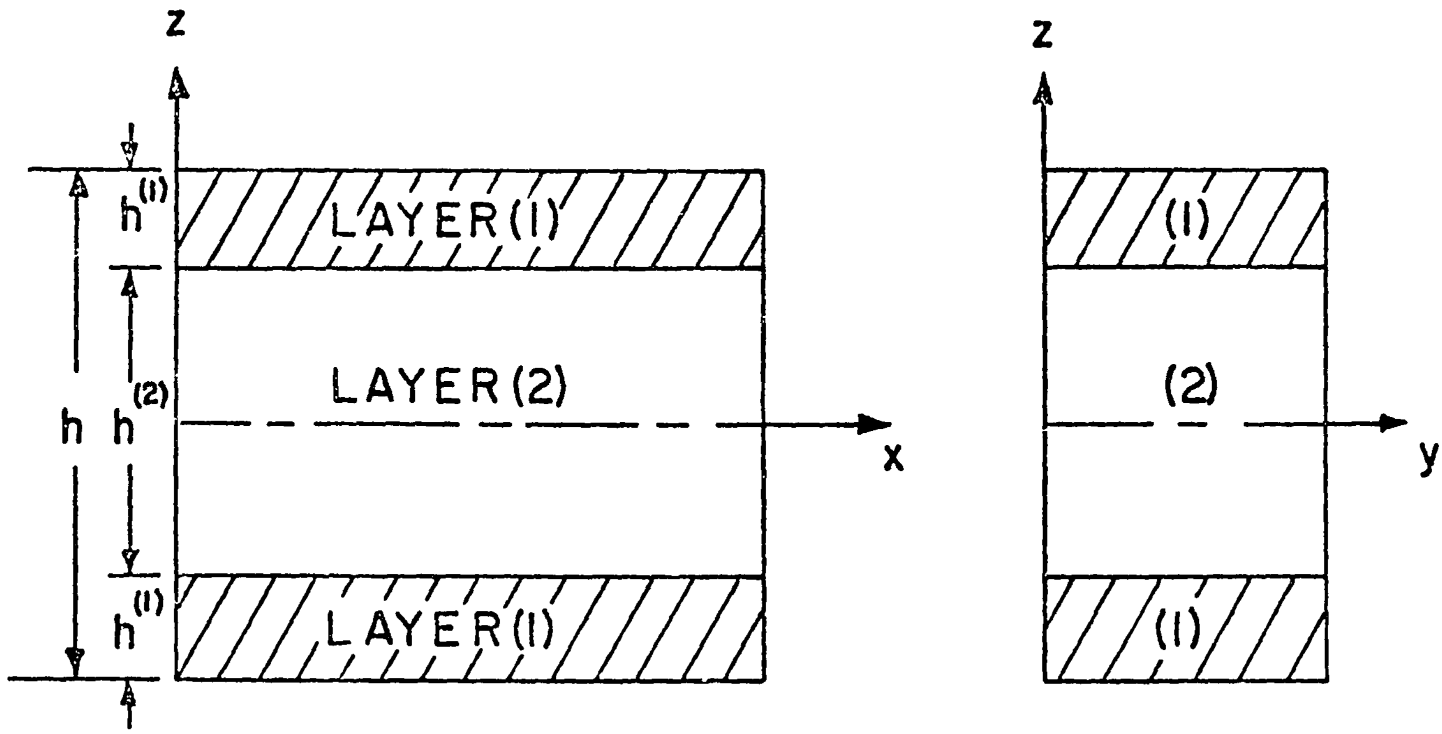

Figure 10. Three-ply symmetrically laminated plate.

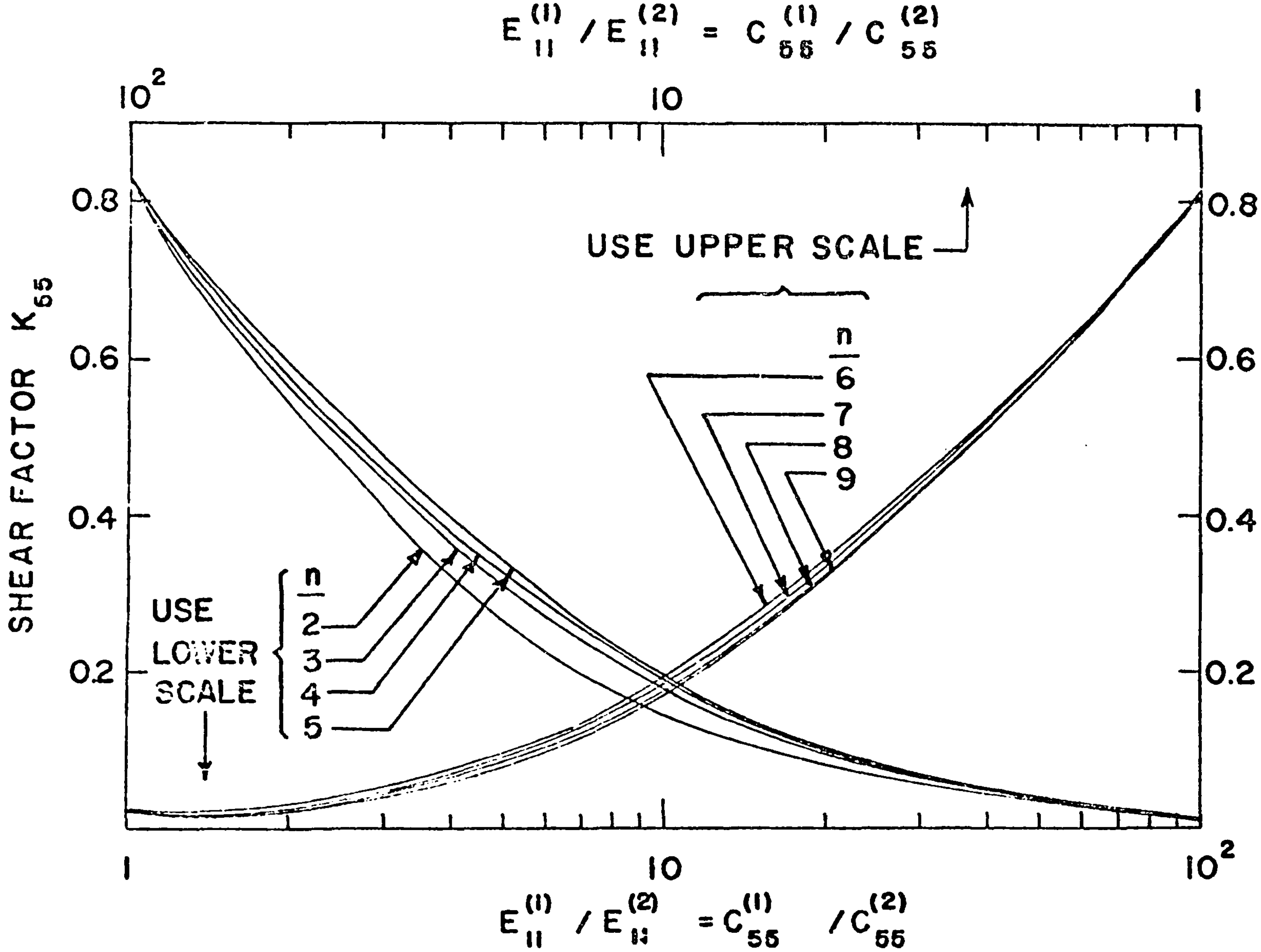

Figure 11. Shear factor for laminates consisting of multiple alternating layers of two materials.

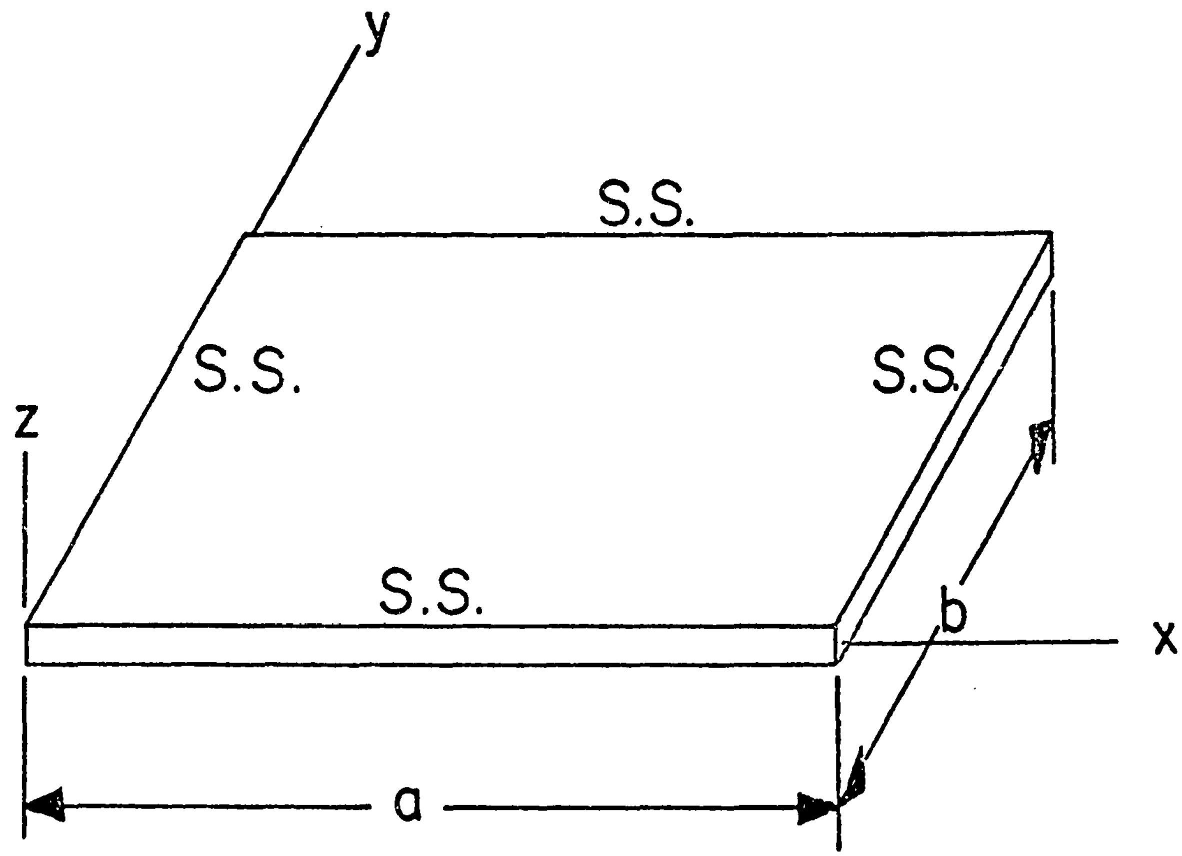

Figure 12. Rectangular plate with simply supported edges.

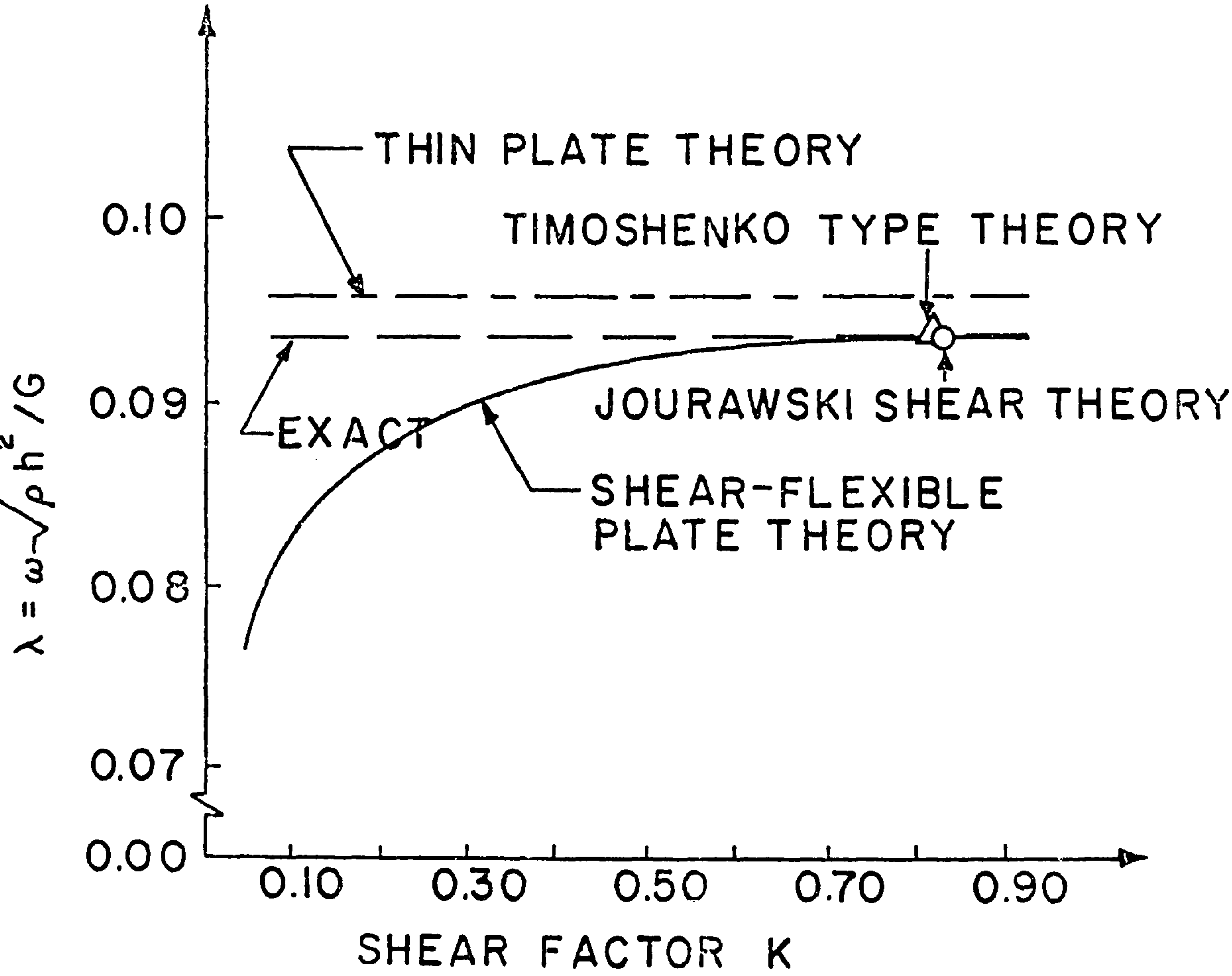

Figure 13. Lowest Eigenvalue (λ) for homogeneous, isotropic plate.

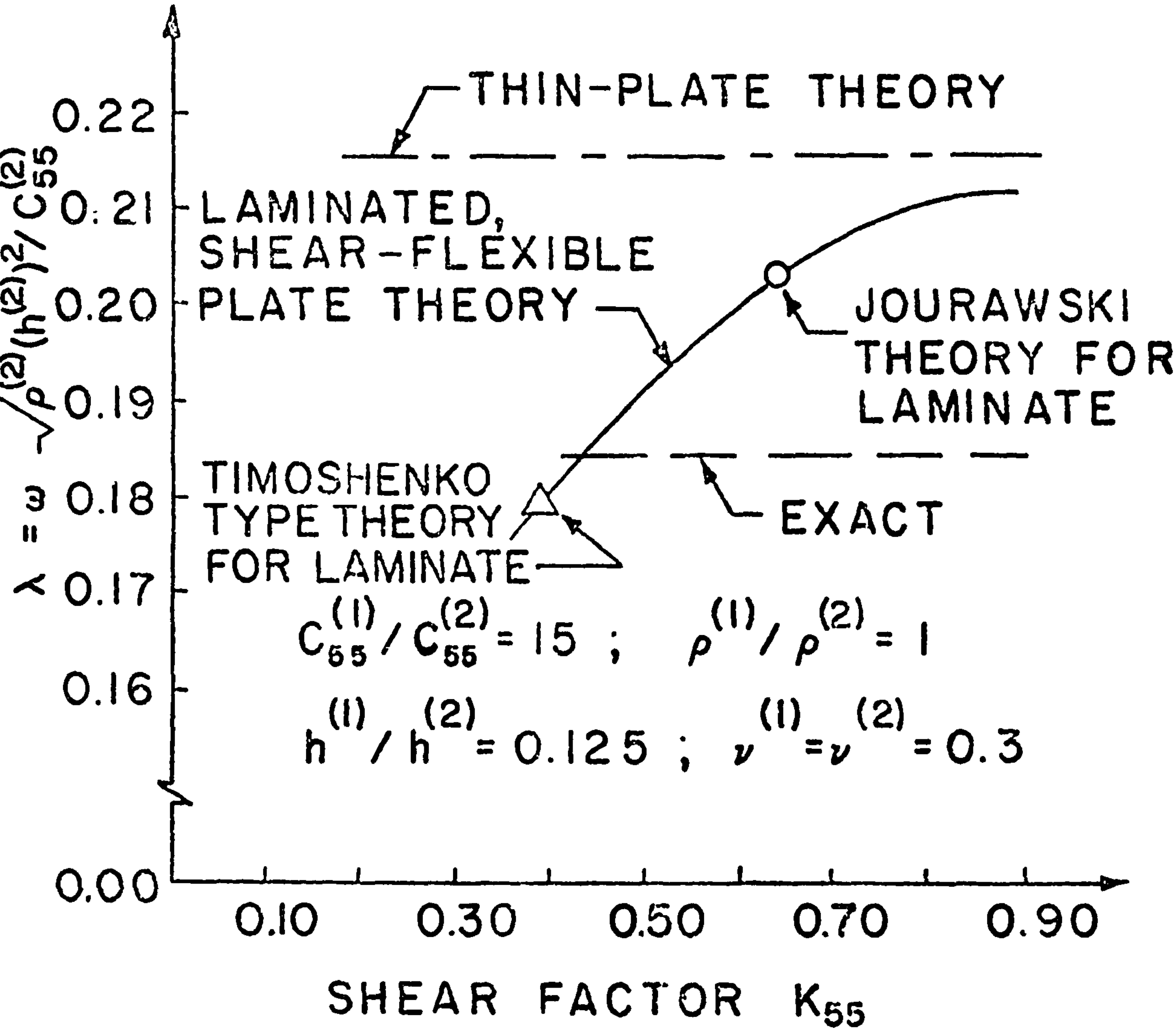

Figure 14. Lowest eigenvalues (λ) for three-ply symmetrically laminated plate.

