THE STAIRCASE DESIGN

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PREFACE

In planning a randomized block experiment we frequently find that we have blocks available which are quite homogeneous, and yet contain different numbers of experimental units. The usual practice is to discard experimental units from the larger blocks making them all equal in size to the smallest one or to discard all blocks that have fewer experimental units than the largest one. The purpose of the design proposed in this paper is to allow usage of these commonly discarded experimental units so as to gain more information about the treatments and, in the first case, to include more treatments in the experiment.

A particular example where this would be useful is an experiment involving animals as experimental units where the blocks consist of litter mates. Let us suppose that we have two litters of size seven, three of size five, and one of size four. Using the staircase design we can include seven treatments and still have the four we are most interested in replicated six times.

Another usage of the design is in analyzing an ordinary randomized block experiment when data are missing in a stairstep pattern. The most probable occurrence of this is a single step, e.g. where one or more treatments are missing in a single block.

I am indebted to Dr. F. A. Graybill for suggesting the problem, and for his advice during the preparation of this paper.

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INTRODUCTION AND NOTATION

Consider a two-way classification model

$$(1.1) \quad Y_{ij} = \mu + \rho_i + \tau_j + e_{ij}, \qquad i = 1, 2, \ldots, a_j, \; j = 1, 2, \ldots, N,$$

where μ, ρ_i, τ_j are constants and e_{ij} is a normal independent variable with mean zero and variance σ^2. Also the j's will be ordered in such a way that $a_j \geq a_{j'}$ for $j < j'$. The purpose of this paper is

1. to derive the least squares method for testing the hypothesis $\tau_1 = \tau_2 = \ldots = \tau_N$ under the model given above and to give the power of this test,
2. to derive the best, linear, unbiased estimates for $\tau_j - \tau_{j'}$, and the variances of these estimates.

First we will separate the j's into subsets such that $a_j = a_{j'}$ if and only if j and j' are in the same subset. Each of these subsets will be called a step. We will designate the number of steps as k.

Let $a_j = M^1$ for $j = 1, 2, \ldots, N_1$,

$a_j = M^2$ for $j = N_1 + 1, N_1 + 2, \ldots, N_1 + N_2$,

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$a_j = M^k$ for $j = N_1 + N_2 + \ldots + N_{k-1} + 1, N_1 + N_2 + \ldots + N_{k-1} + 2, \ldots, N_1 + N_2 + \ldots + N_k$,

where

$$\sum_{\ell=1}^{k} N_\ell = N .$$

Now, let

$$N^p = \sum_{\ell=1}^{p} N_\ell , \qquad N^0 = 0, \qquad M^{k+1} = 0,$$

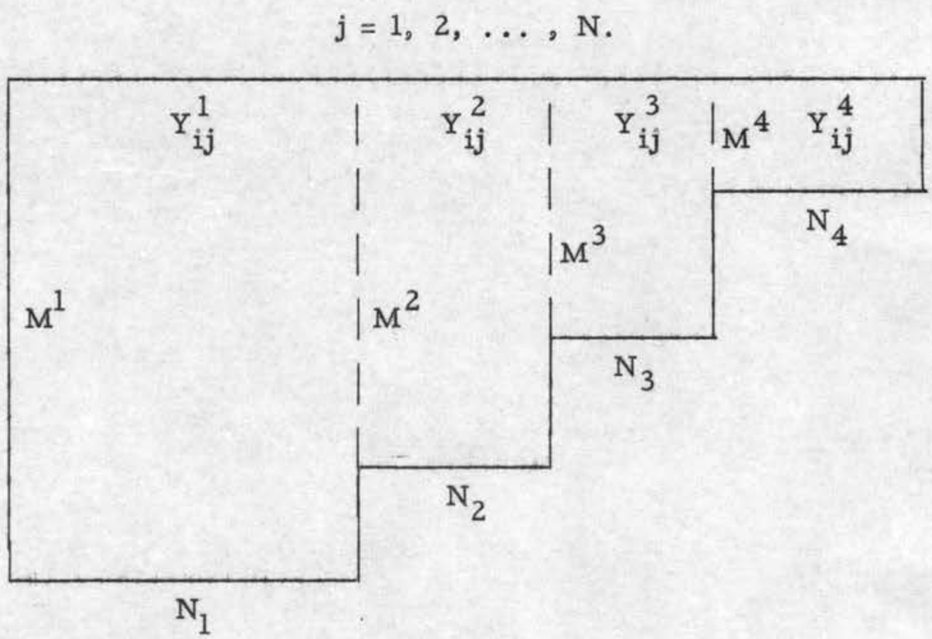
$$Y_{ij} = Y_{ij}^{s} \text{ for } i = 1, 2, \ldots , M^s, \; j = N^{s-1} + 1, N^{s-1} + 2, \ldots , N^s,$$

$$Y_{ij} = Y_{ij}^{\prime s} \text{ for } i = 1, 2, \ldots , M^{s+1}, \; j = 1, 2, \ldots , N^s,$$

$$\tau_j = \tau_j^{s} \quad \text{for } j = N^{s-1} + 1, N^{s-1} + 2, \ldots , N^s,$$

$$\tau_j = \tau_j^{\prime s} \text{ for } j = 1, 2, \ldots , N^s .$$

The following diagram will serve to illustrate some of the notation.



