SMALL RESOLUTIONS OF CLOSURES OF K-ORBITS IN FLAG VARIETIES

By

SCOTT LARSON

Bachelor of Science in Mathematics and Physics
University of Wisconsin – Eau Claire
Eau Claire, Wisconsin
2009

Master of Science in Mathematics
Oklahoma State University
Stillwater, Oklahoma
2013

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Dissertation Approved:

Dr. Roger Zierau

Dissertation Advisor

Dr. Leticia Barchini

Dr. Edward Richmond

Dr. Kaladi Babu

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Name: SCOTT LARSON

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Abstract: We construct explicit proper morphisms $\mu : Z \to Y$, where Y is the closure of a K-orbit in a flag variety. Of particular interest is when μ is a resolution of singularities and when it is a small resolution. We apply our construction to the group $Sp(2n, \mathbf{R})$ and construct a resolution uniformly for every K-orbit closure in an isotropic grassmannian flag variety. This provides a family of small resolutions, and we change the construction to describe more families of small resolutions. We also apply our construction to the group $U(p, q)$ and determine that any of our morphisms which are generically finite, are in fact birational for this group. This enables us to compute many small resolutions, and a simple family of small resolutions is described in terms of combinatorics of clans.

The concept of inducing a small resolution to larger dimensions is introduced. This shows that small resolutions propagate and highlights the importance of determining small resolutions in low rank groups. We apply this to the groups $Sp(2n, \mathbf{R})$ and $U(p, q)$ to obtain many small resolutions.

A repeated obstacle in applications is determining information about the fiber of μ. We provide a fiber dimension formula for a large class of our resolutions, which we call Barbasch-Evens type. When Z is constructed from a K-orbit closure and a single Schubert variety, then we describe fibers of μ isomorphically – enabling us to find more small resolutions and compute Kazhdan-Lusztig-Vogan polynomials of a family of closures of K-orbits.

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CHAPTER I

INTRODUCTION

1.1 Contributions of thesis

This thesis gives explicit constructions of proper algebraic morphisms

$$\mu : Z \to Y \tag{1.1}$$

(q.v. (2.31)), where Y is the closure of a K-orbit in a flag variety. Let G be a connected reductive algebraic group over an algebraically closed field of characteristic zero and define

$$Z = G_0 \times^{R_1} G_1 \times^{R_2} \cdots \times^{R_m} G_m / R_{m+1}, \tag{1.2}$$

where for every $0 \le i \le m$, $G_i \subseteq G$ such that G_0/R_1 is the closure of a K-orbit in a flag variety and for every $1 \le i \le m$, G_i/R_{i+1} is a Schubert variety (q.v. (2.30)). When Z is of the form (1.2), we call (1.1) *type* μ. A simple formula (2.50), depending on combinatorics of K-orbits and Schubert varieties, characterizes when μ is generically finite – an important property for studying local systems.

Describing fibers of μ is a repeated obstacle for applications. Our focus is on resolutions of singularities, which requires finding smooth Z and birational μ. The latter condition is equivalent to the fiber of μ over the open K-orbit being a single point, which is characterized by (5.2). Among these resolutions numerous *small* resolutions (q.v. (2.53)) are singled out, where intuitively, small means there are relatively few large dimensional fibers.

We consider two families of real reductive groups in detail, namely, $Sp(2n, \mathbf{R})$ and $U(p,q)$. Barbasch-Evens [5] describe K-orbits in type A grassmannian flag varieties for the groups $U(p,q)$, $GL(n,\mathbf{H})$, and $GL(n,\mathbf{R})$ and construct resolutions for corresponding K-orbit closures. We follow [5] and describe all K-orbits in generalized grassmannian flag varieties for $Sp(2n,\mathbf{R})$. A uniform construction of a resolution of singularities is described for all corresponding K-orbit closures. This leads to a formula for the dimension of such K-orbits, and in certain cases, provides a small resolution. We describe more families of small resolutions by changing μ (q.v. Theorem 3.3.6). Some of these small resolutions require determining local coordinates on certain K-orbit closures to show that Z is smooth.

Clans are used by Yamamoto [31] to parameterize K-orbits in the full flag variety for $Sp(2n,\mathbf{R})$ and $U(p,q)$. A combinatorial description of smooth K-orbit closures for $U(p,q)$ is given by pattern avoidance in McGovern [20], which we describe simply as concatenating maximum clans from various $U(p',q')$, where $p' \leq p$ and $q' \leq q$ (q.v. Remark 4.3.2). Describing all smooth K-orbit closures for $Sp(2n,\mathbf{R})$ is not as simple and remains an open problem.

Let B be a Borel subgroup of G. Flag varieties X^I of G correspond to subsets of simple reflections $I \subseteq S$. A special case $\mu = \pi$ is given by restricting the projection of flag varieties $X^J \to X^I$, where $J \subseteq I$, to a K-orbit closure $Z \subseteq X^J$. We use π to describe a family of small resolutions for $U(p,q)$ (q.v. Proposition 5.3.2). For the groups $U(2,2)$ and $U(3,2)$, we use π to describe small resolutions for every Y admitting a small resolution of type μ.

If Y is normal, then Zariski's main theorem can be used to show that when all fibers of μ are zero dimensional, then a birational μ is an isomorphism. Thus fiber bundle structures on Y can be viewed as bijective μ; however, we typically obtain fiber bundles directly from π, where the fibers all have large dimension. A particularly useful case arises by projecting to a closed K-orbit, in which case π is always a fiber bundle (q.v. Lemma 3.3.5) and we obtain $Y \cong Z$ (by a corresponding μ). For $Sp(2n,\mathbf{R})$ and $U(p,q)$, all such fiber bundles can be described combinatorially as concatenating clans from lower rank groups.

The K-equivariance of μ enables us to *induce* (q.v. Key Lemma 4.1.1) small resolutions of type μ to small resolutions of higher dimensional varieties (again of type μ). Propagation of small resolutions highlights the importance of determining all small resolutions of type μ in low rank. Moreover, all small resolutions described in this thesis were discovered by first describing μ in low dimensions and generalizing to higher dimensions, using either a unique description for μ or by induction.

When the Schubert varieties used to construct Z are particularly nice (q.v. (5.9)), Z admits a beautiful description as a fibered product of a smooth K-orbit closure with various flag varieties. This description is similar to the presentation found in Gelfand-MacPherson [11] for resolutions of Schubert varieties. We call such Z *Barbasch-Evens type*, since [5] considers many resolutions that may be described isomorphically in this form. We obtain a formula (q.v. (5.16)) for the dimension of any fiber of Barbasch-Evens type.

If the number of Schubert varieties, say m, used to construct Z is one, then we provide a simple description of fibers of μ as intersections of algebraic varieties in a flag variety (q.v. (6.2)). The case $\mu = \pi$ is equivalent to studying $m = 1$ of Barbasch-Evens type, by pulling back to the full flag variety (q.v. §5.3), so we have a dimension formula along with an isomorphic description of all fibers. The group $Sp(6, \mathbf{R})$ admits examples of small resolutions using various techniques introduced throughout the thesis. These examples are all of the form $m = 1$, and we conclude with an example to show that general Schubert varieties are required in the definition of μ to find certain small resolutions.

1.2 Some history

Demazure [10] and Hansen [13] independently construct resolutions of Schubert varieties using Bott-Samelson spaces [7]. These resolutions are B-equivariant morphisms naturally associated to a reduced word of a Weyl group element. The construction given here is in the same spirit.

Resolutions are often defined to be an isomorphism over the smooth locus, but fibers of Demazure-Hansen resolutions are typically too large to satisfy this property. In particular, these resolutions are rarely small. Gelfand-MacPherson [11] generalize Demazure-Hansen resolutions by considering more general parabolic subgroups. Thus Gelfand-MacPherson obtain iterated fiber bundles of flag varieties, where Demazure-Hansen have iterated $\mathbf{P}^1$ fiber bundles. Gelfand-MacPherson use these resolutions to study intersection cohomology of Schubert varieties, leading them to conjecture the decomposition theorem (cf. [11, §2.10]). Collapsing certain parabolic subgroups from Demazure-Hansen resolutions enable Gelfand-MacPherson to describe small resolutions of some Schubert varieties (cf. [11, §5.4]).

Zelevinskiĭ uses Gelfand-MacPherson resolutions to systematically construct small resolutions for any Schubert variety in a grassmannian flag variety. A remark from [32, §6b] indicates that not all Schubert varieties admit small resolutions of this type. Many authors address small resolutions of Schubert varieties using similar constructions. Sankaran-Vanchinathan [24, 25] consider symplectic, orthogonal, and some exceptional types. They provide a fiber dimension formula similar to (5.16) for all fibers of Gelfand-MacPherson resolutions, compute singular loci of Schubert varieties of type E_6, compute Kazhdan-Lusztig polynomials, and show that some Schubert varieties do not admit *any* small resolution. Similar considerations may be used to show that some K-orbit closures for $Sp(2n, \mathbf{R})$ do not admit any small resolution, when there exists normal singular $\mathbf{Q}$-smooth locus.

Perrin [22] considers a more general construction than Gelfand-MacPherson in the same way that Schubert varieties generalize flag varieties. In particular, the morphisms are not necessarily resolutions of singularities since the domain need not be smooth. This generalization enables Perrin to classify which Schubert varieties admit small resolutions for the case of generalized grassmannian Schubert varieties of minuscule type. Similar methods show that the Schubert variety mentioned in [32, §6b] does not admit *any* small resolution since it is locally factorial.

Barbasch-Evens [5] consider a similar construction for K-orbits. In particular, they interpret results of Vogan [28] and Chang [9] to conclude that any K-orbit closure has a natural K-equivariant resolution analogous to Demazure-Hansen for Schubert varieties (cf. [5, §6.5]). Different resolutions for the real groups $U(p,q)$, $GL(n,\mathbf{H})$, and $GL(n,\mathbf{R})$ are described in [5], and they conclude, for $U(p,q)$, that any K-orbit closure in a grassmannian flag variety admits a small resolution. These resolutions may be described as the form $\mu = \pi$, or equivalently, Barbasch-Evens type with $m = 1$ pulled back to the full flag variety.

1.3 Motivation

Natural resolutions mentioned above have proved useful in studying both coherent and constructible sheaves on Schubert varieties and K-orbit closures. Demazure-Hansen resolutions are used to study sections of coherent sheaves and lead to, e.g., Demazure's character formula and normality of all Schubert varieties. One cannot use analogous Chang-Vogan resolutions to show all K-orbit closures are normal, since Barbasch-Evens provide a counter-example to this statement. However, they are successful in showing certain K-orbit closures are normal by using their more general resolution. The fact that these resolutions are K-equivariant allows us to view them as stratified with respect to the Whitney stratification of K-orbits on a flag variety.

Let H be an algebraic group acting algebraically on a smooth irreducible algebraic variety X with finitely many orbits

$$X = \coprod_{x \in \Lambda} \mathcal{S}_x, \tag{1.3}$$

where Λ is a set of representative basepoints. For every $x, y \in \Lambda$, let $x \leq y$ if $\mathcal{S}_x \subseteq X_y$, where X_y is the closure of $\mathcal{S}_y$. Then for every $y \in \Lambda$, we have

(i) $X_y = \coprod_{x \leq y} \mathcal{S}_x$,

(ii) for every $x \leq y$, $x' \in \mathcal{S}_x$, the local rings of X_y at x and x' are isomorphic.

In particular, (1.3) is a Whitney stratification (as in, e.g., [27]). So we can consider intersection cohomology, described in terms of combinatorics of Λ, and compatible with equivariant proper morphisms.

We briefly recall a definition of the intersection cohomology complex $\mathbf{IC}^\bullet$ (with respect to the middle perversity) of X_y (cf. [12, §4]). Let $\mathcal{L}$ be a local system on $\mathcal{S}_y$ with coefficients in a regular noetherian ring with finite Krull dimension. For every $x \leq y$, let $j_x : \{x\} \to X_y$ be inclusion and let d_x be the dimension of $\mathcal{S}_x$. Then $\mathbf{IC}^\bullet$ is the unique constructible complex in a derived category satisfying:

(a) $\mathbf{IC}^\bullet | \mathcal{S}_y \cong \mathcal{L}[d_y]$.

(b) $\mathbf{H}^i(\mathbf{IC}^\bullet) = 0$ for all $i < -d_y$.

(c) For every $x < y$, $H^i(j_x^*(\mathbf{IC}^\bullet)) = 0$ for $i \geq -d_x$.

(d) For every $x < y$, $H^i(j_x^!(\mathbf{IC}^\bullet)) = 0$ for $i \leq -d_x$.

Goresky-MacPherson observe that pushing forward certain constructible complexes under certain algebraic morphisms results in a constructible complex satisfying axioms (a)–(d). In particular, *small resolutions* were defined in [12, §6.2] to ensure that pushing forward a shifted constant sheaf, by a resolution of singularities, gives intersection cohomology. This enigmatic definition applies naturally to any generically finite morphism $\mu : Z \to Y$, where one might consider the *small locus* of μ with respect to a local system $\mathcal{L}$ on $U \subseteq Z$, such that μ restricts to a finite morphism on U, giving the largest Zariski open subset $W \subseteq Y$ such that

$$\mu_* \mathbf{IC}^\bullet(\mathcal{L}) | \mu^{-1}(W) = \mathbf{IC}^\bullet(\mu_* \mathcal{L}) | W. \tag{1.4}$$

When $\mathcal{L}$ is over a field of characteristic zero, it is typical to relax the definition to a *semi-small* morphism, where the decomposition theorem is available. So small resolutions may be considered as examples of morphisms such that the decomposition theorem applies trivially.

While small resolutions were defined by a condition on fiber dimensions, and theoretically computes $\mathbf{IC}^\bullet$, a repeated theme is that applications require further analysis of the fibers. The intriguing $\mathbf{IC}^\bullet$ has many applications to representation theory (cf. [17]), and we list a few regarding reductive groups. Kazhdan-Lusztig [15] defined polynomials $\mathrm{KL}_{y,w} \in \mathbf{Z}[q]$, depending on two elements y, w in a Coxeter group, useful for computing multiplicities in the Jordan-Hölder series of Verma modules. For a Schubert variety X_w of dimension n, we have

$$\mathrm{KL}_{y,w}(q) = \sum_{i \geq 0} \dim(H^{2i-n}(\mathbf{IC}^\bullet_y))q^i, \tag{1.5}$$

where $\mathbf{IC}^\bullet$ is intersection cohomology of X_w. Zelevinskiĭ [32] used (1.5), along with his small resolutions, to compute the Kazhdan-Lusztig polynomials for every Schubert variety in a grassmannian flag variety. This explicit calculation requires determining singular cohomology of all fibers; namely, if $\mu : Z \to X_w$ is a small resolution, then

$$H^{i-n}(\mathbf{IC}^\bullet_y) \cong H^i(\mu^{-1}(y)), \tag{1.6}$$

and $\mathrm{KL}_{y,w}$ is related to the Poincare polynomial of the fiber of μ over y.

Intersection cohomology gives rise to characteristic cycle multiplicities $m_{y,w} \in \mathbf{Z}$ (for $y \leq w$) important for studying Verma modules (as in, e.g., [6]). Bressler-Finkelberg-Lunts [8] used Zelevinskiĭ's small resolutions to compute the characteristic cycle of every Schubert variety in a grassmannian flag variety. This explicit calculation requires computing codifferentials of a morphism $\mu : Z \to X$ of smooth manifolds, where $X_w \subseteq X$ is inclusion of a Schubert variety in a flag variety. This leads to the conclusion that certain multiplicities $m_{y,w}$ are zero. In particular, [8] concludes that in this case all characteristic cycles are trivial.

Intersection cohomology is also important for studying infinite dimensional representations of real reductive groups. However, an added complication compared to Schubert varieties is keeping track of nontrivial local systems on K-orbits. In particular, [18, 28] defines Kazhdan-Lusztig-Vogan polynomials $\mathrm{KLV}_{\gamma,\delta} \in \mathbf{Z}[q]$, where γ, δ are local systems on K-orbits. Similar to the case of $\mathrm{KL}_{y,w}$, the polynomials $\mathrm{KLV}_{\gamma,\delta}$ compute characters of irreducible representations in terms of Langlands' parameters (cf. [28]).

1.4 Description of thesis

Chapter 2 gives a careful construction of a family of algebraic varieties via quotients of actions by algebraic groups. We apply this quotient to K-orbits and Schubert varieties to obtain our main construction Z. There is a natural morphism μ from Z to a flag variety, which defines an action of the monoid $(W, \star)$ on the set of K-orbits in the full flag variety. The monoid action describes the image of μ. We conclude this chapter by considering when μ is a resolution of singularities of a K-orbit closure in a flag variety.

Chapter 3 considers resolutions of type μ for the group $Sp(2n, \mathbf{R})$ corresponding to generalized grassmannian flag varieties. Some linear algebra is recalled to describe the required symplectic geometry. We describe all K-orbits and closure relations in terms of three integers (a, b, c). A resolution is constructed uniformly for all K-orbit closures, and multiple families of small resolutions of type μ are described. We conclude with some examples.

Chapter 4 defines *inducing* a small resolution that is equivariant under a parabolic subgroup of a reductive group G, to a G-equivariant small resolution of certain G-spaces. The combinatorics of clans enables a simple description of corresponding fiber bundle structures, and allows us to induce small resolutions for $Sp(2n, \mathbf{R})$ and $U(p, q)$.

Chapter 5 considers the group $U(p,q)$ in detail. We show that generically finite morphisms of type μ are in fact birational in this case. All resolutions of type μ for $U(p,q)$ have a simplified description, leading to a fiber dimension formula. Low rank examples indicate that there are many small resolutions for this group, and we conclude with a family of small resolutions.

Chapter 6 describes a simplified form of μ in the case where there is a single Schubert variety, which we call $m = 1$. Then fibers of μ admit a simple description as an intersection of two subvarieties of a flag variety. This provides enough information on the fiber to compute all $\mathrm{KLV}_{\gamma,\delta}$ for our family of small resolutions in $U(p,q)$. All small resolutions we found for the group $Sp(6,\mathbf{R})$ can be described by the case $m = 1$. An example is provided to show that general Schubert varieties are required in the definition of μ to find certain small resolutions.

1.5 Real forms and K

We recall without proof two results in the literature (qq.v., Theorem 1.5.1 and Theorem 1.5.8) to provide background for our results and notation.

Let G be a complex connected reductive algebraic group. A *real form* of G is an antiholomorphic Lie group automorphism τ of order 2. We call the fixed point subgroup $G_{\mathbf{R}} = G^\tau$ a *real reductive algebraic group.* A *compact real form* is a real form σ such that G^σ is a maximal compact subgroup of G. The following statement can be found in, e.g., [2, Theorem 3.4].

Theorem 1.5.1 (Cartan)

1. *If τ is a real form of G, then there exists a compact real form σ of G such that $\sigma \circ \tau = \tau \circ \sigma$. The real form σ is unique up to conjugation by $G_{\mathbf{R}}$. The composition $\theta = \sigma \circ \tau$ is an algebraic involution of G, called a* Cartan involution *for τ.*

2. *If θ is an algebraic involution of G, then there exists a compact real form σ of G*

such that $\sigma \circ \theta = \theta \circ \sigma$. The real form σ is unique up to conjugation by $K = G^{\theta}$. The composition $\tau = \sigma \circ \theta$ is a real form of G.

3. *The group $K_{\mathbf{R}} = G_{\mathbf{R}}^{\theta}$ is maximally compact in $G_{\mathbf{R}}$. Its complexification is the reductive algebraic group K.*

The diagram

$$\begin{tikzcd}
& G \arrow[dl, dash, "\tau"'] \arrow[dr, dash, "\theta"] & \\
G_{\mathbf{R}} \arrow[dr, dash, "\theta"'] & & K \arrow[dl, dash, "\tau"] \\
& K_{\mathbf{R}} &
\end{tikzcd}$$

shows containment of subgroups.

Example 1.5.2 *Let τ be a compact real form. Then $\tau = \sigma$, $\theta = \mathrm{id}$, and $G = K$.*

Example 1.5.3 *Let σ_1 be a compact real form for G, so*

$$\sigma(g_1, g_2) = (\sigma_1(g_1), \sigma_1(g_2)) \tag{1.7}$$

is a compact real form of $G \times G$. Define

$$\tau(g_1, g_2) = (\sigma_1(g_2), \sigma_1(g_1)), \tag{1.8}$$

a real form for G such that $\sigma \circ \tau = \tau \circ \sigma$. Then

$$\theta(g_1, g_2) = (g_2, g_1) \tag{1.9}$$

is the corresponding Cartan involution of $G \times G$, $G_{\mathbf{R}} = \{(g, \sigma_1(g)) \mid g \in G\} \cong G$, and $K = \Delta G$ is the diagonal copy of G.

Example 1.5.4 *Let $G = GL(n, \mathbf{C})$. Let*

$$\sigma(g) = {}^t\bar{g}^{-1}, \tag{1.10a}$$

$$\tau(g) = \bar{g}, \tag{1.10b}$$

$$\theta(g) = {}^t g^{-1}, \tag{1.10c}$$

so we have $G_{\mathbf{R}} = GL(n, \mathbf{R})$, $K = O(n, \mathbf{C})$, and $K_{\mathbf{R}} = O(n, \mathbf{R})$.

Example 1.5.5 *Let $G = GL(n, \mathbf{C})$. Let k, p, q be positive integers such that $p + q = n$ and set*

$$I_k = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \qquad I_{p,q} = \begin{bmatrix} I_p & \\ & -I_q \end{bmatrix}, \tag{1.11}$$

where I_k is a $k \times k$ matrix and $I_{p,q}$ is an $n \times n$ matrix. Let

$$\sigma(g) = {}^t\bar{g}^{-1}, \tag{1.12a}$$

$$\tau(g) = I_{p,q}\ {}^t\bar{g}^{-1}\, I_{p,q}, \tag{1.12b}$$

$$\theta(g) = I_{p,q}\, g\, I_{p,q}, \tag{1.12c}$$

so we have $G_{\mathbf{R}} = U(p,q)$, $K = GL(p, \mathbf{C}) \times GL(q, \mathbf{C})$, and $K_{\mathbf{R}} = U(p) \times U(q)$.

