A NEW REPRESENTATION

OF

GEGENBAUER'S POLYNOMIALS

By

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PREFACE

During the past few years Professor E. W. Titt and his students have been developing a general theory of second order linear partial differential equations. In the course of these studies, new expressions for Legendre's and Tchebycheff's derived polynomials were found by A. J. Kainen. This paper is an extension, generalization, and general investigation of these expressions. The general theory may be found in papers 1, 2, 3 and 4 of the bibliography and the specific details leading to these new expressions are in papers 2 and 3.

I wish to thank Professor R. B. Deal, who served as my adviser during the preparation of this paper, and Professor L. Wayne Johnson for their kind interest in all matters pertaining to my work. I am also indebted to the Office of Naval Research and the National Science Foundation for financial support of this work.

TABLE OF CONTENTS

I. INTRODUCTION

In recent years E. W. Titt has been in the process of developing a theory of linear second order partial differential equations, with n independent variables, following the ideas of Green and Volterra. In order to obtain an integrating factor for the normal hyperbolic equation, a potential is integrated over a hyper surface in the interior of the characteristic cone. The analytical treatment of the distributed potential is facilitated by the reduction of the (n-1)-tuple integral to a single integral. This single integral is in the nature of a transform of the original potential, the kernal K of which varies in analytic form from dimension to dimension.

The complete development can be found in the papers of E. W. Titt and others, some of which have not been published [1, 2, 3, 4]. The kernal K is given by the integral

$$K = \int_{\mu}^{S} \frac{\mu}{\theta} \frac{(\theta^2 - \mu^2)^{n-3}}{\theta} \, d\theta$$

where μ and S are non-euclidian distances occuring in the distribution of the potential of a splitting hyperplane. The notation used here is that used by O. P. Sanders [4]. This expression for K was obtained by solving the differential equation

$$\left[D_{SS} + \frac{n-2}{S} D_S\right] K = \frac{\mu}{S^{n-3}} (S^2 - \mu^2)^{n-4} \, 2(n-3). \tag{1.1}$$

The operator on the left is referred to as $\overline{\Delta}$ and the application of this operator plays a large part in the development of the properties of the kernal. If an exponential change of variable, $S = e^t$, $\mu = e^a$, is made in (1.1) the expression for $\bar{\Delta}$ becomes

$$\bar{\Delta} K = (2e^a)^{n-4} e^{-(t-a)} \sinh^{n-4}(t-a).$$

The repeated application of $\bar{\Delta}$ m times to K results in the following expressions for $\bar{\Delta}^m K$

$$\bar{\Delta}^m K(e^t, e^a) = \frac{2^{n-3}(n-3)!}{(n-3-m)!} e^{(n-3-m)a} e^{-mt} \left[D^2-(m-2)^2\right] \ldots \left[D^2-2^2\right] D \sinh^{n-3-m}(t-a)$$

for m even and

$$\bar{\Delta}^m K(e^t, e^a) = \frac{2^{n-3}(n-3)!}{(n-3-m)!} e^{(n-3-m)a} e^{-mt} \left[D^2-(m-2)^2\right] \ldots \left[D^2-3^2\right]\left[D^2-1\right] \sinh^{n-3-m}(t-a)$$

for m odd. In an unpublished paper Kainen shows that the series of operators operating on $\sinh^{n-3m}(t-a)$ are either Legendre's derived polynomials or Tchebycheff's derived polynomials. [2] The results obtained by Kainen are given in the following expressions for Legendre's and Tchebycheff's derived polynomials.

$$P_n^{(\lambda)}(\cosh t) = \frac{(2\lambda)!}{2^\lambda \lambda!(n-\lambda)!} \cdot \frac{1}{\sinh^{2\lambda} t} D(D^2-2^2)(D^2-4^2) \ldots \left[D^2-(n-\lambda-1)^2\right] \sinh^{n+\lambda} t$$

$$= \frac{(-1)^\lambda 2^\lambda \lambda!}{(n-\lambda)!(2\lambda)!} D(D^2-2^2)(D^2-4^2) \ldots \left[D^2-(n+\lambda-1)^2\right] \sinh^{n-\lambda} t$$

for $n+\lambda$ odd,

$$P_n^{(\lambda)}(\cosh t) = \frac{(2\lambda)!}{2^{\lambda}\lambda!(n-\lambda)!} \cdot \frac{1}{\sinh^{2\lambda} t} \; (D^2-1)(D^2-3^2) \ldots \left[D^2-(n-\lambda-1)^2\right] \sinh^{n+\lambda} t$$

$$= \frac{(-1)^{\lambda} 2^{\lambda}\lambda!}{(n-\lambda)!(2\lambda)!} \; (D^2-1)(D^2-3^2) \ldots \left[D^2-(n+\lambda-1)^2\right] \sinh^{n-\lambda} t$$

for $n+\lambda$ even,

$$T_n^{(\lambda)}(\cosh t) = \frac{n 2^{\lambda-1}(\lambda-1)!}{(n-\lambda)!} \cdot \frac{1}{\sinh^{2\lambda-1} t} \; D(D^2-2^2)(D^2-4^2) \ldots \left[D^2-(n-\lambda-1)^2\right] \sinh^{n+\lambda+1} t$$

for $n+\lambda$ odd and

$$T_n^{(\lambda)}(\cosh t) = \frac{n 2^{\lambda-1}(\lambda-1)!}{(n-\lambda)!} \cdot \frac{1}{\sinh^{2\lambda-1} t} \; (D^2-1)(D^2-3^2) \ldots \left[D^2-(n-\lambda-1)^2\right] \sinh^{n+\lambda-1} t$$

for $n+\lambda$ even.

