DEFORMATION ENERGIES AND COLLECTIVE PROPERTIES OF HEAVY NUCLEI

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PREFACE

In this study a closed formula for deformation energies of heavy nuclei has been used to find the potential well in which a nucleus vibrates. This knowledge has permitted a calculation of mass parameters and moments of inertia for even-even nuclei using experimental data from vibrational rather than rotational excited states. The results obtained did not correlate well with results previously calculated from rotational level data. This has been attributed to the simple collective model that was used, and it is hoped that with a more sophisticated model the potential energy curves found will be useful in fathoming the extremely complicated level structure of heavy nuclei.

I would like to thank Dr. Hermann G. Kuemmel for his suggestion of the problem and his guidance during the course of my work.

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CHAPTER I

INTRODUCTION

Nuclear Models

In medium and heavy nuclei the problem of describing the interactions of the various nucleons with one another is so involved that an exact description cannot be made. The job for the nuclear physicist, therefore, becomes one of making a mathematically solvable model of the nucleus which fits all the empirical data.

Many different models have been proposed (1): each of which describes some, but not all, of the experimental evidence. Behind each model lie certain simplifying assumptions about the nucleus; these assumptions basically fit into two categories. The first is that individual nucleons move almost independently in a common nuclear potential. Models based on this type of assumption are known as independent particle models, and an example is the shell model (2). The second group of models are known as strong interaction models. These models are based on the assumption that all of the nucleons are strongly coupled to each other. A well-known model of this type is the liquid drop model (3). In the past several years, a unified nuclear model has been proposed which would take into account both individual and collective motion of the nucleons (4, 5).

The liquid drop model is a comparatively simple model, and some

results obtained with it give the basic trends of certain experimental evidence. However, upon closer examination of the correlation between experiment and liquid drop theory, it is found that maximum discrepancies occur at certain magic numbers. These magic numbers are quite important in that some nuclear properties are found by experiment to be discontinuous at these numbers. These discontinuities probably cannot be explained by strong interaction models. The shell model, on the other hand, treats the nucleus as having a shell structure and correctly predicts the magic numbers as occurring at 2, 8, 14, 20, 28, 50, 82, 126, and 184. Because of this shell structure, nuclei with a neutron or proton number far away from these magic numbers behave quite differently from nuclei with neutron or proton number near closed shells.

A New Mass Formula

The mass M of a nucleus containing Z protons and N neutrons is given by

$$M = Z M_p + N M_n - B$$

Where M_p is the mass of a proton, M_n is the mass of a neutron and B is the binding energy of the nucleus. The binding energy is made up of many terms which attempt to describe the interactions of the nucleons in the nucleus. The most well-known formula for heavy masses ($A > 40$) which agrees with experimental evidence and is simple in form is known as the semi-empirical mass formula. It was first formulated by Weizsäcker (3). A discussion of modern versions of this mass formula can be found in most standard nuclear physics texts. For example, see Evans (6).

The semi-empirical mass formula is based upon the liquid drop model of the nucleus. Consequently, it does not agree with experimental

evidence near closed shells of the nucleus. A better fit to experiment can be obtained if additional parameters which take into account the shell structure are incorporated into the mass formula. This has been done by Kuemmel, et. al. (7), and the agreement with experiment has been found to be very much improved.

One of the terms in this new mass formula has to do with the binding energy brought about by the deformation of the nucleus around a spherical shape. Since this deformation energy is a function of α, the deformation parameter, the potential well of the deformation energy can be plotted. If a nucleus is thought of as oscillating inside this potential well, experimentally known energy levels can then be used to determine various parameters of the nucleus.

CHAPTER II

DEFORMATION ENERGIES OF HEAVY NUCLEI

Deformation Energy Term in the New Mass Formula

In the new mass formula of Kuemmel, et. al. (7), the binding energy brought about by a deformation α is given to second order by:

$$D(\alpha) = -\tfrac{1}{2}\left[e^{-\left(\frac{\alpha}{\alpha_o}\right)^2} - 1\right]\left[\eta_i^N n_i\hat{n}_i + \beta_i^N n_i^2\hat{n}_i + \eta_k^Z z_k\hat{z}_k + \beta_k^Z z_k^2\hat{z}_k\right] - \frac{A}{5}^{2/3}\alpha^2\left[2u_s - u_c\frac{z^2}{A}\right]. \quad (1)$$

In this formula the subscripts i and k refer to the last number in the shell being filled, and the superscripts N and Z refer to neutrons or protons respectively. For example η_{126}^N refers to the parameter η for nuclei with $82 < N \leq 126$. The parameters α_o, η_i^N, β_i^N, η_k^Z, β_k^Z, u_s, and u_c are all found by determining which values for them would give the best fit to experimental mass values. This is done using the complete mass formula in which the deformation energy is only one of many energy terms. The values for these parameters are listed in Table I.

The symbols n_i, $\hat{n}_i$, z_k, and $\hat{z}_k$ represent the number of neutrons in the i^{th} shell, the number of neutron holes in the i^{th} shell, the number of protons in the k^{th} shell, and the number of proton holes in the k^{th} shell. As usual, A and Z stand for the mass number and atomic number of the nucleus in question.

TABLE I

VALUES OF PARAMETERS USED IN DEFORMATION ENERGY TERM OF NEW MASS FORMULA

η^{N}_{126} = 60.00 kev	η^{Z}_{82} = 30.00 kev	$u_s = 1.951 \times 10^4$ kev
		u_c = 753.3 kev
γ^{N}_{126} = -1.000 kev	γ^{Z}_{82} = 30.00 kev	α_o = 1.500 for $82 < N \leq 126$

For a given nucleus, if we let c equal the second bracket in equation (1) and set $b = \frac{A^{2/3}}{5}\left[2u_s - u_c \frac{Z^2}{A}\right]$, our equation takes on the much simpler form:

$$D(\alpha) = +\frac{c}{2}\left[1 - e^{-(\alpha/\alpha_o)^2}\right] - b\alpha^2 \qquad (2)$$

For a given nucleus, the largest value of $D(\alpha)$ will give the largest value of the binding energy and hence the smallest mass. The nucleus will be most stable when this occurs. The value of α giving the minimum mass value will be designated α_{min}; it can be found in the usual way by setting $(\partial D(\alpha)/\partial\alpha)_{\alpha_{min}} = 0$. Using this value for α_{min}, we can find $D(\alpha_{min})$, the value of the deformation energy when the nucleus is most stable.

A plot of $D(\alpha)$ vs. α can then be made for a given nucleus by calculating the values of c and b and using equation (2). Several such plots are shown in Figs. 1 and 2 for even-even nuclei. (An even-even nucleus is one with an even number of both protons and neutrons.) It is seen that the potential curve thus plotted acts very much like an harmonic oscillator near α_{min}. Note that the potential well is deeper the further away the nucleus is from a closed shell.