It may be helpful to note further that $Y_{ij}^{\prime 1}$ is a subset of Y_{ij}^{1}, $Y_{ij}^{\prime 2}$ is a subset of the union of Y_{ij}^{1} and Y_{ij}^{2}, $Y_{ij}^{\prime 3}$ is a subset of the union of Y_{ij}^{1} and Y_{ij}^{2} and Y_{ij}^{3}, etc.

A subscript replaced by a dot indicates the mean of the elements when summed over the range of the replaced subscript, eg.

$$Y'^2_{..} = \frac{\sum_{i=1}^{M^3} \sum_{j=1}^{N^2} Y'^2_{ij}}{M^3 N^2} .$$

Since superscripts are being used in abundance, a Y, M, N, or τ that is raised to a power will always be enclosed in the appropriate brackets.

If, in a summation, the lower limit of summation should exceed the upper limit of summation, the sum will be zero.

The notation used in the section on least squares is that used by Kempthorne (1952), pages 79-82, with the following exceptions. To be consistent with the notation given above, the normal equations are divided by a constant to give them in terms of means instead of totals. $\tilde{\tau}$ is used as a symbol for an estimate of τ because of typing restrictions. Q_j^s will refer to only the Q_j's where $j = N^{s-1} + 1, N^{s-1} + 2, \ldots, N^s$.

U is distributed as $\chi'^2_{p,\lambda}$ will be used to denote a random variable U which is distributed as the non-central χ^2 with p degrees of freedom and non-centrality λ. The probability density function is

$$f(U) = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m e^{-\frac{1}{2}U} U^{m+\frac{p}{2}-1}}{m!\, 2^{m+\frac{p}{2}} \Gamma(m + \frac{p}{2})} .$$

V is distributed as $F'_{p,q,\lambda}$ will be used to denote a random variable V which is distributed as the non-central F with p degrees of freedom, q degrees of freedom, and non-centrality λ. The probability density function is

$$f(V) = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m V^{m+\frac{p}{2}-1}}{m!\, \beta(m+\frac{p}{2}, \frac{q}{2})\, (1+V)^{m+\frac{p}{2}+\frac{q}{2}}} \; .$$

THE TEST FUNCTION AND ITS DISTRIBUTIONAL PROPERTIES

The purpose of this section is to give a test of the hypothesis $\tau_1 = \tau_2 = \ldots = \tau_N$ and to prove the distributional properties of the test function. The proof that this test is the same as that given by the method of least squares will be given in the next section.

Consider the following quadratic forms:

$$q_\ell^1 = \sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} (Y_{ij}^\ell - Y_{i.}^\ell - Y_{.j}^\ell + Y_{..}^\ell)^2 , \qquad \ell = 1, 2, \ldots, k,$$

$$q_\ell^2 = \frac{N^\ell N_{\ell+1}}{N^{\ell+1}} \sum_{i=1}^{M^{\ell+1}} (Y_{i.}^{\prime\ell} - Y_{i.}^{\ell+1} - Y_{..}^{\prime\ell} + Y_{..}^{\ell+1})^2 , \qquad \ell = 1, 2, \ldots, k-1,$$

$$q_\ell^3 = M^\ell \sum_{j=N^{\ell-1}+1}^{N^\ell} (Y_{.j}^\ell - Y_{..}^\ell)^2 , \qquad \ell = 1, 2, \ldots, k,$$

$$q_\ell^4 = \frac{M^{\ell+1} N^\ell N_{\ell+1}}{N^{\ell+1}} (Y_{..}^{\prime\ell} - Y_{..}^{\ell+1})^2 , \qquad \ell = 1, 2, \ldots, k-1,$$

$$q_\ell^5 = \frac{1}{N^\ell} \sum_{i=M^{\ell+1}+1}^{M^\ell} (N^{\ell-1} Y_{i.}^{\prime\ell-1} + N_\ell Y_{i.}^\ell)^2 , \qquad \ell = 1, 2, \ldots, k,$$

$$q_\ell^6 = \sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} (Y_{ij}^\ell)^2 , \qquad \ell = 1, 2, \ldots, k.$$