The principal contribution of this paper is the study of a certain function

$$F = \left[D^2-(a-2)^2\right]\left[D^2-(a-4)^2\right] \ldots \left[D^2-3^2\right]\left[D^2-1\right] \sinh^b x$$

for a odd and

$$F = \left[D^2-(a-2)^2\right]\left[D^2-(a-4)^2\right] \ldots \left[D^2-4^2\right]\left[D^2-2^2\right] D \sinh^b x$$

for a even, which may be written as

$$F = \left[D-(a-2)\right]\left[D-(a-4)\right] \ldots \left[D+(a-4)\right]\left[D+(a-2)\right] \sinh^b x$$

where a is an integer, which is shown to satisfy the differential equation

$$\left[\sinh x \; D_{xx} + (a-b) \cosh x \; D_x - ab \sinh x\right] \quad F = 0.$$

It is shown that various special cases yield the various Legendre and Tchebycheff polynomials that arise in the study of the kernal. The function F turns out to be of interest in itself and its relationships with the Gegenbauer polynomials, the associated Legendre functions, and what might be termed the associated Gegenbauer functions, are studied.

II. THE FUNCTION F

This chapter shows that the function F satisfies the differential equation, discusses the analog with circular functions and discusses the connection with the hypergeometric and Reimann differential equations.

1. To show that the function

$$(2.1)\quad F = \left[D-(a-2)\right]\left[D-(a-4)\right] \ldots \left[D+(a-4)\right]\left[D+(a-2)\right] \sinh^b x$$

is a solution of the differential equation

$$(2.2)\quad \left[\sinh x\, D_{xx} + (a-b) \cosh x\, D_x - ab \sinh x\right] F = 0,$$

F is directly substituted in the equation and the result shown to be equal to zero. Before the substitution is made the form of the equation is changed by the use of the relationships

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

The equation now appears as

$$\left[(e^x - e^{-x})\, D_{xx} + (a-b)(e^x + e^{-x})\, D_x - ab(e^x - e^{-x})\right] F = 0.$$

The coefficients of the exponentials are now factored and the equation appears as

$$(2.3)\quad e^x \left[D_x + a\right]\left[D_x - b\right] F + e^{-x} \left[D_x - a\right]\left[D_x + b\right] F = 0.$$

The next step is the direct substitution of F.

In addition to the ordinary rules for operators the following one easily verified.

(2.4) $$(D \pm n) \sinh^n(x) = n^{\pm x} \sinh^{n-1}(x)$$

Direct substitution of (2.1) on the left hand side of the equation (2.3) gives

$$e^x[D + a][D - b][D - (a-2)][D - (a-4)] \ldots [D + (a-2)] \sinh^b x$$

$$-e^{-x}[D - a][D + b][D - (a-2)][D - (a-4)] \ldots [D + (a-2)] \sinh^b x.$$

The exponentials are now moved to the right after a rearrangement of the operators to give the following equation

$$[D - (a-1)][D - (a-3)][D - (a-5)] \ldots$$
$$[D + (a-3)][D + (a-1)] e^x[D - b] \sinh^b x$$
$$-[D - (a-3)][D - (a-5)] \ldots$$
$$[D + (a-3)][D + (a-1)][D - (a-1)] e^{-x}[D + b] \sinh^b x = 0.$$

The last operator to the right is now allowed to operate on the $\sinh^b x$ according to (2.4). This gives

(2.5) $$[D - (a-1)][D - (a-3)] \ldots$$
$$[D + (a-3)][D + (a-1)] e^x be^{-x} \sinh^{b-1} x$$
$$-[D - (a-1)][D - (a-3)] \ldots$$
$$[D + (a-3)][D + (a-1)] e^{-x} be^x \sinh^{b-1} x = 0.$$

The two terms of the left hand side of equation (2.3) are now equal and opposite in sign and therefore F is a solution of (2.2).

Only one restriction has been made in the work above and that was on a. The middle of F is seen to be as follows if a is an odd integer

$$\left[D - (a-2)\right] \ldots (D-3)(D-1)(D+1)(D+3) \ldots \left[D + (a-2)\right] \tag{2.6}$$

or if a is an even integer

$$\left[D - (a-2)\right] \ldots (D-4)(D-2)D(D+2)(D+4) \ldots \left[D + (a-2)\right].$$

If a is not an integer the continuity of the operators is lost at the middle. This is best seen from an example. Take (2.6) and move both e^{x} and e^{-x} by it from left to right and the middle of the two results appear as follows

$$(D-4)(D-2)D(D+2)$$

and

$$(D-2)D(D+2)(D+4).$$

These two expressions still contain the same factors. For a case where a is not an integer the middle appears, for example, as follows

$$(D-3.5)(D-1.5)(D+1.5)(D+3.5).$$

After e^{x} and e^{-x} are moved by this middle, it appears

$$(D-4.5)(D-2.5)(D+.5)(D+2.5)$$

and

$$(D-2.5)(D-1.5)(D+2.5)(D+4.5).$$

The middle terms no longer match up and the step leading to (2.5) can not be made if a is not an integer.