Example 1.5.6 *Let $G = GL(2n, \mathbf{C})$. Let*

$$S = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \quad J = \begin{bmatrix} & S \\ -S & \end{bmatrix}, \tag{1.13}$$

where S is an $n \times n$ matrix and J is a $2n \times 2n$ matrix. One can replace S with any real nonsingular symmetric matrix to obtain the same construction up to isomorphism; in particular, $S = I$ is often used. Let

$$\sigma(g) = {}^t\bar{g}^{-1}, \tag{1.14a}$$

$$\tau(g) = -J\, \bar{g}\, J, \tag{1.14b}$$

$$\theta(g) = -J\ {}^tg^{-1}\, J, \tag{1.14c}$$

so we have $G_{\mathbf{R}} = GL(n, \mathbf{H})$, $K = Sp(2n, \mathbf{C})$, and $K_{\mathbf{R}} = Sp(n)$.

Example 1.5.7 *Let $G = Sp(2n, \mathbf{C})$ (q.v. Example 1.5.6). Let*

$$\sigma(g) = {}^t\bar{g}^{-1}, \tag{1.15a}$$

$$\tau(g) = I_{n,n}\ {}^t\bar{g}^{-1}\, I_{n,n}, \tag{1.15b}$$

$$\theta(g) = I_{n,n}\, g\, I_{n,n}, \tag{1.15c}$$

so we have $G_{\mathbf{R}} = Sp(2n, \mathbf{R})$, $K = GL(n, \mathbf{C})$, *and* $K_{\mathbf{R}} = U(n)$. *Explicitly,*

$$G^{\theta} = \left\{ \begin{pmatrix} a & 0 \\ 0 & S\ {}^{t}a^{-1}\, S \end{pmatrix} \mid a \in GL(n, \mathbf{C}) \right\}. \tag{1.16}$$

The following fact is well-known and called *Matsuki duality*.

Theorem 1.5.8 ([29, 19]) *$G_{\mathbf{R}}$ and K act on the flag variety of G with finitely many orbits. Orbits of $G_{\mathbf{R}}$ correspond bijectively to orbits of K. The correspondence is a decreasing function on posets defined by closure relations. Furthermore, an orbit of $G_{\mathbf{R}}$ corresponds to an orbit of K precisely when the intersection of orbits is a single $K_{\mathbf{R}}$-orbit.*

CHAPTER II

THE MAIN CONSTRUCTION

2.1 Quotients

The notion of quotient is prominent throughout topology and geometry, and can be quite subtle in algebraic geometry. When a group acts on an algebraic variety, we can quotient the underlying topological space but the natural ringed space structure on the quotient space is not always again an algebraic variety. When the ringed space is an algebraic variety, the quotient is said to *exist*.

Let X be an algebraic variety over k an algebraically closed field of characteristic zero and let H be a linear algebraic group over k. Suppose that X is a H-variety with a right action. Let X/H be the quotient space with the quotient topology, let $\rho: X \to X/H$ be the quotient map, and for any open subset $U \subseteq X/H$ let

$$\mathcal{O}_{X/H}(U) = \left\{ f: U \to k \middle| f \circ \rho|\rho^{-1}(U) \in \mathcal{O}_X(\rho^{-1}(U)) \right\}. \tag{2.1}$$

Thus, $\mathcal{O}_{X/H}(U)$ may be identified with the ring of invariant functions $\mathcal{O}_X(\rho^{-1}(U))^H$ on $\rho^{-1}(U)$. Then X/H is a ringed space but may fail to be an algebraic variety.

Example 2.1.1 *We provide a typical example of such a quotient, and describe the topological space as a fiber bundle. If X is a linear algebraic group and H is a closed subgroup, then X/H is an algebraic variety by, e.g., [26, Theorem 5.5.5]. Let $X = G$ be a connected reductive algebraic group and let $H = R$ be a parabolic subgroup.*

The morphism $\pi: G \to G/R$ is a fiber bundle with fiber R. To see this, let $T \subseteq R$ be a maximal torus, let R^- be the opposite parabolic subgroup, and let U^- be the unipotent *radical. Let $U^- \times R$ act on G by*

$$(u,r)g = ugr^{-1} \tag{2.2}$$

and let $\iota : U^- \times R \to G$ be the orbit map through the identity $e \in G$. The stabilizer of $e \in G$ is given by $U^- \cap R = 1$ so ι is injective. The derivative

$$d\iota_{(e,e)}(x,y) = x - y \tag{2.3}$$

is surjective so ι is dominant by [26, Theorem 4.3.6 (i)]. It follows that ι is an open embedding by [26, Lemma 2.3.3 (i)] and [26, Theorem 5.3.2]. Identify U^- with the image of $\pi \circ \iota$ in G/R, so U^- is open since π is an open map. Define $\sigma : U^- \to G$ by $\sigma(u) = \iota(u,1)$. Then σ is a local section of π, and translating σ by elements of G shows that π has local sections at every point. For $g \in G$, we have $\pi^{-1}(gU^-) \cong gU^- \times R$, so π is a fiber bundle as claimed.

2.1.2 *The definition of $\mathcal{O}_{X/H}$ gives a sheaf of functions on X/H.*

Proof. For each non-empty open subset U of X/H, $\mathcal{O}_{X/H}(U)$ is a k-algebra of k-valued functions. So it remains to check that

(A) If U_i and U_j are non-empty open subsets and $U_i \subseteq U_j$, restriction defines a k-algebra hom $\mathcal{O}_{X/H}(U_j) \to \mathcal{O}_{X/H}(U_i)$.

(B) Let $\{U_i\}_{i \in I}$ be an open covering of the open set U. Suppose that for each $i \in I$ we are given $f_i \in \mathcal{O}_{X/H}(U_i)$ such that if $U_i \cap U_j$ is non-empty, f_i and f_j restrict to the same element of $\mathcal{O}_{X/H}(U_i \cap U_j)$. Then there is $f \in \mathcal{O}_{X/H}(U)$ such that for every $i \in I$, the function f restricts to f_i on U_i.

Let $U_i \subseteq U_j$ and let f be in $\mathcal{O}_{X/H}(U_j)$. Then $f \circ \rho|\rho^{-1}(U_j)$ is in $\mathcal{O}_X(\rho^{-1}(U_j))$. Hence $f \circ \rho|\rho^{-1}(U_i)$ is in $\mathcal{O}_X(\rho^{-1}(U_i))$ since $\mathcal{O}_X$ is a sheaf. It follows that $f|U_i$ is in $\mathcal{O}_{X/H}(U_i)$, so (A) is satisfied.

Let $\{U_i\}_{i\in I}$ be an open cover of U and let $\{f_i\}_{i\in I}$ satisfy the assumption from (B). So there is a function $f: U \to k$ extending $\{f_i\}_{i\in I}$, and it remains to show that f is in $\mathcal{O}_{X/H}(U)$. By definition of $\mathcal{O}_{X/H}$ we have $f_i \circ \rho|\rho^{-1}(U_i)$ in $\mathcal{O}_X(\rho^{-1}(U_i))$ such that

$$f_i \circ \rho|\rho^{-1}(U_i) = f_j \circ \rho|\rho^{-1}(U_j)$$

since $f_i|U_i = f_j|U_j$. Hence there exists F in $\mathcal{O}_X(\rho^{-1}(U))$ such that $F|\rho^{-1}(U_i) = f_i \circ \rho|\rho^{-1}(U_i)$ since $\mathcal{O}_X$ is a sheaf. It follows that $f \circ \rho|\rho^{-1}(U) = F$ is in $\mathcal{O}_X(\rho^{-1}(U))$, so f is in $\mathcal{O}_{X/H}(U)$ and hence (B) is satisfied. ∎

2.1.3 *If $\xi: X \to Y$ is a morphism of ringed spaces such that ξ is constant on every H-orbit, then there exists a unique morphism $\zeta: X/H \to Y$ making the diagram*

$$\begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\rho"'] & & Y \\ & X/H \arrow[ur, "\zeta"'] & \end{tikzcd}$$

commute.

Proof. The map ζ on underlying topological spaces exists and is unique since X/H is given the quotient topology. To show that ζ is a morphism of ringed spaces, let U be open in Y and let f be in $\mathcal{O}_Y(U)$. The hom $\mathcal{O}_Y(U) \to \mathcal{O}_X(\xi^{-1}(U))$ by $f \mapsto f \circ \xi|\xi^{-1}(U)$ satisfies

$$(f \circ \xi|\xi^{-1}(U))(xg) = f(\xi(xg)) = f(\xi(x)) \tag{2.4}$$

by assumption on ξ. Hence there is a hom $\mathcal{O}_Y(U) \to \mathcal{O}_X(\xi^{-1}(U))^H$ by (2.4). The formula

$$f \circ \xi|\xi^{-1}(U) = (f \circ \zeta \circ \rho)|\rho^{-1}(\zeta^{-1}(U)) = (f \circ \zeta|\zeta^{-1}(U)) \circ \rho|\rho^{-1}(\zeta^{-1}(U)) \tag{2.5}$$

shows that $f \mapsto f \circ \zeta|\zeta^{-1}(U)$ is a hom $\mathcal{O}_Y(U) \to \mathcal{O}_{X/H}(\zeta^{-1}(U))$ by definition of $\mathcal{O}_{X/H}(\zeta^{-1}(U))$. ∎

2.1.4 *If $\zeta: Z \to X$ is a fiber bundle with fiber Y and $\xi: \widetilde{X} \to X$ is a morphism, then base change $\zeta': \widetilde{X} \underset{X}{\times} Z \to \widetilde{X}$ of ζ along ξ is a fiber bundle with fiber Y.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X such that for every $i \in I$, we have

$$\alpha_i : \zeta^{-1}(U_i) \cong U_i \times Y \tag{2.6}$$

such that $\zeta_{U_i} : \zeta^{-1}(U_i) \to U_i$ commutes with projection $\mathrm{pr}_{U_i} : U_i \times Y \to U_i$. For every $i \in I$, let $\widetilde{U}_i = \xi^{-1}(U_i)$ so $\{\widetilde{U}_i\}_{i\in I}$ is an open cover of $\widetilde{X}$. We claim that $\widetilde{U}_i$ are trivializable for ζ'.

For every $i \in I$, define $\beta_i : (\zeta')^{-1}(\widetilde{U}_i) \to \widetilde{U}_i \times Y$ by

$$\beta_i(\tilde{x}, z) = (\tilde{x}, \mathrm{pr}_2(\alpha_i(z))) \tag{2.7}$$

and define $\gamma_i : \widetilde{U}_i \times Y \to (\zeta')^{-1}(\widetilde{U}_i)$ by

$$\gamma_i(\tilde{x}, y) = (\tilde{x}, \alpha_i^{-1}(\xi(\tilde{x}), y)). \tag{2.8}$$

Then β_i and γ_i are well-defined algebraic morphisms which are inverse to each other. The claim follows. ∎

We provide the following extension of [26, Lemma 5.5.8]. Recall that R is a parabolic subgroup of a connected reductive group G. Let $\pi : G \to G/R$, $X \subseteq G/R$ a subvariety, and $\widetilde{X} = \pi^{-1}(X) \subseteq G$. Suppose that Y is a variety with a left action of R. Let R act on the right of $\widetilde{X} \times Y$ by

$$(g, y)r = (gr, r^{-1}y) \tag{2.9}$$

and set $\widetilde{X} \times^R Y = (\widetilde{X} \times Y)/R$. Let $\rho : \widetilde{X} \times Y \to \widetilde{X} \times^R Y$ be the quotient morphism. If $g \in \widetilde{X}$ and $y \in Y$ then we denote a point in the quotient by $[g, y] = \rho(g, y)$. Let $\mathrm{pr}_1 : \widetilde{X} \times Y \to \widetilde{X}$ and $\pi_X : \widetilde{X} \to X$ be the obvious projections, and let $\xi = \pi_X \circ \mathrm{pr}_1$. Then ξ is constant on R-orbits so $\zeta : \widetilde{X} \times^R Y \to X$ is well-defined by §2.1.3.

Lemma 2.1.5 *Let R be a parabolic subgroup of a connected reductive group G and $\pi : G \to G/R$. Let $X \subseteq G/R$ a subvariety, $\widetilde{X} = \pi^{-1}(G)$, and $Z = \widetilde{X} \times^R Y$ as above. The quotient Z exists as an algebraic variety. The morphism $\zeta : Z \to Y$ is a fiber bundle with fiber Y.*

Proof. Consider an open subset $U \subseteq Z$, and let $W = \rho^{-1}(U) \subseteq \widetilde{X} \times Y$. Let $\xi_W : W \to U$ by restricting ρ and $\rho_W : W \to W/R$ by quotienting the R-action on W, so $\zeta_W : W/R \to U$ exists by §2.1.3. We show that ζ_W is an isomorphism (of ringed spaces) by showing their sheaves of functions are isomorphic. Let $V \subseteq U$ be an open subset so

$$\begin{aligned}\mathcal{O}_U(V) &\cong \mathcal{O}_{\widetilde{X}\times^R Y}(V)\\ &\cong \mathcal{O}_{\widetilde{X}\times Y}(\rho^{-1}(V))^R\\ &\cong \mathcal{O}_W(\rho^{-1}(V))^R\\ &\cong \mathcal{O}_{W/R}(V)\end{aligned} \tag{2.10}$$

by the last comment in [26, §1.4.2] and (2.1).

To show Z is a prevariety, we need to find an open cover such that $\mathcal{O}_Z$ restricts to an algebraic variety on each open set; we do this by applying (2.10) to trivializable open subsets. Let $\{U_i\}_{i\in I}$ be an open cover of X such that $\widetilde{U}_i = \pi^{-1}(U_i) \cong U_i \times R$ are R-equivariant trivializations in G (cf. Example 2.1.1 and §2.1.4). Then $\left\{\widetilde{U}_i \times Y\right\}_{i\in I}$ is an open cover of $\widetilde{X} \times Y$. It follows that $\left\{\rho(\widetilde{U}_i \times Y)\right\}_{i\in I}$ is an open cover of Z since ρ is an open map. We have

$$\rho(\widetilde{U}_i \times Y) \cong \widetilde{U}_i \times^R Y \cong U_i \times Y \tag{2.11}$$

by (2.10) and the R-equivariant trivializations of $\widetilde{U}_i$. Therefore Z is a prevariety.

The prevariety Z is a variety means the topological space is separated. We show that the morphism $\zeta : Z \to X$ is separated, and it will follow from [14, Corollary 4.6 (b)] that Z is separated since X is separated. Recall that the fibered product (as a topological space) is given by

$$Z \underset{X}{\times} Z = \{(z_1, z_2) \mid \zeta(z_1) = \zeta(z_2)\} \tag{2.12}$$

and the diagonal morphism $\delta_\zeta : Z \to Z \underset{X}{\times} Z$ is given by $\delta_\zeta(z) = (z, z)$. It remains to show that the image of δ_ζ is closed by [14, Corollary 4.2]. Let W_1 and W_2 be open subsets of Z such that $W_i \cong U_i \times Y$ as in (2.11). Then

$$\begin{aligned} W_1 \underset{X}{\times} W_2 &= \{(g_1R/R, y_1, g_2R/R, y_2) \mid g_1R = g_2R\} \\ &= \{(gR/R, y_1, gR/R, y_2)\} \end{aligned} \tag{2.13}$$

in $U_1 \times Y \times U_2 \times Y$. Hence

$$\operatorname{Im}(\delta_\zeta) \cap (W_1 \underset{X}{\times} W_2) = \{(gR/R, y, gR/R, y)\} \tag{2.14}$$

is closed in $W_1 \underset{X}{\times} W_2$ since $(U_1 \cap U_2) \times Y$ is closed in $(U_1 \cap U_2) \times Y \times Y$ under the diagonal closed embedding $Y \to Y \times Y$ (recall that Y is a variety so is separated). Thus $\operatorname{Im}(\delta_\zeta)$ is closed in $Z \underset{X}{\times} Z$ since W_i cover Z. The claim follows. ■

2.2 K-orbits and Schubert varieties

Our main construction uses closures of K-orbits and $m \geq 0$ many Schubert varieties to provide quotients as in Lemma 2.1.5. We note that similar ideas have appeared before, e.g., in Richardson-Springer [23] and it is common to use Bott-Samelson type varieties (q.v. (2.19)) for resolutions. The resulting varieties will take advantage of local sections from Example 2.1.1 to obtain iterated fiber bundles, since local sections gave us a fiber bundle structure in Lemma 2.1.5.

From now on, let G be a connected reductive algebraic group and fix a Borel subgroup B. Let $X = G/B$ be the flag variety of G, let θ be an involutive algebraic automorphism of G, and let G^θ be the fixed point subgroup of θ. Let K be any subgroup of finite index in G^θ. It is well-known that there are finitely many K-orbits and B-orbits on X, and we denote $V = K\backslash G/B$ and $W = B\backslash G/B$. Then

$$X = \coprod_{v \in V} K\dot{v}B/B = \coprod_{w \in W} B\dot{w}B/B, \tag{2.15}$$

where $\dot{v}$ is a representative of the corresponding $(K \times B)$-orbit in G and similarly for $\dot{w}$. We may identify W with the Weyl group of G by Bruhat's lemma (cf. [26, Theorem 8.3.8]).

Let S be the set of simple reflections of W with respect to B. We identify a standard parabolic subgroup $B \subseteq P \subseteq G$ with subsets of simple reflections $\varnothing \subseteq I \subseteq S$ such that for every $s \in I$, P contains $\dot{s}$ in G. In particular, P has semisimple rank $\#I$ and the subgroup W_I of W generated by I may be identified with the Weyl group of a Levi subgroup of P. We write $X^I = G/P$ for the flag variety of G corresponding to I. There are finitely many K-orbits and B-orbits

$$X^I = \coprod_{v \in V^I} K\dot{v}P/P = \coprod_{w \in W^I} B\dot{w}P/P, \tag{2.16}$$

where $V^I \subseteq V$ and $W^I \subseteq W$ by letting $K\dot{v}B/B$ be the unique open orbit in $K\dot{v}P/B$ and similarly for $B\dot{w}B/B$. Then W^I may be identified with the maximum length representative of elements of W/W_I.

Given $v \in V$ and $w \in W$, let

$$G_v = \overline{K\dot{v}B}, \qquad G_w = \overline{B\dot{w}B}, \tag{2.17a}$$

$$X_v = \overline{K\dot{v}B/B}, \qquad X_w = \overline{B\dot{w}B/B}, \tag{2.17b}$$

$$X_v^I = \overline{K\dot{v}P/P}, \qquad X_w^I = \overline{B\dot{w}P/P}, \tag{2.17c}$$

in G, X, and X^I. For any $v \in V$ observe that B stabilizes G_v by right multiplication, and for any $w \in W$, B stabilizes G_w by both left and right multiplication.

Let $v_0 \in V$ and $w_1, \ldots, w_m \in W$. Let R_1 be a standard parabolic subgroup stabilizing G_{v_0} by right multiplication and G_{w_1} by left multiplication, and for every $2 \le i \le m$, let R_i be a standard parabolic subgroup stabilizing $G_{w_{i-1}}$ by right multiplication and G_{w_i} by left multiplication. Then $R_1 \times \cdots \times R_m$ acts on the right of $G_{v_0} \times G_{w_1} \times \cdots \times G_{w_m}$ by

$$(g_0, g_1, \ldots, g_m)(r_1, \ldots, r_m) = (g_0 r_1, r_1^{-1} g_1 r_2, \ldots, r_m^{-1} g_m) \tag{2.18}$$

and we denote the quotient by

$$G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m} = (G_{v_0} \times G_{w_1} \times \cdots \times G_{w_m})/(R_1 \times \cdots \times R_m). \tag{2.19}$$

We call this quotient a *Bott-Samelson variety.*

Remark 2.2.1 *We may replace V by W with no change. The constructions give quotients, fiber bundles, and resolutions with a left B-action instead of left K-action. This leads to resolutions of Schubert varieties as in, e.g., [10, 13, 11, 22].*

Recall for $Z \subseteq G/R$, we write $\widetilde{Z}$ for $\pi^{-1}(Z) \subseteq G$.

Lemma 2.2.2 *Let R_1 and R_2 be any standard parabolic subgroups, $X_0 \subseteq G/R_1$, let $X_1 \subseteq G/R_2$ such that R_1 stabilizes $\widetilde{X}_1$ by left multiplication, and let Y be a variety with a left action of R_2. Then*

$$\left(\widetilde{X}_0 \times \widetilde{X}_1 \times Y\right)/(R_1 \times R_2) \cong \widetilde{X}_0 \times^{R_1} \left(\widetilde{X}_1 \times^{R_2} Y\right). \tag{2.20}$$

Proof. Define

$$\rho_2 : \widetilde{X}_0 \times \widetilde{X}_1 \times Y \to \widetilde{X}_0 \times \left(\widetilde{X}_1 \times^{R_2} Y\right) \tag{2.21}$$

by quotienting by R_2. Then R_1 acts on the left of $\widetilde{X}_1 \times^{R_2} Y$ by assumption. Define

$$\rho_1 : \widetilde{X}_0 \times \left(\widetilde{X}_1 \times^{R_2} Y\right) \to \widetilde{X}_0 \times^{R_1} \left(\widetilde{X}_1 \times^{R_2} Y\right) \tag{2.22}$$

by quotienting by R_1. Let $\xi = \rho_1 \circ \rho_2$ be the morphism of varieties constant on $(R_1 \times R_2)$-orbits. By §2.1.3, there is a well-defined morphism of ringed spaces

$$\zeta : \left(\widetilde{X}_0 \times \widetilde{X}_1 \times Y\right)/(R_1 \times R_2) \to \widetilde{X}_0 \times^{R_1} \left(\widetilde{X}_1 \times^{R_2} Y\right). \tag{2.23}$$

The inverse morphism of ζ is constructed by considering the quotient

$$\widetilde{X}_0 \times \widetilde{X}_1 \times Y \to \left(\widetilde{X}_0 \times \widetilde{X}_1 \times Y\right)/(R_1 \times R_2) \tag{2.24}$$

constant on R_2-orbits, and then the quotient

$$\widetilde{X}_0 \times \left(\widetilde{X}_1 \times^{R_2} Y\right) \to \left(\widetilde{X}_0 \times \widetilde{X}_1 \times Y\right)/(R_1 \times R_2) \tag{2.25}$$

constant on R_1-orbits. ∎

Proposition 2.2.3 $G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}$ *(q.v. (2.19)) exists as an algebraic variety.*

Proof. Let $Z = G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}$ and $Y = G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}$ as in (2.19). Then

$$Y \cong G_{w_1} \times^{R_2} \left(G_{w_2} \times^{R_3} \cdots \times^{R_m} G_{w_m}\right) \tag{2.26}$$

and

$$Z \cong G_{v_0} \times^{R_1} Y \tag{2.27}$$

by Lemma 2.2.2. It follows that Y and Z exist by Lemma 2.1.5. ■

Corollary 2.2.4 *Let $v_0, w_1, \ldots, w_m$ and $R_1, \ldots, R_m$ as in* (2.19)*. Let R be a standard parabolic subgroup stabilizing G_{w_m} by right multiplication. The variety*

$$Z = G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/R \tag{2.28}$$

is an iterated fiber bundle.

