POLYNOMIAL APPROXIMATION OF CR SINGULAR FUNCTIONS

By

TREVOR FANCHER

Bachelor of Science in Mathematics

Oklahoma State University

Stillwater, Oklahoma

2017

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
MASTER OF SCIENCE
May, 2020

POLYNOMIAL APPROXIMATION OF CR SINGULAR FUNCTIONS

Thesis Approved:

Dr. Jiří Lebl

Thesis Adviser

Dr. Alan Noell

Dr. Paul Fili

ACKNOWLEDGMENTS

Thank you, Jiří, without whom I never would have begun.

And, thank you, Sarah, without whom I never would have finished.

Name: TREVOR FANCHER

Date of Degree: MAY, 2020

Title of Study: POLYNOMIAL APPROXIMATION OF CR SINGULAR FUNCTIONS

Major Field: MATHEMATICS

Abstract: We begin by recalling some basic facts about continuity and differentiability in the one real variable setting. Following a quick discussion on what classical mathematicians had in mind when dealing with continuous functions, we present Takagi's everywhere continuous but nowhere differentiable function. After recalling some basic facts about holomorphic functions, we present the theorems of Runge and Mergelyan. We mention why no direct generalization of Mergelyan's result is possible in the context of several complex variables and then move on to the theory of CR functions on CR submanifolds of $\mathbb{C}^n$ in order to state the polynomial approximation theorem of Baouendi–Trèves. We then begin discussion of how much of the Baouendi–Trèves theorem we can recover in the CR singular setting, that is, when our submanifold is no longer CR. For certain functions on a particular CR singular submanifold, we ask the question: Can we find approximating polynomials that are holomorphic in some variables, but perhaps not holomorphic in all variables? We end by answering the question in the affirmative.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

POLYNOMIAL APPROXIMATION IN ONE REAL VARIABLE

We begin by recalling some basic facts about continuity and differentiability in the one real variable setting. Following a quick discussion on what classical mathematicians had in mind when dealing with continuous functions, we present Takagi's everywhere continuous but nowhere differentiable function. Included are plots of Takagi's function as well as plots of a particular polynomial approximation.

1.1 TAKAGI'S NOWHERE DIFFERENTIABLE FUNCTION

In the mathematical discipline of analysis, the most elementary class of functions are those real-valued functions of one real variable, that is, those functions f that map from $\mathbb{R}$ to $\mathbb{R}$. Classification of arbitrary such f forces us to consider examples such as Dirichlet's function, i.e. the function that is 1 on the rationals and 0 on the irrationals, or Thomae's popcorn function, which is discontinuous at all rationals but somehow continuous at all irrationals. By considering a more restrictive class of functions, for example, the continuous functions, one can hope to find common features amongst the entire class that yield interesting perspectives and useful classifications.

Recall that continuous functions are, loosely, those for which a small change in the input variable yields a small change in the corresponding output. More formally, in this setting we can take f being continuous at some point c to mean that $f(x) \to f(c)$ as $x \to c$. Bernard Bolzano and Karl Theodor Wilhelm Weierstrass were the first mathematicians to give a modern definition of limit which allowed them to rigorously work with continuous functions. Having a well specified definition of limit allows us to also ask whether a function is well approximated by an affine function at a given point c, where affine here means a function of the form $x \mapsto m(x-c) + f(c)$ for some slope constant m. If $\frac{f(x)-f(c)}{x-c} \to m$ as $x \to c$, then such an affine function exists. In this case, we say m is the derivative of f at c and such an f is said to be differentiable at c.

A basic property of differentiable functions is that they are continuous. That is, the class of continuous functions already includes the differentiable functions. From the time of the discovery of the derivative by Isaac Newton and Gottfried Wilhelm Leibniz in the late seventeenth-century, many mathematicians thought that a continuous function would be differentiable except outside a set of isolated points. The primary such example where this is true is the absolute value function $x \mapsto |x|$. This function is continuous everywhere and differentiable at all points outside the origin. However, it was Weierstrass who first showed that no such fact is true in general. In fact, Weierstrass went further and gave a class of functions that are continuous *everywhere* but differentiable *nowhere*. This was the first hint that the class of continuous functions are in some sense much larger than the differentiable functions and certainly must contain strange specimens. An example of such a function, although not the one originally given by Weierstrass, was discovered by Teiji Takagi. First published in the 1901 paper [14], the function is defined as follows

$$T(x) := \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n},$$

where $0 \leq \phi(x) \leq 1/2$ is defined to be the distance from x to the nearest integer. The first 3 terms of $T(x)$ are $\phi(x)$, $\phi(2x)/2$, and $\phi(4x)/4$, and each is plotted in Figure 1.1 together with the 10^{th} partial sum of T. Because T is 0 on the integers and periodic with period 1, that is $T(x) = T(x+1)$ for each x, we may restrict our attention to the interval $[0, 1]$. By using what has become known as the Weierstrass M-test, we see that the series T absolutely and uniformly converges on $[0, 1]$ because it is termwise bounded above by the geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$. In fact, this shows that T is continuous on $[0, 1]$ because each term of T is evidently continuous and continuity is preserved under uniform convergence. Furthermore, we can give a heuristic argument that T is differentiable nowhere. From the first 3 plots, we see that the n^{th} term of T is introducing 2^n nondifferentiable peaks into T. Taken as a whole, these peaks are dense on the interval $[0, 1]$ and so, heuristically, the points that are nondifferentiable are at a minimum dense in $[0, 1]$. While this heuristic argument falls short of proving nondifferentiability at, for example, the irrational numbers, it turns out that T is indeed differentiable nowhere. Such a function tells us that the continuous functions defy the naive expectation that continuous functions should be differentiable except outside a small collection of points.

So it was Weierstrass who first discovered that continuous functions can in some sense behave much more wildly than some mathematicians had previously believed. But it was also Weierstrass who discovered that, in another sense, continuous functions are quite well behaved. That is, continuous functions are well approximated by polynomials. Now, from an analytical perspective, polynomials are among the nicest functions. They are real analytic on the entire real line and so are, in particular, infinitely differentiable everywhere. It seems somewhat surprising that so nicely a behaved class of functions as the polynomials could be enough to approximate functions from such a wildly behaved class as the continuous functions. Here, the kind of approximation of which we speak is uniform convergence of functions. Recall that a sequence of real-valued functions (f_n) each defined on some set X is said to uniformly converge to some function f on X if the quantity $m_n = m_n(f, f_n, X) := \sup_{x \in X} |f(x) - f_n(x)|$ is such that $m_n \to 0$ as $n \to \infty$. Here m_n can be thought of as

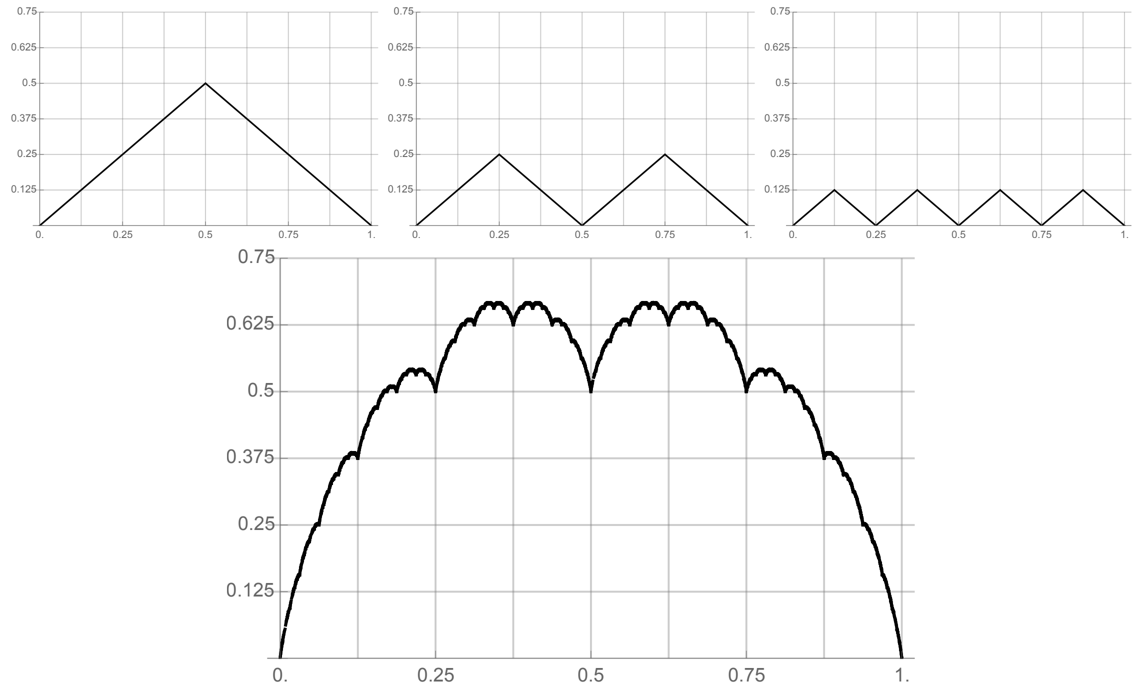

Figure 1.1: Plots of the first 3 terms of the Takagi function and the 10$^{\text{th}}$ partial sum.

representing the pointwise maximum difference between the functions f and f_n across the entire set X. Provided that this maximum difference tends to 0 as n gets large, then the sequence (f_n) can be understood to be well approximating f on the set X.