2. A slight modification in F leads to a function which is a solution of equation (2.2) with circular coefficients, i.e.

$$\text{(2.7)} \qquad \left[\sin x \, D_{xx} + (a-b) \cos x \, D_x - ab \sin x\right] F = 0.$$

The modification in F is that the constant part of the operator is imaginary, and that sinh x becomes sin x.

$$\text{(2.8)} \quad F = \left[D - (a-2)i\right]\left[D - (a-4)i\right] \ldots \left[D + (a-4)i\right]\left[D + (a-2)i\right] \sin^b x.$$

If F is viewed from a slightly different standpoint the change is seen to be even smaller. If the symmetric factors of F in (2.2) are multiplied together F becomes

$$F = \left[D^2 - (a-2)^2\right]\left[D^2 - (a-4)^2\right] \ldots \left[D^2 - 4\right] D \sinh^b x$$

for a even and

$$F = \left[D^2 - (a-2)^2\right]\left[D^2 - (a-4)^2\right] \ldots \left[D^2 - 9\right]\left[D^2 - 1\right] \sinh^b x$$

for a odd. If (2.8) is written this way it becomes

$$F = \left[D^2 + (a-2)^2\right]\left[D^2 + (a-4)^2\right] \ldots \left[D^2 + 4\right] D \sin^b x$$

for a even and

$$F = \left[D^2 + (a-2)^2\right]\left[D^2 + (a-4)^2\right] \ldots \left[D^2 + 9\right]\left[D^2 + 1\right] \sin^b x$$

for a odd.

The fact that this circular F satisfies (2.7) parallels section one closely and only the main points are given here. After the direct substitution (2.7) becomes

$$e^{ix}\left[D + ai\right]\left[D - bi\right]\left[D - (a-2)i\right] \dots \left[D + (a-2)i\right] \sin^b x$$

$$-e^{-ix}\left[D - ai\right]\left[D + bi\right]\left[D - (a-2)i\right] \dots \left[D + (a-2)i\right] \sin^b x = 0.$$

The exponentials are now moved to the right to give

$$\left[D - (a-1)i\right] \dots \left[D + (a-1)i\right] e^{ix} \left[D - bi\right] \sin^b x$$

$$-\left[D - (a-1)i\right] \dots \left[D + (a-1)i\right] e^{-ix} \left[D + bi\right] \sin^b x = 0.$$

A rule analogus to (2.4) is easily verified for sin x

$$\left[D \pm bi\right] \sin^b x = be^{\pm ix} \sin^{b-1} x. \tag{2.9}$$

The application of (2.9) gives

$$\left[D - (a-1)i\right] \dots \left[D + (a-1)i\right] e^{ix}\, be^{-ix} \sin^{b-1} x$$

$$-\left[D - (a-1)i\right] \dots \left[D + (a-1)i\right] e^{-ix} be^{ix} \sin^{b-1} x$$

which is equal to zero showing (2.8) to be a solution of (2.7).

3. If the change of variable $y = \cosh t$ is made in the equation

$$\left[\sinh t\, D_{tt} + (a-b) \cosh t\, D_t - ab \sinh t\right] F(\cosh t) = 0$$

the following equation results

$$\left[(1-y^2)\, D_{yy} + (b-a-1)y\, D_y + ab\right] F(y) = 0. \tag{2.10}$$

The hypergeometric equation is usually written

$$(2.11) \qquad \left[z(1-z)D_{zz} + \left[\gamma - (\alpha+\beta+1)z\right] D_z - \alpha\beta\right] F(\alpha, \beta; \gamma; z) = 0.$$

The change of variable $z = \frac{1+y}{2}$ in (2.11) gives the equation

$$(2.12) \qquad \left[(1-y^2)D_{yy} + \left[2\gamma - (\alpha+\beta+1) - (\alpha+\beta+1)y\right] D_y - \alpha\beta\right] F(\alpha, \beta; \gamma; \tfrac{1+y}{2}) = 0.$$

A comparison of (2.10) and (2.12) shows that by setting

$$\alpha = a, \ \beta = -b, \ \gamma = \frac{a-b+1}{2}$$

that (2.12) reduces to (2.10). One can then express $F(y)$ as follows

$$F(y) = F(a, -b; \frac{a-b+1}{2}; \frac{1+y}{2})$$

or

$$F(\cosh t) = F(a, -b, \frac{a-b+1}{2}, \frac{1+\cosh t}{2}).$$

Since $F(\cosh t)$ is a solution of the hypergeometric differential equation, it is also a solution of Reimann's equation. Expressed in this form

$$F(\cosh t) = P \left| \begin{matrix} 0 & \infty & 1 \\ 0 & a & 0 \\ \frac{3-a+b}{2} & -b & \frac{b-a+1}{2} \end{matrix} \right|$$

III. THE NATURE OF F

1. This chapter is concerned with studying the nature of the solutions of (2.2) and (2.7). For certain values of the parameters the solutions of (2.2) and (2.7) become the trivial solution, being identically equal to zero. The two things affecting the nature of the solution are whether $a > b+2$, $a = b+2$ or $a < b+2$ and whether a and b are odd or even. Each of the twelve possible cases must be investigated. The twelve cases are the combinations of the following two sets of cases.