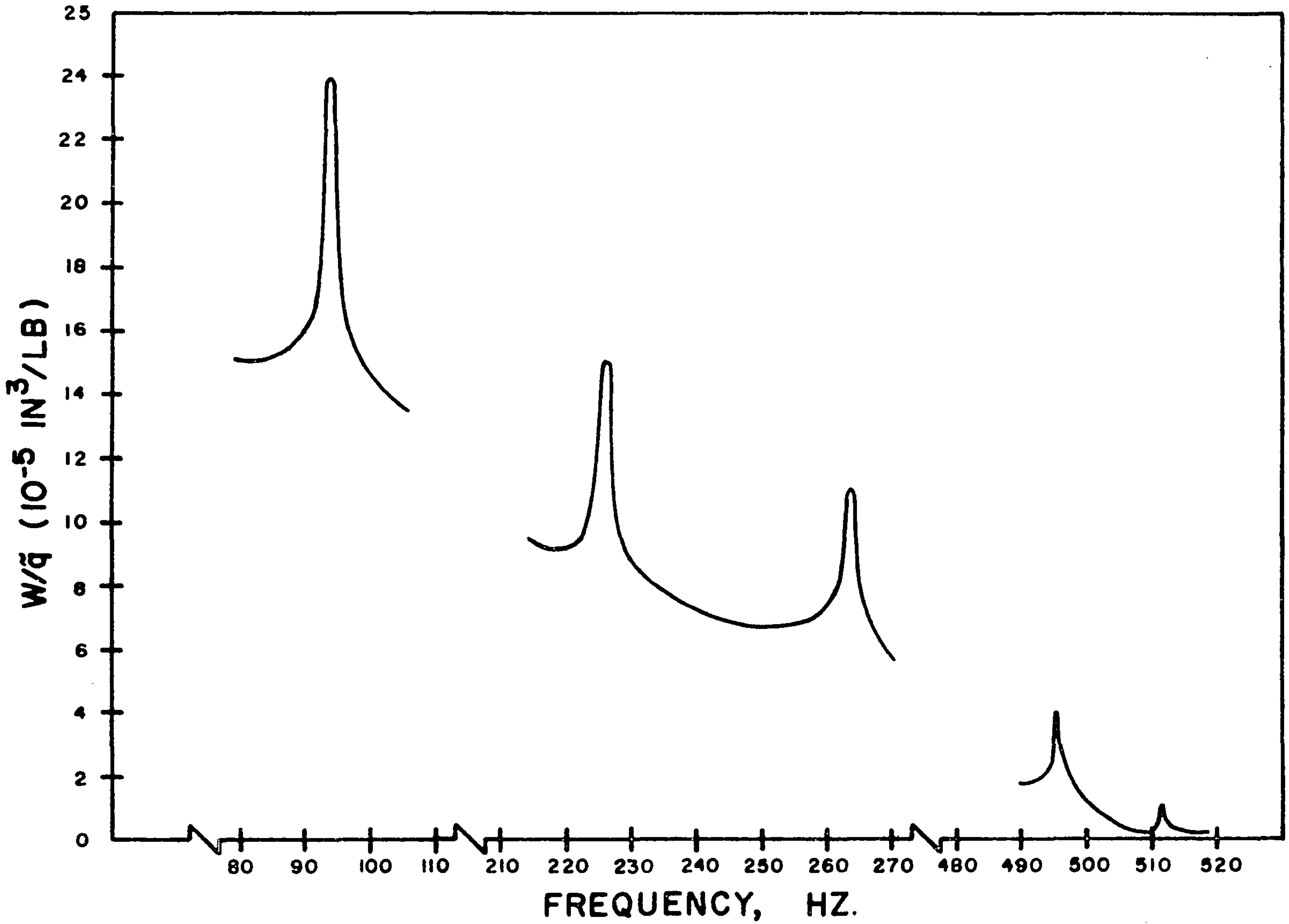

Figure 15. Amplitude frequency response

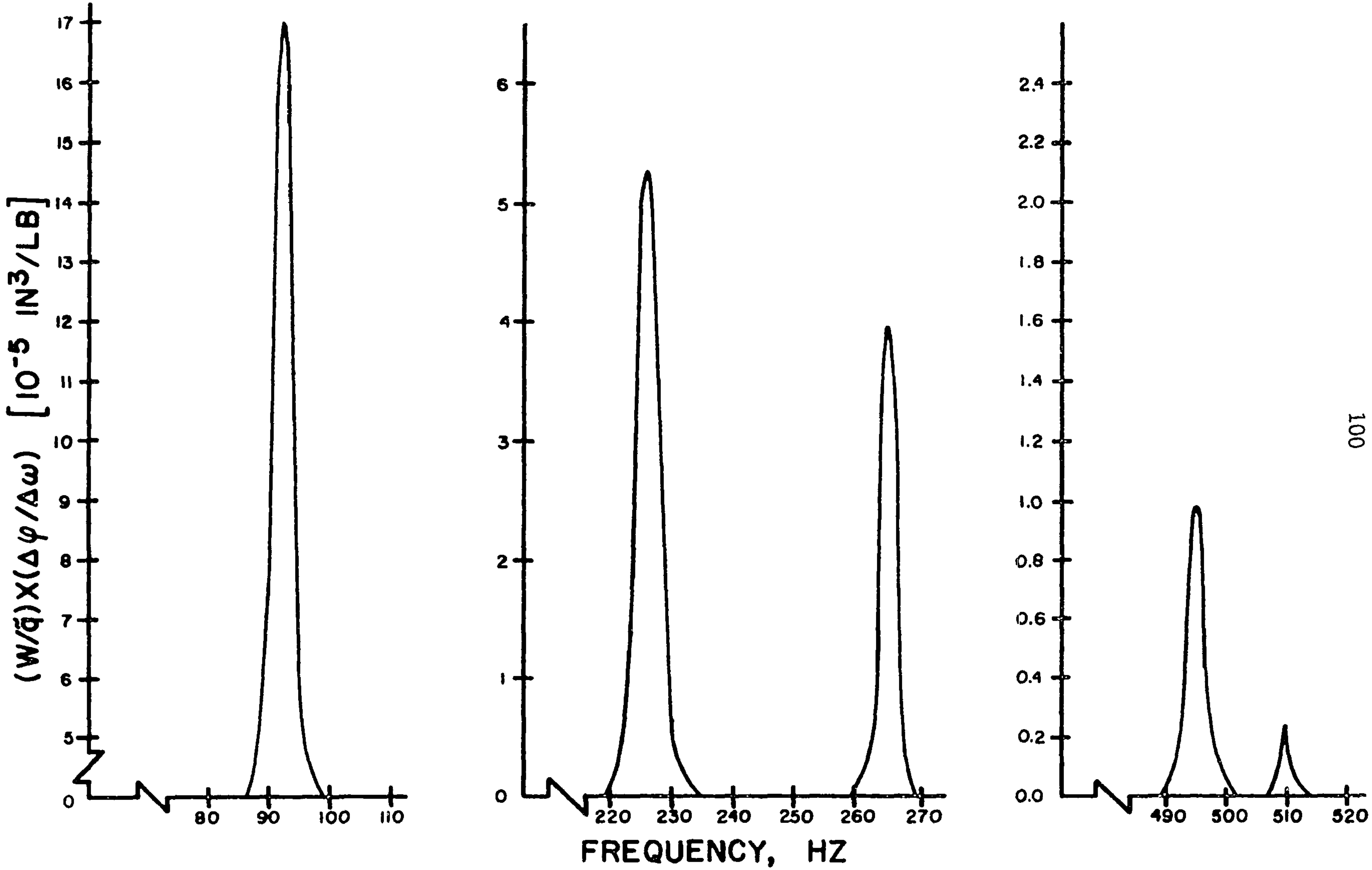

Figure 16. Modified Kennedy-Pancu frequency response

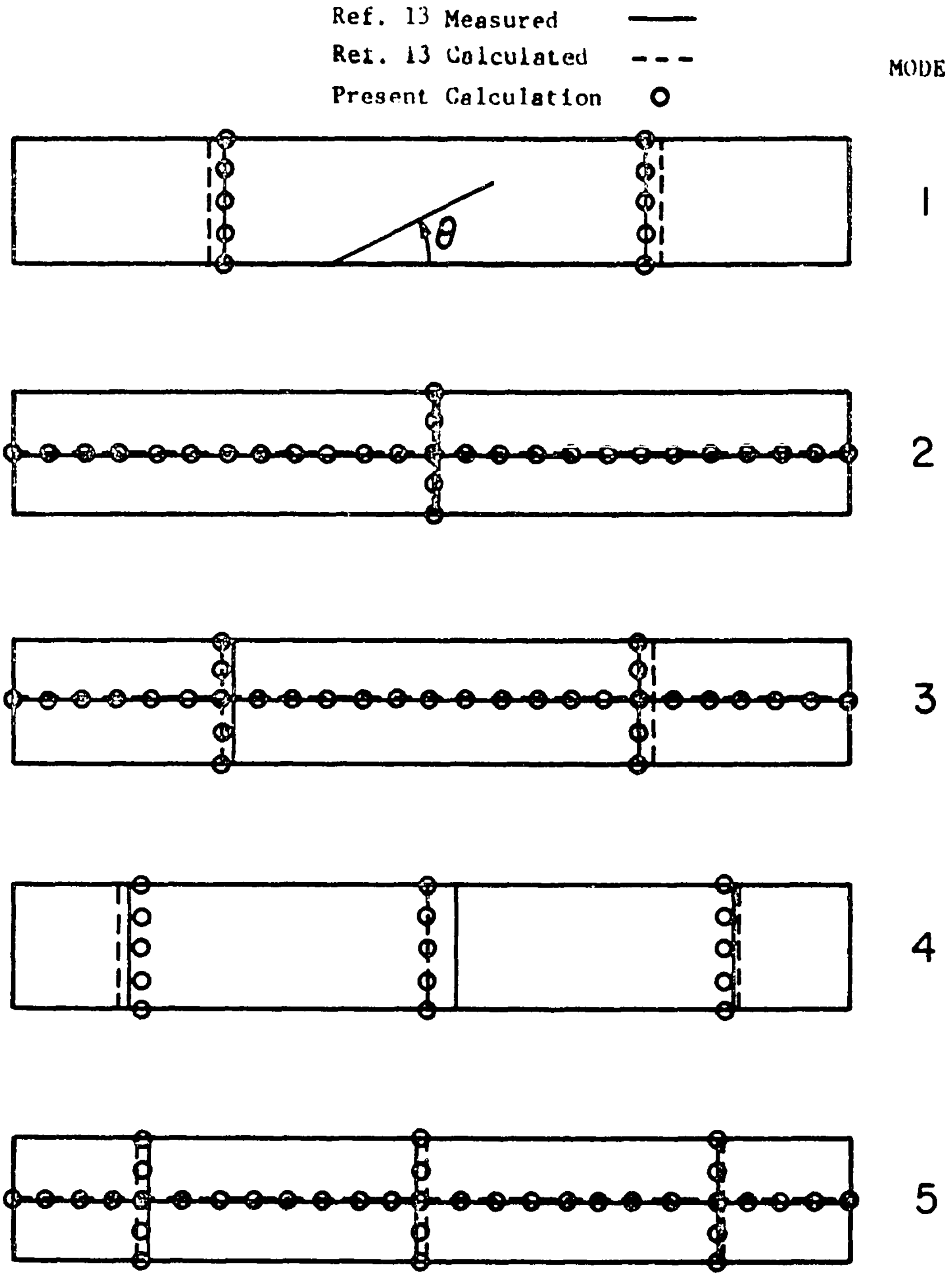

Figure 17. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 0^{o}$

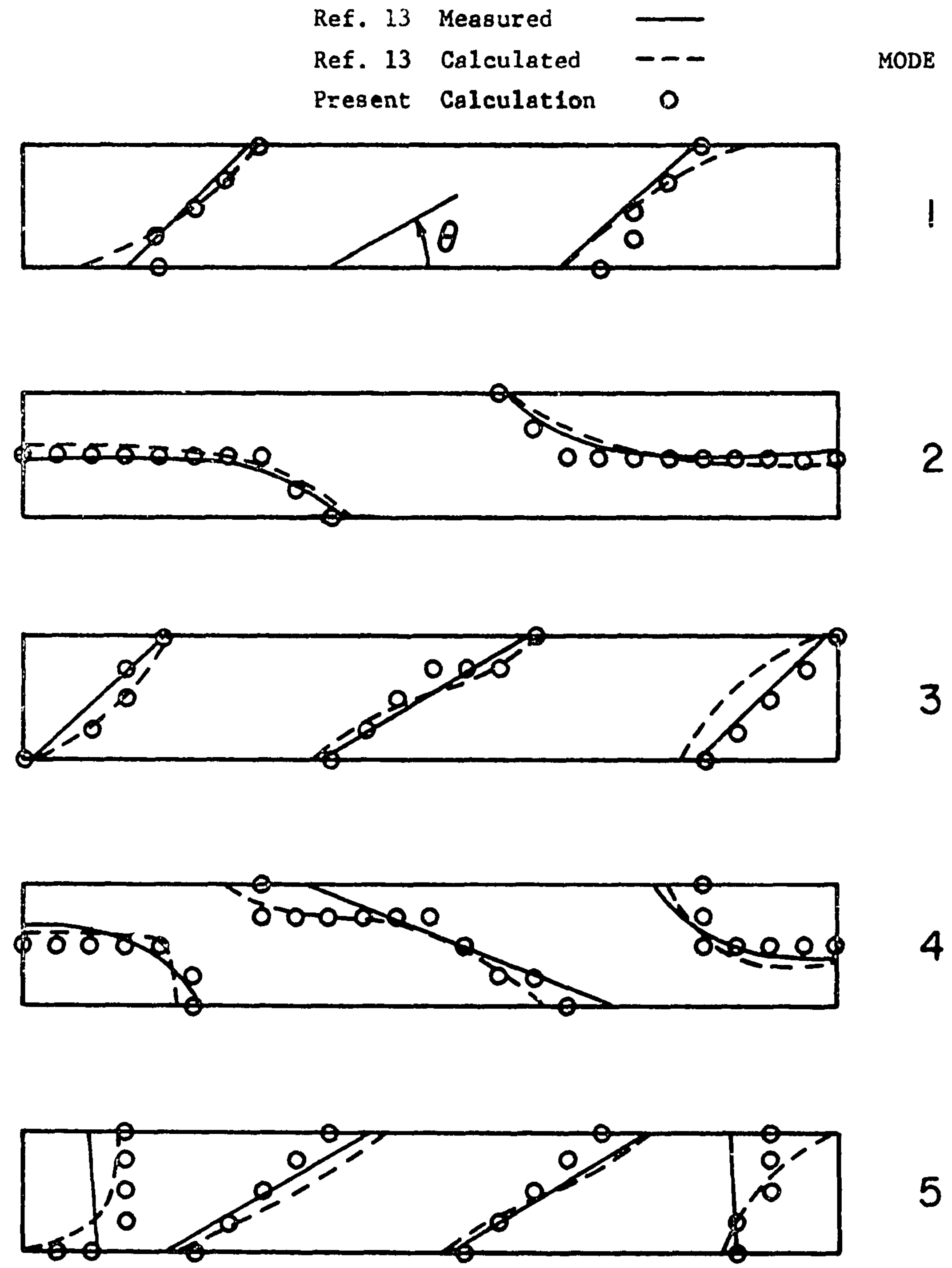

Figure 18. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 10^{\circ}$

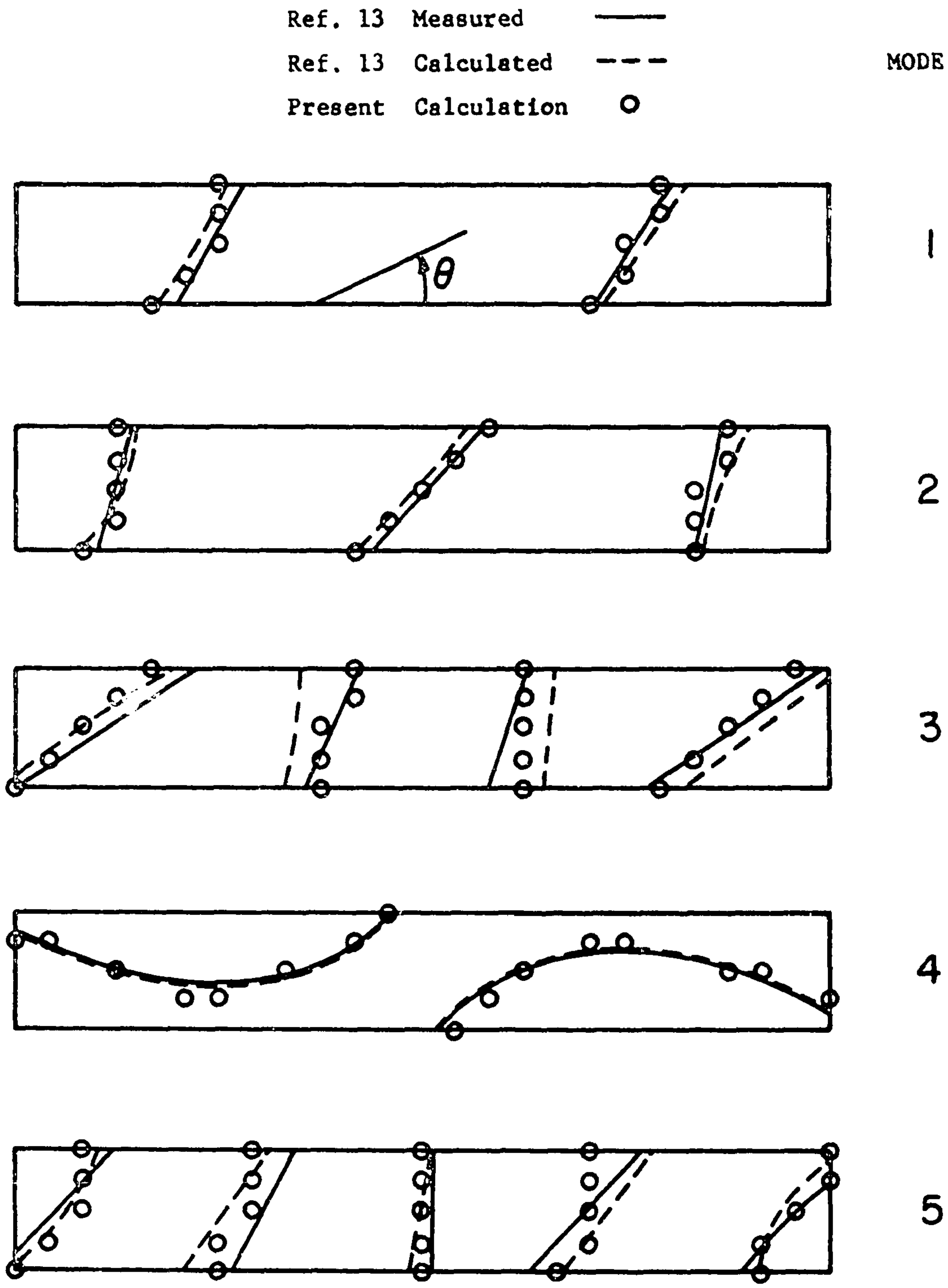

Figure 19. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 30^{\circ}$