Theorem: If

$$(2.1) \qquad V = \frac{\sum_{\ell=1}^{k} q_\ell^3 + \sum_{\ell=1}^{k-1} q_\ell^4}{\sum_{\ell=1}^{k} q_\ell^1 + \sum_{\ell=1}^{k-1} q_\ell^2} \cdot \frac{(M^1-1)(N-1) - \sum_{\ell=2}^{k}(M^1 - M^\ell)(N_\ell)}{N-1}$$

then V is distributed as $F'_{p,\, q,\, \lambda}$, where

$$p = N - 1 , \qquad q = (M^1 - 1)(N-1) - \sum_{\ell=2}^{k}(M^1 - M^\ell)(N_\ell) ;$$

$$(2.2) \qquad \lambda = \sum_{\ell=1}^{k}\left[\frac{M^\ell}{2\sigma^2}\sum_{j=N^{\ell-1}+1}^{N^\ell}(\tau_j^\ell - \tau_\cdot^\ell)^2\right] + \sum_{\ell=1}^{k-1}\left[\frac{M^{\ell+1}N^\ell N_{\ell+1}}{2\sigma^2 N^{\ell+1}}(\tau_\cdot^{\prime\ell} - \tau_\cdot^{\ell+1})^2\right]$$

and $\lambda = 0$ if and only if $\tau_1 = \tau_2 = \ldots = \tau_N$.

Proof: It is clear that

$$q_\ell^1 = \sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} (Y_{ij}^\ell - Y_{i.}^\ell - Y_{.j}^\ell + Y_{..}^\ell)^2 , \qquad \ell = 1, 2, \ldots, k$$

may be written as

$$q_\ell^1 = \sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} \left[(Y_{ij}^\ell)^2 - (Y_{i.}^\ell)^2 - (Y_{.j}^\ell)^2 + (Y_{..}^\ell)^2\right] , \quad \ell = 1, 2, \ldots, k.$$

And

$$q_\ell^3 = M^\ell \sum_{j = N^{\ell-1}+1}^{N^\ell} (Y^\ell_{.j} - Y^\ell_{..})^2 \quad , \qquad \ell = 1, 2, \ldots , k$$

may be written as

$$q_\ell^3 = M^\ell \sum_{j = N^{\ell-1}+1}^{N^\ell} \left[(Y^\ell_{.j})^2 - (Y^\ell_{..})^2 \right] \quad , \qquad \ell = 1, 2, \ldots , k.$$

Adding $q_\ell^1 + q_\ell^3$, we have

$$q_\ell^7 = \sum_{i=1}^{M^\ell} \sum_{j = N^{\ell-1}+1}^{N^\ell} \left[(Y^\ell_{ij})^2 - (Y^\ell_{i.})^2 \right] \quad , \qquad \ell = 1, 2, \ldots , k.$$

Similarly

$$q_\ell^2 = \frac{N^\ell N_{\ell+1}}{N^{\ell+1}} \sum_{i=1}^{M^{\ell+1}} (Y'^\ell_{i.} - Y^{\ell+1}_{i.} - Y'^\ell_{..} + Y^{\ell+1}_{..})^2 \; , \quad \ell = 1, 2, \ldots , k-1$$

may be written as

$$q_\ell^2 = \frac{N^\ell N_{\ell+1}}{N^{\ell+1}} \sum_{i=1}^{M^{\ell+1}} \left[(Y'^\ell_{i.} - Y^{\ell+1}_{i.})^2 - (Y'^\ell_{..} - Y^{\ell+1}_{..})^2 \right] \; , \quad \ell = 1, 2, \ldots , k-1.$$

Adding $q_\ell^2 + q_\ell^4$, we have

$$q_\ell^8 = \frac{N^\ell N_{\ell+1}}{N^{\ell+1}} \sum_{i=1}^{M^{\ell+1}} (Y'^\ell_{i.} - Y^{\ell+1}_{i.})^2 \quad , \qquad \ell = 1, 2, \ldots , k-1$$

which may be written as

$$q_\ell^8 = \frac{N^\ell N_{\ell+1}}{N^{\ell+1}} \sum_{i=1}^{M^{\ell+1}} \left[\frac{N_1 Y_{i.}^1 + N_2 Y_{i.}^2 + \ldots + N_\ell Y_{i.}^\ell}{N^\ell} - Y_{i.}^{\ell+1} \right]^2 ,$$

$$\ell = 1,\ 2,\ \ldots ,\ k-1.$$

Also

$$q_\ell^5 = \frac{1}{N^\ell} \sum_{i=M^{\ell+1}+1}^{M^\ell} (N^{\ell-1} Y_{i.}^{'\ell-1} + N_\ell Y_{i.}^\ell)^2 , \qquad \ell = 1,\ 2,\ \ldots ,\ k$$

$$= \frac{1}{N^\ell} \sum_{i=M^{\ell+1}+1}^{M^\ell} (N_1 Y_{i.}^1 + N_2 Y_{i.}^2 + \ldots + N_\ell Y_{i.}^\ell)^2 , \qquad \ell = 1,\ 2,\ \ldots ,\ k.$$