I. $a = b + 2$	a. a odd b odd
II. $a > b + 2$	b. a even b even
III. $a < b + 2$	c. a odd b even
	d. a even b odd

The following rules of operators will be used

$$(3.1) \qquad \left[D^2 - a^2\right] \sinh^b x = (b+a)(b-a) \sinh^b x + b(b-1) \sinh^{b-2} x$$

and

$$\left[D^2 + a^2\right] \sin^b x = (b+a)(b-a) \sin^b x + b(b-1) \sin^{b-2} x.$$

The function studied first is (2.1), however the form used here will be the following

$$(3.2) \qquad F = \left[D^2 - (a-2)^2\right]\left[D^2 - (a-4)^2\right] \ldots \left[D^2 - 3^2\right]\left[D^2 - 1\right] \sinh^b x$$

for a odd and

$$F = \left[D^2 - (a-2)^2\right]\left[D^2 - (a-4)^2\right] \ldots \left[D^2 - 2^2\right] D \sinh^b x$$

for a even. For a odd there are $\frac{a-1}{2}$ second order factors to the operator and for a even there are $\frac{a-2}{2}$ second order factors and one first order factor to the operator. The trivial cases are treated first.

2. The first trivial case is (I, a) where a = b+ 2 and both are odd. The first term of (3.1) drops out when a = b+ 2. Each application of rule (3.1) on $\sinh^b x$, starting with the operators at the left, reduces the power of sinh x by two. After $\frac{a-3}{2}$ second order operators have been applied to (3.2) the function appears as

$$b!\left[D^2 - 1\right] \sinh x = b!(\sinh x - \sinh x) \equiv 0. \tag{3.4}$$

Therefore the function is identically zero when the parameters of the function fall into case (I, a).

In the same way after the application of $\frac{a-4}{2}$ second order operators for case (I, b) equation (3.3) becomes

$$\frac{b!}{2}\left[D^2 - 4\right] D \sinh^2 x = b!\, D(2) \equiv 0. \tag{3.5}$$

Thus the function is identically zero for both case (I, a) and (I, b).

The next case is (II, a) where $a > b + 2$ and both are odd. This case is the same as (I, a) except that there are additional factors in the operator on the left. The factor corresponding to a = b + 2 is applied first instead of the one on the far left. After $\frac{b-3}{2}$ factors have been applied a result similar to (3.4) is obtained.

$$b!\left[D^2 - (a-2)^2\right]\left[D^2 - (a-4)^2\right] \ldots \left[D^2 - (b+2)^2\right]\left[D^2 - 1\right] \sinh x.$$

After the application of the last operator on the right the function becomes identically zero as in (3.4).

A similar treatment of (II, b) shows that it is also identically zero.

3. The first non trivial case is (II, c) where $a > b + 2$ and a is odd and b is even. After the application of $\frac{b}{2}$ operators the lowest order term is a constant. Since a is odd and b is even none of the terms drop out. The application of more operators does not change the nature of the function since $[D^2 - a^2]$ constant = constant. The function in this case is a polynomial of even powers of sinh x.

The next case follows (II, c) closely. For $a > b + 2$ and a even, b odd - case (II, d) - it is seen that after $\frac{b+1}{2}$ operators are applied that the lowest order term is sinh x. As in case (II, c) none of the terms drop out. The application of the rest of the second order factors does not change the nature of the function which is a polynomial in odd powers of sinh x. The application of the single operator however makes the function become a polynomial in even powers of sinh x (degree b-1) times cosh x.

For the cases where $a < b + 2$ there are no trivial cases since none of the factors are present that lead to trivial cases. When a and b are both odd, case (III, a), the lowest power remaining after all the operators have been applied is $\sinh^{b-a+1} x$. This is easily seen from (3.1). The function is then a polynomial in odd powers of sinh x.

For case (III, b) the lowest term remaining after all the second order operators have been applied is $\sinh^{b-a+2} x$. The single D operator is still left. When it is applied it gives a cosh x factor to each term. The function is then a polynomial of odd powers of sinh x times cosh x.

The last two cases, (III, c) and (III, d), follows the previous cases closely. After the application of the second order operators we have polynomials in even powers of sinh x and odd powers of sinh x, respectively for (III, c) and (III, d). Since a is even for (III, d) there remains a single operator to be applied. The function becomes a polynomial in even powers of sinh x times cosh x.

To summarize the results of this chapter a table of the twelve cases is given using the notation P(x) for a polynomial in x. The polynomials below have either only odd powers or only even powers.

	a odd b odd	a even b even	a odd b even	a even b odd
$a = b+2$	$F \equiv 0$	$F \equiv 0$		
$a > b+2$	$F \equiv 0$	$F \equiv 0$	$P(\sinh^2 x)$	$\cosh x\, P(\sinh^2 x)$
$a < b+2$	$P(\sinh x)$	$\cosh x\, P(\sinh x)$	$P(\sinh^2 x)$	$\cosh x\, P(\sinh^2 x)$

Table I.

4. The function behaves in exactly the same way for the circular sin x. Taking the trivial cases first, it is seen that (I, a) follows exactly and after $\frac{a-3}{2}$ operators F becomes

$$b!\left[D^2 + 1\right] \sin x = b!\left[\sin x - \sin x\right] \equiv 0.$$

Likewise for (I. b) equation (3. 5) becomes

$$\frac{b!}{2}\left[D^2 + 4\right] D \sin^2 x = b!\, D(2) \equiv 0.$$

The same arguments apply here as in section 2 for cases (II, a) and (II, b).

Section 3 could be quoted practically verbatum here for the non trivial cases by replacing sinh x by sin x. The results, if tabled, are exactly the same as in Table 1 except hyperbolic functions are replaced by circular functions.