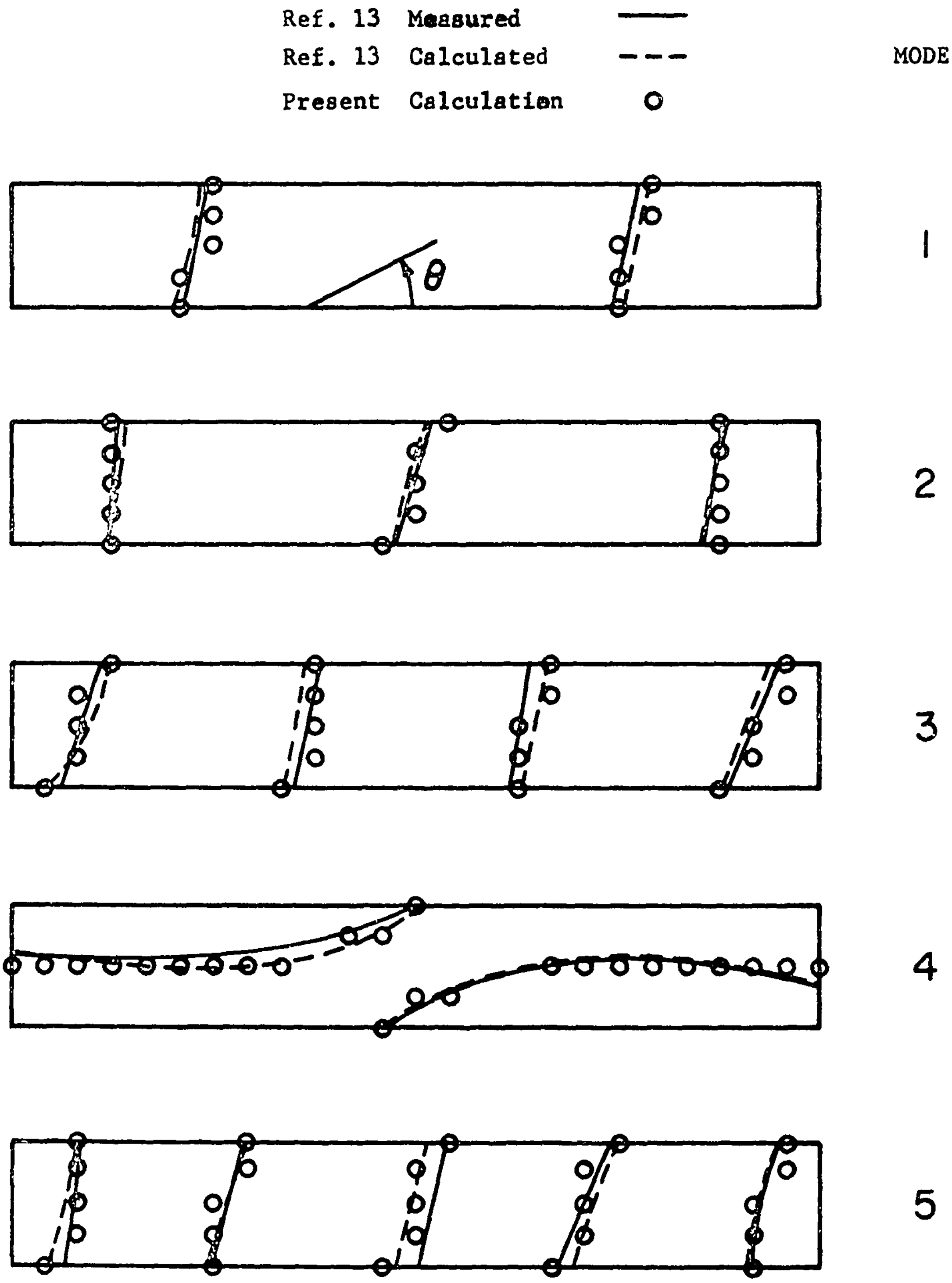

Figure 20. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 45^{\circ}$

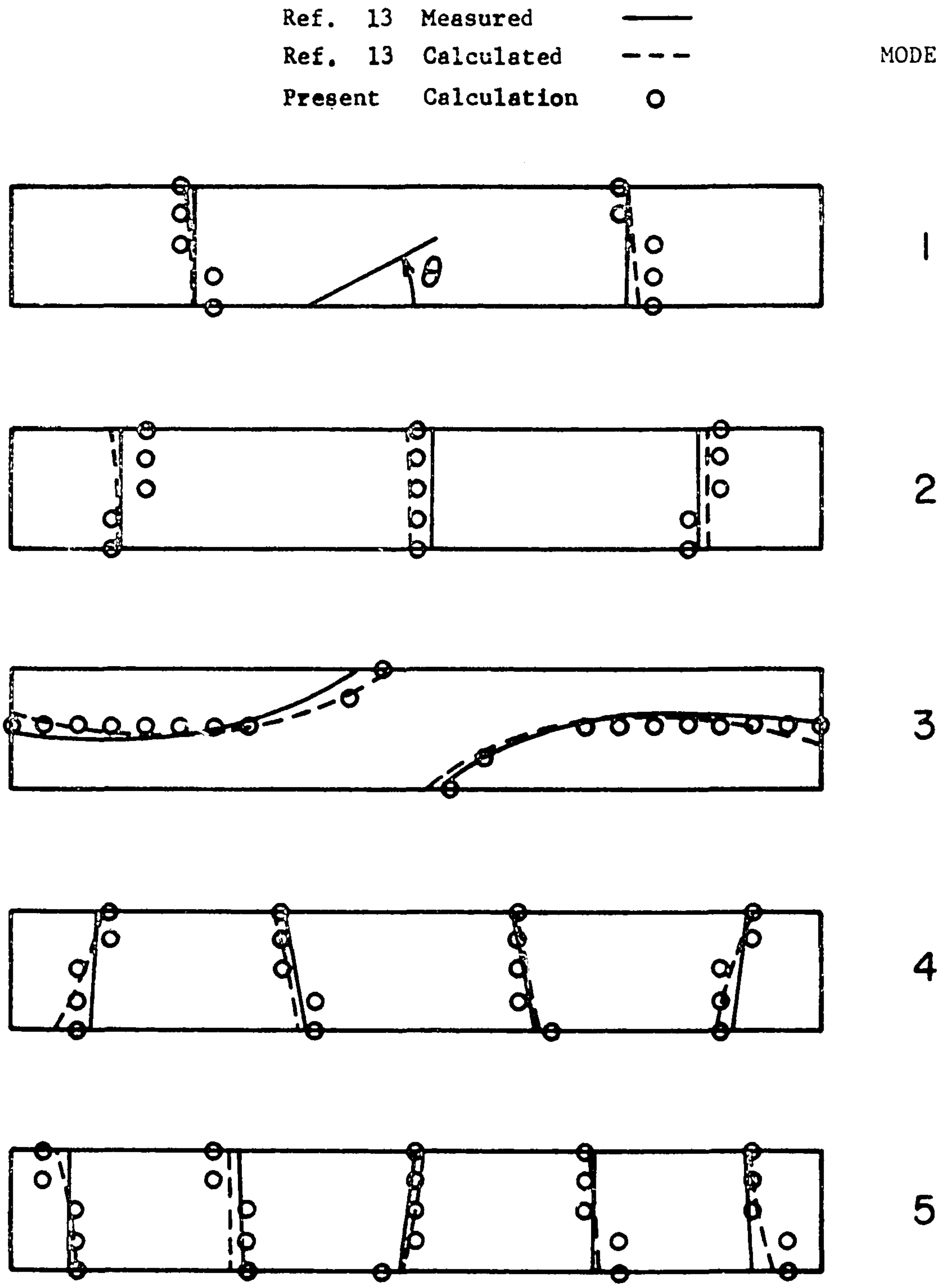

Figure 21. Nodal patterns of the first five modes of composite material plates at angle of orientation θ = 60°

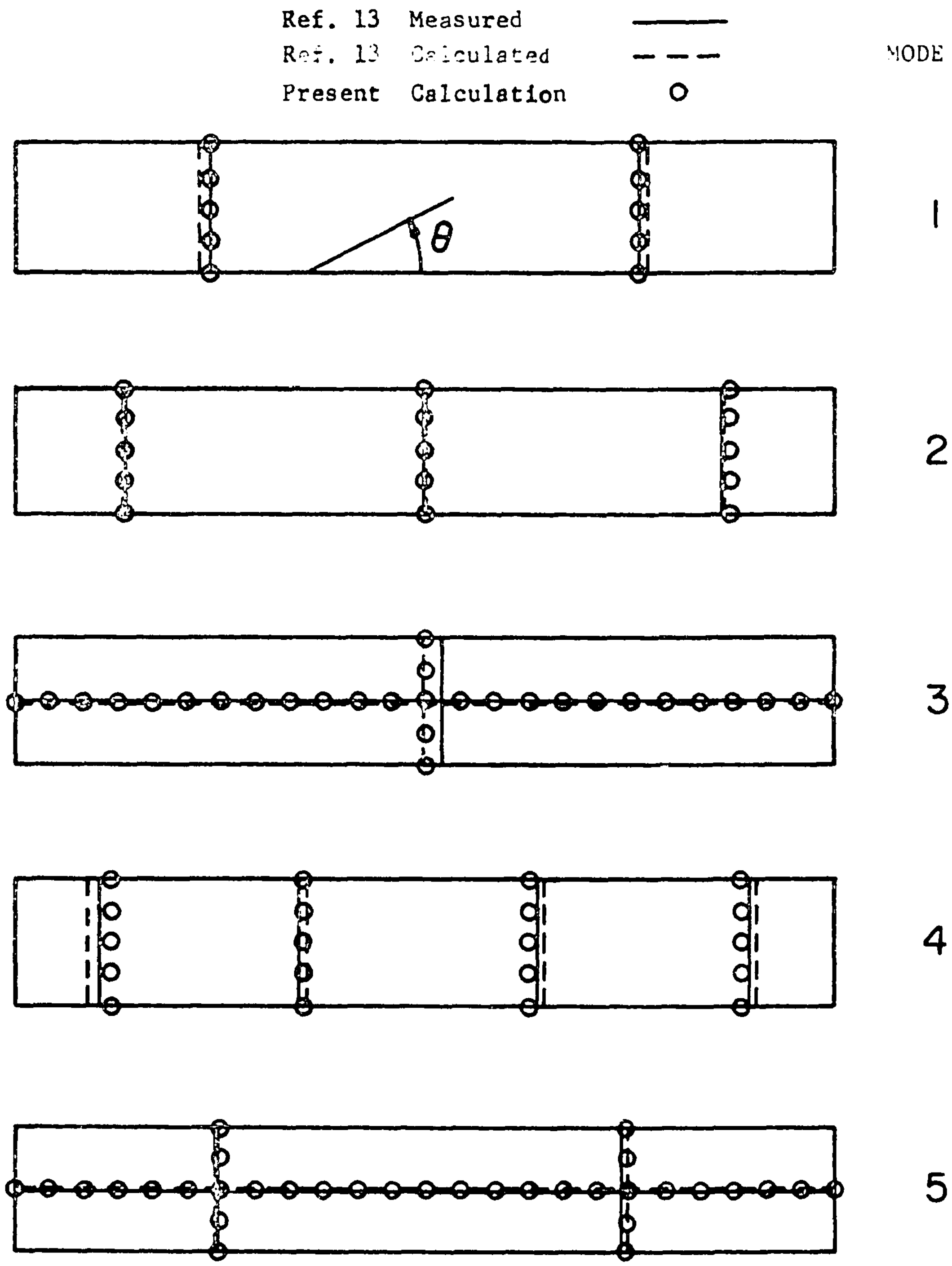

Figure 22. Nodal patterns of the first five modes of composite material plates at angle of orientation θ = 90°

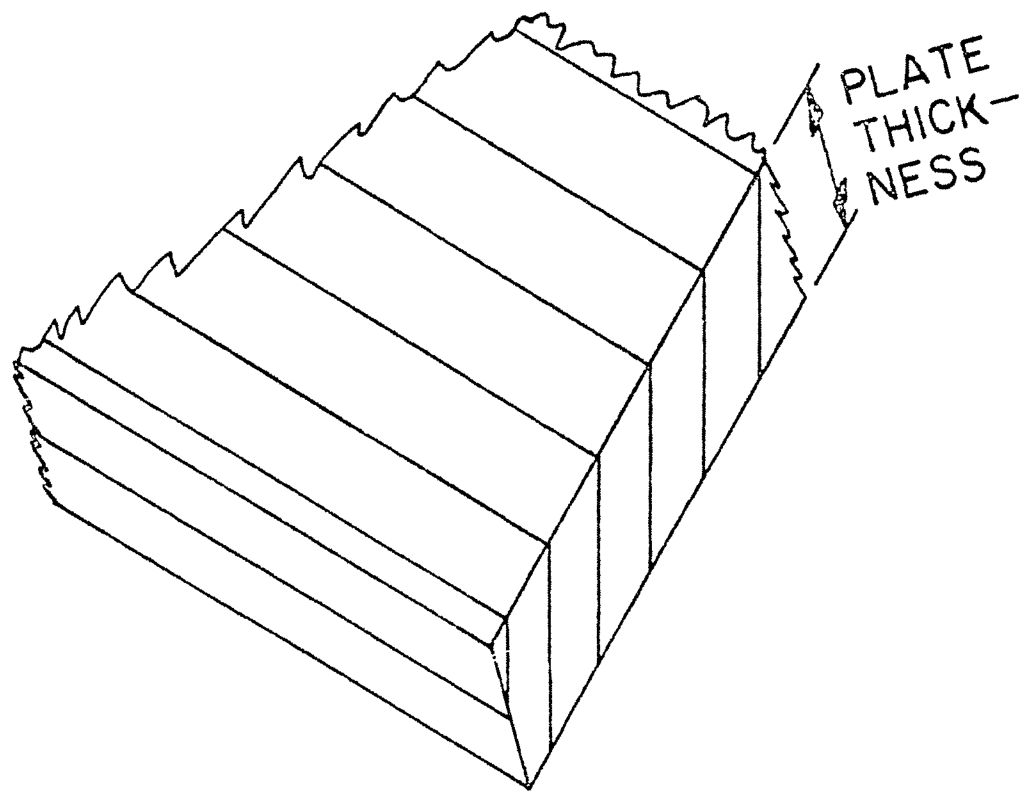

Figure A-1. Shingle-laminated plate.