Now we will add $\sum_{\ell=1}^{k-1} q_\ell^8 + \sum_{\ell=1}^{k} q_\ell^5$ by collecting coefficients of similar terms. First we will collect coefficients of the term

$$\sum_{i=M^{t+1}+1}^{M^t} (Y_{i.}^p)^2 , \qquad p = 1,\ 2,\ \ldots ,\ k, \quad t = p,\ p+1,\ \ldots ,\ k$$

where $p \leqslant t$ or the term does not exist. In the q_ℓ^8, $Y_{i.}^p$ will not occur until $\ell = p-1$. In q_{p-1}^8 the coefficient is

$$\frac{N^{p-1} N_p}{N^p} .$$

In q_p^8 the coefficient is

$$\frac{N^p N_{p+1} (N_p)^2}{N^{p+1} (N^p)^2}$$

which reduces to

$$\frac{(N_p)^2 N_{p+1}}{N^{p+1} N^p} \quad .$$

We continue in this manner until $\ell = t-1$ which will give us the last q_ℓ^8 involving this term because of the upper limit of summation. This coefficient will be

$$\frac{N^{t-1} N_t (N_p)^2}{N^t (N^{t-1})^2}$$

which reduces to

$$\frac{(N_p)^2 N_t}{N^t N^{t-1}} \quad .$$

Also from the q_t^5 we have the coefficient

$$\frac{(N_p)^2}{N^t} \quad .$$

Adding, we have

$$\frac{N^{p-1} N_p}{N^p} + (N_p)^2 \left[\frac{N_{p+1}}{N^{p+1} N^p} + \frac{N_{p+2}}{N^{p+2} N^{p+1}} + \cdots + \frac{N_t}{N^t N^{t-1}} + \frac{1}{N^t} \right] \quad .$$

Examining the last two terms of the sum in brackets, it may be seen to add to

$$\frac{N_t + N^{t-1}}{N^t N^{t-1}} = \frac{N^t}{N^t N^{t-1}} = \frac{1}{N^{t-1}} \quad .$$

Continuing in this manner, it will be seen that the above coefficient reduces to

$$\frac{N^{p-1} N_p}{N^p} + (N_p)^2 \left[\frac{1}{N^p} \right] = \frac{N_p}{N^p} (N^{p-1} + N_p) = N_p \,.$$

In case p = 1 this is slightly irregular since N^{p-1} is zero in that case. However, here we have $N_p = N^p$, so that we still have the coefficient N_p.

We will now collect coefficients of the term

$$\sum_{i=M^{t+1}+1}^{M^t} Y_{i.}^p Y_{i.}^r \qquad \begin{array}{l} p = 1, 2, \ldots, k-1, \\ r = p+1, p+2, \ldots, k, \\ t = r, r+1, \ldots, k. \end{array}$$

In the q_ℓ^8 this term will not occur until $\ell = r-1$. In this case the coefficient is

$$-\frac{2 N^{r-1} N_r N_p}{N^r N^{r-1}}$$

which reduces to

$$-\frac{2 N_r N_p}{N^r} .$$

In q_r^8 the coefficient is

$$\frac{2 N^r N_{r+1} N_p N_r}{N^{r+1} (N^r)^2}$$

which reduces to

$$\frac{2 N_{r+1} N_p N_r}{N^{r+1} N^r} .$$

We continue in this manner until $\ell = t-1$ which again will give the last q_ℓ^8 involving this term. This coefficient will be

$$\frac{2N^{t-1}N_tN_pN_r}{N^t(N^{t-1})^2}$$

which reduces to

$$\frac{2N_tN_pN_r}{N^tN^{t-1}} \quad .$$

Also from q_t^5 we have

$$\frac{2N_pN_r}{N^t} \quad .$$

Adding we have

$$-\frac{2N_pN_r}{N^r} + 2N_pN_r\left[\frac{N_{r+1}}{N^{r+1}N^r} + \frac{N_{r+2}}{N^{r+2}N^{r+1}} + \ldots + \frac{N_t}{N^tN^{t-1}} + \frac{1}{N^t}\right].$$

As before, the sum in brackets is $\frac{1}{N^r}$, and therefore the above coefficient is zero. But these two general terms are the only possible ones involved in the q_ℓ^8 and the q_ℓ^5, so we now have

$$\sum_{\ell=1}^{k-1} q_\ell^8 + \sum_{\ell=1}^{k} q_\ell^5 = \sum_{\ell=1}^{k} q_\ell^9 \quad ,$$

where

$$q_\ell^9 = N_\ell \sum_{i=1}^{M^\ell} (Y_{i.}^\ell)^2 \quad , \qquad \ell = 1, 2, \ldots, k.$$

Now adding $q_\ell^9 + q_\ell^7$ we have

$$q_\ell^6 = \sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} (Y_{ij}^\ell)^2 \quad , \qquad \ell = 1, 2, \ldots, k.$$