IV. F AND GEGENBAUER'S DERIVED POLYNOMIALS

1. In this chapter the function F is shown to include certain classical derived polynomials.

Gegenbauer's polynomial is one of the solutions of the equation

$$(4.1) \qquad \left[(1-x^2)\, D_{xx} - (2\alpha+1)x\, D_x + n(n+2\alpha)\right] C_n^{\alpha}(x) = 0$$

where $C_n^{\alpha}(x)$ is the notation for Gegenbauer's polynomial [5, 6, 7, 8].

If (4.1) is differentiated λ times the following equations result

$$\lambda = 1 \quad \left[(1-x^2)D^3 - 2xD^2 - (2\alpha+1)xD^2 - (2\alpha+1)D + n(n+2\alpha)D\right] C_n^{\alpha}(x) = 0$$

$$\lambda = 2 \quad \left[(1-x^2)D^4 - 4xD^3 - (2\alpha+1)xD^3 - 2D^2 - (2\alpha+1)D^2 + n(n+2\alpha)D^2\right] C_n^{\alpha}(x) = 0$$

$$\lambda = 3 \quad \left[(1-x^2)D^5 - (6 + 2\alpha+1)xD^4 + \left[n(n+2\alpha) - (2\alpha+13)\right]D^3\right] C_n^{\alpha}(x) = 0$$

....

$$\lambda = \lambda \quad \left[(1-x^2)D^{2+\lambda} - (2\lambda+2\alpha+1)xD^{1+\lambda} + \left[n(n+2\alpha) - \lambda(\lambda+2\alpha)\right]D^{\lambda}\right] C_n^{\alpha}(x) = 0$$

or

$$\left[(1-x^2)D^2 - (2\lambda+2\alpha+1)xD^1 + \left[n(n+2\alpha)-\lambda(\lambda+2\alpha)\right]\right] D^{\lambda}\, C_n^{\alpha}(x) = 0.$$

$D^{\lambda}\, C_n^{\alpha}(x)$ is the λth derivative of Gegenbauer's polynomial and is called Gegenbauer's derived polynomial. The notation for Gegenbauer's derived polynomial is $C_n^{\alpha(\lambda)}(x)$ and, from above, it satisfies the differential equation

$$(4.2) \quad \left[(1-x^2)D_{xx} - (2\lambda+2\alpha+1)\, x\, D_x + n(n+2\alpha) - \lambda(\lambda+2\alpha)\right] C_n^{\alpha(\lambda)}(x) = 0.$$

2. To obtain another form of the equation for Gegenbauer's derived polynomials a change of variable is made. The change is $x = \cosh t$ which leads to

$$D_x = \frac{1}{\sinh t} D_t$$

and

$$D_{xx} = \frac{1}{(\sinh t)^2} D_{tt} - \frac{\cosh t}{(\sinh t)^3} D_t.$$

A direct substitution of these values into (4.2) and simplification leads to the differential equation

$$(4.3)\quad \left[\sinh t\, D_{tt} + (2\lambda + 2\alpha)\cosh t\, D_t + [\lambda(\lambda + 2\alpha) - n(n + 2\alpha)]\sinh t\right] C_n^{\alpha(\lambda)}(\cosh t) = 0.$$

Comparing (4.3) with (2.2) and setting $a = \lambda + n + 2\alpha$ and $b = n - \lambda$ leads to a special case of (2.2). Therefore the differential equation for Gegenbauer's derived polynomials is a special case of the differential equation for F. However, saying the equations are the same does not assure that F and $C_n^{\alpha(\lambda)}(\cosh t)$ are the same for certain values of a and b since these equations have two solutions. Since $C_n^{\alpha}(\cosh t)$ is a polynomial, it follows that $C_n^{\alpha(\lambda)}(\cosh t)$ is a polynomial. The other solution of (4.3) is an infinite form [8]. Therefore if F is a polynomial for $a = \lambda + n + 2\alpha$ and $b = n - \lambda$, F can be set equal to $C_n^{\alpha(\lambda)}(\cosh t)$ except for a multiplicative constant. One restriction is immediately apparent and that is n must be an integer and 2α must be an integer since a must be an integer. The case where $2\alpha + n \neq$ an integer, while applicable, will not be considered. It is seen from Table 1. that there are four cases where

F is a polynomial in cosh x. These are cases (II, c), (II, d), (III, c) and (III, d).

From the fact that $a = \lambda + n + 2\alpha$ and $b = n - \lambda$ it follows that $n + \lambda + 2\alpha > n - \lambda$ which implies that $a > b + 2$ if $2\alpha > 0$, so only case II is considered. The case $\lambda = 0$ is not being considered here. From this the following results are obtained

$$C_n^{\alpha(\lambda)}(\cosh t) = kF(\cosh t) \tag{4.4}$$

for $2a > 0$ and odd. F reduces to the trivial solution for 2α even. The above equation may also be written as

$$C_n^{\alpha(\lambda)}(\cosh t) = k\left[D^2 - (n + \lambda + 2\alpha - 2)^2\right] \dots \left[D^2 - 1\right] \sinh^{n-\lambda} t$$

or

$$C_n^{\alpha(\lambda)}(\cosh t) = k\left[D^2 - (n + \lambda + 2\alpha - 2)^2\right] \dots \left[D^2 - 2^2\right] D \sinh^{n-\lambda} t$$

depending on whether $n + \lambda + 2\alpha$ is odd or even.

k is found by comparing the coefficient of the highest order term of $C_n^{\alpha(\lambda)}(\cosh t)$ and the right hand side of the above equations. This constant is of the same type as the one found in the next chapter and will not be computed here.