1.2 WEIERSTRASS'S APPROXIMATION THEOREM

Theorem 1 (Weierstrass)**.** *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then f is the uniform limit of polynomials in x.*

If one held the belief that continuous functions were nondifferentiable on at most a small set, then the preceding theorem of Weierstrass (see [16] and theorem 11.7.1 of [5]) seems only mildly impressive. But when considered alongside the nowhere differentiable continuous functions, such as Takagi's function, at first glance this sort of approximation may seem impossible. Nevertheless, this theorem tells us that any continuous function is, locally at least, uniformly approximable by polynomials. What we mean by locally in this context is uniform convergence on compact intervals. For example, given a continuous function

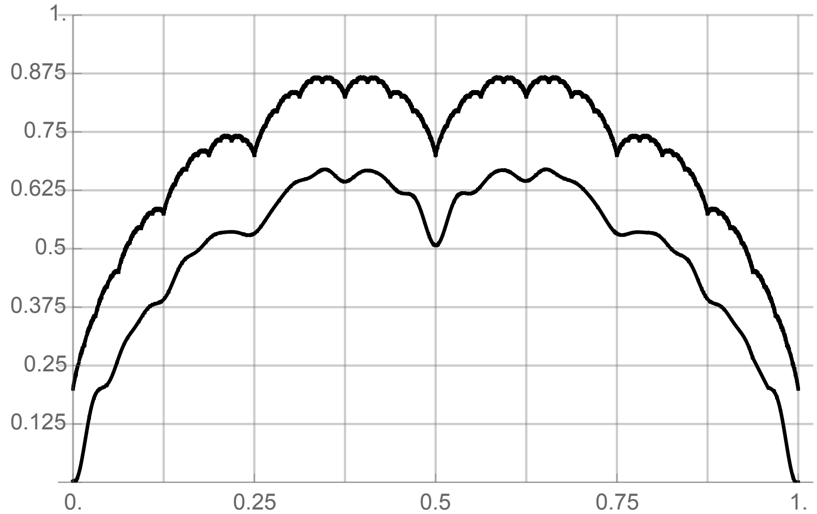

Figure 1.2: Plot of $T(x) + 0.2$ and a polynomial approximation of $T(x)$.

f on some interval, say $(2, 5)$, then for any closed interval inside that interval, say $[2 + \varepsilon, 5 - \varepsilon]$ for however small $\varepsilon > 0$ we wish, we may uniformly approximate f on that closed interval. Taking a closed interval is necessary because, for example, the function $x \mapsto \sin(1/x)$ is continuous on the interval $(0, 1)$ but its rapid oscillation near 0 prevents uniform approximation via polynomials over the entire interval $(0, 1)$.

In Figure 1.2 we give a plot of Takagi's function (shifted up by 0.2 for comparison) and a polynomial approximation of Takagi's function. Weierstrass guarantees the existence of well approximating polynomials but finding them explicitly can be difficult. For example, the polynomial plotted in the figure has degree 350 and largest coefficient approximately 5.52663×10^{58}. The code to polynomialy approximate Takagi's function and the code to generate Figure 1.1 is located in Appendix A.

CHAPTER II

POLYNOMIAL APPROXIMATION IN ONE COMPLEX VARIABLE

We now focus on complex-valued functions of one complex variable. After recalling some basic facts about holomorphic functions, we present a version of Runge's theorem that gives uniform approximation by polynomials. After discussing why the hypothesis of Runge's theorem is necessary by way of example, we do the same for Mergelyan's theorem.

2.1 HOLOMORPHIC FUNCTIONS OF ONE COMPLEX VARIABLE

The complex numbers may be identified with the vector space $\mathbb{R}^2$ along with multiplication of vectors $z = (x, y)$ and $w = (s, t)$ defined as

$$zw = (x, y)(s, t) = (xs - yt, xt + ys) \in \mathbb{R}^2.$$

We write $\mathbb{C}$ for $\mathbb{R}^2$ along with this multiplication of vectors and we call $\mathbb{C}$ the complex plane. If we further identify the standard basis of $\mathbb{R}^2$ as $1 := (1, 0)$ and $i := (0, 1)$, then we may write each vector in $\mathbb{C}$ as a linear combination of 1 and i to get the more familiar notation for $z \in \mathbb{C}$ as $z = 1x + iy = x + iy$. We say x is the real part of z and write

$\operatorname{Re} z = x$ and, similarly, we say y is the imaginary part of z and write $\operatorname{Im} z = y$. Note that the basis vector i is a square root of negative one because $i^2 = (-1, 0) = -1(1, 0) = -1$ under our identifications. It should be noted that $\mathbb{C}$, with normal vector addition and the above definition of multiplication, is an algebraic field. The only nontrivial thing to check is that all complex numbers have inverses and this can be seen by noting that for the arbitrary nonzero complex number $x + iy$ the complex number $\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ is its inverse. Working in a field allows us to divide and with division we are able to write down the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{II.1}$$

for a complex valued function f defined in an open neighborhood around z in $\mathbb{C}$. It is important to note that we take the limit as $h \to 0$ through $\mathbb{C} = \mathbb{R}^2$ and that this is equivalent to $\operatorname{Re} h$ and $\operatorname{Im} h$ both approaching 0 independently of one another. The form of (II.1) reminds us of the definition a single variable real differentiable function and so when f is defined in a neighborhood around z, we say that f is **complex-differentiable** at z if the limit in (II.1) exists. In this case, we write

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

and call the complex number $f'(z)$ the complex-derivative of f at z. When such an f is complex-differentiable at each point of an open set U of $\mathbb{C}$, we say that f is **holomorphic** in U. Our primary reference for general holomorphic function theory in one variable is [15].

Recall that for a real-valued function f of two real variables x and y, we have the partial derivatives

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

where now h is a real number and so the limit is taken on the real line. If we assume f to be complex valued and holomorphic in the neighborhood of the point $z = (x, y)$ we have, by using the aforementioned fact that we may take $h = (\operatorname{Re} h, \operatorname{Im} h) \to 0$ in any way we wish,

$$f'(z) = \lim_{h \in \mathbb{R} \to 0} \frac{f(x+h, y) - f(z)}{h} = \frac{\partial f}{\partial x}(z).$$

Similarly, we also have, now by using the fact that $\frac{1}{i} = -i$,

$$f'(z) = \lim_{h \in \mathbb{R} \to 0} \frac{f(x, y+h) - f(z)}{ih} = -i\frac{\partial f}{\partial y}(z).$$

Adding these two equations gives us that

$$f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z) - i\frac{\partial f}{\partial y}(z)\right).$$

The above observation led Wilhelm Wirtinger to make the definitions

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and so we call $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ the **Wirtinger operators**. These operators allow us to make precise the notion that a holomorphic should depend on z but not on $\overline{z}$. That is, for a complex differentiable function f, we have $f'(z) = \frac{\partial f}{\partial z}(z)$ and $\frac{\partial f}{\partial \overline{z}} = 0$. The equation $\frac{\partial f}{\partial \overline{z}} = 0$ is commonly called the Cauchy–Riemann equations, where the plural is coming because any complex equation may be thought of as two real equations. That is, if we write f separately in its real and imaginary parts as

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions of two real variables such that for each x, y,

$u(x, y) = \mathrm{Re}\,(f(x, y))$ and $v(x, y) = \mathrm{Im}\,(f(x, y))$, then we see that

$$2\frac{\partial}{\partial \overline{z}} f = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Because a complex equation is 0 if and only if its real and imaginary parts are 0, we can say that, at a point where f is real differentiable, $\frac{\partial}{\partial \overline{z}} f = 0$ if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These equations taken together are called the classical Cauchy–Riemann equations. What we have shown so far is that saying a holomorphic f satisfies $\frac{\partial}{\partial \overline{z}} f = 0$ is equivalent to saying that f satisfies the Cauchy–Riemann equations. It turns out that any continuously differentiable complex valued function of a complex variable that satisfies the Cauchy–Riemann equations near a point z is in fact holomorphic near z. And so we say that if a function f satisfies $\frac{\partial}{\partial \overline{z}} f = 0$ at all points near z, then f is holomorphic near z. We have justified the statement that "a function is holomorphic if it does not depend on $\overline{z}$" provided that we interpret what it means to "depend on $\overline{z}$" correctly.