Proof. This follows immediately from Lemma 2.1.5 and Lemma 2.2.2. In particular, if $U_0 \subseteq G_{v_0}/R_1$, and for $1 \le i < m$ if $U_i \subseteq G_{w_i}/R_{i+1}$, if $U_m \subseteq G_{w_m}/R$ are trivializable open subsets, then

$$U_0 \times \cdots \times U_m \subseteq Z \tag{2.29}$$

is an open embedding. ■

Remark 2.2.5 *The variety Z is smooth if and only if X_{v_0} is smooth and for every $1 \le i \le m$, X_{w_i} is smooth. This is a direct consequence of* (2.29) *along with a fiber bundle structure on $X_v \to X_v^J$, for any $v \in V$ such that $G_v P_J = G_v$, with fiber P_J/B.*

2.3 A monoid action

Let $B \subseteq R \subseteq P \subseteq G$ be parabolic subgroups corresponding to subsets of simple reflections $\varnothing \subseteq J \subseteq I \subseteq S$. We define a proper algebraic morphism μ from

$$Z^J = G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/R \tag{2.30}$$

to the flag variety X^I.

Definition 2.3.1 *Define* $\mu: Z^J \to X^I$ *by*

$$\mu[g_0, \ldots, g_m R/R] = g_0 \cdots g_m P/P. \tag{2.31}$$

Proposition 2.3.2 *The map* μ *defined by* (2.31) *is a proper algebraic morphism with image* X_v^I *for some* $v \in V$.

Proof. Consider the morphism

$$\tilde{\varphi}: G \times \cdots \times G \to G \times \cdots \times G \tag{2.32}$$

defined by

$$\tilde{\varphi}(g_0, \ldots, g_m) = (g_0, g_0 g_1, \ldots, g_0 g_1 \cdots g_m). \tag{2.33}$$

The inverse morphism $\tilde{\psi}$ is defined by

$$\tilde{\psi}(g_0, \ldots, g_m) = (g_0, g_0^{-1} g_1, \ldots, g_{m-1}^{-1} g_m). \tag{2.34}$$

Let $\widetilde{Y} \subseteq G \times \cdots \times G$ be the image of the subvariety

$$\tilde{\iota}: G_{v_0} \times G_{w_1} \times \cdots \times G_{w_m} \subseteq G \times \cdots \times G \tag{2.35}$$

under the morphism $\tilde{\varphi}$. The diagram

$$\begin{tikzcd} G_{v_0} \times G_{w_1} \times \cdots \times G_{w_m} \arrow[r, "\tilde{\iota}"] \arrow[d, "\rho"] & \widetilde{Y} \arrow[d, "\pi"] \\ G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/R \arrow[r, "\iota"] & Y \end{tikzcd} \tag{2.36}$$

commutes, where $Y \subseteq G/R_1 \times \cdots \times G/R_m \times G/R$. It follows that ι is an isomorphism since $\rho \circ \tilde{\iota}^{-1}$ factors through π. Therefore Z^J is a projective variety and μ is a morphism between projective varieties. It follows by [14, Theorem 4.9] that μ is proper since μ is projective.

The image of μ is a K-stable (since μ is K-equivariant) closed subvariety of X^I (since μ is proper) so is a union of K-orbit closures in X^I. If K is connected, then the image is a single K-orbit closure by standard irreducibility arguments, but if K is not connected, then we need to argue more carefully.

It is enough to show that given $v_0 \in V$ and $s \in S$, $G_{v_0} P_s/B = G_{v_1}/B$ for some $v_1 \in V$. Let $\pi : G/B \to G/P_s$ so

$$G_{v_0} P_s/B = \pi^{-1}(\pi(G_{v_0}/B)). \tag{2.37}$$

Vogan [28, Lemma 5.1] shows that there is a unique open K-orbit $v_1 \in V$ in (2.37) so the claim follows. In particular, for $v_0 \in V$ and $w \in W$, we have

$$G_{v_0} G_w = G_{v_0} P_{i_1} \cdots P_{i_\ell} = G_{v_1} P_{i_2} \cdots P_{i_\ell} = G_u, \tag{2.38}$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w. Therefore, the image of (2.31) is the closure of a single K-orbit. ■

Remark 2.3.3 *The element $v \in V$ from Proposition 2.3.2 is independent of choice of parabolics R_i and R, subject to* (2.19)*.*

Definition 2.3.4 *Given $v \in V$ and $w \in W$, define $v \star w = u \in V$, where the map*

$$\mu : G_v \times^R X_w \to X_u \tag{2.39}$$

is surjective.

Proposition 2.3.5 *If $V = W$, then $(W, \star)$ is a monoid. In general, the monoid $(W, \star)$ acts on the right of V.*

Proof. This follows directly from the definition since

$$(G_v G_{w_1}) G_{w_2} = G_v (G_{w_1} G_{w_2}) \tag{2.40}$$

by the associative property of multiplication in G. We also have

$$BG_w = G_w = G_w B, \tag{2.41a}$$

$$G_v B = G_v, \tag{2.41b}$$

so $B \subseteq G$ gives the identity element in the monoid. ■

Definition 2.3.6 *Define the function τ from V to subsets of S by*

$$\tau(v) = \{s \in S \mid v \star s = v\}, \tag{2.42}$$

called the τ-invariant *of v.*

Remark 2.3.7 *Let γ be a Langlands parameter with same infinitesimal character as a finite dimensional representation, which may be identified as a K-orbit with irreducible K-equivariant local system (cf. [28]). The* weak τ-invariant $\tau_w(\gamma)$, τ-invariant $\tau(\gamma)$, *and* strong τ-invariant $\tau_s(\gamma)$ *are defined in [28], such that*

$$\tau_s(\gamma) \subseteq \tau(\gamma) \subseteq \tau_w(\gamma). \tag{2.43}$$

By definition,

$$\tau_w(\gamma) = \left\{s \in S \mid \pi^{-1}(\pi(\dot{v}B/B)) \cap K\dot{v}B/B \text{ is infinite}\right\}, \tag{2.44}$$

where v is the underlying K-orbit attached to γ, P is the standard parabolic subgroup corresponding to the simple reflection s, and $\pi : G/B \to G/P$. We have

$$\pi^{-1}(\pi(\dot{v}B/B)) \cap K\dot{v}B/B = \dot{v}P/B \cap K\dot{v}B/B, \tag{2.45}$$

which is the fiber of

$$K\dot{v}B \times^B P/B \to X \tag{2.46}$$

over $\dot{v}B/B$ (q.v. (6.1)*). It follows by counting dimensions that*

$$\tau(v) = \tau_w(\gamma), \tag{2.47}$$

so we are defining the τ-invariant of v as the weak τ-invariant of the underlying K-orbit.

2.4 Resolutions of singularities

For $1 \leq i \leq m$, let R_i be the standard parabolic subgroup corresponding to the subset of simple reflections J_i, and let R be the standard parabolic subgroup corresponding to the subset of simple reflections J. Set

$$Z^J = G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/R \tag{2.48}$$

as in (2.19). For a subset of simple reflections $I \subseteq S$, we let $w_I = \max(W_I)$ the longest element of the corresponding Weyl group.

Proposition 2.4.1 *Let $v = v_0 \star w_1 \star \cdots \star w_m \star w_I$, where P is the standard parabolic subgroup corresponding to the subset of simple reflections I. The map*

$$\mu : Z^J \to X_v^I \tag{2.49}$$

is generically finite if and only if

$$\ell(v) = \ell(v_0) + \sum_{i=1}^{m} \ell(w_i) - \sum_{i=1}^{m} \ell(w_{J_i}) + \ell(w_I) - \ell(w_J), \tag{2.50}$$

where we denote $\ell(u) = \dim(X_u)$ for $u \in V \cup W$.

Proof. The dimension of Z^J is equal to

$$\dim(G_{v_0}/R_1) + \sum_{i=1}^{m-1} \dim(G_{w_i}/R_{i+1}) + \dim(G_{w_m}/R) \tag{2.51}$$

by Corollary 2.2.4. The claim follows immediately since the map

$$\pi : G_u/B \to G_u/R \tag{2.52}$$

is a fiber bundle whenever R stabilizes G_u by right multiplication. ∎

Remark 2.4.2 *If $v \in V$, the length is often defined to be the difference in the dimension of X_v and the dimension of a closed K-orbit, in which case* (2.50) *still holds.*

Definition 2.4.3 *Suppose that $\mu : Z^J \to Y$ as in* (2.49) *is a resolution of singularities (i.e., Z^J is smooth and μ is birational). Then μ is a* small resolution *means for every $r > 0$,*

$$\operatorname{codim}_Y(Y_r) > 2r, \tag{2.53}$$

where

$$Y_r = \left\{ y \in Y \mid \dim(\mu^{-1}(y)) \geq r \right\}, \tag{2.54}$$

and $\operatorname{codim}_Y(\varnothing) = \infty$.

Remark 2.4.4 *In our situation, we may describe a small resolution* μ *via combinatorics of* $v, y \in V$*, where* $Y = X^I_v$*. Namely,* (2.53) *is equivalent to the following statement: for every* $y \in V^I$ *such that* $y \leq v$*,*

$$2d_y < c_y, \tag{2.55}$$

where $d_y = \dim(\mu^{-1}(\dot{y}P/P))$ *and* $c_y = \operatorname{codim}_{X^I_v}(X^I_y)$*.*

Example 2.4.5 *The construction of resolutions in [5] takes on the form of* (2.31)*. Let* $G = GL(n, \mathbf{C})$*,* $G_{\mathbf{R}} = U(p, q)$*, and* $K = GL(p, \mathbf{C}) \times GL(q, \mathbf{C})$*. Given* $1 \leq k < n$*, Barbasch-Evens [5, §2.1] describes* K*-orbit closures in* X^I*, where* I *is the complement of* k *in* $S = \{1, \ldots, n-1\}$*, as follows. The set*

$$V^I_{p,q} = \left\{ (a, b) \in \mathbf{Z}^2_{\geq 0} \mid a \leq p,\ b \leq q,\ a + b \leq k \right\}, \tag{2.56}$$

parameterizes K*-orbits on* X^I*. We have* $(a', b') \leq (a, b)$ *if and only if* $a' \geq a$ *and* $b' \geq b$*. The* K*-orbit* $\mathcal{Q}^{\hat{k}}_{a,b}$ *is closed if and only if* $a + b = k$*, and open if and only if* $a = 0 = b$*.*

Given $(a, b) \in V^I_{p,q}$*,*

$$X^I_{a,b} = \left\{ E^k \in \mathrm{Gr}_k(\mathbf{C}^n) \mid \dim(\mathbf{C}^p \cap E) \geq a,\ \dim(\mathbf{C}^{-q} \cap E) \geq b \right\} \tag{2.57}$$

is the corresponding orbit closure. Define J *to be the complement of* $a+b$ *in* I *and set*

$$Z^J = \{(F, E) \mid F \subseteq E,\ \dim(\mathbf{C}^p \cap F) = a,\ \dim(\mathbf{C}^q \cap F) = b\} \tag{2.58}$$

contained in $\mathrm{Gr}_{a+b}(\mathbf{C}^n) \times \mathrm{Gr}_k(\mathbf{C}^n)$*, as in [5, §2.4]. We have*

$$Z^J = G_{v_0}/R = X^J_{v_0}, \tag{2.59}$$

where $v_0 = (a,b) \in V^{I_0}$ *and* I_0 *is the complement of* $a+b$ *in* S*; i.e.,* Z^J *is the pull-back of a closed* K*-orbit in* X^{I_0}*. Define*

$$\mu : Z^J \to X^I_{a,b}, \qquad \mu(F,E) = E \tag{2.60}$$

as in (2.49)*. Then* μ *is a resolution of singularities (not just generically finite). If* $n-k \geq \max(k,p,q)$ *then [5, §5.3] shows that* μ *is a small resolution.*

Similarly, [5, §5.5] shows that every $X^{\hat{k}}_{a,b}$ *has a small resolution of the form* μ*, with* $m = 0$*.*

CHAPTER III

$Sp(2n, \mathbf{R})$

> "There is little in the papers of Harish-Chandra which is not as important for the symplectic group as in the general case." (V.S. Varadarajan)

3.1 Symplectic geometry

In this section, let $G' = GL(2n, \mathbf{C})$ and let B' be the upper triangular matrices. Define $\theta' : G' \to G'$ by (1.14c). Then $G = Sp(2n, \mathbf{C})$ is the fixed point subgroup of θ' such that $B = G \cap B'$ is a Borel subgroup since B' is θ'-stable. Define $\theta : G \to G$ by (1.15c), so $G_{\mathbf{R}} = Sp(2n, \mathbf{R})$, $K = GL(n, \mathbf{C})$, and $K_{\mathbf{R}} = U(n)$.

We sometimes write $V = V_n$ for the K-orbits in $X = G/B$ to emphasize the rank of G. The involution θ' is identified with the nondegenerate alternating bilinear form on $\mathbf{C}^{2n}$ by

$$\omega(x, y) = {}^t x \, J \, y, \tag{3.1}$$

where J is given by (1.13). Then $Sp(2n, \mathbf{C})$ is equal to the isometry group of ω. We recall some well-known basic facts about nondegenerate alternating forms which are used below.

Proposition 3.1.1 *[4, Theorem 3.5] Let E be a subspace of $\mathbf{C}^{2n}$ and let*

$$E^{\perp} = \left\{ x \in \mathbf{C}^{2n} \mid \forall y \in E, \ \omega(x, y) = 0 \right\}. \tag{3.2}$$

(i) $\mathrm{rad}(\omega|E) = E \cap E^{\perp}$,

(ii) $\dim(E^{\perp}) = 2n - \dim(E)$,

(iii) If $\omega|E$ is nondegenerate, then $\omega|E^\perp$ is nondegenerate and $\mathbf{C}^{2n} = E \oplus E^\perp$.

3.1.2

(i) If $E \subseteq F$, then $F^\perp \subseteq E^\perp$,

(ii) $E^\perp \cap F^\perp = (E+F)^\perp$,

(iii) if E is isotropic (i.e., $E \subseteq E^\perp$), then $\bar{\omega}(\bar{x}, \bar{y}) = \omega(x,y)$ defines a nondegenerate alternating bilinear form on $E^\perp/E$.

We provide another fact often used for computing closures of K-orbits.

3.1.3 *Define the function*

$$\mathrm{rank}\colon M_{m\times n}(\mathbf{C}) \to \mathbf{Z} \tag{3.3}$$

by assigning the rank to each matrix. Then $\overline{\mathrm{rank}^{-1}(r)} = \mathrm{rank}^{-1}([0,r])$.

Proof. Let $I \subseteq \{1,\dots,m\}$ and $J \subseteq \{1,\dots,n\}$ such that $\#I = k = \#J$. Let $d^k_{I,J}$ be the determinant of the $I \times J$ minor. The set $\mathrm{rank}^{-1}([0,r])$ is the vanishing locus of all $d^k_{I,J} = 0$, where $k > r$, since the rank is the same as the determinantal rank. Therefore, $\overline{\mathrm{rank}^{-1}(r)} \subseteq \mathrm{rank}^{-1}([0,r])$.

Let f be a regular function on $M_{m\times n}(\mathbf{C})$ vanishing at all matrices of rank r and let M be a matrix of rank $\ell \le r$. Let A be a row reduction matrix giving a matrix AM with $\mathrm{rank}(AM) = \ell$. It is clear that for E_i the matrix with entry 1 in position (i,i) and 0 in position (i,j) $(i \ne j)$, $N = \sum_{i=\ell+1}^{k} E_i$, and $t \ne 0$,

$$AM + tN \tag{3.4}$$

has rank r. Thus for every $t \ne 0$,

$$M + tA^{-1}N \tag{3.5}$$

has rank r. It follows that f vanishes on all matrices of this form – including $t = 0$ by continuity. Therefore, $\overline{\mathrm{rank}^{-1}(r)} = \mathrm{rank}^{-1}([0,r])$. ∎

3.2 Orbits

In this section, we describe K-orbits and closure relations for all maximal parabolic flag varieties. These orbits are labelled by three integers a, b, c (q.v. Proposition 3.2.4). Let P be a maximal parabolic subgroup of G containing B. If P corresponds to $\hat{k}$, then $G/P \cong \mathrm{Gr}_k^0(\mathbf{C}^{2n})$ identifies with ω-isotropic subspaces of $\mathbf{C}^{2n}$ of dimension k. Let $\mathbf{C}^{\pm n}$ be the ± 1 eigenspace of $I_{n,n}$.

Lemma 3.2.1 *Given an isotropic subspace $0 \subseteq E \subseteq \mathbf{C}^{2n}$ of ω, define $\varepsilon : E \times E \to \mathbf{C}$ by*

$$\varepsilon(x, y) = \omega(\Lambda^n(x), \Lambda^{-n}(y)), \tag{3.6}$$

where $\Lambda^{\pm n} : \mathbf{C}^{2n} \to \mathbf{C}^{\pm n}$ is projection. Then ε is a symmetric bilinear form on E.

Proof. For every x, y in E, we have $\omega(x, y) = 0$ since E is isotropic. Note $\mathbf{C}^n, \mathbf{C}^{-n}$ are also isotropic subspaces of ω. Hence

$$\begin{aligned} 0 &= \omega(x, y) \\ &= \omega(\Lambda^n(x) + \Lambda^{-n}(x), \Lambda^n(y) + \Lambda^{-n}(y)) \\ &= \omega(\Lambda^n(x), \Lambda^{-n}(y)) + \omega(\Lambda^{-n}(x), \Lambda^n(y)) \\ &= \omega(\Lambda^n(x), \Lambda^{-n}(y)) - \omega(\Lambda^n(y), \Lambda^{-n}(x)) \\ &= \varepsilon(x, y) - \varepsilon(y, x) \end{aligned} \tag{3.7}$$

shows ε is symmetric. ■

Given an isotropic subspace E of ω, define

$$a = \dim(\mathbf{C}^n \cap E), \tag{3.8a}$$

$$b = \dim(\mathbf{C}^{-n} \cap E), \tag{3.8b}$$

$$c = \dim(\mathrm{rad}(\varepsilon)) - a - b. \tag{3.8c}$$

Then a, b, c are nonnegative integers which are invariants of each K-orbit in G/P.

Lemma 3.2.2 *If E and F are transverse lagrangian subspaces of ω, then there exists a linear isomorphism $\gamma : E \to F$ sending a basis $(g_1, \ldots, g_n)$ of E to a basis $(f_1, \ldots, f_n)$ of F, where $f_i = \gamma(g_i)$, such that $(g_1, \ldots, g_n, f_1, \ldots, f_n)$ is a symplectic basis of $\mathbf{C}^{2n}$.*

Proof. We proceed by induction on n. Suppose $n = 1$. For every $x \in E$, there exists $y \in F$ such that $\omega(x, y) \neq 0$, since F is a maximal isotropic subspace of ω. Fix $g_1 \in E$ and let $f_1 \in F$ such that $\omega(g_1, f_1) = 1$. Define $\gamma : E \to F$ by linearly extending $\gamma g_1 = f_1$. Hence (g_1, f_1) is a symplectic basis of ω.

Suppose $n > 1$ and assume the lemma is true for every $1 \leq k < n$. Let E and F be transverse lagrangian subspaces of ω. Given $g_n \in E$, there exists $f_n \in F$ such that $\omega(g_n, f_n) = 1$ as above. In particular, ω restricts to nondegenerate forms on $\langle g_n, f_n \rangle$ and $\langle g_n, f_n \rangle^\perp$ by Proposition 3.1.1. Let $E' = E \cap \langle g_n, f_n \rangle^\perp$ and $F' = F \cap \langle g_n, f_n \rangle^\perp$. Then

$$\begin{aligned} \dim(E') &= \dim(E \cap \langle g_n, f_n \rangle^\perp) \\ &= \dim(E) + \dim(\langle g_n, f_n \rangle^\perp) - \dim(E + \langle g_n, f_n \rangle^\perp) \\ &= n + (2n - 2) - (2n - 1) \\ &= n - 1 \end{aligned} \tag{3.9}$$

where the inequality follows from Proposition 3.1.1 and $E + \langle g_n, f_n \rangle^\perp \subseteq \langle g_n \rangle^\perp$. Similarly, $\dim(F') = n - 1$. It follows that E' and F' are lagrangian in $\langle g_n, f_n \rangle^\perp$ of dimension $n - 1$. By induction, there exists a linear isomorphism $\gamma' : E' \to F'$ and a symplectic basis $(g_1, \ldots, g_{n-1}, f_1, \ldots, f_{n-1})$ of $E' + F'$. By construction, we have an orthogonal decomposition

$$\mathbf{C}^{2n} = \langle g_1, f_1 \rangle + \cdots + \langle g_{n-1}, f_{n-1} \rangle + \langle g_n, f_n \rangle \,. \tag{3.10}$$

Define $\gamma : E \to F$ by linearly extending γ' such that $\gamma g_n = f_n$. This gives the symplectic basis $(g_1, \ldots, g_n, f_1, \ldots, f_n)$ of ω. ∎

Lemma 3.2.3 *Let E^k be an isotropic subspace of ω of the form $E = (\mathbf{C}^n \cap E) + (\mathbf{C}^{-n} \cap E)$. Let $(g_1, \ldots, g_a, g_{a+1}, \ldots, g_{a+b})$ be a basis of E respecting the direct sum. Then this* *basis can be extended to a symplectic basis of* $\mathbf{C}^{2n} = \mathbf{C}^n + \mathbf{C}^{-n}$ *also respecting the direct sum.*

Proof. Define $\widetilde{E} = (\mathbf{C}^n \cap E^\perp) + (\mathbf{C}^{-n} \cap E)$. Note $\widetilde{E}$ is isotropic since the summands are isotropic subspaces that are perpendicular to each other. Then

$$\begin{aligned}\dim(\mathbf{C}^n \cap E^\perp) &= \dim((\mathbf{C}^n + E)^\perp) \\ &= 2n - (n+b) \\ &= n - b\end{aligned} \tag{3.11}$$

by e.g., Proposition 3.1.1 and §3.1.2. Hence $\dim(\widetilde{E}) = (n-b) + b = n$ shows $\widetilde{E}$ is a lagrangian subspace.