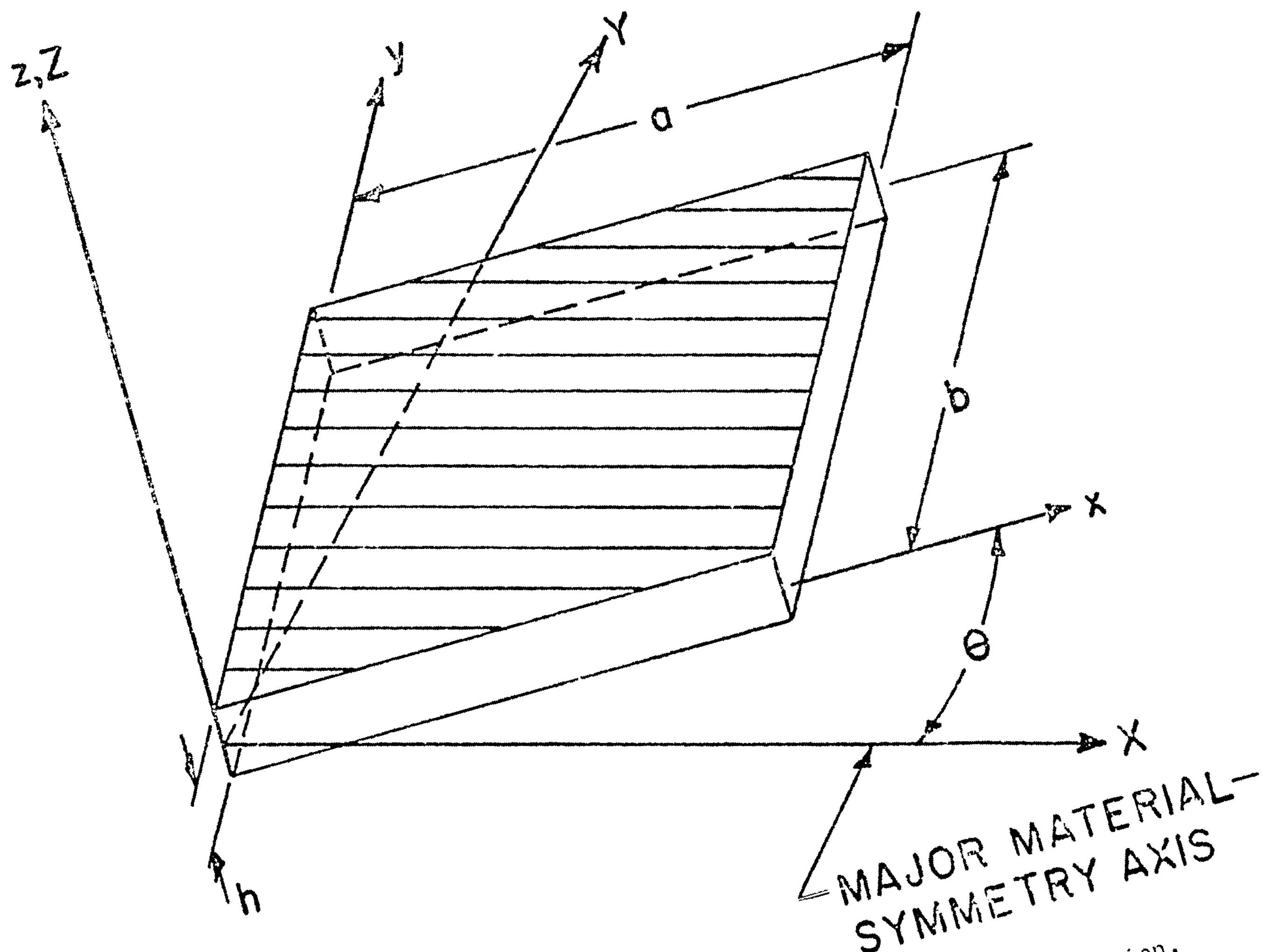

Figure A-2. Major Material-Symmetry Axis Direction.

```
      DIMENSION E(20),G(20),Z(20),A(20,20)
  120 READ(5,10)N
 10   FORMAT(I3)
      IF(N) 130,130,140
  140 NN=N-1
      READ(5,20)(E(J),G(J),Z(J),J=1,N)
20    FORMAT(3F10.5)
      SUMGZ=G(1)*Z(1)
      DO 30 J=2,N
      SUMGZ=SUMGZ+G(J)*(Z(J)-Z(J-1))
30    CONTINUE
      GE2Z5=E(1)**2/(4.*G(1))*(8./15.*Z(1)**5)
      DO 40 J=2,N
40    GE2Z5=GE2Z5+E(J)**2/(4.*G(J))*(Z(J)**4*(Z(J)-Z(J-1))-
     12./3.*Z(J)**2
     2*(Z(J)**3-Z(J-1)**3)+0.2*(Z(J)**5-Z(J-1)**5))
      E2Z4=E(1)*E(1)*Z(1)**4
      DO 50 J=2,N
50    E2Z4=E2Z4+E(J)*E(J)*(Z(J)**2-Z(J-1)**2)**2
      E2Z4=E2Z4-E(1)*E(1)*Z(1)**4
      GE2Z4=1.0/(4.0*G(1))*E2Z4*Z(1)
      DO 60 J=2,NN
      E2Z4=E2Z4-E(J)*E(J)*(Z(J)**2-Z(J-1)**2)**2
60    GE2Z4=GE2Z4+1.0/(4.0*G(J))*E2Z4*(Z(J)-Z(J-1))
      A(1,1)=E(1)*(2./3.)*Z(1)**3
      DO 70 K=2,NN
70    A(1,K)=E(K)*(Z(K)**2*(Z(K)-Z(K-1))-1./3.*(Z(K)**3-Z(K-1)**3))
      DO 71 J=2,N
71    A(J,1)=E(J)*(Z(J)**2-Z(J-1)**2)
      LL=1
      DO 72 K=2,NN
      DO 73 J=2,N
      A(J,K)=A(J+LL,K-LL)
73    CONTINUE
72    LL=LL+1
      GEZ=0.0
```

```
      II=N
      KK=NN
      DO 74 M=1,NN
      AA=0.0
      DO 75 I=1,KK
      L=I+1
      DO 75 J=L,II
      B=A(I,M)*A(J,M)
      AA=AA+B
75    CONTINUE
      C=0.5*G(M)*AA
      GEZ=GEZ+C
      II=II-1
74    KK=KK-1
      SGEZ=GE2Z4+GE2Z5+GEZ
      E1Z3=(1./3.)*E(1)*Z(1)**3
      DO 80 J=2,N
      E1Z3=E1Z3+0.5*E(J)*(Z(J)**2*(Z(J)-Z(J-1))-1./3.*(Z(J)**3-Z(J-1)
     1**3))
80    CONTINUE
      E1Z2Z=0.5*E(1)*Z(1)**2
      DO 90 J=2,N
90    E1Z2Z=E1Z2Z+0.5*E(J)*(Z(J)**2-Z(J-1)**2)
      E1Z2Z=E1Z2Z-0.5*E(1)*Z(1)**2
      AE1Z3=E1Z2Z*Z(1)
      DO 100 J=2,NN
      E1Z2Z=E1Z2Z-0.5*E(J)*(Z(J)**2-Z(J-1)**2)
      AE1Z3=AE1Z3+E1Z2Z*(Z(J)-Z(J-1))
100   CONTINUE
      SE1Z3=(E1Z3+AE1Z3)**2
      SFK=SE1Z3/(SUMGZ*SGEZ)
      WRITE(6,110)N,SFK
110   FORMAT(' N=',I2,5X,'SHEAR FACTOR K =',F7.4//)
      GO TO 120
  130 CALL EXIT
      END
```

```
C
C     PLATE VIBRATION CAL. BY C.C. SIU
C     COMPOSITE SHAPE FACTOR   K11X2,K22X2,K12X2
C     USING AN EXACT ANALYSIS BY S. SRINIVAS TO COMPARE
C     WITH PLATE THEORY INCLUDING TRANSVERSE SHEAR DEFORMATION
C     STIFFNESS COEFFICENTS FOR  3 LAYER
C     1ST LAYER IS SAME AS 3RD LAYER
C
      DOUBLE PRECISION H,H1,H2,RO1,RO2,G1,G2,K11X2,K22X2,K12X2,ANU,AN
      DOUBLE PRECISION C11I,C11II,C12I,C12II,C22I,C22II,C44I,C44II,
     1C45I,C45II,C55I,C55II,C66I,C66II
      DOUBLE PRECISION AS(5,5),AI(5,5)
      DOUBLE PRECISION EVAL(5),EVEC(5,5)
      INTEGER RA,RB,RC,RD,RE
      INTEGER CA,CB,CC,CD,CE
C
C     READ PLATE PROPERTIES
C
      JJJ=1
    5 READ(5,7) K11X2,K22X2,K12X2,ANU,AN
    7 FORMAT(3F10.4,2F10.2)
C
C     WRITE PLATE PROPERTIES
C
      WRITE(6,8)K11X2,K22X2,K12X2,ANU,AN
    8 FORMAT(3F10.4,2F10.2)
C
C     CAL. COEFFICIENT
C
      C11I=1.D0/(1.D0-ANU**2)
      C11II=C11I
      C12I=ANU/(1.D0-ANU**2)
      C12II=C12I
      C22I=C11I
      C22II=C11I
      C44I=1.D0/(2.D0*(1.D0+ANU))
```

```
      C44II=C44I*(1.D0/AN)
      C45I=0.D0
      C45II=0.D0
      C55I=C44I
      C55II=C44II
      C66I=C44I
      C66II=C44II
      PI=3.1415926536D0
      G1=0.0001**0.5*PI
      G2=0.0064**0.5*PI
      H1=1.D0
      H2=8.D0
      H=10.D0
      RO1=1.D0
      RO2=1.D0
      RA=1
      RB=2
      RC=3
      RD=4
      RE=5
      CA=1
      CB=2
      CC=3
      CD=4
       CE=5
C
C     CALCULATE STIFFNESS MATRIX
C
      AS(RA,CA)=25.D0*(C11I/C66II+C66I/C66II)*G1**2
     1+25.D0/16.D0*(C11II/C66II+1.D0)*G2**2
      AS(RA,CB)=25.D0*(C12I/C66II+C66I/C66II)*G1**2
     1+25.D0/16.D0*(C12II/C66II+1.D0)*G2**2
      AS(RA,CC)=0.D0
      AS(RA,CD)=0.D0
      AS(RA,CE)=0.D0
      AS(RB,CA)=25.D0*(C12I/C66II+C66I/C66II)*G1**2
```

```
1+25.D0/16.D0*(C12II/C66II+1.D0)*G2**2
 AS(RB,CB)=25.D0*(C22I/C66II+C66I/C66II)*G1**2
1+25.D0/16.D0*(C22II/C66II+1.D0)*G2**2
 AS(RB,CC)=0.D0
 AS(RB,CD)=0.D0
 AS(RB,CE)=0.D0
 AS(RC,CA)=0.D0
 AS(RC,CB)=0.D0
 AS(RC,CC)=(200.D0*K11X2*(H1/H2)*(C44I/C66II)+200.D0*K22X2*
1(H1/H2)*(C55I/C66II)+400.D0*K12X2*(H1/H2)*(C45I/C66II))*G1**2
2+(25.D0/16.D0*K11X2*(C44II/C66II)+25.D0/16.D0*K22X2*(C55II/C66II)
3+25.D0/8.D0*K12X2*(C45II/C66II))*G2**2
 AS(RC,CD)=20.D0*K11X2*(H1/H2)*(C44I/C66II)*G1
1+(5.D0/4.D0)*K11X2*(C44II/C66II)*G2
 AS(RC,CE)=20.D0*K22X2*(H1/H2)*(C55I/C66II)*G1
1+(5.D0/4.D0)*K22X2*(C55II/C66II)*G2
 AS(RD,CA)=0.D0
 AS(RD,CB)=0.D0
 AS(RD,CC)=20.D0*K11X2*(H1/H2)*(C44I/C66II)*G1
1+(5.D0/4.D0)*K11X2*(C44II/C66II)*G2
 AS(RD,CD)=61.D0/12.D0*(C11I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
1(C11II/C66II+1.D0)*G2**2+K11X2*(2.D0*(H1/H2)*(C44I/C66II)+(C44II
2/C66II))
 AS(RD,CE)=61.D0/12.D0*(C12I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
1(C12II/C66II+1.D0)*G2**2
 AS(RE,CA)=0.D0
 AS(RE,CB)=0.D0
 AS(RE,CC)=20.D0*K22X2*(H1/H2)*(C55I/C66II)*G1
1+(5.D0/4.D0)*K22X2*(C55II/C66II)*G2
 AS(RE,CD)=61.D0/12.D0*(C12I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
1(C12II/C66II+1.D0)*G2**2
 AS(RE,CE)=61.D0/12.D0*(C22I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
1(C22II/C66II+1.D0)*G2**2+K22X2*(2.D0*(H1/H2)*(C55I/C66II)+(C55II
2/C66II))
C
C     CALCULATE INERTIA MATRIX
```

```
C
      AI(RA,CA)=(H/H2)**2*(2.D0*(H1/H2)*R01/R02+1.D0)
      AI(RA,CB)=0.D0
      AI(RA,CC)=0.D0
      AI(RA,CD)=0.D0
      AI(RA,CE)=0.D0
      AI(RB,CA)=0.D0
      AI(RB,CB)=(H/H2)**2*(2.D0*(H1/H2)*R01/R02+1.D0)
      AI(RB,CC)=0.D0
      AI(RB,CD)=0.D0
      AI(RB,CE)=0.D0
      AI(RC,CA)=0.D0
      AI(RC,CB)=0.D0
      AI(RC,CC)=(H/H2)**2*(2.D0*(H1/H2)*R01/R02+1.D0)
      AI(RC,CD)=0.D0
      AI(RC,CE)=0.D0
      AI(RD,CA)=0.D0
      AI(RD,CB)=0.D0
      AI(RD,CC)=0.D0
      AI(RD,CD)=1.D0/12.D0+61.D0/768.D0*(R01/R02)
      AI(RD,CE)=0.D0
      AI(RE,CA)=0.D0
      AI(RE,CB)=0.D0
      AI(RE,CC)=0.D0
      AI(RE,CD)=0.D0
      AI(RE,CE)=1.D0/12.D0+61.D0/768.D0*(R01/R02)
C
C     CALL DNROOT
C
      CALL DNROOT(5,AS,AI,EVAL,EVEC)
C
C     WRITE EIGENVALUES
C
      DO 910 I=1,5
      EVAL(I)=DSQRT(EVAL(I))
  910 WRITE(6,933)I,EVAL(I)
```

```
933 FORMAT('0',40X,'EIGENVALUE(',I2,')=',D16.8)
    JJJ=JJJ+1
    IF(50-JJJ)951,5,5
951 STOP
    END
    SUBROUTINE DNROOT (M,A,B,XL,X)
C   SEE WRITE-UP IN IBM SSP, PAGE 31
    DIMENSION A(1),B(1),XL(1),X(1)
    DOUBLE PRECISION A,B,XL,X,SUMV
C   COMPUTE EIGENVALUES AND EIGENVECTORS OF B
    K=1
    DO 100 J=2,M
    L=M*(J-1)
    DO 100 I=1,J
    L=L+1
    K=K+1
100 B(K)=B(L)
C   THE MATRIX B IS A REAL SYMMETRIC MATRIX
    MV=0
     CALL DEIGEN (B,X,M,MV)
C   FORM RECIPROCALS OF SQUARE ROOT OF EIGENVALUES. THE RESULTS
C   ARE PREMULTIPLIED BY THE ASSOCIATED EIGENVECTORS.
    L=0
    DO 110 J=1,M
    L=L+J
110 XL(J)=1.0/DSQRT(DABS(B(L)))
    K=0
    DO 115 J=1,M
    DO 115 I=1,M
    K=K+1
115 B(K)=X(K)*XL(J)
C   FORM (B**(-1/2))PRIME * A * (B**(-1/2))
    DO 120 I=1,M
    N2=0
    DO 120 J=1,M
    N1=M*(I-1)
```

```
      L=M*(J-1)+I
      X(L)=0.0
      DO 120 K=1,M
      N1=N1+1
      N2=N2+1
  120 X(L)=X(L)+B(N1)*A(N2)
      L=0
      DO 130 J=1,M
      DO 130 I=1,J
      N1=I-M
      N2=M*(J-1)
      L=L+1
      A(L)=0.0
      DO 130 K=1,M
      N1=N1+M
      N2=N2+1
  130 A(L)=A(L)+X(N1)*B(N2)
C     COMPUTE EIGENVALUES AND EIGENVECTORS OF A
      CALL DEIGEN (A,X,M,MV)
      L=0
      DO 140 I=1,M
      L=L+I
  140 XL(I)=A(L)
C     COMPUTE THE NORMALIZED EIGENVECTORS
      DO 150 I=1,M
      N2=0
      DO 150 J=1,M
      N1=I-M
      L=M*(J-1)+I
      A(L)=0.0
      DO 150 K=1,M
      N1=N1+M
      N2=N2+1
  150 A(L)=A(L)+B(N1)*X(N2)
      I=0
      K=0
```

```
      DO 180 J=1,M
      SUMV=0.0
      DO 170 I=1,M
      L=L+1
  170 SUMV=SUMV+A(L)*A(L)
  175 SUMV=DSQRT(SUMV)
      DO 180 I=1,M
      K=K+1
  180 X(K)=A(K)/SUMV
      RETURN
      END
C                                                                       EIGEN001
C     ..................................................................EIGEN002
C                                                                       EIGEN003
C        SUBROUTINE DEIGEN                                              EIGEN004
C                                                                       EIGEN005
C        PURPOSE                                                        EIGEN006
C           COMPUTE EIGENVALUES AND EIGENVECTORS OF A REAL SYMMETRIC    EIGEN007
C           MATRIX                                                      EIGEN008
C                                                                       EIGEN009
C        USAGE                                                          EIGEN010
C           CALL DEIGEN(A,R,N,MV)                                       EIGEN011
C                                                                       EIGEN012
C        DESCRIPTION OF PARAMETERS                                      EIGEN013
C           A - ORIGINAL MATRIX (SYMMETRIC), DESTROYED IN COMPUTATION.  EIGEN014
C               RESULTANT EIGENVALUES ARE DEVELOPED IN DIAGONAL OF      EIGEN015
C               MATRIX A IN DESCENDING ORDER.                           EIGEN016
C           R - RESULTANT MATRIX OF EIGENVECTORS (STORED COLUMNWISE,    EIGEN017
C               IN SAME SEQUENCE AS EIGENVALUES)                        EIGEN018
C           N - ORDER OF MATRICES A AND R                               EIGEN019
C           MV- INPUT CODE                                              EIGEN020
C                   0   COMPUTE EIGENVALUES AND EIGENVECTORS            EIGEN021
C                   1   COMPUTE EIGENVALUES ONLY (R NEED NOT BE         EIGEN022
C                       DIMENSIONED BUT MUST STILL APPEAR IN CALLING    EIGEN023
C                       SEQUENCE)                                       EIGEN024
C                                                                       EIGEN025
```

```
C        REMARKS                                                        EIGEN026
C           ORIGINAL MATRIX A MUST BE REAL SYMMETRIC (STORAGE MODE=1)   EIGEN027
C           MATRIX A CANNOT BE IN THE SAME LOCATION AS MATRIX R         EIGEN028
C                                                                       EIGEN029
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  EIGEN030
C           NONE                                                        EIGEN031
C                                                                       EIGEN032
C        METHOD                                                         EIGEN033
C           DIAGONALIZATION METHOD ORIGINATED BY JACOBI AND ADAPTED     EIGEN034
C           BY VON NEUMANN FOR LARGE COMPUTERS AS FOUND IN 'MATHEMATICALEIGEN035
C           METHODS FOR DIGITAL COMPUTERS', EDITED BY A. RALSTON AND    EIGEN036
C           H.S. WILF, JOHN WILEY AND SONS, NEW YORK, 1962, CHAPTER 7   EIGEN037
C                                                                       EIGEN038
C     ..................................................................EIGEN039
C                                                                       EIGEN040
      SUBROUTINE DEIGEN(A,R,N,MV)                                       EIGEN041
      DIMENSION A(1),R(1)                                               EIGEN042
C                                                                       EIGEN043
C        ...............................................................EIGEN044
C                                                                       EIGEN045
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE  EIGEN046
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION      EIGEN047
C        STATEMENT WHICH FOLLOWS.                                       EIGEN048
C                                                                       EIGEN049
      DOUBLE PRECISION A,R,ANORM,ANRMX,THR,X,Y,SINX,SINX2,COSX,         EIGEN050
     1                 COSX2,SINCS                                      EIGEN051
C                                                                       EIGEN052
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS    EIGEN053
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS      EIGEN054
C        ROUTINE.                                                       EIGEN055
C                                                                       EIGEN056
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO      EIGEN057
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  SQRT IN STATEMENTSEIGEN058
C        40, 68, 75, AND 78 MUST BE CHANGED TO DSQRT.  ABS IN STATEMENT EIGEN059
C        62 MUST BE CHANGED TO DABS.                                    EIGEN060
C                                                                       EIGEN061
```

```
C         ..........................................................EIGEN062
C                                                                   EIGEN063
C         GENERATE IDENTITY MATRIX                                  EIGEN064
C                                                                   EIGEN065