3. The most important special cases of Gegenbauer's polynomials are the Legendre polynomials corresponding to the value $\alpha = 1/2$ [6, 7]. They are denoted by $P_n(x)$. Legendre's derived polynomials are denoted by $P_n^{(\lambda)}(x)$. These are not to be confused with Legendre's associated functions $P_n^{\lambda}(x)$ which are treated in chapter IV. From (4.4) it follows that

$$P_n^{(\lambda)}(\cosh t) = kF(\cosh t)$$

or

$$P_n^{(\lambda)}(\cosh t) = k\left[D^2 - (n+\lambda-1)^2\right] \ldots \left[D^2 - (3)^2\right]\left[D^2 - 1\right] \sinh^{n-\lambda} t$$

for $n+\lambda-1$ odd, or

$$P_n^{(\lambda)}(\cosh t) = k\left[D^2 - (n+\lambda-1)^2\right] \ldots \left[D^2 - (2)^2\right] D \sinh^{n-\lambda} t$$

for $n+\lambda-1$ even. The restriction is made that n be greater than λ.

In order to evaluate k the coefficient of the highest power of this representation is compared with the usual representation of $P_n^{(\lambda)}(x)$. The highest term in the usual definition is [6]

$$\frac{(2n)!}{2^n (n)!(n-\lambda)!} \cosh^{n-\lambda} t.$$

The repeated application of (2.4) gives the highest term of F to be

$$k\frac{(2n-1)!(2\lambda)!(-1)^\lambda}{2^{n+\lambda-1}(n-1)!(\lambda)!} \cosh^{n-\lambda} t.$$

k is solved for from the above equations; its value is found to be

$$k = \frac{(-1)^\lambda 2^\lambda (\lambda)!}{(n-\lambda)!(2\lambda)!}. \tag{4.5}$$

The important special case of $\lambda = 0$, $2\alpha = 1$ gives Legendre's polynomial. This case gives the following values for a and b; $a = n+1$, $b = n$. Since $a < b+2$ and $a \bmod 1 \neq b \bmod 1$ these values correspond to cases (III, c) and (III, d). Since both of these are polynomials in cosh t, F can be set equal to Legendre's polynomial, except for a multiplicative constant, for $a = n+1$ and $b = n$. This leads to the following representation

$$P_n(\cosh t) = \frac{1}{n!}\left[D^2 - (n-1)^2\right] \ldots \left[D^2 - 3^2\right]\left[D^2 - 1\right] \sinh^n t \tag{4.6}$$

for n even and

$$P_n(\cosh t) = \frac{1}{n!}\left[D^2 - (n-1)^2\right] \ldots \left[D^2 - 2^2\right] D \sinh^n t$$

for n odd. For $\lambda = 0$ (4.5) reduces to $\frac{1}{n!}$ as used above.

4. The developement for F(cos t) is analogous to the developement for F(cosh t). If the change of variable $x = \cos t$ is made in (4.3), then equation (4.4) becomes

$$\left[\sin t\, D_{tt} + (2\lambda + 2\alpha)\cos t\, D_t + \left[\lambda(\lambda+2\alpha) - n(n+2\alpha)\right]\right] C_n^{\alpha(\lambda)}(\cos t) = 0.$$

The same arguments as in section two lead to the following equations

$$C_n^{\alpha(\lambda)}(\cos t) = kF(\cos t)$$

for $2\alpha > 0$ and odd,

$$C_n^{\alpha(\lambda)}(\cos t) = k\left[D^2 + (n+\lambda+2\alpha-2)^2\right] \ldots \left[D^2 + 1\right] \sin^{n-\lambda} t$$

for $n+\lambda+2\alpha$ odd and

$$C_n^{\alpha(\lambda)}(\cos t) = k\left[D^2 + (n+\lambda+2\alpha-2)^2\right] \ldots \left[D^2 + 2^2\right] D \sin^{n-\lambda} t$$

for $n+\lambda+2\alpha$ even.

The special case for Legendre's derived polynomials is treated in the same way and the equations of section three become

$$P_n^{(\lambda)}(\cos t) = kF(\cos t)$$

$$P_n^{(\lambda)}(\cos t) = k\left[D^2 + (n+\lambda-1)^2\right] \ldots \left[D^2 + 1\right] \sin^{n-\lambda} t$$

$$P_n^{(\lambda)}(\cos t) = k\left[D^2 + (n+\lambda-1)^2\right] \ldots \left[D^2 + 2^2\right] \sin^{n-\lambda} t.$$

k is determined in the same way as in section three and is found to be the same thing,

$$k = \frac{(-1)^{\lambda} 2^{\lambda} \lambda!}{(n-\lambda)!(2\lambda)!}.$$

Likewise for the special case $\lambda = 0$ (4.6) becomes

$$P_n(\cos t) = \frac{1}{n!}\left[D^2 + (n-1)^2\right] \ldots \left[D^2 + 3^2\right]\left[D^2 + 1\right] \sin^n t$$

for n even and

$$P_n(\cos t) = \frac{1}{n!}\left[D^2 + (n-1)^2\right] \ldots \left[D^2 + 2^2\right] D \sin^n t$$

for n odd.