In order to state the next result, we first recall some basic facts about integration of one-forms over the plane. If $\gamma = (\gamma_1, \gamma_2)\colon [0, 1] \to \mathbb{R}^2$ is a piecewise smooth function with a non-vanishing Jacobian, i.e. $\left[\frac{\partial \gamma}{\partial x}\Big|_t \;\; \frac{\partial \gamma}{\partial y}\Big|_t\right] \neq [0 \quad 0]$ for each $t \in [0, 1]$, then we define $\gamma^* := \gamma([0, 1])$ and call γ^* a **path**. If f and g are integrable functions of two real variables, then we call the formal object $f\,dx + g\,dy$ a one-form. We define integration of the one-form $f\,dx + g\,dy$ over a path γ^* as

$$\int_\gamma f\,dx + g\,dy := \int_0^1 f(\gamma_1(t), \gamma_2(t))\,\frac{\partial \gamma_1}{\partial t}\bigg|_t\,dt + \int_0^1 g(\gamma_1(t), \gamma_2(t))\,\frac{\partial \gamma_2}{\partial t}\bigg|_t\,dt.$$

Let $f = u + iv$ be the decomposition of a complex-valued function f of a complex variable into its real part u and imaginary part v. We wish to integrate f over a path γ^*. To do so, we first make the definition $dz = dx + i\,dy$ so that

$$f\,dz = (u + iv)(dx + i\,dy) = u\,dx - v\,dy + i(v\,dx + u\,dy).$$

Assuming both u and v are integrable, it now makes sense to define the integration of f over a path γ^* as

$$\int_\gamma f\,dz = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

In fact, if we write $\gamma = \gamma_1 + i\gamma_2$ so that $\gamma' = \frac{\partial\gamma}{\partial t} = \frac{\partial\gamma_1}{\partial t} + i\frac{\partial\gamma_2}{\partial t}$, then we may make the further simplification that

$$\int_\gamma f\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt. \tag{II.2}$$

Example 2. We compute the integral of z^n around the boundary $\partial\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| = 1\}$ of the unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| < 1\}$. With the convenient notation $e^z = e^{x+iy} := e^x(\cos(y) + i\sin(y))$, we may parameterize $\partial\mathbb{D}$ by $\gamma(t) = e^{2\pi it}$ for $t \in [0,1]$. By using equation II.2 we have

$$\int_\gamma z^n\,dz = \int_0^1 e^{2\pi int}2\pi ie^{2\pi it}\,dt = 2\pi i\int_0^1 e^{2\pi i(n+1)t}\,dt.$$

Note that, if $n = -1$, then the integral becomes $2\pi i\int_0^1 dt = 2\pi i$. On the other hand, if $n \neq -1$ then

$$\begin{aligned}\int_0^1 e^{2\pi i(n+1)t}\,dt &= \int_0^1 \cos(2\pi(n+1)t)\,dt + i\int_0^1 \sin(2\pi(n+1)t)\,dt\\ &= \left.\frac{\sin(2\pi(n+1)t)}{2\pi(n+1)}\right|_0^1 - i\left.\frac{\cos(2\pi(n+1)t)}{2\pi(n+1)}\right|_0^1 = 0\end{aligned}$$

because both cos and sin are 2π periodic. ////

We are now ready to state a fundamental result of holomorphic functions.

Theorem 3 (Cauchy's Integral Formula)**.** *Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and let γ be a path once around the boundary of $\mathbb{D}$ in the counterclockwise direction so that $\gamma^* = \partial\mathbb{D}$. If f is holomorphic in a neighborhood of the closure of $\mathbb{D}$, then for each $w \in \mathbb{D}$ it holds that*

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dz.$$

It should be noted Cauchy's integral formula (see [4] and theorem 2.5 of [15]) holds in much greater generality. But even as stated above, the theorem is quite remarkable. Notice that the formula is calculating function values $f(w)$ for w inside the unit disc but the function is only evaluated on the relatively small boundary set of the disc. From a measure theory point of view, the values of a function on a set of positive measure are being recovered by integrating a related function over a set of measure 0. This must mean that holomorphic functions are quite restricted as compared to, say, the smooth functions. For example, there exists smooth functions that are identically 0 on the boundary of $\mathbb{D}$ but that are 1 at the origin.

2.2 RUNGE'S POLYNOMIAL APPROXIMATION THEOREM

Before we state Runge's theorem we first point out an important difference of polynomials in z as compared to polynomials in x and y. We say a polynomial P is a polynomial in z if $P = \sum_{j=1}^{n} a_j z^j$ for some natural number n and complex numbers a_j. Note that P is holomorphic over all of $\mathbb{C}$ because $\frac{\partial P}{\partial \bar{z}} = 0$ at each point of $\mathbb{C}$. If we take an arbitrary polynomial Q of two real variables x and y, then it is not necessarily true that Q is holomorphic. For example, $\bar{z} = x - iy$ is such a polynomial. A generalization of Weierstrass's approximation result of Theorem 1 gives polynomials in x and y uniformly converging to any continuous function on a compact subset K of $\mathbb{C}$. In order to use the tools of holomorphic function theory, we wish to approximate by polynomials in z. The first such approximation theorem we present was proved by Carl Runge in 1885 (see [13] and theorem 13.7 of [12]).

Theorem 4 (Runge). *Let f be holomorphic in an open set U of $\mathbb{C}$ and let K be a compact subset of U. If $\mathbb{C} \setminus K$ is connected, then f is the uniform limit of holomorphic polynomials on K.*

We first note that if we drop the requirement that $\mathbb{C} \setminus K$ be connected, then f may be approximated by rational functions in z. The rational approximation version is the more classical statement, but here we are interested in polynomial approximation. By way of the next example we show that $\mathbb{C} \setminus K$ is connected is a necessary condition.

Example 5. In Example 16 of chapter III we will need that $f(z) = \overline{z}$ is not the uniform limit of polynomials on a circle centered at the origin. We can prove this fact while simultaneously showing that $\mathbb{C} \setminus K$ being connected is necessary in Runge's theorem. Fix some $r > 0$ and consider the holomorphic function $g(z) = r/z$ on the open set $U = \mathbb{C} \setminus \{0\}$. Note that, on the compact set $K = \{z \in \mathbb{C} \colon\ |z|^2 = r\}$, we have $g \equiv f$. Furthermore, $\mathbb{C} \setminus K$ is disconnected. Now let p be some polynomial in z. We claim that there is some $z \in K$ such that $|rp(z) - f(z)| \geq \sqrt{r}$. For suppose not. Then $|rp(z) - f(z)| < \sqrt{r}$ for each $z \in K$. Note that, on K,

$$|rp(z) - f(z)| < \sqrt{r} \iff |rp(z) - g(z)| < \sqrt{r} \iff |p(z) - 1/z| < \sqrt{r}/r = 1/\sqrt{r}$$

Now, from our work in Example 2, if we integrate around the circle K of radius $\sqrt{r}$ in the counterclockwise direction, then we have, by continuity of $z \mapsto |1/z - p(z)|$ on K,

$$\begin{aligned} 2\pi = \left| \int_K 1/z\, dz - 0 \right| = \left| \int_K 1/z\, dz - \int_K p(z)\, dz \right| &= \left| \int_K 1/z - p(z)\, dz \right| \\ &\leq \int_K |1/z - p(z)|\, |dz| \\ &< \int_0^{2\pi} (1/\sqrt{r})\sqrt{r}\, dt \\ &= 2\pi, \end{aligned}$$

which is clearly a contradiction. Therefore there is some $z \in K$ such that $|rp(z) - f(z)| \geq \sqrt{r}$ for every polynomial p. But, any sequence of polynomials (q_n) may be written $(r(q_n/r))$ and so the polynomials q_n do not uniformly converge to $f \equiv g$ on K. This shows that $f(z) = \overline{z}$ is not the uniform limit of polynomials in any neighborhood of the origin and, because $g(z) = r/z$ is holomorphic in the neighborhood $\mathbb{C} \setminus \{0\}$ of K, that the condition $\mathbb{C} \setminus K$ is connected is necessary in Runge's theorem. ////

2.3 MERGELYAN'S POLYNOMIAL APPROXIMATION THEOREM

The polynomial approximation version of Runge's theorem discussed in the previous section starts with a holomorphic function on some open set U and gives approximating polynomials on compact subsets K provided that K does not disconnect the plane. We showed that it is necessary that K does not disconnect the plane, but an interesting question remains: What if we start with a function f continuous on a compact set K? When is f the uniform limit of holomorphic polynomials? Mergelyan's theorem (see [11] and theorem 20.5 of [12]) gives us the best possible answer to such a question.

Theorem 6 (Mergelyan)**.** *Let K be a compact subset of $\mathbb{C}$ and let f be a continuous function on K. If f is holomorphic in the interior of K and $\mathbb{C} \setminus K$ is connected, then f is the uniform limit of polynomials in z.*

In Example 5 we have already seen that the condition that $\mathbb{C} \setminus K$ is connected really is necessary. Further, recall that if a sequence of holomorphic functions f_n converge to some holomorphic function f uniformly on compact subsets, then f is itself holomorphic. Therefore the condition of Mergelyan's theorem that f is holomorphic in the interior of K is necessary because the uniform limit of polynomials must necessarily be holomorphic in the interior of K. It is in this sense, namely that each condition placed on f and K is necessary, that we say Mergelyan's gives us the best possible answer to polynomial approximation of holomorphic functions on compact subsets of the complex plane. As an interesting

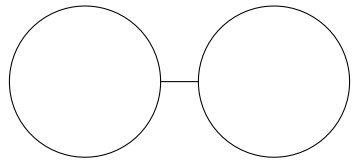

Figure 2.1: Figure of K with $\varepsilon = 1/4$.

application of Mergelyan's theorem, consider the following example.