      IF(MV-1) 10,25,10                                             EIGEN066
   10 IQ=-N                                                         EIGEN067
      DO 20 J=1,N                                                   EIGEN068
      IQ=IQ+N                                                       EIGEN069
      DO 20 I=1,N                                                   EIGEN070
      IJ=IQ+I                                                       EIGEN071
      R(IJ)=0.D+00                                                  EIGEN072
      IF(I-J) 20,15,20                                              EIGEN073
   15 R(IJ)=1.D+00                                                  EIGEN074
   20 CONTINUE                                                      EIGEN075
C                                                                   EIGEN076
C         COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)        EIGEN077
C                                                                   EIGEN078
   25 ANORM=0.D+00                                                  EIGEN079
      DO 35 I=1,N                                                   EIGEN080
      DO 35 J=I,N                                                   EIGEN081
      IF(I-J) 30,35,30                                              EIGEN082
   30 IA=I+(J*J-J)/2                                                EIGEN083
      ANORM=ANORM+A(IA)*A(IA)                                       EIGEN084
   35 CONTINUE                                                      EIGEN085
      IF(ANORM) 165,165,40                                          EIGEN086
   40 ANORM=1.414D+00*DSQRT(ANORM)                                  EIGEN087
      ANRMX=ANORM*1.0D-06/FLOAT(N)                                  EIGEN088
C                                                                   EIGEN089
C         INITIALIZE INDICATORS AND COMPUTE THRESHOLD, THR          EIGEN090
C                                                                   EIGEN091
      IND=0                                                         EIGEN092
      THR=ANORM                                                     EIGEN093
   45 THR=THR/FLOAT(N)                                              EIGEN094
   50 L=1                                                           EIGEN095
   55 M=L+1                                                         EIGEN096
C                                                                   EIGEN097
```

```
C          COMPUTE SIN AND COS                                          EIGEN098
C                                                                       EIGEN099
   60 MQ=(M*M-M)/2                                                      EIGEN100
      LQ=(L*L-L)/2                                                      EIGEN101
      LM=L+MQ                                                           EIGEN102
   62 IF(DABS(A(LM))-THR) 130,65,65                                     EIGEN103
   65 IND=1                                                             EIGEN104
      LL=L+LQ                                                           EIGEN105
      MM=M+MQ                                                           EIGEN106
      X=0.5D+00*(A(LL)-A(MM))                                           EIGEN107
   68 Y=-A(LM)/DSQRT(A(LM)*A(LM)+X*X)                                   EIGEN108
      IF(X) 70,75,75                                                    EIGEN109
   70 Y=-Y                                                              EIGEN110
   75 SINX=Y/DSQRT(2.D+00*(1.D+00+(DSQRT(1.D+00-Y*Y))))                 EIGEN111
      SINX2=SINX*SINX                                                   EIGEN112
   78 COSX=DSQRT(1.0D+00-SINX2)                                         EIGEN113
      COSX2=COSX*COSX                                                   EIGEN114
      SINCS =SINX*COSX                                                  EIGEN115
C                                                                       EIGEN116
C          ROTATE L AND M COLUMNS                                       EIGEN117
C                                                                       EIGEN118
      ILQ=N*(L-1)                                                       EIGEN119
      IMQ=N*(M-1)                                                       EIGEN120
      DO 125 I=1,N                                                      EIGEN121
      IQ=(I*I-I)/2                                                      EIGEN122
      IF(I-L) 80,115,80                                                 EIGEN123
   80 IF(I-M) 85,115,90                                                 EIGEN124
   85 IM=I+MQ                                                           EIGEN125
      GO TO 95                                                          EIGEN126
   90 IM=M+IQ                                                           EIGEN127
   95 IF(I-L) 100,105,105                                               EIGEN128
  100 IL=I+LQ                                                           EIGEN129
      GO TO 110                                                         EIGEN130
  105 IL=L+IQ                                                           EIGEN131
  110 X=A(IL)*COSX-A(IM)*SINX                                           EIGEN132
      A(IM)=A(IL)*SINX+A(IM)*COSX                                       EIGEN133
```

```
      A(IL)=X                                                           EIGEN134
  115 IF(MV-1) 120,125,120                                              EIGEN135
  120 ILR=ILQ+I                                                         EIGEN136
      IMR=IMQ+I                                                         EIGEN137
      X=R(ILR)*COSX-R(IMR)*SINX                                         EIGEN138
      R(IMR)=R(ILR)*SINX+R(IMR)*COSX                                    EIGEN139
      R(ILR)=X                                                          EIGEN140
  125 CONTINUE                                                          EIGEN141
      X=2.D+00*A(LM)*SINCS                                              EIGEN142
      Y=A(LL)*COSX2+A(MM)*SINX2-X                                       EIGEN143
      X=A(LL)*SINX2+A(MM)*COSX2+X                                       EIGEN144
      A(LM)=(A(LL)-A(MM))*SINCS+A(LM)*(COSX2-SINX2)                     EIGEN145
      A(LL)=Y                                                           EIGEN146
      A(MM)=X                                                           EIGEN147
C                                                                       EIGEN148
C        TESTS FOR COMPLETION                                           EIGEN149
C                                                                       EIGEN150
C        TEST FOR M = LAST COLUMN                                       EIGEN151
C                                                                       EIGEN152
  130 IF(M-N) 135,140,135                                               EIGEN153
  135 M=M+1                                                             EIGEN154
      GO TO 60                                                          EIGEN155
C                                                                       EIGEN156
C        TEST FOR L = SECOND FROM LAST COLUMN                           EIGEN157
C                                                                       EIGEN158
  140 IF(L-(N-1)) 145,150,145                                           EIGEN159
  145 L=L+1                                                             EIGEN160
      GO TO 55                                                          EIGEN161
  150 IF(IND-1) 160,155,160                                             EIGEN162
  155 IND=0                                                             EIGEN163
      GO TO 50                                                          EIGEN164
C                                                                       EIGEN165
  160 IF(THR-ANRMX) 165,165,45                                          EIGEN168
C        COMPARE THRESHOLD WITH FINAL NORM                              EIGEN166
C                                                                       EIGEN167
C                                                                       EIGEN169
```

```
C          SORT EIGENVALUES AND EIGENVECTORS
C
  165 IQ=-N
      DO 185 I=1,N
      IQ=IQ+N
      LL=I+(I*I-I)/2
      JQ=N*(I-2)
      DO 185 J=I,N
      JQ=JQ+N
      MM=J+(J*J-J)/2
      IF(A(LL)-A(MM)) 170,185,185
  170 X=A(LL)
      A(LL)=A(MM)
      A(MM)=X
      IF(MV-1) 175,185,175
  175 DO 180 K=1,N
      ILR=IQ+K
      IMR=JQ+K
      X=R(ILR)
      R(ILR)=R(IMR)
  180 R(IMR)=X
  185 CONTINUE
      RETURN
      END
      SUBROUTINE CHECK (I)
      WRITE (3,1) I
    1 FORMAT (' CHECK ',I3)
      RETURN
      END
```

```
C
C      VIBRATION OF LAMINATED ANISOTROPIC PLATES
C      CONSIDERING THICKNESS SHEAR AND DAMPING EFFECTS
C
       DIMENSION S(40,40),T(40)
       DIMENSION ANGLE(8),H(24),RO(24)
       DIMENSION ER11(24),ER22(24),GE11(24),GE22(24)
       DIMENSION VR12(24),VR21(24)
       DIMENSION GR12(24),GR13(24),GR23(24),GG12(24),GG13(24),GG23(24)
       DIMENSION CRT11(24),CIT11(24),CRT12(24),CIT12(24)
       DIMENSION CRT22(24),CIT22(24),CRT66(24),CIT66(24)
       DIMENSION CRT16(24),CIT16(24),CRT26(24),CIT26(24)
       DIMENSION CRT44(24),CIT44(24),CRT45(24),CIT45(24)
       DIMENSION CRT55(24),CIT55(24)
       REAL K11,K22,K12
       READ(5,15)J,NN,INPU1,INPU2
    15 FORMAT(4I5)
       READ(5,32)A,B,K11,K22,K12
    32 FORMAT(5F10.4)
    10 READ(5,35,END=3333)F
    35 FORMAT(F10.4)
       READ(5,36)(ANGLE(I),H(I),H(I+1),RO(I),I=1,J)
    36 FORMAT(4F10.0)
       READ(5,37)(ER11(I),ER22(I),GE11(I),GE22(I),I=1,J)
    37 FORMAT(4E10.4)
       READ(5,39)(VR12(I),VR21(I),I=1,J)
    39 FORMAT(2F10.6)
       READ(5,40)(GR12(I),GR13(I),GR23(I),GG12(I),GG13(I),GG23(I),I=1,J)
    40 FORMAT(6E10.4)
C
C      CALCULATE STIFFNESS,DAMPING, AND DENSITY COEFFICIENTS
C
       SMAR11=0.
       SMAI11=0.
       SMAR12=0.
       SMAI12=0.
```

```
SMAR16=0.
SMAI16=0.
SMAR22=0.
SMAI22=0.
SMAR26=0.
SMAI26=0.
SMAR66=0.
SMAI66=0.
SMAR44=0.
SMAI44=0.
SMAR45=0.
SMAI45=0.
SMAR55=0.
SMAI55=0.
SMBR11=0.
SMBI11=0.
SMBR12=0.
SMBI12=0.
SMBR16=0.
SMBI16=0.
SMBR22=0.
SMBI22=0.
SMBR26=0.
SMBI26=0.
SMBR66=0.
SMBI66=0.
SMDR11=0.
SMDI11=0.
SMDR12=0.
SMDI12=0.
SMDR16=0.
SMDI16=0.
SMDR22=0.
SMDI22=0.
SMDR26=0.
SMDI26=0.
```

```
      SMDR66=0.
      SMDI66=0.
      SUMP0=0.
      SUMP1=0.
      SUMP2=0.
      PI=3.141592653589793
      DO 150 I=1,J
      CR11=ER11(I)/(1.-VR12(I)*VR21(I))
      CI11=ER11(I)/(1.-VR12(I)*VR21(I))*GE11(I)
      CR22=ER22(I)/(1.-VR12(I)*VR21(I))
      CI22=ER22(I)/(1.-VR12(I)*VR21(I))*GE22(I)
      CR12=VR12(I)*ER22(I)/(1.-VR12(I)*VR21(I))
      CR12=VR12(I)*ER22(I)/(1.-VR12(I)*VR21(I))*GE22(I)
      CR44=GR13(I)
      CI44=GR13(I)*GG13(I)
      CR55=GR23(I)
      CI55=GR23(I)*GG23(I)
      CR66=GR12(I)
      CI66=GR12(I)*GG12(I)
      THETA=ANGLE(I)*PI/180.
      C4=COS(THETA)*COS(THETA)*COS(THETA)*COS(THETA)
      S4=SIN(THETA)*SIN(THETA)*SIN(THETA)*SIN(THETA)
      C3S1=COS(THETA)*COS(THETA)*COS(THETA)*SIN(THETA)
      S3C1=SIN(THETA)*SIN(THETA)*SIN(THETA)*COS(THETA)
      C2S2=COS(THETA)*COS(THETA)*SIN(THETA)*SIN(THETA)
      C2=COS(THETA)*COS(THETA)
      S2=SIN(THETA)*SIN(THETA)
      CS=COS(THETA)*SIN(THETA)
      CRT11(I)=CR11*C4+2.*(CR12+2.*CR66)*C2S2+CR22*S4
      CIT11(I)=CI11*C4+2.*(CI12+2.*CI66)*C2S2+CI22*S4
C
C     COEFFICIENT TRANSFORMATION
C
      CRT12(I)=CR12*C4+(CR11+CR22-4.*CR66)*C2S2+CP12*S4
      CIT12(I)=CI12*C4+(CI11+CI22-4.*CI66)*C2S2+CI12*S4
      CRT22(I)=CR22*C4+2.*(CR12+2.*CR66)*C2S2+CR11*S4
```

```
CIT22(I)=CI22*C4+2.*(CI12+2.*CI66)*C2S2+CI11*S4
CRT66(I)=CR66*C4+(CR11+CR22-2.*CR12-2.*CR66)*C2S2+CR66*S4
CIT66(I)=CI66*C4+(CI11+CI22-2.*CI12-2.*CI66)*C2S2+CI66*S4
CRT16(I)=(2.*CR66+CR12-CR11)*C3S1-(2.*CR66+CR12-CR22)*S3C1
CIT16(I)=(2.*CI66+CI12-CI11)*C3S1-(2.*CI66+CI12-CI22)*S3C1
CRT26(I)=-(2.*CR66+CR12-CR22)*C3S1+(2.*CR66+CR12-CR11)*S3C1
CIT26(I)=-(2.*CI66+CI12-CI22)*C3S1+(2.*CI66+CI12-CI11)*S3C1
CRT44(I)=CR44*C2+CR55*S2
CIT44(I)=CI44*C2+CI55*S2
CRT45(I)=CR55*CS-CR44*CS
CIT45(I)=CI55*CS-CI44*CS
CRT55(I)=CR55*C2+CR44*S2
CIT55(I)=CI55*C2+CI44*S2
AAR11=CRT11(I)*(H(I+1)-H(I))
SMAR11=SMAR11+AAR11
AAI11=CIT11(I)*(H(I+1)-H(I))
SMAI11=SMAI11+AAI11
AAR12=CRT12(I)*(H(I+1)-H(I))
SMAR12=SMAR12+AAR12
AAI12=CIT12(I)*(H(I+1)-H(I))
SMAI12=SMAI12+AAI12
AAR22=CRT22(I)*(H(I+1)-H(I))
SMAR22=SMAR22+AAR22
AAI22=CIT22(I)+(H(I+1)-H(I))
SMAI22=SMAI22+AAI22
AAR66=CRT66(I)*(H(I+1)-H(I))
SMAR66=SMAR66+AAR66
AAI66=CIT66(I)*(H(I+1)-H(I))
SMAI66=SMAI66+AAI66
AAR16=CRT16(I)*(H(I+1)-H(I))
SMAR16=SMAR16+AAR16
AAI16=CIT16(I)*(H(I+1)-H(I))
SMAI16=SMAI16+AAI16
AAR26=CRT26(I)*(H(I+1)-H(I))
SMAR26=SMAR26+AAR26
AAI26=CIT26(I)*(H(I+1)-H(I))
```

```
SMAI26=SMAI26+AAI26
AAR44=CRT44(I)*(H(I+1)-H(I))