Let F be a complementary subspace of $\mathbf{C}^n \cap \widetilde{E}$ in $\mathbf{C}^n$ and define $\widetilde{F} = F + (\mathbf{C}^{-n} \cap F^\perp)$. Note $\widetilde{F}$ is isotropic by an above argument and is lagrangian. We have

$$(\mathbf{C}^{-n} \cap E) \cap (\mathbf{C}^{-n} \cap F^\perp) = 0 \tag{3.12}$$

since $\mathbf{C}^{-n} \cap E \cap F^\perp = (\mathbf{C}^{-n} + (E \cap F^\perp)^\perp)^\perp = (\mathbf{C}^{-n} + E^\perp + F)^\perp = (\mathbf{C}^{2n})^\perp$, where the last equality follows since F is complementary to $\mathbf{C}^n \cap \widetilde{E} = \mathbf{C}^n \cap E^\perp$ in $\mathbf{C}^n$. It follows that $\widetilde{E}$ and $\widetilde{F}$ are complementary lagrangian subspaces.

Let $U_1 = (\mathbf{C}^n \cap \widetilde{E}) + (\mathbf{C}^{-n} \cap \widetilde{F})$ and $U_2 = (\mathbf{C}^n \cap \widetilde{F}) + (\mathbf{C}^{-n} \cap \widetilde{E})$. Then $\mathbf{C}^{2n} = U_1 + U_2$ is an orthogonal decomposition so ω restricts to a nondegenerate form on U_1 and U_2 by Proposition 3.1.1. We apply Lemma 3.2.2 to each summand of the orthogonal decomposition

$$\mathbf{C}^{2n} = \big((\mathbf{C}^n \cap \widetilde{E}) + (\mathbf{C}^{-n} \cap \widetilde{F})\big) + \big((\mathbf{C}^n \cap \widetilde{F}) + (\mathbf{C}^{-n} \cap \widetilde{E})\big) \tag{3.13}$$

giving us our claim since $E \subseteq \widetilde{E}$. ■

Proposition 3.2.4 *The* K*-orbits in* $G/P_{\hat{k}}$ *consist of*

$$\mathcal{Q}^{\hat{k}}_{a,b,c} = \left\{E^k \mid \dim(\mathbf{C}^n \cap E) = a,\ \dim(\mathbf{C}^{-n} \cap E) = b,\ \dim(\mathrm{rad}(\varepsilon)) = a+b+c\right\}, \tag{3.14}$$

where a, b, c, ε are defined in Lemma 3.2.1. Moreover,

$$V_n^{\hat{k}} = \left\{(a,b,c) \in \mathbf{Z}_{\geq 0}^3 \mid a+b+c \leq k,\ k+c \leq n\right\} \tag{3.15}$$

parameterizes the nonempty K-orbits.

Proof. Let $E \in \mathcal{Q}_{a,b,c}^{\hat{k}}$. Then $\operatorname{rad}(\varepsilon) \subseteq E$ forces $a+b+c \leq k$. The subspace $E' = E + \Lambda^+(\operatorname{rad}(\varepsilon))$ is isotropic since any $x, y \in E'$ gives

$$\begin{aligned} \omega(x,y) &= \omega(x_1 + \Lambda^+(x_2), y_1 + \Lambda^+(y_2)) \\ &= \omega(\Lambda^-(x_1), \Lambda^+(y_2)) + \omega(\Lambda^+(x_2), \Lambda^-(y_1)) \\ &= 0, \end{aligned} \tag{3.16}$$

where $x_1, y_1 \in E$ and $x_2, y_2 \in \operatorname{rad}(\varepsilon)$. It follows that $k + c \leq n$. So $V_n^{\hat{k}}$ partitions the K-orbits, and we need to show that it parameterizes the K-orbits.

Let $a, b, c \in \mathbf{Z}_{\geq 0}$ such that $a+b+c \leq k$ and $k+c \leq n$. Let $(e_1, \ldots, e_{2n})$ be the standard basis of $\mathbf{C}^{2n}$, and for every $1 \leq i \leq 2n$, let $e_{-i} = e_{2n+1-i}$. Define

$$\begin{aligned} E_{a,b,c}^k = \langle & e_1, \ldots, e_a, e_{-n-1+1}, \ldots, e_{-n-1+b}, e_{a+1} + e_{-n-1+b+1}, \ldots, e_{a+c} + e_{-n-1+b+c}, \\ & e_{a+c+1} + e_{-(a+c+1)}, \ldots, e_{a+c+d} + e_{-(a+c+d)} \rangle, \end{aligned} \tag{3.17}$$

where $d = k - a - b - c$. It follows from

$$-n + b + c \leq -k + b = -(a + c + d) \tag{3.18}$$

that $E_{a,b,c}$ is isotropic. This shows that for every $(a,b,c) \in V_n^{\hat{k}}$, the set $\mathcal{Q}_{a,b,c}^{\hat{k}}$ defined by (3.14) is nonempty. It remains to show that $\mathcal{Q}_{a,b,c}^{\hat{k}}$ is a single K-orbit.

Let $E \in \mathcal{Q}_{a,b,c}^{\hat{k}}$. Then E decomposes into orthogonal one dimensional subspaces for the symmetric bilinear form ε. Let $(g_1, \ldots, g_k)$ be a basis of E such that for every $1 \leq i \leq a+b+c$ and $1 \leq j \leq k$, we have $\varepsilon(g_i, g_j) = 0$, and for every $a+b+c < i, j \leq k$, we have $\varepsilon(g_i, g_j) = \delta_{ij}$ (the Kronecker delta function). We can assume $(g_1, \ldots, g_{a+b})$ is a basis of $(\mathbf{C}^n \cap E) + (\mathbf{C}^{-n} \cap E)$ respecting the direct sum since the radical of ε contains each summand. For $a+b < i \leq k$, we have $g_i = g_i^+ + g_i^-$, where $g_i^\pm$ is nonzero in $\mathbf{C}^{\pm n}$.

We claim that

$$\{g_1, \ldots, g_{a+b}, g^+_{a+b+1}, \ldots, g^+_k, g^-_{a+b+1}, \ldots, g^-_k\} \tag{3.19}$$

is a linearly independent set. Suppose

$$\sum_{i=1}^{a+b} x_i g_i + \sum_{i=a+b+1}^{k} (y_i g_i^+ + z_i g_i^-) = 0, \tag{3.20}$$

from which it follows that the vector

$$\sum_{i=1}^{a+b} x_i g_i + \sum_{i=a+b+1}^{k} y_i (g_i^+ + g_i^-) = \sum_{i=a+b+1}^{k} (y_i - z_i) g_i^- \tag{3.21}$$

is in $\mathbf{C}^{-n} \cap E$. The set $\{g_{a+b+1}, \ldots, g_k\}$ spans a vector space complement of $(\mathbf{C}^n \cap E) + (\mathbf{C}^{-n} \cap E)$ in E so for every $a + b + 1 \le i \le k$, we have $y_i = 0$. Similarly, for every $a + b + 1 \le i \le k$, we have $z_i = 0$ and our claim holds true.

We show that (3.19) can be extended to a symplectic basis of $\mathbf{C}^n + \mathbf{C}^{-n}$ respecting the direct sum. For $1 \le i \le a + b + c$ and $a + b < j \le k$,

$$\begin{aligned} \omega(g_i, g_j^+) &= \omega(\Lambda^n(g_i) + \Lambda^{-n}(g_i), \Lambda^n(g_j^+)) \\ &= -\omega(\Lambda^n(g_j^+ + g_j^-), \Lambda^{-n}(g_i)) \\ &= -\varepsilon(g_j, g_i) \end{aligned} \tag{3.22}$$

is zero since g_i is in the radical of ε; similarly, $\omega(g_i, g_j^-) = 0$. For $a + b < i \le a + b + c$ and $a + b < j \le k$,

$$\begin{aligned} \omega(g_i^+, g_j^-) &= \omega(\Lambda^n(g_i^+ + g_i^-), \Lambda^{-n}(g_j^+ + g_j^-)) \\ &= \varepsilon(g_i, g_j) \end{aligned} \tag{3.23}$$

is zero since g_i in the radical of ε; similarly, $\omega(g_i^-, g_j^+) = 0$. Let F be the span of

$$\{g^+_{a+b+c+1}, \ldots, g^+_k, g^-_{a+b+c+1}, \ldots, g^-_k\}\,. \tag{3.24}$$

Then

$$\omega(g_i^+, g_j^-) = \varepsilon(g_i, g_j) = \delta_{ij} \tag{3.25}$$

shows

$$(g^+_{a+b+c+1}, \dots, g^+_k, g^-_k, \dots, g^-_{a+b+c+1}) \tag{3.26}$$

is a symplectic basis of F. In particular, $\mathbf{C}^{2n} = F + F^\perp$ by Proposition 3.1.1. The set

$$\{g_1, \dots, g_{a+b}, g^+_{a+b+1}, \dots, g^+_{a+b+c}, g^-_{a+b+1}, \dots, g^-_{a+b+c}\} \tag{3.27}$$

spans an isotropic subspace of $F^\perp$. We have

$$\dim(\mathbf{C}^{\pm n} \cap F^\perp) = \dim((\mathbf{C}^{\pm n} + F)^\perp) = n - (k - a - b - c) \tag{3.28}$$

by Proposition 3.1.1. Hence $\mathbf{C}^{\pm n} \cap F^\perp$ are complementary lagrangian subspaces of $F^\perp$. We use Lemma 3.2.3 to extend (3.27) to a symplectic basis of $(\mathbf{C}^n \cap F^\perp) + (\mathbf{C}^{-n} \cap F^\perp)$ while preserving the direct sum.

We can apply this construction to the basepoint (3.17) giving a symplectic matrix γ between bases. Then γ preserves $\mathbf{C}^{\pm n}$ and hence $\gamma \in K$. It is clear that γ takes the basepoint (3.17) to E so $\mathcal{Q}^{\hat{k}}_{a,b,c}$ is a single K-orbit. ∎

Corollary 3.2.5 *The closure $X^{\hat{k}}_{a,b,c}$ of $\mathcal{Q}^{\hat{k}}_{a,b,c}$ in $G/P_{\hat{k}}$ is given by*

$$\left\{E^k \mid \dim(\mathbf{C}^n \cap E) \geq a,\ \dim(\mathbf{C}^{-n} \cap E) \geq b,\ \dim(\mathrm{rad}(\varepsilon)) \geq a + b + c\right\}. \tag{3.29}$$

Therefore, $(V^{\hat{k}}, \leq)$ is given by $(a', b', c') \leq (a, b, c)$ if and only if $a' \geq a$, $b' \geq b$, and $a' + b' + c' \geq a + b + c$.

Proof. The set

$$Y = \{E \in G/P \mid \dim(\mathrm{rad}(\varepsilon)) \geq a + b + c\} \tag{3.30}$$

is seen to be closed in G/P as follows. Define

$$\pi : G \to G/P, \qquad \pi(g) = gP/P \tag{3.31}$$

and

$$\gamma : G \to \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^k, (\mathbf{C}^k)^\vee), \qquad \gamma(g)(x)(y) = \varepsilon(gx, gy). \tag{3.32}$$

We have

$$\pi^{-1}(Y) = \{g \in G \mid \dim(\ker(\gamma(g))) \geq a+b+c\} \tag{3.33a}$$

$$= \{g \in G \mid \operatorname{rank}(\gamma(g)) \leq k-a-b-c\} \tag{3.33b}$$

which is a Zariski closed subset by §3.1.3. Therefore, Y is closed in G/P.

A similar argument applied to the map

$$\eta : G \to \operatorname{Hom}_{\mathbf{C}}(\mathbf{C}^k, \mathbf{C}^{-n}), \qquad \eta(g) = \Lambda^{-n} \circ g \circ \iota, \tag{3.34}$$

where $\iota : \mathbf{C}^k \to \mathbf{C}^{2n}$ is inclusion, shows

$$\{E \in G/P \mid \dim(\mathbf{C}^n \cap E) \geq a\} \tag{3.35}$$

is closed. It follows that $\overline{\mathcal{Q}_{a,b,c}} \subseteq X^{\hat{k}}_{a,b,c}$.

Let $(a', b', c') \in V^{\hat{k}}$ such that $a' = a$, $b' = b$, and $c' = c+1$. Define $g \in G$ by

$$e_{a+c+1} \mapsto (e_{a+c+1} + e_{-(n+b+c+1)})/\sqrt{2} \tag{3.36a}$$

$$e_{-(n+b+c+1)} \mapsto (e_{a+c+1} - e_{-(n+b+c+1)})/\sqrt{2} \tag{3.36b}$$

$$e_{-(a+c+1)} \mapsto (e_{-(a+c+1)} + e_{n+b+c+1})/\sqrt{2} \tag{3.36c}$$

$$e_{n+b+c+1} \mapsto (e_{-(a+c+1)} - e_{n+b+c+1})/\sqrt{2} \tag{3.36d}$$

such that g fixes all remaining standard basis vectors. Then $g \in K$ and

$$\begin{aligned} &\langle e_1, \ldots, e_a, e_{n+1}, \ldots, e_{n+b}, e_{a+1} + e_{n+b+1}, \ldots, e_{a+c} + e_{n+b+c} \\ &e_{a+c+1} + e_{-(n+b+c+1)} + e_{-(a+c+1)} + e_{n+b+c+1}, \\ &e_{a+c+2} + e_{-(a+c+2)}, \ldots, e_{a+c+d} + e_{-(a+c+d)}\rangle \end{aligned} \tag{3.37}$$

is seen to be $gE_{a,b,c} \in \mathcal{Q}_{a,b,c}$. Let $\lambda : \mathbf{C}^\times \to T$ be the cocharacter such that coordinates $a+c+1$ and $-(n+b+c+1)$ are t^{-1}, coordinates $-(a+c+1)$ and $n+b+c+1$ are t, and remaining coordinates are trivial. The morphism $\alpha : \mathbf{C}^\times \to \operatorname{Gr}^0_k(\mathbf{C}^{2n})$ by

$$\alpha(t) = \lambda(t) g E_{a,b,c} \tag{3.38}$$

extends to all of $\mathbf{C}$ with $\alpha(0) = E_{a,b,c+1}$. Therefore, $\mathcal{Q}_{a,b,c+1} \subseteq \overline{\mathcal{Q}_{a,b,c}}$.

Let $(a',b',c') \in V^{\hat{k}}$ such that $a' = a+1$, $b' = b$, and $c' = c-1$. Let $\lambda : \mathbf{C}^\times \to T$ be the cocharacter such that coordinate $a+c+1$ is t^{-1}, coordinate $-(a+c+1)$ is t^{-1}, and remaining coordinates are trivial. The morphism $\alpha : \mathbf{C}^\times \to \mathrm{Gr}_k^0(\mathbf{C}^{2n})$ by (3.38), such that $g = 1$, extends to $\mathbf{C}$ with $\alpha(0) = E_{a+1,b,c-1}$. Therefore, $\mathcal{Q}_{a+1,b,c-1} \subseteq \overline{\mathcal{Q}_{a,b,c}}$ and similarly for $\mathcal{Q}_{a,b+1,c-1}$. The claim follows. ∎

3.2.6 *As a finite poset, if $k \leq \frac{n}{2}$, then for every $n' \geq n$, $V_n^{\hat{k}} = V_{n'}^{\hat{k}}$.*

Proof. If $k \leq \frac{n}{2}$, then $c \leq a+b+c \leq k \leq \frac{n}{2} \leq n-k$ always holds. Therefore,

$$V_n^{\hat{k}} = \left\{(a,b,c) \in \mathbf{Z}_{\geq 0}^3 \mid a+b+c \leq k\right\} \tag{3.39}$$

is independent of n. The poset structure agrees by Corollary 3.2.5.

If $k > \frac{n}{2}$, then $(0,0,n+1-k) \in V_{n+1}^{\hat{k}} \setminus V_n^{\hat{k}}$, since $n+1 \leq 2k$. This shows that the inequality is sharp. ∎

3.3 Resolutions

Lemma 3.3.1 *Let $(a,b,c) \in V^{\hat{k}}$; i.e., $\mathcal{Q}_{a,b,c}^{\hat{k}}$ is nonempty. Given $E^k \in \mathcal{Q}_{a,b,c}^{\hat{k}}$, there exists a unique $F^{k+c} \in \mathrm{Gr}_{k+c}^0(\mathbf{C}^{2n})$ such that $\dim(\mathbf{C}^n \cap F^{k+c}) \geq a+c$, $\dim(\mathbf{C}^{-n} \cap F^{k+c}) \geq b+c$, and $E^k \subseteq F^{k+c}$.*

Proof. The statement of the lemma is K-invariant so assume E is given by (3.17). For existence, let $F = E + \Lambda^+(\mathrm{rad}(\varepsilon))$. Then

$$F = \left\langle e_1, \dots, e_{a+c}, e_{-n}, \dots, e_{-n+b+c}, e_{a+c+1} + e_{-(a+c+1)}, \dots, e_{a+c+d} + e_{-(a+c+d)}\right\rangle \tag{3.40}$$

of dimension $a+b+2c+d = k+c$, so F satisfies the desired properties.

For uniqueness, we show that any F agrees with (3.40). Let $F = E + E'$ for some E' of dimension c. The relation $\dim(\mathbf{C}^n \cap F) = \dim(\mathbf{C}^n \cap E) + c$ allows us to assume that $E' \subseteq \mathbf{C}^n$. Similarly, $F = E + E''$ for some $E'' \subseteq \mathbf{C}^{-n}$. Then $\dim(E \cap (E' + E'')) = c$ since

$\dim(F) = \dim(E) + c$. We have $E \cap (E' + E'') \subseteq \mathrm{rad}(\varepsilon)$ since it is contained in E and the radical for F. It follows that $F = E + \Lambda^+(\mathrm{rad}(\varepsilon))$ since $E' = \Lambda^+(E \cap (E' + E'')) \subseteq \Lambda^+(\mathrm{rad}(\varepsilon))$ and $\dim(E + \Lambda^+(\mathrm{rad}(\varepsilon))) = k + c = \dim(F)$. ■

Theorem 3.3.2 *Let* $(a,b,c) \in V_n^{\hat{k}}$. *Set* $Z_{a,b,c}^{\hat{k}} \subseteq \mathrm{Gr}_{a+b+2c}^0(\mathbf{C}^{2n}) \times \mathrm{Gr}_{k+c}^0(\mathbf{C}^{2n}) \times \mathrm{Gr}_k^0(\mathbf{C}^{2n})$ *by the containment relations*

$$\begin{tikzcd}[arrows=dash]
\mathbf{C}^{2n} \arrow[d] & \\
F^{k+c} \arrow[d] \arrow[dr] & \\
F^{a+b+2c} \arrow[d] & E^k \arrow[dl] \\
0 &
\end{tikzcd} \tag{3.41}$$

such that $\dim(\mathbf{C}^n \cap F^{a+b+2c}) = a + c$ *and* $\dim(\mathbf{C}^{-n} \cap F^{a+b+2c}) = b + c$. *Then* $\mathrm{pr}: Z_{a,b,c}^{\hat{k}} \to X_{a,b,c}^{\hat{k}}$ *by* $(F^{a+b+2c}, F^{k+c}, E^k) \mapsto E^k$ *is a resolution of singularities.*

Proof. Let $I_0 = \{1, \dots, n\} \smallsetminus \{a + b + 2c\}$, $v_0 = (a+c, b+c) \in V^{I_0}$, and $G_{v_0} = \overline{K\dot{v}_0 B}$, where v_0 is identified with the maximal length representative in V. Let $J_1 = I_0 \smallsetminus \{k + c\}$, $I_1 = \{1, \dots, n\} \smallsetminus \{k + c\}$, $J = I_1 \smallsetminus \{k\}$, and $I = \{1, \dots, n\} \smallsetminus \{k\}$. Let R_1 correspond to J_1, R correspond to J, P correspond to I, and define $\varphi: G_{v_0} \times^{R_1} P_{I_1}/R \to Z_{a,b,c}^{\hat{k}}$ by

$$\varphi[g_0, g_1 R/R] = (g_0 \mathbf{C}^{a+b+2c}, g_0 \mathbf{C}^{k+c}, g_0 g_1 \mathbf{C}^k). \tag{3.42}$$

The relation $G_{v_0}/P_{I_0} = X_{v_0}^{I_0}$ shows that for every $g_0 \in G_{v_0}$, we have $\dim(\mathbf{C}^n \cap g_0 \mathbf{C}^{a+b+2c}) \geq a + c$ and $\dim(\mathbf{C}^{-n} \cap g_0 \mathbf{C}^{a+b+2c}) \geq b + c$. The pull-back $X_{v_0}^{J_1} \subseteq X^{J_1}$ of $X_{v_0}^{I_0} \subseteq X^{I_0}$ by $X^{J_1} \to X^{I_0}$ shows that $X_{v_0}^{J_1}$ is given by

$$\left\{F^{a+b+2c} \subseteq F^{k+c} \mid \dim(\mathbf{C}^n \cap F^{a+b+2c}) \geq a + c,\ \dim(\mathbf{C}^{-n} \cap F^{a+b+2c}) \geq b + c\right\}. \tag{3.43}$$

The stabilizer of $\mathbf{C}^{k+c}$ is P_{I_1}, so for every $g_1 \in P_{I_1}$, we have $g_0 \mathbf{C}^{k+c} = g_0 g_1 \mathbf{C}^{k+c}$. Thus for every $g_0 \in G_{v_0}$ and $g_1 \in P_{I_1}$, we have $g_0 \mathbf{C}^{a+b+2c} \subseteq g_0 \mathbf{C}^{k+c}$ and $g_0 g_1 \mathbf{C}^k \subseteq g_0 g_1 \mathbf{C}^{k+c} = g_0 \mathbf{C}^{k+c}$. It follows that φ is a well-defined function, since R stabilizes $\mathbf{C}^k$. The morphism φ agrees with (2.36), so it is an isomorphism onto $Z^{\hat{k}}_{a,b,c}$. Thus $Z^{\hat{k}}_{a,b,c}$ is a smooth algebraic variety.