Thus we have shown that

$$(2.3)\quad \sum_{\ell=1}^{k} q_\ell^1 + \sum_{\ell=1}^{k-1} q_\ell^2 + \sum_{\ell=1}^{k} q_\ell^3 + \sum_{\ell=1}^{k-1} q_\ell^4 + \sum_{\ell=1}^{k} q_\ell^5 = \sum_{\ell=1}^{k} q_\ell^6 \ .$$

Now it is easily shown that the rank of q_ℓ^1 is $(M^\ell - 1)(N_\ell - 1)$, the rank of q_ℓ^2 is $(M^{\ell+1} - 1)$, the rank of q_ℓ^3 is $(N_\ell - 1)$, the rank of q_ℓ^4 is 1, and the rank of q_ℓ^5 is $(M^\ell - M^{\ell+1})$. Adding we see that

$$\sum_{\ell=1}^{k} (M^\ell - 1)(N_\ell - 1) + \sum_{\ell=1}^{k-1} (M^{\ell+1} - 1) + \sum_{\ell=1}^{k} (N_\ell - 1)$$

$$+ (k-1) + \sum_{\ell=1}^{k-1} (M^\ell - M^{\ell+1}) + (M^k - M^{k+1}) = \sum_{\ell=1}^{k} M^\ell N_\ell \ .$$

Thus we have the fact that the sum of the ranks of the quadratic forms on the left of (2.3) above is equal to the number of squared observations on the right. We may now invoke a theorem proved by Madow (1940) showing the quadratic forms to be independent, and verifying the follow- distributions.

1. $\frac{q_\ell^1}{\sigma^2}$ is distributed as $\chi'^2_{p,\lambda}$, where $p = (M^\ell - 1)(N_\ell - 1)$, $\lambda = 0$,

since

$$E(Y_{ij}^\ell - Y_{i.}^\ell - Y_{.j}^\ell + Y_{..}^\ell) = 0 \ .$$

2. $\frac{q_\ell^2}{\sigma^2}$ is distributed as $\chi'^2_{p,\lambda}$, where $p = (M^{\ell+1} - 1)$, $\lambda = 0$,

since

$$E(Y_{i.}'^\ell - Y_{i.}^{\ell+1} - Y_{..}'^\ell + Y_{..}^{\ell+1}) = 0 \ .$$

3. $\frac{q_\ell^3}{\sigma^2}$ is distributed as $\chi'^2_{p,\lambda}$, where $p = (N_\ell - 1)$,

$$\lambda = \frac{M^{\ell}}{2\sigma^2} \sum_{j = N^{\ell-1}+1}^{N^{\ell}} (\tau_j^{\ell} - \tau_{.}^{\ell})^2 ,$$

since

$$E(Y_{.j}^{\ell} - Y_{..}^{\ell}) = \tau_j^{\ell} - \tau_{.}^{\ell} .$$

4. $\frac{q_\ell^4}{\sigma^2}$ is distributed as $\chi'^2_{p,\lambda}$, where $p = 1$,

$$\lambda = \frac{M^{\ell+1} N^{\ell} N_{\ell+1}}{2\sigma^2 N^{\ell+1}} (\tau_{.}'^{\ell} - \tau_{.}^{\ell+1})^2 ,$$

since

$$E(Y_{..}'^{\ell} - Y_{..}^{\ell+1}) = \tau_{.}'^{\ell} - \tau_{.}^{\ell+1} .$$

We will make use of the following theorems which are proved by Tang (1938).

Theorem 1: If A_1 and A_2 are independent and if A_1 is distributed as $\chi'^2_{a_1,\lambda_1}$ and A_2 is distributed as $\chi'^2_{a_2,\lambda_2}$ then $A_1 + A_2$ is distributed as $\chi'^2_{a_1+a_2,\lambda_1+\lambda_2}$.

Theorem 2: If A_1 and A_2 are independent and if A_1 is distributed as $\chi'^2_{a_1,\lambda_1}$ and if A_2 is distributed as $\chi'^2_{a_2,\lambda_2}$ and if $\lambda_2 = 0$, then

$$\frac{A_1}{A_2} \cdot \frac{a_2}{a_1}$$

is distributed as F'_{a_1,a_2,λ_1}. If $\lambda_1 = 0$, this reduces to Snedecor's F.

Now, by Theorem 1, we have

$$\frac{1}{\sigma^2}\left[\sum_{\ell=1}^{k} q_\ell^1 + \sum_{\ell=1}^{k-1} q_\ell^2\right]$$

is distributed as $\chi'^2_{p,\lambda}$, where

$$p = \sum_{\ell=1}^{k} (M^{\ell}-1)(N_{\ell}-1) + \sum_{\ell=1}^{k-1} (M^{\ell+1}-1)$$

$$= (M^{1}-1)(N_{1}-1) + \sum_{\ell=2}^{k} (M^{\ell}-1)(N_{\ell})$$

$$= (M^{1}-1)(N_{1}-1) + \sum_{\ell=2}^{k} \left[(M^{1}-1)(N_{\ell}) - (M^{1}-M^{\ell})(N_{\ell})\right]$$

$$= (M^{1}-1)(N-1) - \sum_{\ell=2}^{k} (M^{1}-M^{\ell})(N_{\ell}) \quad ,$$

and $\lambda = 0$.