V. F AND GEGENBAUER'S ASSOCIATED FUNCTIONS

1. In this chapter the function F is shown to include certain classical associated functions.

Gegenbauer's associated function is defined analogously to Legendre's associated function as

$$C_n^{\alpha\lambda}(x) = (x^2-1)^{\frac{(\lambda+2\alpha-1)}{2}} C_n^{\alpha(\lambda)}(x) \tag{5.1}$$

for $x>1$ and

$$C_n^{\alpha\lambda}(x) = (1-x^2)^{\frac{(\lambda+2\alpha-1)}{2}} C_n^{\alpha(\lambda)}(x)$$

for $x<1$. $C_n^{\alpha(\lambda)}(x)$ satisfies the equation

$$\left[(1-x^2)D_{xx} - (2\lambda+2\alpha+1)xD_x + \left[n(n+2\alpha) - \lambda(\lambda+2\alpha)\right]\right] C_n^{\alpha(\lambda)}(x) = 0$$

or, after the change of variable $x = \cosh t$, the equation

$$\left[\sinh t\, D_{tt} + (2\lambda+2\alpha) \cosh t\, D_t - \left[n(n+2\alpha) - \lambda(\lambda+2\alpha)\right] \sinh t\right] C_n^{\alpha(\lambda)}(\cosh t) = 0. \tag{5.2}$$

The substitution $u = \sinh^{(2\lambda+2\alpha-1)} t\, C_n^{\alpha(\lambda)}(\cosh t)$ in (5.2) gives

$$\left[\sinh t\, D_{tt} - (2\lambda+2\alpha-2) \cosh t\, D_t + \left[\lambda(\lambda+2\alpha-1) + 1 - 2\alpha - n(n+2\alpha)\right]\sinh t\right] u = 0. \tag{5.3}$$

Since

$$u = \sinh^{\lambda} t\, C_n^{\alpha\lambda}(\cosh t)$$

equation (5.3) can be written as

$$(5.4)\quad \left[\sinh t\, D_{tt} + \left[(n-\lambda+1) - (n+\lambda+2\alpha-1)\right] \cosh t\, D_t - \right.$$

$$\left.(n-\lambda+1)(n+\lambda+2\alpha-1) \sinh t\right] \sinh^{\lambda} t\, C_n^{\alpha\lambda}(\cosh t) = 0.$$

2. Equation (5.4) is of the form studied in the first two chapters so $F(\cosh t)$ and $\sinh t\, C_n^{\alpha\lambda}(\cosh t)$ are solutions of the same differential equation. The values of a and b in terms of α and λ are

$$a = n-\lambda+1, \quad b = n+\lambda+2\alpha-1.$$

Since $a < b+2$ for $\lambda+2\alpha > 0$, $F(\cosh t)$ falls in category III. All four cases of III may occur for different values of n, λ and α. In order to determine if $F(\cosh t)$ and $\sinh^{\lambda} t\, C_n^{\alpha\lambda}(\cosh t)$ are equal, the nature of the two functions must be studied.

From (5.1) it is seen that

$$\sinh^{\lambda} t\, C_n^{\alpha\lambda}(\cosh t) = \sinh^{2\lambda+2\alpha-1}\, C_n^{\alpha(\lambda)}(\cosh t).$$

$C_n^{\alpha(\lambda)}(\cosh t)$ is a polynomial in $\cosh t$ and 2λ is even. The determining factor is then $2\alpha-1$. For the two cases $2\alpha-1$ odd and $2\alpha-1$ even, it is seen that

$$\sinh^{\lambda} t\, C_n^{\alpha\lambda}(\cosh t) = \sinh t\, P(\cosh t)$$

for $2\alpha-1$ odd and

$$\sinh^{\lambda} t\, C_n^{\alpha\lambda}(\cosh t) = P(\cosh t)$$

for $2\alpha-1$ even. If a and b are compared modulus 1,

$$a \bmod 1 = b \bmod 1$$

for $2\alpha-1$ odd and

$$a \bmod 1 \neq b \bmod 1$$

for $2\alpha-1$ even. These equations imply from Table 1 that

$$F(\cosh t) = \sinh t\, P(\cosh t)$$

for $2\alpha-1$ odd and

$$F(\cosh t) = P(\cosh t)$$

for $2\alpha-1$ even. If these equations are compared, it is seen that $F(\cosh t)$ and $\sinh^{\lambda} t\, C_n^{\alpha\lambda}(\cosh t)$ are the same solution of (5.4). Therefore the following expressions can be written for $C_n^{\alpha\lambda}(\cosh t)$

$$C_n^{\alpha\lambda}(\cosh t) = \frac{k}{\sinh^{\lambda} t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2 - 3^2\right]\left[D^2 - 1\right] \sinh^{n+\lambda+2\alpha-1} t$$

for $n-\lambda$ even and

$$C_n^{\alpha\lambda}(\cosh t) = \frac{k}{\sinh^{\lambda} t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2 - 2^2\right] D \sinh^{n+\lambda+2\alpha-1} t$$

for $n-\lambda$ odd. k is found by comparing the coefficient of the highest order term of $C_n^{\alpha\lambda}(\cosh t)$ and the right hand side of the above equation. The coefficient of the highest order term in $C_n^{\alpha\lambda}(\cosh t)$ is $\frac{2^n (\alpha)_n}{(n-\lambda)!}$ [11] and the coefficient of the highest order term on the right is seen to be