The diagram

$$\begin{tikzcd} G_{v_0} \times^{R_1} P_{I_1}/R \arrow[rr, "\varphi"] \arrow[dr, "\mu"'] & & Z^{\hat{k}}_{a,b,c} \arrow[dl, "\mathrm{pr}"] \\ & X^{\hat{k}}_{a,b,c} & \end{tikzcd} \tag{3.44}$$

commutes, where μ is given by (2.31). Thus pr is a proper algebraic morphism. By Lemma 3.3.1, pr is birational. Therefore, pr is a resolution of singularities. ∎

Corollary 3.3.3 *Let $(a,b,c) \in V^{\hat{k}}_n$. Then*

$$\dim(\mathcal{Q}^{\hat{k}}_{a,b,c}) = 2kn + \frac{d}{2} - (a+b)(n-k) - \frac{(a^2+b^2+c^2+3k^2)}{2}, \tag{3.45}$$

where $d = k-a-b-c$. Moreover, if $(a',b',c') \le (a,b,c)$ in $V^{\hat{k}}_n$, then

$$(n-k+\frac{1+a'+a}{2})(a'-a) + (n-k+\frac{1+b'+b}{2})(b'-b) + (\frac{1+c'+c}{2})(c'-c) \tag{3.46}$$

is the codimension of $\mathcal{Q}^{\hat{k}}_{a',b',c'}$ in $X^{\hat{k}}_{a,b,c}$.

Proof. The morphism $\mu : Z^{\hat{k}}_{a,b,c} \to X^{\hat{k}}_{a,b,c}$ from Theorem 3.3.2 is birational so

$$\dim(\mathcal{Q}^{\hat{k}}_{a,b,c}) = \dim(X^{\hat{k}}_{a,b,c}) = \dim(Z^{\hat{k}}_{a,b,c}). \tag{3.47}$$

By (3.44), we have

$$\begin{aligned} \dim(Z^{\hat{k}}_{a,b,c}) &= \dim(G_{v_0} \times^{R_1} P_{I_1}/R) \\ &= \dim(X^{J_1}_{v_0}) + \dim(P_{I_1}/R) \\ &= \dim(X^{I_0}_{a+c,b+c}) + \dim(P_{I_0}/R_1) + \dim(P_{I_1}/R) \\ &= (a+c)(n-a-c) + (b+c)(n-a-c-b-c) \\ &\quad + 2d(n-k-c) + \frac{d(d+1)}{2} + ck \end{aligned} \tag{3.48}$$

and hence (3.45) and (3.46) follow. ∎

Lemma 3.3.4 *If $n \ge 2$, then $X^{\hat{1}}_{0,0,1}$ is smooth.*

Proof. Let $k = 1$, so $k \le \frac{n}{2}$ and $V^{\hat{1}} = \{(1,0),(0,1),(0,0,1),(0,0)\}$ by (3.39). Then

$$X^{\hat{1}}_{0,0,1} = \left\{E^1 \subseteq \mathbf{C}^{2n} \mid \dim(\mathrm{rad}(\varepsilon)) \ge 1\right\}. \tag{3.49}$$

Identifying $X^{\hat{1}}$ with $\mathbf{P}^{2n-1}$ gives

$$\begin{aligned} X^{\hat{1}}_{0,0,1} &= \left\{E^1 \in \mathbf{P}^{2n-1} \mid \omega(\Lambda^n(E), \Lambda^{-n}(E)) = 0\right\} \\ &= \left\{[x_1, \dots, x_n, x_{-n}, \dots, x_{-1}] \in \mathbf{P}^{2n-1} \mid x_1 x_{-1} + \dots + x_n x_{-n} = 0\right\} \end{aligned} \tag{3.50}$$

which is a smooth hypersurface by the jacobian criterion. ∎

Any G-equivariant morphism onto a G-orbit is a locally trivial fibration in the étale topology (cf. [16]), and the following gives an explicit description of a locally trivial fiber bundle in the Zariski topology when projecting to a closed K-orbit.

Lemma 3.3.5 *Let $\zeta : Z \to Y$ be a K-equivariant morphism onto a complete K-orbit $Y \cong K/H$. Then*

$$Z \cong K \times^H \zeta^{-1}(y), \tag{3.51}$$

where $H = \mathrm{Stab}_K(y)$ for some $y \in Y$.

Proof. Let $h \in H$ and $z \in \zeta^{-1}(y)$. Then

$$\zeta(hz) = h\zeta(z) = hy = y \tag{3.52}$$

shows H acts on $\zeta^{-1}(y)$. Define

$$\varphi : K \times^H \zeta^{-1}(y) \to Z, \qquad \varphi[k, z] = kz. \tag{3.53}$$

We can take $U \subseteq K/H$ such that there exists a local section $\sigma : U \to K$, since $H \subseteq K$ is a parabolic subgroup by completeness of K/H. Let $\pi : K \to K/H$ by $\pi(k) = ky$. Then φ is locally

$$\varphi_U : U \times \zeta^{-1}(y) \to \zeta^{-1}(U), \qquad \varphi_U(x, z) = \sigma(x)z, \tag{3.54}$$

where

$$
\begin{aligned}
\zeta(\sigma(ky)z) &= \sigma(ky)\zeta(z) \\
&= \sigma(ky)y \\
&= \pi(\sigma(ky)) \\
&= ky
\end{aligned}
\tag{3.55}
$$

shows φ_U is well-defined. Then

$$
\psi_U : \zeta^{-1}(U) \to U \times \zeta^{-1}(y), \qquad \psi_U(z) = (\zeta(z), \sigma(\zeta(z))^{-1}z) \tag{3.56}
$$

gives the inverse morphism. ■

Theorem 3.3.6 *The following families of $(a,b,c) \in V_n^{\hat{k}}$ admit small resolutions.*

(A) $k = a+b+c = n-1,\ c = 1.$

(B) $c = 0,\ 0 < a+b < k,\ a < n-k+1,\ b < n-k+1.$

(C) $k = a+b+c,$

$$
n < \begin{cases} 2a+b+2c+1, & \text{if } a \le b, \\ a+2b+2c+1, & \text{if } a \ge b. \end{cases} \tag{3.57}
$$

(D) $k = a+b+c,\ c = 1,\ k + \max(a,b) < n.$

(E) $k = a+b+c,$

$$
n < \begin{cases} 2a+5, & \text{if } b = 0, \\ 2b+5, & \text{if } a = 0. \end{cases} \tag{3.58}
$$

Proof of (A). Let μ be the resolution given in Theorem 3.3.2. Suppose $(a,b,c) \in V_n^{\hat{k}}$ satisfies $k = n-1$, $c = 1$, and $n = a+b+2$. It follows that $k = n-1 = a+b+1 = a+b+c$, and that any $(a',b',c') < (a,b,c)$ must satisfy $a'+b' = a+b+1$ with $c' = 0$. We have

$$
\mu^{-1}(E_{a+1,b}^{n-1}) = \left\{ F^n \in \mathrm{Gr}_n^0(\mathbf{C}^{2n}) \mid E \subseteq F,\ \dim(\mathbf{C}^{-n} \cap F) = b+1 \right\}, \tag{3.59}
$$

which is a single point since $F \subseteq E^{\perp}$ and $\dim(\mathbf{C}^{-n} \cap E^{\perp}) = n - (a+1) = b+1$. Therefore, μ is bijective. ∎

Proof (B). Let μ be the resolution given in Theorem 3.3.2. Suppose $c = 0$, $0 < a+b < k$, $a < n-k+1$, and $b < n-k+1$. For every $E \in X_{a,b}$, we have

$$\mu^{-1}(E) \cong \mathrm{Gr}_a(\mathbf{C}^n \cap E) \times \mathrm{Gr}_b(\mathbf{C}^{-n} \cap E) \tag{3.60}$$

by (3.41). Thus if $E \in \mathcal{Q}_{a',b',c'}$, we have

$$\dim(\mu^{-1}(E)) = a(a'-a) + b(b'-b). \tag{3.61}$$

By (3.46), if $(a',b') < (a,b)$, then $c_{a',b'} - 2d_{a',b'}$ (q.v. (2.55)) equals

$$\begin{aligned}
&(n-k+\frac{1}{2}+\frac{a'}{2}+\frac{a}{2})(a'-a) + (n-k+\frac{1}{2}+\frac{b'}{2}+\frac{b}{2})(b'-b) - 2a(a'-a) - 2b(b'-b) \\
&= (n-k+\frac{1}{2}+\frac{a'}{2}-\frac{3a}{2})(a'-a) + (n-k+\frac{1}{2}+\frac{b'}{2}-\frac{3a}{2})(b'-b) \\
&= (n-k+\frac{1}{2}+\frac{a'-a}{2}-a)(a'-a) + (n-k+\frac{1}{2}+\frac{b'-b}{2}-b)(b'-b)
\end{aligned} \tag{3.62}$$

which is always positive under conditions (B). It follows from (3.46) and (3.61) that if $(a',b',c') < (a,b)$, then (2.55) is satisfied and hence μ is small.

In particular, if $c = 0$ and $k \leq \frac{n}{2}$, then μ is small. ∎

Proof of (C). Let μ be the resolution given in Theorem 3.3.2. Suppose $k = a+b+c$ so we have $E = \mathrm{rad}(\varepsilon)$ and $k = a+b+2c$. Then $\mu^{-1}(E)$ is equal to

$$\left\{F \in \mathrm{Gr}^0_{k+c}(\mathbf{C}^{2n}) \mid E \subseteq F,\ \dim(\mathbf{C}^n \cap F) = a+c,\ \dim(\mathbf{C}^{-n} \cap F) = b+c\right\}. \tag{3.63}$$

For $E \in \mathrm{Gr}^0_k(\mathbf{C}^{2n})$, let $E_+ = E + \Lambda^+(E)$ so for every $F \in \mu^{-1}(E)$, we have $E_+ \subseteq F$ by (3.63); in particular, E_+ is isotropic. We show that if $E \in \mathcal{Q}_{a',b',c'}$, then (3.63) is isomorphic to $X_{a+c-a'-c',b+c-b'-c'}$ in $\mathrm{Gr}^0_{c-c'}(\mathbf{C}^{2(n-k-c')})$ – a smooth variety with dimension given by (3.45).

We choose to identify $\mathbf{C}^{2(n-k-c')}$ with $E_+^\perp/E_+$ and symplectic form given by §3.1.2 (iii). Then $\mathbf{C}^{n-k-c'} \cong (\mathbf{C}^n \cap E_+^\perp) + E_+$ and $\mathbf{C}^{-(n-k-c')} \cong (\mathbf{C}^{-n} \cap E_+^\perp) + E_+$ provide transverse lagrangian subspaces since they are isotropic and

$$E_+ = (\mathbf{C}^n \cap E_+) + (\mathbf{C}^{-n} \cap E_+), \qquad E_+^\perp = (\mathbf{C}^n \cap E_+^\perp) + (\mathbf{C}^{-n} \cap E_+^\perp) \tag{3.64}$$

by definition of E_+. Define

$$\eta : \mu^{-1}(E) \to X_{a+c-a'-c', b+c-b'-c'}, \qquad \eta(F) = F + E_+, \tag{3.65}$$

which is well-defined since $E_+ \subseteq F \subseteq E_+^\perp$ and

$$\dim(\mathbf{C}^n \cap E_+) = a' + c', \qquad \dim(\mathbf{C}^{-n} \cap E_+) = b' + c' \tag{3.66}$$

give the correct dimensions. Therefore, for $E \in \mathcal{Q}_{a',b',c'}$, it follows from (3.45) that $\dim(\mu^{-1}(E))$ equals

$$\begin{aligned}
&(2(n-k-c') - (n-k-c) - \frac{3(c-c')}{2})(c-c') - \frac{(a+c-a'-c')^2 + (b+c-b'-c')^2}{2} \\
&= (n-k-\frac{c+c'}{2})(c-c') - \frac{(a-a')^2}{2} - \frac{(b-b')^2}{2} + ((a-a') + (b-b') + (c-c'))(c-c') \\
&= (n-k+a-a'+b-b'+\frac{c-3c'}{2})(c-c') - \frac{(a'-a)^2}{2} - \frac{(b'-b)^2}{2}.
\end{aligned} \tag{3.67}$$

Then $2(c_{a',b',c'} - 2d_{a',b',c'})$ equals

$$\begin{aligned}
&(2n-3a-2b-2c+1+3a')(a'-a) + (2n-2a-3b-2c+1+3b')(b'-b) \\
&+ (4n-c-4a'-4b'-5c'+1)(c'-c).
\end{aligned} \tag{3.68}$$

If $(a',b',c') = (a+1,b,c-1)$, then (3.68) is equal to

$$-2n + 4a + 2b + 4c + 2 \tag{3.69}$$

which is positive if and only if

$$n < 2a + b + 2c + 1. \tag{3.70}$$

Let $(a', b', c') < (a, b, c)$ so there exist $i, j \geq 0$ such that $a' = a + i$, $b' = b + j$, $c' = c - i - j$, and $i + j > 0$. Then (3.68) equals

$$-2n(i+j) + 4c(i+j) + i^2 + j^2 + (i-j)^2 + 4(a+b)(i+j) - 2bi - 2aj. \tag{3.71}$$

Assuming (3.70) forces (3.71) to be strictly larger than

$$2j(b-a) + i(i-2) + j(j-2) + (i-j)^2 \tag{3.72}$$

which is nonnegative for $a \leq b$ (a simple check shows (3.71) is positive for $a = b$ and $i = 1 = j$). ■

Proof of (D). Suppose that $c = 1$ and $k = a + b + c$. Set

$$Z = \left\{F^{a+b} \subseteq E^k \mid \dim(\mathbf{C}^n \cap F) = a,\ \dim(\mathbf{C}^{-n} \cap F) = b,\ E = \operatorname{rad}(\varepsilon)\right\} \tag{3.73}$$

contained in $\mathrm{Gr}^0_{a+b}(\mathbf{C}^{2n}) \times \mathrm{Gr}^0_k(\mathbf{C}^{2n})$. Define $\mu : Z \to X^{\hat{k}}_{a,b,c}$ by $\mu(F, E) = E$.

Let $E \in \mathcal{Q}_{a',b',c'}$ so

$$\begin{aligned} \mu^{-1}(E) &= \left\{F \in \mathrm{Gr}^0_{a+b}(\mathbf{C}^{2n}) \mid F \subseteq E,\ \dim(\mathbf{C}^n \cap F) = a,\ \dim(\mathbf{C}^{-n} \cap F) = b\right\} \\ &\cong \mathrm{Gr}_a(\mathbf{C}^n \cap E) \times \mathrm{Gr}_b(\mathbf{C}^{-n} \cap E) \end{aligned} \tag{3.74}$$

and we have

$$\dim(\mu^{-1}(E)) = a(a' - a) + b(b' - b). \tag{3.75}$$

In particular, the fiber of μ over $E \in \mathcal{Q}_{a,b,c}$ is a single point; hence μ is birational.

To show Z is smooth, let $I' = \{1, \ldots, a+b-1, a+b+1, \ldots, n\} \subseteq S$ and define

$$\zeta : Z \to X^{I'}_{a,b}, \qquad \zeta(F, E) = F \tag{3.76}$$

into $X^{I'} = \mathrm{Gr}^0_{a+b}(\mathbf{C}^{2n})$. Then ζ is a fiber bundle by Lemma 3.3.5 since ζ surjects onto a closed K-orbit equivariantly. For $(F, E) \in Z$, we have

$$\dim(\mathbf{C}^n \cap F^\perp) = n - b, \qquad \dim(\mathbf{C}^{-n} \cap F^\perp) = n - a \tag{3.77}$$

so

$$F^\perp/F = ((\mathbf{C}^n \cap F^\perp) + F) + ((\mathbf{C}^{-n} \cap F^\perp) + F) \tag{3.78}$$

is a decomposition into transverse lagrangian subspaces under $\bar{\omega}$ (q.v. §3.1.2 (iii)). Thus we can define a symmetric bilinear form on every $\bar{E} \in \mathrm{Gr}_1(F^\perp/F)$ by

$$\bar{\varepsilon}(\bar{x}, \bar{y}) = \bar{\omega}(\bar{\Lambda}^+(\bar{x}), \bar{\Lambda}^-(\bar{y})) \tag{3.79}$$

where $\bar{\Lambda}^\pm$ are projection homs defined by (3.78). For every $(F, E) \in Z$, we have $E = \mathrm{rad}(\varepsilon)$ if and only if $\bar{E} = \mathrm{rad}(\bar{\varepsilon})$. Then

$$\begin{aligned} \zeta^{-1}(F) &= \left\{E \in \mathrm{Gr}_k^0(\mathbf{C}^{2n}) \mid F \subseteq E,\ E = \mathrm{rad}(E)\right\} \\ &\cong \{E \in \mathrm{Gr}_1(F^\perp/F) \mid E = \mathrm{rad}(E)\} \\ &\cong X^{\hat{1}}_{0,0,1}, \end{aligned} \tag{3.80}$$

where $(0,0,1) \in V^{\hat{1}}_{n-a-b}$ and $n - a - b \geq 2$ since $c = 1$. It follows that Z is smooth by Lemma 3.3.4.

If $(a', b', c') = (a+1, b, 0)$, then $c_{a',b',c'} - 2d_{a',b',c'}$ equals

$$(n - a - b - 1 + a + 1) + 1 - 2a = n - 2a - b - 1 \tag{3.81}$$

and similarly

$$c_{a',b',c'} - 2d_{a',b',c'} = n - a - 2b - 1 \tag{3.82}$$

for $(a', b', c') = (a, b+1, 0)$. The claim follows.

In particular, if $c = 1$, $k = a + b + c$, and $k \leq \dfrac{n}{2}$, then μ is small. ■

Proof of (E). Suppose that $k = a + b + c$. If $b = 0$, then set

$$Z = \left\{E^k \subseteq F^n \mid \dim(\mathbf{C}^n \cap F) = n - c,\ \dim(\mathbf{C}^{-n} \cap F) = c\right\} \tag{3.83}$$

and if $a = 0$, then set

$$Z = \left\{E^k \subseteq F^n \mid \dim(\mathbf{C}^n \cap F) = c,\ \dim(\mathbf{C}^{-n} \cap F) = n - c\right\} \tag{3.84}$$

contained in $\mathrm{Gr}_k^0(\mathbf{C}^{2n}) \times \mathrm{Gr}_n^0(\mathbf{C}^{2n})$.

Without loss of generality, assume that $b = 0$. Let $I_0 = \{1, \ldots, n-1\}$. Then

$$Z = G_{n-c,c}/R, \tag{3.85}$$

where $G_{n-c,c}$ is the preimage of $X_{n-c,c}^{I_0}$ in G and R is the standard parabolic subgroup corresponding to $J = I_0 \cap I$. That is, Z is the pull-back of a closed K-orbit in X^{I_0}, so Z is smooth.

If $E \in \mathcal{Q}_{a',b',c'}$, then

$$\mu^{-1}(E) \cong \mathrm{Gr}_{a'-a}(\mathbf{C}^{-n} \cap E'^{\perp}) \tag{3.86}$$

of dimension

$$(a'-a)(n-2a'+a-c). \tag{3.87}$$

In particular, if $E \in \mathcal{Q}_{a,b,c}$, then $\mu^{-1}(E)$ is a point so μ is birational.

If $E \in \mathcal{Q}_{a',b',c'}$, then $c_{a',b',c'} - 2d_{a',b',c'}$ is

$$\begin{aligned}
&(n-a-c+\frac{1+a'+a}{2})(a'-a) + (n-a-c+\frac{1+b'}{2})(b') + (\frac{1+c'+c}{2})(c'-c) \\
&-2(a'-a)(n-2a'+a-c) \\
&= (-n-\frac{5a}{2}+c+\frac{9a'}{2}+\frac{1}{2})(a'-a) + b'(n-a-c+\frac{1+b'}{2}) + (\frac{1+c'+c}{2})(c'-c).
\end{aligned} \tag{3.88}$$

The expression (3.88) is positive if

$$n > -1 + a + 2c \tag{3.89a}$$

$$n < 2a + 5. \tag{3.89b}$$

Indeed, the relations (3.89) are forced by $(a', b', c') = (a+1, 0, c-1)$ and $(a', b', c') = (a, 1, c-1)$, and the general case follows immediately by substituting (3.89) into (3.88). Observe that (3.89a) is superfluous since $a + 2c = k + c \leq n$ always holds true. ■

Remark 3.3.7 *It is worth noting that all small resolutions described in Theorem 3.3.6 have smooth fibers.*

3.4 Examples

The small resolutions from Theorem 3.3.6 were constructed by generalizing all small resolutions found for $n \leq 5$. In turn, we recover all examples from Theorem 3.3.6 and describe them in this section.

Let $n = 1$. All $X_{a,b}$ are smooth since $\mathcal{Q}_{a,b}$ is open or closed.

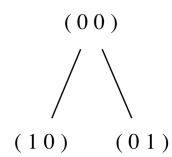

Figure 3.1: $n = 1$, $k = 1$

Let $n \geq 2$ and $k = 1$. Then $X_{a,b,c}$ is always smooth by Lemma 3.3.4.

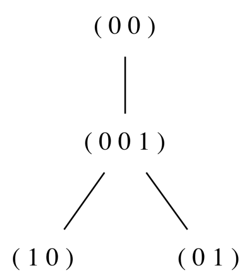

Figure 3.2: $n \geq 2$, $k = 1$

Let $n = 2$ and $k = 2$. The only small resolutions are for open or closed orbits.