Example 7. Fix some $\varepsilon > 0$. Put $F = \{z\colon |z - (-1 - \varepsilon)| \le 1\}$, $G = [-\varepsilon, \varepsilon]$, and $H = \{z\colon |z - (1 + \varepsilon)| \le 1\}$. Now put $K = F \cup G \cup H$. See Figure 2.1. Let f be holomorphic in a neighborhood of F, let h be holomorphic in a neighborhood of H, and let g be some continuous function on G such that $g(-\varepsilon) = f(-\varepsilon)$ and $g(\varepsilon) = h(\varepsilon)$. Put

$$k(z) = \begin{cases} f(z) & \text{if } z \in F, \\ g(z) & \text{if } z \in G, \text{ and} \\ h(z) & \text{if } z \in H, \end{cases}$$

so that k is continuous on K and holomorphic in the interior of K. By Mergelyan's theorem there exists a sequence of holomorphic polynomials that uniformly converge to k on all of K. To see why such an approximation is remarkable, consider $f(z) = e^{1/z}$, $h(z) = e^z$, and g any continuous function (of which there are many exotic choices) equal to $f(-\varepsilon)$ at $-\varepsilon$ and equal to $h(\varepsilon)$ at ε. By making ε small we may make $f(-\varepsilon)$ as large as we wish while simultaneously making $h(\varepsilon)$ as close to 1 as we wish. Even still, Mergelyan's theorem gives us uniformly approximating polynomials for each $\varepsilon > 0$. ////

CHAPTER III

POLYNOMIAL APPROXIMATION IN SEVERAL COMPLEX VARIABLES

We begin this chapter by developing enough of the theory of several complex variables and submanifolds of $\mathbb{C}^n$ in order to state the polynomial approximation theorem of Baouendi–Trèves. After starting with the basic definitions of holomorphic functions of several complex variables, we then discuss the basic results that generalize from the one variable case. No direct generalization of Mergelyan's result stated in Theorem 6 is possible in the context of several complex variables and so we take a detour in order to build the machinery of real submanifolds of complex space and their respective tangent spaces. A particular class of these submanifolds we will call CR submanifolds and we will study those functions that are killed by the antiholomorphic tangential CR equations of these CR submanifolds. Such functions are known as CR functions and the theorem of Baouendi–Trèves tells us that they may be approximated locally by holomorphic polynomials. We then begin discussion of how much of the Baouendi–Trèves theorem we can recover in the CR singular setting, that is, when our submanifold is no longer CR.

3.1 HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

We wish now to discuss complex valued functions of several complex variables. Our primary reference for the general theory of several complex variables is [6]. Fix some open set U from $\mathbb{C}^n$ and some complex valued function f defined on U. We say that f is **holomorphic** in U if it is locally bounded and, for each $z = (z_1, \ldots, z_n) \in U$ and each $j = 1, \ldots, n$, it holds that

$$\lim_{h \in \mathbb{C} \to 0} \frac{f(z_1, \ldots, z_j + h, \ldots, z_n) - f(z_1, \ldots, z_n)}{h}$$

exists as a complex number. If we let e_j be the standard basis vector of $\mathbb{C}^n$ with 1 in the j^{th} component and a 0 in the other components, then we may write the limit as $\lim_{h \to 0} \frac{f(z+he_j)-f(z)}{h}$. Two things should be noted about this definition; the first is that, by a theorem of Friedrich Hartogs, one may drop the locally bounded hypothesis and obtain the same class of functions; the second is that, a holomorphic function of several variables is a holomorphic function of a single variable in each variable individually. That is, if near a point $p \in U$ we define $g(z) = f(p + ze_j)$, then we know that the limit

$$g'(z) = \lim_{h \to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h \to 0} \frac{f((p + ze_j) + he_j) - f(p + ze_j)}{h}$$

exists. This means that g is a single variable holomorphic function and so, as before, we have that

$$g' = \frac{\partial g}{\partial z} \quad \text{and} \quad \frac{\partial g}{\partial \overline{z}} = 0.$$

We generalize the Wirtinger operators from one variable to several variables with the notation

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

so that we may simply write $\frac{\partial f}{\partial z_j}$ for what we have been calling g' and so that we may say a several variable holomorphic function f satisfies $\frac{\partial f}{\partial \bar{z}_j} = 0$ for each $j = 1, \ldots, n$.

3.2 GENERALIZING MERGELYAN VIA SUBMANIFOLDS AND CR FUNCTIONS

One might hope Mergelyan's theorem has a direct generalization to several variables. But any direct generalization of Mergelyan's theorem will be presented with the issue of dealing with the different way that subsets of $\mathbb{C}^n$, when $n > 1$, interact with the complex structure of $\mathbb{C}^n$ as compared to $\mathbb{C}$. In $\mathbb{C}$ this turns out to be a topological concern because the complex structure of $\mathbb{C}$ only sees the interior of sets. This is why in Mergelyan's theorem it was enough to suppose that the function we wished to approximate was holomorphic on the interior of the compact set. This assumption will not work in higher dimensions because the complex structure of higher dimensional complex spaces may see sets with no interior. One way to deal with this is to introduce the notion of real submanifolds of complex space and their tangent spaces. This will allow us to talk about the geometry of subsets of higher dimensional complex space and how those subsets interact with the complex structure there. From there we will discuss a new class of functions, the CR functions, defined on a particular class of these submanifolds. In the next section we will present the Baouendi–Trèves polynomial approximation theorem of these CR functions.

3.2.1 REAL SUBMANIFOLDS AND THEIR TANGENT SPACES

We begin with an introduction to submanifolds. Our primary source for the theory of submanifolds is [10]. We say a set $M \subset \mathbb{R}^n$ endowed with the subspace topology is a **smooth embedded real submanifold of $\mathbb{R}^n$ with real codimension** k, or just a **real submanifold** when the rest is clear from context, if for each point p in M there is a neighborhood U of p and a smooth function $\rho \colon U \to \mathbb{R}^k$ such that the zero set of ρ is $U \cap M$ and the real Jacobian matrix of ρ is full rank on U. We call ρ a **defining function** for M

near p. Because the Jacobian of a defining function is everywhere full rank, we may apply the implicit function theorem to ρ to write some neighborhood of p in M as the graph of some $\mathbb{R}^k$-valued smooth function of $n - k$ real variables. This allows us to think of real submanifolds of $\mathbb{R}^n$ as those subsets of $\mathbb{R}^n$ that are locally the graph of a smooth function.

Example 8. The unit circle $\partial\mathbb{D} = \{z = x + iy \in \mathbb{C} \colon 1 = |z| = x^2 + y^2\}$ is a smooth real submanifold of $\mathbb{C}$ with real codimension 1 because the function $\rho \colon \partial\mathbb{D} \to \mathbb{R}$ given by $\rho(x, y) := 1 - x^2 - y^2$ is 0 exactly on $\partial\mathbb{D}$ and its Jacobian matrix $[-2x \quad -2y]$ does not vanish on a neighborhood of $\partial\mathbb{D}$. Note that $\partial\mathbb{D}$ is an example of a real submanifold having a so called global defining function because the neighborhood U of any point in our definition of real submanifold may be taken as the entire submanifold. We do not in general require defining functions to be global because the results we are interested in are of a local nature. ////

In three real dimensions a surface is the graph Γ of a real-valued function f of two real variables. Provided f is smooth we have that $\rho(x, y, z) := z - f(x, y)$ is a defining function because ρ is 0 exactly on Γ and its Jacobian is $[-f_x \quad -f_y \quad 1]$, which clearly never vanishes. Real codimension 1 submanifolds of $\mathbb{R}^n$ are of enough general interest that we call these submanifolds **real hypersurfaces**. So, for example, the unit circle of Example 8 is a real hypersurface of $\mathbb{R}^2$.

Going forward, we will be interested in real submanifolds of $\mathbb{C}^n = \mathbb{R}^{2n}$ of codimension $k > 1$. Generically, these may be thought of as the transversal intersection of k real hypersurfaces in $\mathbb{C}^n$. Transversal intersection, as opposed to tangential intersection, is enforced by the condition that the differentials of the defining functions form a linearly independent set. That is, if $\rho_1, \ldots, \rho_k$ define k real hypersurfaces in $\mathbb{C}^n$, then, in order to say their common zero set is a real submanifold of codimension k, we require that the real Jacobian of $\rho := (\rho_1, \ldots, \rho_k)$ to be full rank, which is equivalent to saying that differentials $d\rho_1, \ldots, d\rho_k$ form a linearly independent set at each point of their domain of definition.

Example 9. Consider the set $M = \{(z, w) \in \mathbb{C}^2 \colon\ w = z\overline{z}\}$. Because $z\overline{z}$ is real, M is given by $0 = \operatorname{Im} w = \frac{1}{2i}(w - \overline{w})$ and $\operatorname{Re} w = z\overline{z}$. This means that if we put $\rho_1 = -2 \operatorname{Im} w = i(w - \overline{w})$ and $\rho_2 = z\overline{z} - \frac{1}{2}(w + \overline{w})$, then the zero sets of ρ_1 and ρ_2 intersect to give M. To check that this intersection is transversal, we first compute that $d\rho_1 = i\, dw - i\, d\overline{w}$ and $d\rho_2 = \overline{z}\, dz + z\, d\overline{z} - \frac{1}{2}\, dw - \frac{1}{2}\, d\overline{w}$. Notice that the different signs on dw and $d\overline{w}$ in the equations tell us that $d\rho_1$ and $d\rho_2$ are linearly independent, which means they intersect transversally. That is, taken together as the single function $\rho = (\rho_1, \rho_2)$, they form a defining function of M, which means M is a real submanifold of codimension 2. ////