$$\frac{2^{\lambda-n+1}(2n+2\alpha-2)!\left(\lambda+\frac{2\alpha-1}{2}\right)!}{\left(n+\frac{2\alpha-3}{2}\right)!(2\lambda+2\alpha-1)!}$$

for 2α odd and

$$\frac{2^{n-\lambda}(n+\alpha-1)!}{(\lambda+\alpha-1)!}$$

for 2α even. The above two values give two corresponding values of k

$$k = \frac{2^{2n-\lambda+1}(n+\frac{2\alpha-3}{2})!(2\lambda+2\alpha-1)!(\alpha)n}{(n-\lambda)!(2n+2\alpha-2)!(n+\frac{2\alpha-1}{2})!}$$

for 2α odd and

$$k = \frac{2\lambda(\lambda+\alpha-1)!(\alpha)n}{(n-\lambda)!(n+\alpha-1)!}$$

for 2α even.

3. The most important of Gegenbauer's associated functions is for the value of $\alpha = 1/2$. This is Legendre's associated function [5]. It is given by

$$P_n^{\lambda}(\cosh t) = \frac{(2\lambda)!}{2^{\lambda}\lambda!(n-\lambda)!\sinh^{\lambda} t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2-2^2\right]D \sinh^{n+\lambda} t$$

for $n+\lambda$ odd and

$$P_n^{\lambda}(\cosh t) = \frac{(2\lambda)!}{2^{\lambda}\lambda!(n-\lambda)!\sinh^{\lambda} t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2 - 1\right] \sinh^{n+\lambda} t$$

for $n+\lambda$ even. The above equations reduce to Legendre's polynomials for $\lambda = 0$. The representations are

$$P_n(\cosh t) = \frac{1}{n!}\left[D^2 - (n-1)^2\right] \ldots \left[D^2 - 2^2\right]D \sinh^n t$$

for n odd and

$$P_n(\cosh t) = \frac{1}{n!}\left[D^2 - (n-1)^2\right] \ldots \left[D^2 - 3^2\right]\left[D^2 - 1\right] \sinh^n t$$

for n even. The expressions above are exactly the same as the ones found in section 3 of chapter four.

4. Two other important special cases of Gegenbauer's associated functions exist and they correspond to the values $\alpha = 0$ and $\alpha = 1$. They are Tchebycheff's associated functions of the first and second kind, respectively [6, 7]. Tchebycheff's associated function of the first kind is given by the expression

$$T_n^\lambda(\cosh t) = \frac{2^\lambda n(\lambda-1)!}{(n-\lambda)!\sinh^\lambda t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2 - 2^2\right] D \sinh^{n+\lambda-1} t$$

for $n+\lambda$ odd and

$$T_n^\lambda(\cosh t) = \frac{2^\lambda n(\lambda-1)!}{(n-\lambda)!\sinh^\lambda t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2 - 3^2\right]\left[D^2 - 1\right] \sinh^{n+\lambda-1} t$$

for $n+\lambda$ even. Tchebycheff's associated function of the second kind is given by the expressions

$$U_n^\lambda(\cosh t) = \frac{2^\lambda (n+1)(\lambda)!}{(n-\lambda)!\sinh^\lambda t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2 - 2^2\right] D \sinh^{n+\lambda+1} t$$

for $n+\lambda$ odd and

$$U_n^\lambda(\cosh t) = \frac{2^\lambda (n+1)(\lambda)!}{(n-\lambda)!\sinh^\lambda t}\left[D^2 - (n-\lambda-1)^2\right] \ldots \left[D^2 - 3^2\right]\left[D^2 - 1\right] \sinh^{n+\lambda+1} t$$

for $n+\lambda$ even. The expressions for these functions reduce to identically zero for $\lambda = 0$.

5. The results of the chapter for circular functions parallel closely the results for hyperbolic functions. The only changes are that hyperbolic functions be replaced by circular ones and that terms

of the kind $D^2 - a^2$ be replaced by $D^2 + a^2$. A typical result is

$$P_n^{\lambda}(\cos t) = \frac{(2\lambda)!}{2^{\lambda}\lambda!(n-\lambda)!\sin^{\lambda} t}\left[D^2 + (n-\lambda-1)^2\right] \dots \left[D^2 + 2^2\right] D \sin^{n+\lambda} t$$

for $n+\lambda$ odd.

VI. SUMMARY

In this paper the function F has been defined and studied. It was shown to be the solution of a certain differential equation. Secondly, F was studied as a function of its parameters a and b and the analytic nature of F was found for various combinations of values of a and b. Then relationships between F and the classical Gegenbauer functions were derived through the differential equation from two standpoints. One was from the standpoint of derived polynomials and the other was from associated functions. Throughout the paper the case of F with a circular argument has been treated parallel to that of F with a hyperbolic argument.

Current investigations of F are continuing along the following lines. The nature of F as related to the classical functions for other values of a and b is desired. It can be shown that F is always either a Gegenbauer polynomial or the derivative of a Gegenbauer polynomial, if n is allowed to take on non integral values. Second the developement of known properties of Gegenbauer's function is to be made from the standpoint of F. This work is well under way. Finally, the developement of new properties of Gegenbauer's function is to be investigated. This work has barely been touched. A more convenient notation would perhaps replace the parameter a by a parameter whose value is a-2.

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