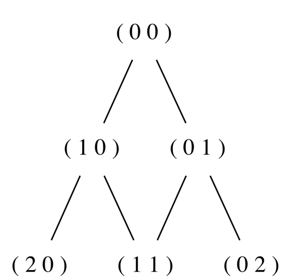

Figure 3.3: $n = 2$, $k = 2$

Let $n = 3$ and $k = 2$. If $a+b = 1$ and $c = 1$, then the morphism defined by Theorem 3.3.6 (A) is small (and bijective). If $c = 0$, then for any a, b, the morphism defined by Theorem 3.3.6 (B) is small.

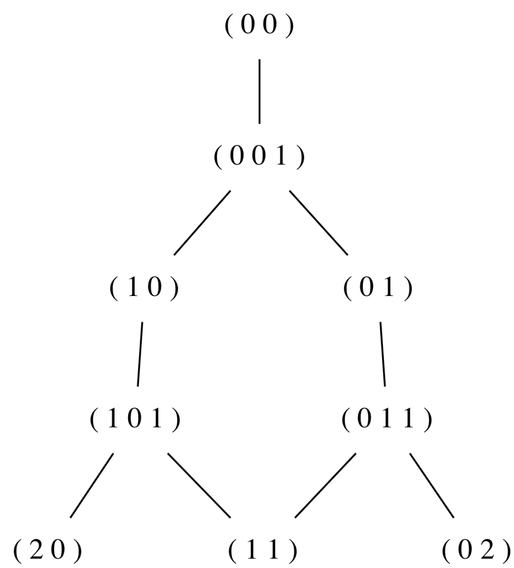

Figure 3.4: $n = 3$, $k = 2$

Let $n = 3$ and $k = 3$. The only small resolutions are for open or closed orbits.

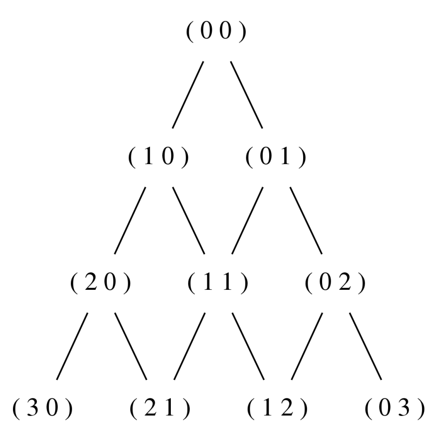

Figure 3.5: $n = 3$, $k = 3$

Let $n = 4$ and $k = 2$ so $k \le \frac{n}{2}$. If $c = 0$, then for any a, b, the morphism defined by Theorem 3.3.6 (B) is small. If $(a,b,c) = (0,0,2)$, then the morphism defined by Theorem 3.3.6 (C) is small. If $a + b = 1 = c$, then the morphism defined by Theorem 3.3.6 (D) is small. Type (B) and (D) are small for every $n \ge 4$ but type (C) fails to be small for $n > 4$.

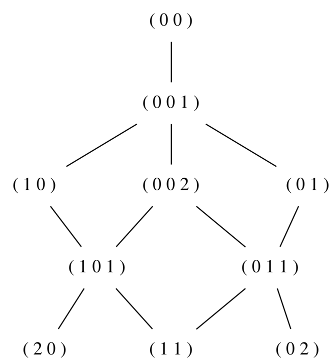

Figure 3.6: $n \ge 4$, $k = 2$

Let $n = 4$ and $k = 3$. If $a+b = 2$ and $c = 1$, then the morphism defined by Theorem 3.3.6 (A) is small (and bijective). If $c = 0$ and $a, b \le 1$, then the morphism defined by Theorem 3.3.6 (B) is small.

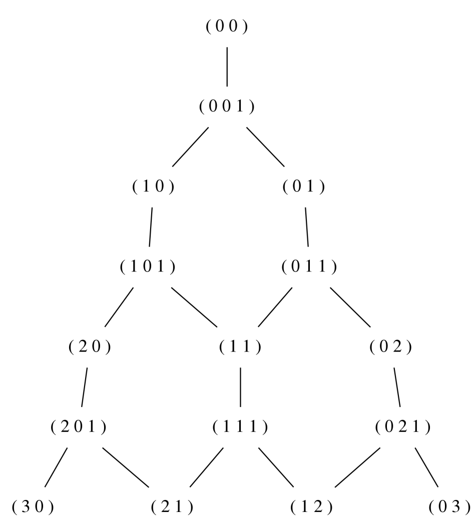

Figure 3.7: $n = 4$, $k = 3$

Let $n = 4$ and $k = 4$. The only small resolutions are for open or closed orbits.

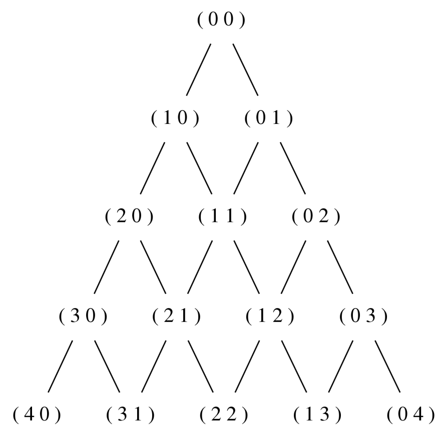

Figure 3.8: $n = 4$, $k = 4$

Let $n = 5$ and $k = 3$. If $c = 0$, then the morphism defined by Theorem 3.3.6 (B) is small. If $(a, b, c) \in \{(1,0,2), (0,1,2), (1,1,1)\}$, then the morphism defined by Theorem 3.3.6 (C) is small. If $(a, b, c) \in \{(2,0,1), (0,2,1)\}$, then the morphism by Theorem 3.3.6 (E) is small.

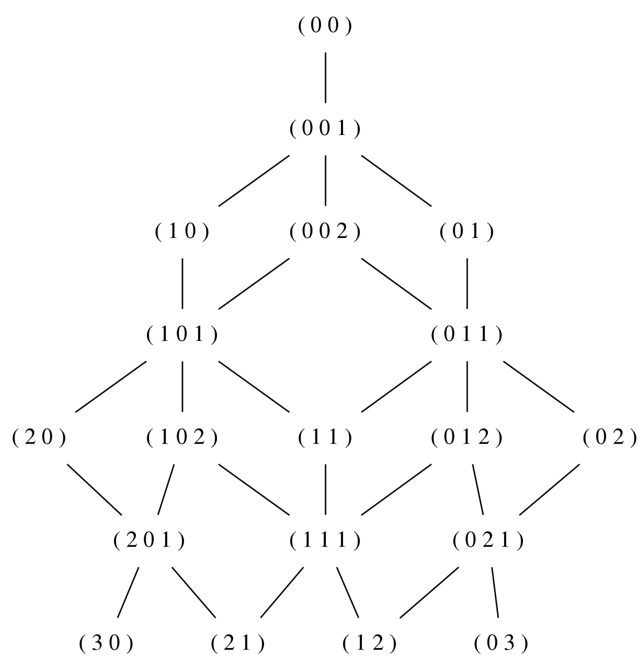

Figure 3.9: $n = 5$, $k = 3$

Let $n = 5$ and $k = 4$. If $a+b = 3$ and $c = 1$, then the morphism defined by Theorem 3.3.6 (A) is small (and bijective). If $(a, b) \in \{((1,1),(1,0),(0,1)\}$, then the morphism defined by Theorem 3.3.6 (B) is small.

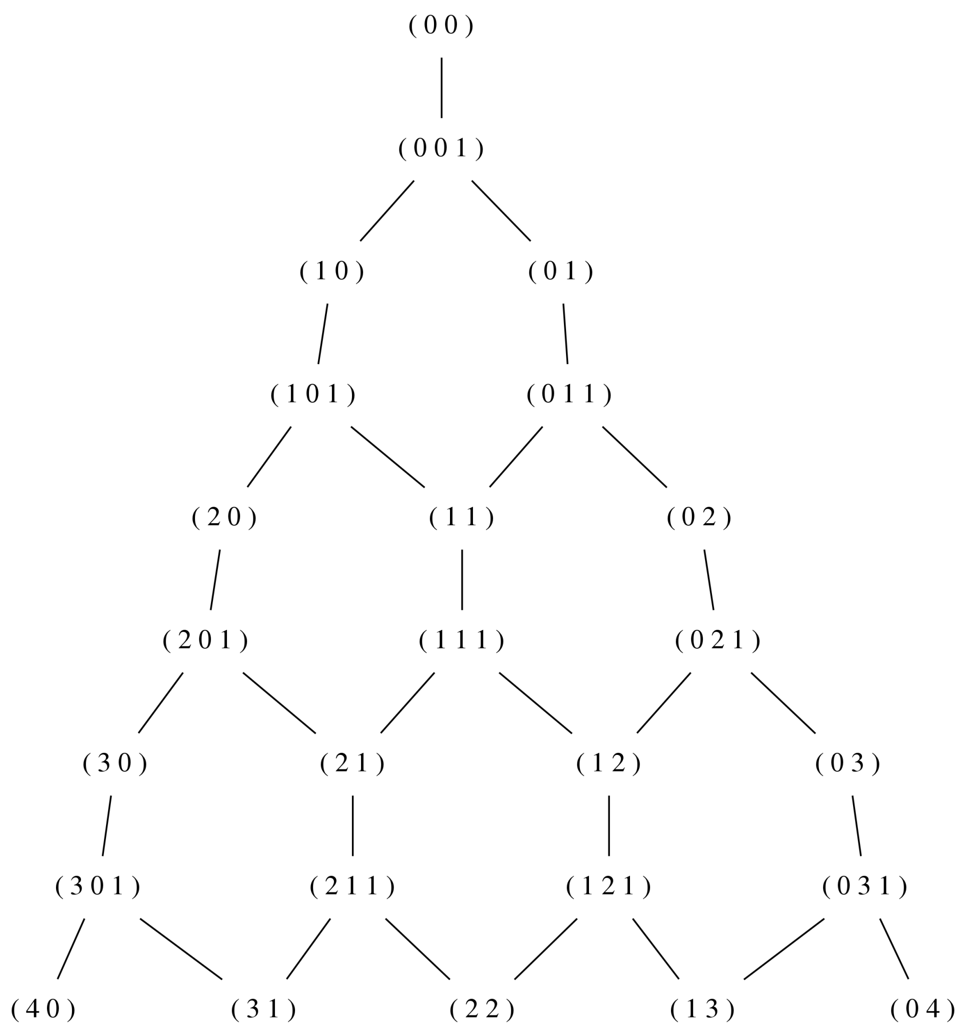

Figure 3.10: $n = 5$, $k = 4$

Let $n = 5$ and $k = 5$. The only small resolutions are for open or closed orbits.

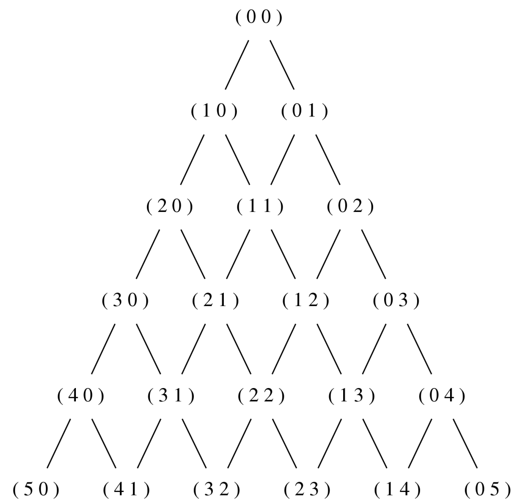

Figure 3.11: $n = 5$, $k = 5$

Table 3.1: Row n and column k gives lower bound ratio of (a, b, c) in $V_n^{\hat{k}}$ with small resolution.

1											
1	.6666										
1	.8888	.5000									
1	.9000	.6875	.4000								
1	.8000	.7894	.5200	.3333							
1	.8000	.8000	.6451	.4166	.2857						
1	.8000	.6500	.7058	.5217	.3469	.2500					
1	.8000	.6500	.6285	.5576	.4218	.2968	.2222				
1	.8000	.6500	.5428	.5636	.4729	.3529	.2592	.2000			
1	.8000	.6500	.5428	.5178	.5000	.4000	.3027	.2300	.1818		
1	.8000	.6500	.5428	.4642	.4698	.4181	.3384	.2647	.2066	.1666	
1	.8000	.6500	.5428	.4642	.4404	.4224	.3655	.2926	.2349	.1875	.1538
1	.8000	.6500	.5428	.4642	.4047	.4033	.3677	.3189	.2574	.2110	.1715
1	.8000	.6500	.5428	.4642	.4047	.3833	.3664	.3300	.2782	.2295	.1914
1	.8000	.6500	.5428	.4642	.4047	.3583	.3536	.3238	.2948	.2464	.2068
1	.8000	.6500	.5428	.4642	.4047	.3583	.3393	.3240	.2969	.2629	.2208
1	.8000	.6500	.5428	.4642	.4047	.3583	.3212	.3150	.2898	.2644	.2345
1	.8000	.6500	.5428	.4642	.4047	.3583	.3212	.3045	.2907	.2674	.2456
1	.8000	.6500	.5428	.4642	.4047	.3583	.3212	.2909	.2842	.2627	.2428
1	.8000	.6500	.5428	.4642	.4047	.3583	.3212	.2909	.2762	.2638	.2436

CHAPTER IV

INDUCING SMALL RESOLUTIONS

We describe a general process for inducing small resolutions from varieties stable under a parabolic subgroup to G-spaces. This construction provides many new small resolutions of K-orbit closures, and leads us to describe fiber bundle structures. The combinatorial structure of clans provides a simple description of fiber bundles for $U(p,q)$ and $Sp(2n,\mathbf{R})$. We view induction as highlighting the importance for describing small resolutions of K-orbit closures in low rank, since small resolutions propagate. We conclude with examples for $U(p,q)$ and $Sp(2n,\mathbf{R})$.

4.1 The general case

Key Lemma 4.1.1 *Let G be a connected reductive group, P a parabolic subgroup, X and Y algebraic varieties with algebraic actions of P, and $\xi : Y \to X$ a P-equivariant algebraic morphism. If ξ is a small resolution, then*

$$\nu : G \times^P Y \to G \times^P X, \quad \nu[g,y] = [g, \xi(y)] \tag{4.1}$$

is a small resolution.

Proof. Let $\pi : G \to G/P$, $U \subseteq G/P$, $\widetilde{U} = \pi^{-1}(U) \subseteq G$, and $\pi_U = \pi|\widetilde{U}$. Suppose U is such that π_U is trivializable, so $\widetilde{U} \cong U \times P$. Then ν is locally of the form

$$(\mathrm{id}, \xi) : U \times Y \to U \times X \tag{4.2}$$

by (2.11). If $U_X \subseteq X$ and $U_Y \subseteq Y$ are isomorphic by the birational morphism ξ then $U \times U_Y \cong U \times U_X$ shows that ν is birational. It follows from [14, Corollary 4.8 (d),(f)] that ν is proper. The space $G \times^P Y$ is smooth by (4.2). Therefore ν is a resolution of singularities that is small by (4.2). ∎

Remark 4.1.2 *We apply Key Lemma 4.1.1 by constructing isomorphisms from G-spaces to varieties of the form $G \times^P X$. We call this process* inducing *a small resolution from P to G.*

4.2 Clans

Let $W = S_n$ be the Weyl group of $GL(n, \mathbf{C})$ realized as permutations of $\{1, \ldots, n\}$. A *clan* is an involution in W such that the fixed points are decorated with $+$ or $-$. For example, we can write

$$v = (\ 1\ 2\ \ +\ \ -\ 1\ 2\ ) \tag{4.3}$$

for the involution $1 \mapsto 5$, $2 \mapsto 6$, and fixing 3,4. In this example, $v = (\ 2\ 1\ \ +\ \ -\ 2\ 1)$ but does not equal $(\ 1\ 2\ \ -\ \ +\ 1\ 2\ )$. Let V denote the set of all clans. Given $v \in V$, let ${}^t v$ denote the clan that reverses the entries of v and let $-v$ denote the clan that switches $\pm$ with $\mp$. For example, if v is defined as in (4.3), then

$${}^t v = (\ 2\ 1\ \ -\ \ +\ 2\ 1\ ) = (\ 1\ 2\ \ -\ \ +\ 1\ 2\ ) = -v. \tag{4.4}$$

For any $v \in V$ and $1 \leq k \leq n$, define

$$v(k, \pm) = \#\left\{1 \leq i \leq k \mid v_i = \pm\right\} + \#\left\{1 \leq i < j \leq k \mid v(i) = j\right\}, \tag{4.5}$$

where $v(i) = j$ as an involution.

Proposition 4.2.1 ([31]) *The set*

$$V_{p,q} = \{v \in V \mid v(n,+) - v(n,-) = p - q\} \tag{4.6}$$

parameterizes K-orbits on G/B for $U(p,q)$. The set

$$V_n = \{v \in V_{n,n} \mid {}^t v = -v\} \tag{4.7}$$

parameterizes K-orbits on G/B for $Sp(2n, \mathbf{R})$.

Remark 4.2.2 *From now on, we let the pair (G, K) be defined by $v \in V$. In particular, if $v \in V_{p,q}$, then $(G, K) = (GL_n, GL_p \times GL_q)$, and if $v \in V_n$, then $(G, K) = (Sp_{2n}, GL_n)$. We also let X_v be the K-orbit closure in the flag variety defined by v. In particular, if $v \in V$ and $v' \in V'$ then X_v and $X_{v'}$ are subvarieties of distinct flag varieties.*

4.3 Fiber bundles

Let $\mathbf{C}^\bullet$ be the standard flag corresponding to the upper triangular Borel B and let $\mathbf{C}^{-\bullet}$ correspond to the opposite Borel B^-.

Proposition 4.3.1 *Suppose that $v \in V_{p,q}$ is of the form*

$$v = (v^1, v^2), \tag{4.8}$$

where $v^i \in V_{p_i,q_i}$ for some $n = n_1 + n_2$ and $n_i = p_i + q_i$. Then

$$X_v \cong K \times^{P_K} (X_{v^1} \times X_{v^2}), \tag{4.9}$$

where (G, K) corresponds to $v \in V_{p,q}$, P is a parabolic subgroup of G, and $P_K = K \cap P$ is a parabolic subgroup of K.

Proof. By [31, Proposition 2.2.6], if $F^\bullet \in \mathcal{Q}_v$, then

$$\dim(\mathbf{C}^p \cap F^k) = v(k, +), \tag{4.10a}$$

$$\dim(\mathbf{C}^{-q} \cap F^k) = v(k, -). \tag{4.10b}$$

In particular, for any $v \in V_{p',q'}$, we have

$$v(n', +) = p', \quad v(n', -) = q', \tag{4.11}$$

where $n' = p' + q'$. It follows that for every $F^\bullet \in \mathcal{Q}_v$, we have

$$\dim(\mathbf{C}^p \cap F^{n_1}) = p_1, \quad \dim(\mathbf{C}^{-q} \cap F^{n_1}) = q_1. \tag{4.12}$$

By Example 2.4.5, $\mathcal{Q}_v$ (and X_v) maps to the closed K-orbit $X^I_{p_1,q_1}$, where $I = \hat{n}_1$. By Lemma 3.3.5, we have $X_v \cong K \times^{P_K} \pi^{-1}(E)$, where $\pi : X_v \to X^I_{p_1,q_1}$ is projection,

$$E = \langle e_1, \dots, e_{p_1}, e_{p+1}, \dots, e_{p+q_1} \rangle \tag{4.13}$$

is a basepoint, $P = \mathrm{Stab}_G(E)$, and $P_K = K \cap P$.

Define

$$\eta : \pi^{-1}(E) \to X_{v_1} \times X_{v_2}, \quad \eta(F^\bullet) = (\alpha(F^\bullet), \beta(F^\bullet)), \tag{4.14}$$

where

$$\alpha(F^\bullet) = (0 \subseteq F^1 \subseteq \cdots \subseteq F^{n_1-1} \subseteq E) \tag{4.15a}$$

$$\beta(F^\bullet) = (E \subseteq F^{n_1+1} \subseteq \cdots \subseteq F^{n-1} \subseteq \mathbf{C}^n) \tag{4.15b}$$

map to flag varieties for $GL(E)$ and $GL(\mathbf{C}^n/E)$ respectively. Then η is an algebraic morphism with inverse given by

$$\eta^{-1}(A^\bullet, B^\bullet) = (0 \subseteq A^1 \subseteq \cdots \subseteq A^{n_1-1} \subseteq E \subseteq B^1 \subseteq \cdots \subseteq B^{n_2-1} \subseteq \mathbf{C}^n). \tag{4.15c}$$

Note the image of η is well-defined by considering direct sum decompositions

$$E = (\mathbf{C}^p \cap E) + (\mathbf{C}^q \cap E), \quad \mathbf{C}^n/E = (\mathbf{C}^p + E) + (\mathbf{C}^q + E). \tag{4.16}$$

with each stabilizer giving the corresponding 'K'. ■

Remark 4.3.2 *Suppose that $v \in V_{p,q}$ is of the form*

$$v = (v^1, \dots, v^h), \tag{4.17}$$

where for every $1 \le i \le h$, we have $v_i = \max(V_{p_i,q_i})$. It follows immediately from Proposition 4.3.1 that X_v is smooth, and conversely any smooth X_v is of the form (4.17) by [20] (providing a pattern avoidance *condition for smoothness).*

Proposition 4.3.3 *Suppose that $v \in V_n$ is of the form*

$$v = (v^1, v^2, -{}^t v^1), \tag{4.18}$$

where $v^1 \in V_{p_1,q_1}$ and $v^2 \in V_{n_2}$ for some $n = n_1 + n_2$ and $n_1 = p_1 + q_1$. Then

$$X_v \cong K \times^{P_K} (X_{v^1} \times X_{v^2}), \tag{4.19}$$

where (G, K) corresponds to $v \in V_n$, P is a parabolic subgroup of G, and $P_K = K \cap P$ is a parabolic subgroup of K.