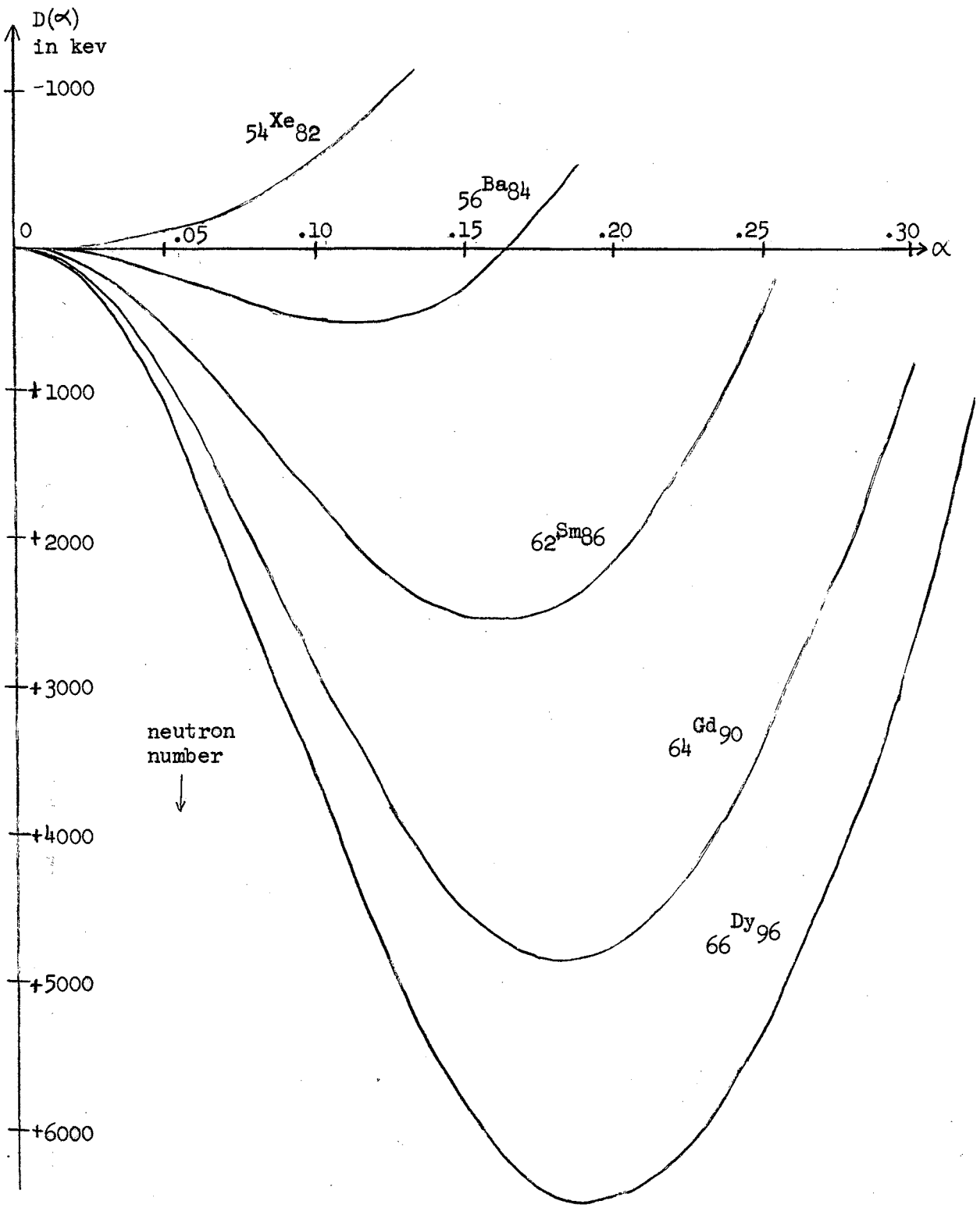


Fig. 1: Deformation energy vs. deformation parameter α for several nuclei in lower half of neutron shell ending with N = 126.

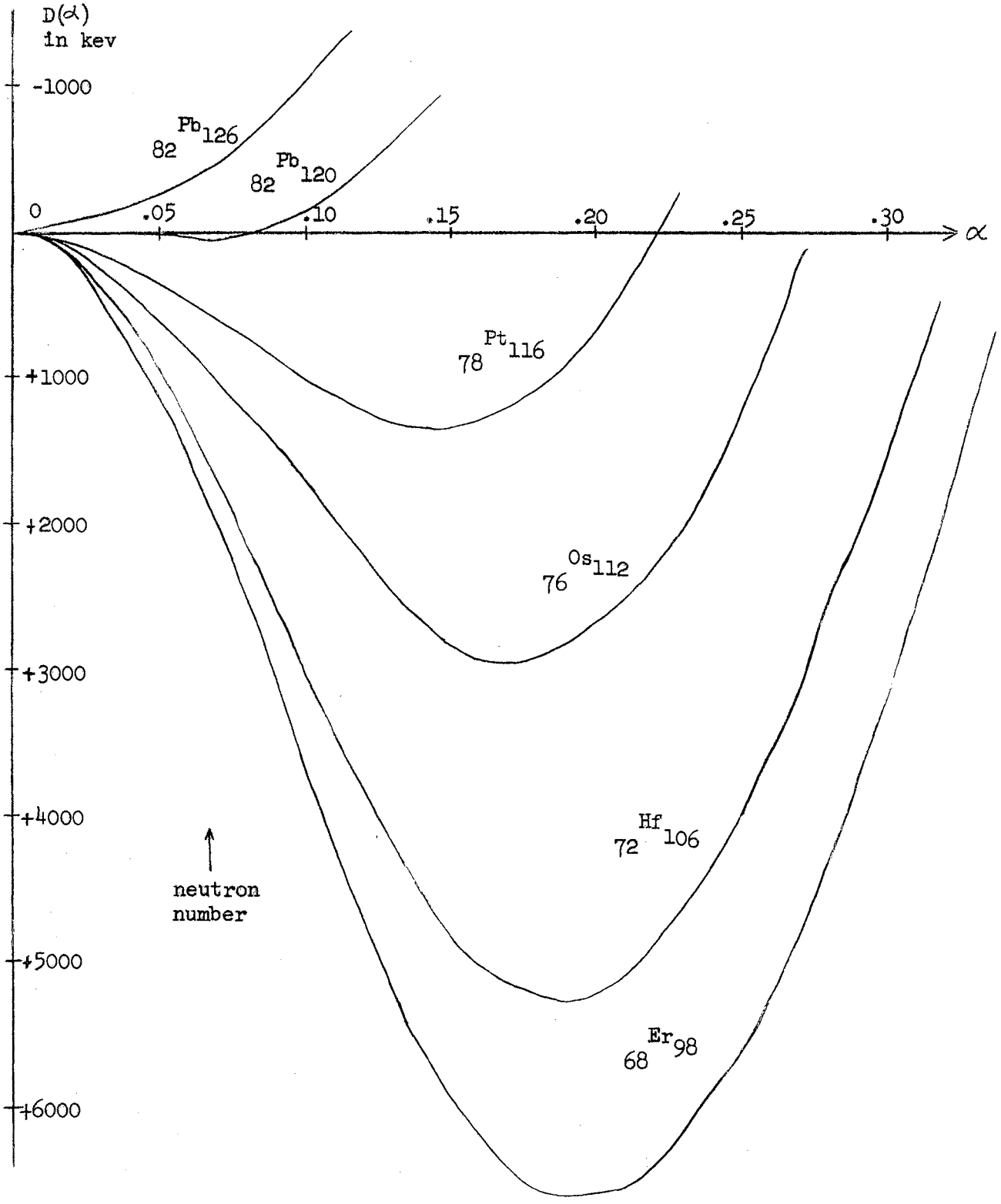


Fig. 2: Deformation energy vs. deformation parameter α for several nuclei in upper half of neutron shell ending with N = 126.

Harmonic Oscillator Approximation

If we take a Taylor series expansion of $D(\alpha)$ around α_{min} we get

$$D(\alpha) = D(\alpha_{min}) + (\alpha - \alpha_{min})D'(\alpha_{min}) + \frac{(\alpha - \alpha_{min})^2}{2!} D''(\alpha_{min}) + \frac{(\alpha - \alpha_{min})^3}{3!} D'''(\alpha_{min}) + \cdots , \quad (3)$$

where a superscript prime indicates that a derivative has been taken with respect to α. $D(\alpha_{min})$ can be arbitrarily set equal to zero, and D' at (α_{min}) equals zero. If we neglect all terms of the order of $(\alpha - \alpha_{min})^3$ or higher, we get the harmonic oscillator approximation

$$D(\alpha) = \frac{(\alpha - \alpha_{min})^2}{2} D''(\alpha_{min}) . \quad (4)$$

The derivation of $D''(\alpha_{min})$ is as follows. From equation (2), $D'(\alpha)$ is found to be

$$D'(\alpha) = c \frac{\alpha}{\alpha_o^2} e^{-(\alpha/\alpha_o)^2} - 2\alpha b \quad (5)$$

Setting this equal to zero gives us

$$\alpha_{min}^2 = \alpha_o^2 \ln \left[\frac{c}{2\alpha_o^2 b}\right] \quad (6)$$