We wish to study how real submanifolds interact with the complex structure of $\mathbb{C}^n = \mathbb{R}^{2n}$. What we mean here by complex structure is the complex linear space spanned by the antiholomorphic vectors, i.e. $\frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}$. These, along with the holomorphic vectors, will form the basis of the complexified tangent space of $\mathbb{C}^n$ at a point p, which will be denoted as $\mathbb{C}T_p\mathbb{C}^n$. Before defining $\mathbb{C}T_p\mathbb{C}^n$ we first define $\mathbb{R}T_p\mathbb{C}^n$, **the real tangent space of $\mathbb{C}^n$ at a point** p, to be the real span of the standard partial differentiation operators. That is, if our coordinates are $z_j = x_j + iy_j$ for $j = 1, 2, \ldots, n$, then

$$\mathbb{R}T_p\mathbb{C}^n := \left\langle \frac{\partial}{\partial x_1}\bigg|_p, \ldots, \frac{\partial}{\partial x_n}\bigg|_p, \frac{\partial}{\partial y_1}\bigg|_p, \ldots, \frac{\partial}{\partial y_n}\bigg|_p \right\rangle_{\mathbb{R}}.$$

This means that $\mathbb{R}T_p\mathbb{C}^n$ is a $2n$-dimensional real vector space consisting of vectors of the form

$$X_p = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j}\bigg|_p + b_j \frac{\partial}{\partial y_j}\bigg|_p \right),$$

where the a_j and b_j are real. Because differentiation is linear, these vectors $X_p \in \mathbb{R}T_p\mathbb{C}^n$ act on real-valued functions f defined on subsets of $\mathbb{C}^n$ in the natural way, that is

$$X_p f = \sum_{j=1}^{n} \left(a_j \frac{\partial f}{\partial x_j}\bigg|_p + b_j \frac{\partial f}{\partial y_j}\bigg|_p \right) \in \mathbb{R}.$$

If M is real submanifold with defining function $\rho = (\rho_1, \ldots, \rho_k)$, then we say that a nonzero vector X_p is **tangent** to M at p if $X_p\rho_j = 0$ for $j = 1, 2, \ldots, k$. Recall the unit circle of Example 8 with the single defining equation $\rho = 1 - x^2 - y^2$. If we set $X_{(x,y)} = -y \left.\frac{\partial}{\partial x}\right|_{(x,y)} + x \left.\frac{\partial}{\partial y}\right|_{(x,y)}$, then

$$X_{(x,y)}(1 - x^2 - y^2) = 2xy - 2xy = 0,$$

which shows that $X_{(x,y)}$ is tangent to M at each point (x, y). In fact, this particular tangent vector kills the defining function ρ at every point away from the origin, but in general we only require tangent vectors to kill the defining function at points p on M. Note that if we write $X_{(x,y)}$ in coordinates using the ordered basis $\left(\left.\frac{\partial}{\partial x}\right|_{(x,y)}, \left.\frac{\partial}{\partial y}\right|_{(x,y)}\right)$ then $X_{(x,y)} = (-y, x)$, which is a 90° counterclockwise rotation of the radial unit vector $(x, y) \in \mathbb{R}^2$. So if we imagine the vector $(-y, x)$ with its tail at the head of (x, y) on the unit circle, then we can see why we are geometrically justified in calling $X_{(x,y)}$ a tangent vector.

Clearly a linear combination of tangent vectors is again a tangent vector. If M is a real submanifold of $\mathbb{C}^n$ and p is a point of M then we call the real linear subspace of $\mathbb{R}T_p\mathbb{C}^n$ formed by the vectors tangent to M at p the **real tangent space to** M **at** p and denote it by $\mathbb{R}T_pM$. And so we have that, for $\rho = (\rho_1, \ldots, \rho_k)$ a defining function of M near p,

$$\mathbb{R}T_pM = \left\{X_p \in \mathbb{R}T_p\mathbb{C}^n \colon X_p\rho_1 = \cdots = X_p\rho_k = 0\right\}.$$

We now define $\mathbb{C}T_p\mathbb{C}^n$, **the complexified tangent space of** $\mathbb{C}^n$ **at** p, in a similar fashion to how we defined the real tangent space, but now we take the complex span of the standard partial differentiation operators $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$ on $\mathbb{C}^n$. That is,

$$\mathbb{C}T_p\mathbb{C}^n := \left\langle \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p, \left.\frac{\partial}{\partial y_1}\right|_p, \ldots, \left.\frac{\partial}{\partial y_n}\right|_p \right\rangle_{\mathbb{C}}.$$

Now we use the vectors

$$\left.\frac{\partial}{\partial z_j}\right|_p := \frac{1}{2}\left(\left.\frac{\partial}{\partial x_j}\right|_p - i\left.\frac{\partial}{\partial y_j}\right|_p\right) \quad \text{and} \quad \left.\frac{\partial}{\partial \bar{z}_j}\right|_p := \frac{1}{2}\left(\left.\frac{\partial}{\partial x_j}\right|_p + i\left.\frac{\partial}{\partial y_j}\right|_p\right)$$

and a change of basis for $\mathbb{C}T_p\mathbb{C}^n$ to say

$$\mathbb{C}T_p\mathbb{C}^n = \left\langle \left.\frac{\partial}{\partial z_1}\right|_p, \ldots, \left.\frac{\partial}{\partial z_n}\right|_p, \left.\frac{\partial}{\partial \bar{z}_1}\right|_p, \ldots, \left.\frac{\partial}{\partial \bar{z}_n}\right|_p \right\rangle_{\mathbb{C}}.$$

And so for X_p a vector in $\mathbb{C}T_p\mathbb{C}^n$ we can write

$$X_p = \sum_{j=1}^{n}\left(a_j \left.\frac{\partial}{\partial z_j}\right|_p + b_j \left.\frac{\partial}{\partial \bar{z}_j}\right|_p\right),$$

where now the a_j and b_j are complex numbers. If each b_j is zero, then we call X_p a **holomorphic tangent vector** and if each a_j is zero, then we call X_p an **antiholomorphic tangent vector**. We denote by $T_p^{(1,0)}\mathbb{C}^n$ the complex linear subspace of $\mathbb{C}T_p\mathbb{C}^n$ consisting of the holomorphic tangent vectors and we denote by $T_p^{(0,1)}\mathbb{C}^n$ the subspace consisting of the antiholomorphic tangent vectors. That is,

$$T_p^{(1,0)}\mathbb{C}^n := \left\{\sum_{j=1}^{n} a_j \left.\frac{\partial}{\partial z_j}\right|_p : a_j \in \mathbb{C}\right\} \quad \text{and} \quad T_p^{(0,1)}\mathbb{C}^n := \left\{\sum_{j=1}^{n} b_j \left.\frac{\partial}{\partial \bar{z}_j}\right|_p : b_j \in \mathbb{C}\right\}.$$

Similar to how we defined the real tangent space of a real submanifold of M, we may also define $\mathbb{C}T_pM$, **the complexified tangent space of M at p**, to be those vectors of the complexified tangent space of $\mathbb{C}^n$ that kill a defining function $\rho = (\rho_1, \ldots, \rho_k)$ of M near p. That is,

$$\mathbb{C}T_pM = \left\{X_p \in \mathbb{C}T_p\mathbb{C}^n \colon X_p\rho_1 = \cdots = X_p\rho_k = 0\right\}.$$

We wish to keep track of the subspace of these holomorphic and antiholomorphic vectors that are tangent to a real submanifold M. So we define

$$T_p^{(1,0)} M := \mathbb{C}T_p M \cap T_p^{(1,0)} \mathbb{C}^n \quad \text{and} \quad T_p^{(0,1)} M := \mathbb{C}T_p M \cap T_p^{(0,1)} \mathbb{C}^n.$$

Now that we have the requisite definitions of real submanifolds and their tangent spaces, let us see some examples.

Example 10. Consider the set $M = \{(z, w) \in \mathbb{C}^2 : w = 0\}$. Because the equation $w = 0$ is two real equations, if we set $\rho_1 = 2 \operatorname{Re} w = w + \overline{w}$ and $\rho_2 = -2 \operatorname{Im} w = i(w - \overline{w})$, then $\rho = (\rho_1, \rho_2)$ is zero exactly on M. Further, $d\rho_1 = dw + d\overline{w}$ and $d\rho_2 = i\, dw - i\, d\overline{w}$ are clearly linearly independent and so ρ is a defining function for M. To compute the complexified tangent space to M, first fix some point p of M and some X_p in $\mathbb{C}T_p\mathbb{C}^2$. We may write

$$X_p = a \left.\frac{\partial}{\partial z}\right|_p + b \left.\frac{\partial}{\partial \overline{z}}\right|_p + c \left.\frac{\partial}{\partial w}\right|_p + d \left.\frac{\partial}{\partial \overline{w}}\right|_p$$

for complex numbers a, b, c, and d. We have

$$X_p \rho_1 = c + d \quad \text{and} \quad X_p \rho_2 = ic - id$$

and so if both equations are 0, then we have that $c = -d$ and $ic = id$, which says that both c and d are 0. Because a and b were eliminated, we have that a and b are free. That is,

$$\mathbb{C}T_p M = \left\langle \left.\frac{\partial}{\partial z}\right|_p, \left.\frac{\partial}{\partial \overline{z}}\right|_p \right\rangle_{\mathbb{C}}.$$

And so we have the complexified tangent space of M at each point. If we intersect with the holomorphic and antiholomorphic tangent vectors of $\mathbb{C}^2$, then we have

$$T_p^{(1,0)} M = \left\langle \left.\frac{\partial}{\partial z}\right|_p \right\rangle_{\mathbb{C}} \quad \text{and} \quad T_p^{(0,1)} M = \left\langle \left.\frac{\partial}{\partial \overline{z}}\right|_p \right\rangle_{\mathbb{C}}.$$

////

We can think of this last example as $\mathbb{R}^2$ living in the z-plane of $\mathbb{C}^2$. But there are others ways to place a copy of $\mathbb{R}^2$ into $\mathbb{C}^2$. As we will see in the next example, the way that $\mathbb{R}^2$ is placed in $\mathbb{C}^2$ affects the resulting tangent spaces.