Proof. The proof follows from the proof of Proposition 4.3.1, the description of closed K-orbits in Corollary 3.2.5, and the fact that X_v is the intersection of X with the corresponding K-orbit closure for $v \in V_{n,n}$. Explicitly, we have X_{v^1} in the flag variety for $GL(E)$ (where E is isotropic of dimension n_1) and X_{v^2} in the flag variety for $Sp(E^\perp/E)$. ∎

Theorem 4.3.4 *Suppose that $v \in V_{p,q}$ or $v \in V_n$ is of the form (4.8) or (4.18) such that there exists a small resolution of the form (2.31) for X_{v^1} and X_{v^2}. Then there exists a small resolution of X_v.*

Proof. We provide a proof for the case $v \in V_{p,q}$ since the case $v \in V_n$ is similar. To induce the small resolution of $X_{v_1} \times X_{v_2}$ from P_K to K (q.v. Key Lemma 4.1.1), we must show that there is an action of P_K which makes the resolution equivariant. Note the action of P_K on $X_{v_1} \times X_{v_2}$ is given by

$$g(x_1, x_2) = ((g|E)x_1, gx_2), \tag{4.20}$$

which is well-defined since $P_K \subseteq \mathrm{Stab}_G(E)$. Moreover, (4.20) gives an action of P_K on each flag variety. We have $P_K \subseteq K$ so all direct sums in (4.16) are preserved by this action. It follows that for any $u_i \in V_{p_i,q_i}$, P_K stabilizes $X_{u_1} \times X_{u_2}$.

Let

$$Z_i = G_{v_0^i} \times^{R_1^i} G_{w_1^i} \times^{R_2^i} \cdots \times^{R_{m_i}^i} X_{w_{m_i}^i} \tag{4.21}$$

be the small resolutions given by assumption. Then P_K acts on $Z_1 \times Z_2$ by

$$g(z_1, z_2) = ((g|E)z_1, gz_2), \tag{4.22}$$

where multiplication occurs in the first factor of Z_i. Therefore, the small resolution of $X_{v_1} \times X_{v_2}$ is P_K-equivariant. $\blacksquare$

CHAPTER V

$U(p,q)$

5.1 Generically homogeneous fibers

The following lemma gives a method for determining certain $v \in V$ have the property that every morphism μ satisfying (2.50) is birational. This condition is useful for general V and we apply it below to $V_{p,q}$.

Lemma 5.1.1 *Let $v = v_0 \star w_1 \star \cdots \star w_m$,*

$$Z = G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/B, \tag{5.1}$$

and define $\mu : Z \to X_v$ by multiplication. If μ satisfies (2.50), then for every $x \in \mathcal{Q}_v$ and for every $z \in \mu^{-1}(x)$ we have

$$\mu^{-1}(x) \cong K_x/K_z, \tag{5.2}$$

where, $K_z = \mathrm{Stab}_K(z)$ and $K_x = \mathrm{Stab}_K(x)$. In particular, we have

$$K_x^0 \subseteq K_z \subseteq K_x, \tag{5.3}$$

where K_x^0 is the connected component of K_x.

Proof. K-equivariance of the morphism μ forces the generically finite subset of X_v to contain the open orbit $\mathcal{Q}_v$, so for every $x \in \mathcal{Q}_v$ we have a finite set $\mu^{-1}(x)$ (of constant cardinality). If $z \in \mu^{-1}(x)$, then

$$\dim(Kz) = \dim(Kx) \tag{5.4}$$

since $\dim(Kz) \geq \dim(Kx)$ holds by surjectivity of Kz onto Kx and equality holds by generic finiteness. Let $Y_1, \ldots, Y_k$ be the (finitely many) orbits mapping to $\mathcal{Q}_v$ and let Y be the preimage of $\mathcal{Q}_v$. Then Y is an irreducible variety since it is an open subset of the irreducible Z. We have

$$Y = \bigcup_{i=1}^{k} \overline{Y_i}, \tag{5.5}$$

where the closure is taken in Y. It follows by irreducibility that for every $1 \leq i \leq k$, $Y = \overline{Y_i}$. Therefore Y_1 is open in Y which forces $k = 1$ by equidimension of Y_i.

If $\mu(z) = x$, then base change

$$\begin{tikzcd} \mu^{-1}(\mathcal{Q}_v) = Y \cong K/K_z \arrow[r] \arrow[d] & K/K_x \arrow[d] \\ Z \arrow[r, "\mu"] & X_v \end{tikzcd} \tag{5.6}$$

gives the generic fiber of μ equal to (5.2). Then (5.3) follows directly from the fact that K_x/K_z is finite. ■

We often use Lemma 5.1.1 when $v \in V$ has a trivial component group; i.e., for every $x \in \mathcal{Q}_v$, we have $K_x^0 = K_x$. This holds in particular for every $v \in V_{p,q}$.

We recall a couple of formulas on the combinatorics of clans to help describe small resolutions for $U(p,q)$. For any $v \in V_{p,q}$, we have

$$\ell(v) = \sum_{\substack{v_i = v_j \in \mathbf{N} \\ i<j}} \left(j - i - \#\{a \in \mathbf{N} \mid v_s = v_t = a \text{ for } s < i < t < j\}\right) \tag{5.7}$$

by [31, Definition 2.3.7]. It follows from [21, §2] that for every $v \in V_{p,q}$ and $s_i \in S \smallsetminus \tau(v)$, we have

$$v \star s_i = (\ v_1 \ \cdots \ v_{i-1} \ v_i' \ v_{i+1}' \ v_{i+2} \ \cdots \ v_n\ ) \tag{5.8}$$

where (v_i', v_{i+1}') are matching integers if $(v_i, v_{i+1}) = (\pm, \mp)$ and otherwise $(v_i', v_{i+1}') = (v_{i+1}, v_i)$.

Remark 5.1.2 *For every $v \in V_{p,q}$, one can use [30, Corollary 1.3] to describe X_v explicitly in terms of flags. To describe small resolutions of the form μ, we need only* *the monoid action described explicitly by* (5.8)*, and an explicit description of smooth* X_v *which are described explicitly by Proposition 4.3.1.*

5.2 Fiber dimensions for maps of Barbasch-Evens type

We give a formula (5.16a) for fiber dimensions of any resolution of the form (2.31) such that for every $1 \le i \le m$, G_{w_i} is a parabolic subgroup; i.e.,

$$\mu : G_{v_0} \times^{R_1} P_{I_1} \times^{R_2} \cdots \times^{R_m} P_{I_m}/B \to X_v \tag{5.9}$$

for subsets of simple reflections $I_1, \ldots, I_m$. All resolutions we construct for $U(p,q)$ will be of the form (5.9), and when combined with Lemma 5.1.1 enables us to compute many examples of small resolutions.

Lemma 5.2.1 *Let*

$$\widetilde{Z} = G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}. \tag{5.10}$$

The diagram

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\mu'} & G_v \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\mu} & X_v \end{array} \tag{5.11}$$

is the pull-back of the morphism μ *to* G*.*

Proof. Let

$$\xi : \widetilde{Z} \to Z \underset{X_v}{\times} G_v \tag{5.12}$$

be the morphism provided by the universal property of pull-back. Define

$$\eta : Z \underset{X_v}{\times} G_v \to \widetilde{Z} \tag{5.13a}$$

by

$$\eta([g_0, \ldots, g_m B/B], g) = [g_0, \ldots, g_{m-1}, g_{m-1}^{-1} \cdots g_0^{-1} g]. \tag{5.13b}$$

Then η is a well-defined algebraic morphism by considering the universal property of quotient applied to

$$G_{v_0} \times G_{w_1} \cdots \times G_{w_m} \underset{X_v}{\times} G_v \to \widetilde{Z} \tag{5.14}$$

with the same formula as (5.13b). Notice if $g_0 \in G_{v_0}$, for every $1 \le i \le m$, $g_i \in G_{w_i}$, and $g_0 \cdots g_m B/B = gB/B \in X_v$, then

$$g_m B = g_{m-1}^{-1} \cdots g_0^{-1} g B \tag{5.15}$$

is contained in G_{w_m}, so the image of η indeed lies in $\widetilde{Z}$. Then ξ and η are inverse algebraic morphisms. ∎

In the following proposition, for $v \in V$, let $\ell(v)$ be the length of v; i.e., the difference between the dimension of the K-orbit corresponding to v and a closed K-orbit. A useful fact is that for any $v \in V$ such that $K \times R$ acts on G_v, the $(K \times R)$-orbits on G_v correspond to the $(K \times P)$-orbits on $G_v \times^R P$ (as in, e.g., [3, Lemma 6.15]), where $R \subseteq P$.

Proposition 5.2.2 *Let μ be a resolution of X_v of the form (5.9). If $y \le v$, then*

$$\dim(\mu^{-1}(\dot{y}B/B)) = \max\{d_x \mid x \le u,\ x \star w_J = x,\ x \star w_{I_m} = y \star w_{I_m}\}, \tag{5.16a}$$

where

$$d_x = \ell(x) + \dim(\nu^{-1}(\dot{x}B/B)) + \ell(w_{I_m}) - \ell(w_J) - \ell(x \star w_{I_m}), \tag{5.16b}$$

$u = v_0 \star w_{I_1} \star \cdots \star w_{I_{m-1}}$, ν is the corresponding morphism to X_u given by (2.31), and J is the subset of simple reflections corresponding to R_m.

In particular, the fiber dimension for every $y \le v$ can be computed recursively by (5.16a).

Proof. We choose to describe the fibers of the pull-back

$$\mu' : G_{v_0} \times^{R_1} P_{I_1} \times^{R_2} \cdots \times^{R_m} P_{I_m} \to G_v \tag{5.17}$$

of μ to G, as in Lemma 5.2.1. Notice that $K \times P_{I_m}$ acts on $G_v = \mathrm{Im}(\mu')$, and μ' is equivariant under this action. Consider the morphism

$$\nu' : G_{v_0} \times^{R_1} P_{I_1} \times^{R_2} \cdots \times^{R_{m-1}} P_{I_{m-1}} \to G_u. \tag{5.18}$$

Let $\mathcal{Q}$ be a $(K \times R_m)$-orbit in G_u and

$$\widetilde{\mathcal{Q}} = (\nu')^{-1}(\mathcal{Q}). \tag{5.19}$$

Then μ' restricts to a morphism

$$\mu_{\mathcal{Q}} : \widetilde{\mathcal{Q}} \times^{R_m} P_{I_m} \to \mathcal{Q}P_{I_m} \tag{5.20}$$

onto a single $(K \times P_{I_m})$-orbit in G_v. Therefore, for every $\dot{y} \in G_v$, we have

$$\dim((\mu')^{-1}(\dot{y})) = \max\left\{\dim(\mu_{\mathcal{Q}}^{-1}(\dot{y})) \mid \mathcal{Q} \in K\backslash G_u/R_m,\ \mathcal{Q}P_{I_m} = K\dot{y}P_{I_m}\right\} \tag{5.21}$$

since the collection of domains of all $\mu_{\mathcal{Q}}$ give a finite partition of the domain of μ'. For every $\mathcal{Q} \in K\backslash G_u/R_m$, we have

$$\dim(\mu_{\mathcal{Q}}^{-1}(\dot{y})) = \dim(\widetilde{\mathcal{Q}} \times^{R_m} P_{I_m}) - \dim(\mathcal{Q}P_{I_m}) \tag{5.22a}$$

$$= \dim(\widetilde{\mathcal{Q}}) + \dim(P_{I_m}/R_m) - \dim(\mathcal{Q}P_{I_m}) \tag{5.22b}$$

since the $(K \times P_{I_m})$-equivariant morphism $\mu_{\mathcal{Q}}$ has equidimensional fibers and $\widetilde{\mathcal{Q}} \times^{R_m} P_{I_m} \to P_{I_m}/R_m$ is a fiber bundle by Lemma 2.1.5. For every $\dot{x} \in \mathcal{Q}$, we have

$$\dim(\widetilde{\mathcal{Q}}) = \dim(\mathcal{Q}) + \dim((\nu')^{-1}(\dot{x})) \tag{5.23}$$

since $\nu'|\widetilde{\mathcal{Q}}$ is $(K \times R_m)$-equivariant onto the $(K \times R_m)$-orbit. Therefore (5.16a) follows immediately by observing that for any $u \leq v$ in V such that G_v is $(K \times P_I)$-stable for some standard parabolic subgroup P_I, the open K-orbit in $K\dot{u}P_I/B$ corresponds to $u \star w_I$. ■

5.3 A family of small resolutions

We consider small resolutions of the form

$$\mu : G_v \times^R P/B \to X_u, \tag{5.24}$$

where $v \in V$ gives a smooth X_v, P is a parabolic subgroup corresponding to I, and R is a parabolic subgroup corresponding to J. This is equivalent to considering all small resolutions of the form

$$\pi : X_v^J \to X_u^I, \tag{5.25}$$

given by projection, but we prefer to pull-back π to X and consider μ instead.

Applying (5.16a) to (5.24) gives, for $y \leq v$,

$$\dim(\mu^{-1}(\dot{y}B/B)) = \dim(K\dot{y}R/B) + \dim(P/B) - \dim(K\dot{y}P/B). \tag{5.26}$$

Then (5.26) provides a formula for all fiber dimensions since every $x \leq u$ is in the same $(K \times P)$-orbit for some $y \leq v$; i.e., $K\dot{x}P = K\dot{y}P$. We can take advantage of the simple description of smooth $v \in V_{p,q}$ given by Remark 4.3.2, which we will carry out explicitly for P corresponding to any set of simple reflections that pairwise commute. We conclude this section by describing all small resolutions of the form (2.31) for $U(2,2)$ and $U(3,2)$.

Lemma 5.3.1 *Let*

$$\mu : G_v \times^B P/B \to X_u, \tag{5.27}$$

where G_v is smooth and P corresponds to the simple reflection s. Then μ is a small resolution if and only if for every $y \leq v$ such that $\ell(y) = \ell(v) - 1$, we have

$$s \notin \tau(y) \cup \tau(v). \tag{5.28}$$

Proof. It is clear by (2.50) and (5.2) that μ is birational if and only if $s \notin \tau(v)$. It remains to show that μ is small if and only if $s \notin \tau(y)$, for every $y \leq v$ such that

$\ell(y) = \ell(v) - 1$. By (5.26), $\dim(\mu^{-1}(\dot{y}B/B))$ is either zero or one. So μ is small if and only if every codimension two orbit $x \le u$ has a zero dimensional fiber. This is equivalent to every codimension one orbit $y \le v$ giving a finite morphism

$$K\dot{y}B \times^B P/B \cong K\dot{y}P/B, \tag{5.29}$$

which in turn is equivalent to $s \notin \tau(y)$ as claimed. ■

We can make Lemma 5.3.1 explicit on the level of clans as follows. Let $v \in V_{p,q}$ be a clan such that X_v is smooth, so $v = (v^1, \dots, v^h)$ such that for every $1 \le i \le h$, we have $v^i = \max(V_{p_i,q_i})$ by Remark 4.3.2. It suffices to consider the case $h = 2$, so $v = (v^1, v^2)$. If $s \notin \tau(v)$, then $v \star s = (u^1, u^2)$ can be described in cases. We always have v^1 and u^1 agreeing for all but the last entry, and v^2 and u^2 agree in all but the first entry. If the last entry of v^1 and the first entry of v^2 are opposite signs, then the last entry of u^1 is the same integer as the first entry of u^2. In any other case, the last entry of u^1 equals the first entry of v^2 and the first entry of u^2 equals the last entry of v^1.

In the case where the last entry of v^1 is the opposite sign as v^2, then (5.27) is an isomorphism since v is necessarily closed and u is necessarily of length one; in particular, (5.27) is a small resolution. It remains to consider the case where the last entry of u^1 is equal to the first entry of v^2 and the first entry of u^2 is equal to the last entry of u^1. There are five cases to consider:

$$(v^1, v^2) = \begin{cases} (\dots +, 1 \dots), & 1, \\ (\dots -, 1 \dots), & 2, \\ (\dots 1, + \dots), & 3, \\ (\dots 1, - \dots), & 4, \\ (\dots 1, 2 \dots), & 5. \end{cases} \tag{5.30}$$

In case 1, $v^1 = (+ \cdots +)$, so s satisfies (5.28) if and only if $v^2 \neq (1 + \cdots + 1)$ (allowing possibly zero + signs). Similarly for case 2, $v^1 = (- \cdots -)$, so s satisfies (5.28) if and only if $v^2 \neq (1 - \cdots - 1)$. Case 3 is determined by switching the roles of v^1 and v^2 for case 1, and case 4 is determined by considering case 2. Case 5 always forces s to satisfy (5.28).

Proposition 5.3.2 *Let*

$$\mu : G_v \times^B P/B \to X_u, \tag{5.31}$$

where G_v is smooth, and P corresponds to a subset of simple reflections I that pairwise commute. Then μ is a small resolution if and only if for every $s \in I$ and $y \leq v$ such that $\ell(y) = \ell(v) - 1$, we have

$$s \notin \tau(y) \cup \tau(v). \tag{5.32}$$

Proof. For every $y \leq v$, let $d_y = \dim(\mu^{-1}(\dot{y}B/B))$. Then

$$d_y = \#(I \cap \tau(y)) \tag{5.33}$$

by (5.26). It follows immediately that a small resolution satisfies (5.32). For every $y \leq v$, we can write $y = (y^1, \dots, y^h)$ and $v = (v^1, \dots, v^h)$, where for every $1 \leq j \leq h$, we have $y^j \leq v^j$. For every $1 \leq j \leq h$, let $c_y^j = \ell(v^j) - \ell(y^j)$. Then the codimension of X_y in X_v is $\sum_j c_y^j$ and the codimension of $X_{y \star w_I}$ in X_u is

$$c_y = \sum_j c_y^j + d_y. \tag{5.34}$$

So μ is a small resolution if and only if for every $y < v$, we have

$$\sum_j c_y^j > d_y. \tag{5.35}$$

Suppose that (5.32) is satisfied. We show μ is small by induction on d_y. If $d_y = 1$, then the codimension of X_y in X_v is at least two by (5.32). Suppose $d_y = a$, and that (5.35) holds true whenever the dimension is less than a. We can assume that $y < v$ is maximal such that $d_y = a$, since a smaller $x < y$ with $d_x = d_y$ gives $\sum_j c_x^j > \sum_j c_y^j$.

Suppose that $y < z \leq v$ such that $\ell(z) = \ell(y)+1$. Then by the induction hypothesis, we have $\sum_j c_z^j > d_z$. Let $z = (\dots, z^{\alpha-1}, z^\alpha, z^{\alpha+1}, \dots)$ such that $\ell(z^\alpha) = \ell(y^\alpha) + 1$.

Decreasing the length of z^α by one affects the first and last entries of z in a controlled manner: a sign $\pm$ must remain fixed and if both endpoints are integers, then at most one integer becomes a sign, except for the case where $z^\alpha = (11)$. This claim follows directly from (5.7). It is clear from (5.8) that d_y depends only on the first and last entry of each y^j being a $+$, $-$, or integer. It suffices to assume that $z^\alpha = (11)$ since any other case satisfies $d_y = d_z + 1$, in which case

$$\sum_j c_y^j = \sum_j c_z^j + 1 > d_z + 1 = d_y. \tag{5.36}$$

Then $y^\alpha = (+-)$ or $y^\alpha = (-+)$; assume that $y^\alpha = (+-)$.

We need only consider the case that $y^{\alpha-1} = z^{\alpha-1}$ exists and ends with a $+$, and $y^{\alpha+1} = z^{\alpha+1}$ exists and starts with a $-$, since any other condition satisfies $d_y = d_z + 1$ and thus (5.36) holds true.

Suppose v contains a consecutive string of $v^j = (11)$, e.g., $v = (\dots, 11, 22, 33, \dots)$. Then we can instead consider $v' = (\dots, 11, \dots)$, since $y = (\dots, +-, -+, +-, \dots)$ has the same difference $\sum_j c_y^j - d_y$ as does $\sum_j c_{y'}^j - d_{y'}$, where $y' = (\dots, +-, \dots)$. So we can assume that $z^{\alpha-1} \neq (11)$ and also $z^{\alpha+1} \neq (11)$.

By our previous assumption, we can assume that $z^j = v^j$ for $j \neq \alpha - 1, \alpha, \alpha + 1$, the first entry of $z^{\alpha-1}$ is the first entry of $v^{\alpha-1}$, and the last entry of $z^{\alpha+1}$ is the last entry of $v^{\alpha+1}$. This is equivalent to assuming $h = 3$, in which case $d_y \leq 2$. The fact that $z^\alpha = (11)$ forces the last entry of $v^{\alpha-1}$ and the first entry of $v^{\alpha+1}$ to be an integer by (5.32). If $d_y = 2$, then c_y^1 and c_y^3 are at least one since this is required for $d_y = \#(I \cap \tau(y)) = 2$. We have

$$d_y = 2 < 3 = \sum_j c_y^j \tag{5.37}$$

and the claim follows. ∎

Example 5.3.3 *Suppose* $v = (1{+}1, 22, 3{-}3)$, $z = (11{+}, 22, -33)$, *and* $y = (11{+}, {+}{-}, -22)$. *Then* $d_z = 0$, $\sum_j c_z^j = 2$, $d_y = 2$, *and* $\sum_j c_y^j = 3$. *In particular,* d_y *can jump by two in large codimension.*

Example 5.3.4 *To show that Proposition 5.3.2 is a bit delicate, we provide a counterexample if* G_v *is not assumed to be smooth. This example was found using Atlas software [1]. The first example arises for the group* $U(4,4)$ *with* $v = (\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ )$ *and* $I = \{3,5\}$. *Then for every* $y \leq v$ *such that* $\ell(y) = \ell(v) - 1$, *we have* $s_3, s_5 \notin \tau(y) \cup \tau(v)$ *(as checked with Atlas); i.e., for each* $i = 3,5$, $G_v \times^B P_i/B \to X_{u_i}$ *is small. However,* $G_v \times^B P_{\{3,5\}}/B \to X_u$ *is not small, where* $u = (\ 1\ 2\ 3\ 2\ 4\ 3\ 4\ 1\ )$.

5.4 Examples

Let $n = 4$ and $p = 2 = q$. We can use Lemma 5.3.1 to construct small resolutions for every singular X_u. There are three clans corresponding to singular varieties, and the corresponding v giving a small resolution is listed in the table below.