Differentiating equation (5) once more gives

$$D''(\alpha) = \frac{c}{\alpha_o^2} e^{-(\alpha/\alpha_o)^2}\left[1 - \frac{2\alpha^2}{\alpha_o^2}\right] - 2b \quad (7)$$

and

$$D''(\alpha_{min}) = -4(b) \ln\left[\frac{c}{2\alpha_o^2 (b)}\right] \quad (8)$$

If we think of the nucleus as vibrating in a potential well $V(\alpha)$ (see Figs. 1 and 2), we can associate a spring constant k with the harmonic

oscillator approximation. The potential energy V then becomes

$$V = \tfrac{1}{2}k\,(\alpha - \alpha_{min})^2 \quad , \tag{9}$$

where

$$k = -D''(\alpha_{min}) \tag{10}$$

It will be shown in Chapter III that the change of variable $(\alpha - \alpha_{min}) = \left(\frac{5}{4\pi}\right)^{\frac{1}{2}}\beta$ is desirable. This gives

$$V = \tfrac{1}{2}\left(\frac{5}{4\pi}\ k\right)\beta^2 = \tfrac{1}{2}k_o\beta^2 \quad . \tag{11}$$

where

$$k_o = \frac{5}{4\pi}\,k \tag{12}$$

Additional Expansion Terms

For relatively large values of $\alpha - \alpha_{min}$ it is not possible to neglect the higher order terms in equation (3). Therefore, we will investigate the cubic and quartic terms. To do this, we have to calculate $D'''(\alpha_{min})$ and $D''''(\alpha_{min})$. Differentiating equation (7) gives

$$D'''(\alpha) = -2\,\frac{c}{\alpha_o^4}\,\alpha\, e^{-(\alpha/\alpha_o)^2}\left[3 - 2\left(\frac{\alpha}{\alpha_o}\right)^2\right] \quad ; \tag{13}$$

and, using equation (6), we get

$$D'''(\alpha_{min}) = -4\,\frac{(b)}{\alpha_o}\left(\ln\left[\frac{c}{2\alpha_o^2\, b}\right]\right)^{\frac{1}{2}} x\left[3 - 2\ln\frac{c}{2\alpha_o^2\,(b)}\right] \quad . \tag{14}$$

Differentiating equation (13) gives

$$D''''(\alpha) = \frac{2c}{\alpha_o^4}\ e^{-(\alpha/\alpha_o)^2}\left[-4\left(\frac{\alpha}{\alpha_o}\right)^4 + 12\left(\frac{\alpha}{\alpha_o}\right)^2 - 3\right] \quad ; \tag{15}$$

and, using equation (6), we get

$$D''''(\alpha_{min}) = -4\,\frac{(b)}{\alpha_o^2}\left\{4\left(\ln\left[\frac{c}{2\alpha_o^2\,(b)}\right]\right)^2 - 12\,\ln\left[\frac{c}{2\alpha_o^2\,(b)}\right] + 3\right\} \quad . \tag{16}$$

If we set

$$k' = -\frac{1}{6}\,D'''(\alpha_{min}) \tag{17}$$

and

$$k'' = -\frac{1}{24}\,D''''(\alpha_{min}) \quad , \tag{18}$$

we get

$$V = \tfrac{1}{2}\,k\,(\alpha - \alpha_{min})^2 + k'(\alpha - \alpha_{min})^3 + k''(\alpha - \alpha_{min})^4 \quad ; \tag{19}$$

and once again setting $\alpha - \alpha_{min} = \sqrt{\frac{5}{4\pi}}\,\beta$, we obtain

$$V = \tfrac{1}{2}\,k_o\beta^2 + k_o'\beta^3 + k_o''\beta^4 \tag{20}$$

where

$$k_o' = \left(\frac{5}{4\pi}\right)^{3/2} k' \quad \text{and} \quad k_o'' = \left(\frac{5}{4\pi}\right)^2 k'' \tag{21}$$

In order to get an idea of the magnitude of the numbers involved, Table II gives the values of b, c, α_{min}, k_o, k_o', and k_o'' for several even-even nuclei.

Figures 3 and 4 show examples of the potential curve, the harmonic oscillator approximation, and the curve when the cubic and quartic terms are used in the expansion. We see that the harmonic oscillator approximation is a pretty good one in the range spanned by the first two energy levels. These energy levels will be discussed in Chapter III.

TABLE II

VALUES OF b, c, α_{min}, k_o, k_o', and k_o''
FOR SOME HEAVY EVEN-EVEN NUCLEI

Nucleus	α_{min}	b x 10^4 kev	c x 10^4 kev	k_o x 10^5 kev	k_o' x 10^5 kev	k_o'' x 10^5 kev
Sm_{86}	.161	10.9	1.59	2.01	.90	-7.07
Sm_{88}	.173	11.2	1.90	2.36	.51	-7.71
Gd_{88}	.177	10.7	1.92	2.35	.33	-7.45
Sm_{90}	.180	11.4	2.16	2.61	.18	-7.98
Gd_{90}	.183	10.9	2.18	2.59	.02	-7.67
Gd_{92}	.187	11.2	2.39	2.76	-.18	-7.84
Er_{98}	.195	10.8	2.64	2.91	-.59	-10.0
Hf_{106}	.187	10.8	2.29	2.67	-.15	-7.60
Os_{112}	.168	10.4	1.65	2.08	.64	-7.05
Pt_{116}	.143	10.3	1.16	1.50	1.30	-5.60

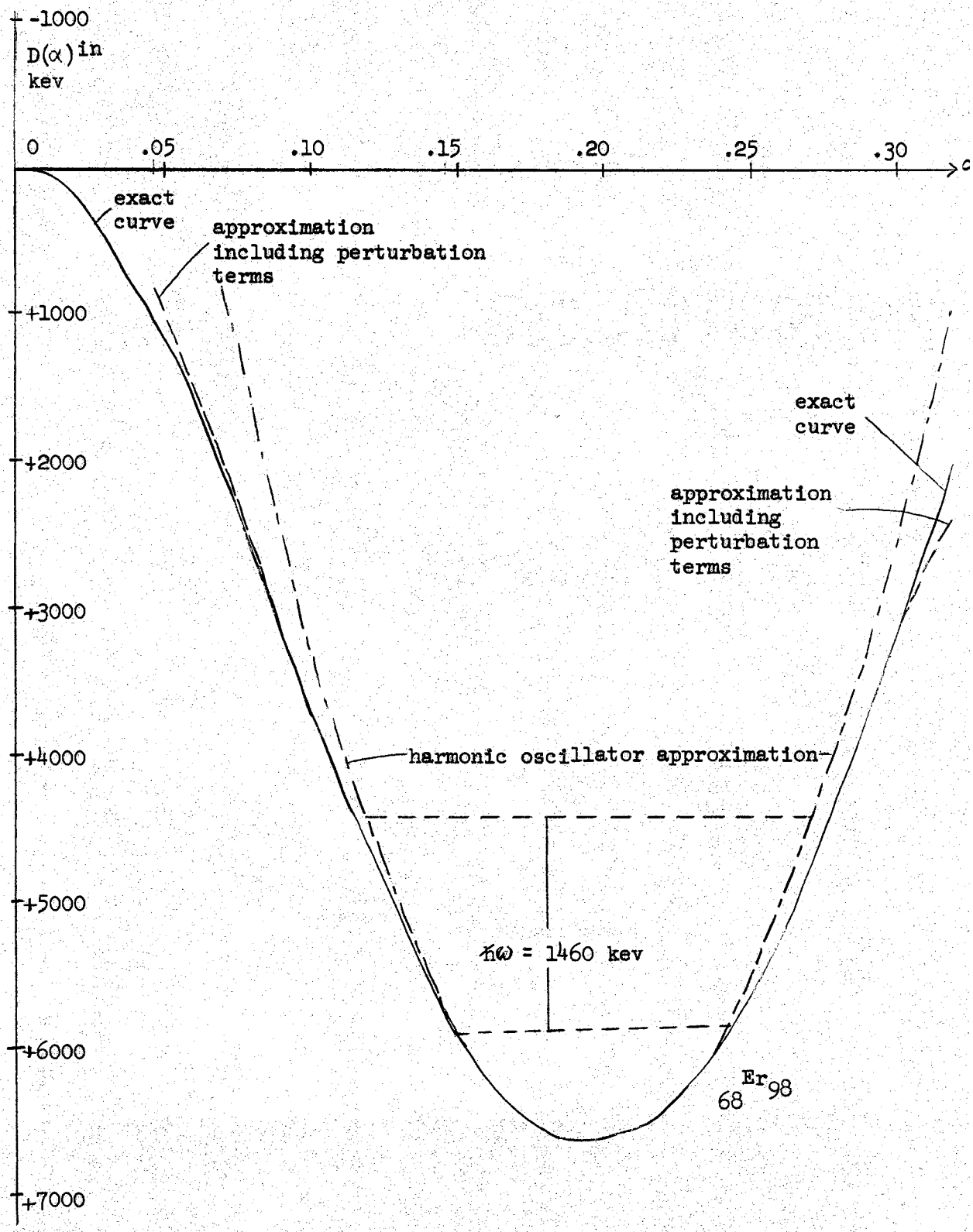


Fig. 3: Deformation energies vs. α showing exact curve, harmonic oscillator approximation and approximation including perturbation terms for $_{68}Er_{98}$.