Also, we have

$$\frac{1}{\sigma^2}\left[\sum_{\ell=1}^{k} q_{\ell}^{3} + \sum_{\ell=1}^{k-1} q_{\ell}^{4}\right]$$

is distributed as $\chi'^2_{p,\lambda}$, where

$$p = \sum_{\ell=1}^{k} (N_{\ell}-1) + (k-1) = N-1 \ ,$$

$$\lambda = \sum_{\ell=1}^{k}\left[\frac{M^{\ell}}{2\sigma^2}\sum_{j=N^{\ell-1}+1}^{N^{\ell}} (\tau_j^{\ell} - \tau_{.}^{\ell})^2\right] + \sum_{\ell=1}^{k-1} \frac{M^{\ell+1} N^{\ell} N_{\ell+1}}{2\sigma^2 N^{\ell+1}} (\tau_{.}^{\prime\ell} - \tau_{.}^{\ell+1})^2$$

Thence, by Theorem 2, we have V as defined in (2.1) above is distributed as $F'_{p,q,\lambda}$, where p, q, and λ are as defined in (2.2) above.

Now it is clear that $\lambda = 0$ if and only if $\tau_1 = \tau_2 = \ldots = \tau_N$ since λ is a sum of non-negative terms and can be zero if and only if each term of the sum is zero. Therefore to test the hypothesis $\tau_1 = \tau_2 = \ldots = \tau_N$ we use V as Snedecor's F with p degrees of freedom and q degrees of freedom, where p and q are as defined in (2.2).

THE ANALYSIS OF VARIANCE

In the last section we showed that the test function V could be used to test the hypothesis $\tau_1 = \tau_2 = \dots = \tau_N$. We will now show that V can be derived by the method of least squares. The model can be considered as a two-way classification model with unequal numbers in the subclasses. In this case the quantities in Table 3.1 are calculated.

TABLE 3.1

Due to	df	Sum of Squares
Blocks ignoring treatments	$M^1 - 1$	$\sum_i N_{i.}(Y_{i..})^2 - N_{..}(Y_{...})^2$
Treatments eliminating blocks	$N - 1$	$\sum_j Q_j \hat{\tau}_j$
Error	$N_{..} - M^1 - N + 1$	By subtraction
Total	$N_{..} - 1$	$\sum_{ijk}(Y_{ijk})^2 - N_{..}(Y_{...})^2$

If we now denote the mean square for treatments eliminating blocks by T and the mean square for error by E, then $W = \frac{T}{E}$ is the test function used to test the hypothesis $\tau_1 = \tau_2 = \dots = \tau_N$. We will now show that V is equal to W.

In the above table, the Total S. S. - the Block ignoring treatments S. S. is equal to

$$\sum_{\ell=1}^{k} q_\ell^6 - \sum_{\ell=1}^{k} q_\ell^5 .$$

It remains only to show that

$$\sum_{j=1}^{N} Q_j \tilde{\tau}_j = \sum_{\ell=1}^{k} q_\ell^3 + \sum_{\ell=1}^{k-1} q_\ell^4$$

and the rest follows by subtraction.

We have the following system of normal equations:

(3.1.1) $\quad Y_{i.} = \tilde{\mu} + \tilde{\rho}_i + \tilde{\tau}_{.}^{\prime 1} \quad , i = M^2 + 1, M^2 + 2, \ldots, M^1$

(3.1.2) $\quad Y_{i.} = \tilde{\mu} + \tilde{\rho}_i + \tilde{\tau}_{.}^{\prime 2} \quad , i = M^3 + 1, M^3 + 2, \ldots, M^2$

.

.

(3.1.k-1) $\quad Y_{i.} = \tilde{\mu} + \tilde{\rho}_i + \tilde{\tau}_{.}^{\prime k-1} \quad , i = M^k + 1, M^k + 2, \ldots, M^{k-1}$

(3.1.k) $\quad Y_{i.} = \tilde{\mu} + \tilde{\rho}_i + \tilde{\tau}_{.} \quad , i = 1, 2, \ldots, M^k$

(3.2.1) $\quad Y_{.j}^1 = \tilde{\mu} + \tilde{\rho}_{.}^1 + \tilde{\tau}_j^1 \quad , j = 1, 2, \ldots, N^1$

(3.2.2) $\quad Y_{.j}^2 = \tilde{\mu} + \tilde{\rho}_{.}^2 + \tilde{\tau}_j^2 \quad , j = N^1 + 1, N^1 + 2, \ldots, N^2$

.