Table 5.1: Small resolutions for $U(2,2)$

u	v
(1 + - 1)	(1 + 1 -)
(1 - + 1)	(1 - 1 +)
(1 2 1 2)	(1 1 2 2)

Let $n = 5$, $p = 3$, and $q = 2$. We use Proposition 5.3.2 along with two ad-hoc examples to compute small resolutions of the form (5.24) for $U(3,2)$. In fact this gives all clans u in $V_{3,2}$ such that a singular X_u has a small resolution of the form (2.31). The only clan in $V_{3,2}$ without a small resolution of the form (2.31) is $u = (\ 1\ 2\ +\ 1\ 2\ )$.

Explicitly, for $u \in V$, let P be the largest parabolic subgroup stabilizing G_u by right multiplication, let R be the intersection of P with the largest parabolic subgroup stabilizing G_v by right multiplication, and let $Z = G_v \times^R P/B$. Then μ given by (2.31) is a small resolution of X_u.

Table 5.2: Small resolutions for $U(3,2)$

u	v
(+ 1 2 1 2)	(+ 1 1 2 2)
(1 2 1 2 +)	(1 1 2 2 +)
(+ 1 − + 1)	(+ − 1 + 1)
(+ 1 + − 1)	(+ + 1 − 1)
(1 + − 1 +)	(+ 1 − 1 +)
(1 − + 1 +)	(− 1 + 1 +)
(1 2 1 + 2)	(1 1 2 + 2)
(1 + 2 1 2)	(1 + 1 2 2)
(1 + − + 1)	(+ 1 − 1 +)
(1 − + + 1)	(− 1 + + 1)
(1 + + − 1)	(1 + + 1 −)
(1 + 2 2 1)	(1 + + 1 −)
(1 2 2 + 1)	(− 1 + + 1)

The first six rows of Table 5.2 can be induced from small resolutions for $U(2,2)$. The first eleven rows of Table 5.2 can be computed directly from Proposition 5.3.2. The last two rows of Table 5.2 are computed in the following example.

Example 5.4.1 *We compute the last row of Table 5.2, since the second to last row is similar. Let* $u = (\ 1\ 2\ 2 + 1\ )$, $v = (\ -\ 1 + + 1\ )$, *P the parabolic subgroup corresponding to* $\{1,2\}$, *and R the parabolic subgroup corresponding to* $\{2\}$. *Let* $Z = G_v \times^R P/B$ *and recall the isomorphism*

$$\varphi : Z \to X_v^{\{2\}} \underset{X^{\{1,2\}}}{\times} X \tag{5.38}$$

by (2.36)*. We have*

$$X_v = \left\{F^\bullet \in X \mid F^1 \subseteq \mathbf{C}^{-2}\right\} \tag{5.39}$$

so Z is isomorphic to

$$\begin{tikzcd}[arrows=dash]
 & \mathbf{C}^5 \arrow[d] \arrow[ddl] & \\
 & F^4 \arrow[d] & \\
 & F^3 \arrow[dd] \arrow[dr] & \\
\mathbf{C}^{-2} \arrow[dr] & & E^2 \arrow[d] \\
 & F^1 \arrow[d] & E^1 \arrow[dl] \\
 & 0 &
\end{tikzcd} \tag{5.40}$$

where a line between subspaces indicates containment. Define $\mu\colon Z \to X_v$ as usual, so we have

$$\mu^{-1}(E^\bullet) \cong \mathrm{pr}^{-1}(E^\bullet), \tag{5.41}$$

where $\mathrm{pr}\colon \varphi(Z) \to X_v$ *projects to the rightmost flag. So*

$$\mu^{-1}(E^\bullet) = \mathrm{Gr}_1(\mathbf{C}^{-2} \cap F^3) \tag{5.42}$$

which is $\mathbf{P}^1$ if $\mathbf{C}^{-2} \subseteq F^3$ and is a point otherwise. It follows that

$$Y = \left\{E^\bullet \in X \mid \mathbf{C}^{-2} \subseteq F^3\right\} \tag{5.43}$$

contains the set of points in X_u with positive dimensional fiber; in fact the containment is equality since any point of Y can be placed as a rightmost flag in (5.40)*. Therefore μ is small since* $\dim(X_u) = 9$ *and* $\dim(Y) = \dim(X(\ 1\ -\ 1,\ +\ +\ )) = 6$.

CHAPTER VI

FIBERS OF μ

6.1 A single Schubert variety

Let $u, v \in V$ and $w \in W$ such that $v \star w = u$. We consider morphisms of the form

$$\mu : G_v \times^R X_w \to X_u, \tag{6.1}$$

where R is a parabolic subgroup corresponding to a subset of simple reflections J.

Proposition 6.1.1 *If μ is of the form* (6.1)*, then for every $y \leq u$, we have*

$$\mu^{-1}(\dot{y}B/B) \cong X_v^J \cap \dot{y} X_{w^{-1}}^J. \tag{6.2}$$

Proof. Define

$$\varphi : G_v \times^R X_w \to X_v^J \times X_u \tag{6.3}$$

by (2.36). Define

$$\mathrm{pr} : X_v^J \times X_u \to X_u \tag{6.4}$$

by projection. We have

$$\mu^{-1}(\dot{y}B/B) \cong \mathrm{pr}^{-1}(\dot{y}B/B) \tag{6.5a}$$

$$= \left\{ gR/R \in X_v^J \mid g^{-1}\dot{y} \in G_w \right\} \tag{6.5b}$$

$$= \left\{ gR/R \in X_v^J \mid g \in \dot{y} G_{w^{-1}} \right\} \tag{6.5c}$$

$$= \left\{ gR/R \in X_v^J \mid g \in \dot{y} X_{w^{-1}}^J \right\} \tag{6.5d}$$

$$= X_v^J \cap \dot{y} X_{w^{-1}}^J \tag{6.5e}$$

as claimed. ∎

6.2 Barbasch-Evens type again

If $w = w_I$ in (6.1) for some subset of simple reflections I, then μ is of Barbasch-Evens type. We obtain the following Corollary immediately from (5.26) and (6.2).

Corollary 6.2.1 *Let $u, v \in V$, $I \subseteq S$ such that $v \star w_I = v$, and*

$$\mu : G_v \times^R P_I/B \to X_u. \tag{6.6}$$

Then for every $y \leq v$, we have

$$\dim(X_v^J \cap \dot{y}P_I/R) = \dim(K\dot{y}R/B) + \dim(P_I/B) - \dim(K\dot{y}P_I/B). \tag{6.7}$$

Proposition 6.2.2 *Let $u, v \in V_{p,q}$ and $I \subseteq S$ satisfy §5.3.2. Then for every $y \leq v$, we have*

$$\mathrm{KLV}_{y,u}(t) = (1+t)^{d_y}, \tag{6.8}$$

where $d_y = \#(I \cap \tau(y))$. By $(K \times P_I)$-equivariance, this describes all KLV polynomials for the clan u.

Proof. Recall that $\mu : G_v \times^B P_I/B \to X_u$ is of the form (6.1). For every $y \leq u$, we have

$$\mu^{-1}(\dot{y}B/B) \cong X_v \cap \dot{y}P_I/B \tag{6.9}$$

by (6.2). We have

$$P_I/B = \prod_{s \in I} P_s/B \tag{6.10}$$

since I consists of pairwise commuting simple reflections. It follows that

$$\mu^{-1}(\dot{y}B/B) \cong \prod_{s \in I} X_v \cap \dot{y}P_s/B. \tag{6.11}$$

By Zariski's main theorem, (6.11) is a connected variety contained in $\dot{y}P_s/B \cong \mathbf{P}^1$. It follows that

$$\mu^{-1}(\dot{y}B/B) \cong \prod_{s \in I} \mathbf{P}^1 \tag{6.12}$$

which has Poincare polynomial given by (6.8). ■

6.3 $Sp(6,\mathbf{R})$

We describe small resolutions of the form (6.1) for $Sp(6,\mathbf{R})$. Some of these resolutions are constructed by pulling back a small resolution from §III and others were found by ad-hoc methods. We provide a diagram for the Bruhat order of V_3 in Figure 6.1 which was generated by Atlas software [1] and checked using the following description of the monoid action.

If $v \in V_n$, then by [31, §3.4.1], we have

$$\ell(v) = \frac{1}{2}\left(\ell'(v) + \#\left\{t \in \mathbf{N} \mid c_s = c_t \in \mathbf{N},\ s \le n < t \le 2n+1-s\right\}\right), \tag{6.13}$$

where $\ell'(v)$ is the length of the clan $v \in V_{n,n}$. It follows from [31, §3.3.1] that for every $v \in V_n$ and $s_i \in S \smallsetminus \tau(v)$, we have one of the following two relations:

$$v \star s_i = \begin{cases} v \star s_i', & s_i' \in S' \smallsetminus \tau'(v), \\ v \star s_i' \star s_{2n-i}', & s_i', s_{2n-i}' \in S' \smallsetminus \tau'(v), \end{cases} \tag{6.14}$$

where $v \in V_{n,n}$, S' is the set of simple reflections for $G' = GL(2n,\mathbf{C})$, and $\tau'(v) \subseteq S'$. We also describe a formula for projecting $X_v \subseteq X$ to $X^{\hat{k}}$. Given $v \in V_n$, let $\pi(v) = (a,b,c)$ be defined by

$$a = \#\left\{1 \le i \le k \mid v_i = +\right\} + \#\left\{1 \le i < j \le k \mid v_i = v_j \in \mathbf{N}\right\}, \tag{6.15a}$$

$$b = \#\left\{1 \le i \le k \mid v_i = -\right\} + \#\left\{1 \le i < j \le k \mid v_i = v_j \in \mathbf{N}\right\}, \tag{6.15b}$$

$$c = \#\left\{1 \le i \le k \mid v_i = v_j \in \mathbf{N},\ k < j,\ i \ne 2n+1-j\right\}. \tag{6.15c}$$

The relation $\pi(X_v) = X^{\hat{k}}_{\pi(v)}$ follows from [31, 3.2.11], where π is projection of flag varieties. We recall [31, 3.2.11] to work with basepoints. For any $v \in V_n$, let $g \in G_v$

such that column i is given by

$$g_i = \begin{cases} e_i, & v_i = -,\ i \le n, \\ e_{2n+1-i}, & v_i = -,\ i > n, \\ e_i, & v_i = +,\ i \le n, \\ -e_{2n+1-i}, & v_i = +,\ i > n, \\ \frac{1}{\sqrt{2}}(e_i + e_{2n+1-j}), & v(i) = j,\ i < j \le n, \\ \frac{1}{\sqrt{2}}(e_i + e_j), & v(i) = j,\ i \le n < j, \\ -\frac{1}{\sqrt{2}}(e_{2n+1-i} + e_j), & v(i) = j,\ n < i < j, \\ \frac{1}{\sqrt{2}}(e_{2n+1-i} - e_j), & v(i) = j,\ j < i \le n, \\ \frac{1}{\sqrt{2}}(e_i - e_j), & v(i) = j,\ j \le n < i, \\ \frac{1}{\sqrt{2}}(e_i - e_{2n+1-j}), & v(i) = j,\ n < j < i, \end{cases} \tag{6.16}$$

where $v(i) = j$ as an involution.

Pull-back small resolutions

We list all small resolutions for $Sp(6, \mathbf{R})$ obtained by pulling back small resolutions from §III. We recall the following result of Sankaran-Vanchinathan [24, Theorem 2.4].

Proposition 6.3.1 ([24]) *Let $\zeta : \widetilde{Y} \to Y$ be a fiber bundle with smooth fibers between irreducible varieties. If $\mu : Z \to Y$ is a small resolution, then $\mu' : Z \times_Y \widetilde{Y} \to \widetilde{Y}$ is a small resolution.*

Let $u = (\ 1\ +\ 2\ 1\ -\ 2\ )$ so $\tau(u) = \{1, 3\}$ by (6.14). Define $\pi : X_u \to X^{\hat{2}}_{1,0,1}$, so $X_u = \pi^{-1}(\pi(X_u))$. Define $Z = X^{\hat{3}}_{2,1}$ and $\mu : Z \to X^{\hat{2}}_{1,0,1}$. Then μ is the small resolution given in Theorem 3.3.6 (A). The diagram

$$\begin{tikzcd} G_{2,1} \times^{P_{\{1\}}} P_{\{1,3\}}/B \arrow[r, "\mu'"] \arrow[d, "\pi'"] & X_u \arrow[d, "\pi"] \\ Z \arrow[r, "\mu"] & X^{\hat{2}}_{1,0,1} \end{tikzcd} \tag{6.17}$$

displays μ' as a base change of μ with respect to π. It follows that μ' is a small resolution since π is a fiber bundle. Similarly,

$$G_{1,2} \times^{P_{\{1\}}} P_{\{1,3\}}/B \to X_v \tag{6.18}$$

is a small resolution, where $v = (\ 1\ -\ 2\ 1\ +\ 2\ )$.

Let $u = (\ 1\ +\ 2\ 2\ -\ 1\ )$ so $\tau(u) = \{1,3\}$ by (6.14). Define $\pi : X_u \to X_{1,0}^{\{1,3\}}$, so $X_u = \pi^{-1}(\pi(X_u))$. Define $Z = X_{1,0}^{\{2,3\}}$ and $\mu : Z \to X_{1,0}^{\{1,3\}}$. Then μ is a small resolution given in Theorem 3.3.6 (B). The diagram

$$\begin{tikzcd} G_{1,0} \times^{P_3} P_{\{1,3\}}/B \arrow[r, "\mu'"] \arrow[d, "\pi'"] & X_u \arrow[d, "\pi'"] \\ Z \arrow[r, "\mu"] & X_{1,0}^{\{1,3\}} \end{tikzcd} \tag{6.19}$$

displays μ' as a base change of μ with respect to π. So μ is a small resolution of X_u and

$$G_{0,1} \times^{P_3} P_{\{1,3\}}/B \to X_v \tag{6.20}$$

is a small resolution, where $v = (\ 1\ -\ 2\ 2\ +\ 1\ )$.

More small resolutions

Let $u = (\ 1\ 2\ 1\ 3\ 2\ 3\ )$ and $v = (\ 1\ 1\ 2\ 2\ 3\ 3\ )$. Then $v \star s_2 = u$ by (6.14). Define

$$\mu : G_v \times^B P_2/B \to X_v \tag{6.21}$$

which is generically finite by (2.50). One can show that μ is birational using root types (q.v. [28]), and we will show this directly. Let $F^\bullet \in \mathcal{Q}_u$ be the basepoint corresponding to the basis

$$(e_1 + e_5,\ e_3 + e_4,\ e_1,\ e_2 + e_6,\ e_3,\ e_2)\,. \tag{6.22}$$

Then

$$\mu^{-1}(F^\bullet) \cong \left\{E^2 \subseteq \mathbf{C}^6 \mid F^1 \subseteq E^2 \subseteq F^3,\ \dim(\mathbf{C}^{\pm 3} \cap E^2) = 1\right\} \tag{6.23}$$

is a point, since $E^2 = \langle e_1, e_5\rangle$. It follows that μ is birational. Then Lemma 5.3.1 shows that μ is a small resolution.

Let $u = (\ 1\ 2\ 3\ 1\ 2\ 3\ )$, $v = (\ 1\ 1\ 2\ 2\ 3\ 3\ )$, and $w = s_3 s_2 s_3$. Then $v \star w = u$ by (6.14). Define

$$\mu : G_v \times^{P_3} X_w \to X_u. \tag{6.24}$$

For every $y \le u$, we have

$$\mu^{-1}(\dot{y}B/B) \cong X_v^{\{3\}} \cap \dot{y} X_{w^{-1}}^{\{3\}} \tag{6.25}$$

by (6.2), where

$$X_v^{\{3\}} = \left\{F^\bullet \mid \dim(\mathbf{C}^{\pm 3} \cap F^2) = 1\right\}, \tag{6.26a}$$

$$\dot{y} X_w^{\{3\}} = \left\{F^\bullet \mid \dot{y}^{-1}\mathbf{C}^1 = F^1,\ F^2 \subseteq \dot{y}^{-1}\mathbf{C}^4\right\}, \tag{6.26b}$$

where a direct calculation computes X_w as the image of $P_3 \times^B P_2 \times^B P_3/B$. If $y = u$, then

$$\mu^{-1}(\dot{y}B/B) \cong \left\{F^\bullet \mid \dim(\mathbf{C}^{\pm 3} \cap F^2) = 1,\ \langle e_1 + e_4\rangle = F^1,\ F^2 \subseteq \dot{y}^{-1}\mathbf{C}^4\right\} \tag{6.27a}$$

$$= \{\langle e_1, e_4\rangle\} \tag{6.27b}$$

shows that μ is birational. If $\dot{y} \in N_G(T)$, then

$$\mu^{-1}(\dot{y}B/B) \cong \begin{cases} \mathbf{P}^1, & \mathbf{C}^3 \subseteq \dot{y}^{-1}\mathbf{C}^4,\ \dot{y}^{-1}\mathbf{C}^1 \subseteq \mathbf{C}^{-3}, \\ \mathbf{P}^1, & \mathbf{C}^{-3} \subseteq \dot{y}^{-1}\mathbf{C}^4,\ \dot{y}^{-1}\mathbf{C}^1 \subseteq \mathbf{C}^3, \\ \text{pt}, & \text{else}. \end{cases} \tag{6.28}$$

Considering basepoints of closed K-orbits shows that

$$d_y = \begin{cases} 1, & y = (\ +\ +\ -\ +\ -\ -\ ), \\ 1, & y = (\ -\ -\ +\ -\ +\ +\ ), \\ 0, & \text{else}, \end{cases} \tag{6.29}$$

where upper-semicontinuity forces $d_y = 0$ for all remaining $y \le u$ (q.v. Figure 6.1).

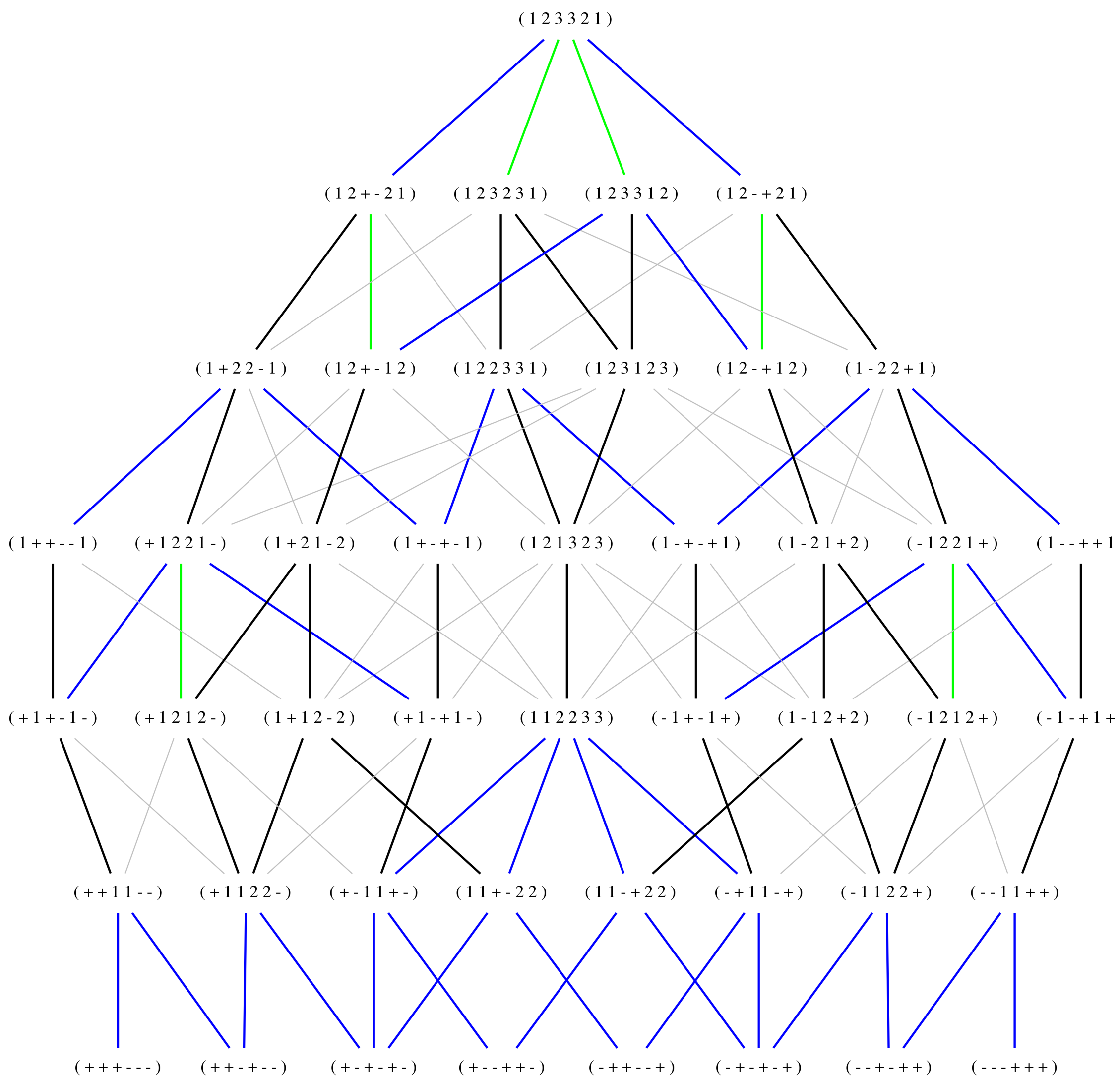

Figure 6.1: Bruhat order for $Sp(6, \mathbf{R})$

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VITA

Scott Larson

Candidate for the Degree of

Doctor of Philosophy

Dissertation: SMALL RESOLUTIONS OF CLOSURES OF K-ORBITS IN FLAG VARIETIES

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Mathematics at Oklahoma State University, Stillwater, Oklahoma in May, 2020.

Completed the requirements for the Master of Science in Mathematics at Oklahoma State University, Stillwater, Oklahoma in 2013.

Completed the requirements for the Bachelor of Science in Mathematics and Physics at University of Wisconsin – Eau Claire, Eau Claire, Wisconsin in 2009.