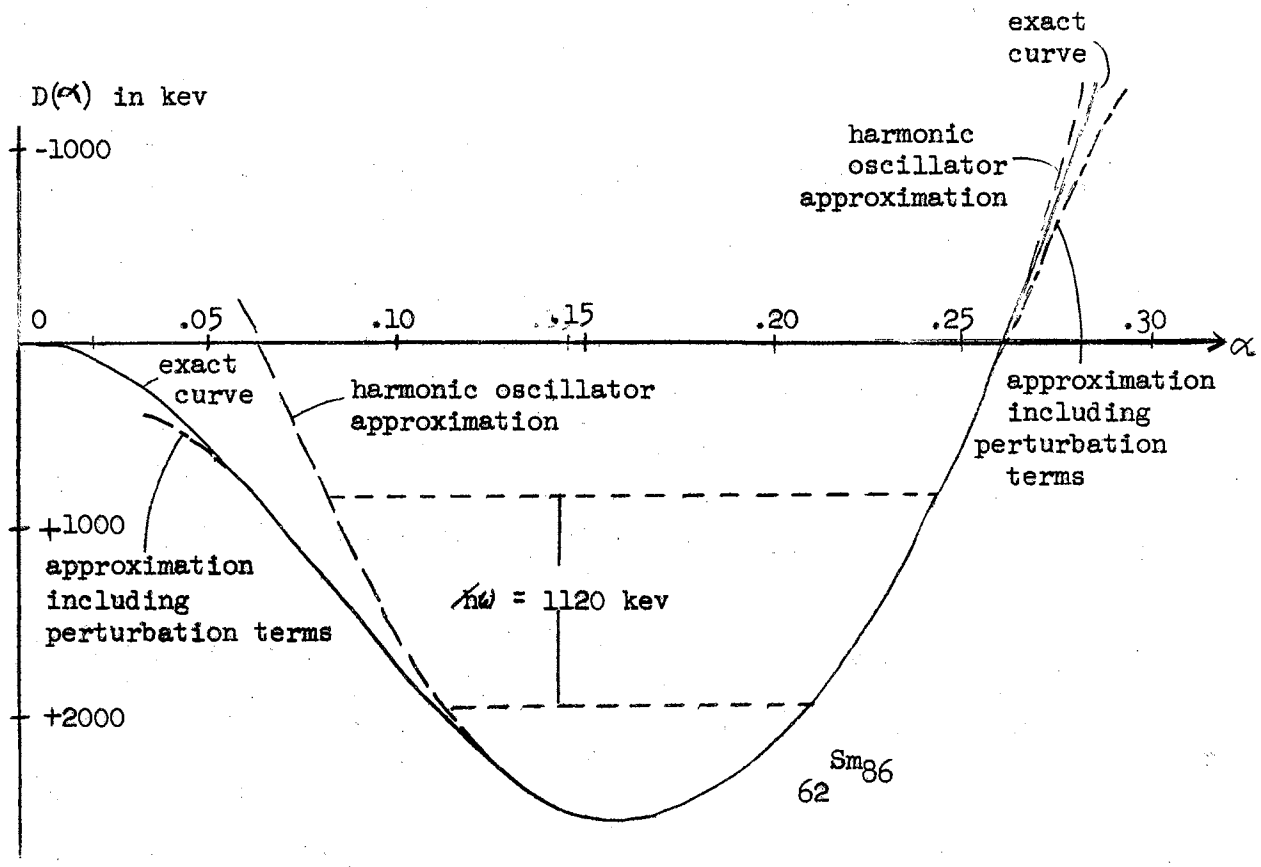


Fig. 4: Deformation energy vs. α for $^{86}_{62}Sm$ showing exact curve, harmonic oscillator approximation and approximation including perturbation terms.

CHAPTER III

MASS PARAMETERS OF DEFORMED HEAVY EVEN-EVEN NUCLEI

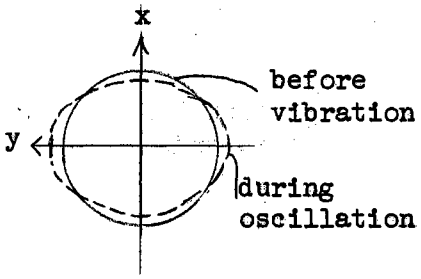
Theory of Rotational and Vibrational Excited States

It is a well known fact that in nuclei, neutrons tend to pair up with neutrons and protons tend to pair up with protons. Hence, if a nucleus has an odd number of protons or neutrons or both, the individual motion of the unpaired nucleons is very important. However, in even-even nuclei, single particle motion is of much less importance; and in the strong interaction approximation the excited states of such nuclei are thought of as coming about entirely because of the rotation or vibration of the nucleus as a whole. Using such a model, it has been found that the important features, though not the details, of excited states of even-even nuclei can be explained (8). A brief discussion of the collective modes of excitation for even-even nuclei will now be given.

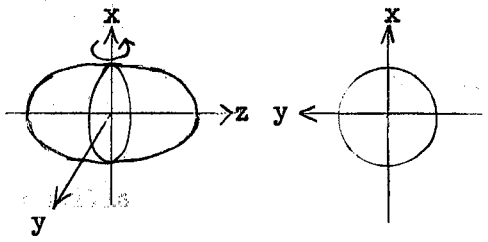
Heavy even-even nuclei near closed neutron shells are regarded as spherical whereas those far from magic numbers are deformed into prolate spheroids (cigar-shaped objects). The proton shells are approximately in phase with the neutron shells for heavy nuclei. (For example, ${}_{68}Er^{166}_{98}$ is far away from both a proton and a neutron magic number while ${}_{82}Pb^{208}_{126}$ has both a proton and a neutron magic number.) Irrotational flow is assumed within the nucleus, so for spherical nuclei the only possible exciting

collective mode is vibrational in nature. The spin-parity sequence of the first few levels is 0^+, 2^+, and then a triplet level with 0^+, 2^+, and 4^+. These levels are equally spaced. In deformed nuclei there are both rotational and vibrational modes of excitation. The rotational spin-parity sequence begins with the 0^+ ground state and the succeeding levels are 2^+, 4^+, 6^+, 8^+, ... The spacing between the levels increases linearly as the spin I increases. There are several possible vibrational modes for deformed nuclei, but the two most important are vibrations along the major axis with cylindrical symmetry maintained and vibrations which destroy the cylindrical symmetry but do not change the length of the major axis. The former type are classified as β- vibrations, the latter as γ-vibrations. Schematic drawings of the lowest collective excitations for spherical and deformed nuclei are shown in Fig. 5. For each vibrational mode there are several rotational excited states. The first β-vibrational mode has a rotational band characterized by spins and parities of 0^+, 2^+, 4^+, ... The first γ-vibrational mode has a rotational band with 2^+, 3^+, 4^+, ... as the corresponding spins and parities. Each level in a deformed nucleus can be thought of as corresponding to a level in a spherical nucleus. The first few theoretical levels in each type of nucleus are shown in Fig. 6. Some of the related levels are connected by dashed lines.