Example 11. Consider $M = \{(z,w) \in \mathbb{C}^2 \colon \operatorname{Im} z = 0 \text{ and } \operatorname{Im} w = 0\}$. Note that M is again a copy of $\mathbb{R}^2$ as in Example 10. Now we check that $\rho_1 = i(z-\overline{z})$ and $\rho_2 = i(w-\overline{w})$ together form a defining function for M. We have that $\rho = (\rho_1, \rho_2)$ is 0 exactly on M because ρ_1 and ρ_2 are constant multiples of $\operatorname{Im} z$ and $\operatorname{Im} w$, respectively. Because $d\rho_1 = i\,dz - i\,d\overline{z}$ and $d\rho_2 = i\,dw - i\,d\overline{w}$ we see that these are linearly independent. Now we compute the tangent space of M. Fix some point p of M and some $X_p = a\,\frac{\partial}{\partial z}\big|_p + b\,\frac{\partial}{\partial \overline{z}}\big|_p + c\,\frac{\partial}{\partial w}\big|_p + d\,\frac{\partial}{\partial \overline{w}}\big|_p$ in $\mathbb{C}T_p\mathbb{C}^2$. We have $X_p\rho_1 = ia - ib$ and $X_p\rho_2 = ic - id$. If this X_p is tangent, then both of these are zero so we have $a = b$ and $c = d$. Note that this means no nontrivial holomorphic nor antiholomorphic vector is tangent to M. That is, $T_p^{(1,0)}M = T_p^{(0,1)}M = \{0\}$. ////

The previous two examples taken together show that it is not enough to consider the real geometry of our submanifolds. We must also see how our submanifold interacts with the complex structure of the space in which it lives.

3.2.2 CR SUBMANIFOLDS AND CR FUNCTIONS

Let M be a real submanifold of $\mathbb{C}^n$. If the dimension of the antiholomorphic tangent space of M at p is constant as p varies, then we say M is a **CR submanifold**. That is, M is a CR submanifold if the map $p \mapsto \dim_{\mathbb{C}} T_p^{(0,1)}M$ is constant. We call $\dim_{\mathbb{C}} T_p^{(0,1)}M$ of a CR submanifold M the **CR dimension** of M. Our primary source for the theory of CR submanifolds is [1].

The two Examples 10 and 11 are both CR submanifolds because their antiholomorphic tangent spaces have constant dimension from point to point. Example 10 has CR dimension 1 and Example 11 has CR dimension 0. Let us see that the real submanifold of Example 9 is not a CR submanifold.

Example 12. Recall that $M = \{w = z\overline{z}\}$ in $\mathbb{C}^2$ is a real submanifold with defining equations $\rho_1 = i(w - \overline{w})$ and $\rho_2 = z\overline{z} - \frac{1}{2}(w + \overline{w})$. To calculate the tangent spaces, fix some point $p = (z, w)$ on M and let $X_p = a \frac{\partial}{\partial z}\big|_p + b \frac{\partial}{\partial \overline{z}}\big|_p + c \frac{\partial}{\partial w}\big|_p + d \frac{\partial}{\partial \overline{w}}\big|_p$ be a vector in $\mathbb{C}T_p\mathbb{C}^2$. If X_p is tangent, then $0 = X_p\rho_1 = ic - id$ and $0 = X_p\rho_2 = a\overline{z} + bz - \frac{1}{2}c - \frac{1}{2}d$. The equation $0 = ic - id$ says that $c = d$ and so the other equation becomes $c = a\overline{z} + bz$. And so we see that $\mathbb{C}T_pM$ is generated by the two vectors $A_p := \frac{\partial}{\partial z}\big|_p + \overline{z}\left(\frac{\partial}{\partial w}\big|_p + \frac{\partial}{\partial \overline{w}}\big|_p\right)$ and $B_p := \frac{\partial}{\partial \overline{z}}\big|_p + z\left(\frac{\partial}{\partial w}\big|_p + \frac{\partial}{\partial \overline{w}}\big|_p\right)$. If $p = 0$, then $B_p = \frac{\partial}{\partial \overline{z}}\big|_p$ is antiholomorphic so that $T_0^{(0,1)}M = \left\langle \frac{\partial}{\partial \overline{z}}\big|_p \right\rangle_{\mathbb{C}}$ but if $p \neq 0$ then we see that $T_p^{(0,1)}M$ is trivial because only the only tangent vectors are linear combinations of both holomorphic and antiholomorphic vectors. And so we see $p \mapsto \dim_{\mathbb{C}} T_p^{(0,1)}$ is not constant so that M is not a CR submanifold. ////

One reason we needed to define submanifolds and their tangent spaces is so that we could see how sets with no interior interact with the complex structure of $\mathbb{C}^n$. Now we want to study smooth functions on these submanifolds. An issue immediately arises because in order to say a function is smooth we want to say that we can take as many derivatives as we wish, but as of yet we only have derivatives at points in the interior of a set. One way of dealing with this issue is to say that a function f defined on a set S is **smooth on** S, or just **smooth**, if there exists an open set U containing S and a function F defined on U such that F restricted to S is equivalent to f and that F is smooth in the normal sense that all partial derivatives of all orders exists. In our setting, this means we will be studying functions on real submanifolds M that extend to a smooth function on some open neighborhood of M.

This definition of a smooth function of a real submanifold M may at first seem to not work because if a function has a smooth extension around M, then it will have many. Consider the real submanifold $M = \{w = 0\} \subset \mathbb{C}^2$ from Example 10. Then the entire function $f : (z, w) \mapsto z$ is a smooth on M because it extends to be smooth on all of $\mathbb{C}^2$, but the entire function $g : (z, w) \mapsto z + w$ is also smooth on M and in fact $f \equiv g$ on M. This seems to be a problem because, even if they are equivalent on M, we have $\frac{\partial f}{\partial w} = 0 \neq 1 = \frac{\partial g}{\partial w}$ and so their derivatives, even on M, are not equivalent. This turns out to not be problem as long as we only take tangential derivatives. That is, if F and G are smooth extensions of f, then $X_p F = X_p G$ provided that X_p is a tangent vector. The issue with the functions f and g above arose because $\frac{\partial}{\partial w}\big|_p$ is not in the tangent space of M for any point p, as we saw in Example 10.

Now that we know what smooth functions on a real submanifold are and we also know how to take their tangential derivatives, we can further restrict our attention to the class of functions for which the Baouendi–Trèves theorem applies. If M is CR submanifold and f is smooth on M, then we say that f is a **smooth CR function**, or just **CR**, if $X_p f = 0$ for each point p in M and each vector X_p in the antiholomorphic tangent space of M. That is, if $X_p f = 0$ for each p in M and X_p in $T_p^{(0,1)}M$, then f is CR. We call such functions CR because we think of $\frac{\partial}{\partial \bar{z}_j} = 0$ for $j = 1, \ldots, n$ as encapsulating the tangential CR equations in $\mathbb{C}^n$ just as we saw $\frac{\partial}{\partial \bar{z}} = 0$ encapsulates the CR equations in one variable. Therefore, the CR functions on M are precisely those functions that satisfy the tangential CR equations on M.

3.3 THE BAOUENDI–TRÈVES THEOREM

Our next polynomial approximation result, the theorem of Baouendi–Trèves (see [3] and theorem 2.4.1 of [1]; also, for a proof restricted to hypersurface type CR submanifolds, see theorem 3.3.1 of [6]), applies in the setting of CR functions on CR submanifolds.

Theorem 13 (Baouendi–Trèves)**.** *Let M be a CR submanifold and $p \in M$. There exists a compact neighborhood $K \subset M$ of p such that each CR function f is the uniform limit of holomorphic polynomials on K.*

We notice that, unlike in the case of the theorems of Weierstrass, Runge, and Mergelyan, Baouendi–Trèves does not allow one to choose the compact set where the polynomials approximate the function. That is, this theorem is giving us a compact K that depends on both M and the point around which the function f is to approximated. It may be that K is very small. But some sort of restriction on K is necessary because, for example, Baouendi–Trèves applies for the CR submanifold $\partial\mathbb{D} \subset \mathbb{C}$ and on $\partial\mathbb{D}$ the function $1/z$ is CR. We have already seen that no sequence of holomorphic polynomials may approximate $1/z$ uniformly on $\partial\mathbb{D}$ and so, with this simple example, we see why the K in Baouendi–Trèves can not possibly be arbitrarily chosen.

The example of $1/z$ on $\partial\mathbb{D}$ also reveals how Baouendi–Trèves is weaker than Mergelyan's theorem in the setting where they both apply. Mergelyan only asks that the complement of K be connected. That is, we could remove an arbitrarily small open arc from $\partial\mathbb{D}$ and Mergelyan would give polynomial approximation. But Baouendi–Trèves gives us the K and so we have no control over the size of the set on which the polynomials approximate our function.