SMAR44=SMAR44+AAR44
AAI44=CIT44(I)*(H(I+1)-H(I))
SMAI44=SMAI44+AAI44
AAR45=CRT45(I)*(H(I+1)-H(I))
SMAR45=SMAR45+AAR45
AAI45=CIT45(I)*(H(I+1)-H(I))
SMAI45=SMAI45+AAI45
AAR55=CRT55(I)*(H(I+1)-H(I))
SMAR55=SMAR55+AAR55
AAI55=CIT55(I)*(H(I+1)-H(I))
SMAI55=SMAI55+AAI55
BBR11=CRT11(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR11=SMBR11+BBR11
BBI11=CIT11(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI11=SMBI11+BBI11
BBR12=CRT12(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR12=SMBR12+BBR12
BBI12=CIT12(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI12=SMBI12+BBI12
BBR22=CRT22(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR22=SMBR22+BBR22
BBI22=CIT22(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI22=SMBI22+BBI22
BBR66=CRT66(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR66=SMBR66+BBR66
BBI66=CIT66(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI66=SMBI66+BBI66
BBR16=CRT16(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR16=SMBR16+BBR16
BBI16=CIT16(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI16=SMBI16+BBI16
BBR26=CRT26(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR26=SMBR26+BBR26
BBI26=CIT26(I)*(H(I+1)*H(I+1)-H(I)*H(I))
```

```
      SMBI26=SMBI26+BBI26
      DDR11=CRT11(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR11=SMDR11+DDR11
      DDI11=CIT11(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI11=SMDI11+DDI11
      DDR12=CRT12(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR12=SMDR12+DDR12
      DDI12=CIT12(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI12=SMDI12+DDI12
      DDR22=CRT22(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR22=SMDR22+DDR22
      DDI22=CIT22(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI22=SMDI22+DDI22
      DDR66=CRT66(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR66=SMDR66+DDR66
      DDI66=CIT66(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI66=SMDI66+DDI66
      DDR16=CRT16(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR16=SMDR16+DDR16
      DDI16=CIT16(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI16=SMDI16+DDI16
      DDR26=CRT26(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR26=SMDR26+DDR26
      DDI26=CIT26(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI26=SMDI26+DDI26
      PP0=RO(I)*(H(I+1)-H(I))
      SUMP0=SUMP0+PP0
      PP1=RO(I)*(H(I+1)*H(I+1)-H(I)*H(I))
      SUMP1=SUMP1+PP1
      PP2=RO(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
  150 SUMP2=SUMP2+PP2
      AR11=SMAR11
      AI11=SMAI11
      AR12=SMAR12
      AI12=SMAI12
      AR22=SMAR22
```

```
AI22=SMAI22
AR66=SMAR66
AI66=SMAI66
AR16=SMAR16
AI16=SMAI16
AR26=SMAR26
AI26=SMAI26
AR44=SMAR44
AI44=SMAI44
AR45=SMAR45
AI45=SMAI45
AR55=SMAR55
AI55=SMAI55
BR11=0.5*SMBR11
BI11=0.5*SMBI11
BR12=0.5*SMBR12
BI12=0.5*SMBI12
BR22=0.5*SMBR22
BI22=0.5*SMBI22
BR66=0.5*SMBR66
BI66=0.5*SMBI66
BR16=0.5*SMBR16
BI16=0.5*SMBI16
BR26=0.5*SMBR26
BI26=0.5*SMBI26
DR11=SMDR11/3.
DI11=SMDI11/3.
DR12=SMDR12/3.
DI12=SMDI12/3.
DR22=SMDR22/3.
DI22=SMDI22/3.
DR66=SMDR66/3.
DI66=SMDI66/3.
DR16=SMDR16/3.
DI16=SMDI16/3.
DR26=SMDR26/3.
```

```
      DI26=SMDI26/3.
      P0=SUMP0
      P1=0.5*SUMP1
      P2=SUMP2/3.
C
C     WRITE MATERIAL PROPERTIES AND STIFFNESS COEFFICIENTS
C     CONSIDERING THICKNESS SHEAR AND DAMPING EFFECTS
      WRITE (6,50) AR11,BR11,DR11
   50 FORMAT('0',18X,'AR11= ',E12.5,5X, 'BR11= ',E12.5,5X,
     1'DR11= ',E12.5)
      WRITE(6,55)AI11,BI11,DI11
   55 FORMAT('0',18X,'AI11= ',E12.5,5X, 'BI11= ',E12.5,5X,
     1'DI11= ',E12.5)
      WRITE(6,60)AR12,BR12,DR12
   60 FORMAT('0',18X,'AR12= ',E12.5,5X, 'BR12= ',E12.5,5X,
     1'DR12= ',E12.5)
      WRITE(6,65)AI12,BI12,DI12
   65 FORMAT('0',18X,'AI12= ',E12.5,5X, 'BI12= ',E12.5,5X,
     1'DI12= ',E12.5)
      WRITE(6,70)AR22,BR22,DR22
   70 FORMAT('0',18X,'AR22= ',E12.5,5X, 'BR22= ',E12.5,5X,
     1'DR22= ',E12.5)
      WRITE(6,75)AI22,BI22,DI22
   75 FORMAT('0',18X,'AI22= ',E12.5,5X, 'BI22= ',E12.5,5X,
     1'DI22= ',E12.5)
      WRITE(6,80)AR66,BR66,DR66
   80 FORMAT('0',18X,'AR66= ',E12.5,5X, 'BR66= ',E12.5,5X,
     1'DR66= ',E12.5)
      WRITE(6,85)AI66,BI66,DI66
   85 FORMAT('0',18X,'AI66= ',E12.5,5X, 'BI66= ',E12.5,5X,
     1'DI66= ',E12.5)
      WRITE(6,90)AR16,BR16,DR16
   90 FORMAT('0',18X,'AR16= ',E12.5,5X, 'BR16= ',E12.5,5X,
     1'DR16= ',E12.5)
      WRITE(6,95)AI16,BI16,DI16
   95 FORMAT('0',18X,'AI16= ',E12.5,5X, 'BI16= ',E12.5,5X,
```

```
     1'DI16= ',E12.5)
      WRITE(6,100)AR26,BR26,DR26
  100 FORMAT('0',18X,'AR26= ',E12.5,5X, 'BR26= ',E12.5,5X,
     1'DR26= ',E12.5)
      WRITE(6,105)AI26,BI26,DI26
  105 FORMAT('0',18X,'AI26= ',E12.5,5X, 'BI26= ',E12.5,5X,
     1'DI26= ',E12.5)
      WRITE(6,110)AR44,AR55,AR45
  110 FORMAT('0',18X,'AR44= ',E12.5,5X, 'AR55= ',E12.5,5X,
     1'AR45= ',E12.5)
      WRITE(6,115)AI44,AI55,AI45
  115 FORMAT('0',18X,'AI44= ',E12.5,5X, 'AI55= ',E12.5,5X,
     1'AI45= ',E12.5)
      WRITE(6,116)P0,P1,P2
  116 FORMAT('0',18X,'P0=',E12.5,5X,'P1=',E12.5,5X,'P2='
     1,E12.5)
      WRITE(6,120)
  120 FORMAT('1')
      W2=(2.*PI*F)**2
C
C     CALCULATE A MATRIX FOR FREE BOUNDARY CONDITIONS
C     CALCULATE SUBMATRIX
C     URUR,URVR,URWR,URXR,URYR,URUI,URVI,URWI,URXI,URYI
      JJUR=0
      DO 500 K1=1,NN
      K=K1+INPU1
      DO 500 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
```

```
      JXI=8*NN**2
      JYI=9*NN**2
      JJUR=JJUR+1
      DO 500 M1=1,NN
      M=M1+INPU1
      DO 500 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI8MK,CI8NL
      IF(M-K)502,504,502
  502 CI8MK=0.
      GO TO 505
  504 CI8MK=2.*M*K*PI**2
  505 CONTINUE
      IF(N-L)506,508,506
  506 CI8NL=0.
      GO TO 509
  508 CI8NL=2.*N*L*PI**2
  509 CONTINUE
C
C     INTEGRAL CI1MK, CI1NL
C
      IF(M-K)512,514,512
  512 CI1MK=1.
      GO TO 515
  514 CI1MK=1.5
  515 CONTINUE
      IF(N-L)516,518,516
  516 CI1NL=1.
      GO TO 519
  518 CI1NL=1.5
  519 CONTINUE
C
C     INTEGRAL CI6MK,CI6NL
C
```

```
      IF(M-K)522,524,522
  522 CI6MK=0.
      GO TO 525
  524 CI6MK=0.
  525 CONTINUE
      IF(N-L)526,528,526
  526 CI6NL=0.
      GO TO 529
  528 CI6NL=0.
  529 CONTINUE
C
C     INTEGRAL CI5MK,CI5NL
C
      CI5MK=CI6MK
      CI5NL=CI6NL
C
C     INTEGRAL CI4MK,CI4NL
C
      CI4MK=CI6MK
      CI4NL=CI6NL
C
C     INTEGRAL CI10MK,CI10NL
C
      CI10MK=CI8MK
      CI10NL=CI8NL
C
C     INTEGRAL CI3MK,CI3NL
C
      CI3MK=CI1MK
      CI3NL=CI1NL
C
C     INTEGRAL CI44MK,CI44NL
C
      CI44MK=CI8MK
      CI44NL=CI8NL
C
```

```
C       INTEGRAL CI31MK,CI31NL
C
        CI31MK=CI1MK
        CI31NL=CI1NL
C
C       INTEGRAL CI34MK,CI34NL
C
        CI34MK=CI4MK
        CI34NL=CI4NL
C
C       INTEGRAL CI33MK, CI33NL
C
        CI33MK=CI4MK
        CI33NL=CI4NL
C
C       INTEGRAL CI37MK, CI37NL
C
        CI37MK=CI4MK
        CI37NL=CI4NL
C
//216 CI17NL=1.
        JYI=JYI+1
  800 S(JJXR,JYI)=-(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
     1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*B*K12*AI45*CI23MK
     2*CI23NL)
C       CALCULATE SUBMATRIX
C       YRUR,YRVR,YRWR,YRXR,YRYR,YRUI,YRVI,YRWI,YRXI,YRYI
        JJYR=4*NN**2
        DO 900 K1=1,NN
        K=K1+INPU1
        DO 900 L2=1,NN
        L=L2+INPU2
        JUR=0
        JVR=1*NN**2
        JWR=2*NN**2
        JXR=3*NN**2
```

```
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJYR=JJYR+1
      DO 900 M1=1,NN
      M=M1+INPU1
      DO 900 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI37MK, CI37NL
C
      IF(M-K)902,904,902
  902 CI37MK=0.
      GO TO 905
  904 CI37MK=0.
  905 CONTINUE
      IF(N-L)906,908,906
  906 CI37NL=0.
      GO TO 909
  908 CI37NL=0.
  909 CONTINUE
C
C     INTEGRAL CI36MK, CI36NL
C
      CI36MK=CI37MK
      CI36NL=CI37NL
C
C     INTEGRAL CI55MK, CI55NL
C
      IF(M-K)912,914,912
  912 CI55MK=0.
      GO TO 915
```

```
  914 CI55MK=2.*M*K*PI**2
  915 CONTINUE
      IF(N-L)916,918,916
  916 CI55NL=0.
      GO TO 919
  918 CI55NL=2.*N*L*PI**2
  919 CONTINUE
C
C     INTEGRAL CI35MK, CI35NL
C
      IF(M-K)922,924,922
  922 CI35MK=1.
      GO TO 925
  924 CI35MK=1.5
  925 CONTINUE
      IF(N-L)926,928,926
  926 CI35NL=1.0
      GO TO 929
  928 CI35NL=1.5
  929 CONTINUE
C
C     INTEGRAL CI41MK, CI41NL
C
      CI41MK=CI35MK
      CI41NL=CI35NL
C
C     INTEGRAL CI57MK, CI57NL
C
      CI57MK=CI55MK
      CI57NL=CI55NL
C
C     INTEGRAL CI42MK, CI42NL
C
      CI42MK=CI37MK
      CI42NL=CI37NL
C
```

```
C      INTEGRAL CI43MK, CI43NL
C
       CI43MK=CI37MK
       CI43NL=CI37NL
C
C      INTEGRAL CI15MK, CI15NL
C
       CI15MK=0.
       CI15NL=0.
C
C      INTEGRAL CI12MK, CI12NL
C
       CI12MK=CI35MK
       CI12NL=CI35NL
C
C      INTEGRAL CI24MK, CI24NL
C
       CI24MK=CI37MK
       CI24NL=CI37NL
C
C      INTEGRAL CI25MK, CI25NL
C
       CI25MK=CI37MK
       CI25NL=CI37NL
C
C      INTEGRAL CI23MK, CI23NL
C
       CI23MK=CI35MK
       CI23NL=CI35NL
C
C      INTEGRAL CI30MK, CI30NL
C
       CI30MK=CI55MK
       CI30NL=CI55NL
C
C      INTEGRAL CI22MK, CI22NL
```

```
C
      CI22MK=CI35MK
      CI22NL=CI35NL
C
C     INTEGRAL CI29MK, CI29NL
C
      CI29MK=CI55MK
      CI29NL=CI55NL
C
C     INTEGRAL CI27MK, CI27NL
C
      CI27MK=CI37MK
      CI27NL=CI37NL
      S(JJYR,JUR)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
     1*BR26*CI35MK*CI55NL+BR66*CI36MK*CI37NL
      JVR=JVR+1
      S(JJYR,JVR)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK
     2*CI41NL
      JWR=JWR+1
      S(JJYR,JWR)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
      JXR=JXR+1
      S(JJYR,JXR)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
     1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
     2*CI23NL
      JYR=JYR+1
      S(JJYR,JYR)=(A/B)*DR22*CI22MK*CI29NL+2.*DR26*CI27MK*CI27NL+(B/A)
     1*DR66*CI29MK*CI22NL+A*B*K22*AR55*CI22MK*CI22NL-W2*A*B*P2*CI22MK
     2*CI22NL
      JUI=JUI+1
      S(JJYR,JUI)=-(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
     1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
      JVI=JVI+1
      S(JJYR,JVI)=-((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
     1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
      JWI=JWI+1
```

```
      S(JJYR,JWI)=-(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
      JXI=JXI+1
      S(JJYR,JXI)=-(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
     1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*3*K12*AI45*CI23MK
     2*CI23NL)
      JYI=JYI+1
  900 S(JJYR,JYI)=-((A/B)*DI22*CI22MK*CI29NL+2.*DI26*CI27MK*CI27NL+(B/A)
     1*DI66*CI29MK*CI22NL+A*B*K22*AI55*CI22MK*CI22NL)
C     CALCULATE SUBMATRIX
C     UIUR,UIVR,UIWR,UIXR,UIYR,UIUI,UIVI,UIWI,UIXI,UIYI
      JJUI=5*NN**2
      DO 1000 K1=1,NN
      K=K1+INPU1
      DO 1000 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJUI=JJUI+1
      DO 1000 M1=1,NN
      M=M1+INPU1
      DO 1000 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI8MK,CI8NL
      IF(M-K)1002,1004,1002
 1002 CI8MK=0.
      GO TO 1005
```

```
 1004 CI8MK=2.*M*K*PI**2
 1005 CONTINUE
      IF(N-L)1006,1008,1006
 1006 CI8NL=0.
      GO TO 1009
 1008 CI8NL=2.*N*L*PI**2
 1009 CONTINUE
C
C     INTEGRAL CI1MK, CI1NL
C
      IF(M-K)1012,1014,1012
 1012 CI1MK=1.
      GO TO 1015
 1014 CI1MK=1.5
 1015 CONTINUE
      IF(N-L)1016,1018,1016
 1016 CI1NL=1.
      GO TO 1019
 1018 CI1NL=1.5
 1019 CONTINUE
C
C     INTEGRAL CI6MK,CI6NL
C
      IF(M-K)1022,1024,1022
 1022 CI6MK=0.
      GO TO 1025
 1024 CI6MK=0.
 1025 CONTINUE
      IF(N-L)1026,1028,1026
 1026 CI6NL=0.
      GO TO 1029
 1028 CI6NL=0.
 1029 CONTINUE
C
C     INTEGRAL CI5MK,CI5NL
C
```

```
      CI5MK=CI6MK
      CI5NL=CI6NL
C
C     INTEGRAL CI4MK,CI4NL
C
      CI4MK=CI6MK
      CI4NL=CI6NL
C
C     INTEGRAL CI10MK,CI10NL
C
      CI10MK=CI8MK
      CI10NL=CI8NL
C
C     INTEGRAL CI3MK,CI3NL
C
      CI3MK=CI1MK
      CI3NL=CI1NL
C
C     INTEGRAL CI44MK,CI44NL
C
      CI44MK=CI8MK
      CI44NL=CI8NL
C
C     INTEGRAL CI31MK,CI31NL
C
      CI31MK=CI1MK
      CI31NL=CI1NL
C
C     INTEGRAL CI34MK,CI34NL
C
      CI34MK=CI4MK
      CI34NL=CI4NL
C
C     INTEGRAL CI33MK, CI33NL
C
      CI33MK=CI4MK
```

```
      CI33NL=CI4NL
C
C     INTEGRAL CI37MK, CI37NL
C
      CI37MK=CI4MK
      CI37NL=CI4NL
C
C     INTEGRAL CI36MK, CI36NL
C
      CI36MK=CI4MK
      CI36NL=CI4NL
C
C     INTEGRAL CI55MK, CI55NL
C
      CI55MK=CI8MK
      CI55NL=CI8NL
C
C     INTEGRAL CI35MK, CI35NL
C
      CI35MK=CI1MK
      CI35NL=CI1NL
      S(JJUI,JUR)=+((B/A)*AI11*CI8MK*CI1NL+2.*AI16*CI6MK*CI6NL+(A/B)
     1*AI66*CI1MK*CI8NL)
      JVR=JVR+1
      S(JJUI,JVR)=+(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
     1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
      JWR=JWR+1
      S(JJUI,JWR)=0.
      JXR=JXR+1
      S(JJUI,JXR)=+((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL+BI16
     1*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
      JYR=JYR+1
      S(JJUI,JYR)=+(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
     1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