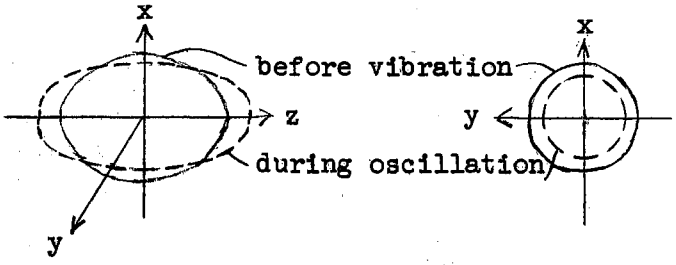
If the nucleus is thought of as oscillating in the potential well due to the deformation energy term in the new mass formula, a correspondence can be made between vibrational levels in this well and β-vibrational levels in deformed nuclei. This has been done for the first energy level in the potential well. The first β-vibrational level corresponds to this, and it is easily found using experimental data



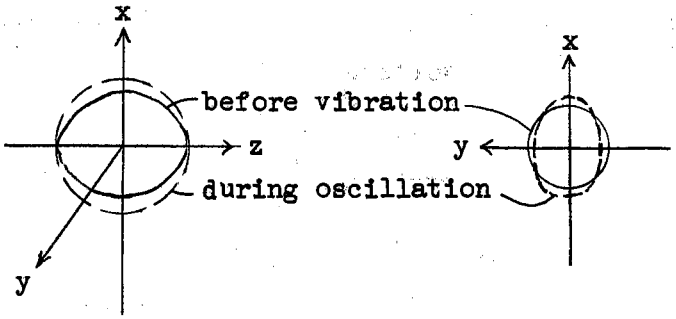
(a) Vibrational modes of a spherical nucleus. Nucleus vibrates about spherical shape.



(b) Rotational modes of a deformed nucleus. Nucleus has cylindrical symmetry and rotation is about x-axis.



(c) β-vibrational modes of a deformed nucleus. Nucleus vibrates about a prolate spheroid shape but retains cylindrical symmetry.



(d) γ-vibrational modes of a deformed nucleus. Nucleus vibrates about cylindrical symmetry, but length of major axis is unchanged.

Fig. 5: Schematic representation of lowest collective excitations in spherical and deformed nuclei.

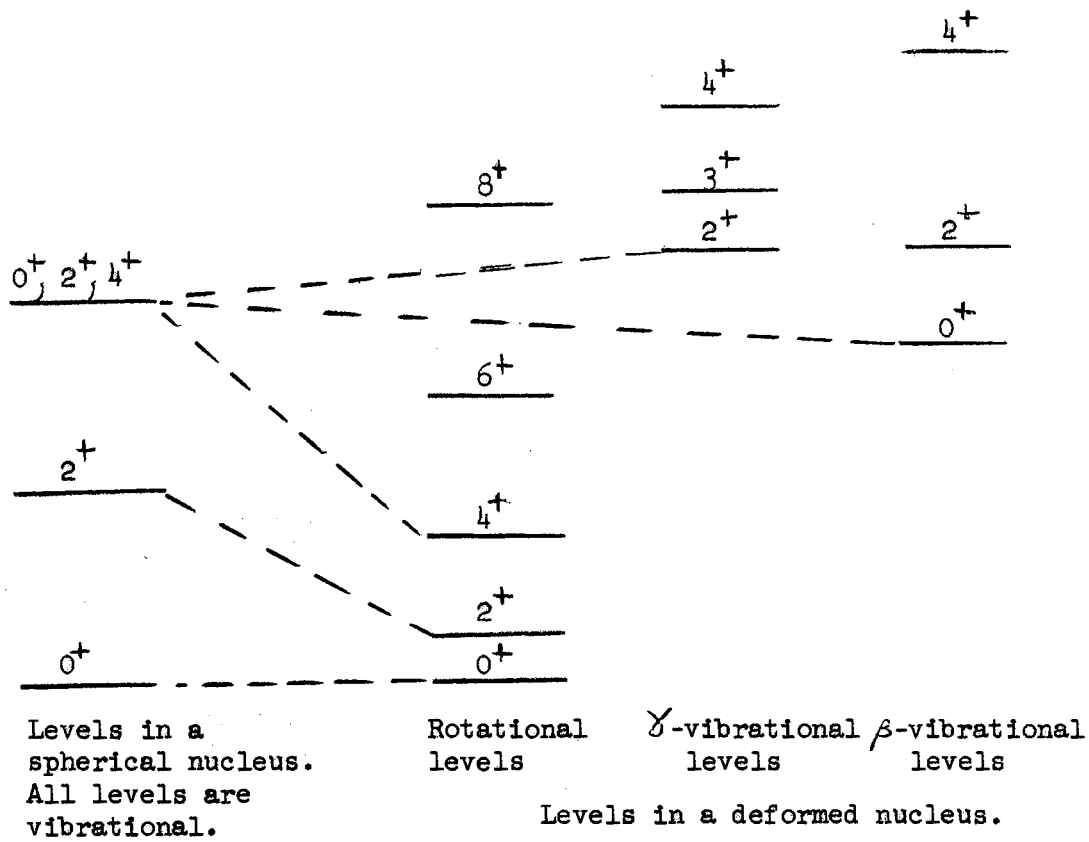


Fig. 6: Rotational and vibrational levels in spherical and deformed nuclei. (Dashed lines identify some of the lower levels in a deformed nucleus with corresponding levels in a spherical nucleus.)

because, as can be seen from Fig. 6, it is the first excited state with spin-parity 0^+.

Tabulation of Experimental Data for First Excited 0^+ Level

Excited 0^+ levels have been found in many heavy even-even nuclei. In some nuclei two or more such levels have been found, but we are interested only in the lowest one. A tabulation of the energies corresponding to these levels and the sources from which the data were obtained is found in Table III for some nuclei with $82 < N \leq 126$.

Mass Parameters of Heavy Even-Even Nuclei

If the nucleus is treated as a deformed liquid drop, its excitation energy comes about because of vibrations and rotations. The total classical hamiltonian is then

$$H_{classical} = T_{rotational} + T_{vibrational} + V \quad . \tag{22}$$

If the nuclear motion is described as being only oscillation within the potential well caused by the deformation energy term in the mass formula, the hamiltonian reduces to

$$H_{cl} = T_{vib} + V \quad . \tag{23}$$

According to Bohr (4), V is given by

$$V = \tfrac{1}{2} k_0 \beta^2 \quad , \tag{24}$$

where β is a deformation parameter related to $\alpha - \alpha_{min}$ (to first order) by the equation

$$\beta = \left(\frac{4\pi}{5}\right)^{\frac{1}{2}} (\alpha - \alpha_{min}) \quad . \tag{25}$$

T_{vib} is given by Bohr (4) as

$$T_{vib} = \tfrac{1}{2} B \left(\dot{\beta}^2 + \beta^2 \dot{\gamma}^2\right) \quad , \tag{26}$$

TABLE III

ENERGIES OF FIRST EXCITED 0^+ LEVELS IN SOME HEAVY EVEN-EVEN NUCLEI

Nucleus	Energy of First Excited 0^+ Level (kev)	Source
$_{62}Sm^{148}_{86}$	1120	Kenefick, R.A. & R. K. Sheline. Phys. Rev., Vol. 133, (1964), p. B25.
$_{62}Sm^{150}_{88}$	740	Lutsenko, N.P.*; Vol. 47, (1963), p. 42.
$_{64}Gd^{152}_{88}$	615	N.D.S.**
$_{62}Sm^{152}_{90}$	685	N.D.S.**
$_{64}Gd^{154}_{90}$	681	Lutsenko, N.P.*; Vol. 47, (1963), p. 42.
$_{64}Gd^{156}_{92}$	1010	Thosar, et. al., N.P.* Vol. 41, (1963), p. 386
$_{68}Er^{166}_{98}$	1460	N.D.S.**
$_{72}Hf^{178}_{106}$	1197	Verheul, et. al., N.P.*; Vol. 42, (1963), p. 551.
$_{76}Os^{188}_{112}$	1087	Marklund, et. al., N.P.,* Vol. 15, (1960), p. 533.
$_{78}Pt^{194}_{116}$	1267	N.D.S.**

* N.P. = Nuclear Physics.