3.4 THE CR SINGULAR SETTING

The Baouendi–Trèves theorem of the preceding section tells us that all CR functions are locally the uniform limit of polynomials. We wish now to study how much of this result we can recover if we drop the assumption that our submanifold is CR. If M is a smooth connected embedded real submanifold of $\mathbb{C}^n$ that is not CR, then we say that M is a **CR singular** submanifold. The study of CR singular submanifolds began with [2]. Recent progress on the study of CR submanifolds may be found in, for example, [7], [8], [9], and the references within.

By definition of CR submanifold, we have that in a CR singular submanifold the dimension of the antiholomorphic tangent space of M is not constant. Because our definition of CR function requires a CR submanifold, we need to expand the class of functions under consideration. What follows is an attempt at this generalization.

3.4.1 THE CR CONDITION IN THE SINGULAR SETTING

Fix some CR singular submanifold M. If a point p of M has an open neighborhood $U \subset M$ such that U is a CR submanifold, then we say that p is a **CR point** of M. Equivalently, p is a CR point if locally near p the map $q \mapsto \dim_{\mathbb{C}} T_q^{(0,1)} M$ is constant. We have the following proposition.

Proposition 14. *The set of CR points of M is both open and dense in M.*

Proof. By definition of CR points, all points near a CR point are also CR points and so the set of CR points of M is an open set in M. To check that the CR points are dense in M, select some non-CR point p. Consider the map $d(q) = \dim_{\mathbb{C}} T_q^{(0,1)} M$ on M. Recall that $T_q^{(0,1)} M$ is defined as the anti-holomorphic vectors annihilated by a defining function $\rho = (\rho_1, \ldots, \rho_d)$ of M. This means that $d(q)$ is the nullity (that is, the dimension of the kernel) of

$$\begin{bmatrix} \frac{\partial \rho_1}{\partial \bar{z}_1}(q) & \cdots & \frac{\partial \rho_1}{\partial \bar{z}_n}(q) \\ \vdots & \ddots & \vdots \\ \frac{\partial \rho_d}{\partial \bar{z}_1}(q) & \cdots & \frac{\partial \rho_d}{\partial \bar{z}_n}(q) \end{bmatrix}$$

and it is well known that the nullity of a matrix of continuous entries is an upper semicontinuous function. Select some neighborhood U of p and note that $d(U)$ is a finite set of nonnegative integers. Select some $x \in U$ such that $d(x) = \min d(U)$. Because d is upper semicontinuous, we have a smaller neighborhood $V \subseteq U$ of x such that $d(q) \leq d(x)$ for each $q \in V$. But, by definition of x, we have $d(q) \geq d(x)$ on V and so $d \equiv d(x)$ on V. That is, x is a CR point of M in the arbitrary neighborhood U of the non-CR point p. This gives us a sequence of CR points converging to p. Therefore, the set of CR points of M is dense, as desired. □

We say a smooth function f on M is **CR at CR points** if, near each CR point p with neighborhood $U \subset M$ such that U is a CR submanifold, we have that $f|_U$ is a CR function on U. We may characterize the f that are CR at CR points by the following proposition.

Proposition 15. *Let $\Gamma(\mathbb{C}TM)$ denote the set of smooth sections of the bundle $\mathbb{C}TM$ and put $K = \{X \in \Gamma(\mathbb{C}TM) \colon X_p \in T_p^{(0,1)}M$ for each $p \in M\}$. Suppose f is a smooth function on M. Then f is CR at CR points if and only if $Xf \equiv 0$ for each $X \in K$.*

Proof. Suppose f is CR at CR points. Fix some $X \in K$ and some $p \in M$. If p is a CR point of M then, because f is CR at CR points, we have $X_p f = 0$ by definition of f being a CR function. Now suppose p is not a CR point. By Proposition 14 we have a sequence (p_n) of CR points converging to p. Because X is a smooth section we have $\lim X_{p_n} = \lim X_p$ and because each p_n is a CR point we have $X_{p_n} f \equiv 0$. Hence,

$$X_p f = \lim X_{p_n} f = \lim 0 = 0$$

because f is smooth and so has a continuous first derivative. This shows $Xf \equiv 0$, as desired.

Now we prove the converse. Suppose $Xf \equiv 0$ for each $X \in K$. Select some CR point p of M. We have some open neighborhood $U \subset M$ of p such that U is a CR submanifold. Select some antiholomorphic tangent vector $L_q \in T_q^{(0,1)}U$ for an arbitrary $q \in U$. To show that f is CR on U, we need to show that $L_q f = 0$.

Because U is a CR submanifold, we have that $T^{(0,1)}U$ is a vector bundle and so we may extend L_q to a vector field $L \in T^{(0,1)}U$, perhaps first by replacing U with a smaller neighborhood of q. We now wish to extend L to a vector field in K. To do this, fix some compact neighborhood $B \subset U$ of q and some large compact neighborhood $A \subset U$ of B (that is, A is compactly inside of U and B is compactly inside of the interior of A). We may now select a smooth bump function ϕ such that $\phi \equiv 1$ on B and $\phi \equiv 0$ on $M \setminus A$. Put $X_q = \phi(q) L_q$ for each $q \in M$. Note that we take this to mean $X_q \equiv 0$ where $\phi \equiv 0$ even though, strictly speaking, L_q may not be defined there. Also, in the set $A \setminus B$ where ϕ transitions from 0 to 1 we have that X_q is an antiholomorphic tangent vector because it is just a scalar multiple of L_q. We have shown $X \in K$. Hence, near p we have

$$L_p f = \lim_{q \to p} \phi(q)^{-1} X_q f = \lim_{q \to p} 1 X_q f = \lim_{q \to p} 1 \cdot 0 = 0,$$

as required to show that f is CR at CR points. □

While CR at CR points is a natural generalization of the CR condition to CR singular submanifolds, a function f being CR at CR points is not a restrictive enough class of functions if what we wish is to allow for local polynomial approximation. Consider the following example.

Example 16. Let M be as in Example 12 (so $M = \{w = z\bar{z}\} \subset \mathbb{C}^2$). Recall that we determined $T_p^{(0,1)}M$ to be trivial if $p \neq 0$ and $T_0^{(0,1)}M$ is the span of $\frac{\partial}{\partial \bar{z}}\big|_0$. Let $f(z, w) = \bar{z}$ and let $X \in K$ where K is the set of smooth sections from Proposition 15. Then $X \equiv 0$ on $M \setminus \{0\}$ because $T_p^{(0,1)}M = \{0\}$ for each non-origin point p. Therefore, by continuity, $X \equiv 0$ on M. This means $Xf \equiv 0$ on M and so f is CR at CR points by Proposition 15.

Suppose to the contrary that f is the uniform limit of holomorphic polynomials (p_n) on some neighborhood of the origin

$$U = B_{3r}(0) \cap M = \{(z, w) \colon \|(z, w)\|^2 < 3r\} \cap M.$$

By possibly taking a smaller neighborhood, we may assume $r < 1$. Note that on M we have $w = z\bar{z} = |z|^2$ so that

$$U = \{(z, w) \colon \|(z, w)\|^2 = |z|^2 (1 + |z|^2) < 3r\} \cap M$$

Put $V = \{(z,w)\colon\ |z|^2 < r\} \cap M$ and notice that for $(z,w) \in V$ we have

$$|z|^2 (1 + |z|^2) < r(1 + r) = r^2 + r < 2r$$

because $0 < r < 1$. We have shown that V is a subset of $B_{2r}(0) \cap M$ and so the boundary of V is a subset of U. Notice that boundary of V may be written

$$\partial V = \{(z,w)\colon\ |z|^2 = r \text{ and } w = |z|^2\}.$$

On ∂V we have that $w = r$ so that if we put $q_n(z) = p_n(z,r)$ then the (q_n) are polynomials in z equivalent to (p_n) on ∂V. That is, the polynomials (q_n) uniformly converge to $z \mapsto \overline{z}$ on the circle $\{|z|^2 = r\}$. As expected, this contradicts our result from Example 5. ////

The preceding example tells us that those functions that are CR at CR points is not quite the correct set of functions to study if we are interested in local holomorphic polynomial approximation. There are simply too many functions that are CR at CR points.

3.4.2 A POLYNOMIAL APPROXIMATION IN THE SINGULAR SETTING

In the preceding subsection, we saw a naive attempt at generalizing the CR condition to non-CR submanifolds. If our goal is holomorphic polynomial approximation, then the attempted generalization failed. However, recall a generalization of Weierstrass's theorem gives non-holomorphic approximation. A fair question in the CR singular setting is: Can we at least do better than Weierstrass? A particular way to rephrase this question is: For certain functions on certain CR singular submanifolds, can we find a polynomial approximation that is holomorphic in some variables, but perhaps not holomorphic in all variables? What follows is an example where the answer to the preceding question is yes.