      JUI=JUI+1
      S(JJUI,JUI)=(B/A)*AR11*CI8MK*CI1NL+2.*AR16*CI6MK*CI6NL+(A/B)
```

```
     1*AR66*CI1MK*CI8NL-W2*A*B*P0*CI1MK*CI1NL
      JVI=JVI+1
      S(JJUI,JVI)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
     1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
      JWI=JWI+1
      S(JJUI,JWI)=0.
      JXI=JXI+1
      S(JJUI,JXI)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL
     1+BR16*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*P1*CI31MK
     2*CI31NL
      JYI=JYI+1
 1000 S(JJUI,JYI)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
     1*BR26*CI35MK*CI55NL+BR66*CI36MK*CI37NL
C     CALCULATE SUBMATRIX
C     VIUR,VIVR,VIWR,VIXR,VIYR,VIUI,VIVI,VIWI,VIXI,VIYI
      JJVI=6*NN**2
      DO 1100 K1=1,NN
      K=K1+INPU1
      DO 1100 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJVI=JJVI+1
      DO 1100 M1=1,NN
      M=M1+INPU1
      DO 1100 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
```

```
C
C      INTEGRAL CI4MK, CI4NL
C
       IF(M-K)1102,1104,1102
 1102  CI4MK=0.
       GO TO 1105
 1104  CI4MK=0.
 1105  CONTINUE
       IF(N-L)1106,1108,1106
 1106  CI4NL=0.
       GO TO 1109
 1108  CI4NL=0.
 1109  CONTINUE
C
C      INTEGRAL CI5MK, CI5NL
C
       CI5MK=CI4MK
       CI5NL=CI4NL
C
C      INTEGRAL CI3MK, CI3NL
C
       IF(M-K)1112,1114,1112
 1112  CI3MK=1.
       GO TO 1115
 1114  CI3MK=1.5
 1115  CONTINUE
       IF(N-L)1116,1118,1116
 1116  CI3NL=1.
       GO TO 1119
 1118  CI3NL=1.5
 1119  CONTINUE
C
C      INTEGRAL CI10MK, CI10NL
C
       IF(M-K)1122,1124,1122
 1122  CI10MK=0.
```

```
      GO TO 1125
 1124 CI10MK=2.*M*K*PI**2
 1125 CONTINUE
      IF(N-L)1126,1128,1126
 1126 CI10NL=0.
      GO TO 1129
 1128 CI10NL=2.*N*L*PI**2
 1129 CONTINUF
C
C     INTEGRAL CI2MK, CI2NL
C
      CI2MK=CI3MK
      CI2NL=CI3NL
C
C     INTEGRAL CI9MK, CI9NL
C
      CI9MK=CI10MK
      CI9NL=CI10NL
C
C     INTEGRAL CI7MK, CI7NL
C
      CI7MK=CI4MK
      CI7NL=CI4NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI4MK
      CI39NL=CI4NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI4MK
      CI40NL=CI4NL
C
C     INTEGRAL CI56MK, CI56NL
C
```

```
      CI56MK=CI10MK
      CI56NL=CI10NL
C
C     INTEGRAL CI38MK,CI38NL
C
      CI38MK=CI3MK
      CI38NL=CI3NL
C
C     INTEGRAL CI41MK, CI41NL
C
      CI41MK=CI3MK
      CI41NL=CI3NL
C
C     INTEGRAL CI57MK, CI57NL
C
      CI57MK=CI10MK
      CI57NL=CI10NL
C
C     INTEGRAL CI42MK, CI42NL
C
      CI42MK=CI4MK
      CI42NL=CI4NL
C
C     INTEGRAL CI43MK, CI43NL
C
      CI43MK=CI4MK
      CI43NL=CI4NL
      S(JJVI,JUR)=+(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
     1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
      JVR=JVR+1
      S(JJVI,JVR)=+((A/B)*AI22*CI2MK*CI9NL+2.*AI26*CI7MK*CI7NL+(B/A)
     1*AI66*CI9MK*CI2NL)
      JWR=JWR+1
      S(JJVI,JWR)=0.
      JXR=JXR+1
      S(JJVI,JXR)=+(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
```

```
     1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
      JYR=JYR+1
      S(JJVI,JYR)=+((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
     1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
      JUI=JUI+1
      S(JJVI,JUI)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
     1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
      JVI=JVI+1
      S(JJVI,JVI)=(A/B)*AR22*CI2MK*CI9NL+2.*AR26*CI7MK*CI7NL+(B/A)
     1*AR66*CI9MK*CI2NL-W2*A*B*P0*CI2MK*CI2NL
      JWI=JWI+1
      S(JJVI,JWI)=0.
      JXI=JXI+1
      S(JJVI,JXI)=BR12*CI39MK*CI40NL+(B/A)*BR16*CI56MK*CI38NL+(A/B)
     1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
      JYI=JYI+1
 1100 S(JJVI,JYI)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK*CI41NL
C     CALCULATE SUBMATRIX
C     WIUR,WIVR,WIWR,WIXR,WIYR,WIUI,WIVI,WIWI,WIXI,WIYI
      JJWI=7*NN**2
      DO 1200 K1=1,NN
      K=K1+INPU1
      DO 1200 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJWI=JJWI+1
```

```
      DO 1200 M1=1,NN
      M=M1+INPU1
      DO 1200 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI20MK, CI20NL
C
      IF(M-K)1202,1204,1202
 1202 CI20MK=0.
      GO TO 1205
 1204 CI20MK=2.0*M*K*PI**2
 1205 CONTINUE
      IF(N-L)1206,1208,1206
 1206 CI20NL=0.
      GO TO 1209
 1208 CI20NL=2.0*N*L*PI**2
 1209 CONTINUE
C
C     INTEGRAL CI17MK,CI17NL
C
      IF(M-K)1212,1214,1212
 1212 CI17MK=1.
      GO TO 1215
 1214 CI17MK=1.5
 1215 CONTINUE
      IF(N-L)1216,1218,1216
C     INTEGRAL CI36MK, CI36NL
C
      CI36MK=CI4MK
      CI36NL=CI4NL
C
C     INTEGRAL CI55MK, CI55NL
C
      CI55MK=CI8MK
      CI55NL=CI8NL
```

```
C
C      INTEGRAL CI35MK, CI35NL
C
       CI35MK=CI1MK
       CI35NL=CI1NL
       S(JJUR,JUR)=(B/A)*AR11*CI8MK*CI1NL+2.*AR16*CI6MK*CI6NL+(A/B)
      1*AR66*CI1MK*CI8NL-W2*A*B*P0*CI1MK*CI1NL
       JVR=JVR+1
       S(JJUR,JVR)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
      1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
       JWR=JWR+1
       S(JJUR,JWR)=0.
       JXR=JXR+1
       S(JJUR,JXR)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL
      1+BR16*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*P1*CI31MK
      2*CI31NL
       JYR=JYR+1
       S(JJUR,JYR)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
      1*BR26*CI35MK*CI55NL+BR66*CI36MK*CI37NL
       JUI=JUI+1
       S(JJUR,JUI)=-((B/A)*AI11*CI8MK*CI1NL+2.*AI16*CI6MK*CI6NL+(A/B)
      1*AI66*CI1MK*CI8NL)
       JVI=JVI+1
       S(JJUR,JVI)=-(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
      1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
       JWI=JWI+1
       S(JJUR,JWI)=0.
       JXI=JXI+1
       S(JJUR,JXI)=-((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL+BI16
      1*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
       JYI=JYI+1
   500 S(JJUR,JYI)=-(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
      1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
C      CALCULATE SUBMATRIX
C      VRUR,VRVR,VRWR,VRXR,VRYR,VRUI,VRVI,VRWI,VRXI,VRYI
       JJVR=1*NN**2
```

```
      DO 600 K1=1,NN
      K=K1+INPU1
      DO 600 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJVR=JJVR+1
      DO 600 M1=1,NN
      M=M1+INPU1
      DO 600 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI4MK, CI4NL
C
      IF(M-K)602,604,602
  602 CI4MK=0.
      GO TO 605
  604 CI4MK=0.
  605 CONTINUE
      IF(N-L)606,608,606
  606 CI4NL=0.
      GO TO 609
  608 CI4NL=0.
  609 CONTINUE
C
C     INTEGRAL CI5MK, CI5NL
C
```

```
      CI5MK=CI4MK
      CI5NL=CI4NL
C
C     INTEGRAL CI3MK, CI3NL
C
      IF(M-K)612,614,612
  612 CI3MK=1.
      GO TO 615
  614 CI3MK=1.5
  615 CONTINUE
      IF(N-L)616,618,616
  616 CI3NL=1.
      GO TO 619
  618 CI3NL=1.5
  619 CONTINUE
C
C     INTEGRAL CI10MK, CI10NL
C
      IF(M-K)622,624,622
  622 CI10MK=0.
      GO TO 625
  624 CI10MK=2.*M*K*PI**2
  625 CONTINUE
      IF(N-L)626,628,626
  626 CI10NL=0.
      GO TO 629
  628 CI10NL=2.*N*L*PI**2
  629 CONTINUE
C
C     INTEGRAL CI2MK, CI2NL
C
      CI2MK=CI3MK
      CI2NL=CI3NL
C
C     INTEGRAL CI9MK, CI9NL
C
```

```
      CI9MK=CI10MK
      CI9NL=CI10NL
C
C     INTEGRAL CI7MK, CI7NL
C
      CI7MK=CI4MK
      CI7NL=CI4NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI4MK
      CI39NL=CI4NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI4MK
      CI40NL=CI4NL
C
C     INTEGRAL CI56MK, CI56NL
C
      CI56MK=CI10MK
      CI56NL=CI10NL
C
C     INTEGRAL CI38MK,CI38NL
C
      CI38MK=CI3MK
      CI38NL=CI3NL
C
C     INTEGRAL CI41MK, CI41NL
C
      CI41MK=CI3MK
      CI41NL=CI3NL
C
C     INTEGRAL CI57MK, CI57NL
C
      CI57MK=CI10MK
```

```
      CI57NL=CI10NL
C
C     INTEGRAL CI42MK, CI42NL
C
      CI42MK=CI4MK
      CI42NL=CI4NL
C
C     INTEGRAL CI43MK, CI43NL
C
      CI43MK=CI4MK
      CI43NL=CI4NL
      S(JJVR,JUR)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
     1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
      JVR=JVR+1
      S(JJVR,JVR)=(A/B)*AR22*CI2MK*CI9NL+2.*AR26*CI7MK*CI7NL+(B/A)
     1*AR66*CI9MK*CI2NL-W2*A*B*P0*CI2MK*CI2NL
      JWR=JWR+1
      S(JJVR,JWR)=0.
      JXR=JXR+1
      S(JJVR,JXR)=BR12*CI39MK*CI40NL+(B/A)*BR16*CI56MK*CI38NL+(A/B)
     1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
      JYR=JYR+1
      S(JJVR,JYR)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK*CI41NL
      JUI=JUI+1
      S(JJVR,JUI)=-(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
     1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
      JVI=JVI+1
      S(JJVR,JVI)=-((A/B)*AI22*CI2MK*CI9NL+2.*AI26*CI7MK*CI7NL+(B/A)
     1*AI66*CI9MK*CI2NL)
      JWI=JWI+1
      S(JJVR,JWI)=0.
      JXI=JXI+1
      S(JJVR,JXI)=-(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
     1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
      JYI=JYI+1
```

```
  600 S(JJVR,JYI)=-((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
     1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
C     CALCULATE SUBMATRIX
C     WRUR,WRVR,WRWR,WRXR,WRYR,WRUI,WRVI,WRWI,WRXI,WRYI
      JJWR=2*NN**2
      DO 700 K1=1,NN
      K=K1+INPU1
      DO 700 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJWR=JJWR+1
      DO 700 M1=1,NN
      M=M1+INPU1
      DO 700 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI20MK, CI20NL
C
      IF(M-K) 702,704,702
  702 CI20MK=0.
      GO TO 705
  704 CI20MK=2.0*M*K*PI**2
  705 CONTINUE
      IF (N-L) 706,708,706
  706 CI20NL=0.
      GO TO 709
```

```
  708 CI20NL=2.0*N*L*PI**2
  709 CONTINUE
C
C      INTEGRAL CI17MK,CI17NL
C
      IF(M-K)712,714,712
  712 CI17MK=1.
      GO TO 715
  714 CI17MK=1.5
  715 CONTINUE
      IF(N-L)716,718,716
  716 CI17NL=1.
      GO TO 719
  718 CI17NL=1.5
  719 CONTINUE
C
C      INTEGRAL CI18MK ,CI18NL
C
      CI18MK=0.
      CI18NL=0.
C
C      INTEGRAL CI13MK, CI13NL
C
      CI13MK=0.
      CI13NL=0.
C
C      INTFGRAL CI11MK, CI11NL
C
      CI11MK=CI17MK
      CI11NL=CI17NL
C
C      INTEGRAL CI15MK, CI15NL
C
      CI15MK=CI13MK
      CI15NL=CI13NL
C
```

```
C       INTEGRAL CI12MK, CI12NL
C
        CI12MK=CI11MK
        CI12NL=CI11NL
        S(JJWR,JUR)=0.
        JVR=JVR+1
        S(JJWR,JVR)=0.
        JWR=JWR+1
        S(JJWR,JWR)=(B/A)*K11*AR44*CI20MK*CI17NL+2.*K12*AR45*CI18MK
       1*CI18NL+(A/B)*K22*AR55*CI17MK*CI20NL-W2*A*B*P0*CI17MK*CI17NL
        JXR=JXR+1
        S(JJWR,JXR)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
        JYR=JYR+1
        S(JJWR,JYR)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
        JUI=JUI+1
        S(JJWR,JUI)=0.
        JVI=JVI+1
        S(JJWR,JVI)=0.
        JWI=JWI+1
        S(JJWR,JWI)=-((B/A)*K11*AI44*CI20MK*CI17NL+2.*K12*AI45*CI18MK
       1*CI18NL+(A/B)*K22*AI55*CI17MK*CI20NL)
        JXI=JXI+1
        S(JJWR,JXI)=-(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK*CI13NL)
        JYI=JYI+1
  700   S(JJWR,JYI)=-(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
C       CALCULATE SUBMATRIX
C       XRUR,XRVR,XRWR,XRXR,XRYR,XRUI,XRVI,XRWI,XRXI,XRYI
        JJXR=3*NN**2
        DO 800 K1=1,NN
        K=K1+INPU1
        DO 800 L2=1,NN
        L=L2+INPU2
        JUR=0
        JVR=1*NN**2
        JWR=2*NN**2
        JXR=3*NN**2
```

```
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJXR=JJXR+1
      DO 800 M1=1,NN
      M=M1+INPU1
      DO 800 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI44MK, CI44NL
C
      IF(M-K)802,804,802
  802 CI44MK=0.
      GO TO 805
  804 CI44MK=2.*M*K*PI**2
  805 CONTINUE
      IF(N-L)806,808,806
  806 CI44NL=0.
      GO TO 809
  808 CI44NL=2.*N*L*PI**2
  809 CONTINUE
C
C     INTEGRAL CI31MK, CI31NL
C
      IF(M-K)812,814,812
  812 CI31MK=1.
      GO TO 815
  814 CI31MK=1.5
  815 CONTINUE
      IF(N-L)816,818,816
  816 CI31NL=1.
      GO TO 819
```

```
  818 CI31NL=1.5
  819 CONTINUE
C
C     INTEGRAL CI33MK, CI33NL
C
      IF(M-K)822,824,822
  822 CI33MK=0.
      GO TO 825
  824 CI33MK=0.
  825 CONTINUE
      IF(N-L)826,828,826
  826 CI33NL=0.
      GO TO 829
  828 CI33NL=0.
  829 CONTINUE
C
C     INTEGRAL CI34MK, CI34NL
C
      CI34MK=CI33MK
      CI34NL=CI33NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI33MK
      CI39NL=CI33NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI33MK
      CI40NL=CI33NL
C
C     INTEGRAL CI56MK, CI56NL
C
      CI56MK=CI44MK
      CI56NL=CI44NL
C
```

```
C       INTEGRAL CI38MK, CI38NL
C
        CI38MK=CI31MK
        CI38NL=CI31NL
C
C       INTEGRAL CI13MK, CI13NL
C
        CI13MK=0.
        CI13NL=0.
C
C       INTEGRAL CI11MK, CI11NL
C
        CI11MK=CI31MK
        CI11NL=CI31NL
C
C       INTEGRAL CI28MK, CI28NL
C
        CI28MK=CI44MK
        CI28NL=CI44NL
C
C       INTEGRAL CI21MK, CI21NL
C
        CI21MK=CI31MK
        CI21NL=CI31NL
C
C       INTEGRAL CI26MK, CI26NL
C
        CI26MK=CI33MK
        CI26NL=CI33NL
C
C       INTEGRAL CI25MK, CI25NL
C
        CI25MK=CI33MK
        CI25NL=CI33NL
C
C       INTEGRAL CI24MK, CI24NL
```

```
C
      CI24MK=CI33MK
      CI24NL=CI33NL
C
C     INTEGRAL CI30MK, CI30NL
C
      CI30MK=CI44MK