**N.D.S. = Nuclear Data Sheets, National Academy of Sciences, National Research Council, Washington, D.C.

where γ is a shape parameter which describes the deviation from rotational symmetry. We are concerned only with β vibrations; this means that rotational symmetry is maintained and $\dot{\gamma} = 0$. This gives us

$$H_{cl} = \tfrac{1}{2} B\dot{\beta}^2 + \tfrac{1}{2}k_o\beta^2 \quad ; \tag{27}$$

and we see that $B\dot{\beta}$ and β are canonically conjugate variables since $B\dot{\beta} = \frac{\partial L}{\partial \dot{\beta}}$ where $L = T_{vib} - V$. The quantum mechanical form of the harmonic oscillator hamiltonian corresponding to this is

$$H_{q.\ m.} = \tfrac{1}{2}\frac{p^2}{B} + \tfrac{1}{2}k_o\, q^2 \quad , \tag{28}$$

where p and q obey the commutation relation

$$[p, q] = \hbar/i \quad . \tag{29}$$

If we take $q = \beta$, then

$$p = \frac{\hbar}{i}\frac{\partial}{\partial \beta} \quad ; \tag{30}$$

and our quantum mechanical hamiltonian becomes

$$H_{q.\ m.} = -\tfrac{1}{2}\frac{\hbar^2}{B}\frac{\partial^2}{\partial \beta^2} + \tfrac{1}{2}k_o\beta^2 \quad . \tag{31}$$

This is the standard quantum mechanical harmonic oscillator hamiltonian. We will call B the mass parameter since it takes on the role of the mass in the harmonic oscillator solution.

Schroedinger's equation using $H_{q.\ m.}$ is

$$H_{q.\ m.}\psi_n = E_n\psi_n. \tag{32}$$

The solution to this, using equation (31), can be found in basic texts on Quantum mechanics, e.g. reference (9). The solution yields the wave functions ψ_n and the energies. The energies are given by

$$E_n = \hbar\left(\frac{k_o}{B}\right)^{\frac{1}{2}}(n + \tfrac{1}{2}) \qquad n = \text{integer} \quad . \tag{33}$$

The first excited 0^+ level in our model is the difference between the E_0 and the E_1 energy levels. Hence

$$E_{0+} = E_1 - E_0 = \frac{3}{2}\hbar\left(\frac{k_0}{B}\right)^{\frac{1}{2}} - \frac{1}{2}\hbar\left(\frac{k_0}{B}\right)^{\frac{1}{2}} = \hbar\left(\frac{k_0}{B}\right)^{\frac{1}{2}} . \tag{34}$$

Since we know k_0 from our deformation energy expression, we can find B or $B/\hbar^2$ using equation (34). We get

$$\frac{B}{\hbar^2} = \frac{k_0}{E_{0+}^2} . \tag{35}$$

Table IV shows some values for the mass parameter $B/\hbar^2$ calculated from equation (35) for some deformed heavy even-even nuclei.

TABLE IV

VALUES OF $\frac{B}{\hbar^2} = \frac{k_0}{E_{0+}^2}$ FOR SOME HEAVY EVEN-EVEN NUCLEI

Nucleus	k_0 ($\times 10^5$ kev)	E_{0+} (kev)	$\frac{B}{\hbar^2}$ $\left(\frac{1}{\text{kev}}\right)$
Sm_{86}	2.00	1120	.160
Sm_{88}	2.35	740	.430
Gd_{88}	2.35	615	.622
Sm_{90}	2.60	685	.554
Gd_{90}	2.59	681	.558
Gd_{92}	2.76	1010	.270
Er_{98}	2.91	1460	.136
Hf_{106}	2.66	1197	.185
Os_{112}	2.08	1087	.176
Pt_{116}	1.50	1267	.094

Perturbation Corrections

If, instead of a harmonic oscillator potential, we use one which includes higher terms (equation 20), our hamiltonian becomes

$$H = H_o + H' \quad , \tag{36}$$

where

$$H_o = -\frac{\hbar^2}{2B}\frac{\partial^2}{\partial\beta^2} + \tfrac{1}{2} k_o \beta^2 \quad , \tag{37}$$

and

$$H' = k_o' \beta^3 + k_o'' \beta^4 \quad . \tag{38}$$

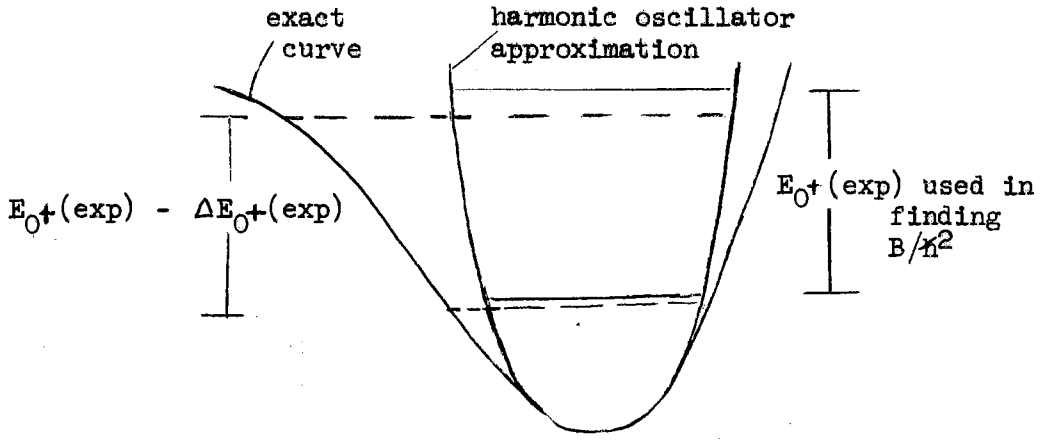
In the region in which we are interested, H' is small compared to the second term in H_o. This can be seen by looking at Figs. 3 and 4. Because of this, we can use standard time independent perturbation theory to find the change which this would bring about in E_{0+} $(=E_1-E_0)$. Since the perturbation terms make the well more shallow, the energy E_{0+} is lowered by a small amount ΔE_{0+} . Since we want E_{0+} found by experiment to equal our E_{0+} found using the perturbing terms, the value used in the harmonic oscillator approximation should be slightly higher than E_{0+} found experimentally. We raise E_{0+} used in the harmonic oscillator approximation by multiplying it by the factor $E_{0+}(\text{exp})/\left[E_{0+}(\text{exp}) - \Delta E_{0+}(\text{exp})\right]$. These relationships are shown schematically in Fig. 7. E_{0+}(corrected) then becomes

$$E_{0+}(\text{cor}) = E_{0+}(\text{exp})\left(\frac{E_{0+}(\text{exp})}{E_{0+}(\text{exp}) - \Delta E_{0+}(\text{exp})}\right) \tag{39}$$

and our corrected value for $B/\hbar^2$ is

$$\frac{B(\text{cor})}{\hbar^2} = \frac{k_o}{E_{0+}^2(\text{cor})} \tag{40}$$

We now proceed with calculating $\Delta E_{0+}(\text{exp})$.



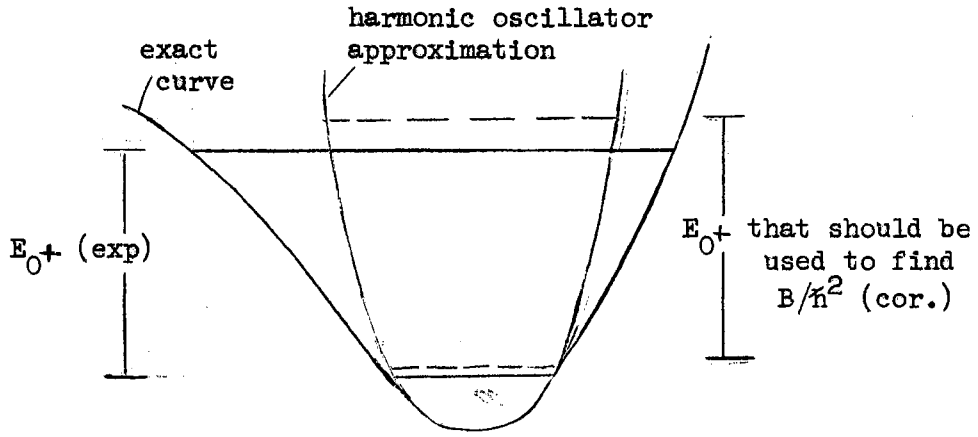


Fig. 7: Relation of E_{0^+} measured experimentally to the E_{0^+} which should be used to calculate $B/\hbar^2$ in the harmonic oscillator approximation.