Example 17. Put $M = \{\zeta = \bar{z}w\} \subset \mathbb{C}^3_{(z,w,\zeta)}$ and

$$\begin{aligned}
\rho_1 &= 2\,\mathrm{Re}(\bar{z}w - \zeta) = \bar{z}w + z\bar{w} - \zeta - \bar{\zeta} &&\implies &&\bar{\partial}\rho_1 = w\,d\bar{z} + z\,d\bar{w} - d\bar{\zeta}\\
\rho_2 &= -2\,\mathrm{Im}(\bar{z}w - \zeta) = i(\bar{z}w - z\bar{w} - \zeta + \bar{\zeta}) &&\implies &&\bar{\partial}\rho_2 = iw\,d\bar{z} - iz\,d\bar{w} + i\,d\bar{\zeta}.
\end{aligned}$$

We see that (ρ_1, ρ_2) is zero exactly on M and, further, that at a point $p = (z, w, \zeta)$,

$$T_p^{(0,1)}M = \begin{cases} \left\langle \frac{\partial}{\partial \bar{w}}\Big|_p + z\,\frac{\partial}{\partial \bar{\zeta}}\Big|_p,\ \ \frac{\partial}{\partial \bar{z}}\Big|_p \right\rangle & \text{if } w = 0, \text{ and} \\ \left\langle \frac{\partial}{\partial \bar{w}}\Big|_p + z\,\frac{\partial}{\partial \bar{\zeta}}\Big|_p \right\rangle & \text{if } w \neq 0. \end{cases}$$

That is, M is a CR singular submanifold because the dimension of its antiholomorphic tangent space is not constant. Also notice that $M \setminus \{w = 0\}$ is a CR submanifold and so $M \setminus \{w = 0\}$ are the CR points of M. Fix some smooth function f on M that is CR at CR points. We now put coordinates on M in such a way that we may use Baouendi–Trèves to produce a polynomial approximation of f.

Let $X = \mathbb{R}^2_{(x,y)} \times \mathbb{C}_\xi$ and notice that X is a CR submanifold whose antiholomorphic tangent space is spanned at each point by $\frac{\partial}{\partial \bar{\xi}}\big|_{(x,y,\xi)}$. Now define $\phi\colon X \to \mathbb{C}^3_{(z,w,\zeta)}$ by

$$\phi(x, y, \xi) = (x + iy, \xi, (x - iy)\xi)$$

so that the image of ϕ is M. Also, on M we see that ϕ has inverse

$$\psi(z, w, \zeta) = \phi|_M^{-1}(z, w, \zeta) = (\mathrm{Re}\, z, \mathrm{Im}\, z, w)\,.$$

The ϕ coordinates on M are nice in the sense that, for a point $p \in M$, we have

$$D\phi_{\psi(p)}\left(\frac{\partial}{\partial \bar{\xi}}\bigg|_{\psi(p)} \right) = \frac{\partial}{\partial \bar{w}}\bigg|_p + z\,\frac{\partial}{\partial \bar{\zeta}}\bigg|_p\,.$$

That is, ϕ pushes the antiholomorphic tangent space of X to that of $M \setminus \{w = 0\}$. Select some $\psi(p) \in X$. Because f is CR at CR points of M, by Proposition 15, we have that

$$\frac{\partial}{\partial \bar{\xi}}\bigg|_{\psi(p)} f \circ \phi = D\phi_{\psi(p)} \left(\frac{\partial}{\partial \bar{\xi}}\bigg|_{\psi(p)} \right) f = \left(\frac{\partial}{\partial \overline{w}}\bigg|_{p} + z \, \frac{\partial}{\partial \bar{\zeta}}\bigg|_{p} \right) f = 0.$$

We have shown that $f \circ \phi$ is a CR function on the CR submanifold X. And so, by Baouendi–Trèves (Theorem 13), there is a compact subset $K \subset X$ and a sequence of holomorphic polynomials (q_n) that uniformly converge to $f \circ \phi$ on K. That is, for each $\varepsilon > 0$ we have an N such that for all $(x, y, \xi) \in K$ and $n \geq N$,

$$|q_n(x, y, \xi) - f \circ \phi(x, y, \xi)| < \varepsilon. \tag{III.1}$$

For each n define $p_n \colon M \to \mathbb{C}$ by

$$p_n(z, w, \zeta) = q_n \circ \psi(z, w, \zeta) = q_n(\operatorname{Re} z, \operatorname{Im} z, w). \tag{III.2}$$

We claim that the (p_n) uniformly converge to f on $\phi(K)$. Fix $\varepsilon > 0$ and select N so that (III.1) holds for each $(x, y, \xi) \in K$ and $n \geq N$. Now let $(z, w, \zeta) \in \phi(K)$ and $n \geq N$ be given. By applying ψ, this means $(x, y, \xi) = (\operatorname{Re} z, \operatorname{Im} z, w)$ is in K. We have

$$\begin{aligned} |p_n(z, w, \zeta) - f(z, w, \zeta)| &= |p_n \circ \phi(x, y, \xi) - f \circ \phi(x, y, \xi)| \\ &= |q_n \circ \psi \circ \phi(x, y, \xi) - f \circ \phi(x, y, \xi)| \\ &= |q_n(x, y, \xi) - f \circ \phi(x, y, \xi)| < \varepsilon. \end{aligned}$$

Therefore, the p_n uniformly converge to f on $\phi(K)$. Furthermore, we notice from (III.2) that, because the q_n are holomorphic polynomials, the p_n are holomorphic in w but not in z. That is, we have answered our question preceding this example in the affirmative. Weierstrass would have given us polynomials approximating f that were holomorphic in neither z nor w. ////

The example suggests that when looking at a particular kind of CR singular submanifold, then we may exploit Baouendi–Trèves to get approximating polynomials that are holomorphic in at least some of the variables. We end with the following list of questions that arise from examination of Example 17 in the context of our previous discussion of CR singular submanifolds.

- What is a reasonable class of functions to study on CR singular submanifolds if what we want is holomorphic polynomial approximation?
- Do we need to place additional conditions on the submanifolds to achieve holomorphic polynomial approximations?
- In the CR singular setting, how much of the theorem of Baouendi–Trèves can we recover?

CHAPTER A

CODE TO GENERATE FIGURES

Mathematica 12 code to generate Figure 1.1

```
<< FunctionApproximations`

phi[x_] := Abs[x − Round[x]];
tn[x_, n_] := phi[2^n x]/2^n;
ptn[x_, n_] := Sum[tn[x, j], {j, 0, n}];

f[x_] := ptn[x, 10];
a = −3;
b = 4;
nn = 350;
epsilon = .001;
xpoints =
Table[a + n (b − a)/nn + RandomReal[{−epsilon, epsilon}],
{n, 0, nn}];
ypoints = f /@ xpoints;
```

```
points = Transpose@{xpoints, ypoints};
p = InterpolatingPolynomial[points, x];
Length[CoefficientList[p, x]] - 1
ListLogPlot[CoefficientList[p, x], PlotRange -> Full]
CoefficientList[p, x][[1 ;; 3]]
CoefficientList[p, x][[-3 ;; -1]]
Max[CoefficientList[p, x]]
Min[CoefficientList[p, x]]
xticks = Table[N[j], {j, 0, 1, 1/4}];
yticks = Table[N[j], {j, 0, 1, 1/8}];
Plot[{p, f[x] + .2}, {x, 0, 1},
 PlotRange -> {0, 1},
 Ticks -> {xticks, yticks},
 PlotStyle -> Black,
 GridLines -> {yticks, yticks},
 WorkingPrecision -> MachinePrecision,
 MaxRecursion -> 15]
```

BIBLIOGRAPHY

[1] M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999. MR1668103

[2] Errett Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–21. MR200476

[3] M. S. Baouendi and F. Trèves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math. (2) **113** (1981), no. 2, 387–421, DOI 10.2307/2006990. MR607899

[4] Augustin-Louis Cauchy, *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*, Debure frères (1823).

[5] Jiří Lebl, *Basic Analysis II: Introduction to Real Analysis* (2019), `http://jirka.org/ra`, version 5.2.

[6] ———, *Tasty Bits of Several Complex Variables* (2019), `http://jirka.org/scv`, version 3.2.

[7] Jiří Lebl, Alan Noell, and Sivaguru Ravisankar, *Codimension two CR singular submanifolds and extensions of CR functions*, J. Geom. Anal. **27** (2017), no. 3, 2453–2471, DOI 10.1007/s12220-017-9767-6. MR3667437

[8] ———, *On Lewy extension for smooth hypersurfaces in* $\mathbb{C}^n \times \mathbb{R}$, Trans. Amer. Math. Soc. **371** (2019), no. 9, 6581–6603, DOI 10.1090/tran/7605. MR3937338

[9] ———, *On the Levi-flat Plateau problem*, Complex Anal. Synerg. **6** (2020), no. 1, Paper No. 3, 15, DOI 10.1007/s40627-019-0040-6. MR4052029

[10] John M. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR2954043

[11] S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, Amer. Math. Soc. Translation **1954** (1954), no. 101, 99. MR0060015

[12] Walter Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. MR924157

[13] C. Runge, *Über die analytische Darstellung willkürlicher Functionen*, Acta Mathematica **7** (1885).

[14] Teiji Takagi, *A Simple Example of the Continuous Function without Derivative*, Tokyo Sugaku-Butsurigakkwai Hokoku **1** (1901), 176–177, DOI 10.11429/subutsuhokoku1901.1.F176.

[15] David C. Ullrich, *Complex made simple*, Graduate Studies in Mathematics, vol. 97, American Mathematical Society, Providence, RI, 2008. MR2450873

[16] K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin (1885).

VITA

Trevor Fancher

Candidate for the Degree of

Master of Science

Thesis: POLYNOMIAL APPROXIMATION OF CR SINGULAR FUNCTIONS

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Master of Science in Mathematics at Oklahoma State University, Stillwater, Oklahoma in May, 2020.

Completed the requirements for the Bachelor of Science in Mathematics at Oklahoma State University, Stillwater, Oklahoma in May, 2017.