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(3.2.k-1) $\quad Y_{.j}^{k-1} = \tilde{\mu} + \tilde{\rho}_{.}^{k-1} + \tilde{\tau}_j^{k-1} \quad , j = N^{k-2} + 1, N^{k-2} + 2, \ldots, N^{k-1}$

(3.2.k) $\quad Y_{.j}^k = \tilde{\mu} + \tilde{\rho}_{.}^k + \tilde{\tau}_j^k \quad , j = N^{k-1} + 1, N^{k-1} + 2, \ldots, N^k$

where

$$\hat{\rho}^{\ell}_{.} = \frac{\sum_{i=1}^{M^{\ell}} \hat{\rho}_i}{M^{\ell}} .$$

Imposing the linear restriction $\hat{\tau}_{.} = 0$, we find from (3.1.k) that

$$\hat{\mu} + \hat{\rho}_i = Y_{i.} \quad , \qquad i = 1, 2, \ldots, M^k .$$

Substituting this into (3.2.k) we have

$$\hat{\tau}_j^{k} = Y^{k}_{.j} - \frac{N^{k-1} Y'^{k-1}_{..} + N_k Y^{k}_{..}}{N} \quad , \; j = N^{k-1} + 1, N^{k-1} + 2, \ldots, N^k$$

Now since

$$\sum_{j=1}^{N^{k-1}} \hat{\tau}_j = - \sum_{j=N^{k-1}+1}^{N} \hat{\tau}_j$$

under the restriction, $\hat{\tau}_{.} = 0$, we may now substitute back and solve (3.1.k-1) obtaining

$$\hat{\mu} + \hat{\rho}_i = Y_{i.} + \frac{N_k}{N^{k-1}} \cdot \frac{N^{k-1}}{N} (Y^{k}_{..} - Y'^{k-1}_{..})$$

$$= Y_{i.} + \frac{N_k}{N} (Y^{k}_{..} - Y'^{k-1}_{..}) \; , \quad i = M^k + 1, M^k + 2, \ldots, M^{k-1}$$

Substituting back into (3.2.k-1) we get

$$\tilde{\tau}_j^{k-1} = Y^{k-1}_{.j} - \frac{M^k}{M^{k-1}} \left[\frac{N^{k-1} Y'^{k-1}_{..} + N_k Y^{k}_{..}}{N} \right] - \frac{\sum_{i=M^k+1}^{M^{k-1}} Y_{i.}}{M^{k-1}}$$

$$- \frac{M^{k-1} - M^k}{M^{k-1}} \cdot \frac{N_k}{N} (Y^{k}_{..} - Y'^{k-1}_{..})$$

$$= Y^{k-1}_{.j} - \frac{M^k}{M^{k-1}} Y'^{k-1}_{..} - \frac{\sum_{i=M^k+1}^{M^{k-1}} Y_{i.}}{M^{k-1}} - \frac{N_k}{N}\left(Y^k_{..} - Y'^{k-1}_{..}\right)$$

$$= Y^{k-1}_{.j} - \frac{N^{k-2} Y'^{k-2}_{..} + N_{k-1} Y^{k-1}_{..}}{N^{k-1}} - \frac{N_k}{N}\left(Y^k_{..} - Y'^{k-1}_{..}\right),$$

$$j = N^{k-2} + 1,\ N^{k-2} + 2,\ \ldots,\ N^{k-1}.$$

Finishing the solution in this manner, we obtain

$$\hat{\tau}^p_j = Y^p_{.j} - \frac{N^{p-1} Y'^{p-1}_{..} + N_p Y^p_{..}}{N^p} - \sum_{\ell = p+1}^{k} \frac{N_\ell}{N^\ell}\left(Y^\ell_{..} - Y'^{\ell-1}_{..}\right),$$

$$j = N^{p-1} + 1,\ N^{p-1} + 2,\ \ldots,\ N^p,\quad p = 1,\ 2,\ \ldots,\ k$$

which may be written as

$$\hat{\tau}^p_j = Y^p_{.j} - Y^p_{..} - \frac{N^{p-1}}{N^p}\left(Y'^{p-1}_{..} - Y^p_{..}\right) - \sum_{\ell = p+1}^{k} \frac{N_\ell}{N^\ell}\left(Y^\ell_{..} - Y'^{\ell-1}_{..}\right),$$

$$j = N^{p-1} + 1,\ N^{p-1} + 2,\ \ldots,\ N^p,\quad p = 1,\ 2,\ \ldots,\ k.$$

Now

$$Q^p_j = M^p Y^p_{.j} - \sum_{i=1}^{M^p} Y_{i.},$$

$$j = N^{p-1} + 1,\ N^{p-1} + 2,\ \ldots,\ N^p,\quad p = 1,\ 2,\ \ldots,\ k.$$

But

$$\frac{\sum_{i=1}^{M^p} Y_{i.}}{M^p} = \frac{M^k}{M^p} \cdot \frac{N^{k-1} Y'^{k-1}_{..} + N_k Y^k_{..}}{N^k} + \frac{\sum_{i=M^k+1}^{M^p} Y_{i.}}{M^p}$$