Perturbation theory is discussed in most basic texts on quantum mechanics (9), and the first order energy correction is given by

$$\Delta E_n = \int_{-\infty}^{\infty} \psi_n^* H' \psi_n \, d\beta \quad , \tag{41}$$

where ψ_n is found as part of the solution to equation (32) using the hamiltonian H_o, and ψ_n^* is the complex conjugate of ψ_n. Since all of our ψ_n are real ($\psi_n^* = \psi_n$) and $H'\psi_n = \psi_n H'$, we have

$$\Delta E_n = \int_{-\infty}^{\infty} \psi_n^2 \, H' d\beta \quad . \tag{42}$$

The ψ_n are given by the formula

$$\psi_n = \left(\frac{a^{\frac{1}{2}}}{2^n n! \, (\pi)^{\frac{1}{2}}} \right)^{1/2} e^{-\frac{1}{2}a\beta^2} \, H_n(a^{\frac{1}{2}}\beta) \quad , \tag{43}$$

where $H_n(a^{\frac{1}{2}}\beta)$ is the Hermite polynomial of degree n, and $a = \frac{\sqrt{k_o B}}{\hbar} = \frac{k_o}{E_{o^+}}$.

The first few ψ_n are

$$\begin{aligned}
\psi_0 &= \left(\frac{a}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}a\beta^2} \\
\psi_1 &= \left(\frac{4a^3}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}a\beta^2} \, \beta \\
\psi_2 &= \left(\frac{a}{4\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}a\beta^2} \, (2a\beta^2 - 1) \\
\psi_3 &= \left(\frac{a}{9\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}a\beta^2} \, (2a^{3/2}\beta^3 - 3a^{\frac{1}{2}}\beta) \\
\psi_4 &= \left(\frac{a}{36\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}a\beta^2} \, (4a^2\beta^4 - 12a\beta^2 + 3)
\end{aligned} \tag{44}$$

We see that the ψ_n are either even or odd functions, depending on whether n is even or odd; therefore, ψ_n^2 is always an even function. H' is given by equation (38), and it is seen that it is made up of an odd function plus an even one. The integral of equation (42) can be written

$$\Delta E_n = k_o' \int_{-\infty}^{\infty} \psi_n^2 \beta^3 \, d\beta + k_o'' \int_{-\infty}^{\infty} \psi_n^2 \beta^4 \, d\beta \tag{45}$$

Since the first integrand is odd, the first integral is zero, and to first order,

$$\Delta E_n^{(1)} = k_o'' \int_{-\infty}^{\infty} \psi_n^2 \beta^4 \, d\beta \quad . \tag{46}$$

The first order approximation does not take into account the β^3 term which is in general more important than the β^4 term. So we need the second order correction in order to see what contribution the β^3 term will make to ΔE_n. The second order contribution is

$$\Delta E_n^{(2)} = \sum_{\substack{k=o \\ k \neq n}}^{\infty} \frac{\left| \int_{-\infty}^{\infty} \psi_n^* \, H' \, \psi_k \, d\beta \right|^2}{E_n - E_k} \quad . \tag{47}$$

If we again break H' up into two parts, we get

$$\int_{-\infty}^{\infty} \psi_n^* \, H' \, \psi_k \, d\beta = k_o' \int_{-\infty}^{\infty} \psi_n^* \, \psi_k \beta^3 d\beta + k_o'' \int_{-\infty}^{\infty} \psi_n^* \, \psi_k \beta^4 \, d\beta \tag{48}$$

In these integrals, if n+k is even, the first integral is zero, whereas if n+k is odd, the second integral is zero. This occurs because of the even-odd properties of the resulting integrands.

Since we are interested in $\Delta E_0 + (= \Delta E_1 - \Delta E_0)$, we will now compute ΔE_1 and ΔE_0. It would be very laborious to compute more than a few of the terms in the summation of equation (47). Therefore, we will stop after the first four terms. This induces an error into our calculation, but it is about the same order of magnitude as neglecting the β^5 and other higher order terms in our expansion of V. Our expressions for ΔE_1 and ΔE_0 are

$$\Delta E_0 = k_o'' \int_{-\infty}^{\infty} \psi_o^2 \beta^4 d\beta + \sum_{k=1}^{4} \frac{\left| \int_{-\infty}^{\infty} \psi_o \, \psi_k \, H' d\beta \right|^2}{E_0 - E_k} \tag{49}$$

$$\Delta E_1 = k_o'' \int_{-\infty}^{\infty} \psi_1^2 \beta^4 d\beta + \sum_{\substack{k=0 \\ k \neq 1}}^{4} \frac{\left| \int_{-\infty}^{\infty} \psi_1 \psi_k H' d\beta \right|^2}{E_1 - E_k} \tag{50}$$

Some of the values of the integrals $I_{jk} = \int_{-\infty}^{\infty} \psi_j \psi_k H' d\beta$ involved are given in equation (51).

$$I_{00} = \frac{3}{4} \frac{k_o''}{a^2} \qquad I_{11} = \frac{15}{4} \frac{k_o''}{a^2}$$

$$I_{01} = \frac{3}{4} (2)^{\frac{1}{2}} \frac{k_o'}{a^{3/2}} \qquad I_{10} = \frac{3}{4} (2)^{\frac{1}{2}} \frac{k_o'}{a^{3/2}}$$

$$I_{02} = \frac{3}{2^{1/2}} \frac{k_o''}{a^2} \qquad I_{12} = \frac{3k_o'}{a^{3/2}} \tag{51}$$

$$I_{03} = \frac{3}{2 \cdot 3^{\frac{1}{2}}} \frac{k_o'}{a^{3/2}} \qquad I_{13} = \left(\frac{75}{2}\right)^{\frac{1}{2}} \frac{k_o''}{a^2}$$

$$I_{04} = 6^{\frac{1}{2}} \frac{k_o''}{a^2} \qquad I_{14} = 2 \cdot 3^{\frac{1}{2}} \frac{k_o'}{a^{3/2}}$$

The expressions for ΔE_1 and ΔE_0 using equation (51) and $a = k_o/E_{0+}$ are

$$\Delta E_1 = \frac{15}{4} k_o'' \frac{E_{0+}^2}{k_o^2} + \frac{9}{8} k_o'^2 \frac{E_{0+}^2}{k_o^3} - 9k_o'^2 \frac{E_{0+}^2}{k_o^3} - \frac{75}{4} \frac{k_o''^2 E_{0+}^2}{k_o^4} - 4 \frac{k_o'^2 E_{0+}^2}{k_o^3} \tag{52}$$

$$\Delta E_0 = \frac{3}{4} k_o'' \frac{E_{0+}^2}{k_o^2} - \frac{9}{8} \frac{k_o'^2 E_{0+}^2}{k_o^3} - \frac{9}{4} \frac{k_o''^2 E_{0+}^3}{k_o^4} - \frac{1}{4} k_o'^2 \frac{E_{0+}^2}{k_o^3} - \frac{3}{2} k_o''^2 \frac{E_{0+}^3}{k_o^4} \tag{53}$$

and E_{0+} is changed by an amount

$$\Delta E_1 - \Delta E_0 = 3 \frac{k_o'' E_{0+}^2}{k_o^2} - \frac{21}{2} \frac{k_o'^2 E_{0+}^2}{k_o^3} - 15 \frac{k_o''^2 E_{0+}^3}{k_o^4} \qquad (54)$$

Since k_o'' is negative (see Table II), $\Delta E_1 - \Delta E_0$ is negative, which means that E_{0+} is lowered. Using equations (54), (39), and (40), we can calculate better values of $B/\hbar^2$. Some of these values are given in Table V. In Chapter IV, when we refer to $B/\hbar^2$, we will be referring to the new values given in Table V.