      CI30NL=CI44NL
C
C     INTEGRAL CI23MK, CI23NL
C
      CI23MK=CI31MK
      CI23NL=CI31NL
      S(JJXR,JUR)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL+BR16
     1*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*P1*CI31MK*CI31NL
      JVR=JVR+1
      S(JJXR,JVR)=BR12*CI39MK*CI40NL+(B/A)*BR16*CI56MK*CI38NL+(A/B)
     1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
      JWR=JWR+1
      S(JJXR,JWR)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
      JXR=JXR+1
      S(JJXR,JXR)=(B/A)*DR11*CI28MK*CI21NL+2.*DR16*CI26MK*CI26NL+(A/B)
     1*DR66*CI21MK*CI28NL+A*B*K11*AR44*CI21MK*CI21NL-W2*A*B*P2*CI21MK
     2*CI21NL
      JYR=JYR+1
      S(JJXR,JYR)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
     1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
     2*CI23NL
      JUI=JUI+1
      S(JJXR,JUI)=-((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL
     1+BI16*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
      JVI=JVI+1
      S(JJXR,JVI)=-(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
     1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
      JWI=JWI+1
      S(JJXR,JWI)=-(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK
```

```
     1*CI13NL)
      JXI=JXI+1
      S(JJXR,JXI)=-((B/A)*DI11*CI28MK*CI21NL+2.*DI16*CI26MK*CI25NL
     1+(A/B)*DI66*CI21MK*CI28NL+A*B*K11*AI44*CI21MK*CI21NL)
      GO TO 1219
 1218 CI17NL=1.5
 1219 CONTINUE
C
C     INTEGRAL CI18MK ,CI18NL
C
      CI18MK=0.
      CI18NK=0.
C
C     INTEGRAL CI13MK, CI13NL
C
      CI13MK=0.
      CI13NL=0.
C
C     INTEGRAL CI11MK, CI11NL
C
      CI11MK=CI17MK
      CI11NL=CI17NL
C
C     INTEGRAL CI15MK, CI15NL
C
      CI15MK=CI13MK
      CI15NL=CI13NL
C
C     INTEGRAL CI12MK, CI12NL
C
      CI12MK=CI11MK
      CI12NL=CI11NL
      S(JJWI,JUR)=0.
      JVR=JVR+1
      S(JJWI,JVR)=0.
      JWR=JWR+1
```

```
      S(JJWI,JWR)=+((B/A)*K11*AI44*CI20MK*CI17NL+2.*K12*AI45*CI18MK
     1*CI18NL+(A/B)*K22*AI55*CI17MK*CI20NL)
      JXR=JXR+1
      S(JJWI,JXR)=+(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK*CI13NL)
      JYR=JYR+1
      S(JJWI,JYR)=+(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
      JUI=JUI+1
      S(JJWI,JUI)=0.
      JVI=JVI+1
      S(JJWI,JVI)=0.
      JWI=JWI+1
      S(JJWI,JWI)=(B/A)*K11*AR44*CI20MK*CI17NL+2.*K12*AR45*CI18MK
     1*CI18NL+(A/B)*K22*AR55*CI17MK*CI20NL-W2*A*B*P0*CI17MK*CI17NL
      JXI=JXI+1
      JYI=JYI+1
      S(JJWI,JXI)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
 1200 S(JJWI,JYI)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
C     CALCULATE SUBMATRIX
C     XIUR,XIVR,XIWR,XIXR,XIYR,XIUI,XIVI,XIWI,XIXI,XIYI
      JJXI=8*NN**2
      DO 1300 K1=1,NN
      K=K1+INPU1
      DO 1300 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJXI=JJXI+1
      DO 1300 M1=1,NN
```

```
      M=M1+INPU1
      DO 1300 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI44MK, CI44NL
C
      IF(M-K)1302,1304,1302
 1302 CI44MK=0.
      GO TO 1305
 1304 CI44MK=2.*M*K*PI**2
 1305 CONTINUE
      IF(N-L)1306,1308,1306
 1306 CI44NL=0.
      GO TO 1309
 1308 CI44NL=2.*N*L*PI**2
 1309 CONTINUE
C
C     INTEGRAL CI31MK, CI31NL
C
      IF(M-K)1312,1314,1312
 1312 CI31MK=1.
      GO TO 1315
 1314 CI31MK=1.5
 1315 CONTINUE
      IF(N-L)1316,1318,1316
 1316 CI31NL=1.
      GO TO 1319
 1318 CI31NL=1.5
 1319 CONTINUE
C
C     INTEGRAL CI33MK, CI33NL
C
      IF(M-K)1322,1324,1322
 1322 CI33MK=0.
      GO TO 1325
```

```
 1324 CI33MK=0.
 1325 CONTINUE
      IF(N-L)1326,1328,1326
 1326 CI33NL=0.
      GO TO 1329
 1328 CI33NL=0.
 1329 CONTINUE
C
C     INTEGRAL CI34MK, CI34NL
C
      CI34MK=CI33MK
      CI34NL=CI33NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI33MK
      CI39NL=CI33NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI33MK
      CI40NL=CI33NL
C
C     INTEGRAL CI56MK, CI56NL
C
      CI56MK=CI44MK
      CI56NL=CI44NL
C
C     INTEGRAL CI38MK, CI38NL
C
      CI38MK=CI31MK
      CI38NL=CI31NL
C
C     INTEGRAL CI13MK, CI13NL
C
      CI13MK=0.
```

```
      CI13NL=0.
C
C     INTEGRAL CI11MK, CI11NL
C
      CI11MK=CI31MK
      CI11NL=CI31NL
C
C     INTEGRAL CI28MK, CI28NL
C
      CI28MK=CI44MK
      CI28NL=CI44NL
C
C     INTEGRAL CI21MK, CI21NL
C
      CI21MK=CI31MK
      CI21NL=CI31NL
C
C     INTEGRAL CI26MK, CI26NL
C
      CI26MK=CI33MK
      CI26NL=CI33NL
C
C     INTEGRAL CI25MK, CI25NL
C
      CI25MK=CI33MK
      CI25NL=CI33NL
C
C     INTEGRAL CI24MK, CI24NL
C
      CI24MK=CI33MK
      CI24NL=CI33NL
C
C     INTEGRAL CI30MK, CI30NL
C
      CI30MK=CI44MK
      CI30NL=CI44NL
```

```
C
C      INTEGRAL  CI23MK,  CI23NL
C
       CI23MK=CI31MK
       CI23NL=CI31NL
       S(JJXI,JUR)=+((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL
      1+BI16*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
       JVR=JVR+1
       S(JJXI,JVR)=+(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
      1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
       JWR=JWR+1
       S(JJXI,JWR)=+(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK
      1*CI13NL)
       JXR=JXR+1
       S(JJXI,JXR)=+((B/A)*DI11*CI28MK*CI21NL+2.*DI16*CI26MK*CI26NL
      1+(A/B)*DI66*CI21MK*CI28NL+A*B*K11*AI44*CI21MK*CI21NL)
       JYR=JYR+1
       S(JJXI,JYR)=+(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
      1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*B*K12*AI45*CI23MK
      2*CI23NL)
       JUI=JUI+1
       S(JJXI,JUI)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL+BR16
      1*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*PI*CI31MK*CI31NL
       JVI=JVI+1
       S(JJXI,JVI)=BR12*CI39MK*CI40NL+(B/A)*BR15*CI56MK*CI38NL+(A/B)
      1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
       JWI=JWI+1
       S(JJXI,JWI)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
       JXI=JXI+1
       S(JJXI,JXI)=(B/A)*DR11*CI28MK*CI21NL+2.*DR16*CI26MK*CI26NL+(A/B)
      1*DR66*CI21MK*CI28NL+A*B*K11*AR44*CI21MK*CI21NL-W2*A*B*P2*CI21MK
      2*CI21NL
       JYI=JYI+1
 1300  S(JJXI,JYI)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
      1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
      2*CI23NL
```

```
C       CALCULATE SUBMATRIX
C       YIUR,YIVR,YIWR,YIXR,YIYR,YIUI,YIVI,YIWI,YIXI,YIYI
        JJYI=9*NN**2
        DO 1400 K1=1,NN
        K=K1+INPU1
        DO 1400 L2=1,NN
        L=L2+INPU2
        JUR=0
        JVR=1*NN**2
        JWR=2*NN**2
        JXR=3*NN**2
        JYR=4*NN**2
        JUI=5*NN**2
        JVI=6*NN**2
        JWI=7*NN**2
        JXI=8*NN**2
        JYI=9*NN**2
        JJYI=JJYI+1
        DO 1400 M1=1,NN
        M=M1+INPU1
        DO 1400 N2=1,NN
        N=N2+'NPU2
        JUR=JUR+1
C
C       INTEGRAL CI37MK, CI37NL
C
        IF(M-K)1402,1404,1402
 1402   CI37MK=0.
        GO TO 1405
 1404   CI37MK=0.
 1405   CONTINUE
        IF(N-L)1406,1408,1406
 1406   CI37NL=0.
        GO TO 1409
 1408   CI37NL=0.
 1409   CONTINUE
```

```
C
C       INTEGRAL CI36MK, CI36NL
C
        CI36MK=CI37MK
        CI36NL=CI37NL
C
C       INTEGRAL CI55MK, CI55NL
C
        IF(M-K)1412,1414,1412
 1412   CI55MK=0.
        GO TO 1415
 1414   CI55MK=2.*M*K*PI**2
 1415   CONTINUE
        IF(N-L)1416,1418,1416
 1416   CI55NL=0.
        GO TO 1419
 1418   CI55NL=2.*N*L*PI**2
 1419   CONTINUE
C
C       INTEGRAL CI35MK, CI35NL
C
        IF(M-K)1422,1424,1422
 1422   CI35MK=1.
        GO TO 1425
 1424   CI35MK=1.5
 1425   CONTINUE
        IF(N-L)1426,1428,1426
 1426   CI35NL=1.0
        GO TO 1429
 1428   CI35NL=1.5
 1429   CONTINUE
C
C       INTEGRAL CI41MK, CI41NL
C
        CI41MK=CI35MK
        CI41NL=CI35NL
```

```
C
C      INTEGRAL CI57MK, CI57NL
C
       CI57MK=CI55MK
       CI57NL=CI55NL
C
C      INTEGRAL CI42MK, CI42NL
C
       CI42MK=CI37MK
       CI42NL=CI37NL
C
C      INTEGRAL CI43MK, CI43NL
C
       CI43MK=CI37MK
       CI43NL=CI37NL
C
C      INTEGRAL CI15MK, CI15NL
C
       CI15MK=0.
       CI15NL=0.
C
C      INTEGRAL CI12MK, CI12NL
C
       CI12MK=CI35MK
       CI12NL=CI35NL
C
C      INTEGRAL CI24MK, CI24NL
C
       CI24MK=CI37MK
       CI24NL=CI37NL
C
C      INTEGRAL CI25MK, CI25NL
C
       CI25MK=CI37MK
       CI25NL=CI37NL
C
```

```
C       INTEGRAL CI23MK, CI23NL
C
        CI23MK=CI35MK
        CI23NL=CI35NL
C
C       INTEGRAL CI30MK, CI30NL
C
        CI30MK=CI55MK
        CI30NL=CI55NL
C
C       INTEGRAL CI22MK, CI22NL
C
        CI22MK=CI35MK
        CI22NL=CI35NL
C
C       INTEGRAL CI29MK, CI29NL
C
        CI29MK=CI55MK
        CI29NL=CI55NL
C
C       INTEGRAL CI27MK, CI27NL
C
        CI27MK=CI37MK
        CI27NL=CI37NL
        S(JJYI,JUR)=+(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
       1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
        JVR=JVR+1
        S(JJYI,JVR)=+((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
       1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
        JWR=JWR+1
        S(JJYI,JWR)=+(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
        JXR=JXR+1
        S(JJYI,JXR)=+(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
       1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*B*K12*AI45*CI23MK
       2*CI23NL)
        JYR=JYR+1
```

```
      S(JJYI,JYR)=+((A/B)*DI22*CI22MK*CI29NL+2.*DI26*CI27MK*CI27NL+(B/A)
     1*DI66*CI29MK*CI22NL+A*B*K22*AI55*CI22MK*CI22NL)
      JUI=JUI+1
      S(JJYI,JUI)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
     1*BR26*CI35MK*CI55NL+BR66*CI36MK*CI37NL
      JVI=JVI+1
      S(JJYI,JVI)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK
     2*CI41NL
      JWI=JWI+1
      S(JJYI,JWI)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
      JXI=JXI+1
      S(JJYI,JXI)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
     1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
     2*CI23NL
      JYI=JYI+1
 1400 S(JJYI,JYI)=(A/B)*DR22*CI22MK*CI29NL+2.*DR26*CI27MK*CI27NL+(B/A)
     1*DR66*CI29MK*CI22NL+A*B*K22*AR55*CI22MK*CI22NL-W2*A*B*P2*CI22MK
     2*CI22NL
C     CALCULATE B MATRIX
      IJUR=0
      IJVR=1*NN**2
      IJWR=2*NN**2
      IJXR=3*NN**2
      IJYR=4*NN**2
      IJUI=5*NN**2
      IJVI=6*NN**2
      IJWI=7*NN**2
      IJXI=8*NN**2
      IJYI=9*NN**2
      DO 1500 M1=1,NN
      M=M1+INPU1
      DO 1500 N2=1,NN
      N=N2+INPU2
      IJUR=IJUR+1
C
```

```
C       INTEGRAL CIMK, CINL
C
        CIMK=1.
        CINL=1.
        T(IJUR)=0.
        IJVR=IJVR+1
        T(IJVR)=0.
        IJWR=IJWR+1
        T(IJWR)=1.*A*B*CIMK*CINL
        IJXR=IJXR+1
        T(IJXR)=0.
        IJYR=IJYR+1
        T(IJYR)=0.
        IJUI=IJUI+1
        T(IJUI)=0.
        IJVI=IJVI+1
        T(IJVI)=0.
        IJWI=IJWI+1
        T(IJWI)=0.
        IJXI=IJXI+1
        T(IJXI)=0.
        IJYI=IJYI+1
 1500   T(IJYI)=0.
C
C       CALL SIMQ
C
        NI=10*NN**2
        CALL SIMQ(S,T,NI,KS)
        WRITE(6,2050)KS
 2050   FORMAT('      KS=',I3//)
        KN1=2*NN**2+1
        KN2=3*NN**2
        KN3=7*NN**2+1
        KN4=8*NN**2
        WRITE(6,2055) (T(J),J=KN1,KN2)
        WRITE(6,2055) (T(I),I=KN3,KN4)
```

```
 2055 FORMAT(E12.5)
      WRITE(6,2067)F
 2067 FORMAT('    FQ=',F10.3)
      GO TO 10
 3333 CALL EXIT
      END
C     PEAK-AMPLITUDE AND MODIFIED KENNEDY-PANCU METHODS
C     WR(J)=T(J) AND WI(I)=T(I)
      REAL M1PIA,N1PIB,M2PIA,N2PIB,M3PIA,N3PIB,M4PIA,N4PIB
   10 READ(5,20)FQ
   20 FORMAT(F10.4)
      IF(FQ)80,80,25
   25 READ(5,30)WR1,WR2,WR3,WR4
   30 FORMAT(4F10.4)
      READ(5,40)WI1,WI2,WI3,WI4
   40 FORMAT(4F10.4)
      READ(5,50)M1,M2,M3,M4,N1,N2,N3,N4
   50 FORMAT(8I5)
      READ(5,55)A,B
   55 FORMAT(2F10.4)
      PI=3.1416
      M1PIA=M1*PI*A
      M2PIA=M2*PI*A
      M3PIA=M3*PI*A
      M4PIA=M4*PI*A
      N1PIB=N1*PI*B
      N2PIB=N2*PI*B
      N3PIB=N3*PI*B
      N4PIB=N4*PI*B
      WRT=WR1*COS(M1PIA)*COS(N1PIB)+WR2*COS(M2PIA)*COS(N2PIB)
     1+WR3*COS(M3PIA)*COS(N3PIB)+WR4*COS(M4PIA)*COS(N4PIB)
      WIT=WI1*COS(M1PIA)*COS(N1PIB)+WI2*COS(M2PIA)*COS(N2PIB)
     1+WI3*COS(M3PIA)*COS(N3PIB)+WI4*COS(M4PIA)*COS(N4PIB)
      WT=(WRT**2+WIT**2)**0.5
      WXOY1=ABS(WT)
      RW=WIT/WRT
```

```
      PHA=ATAN(RW)
      DELS=WT*PHA
      WRITE(6,60)PHA,DELS,FQ
   60 FORMAT('    PHA=',F10.4,'    DELS=',F10.4,'    FQ=',F10.4)
      WRITE(6,65) WXOY1
   65 FORMAT('    WXOY1',E12.5)
      GO TO 10
   80 CALL EXIT
      END
C     NODAL PATTERNS FOR ANG=0,10,30,45,60,90
      WRITE(6,35)
   35 FORMAT('      X      ','      Y      ','      W      '//)
   10 READ(5,20,END=220)WM1N1,WM1N2,WM2N1,WM2N2
   20 FORMAT(4F10.4)
      READ(5,30)M1,N1,M2,N2
   30 FORMAT(4I5)
      PI=3.1416
      DO 100 J=1,5
      Y=(J-1)*0.4
      B=Y/1.6
      X=-0.5
   36 X=X+0.5
      A=X/12.
      M1PIA=M1*PI*A
      N1PIB=N1*PI*B
      M2PIA=M2*PI*A
      N2PIB=N2*PI*B
      W=WM1N1*COS(M1PIA)*COS(N1PIB)+WM1N2*COS(M1PIA)*COS(N2PIB)
     1+WM2N1*COS(M2PIA)*COS(N1PIB)+WM2N2*COS(M2PIA)*COS(N2PIB)
      W1=ABS(W)
      IF(W1.LT.0.01)WRITE(6,40)A,B,W1
      IF(A.LT.1.0)GO TO 36
   40 FORMAT(3F10.4)
  100 CONTINUE
      GO TO 10
  220 CALL EXIT
```