TABLE V

VALUES FOR $B/\hbar^2$ USING E_{0+} INCLUDING PERTURBING TERMS FOR SOME HEAVY EVEN-EVEN NUCLEI

Nucleus	$B/\hbar^2\left(\frac{1}{\text{kev}}\right)$ from Table IV	$\Delta E_{0+}(\text{exp})$ (kev)	$\left(\frac{E_{0+}(\text{exp})}{E_{0+}(\text{exp}) - \Delta E_{0+}(\text{exp})}\right)^2$	$B/\hbar^2\left(\frac{1}{\text{kev}}\right)$ using perturbing terms
Sm_{86}	.160	87	1.17	.136
Sm_{88}	.430	25	1.07	.402
Gd_{88}	.622	16	1.05	.591
Sm_{90}	.554	17	1.05	.528
Gd_{90}	.558	17	1.05	.532
Gd_{92}	.270	33	1.07	.254
Er_{98}	.136	86	1.13	.121
Hf_{106}	.185	49	1.10	.169
Os_{112}	.176	69	1.14	.154
Pt_{116}	.094	230	1.49	.063

CHAPTER IV

HYDRODYNAMICAL PROPERTIES

Mass Parameters

The classical value of the mass parameter B for an incompressible nucleus of constant density is given by Bohr and Mottelson (5) to be

$$B_\lambda = \frac{1}{\lambda} \frac{3}{4\pi} A M R_o^2 \quad . \tag{55}$$

Irrotational flow was assumed to get this formula. In the formula, λ is the order of deformation and equals two for our model, A is the mass number, M is the mass of one nucleon, and $R_o = 1.4 \times 10^{-13} \times A^{1/3}$ cm. To compare with our calculated mass parameter, we need $B/\hbar^2$. This is given by

$$\frac{B}{\hbar^2} = \frac{1}{\hbar^2} \frac{1}{2} \frac{3}{4\pi} A \frac{Mc^2}{c^2} R_o^2 \quad . \tag{56}$$

The constants involved are the following:

$$\begin{aligned} \hbar &= .6582 \times 10^{-15} \text{ ev-sec} \\ R_o &= 1.4 \times 10^{-13} \times A^{1/3} \text{ cm} \\ Mc^2 &= 9.38 \times 10^8 \text{ ev} \\ c^2 &= 9.00 \times 10^{20} \text{ cm/sec} \end{aligned} \tag{57}$$

This reduces equation (56) to

$$\frac{B}{\hbar^2} = .563 \times 10^{-5} A^{5/3} \left(\frac{1}{\text{kev}}\right) \tag{58}$$

Table VI compares $B/\hbar^2$ computed classically with the values found in

Table V. Of course, the classical description is a very crude one since it does not take into accout shell effects or individual motions of the nucleons. We see from Table VI that our mass parameters calculated using experimental 0^+ levels and the deformation potential well are from 2 to 25 times larger than the classical mass parameters.

TABLE VI

COMPARISON OF CLASSICAL MASS PARAMETERS WITH THOSE FOUND IN TABLE V FOR SOME HEAVY EVEN-EVEN NUCLEI

Nucleus	$B/\hbar^2$ (1/kev) Computed Classically	$B/\hbar^2$ (1/kev) From Table V	B from Table V / B Classical
Sm_{86}	.0232	.136	6
Sm_{88}	.0238	.402	17
Gd_{88}	.0244	.591	24
Sm_{90}	.0244	.528	22
Gd_{90}	.0249	.532	21
Gd_{92}	.0252	.254	10
Er_{98}	.0283	.121	4
Hf_{106}	.0317	.169	5
Os_{112}	.0348	.154	4
Pt_{116}	.0366	.063	2

Moments of Inertia

According to Bohr and Mottelson (5), the moment of inertia of the nucleus is given by

$$I = 3B\beta^2 \quad , \tag{59}$$

where β is the equilibrium deformation, that is, the deformation which corresponds to α_{min} in Table II. The β_{min} have been previously computed from rotational spectra by Elbeck, et. al. (10). The values which they found are compared with those found in the present work in Table VII. Next using our mass parameters, we can find I. These have also been found using only rotational spectra by Marklund, et. al. (11). In Table VII a comparison is made between the moments of inertia found by the two methods.

TABLE VII

COMPARISON OF EQUILIBRIUM DEFORMATIONS AND MOMENTS OF INERTIA FROM ROTATIONAL SPECTRA WITH THOSE FOUND IN THE PRESENT WORK FOR SOME HEAVY EVEN-EVEN NUCLEI

Nucleus	β_{min}	β_{min} From (10)	$\frac{3\hbar^2}{I} = \frac{\hbar^2}{B\beta_{min}^2}\left(\frac{1}{kev}\right)$ Using Present Work	$\frac{3\hbar^2}{I}\left(\frac{1}{kev}\right)$ From (11)
Sm_{86}	.256		112.1	
Sm_{88}	.274	.184	32.5	
Gd_{88}	.280		21.6	
Sm_{90}	.286	.290	23.2	126.8
Gd_{90}	.291	.280	22.3	128.1
Gd_{92}	.297	.320	44.6	89.0
Er_{98}	.309	.323	88.6	81.0
Hf_{106}	.296		67.8	93.6
Os_{112}	.266		91.3	160.0
Pt_{116}	.227		303.4	

Conclusions

Before drawing any conclusions, a review of the basic method employed will be made in which the various assumptions will be pointed out. We began by taking the deformation energy term in the new mass formula and applying it to cases in which the nucleus vibrated around an equilibrium deformation. When we did this, we assumed that single particle interactions could be neglected and that cylindrical symmetry was maintained during these vibrations. We would expect the single particle interactions to be most important near closed shells, so in this region a good agreement with other data should not be expected. We next assumed that the lowest β-vibrational mode came about by oscillating in the potential well caused by the deformation energy curve. When this was done, we neglected all rotational-vibrational and β-vibrational -γ- vibrational interactions. Our last assumption was that the classical hydrodynamical expression for the moment of inertia was valid. Moments of inertia from our model were found using only β-vibrational states. They were then compared with moments of inertia found from rotational spectra. Our expression for the deformation energy term enabled us to calculate for the first time a potential well in which the nucleus vibrated. This in turn made it possible for us to calculate for the first time, moments of inertia using β-vibrations.

As seen from Table VII, for several nuclei our values for β_{min} obtained from the mass formula are in good agreement with those found from rotational spectra. However, even for these nuclei the moments of inertia do not compare favorably (e.g. Hf_{106}). For the isotopes with N = 90 or less this disagreement can be justified by saying that these

are too near closed shells to be analyzed like strongly deformed nuclei. For the strongly deformed Er_{98} we get a much better agreement, but even strongly deformed Hf_{106} is not in good agreement. Since there is such poor correlation, either our method of finding B is not valid, or the hydrodynamical expression for I is incorrect. Of course, to some extent, both involve assumptions which neglect potentially important factors.

To study more precisely moments of inertia and other nuclear properties using our method, the following refinements need to be made. In our method of finding B we assumed that the first excited 0^+ level was a purely β-vibrational mode unaffected by γ-vibrations, rotations, or single particle interactions. Sheline (8) has indicated that the single particle interactions have a significant effect on the vibrational levels. The present study indicates that this is an accurate description of the true situation. Secondly, the simple moment of inertia formula obtained from the hydrodynamical model will need to be refined. It has already been shown (11) that if the simple formula for rotational excited states, namely

$$E_{excited} = E_0 + E_1 I(I+1) + E_2 I^2(I+1)^2 \quad , \tag{60}$$

is used, the E_2 term is 2 to 4 times smaller than that calculated theoretically using the hydrodynamical model. A detailed study of E_2 from rotational states would give an indication as to how the simple hydrodynamical expression for moments of inertia should be modified. Of course, these considerations complicate our simple method immensely. However, to get accurate results, they must be taken into account.

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