$$= \frac{M^k N_k}{M^p N^k}\left(Y^{k}_{..} - Y'^{k-1}_{..}\right) + \frac{M^k}{M^p} Y'^{k-1}_{..} + \frac{\sum_{i=M^k+1}^{M^{k-1}} Y_{i.}}{M^p} + \frac{\sum_{i=M^{k-1}+1}^{M^p} Y_{i.}}{M^p}$$

$$= \frac{M^k N_k}{M^p N^k}\left(Y^{k}_{..} - Y'^{k-1}_{..}\right) + \frac{M^{k-1}}{M^p} \cdot \frac{N^{k-2} Y'^{k-2}_{..} + N_{k-1} Y^{k-1}_{..}}{N^{k-1}} + \frac{\sum_{i=M^{k-1}+1}^{M^p} Y_{i.}}{M^p}$$

$$= \frac{M^k N_k}{M^p N^k}\left(Y^{k}_{..} - Y'^{k-1}_{..}\right) + \frac{M^{k-1} N_{k-1}}{M^p N^{k-1}}\left(Y^{k-1}_{..} - Y'^{k-2}_{..}\right) + \frac{M^{k-1}}{M^p} Y'^{k-2}_{..} + \frac{\sum_{i=M^{k-1}+1}^{M^p} Y_{i.}}{M^p}$$

Continuing in this fashion, we have

$$\frac{\sum_{i=1}^{M^p} Y_{i.}}{M^p} = \sum_{\ell=p+1}^{k} \frac{M^\ell N_\ell}{M^p N^\ell}\left(Y^{\ell}_{..} - Y'^{\ell-1}_{..}\right) + \frac{M^p N_p}{M^p N^p}\left(Y^{p}_{..} - Y'^{p-1}_{..}\right) + \frac{M^p}{M^p} Y'^{p-1}_{..}$$

$$= \sum_{\ell=p+1}^{k} \frac{M^\ell N_\ell}{M^p N^\ell}\left(Y^{\ell}_{..} - Y'^{\ell-1}_{..}\right) + \frac{N^{p-1}}{N^p}\left(Y'^{p-1}_{..} - Y^{p}_{..}\right) + Y^{p}_{..}$$

$p = 1, 2, \ldots, k.$

Thence

$$Q_j^p = M^p \left[Y^p_{.j} - Y^p_{..} - \frac{N^{p-1}}{N^p} \left(Y'^{p-1}_{..} - Y^p_{..} \right) - \sum_{\ell = p+1}^{k} \frac{M^\ell N_\ell}{M^p N^\ell} \left(Y^\ell_{..} - Y'^{\ell-1}_{..} \right) \right], \quad p = 1, 2, \ldots, k .$$

And

$$\sum_{j=1}^{N} Q_j \hat{\tau}_j = \sum_{p=1}^{k} \sum_{j = N^{p-1}+1}^{N^p} Q_j^p \hat{\tau}_j^p = \sum_{p=1}^{k} \left[M^p \sum_{j = N^{p-1}+1}^{N^p} \left(Y^p_{.j} - Y^p_{..} \right)^2 \right]$$

$$+ \sum_{p=1}^{k} M^p N_p \left[\frac{N^{p-1}}{N^p} \left(Y'^{p-1}_{..} - Y^p_{..} \right) + \sum_{\ell = p+1}^{k} \frac{M^\ell N_\ell}{M^p N^\ell} \left(Y^\ell_{..} - Y'^{\ell-1}_{..} \right) \right] \cdot \left[\frac{N^{p-1}}{N^p} \left(Y'^{p-1}_{..} - Y^p_{..} \right) + \sum_{\ell = p+1}^{k} \frac{N_\ell}{N^\ell} \left(Y^\ell_{..} - Y'^{\ell-1}_{..} \right) \right] .$$

Collecting coefficients of $\left(Y'^r_{..} - Y^{r+1}_{..} \right)^2$, we have

$$\frac{M^{r+1} N_{r+1} (N^r)^2}{(N^{r+1})^2} + \frac{M^r N_r M^{r+1} (N_{r+1})^2}{M^r (N^{r+1})^2} + \ldots + \frac{M^1 N_1 M^{r+1} (N_{r+1})^2}{M^1 (N^{r+1})^2} .$$

Combining all but the first term gives

$$\frac{M^{r+1} N_{r+1} (N^r)^2}{(N^{r+1})^2} + \frac{N^r M^{r+1} (N_{r+1})^2}{(N^{r+1})^2} = \frac{M^{r+1} N_{r+1} N^r (N^r + N_{r+1})}{(N^{r+1})^2}$$

$$= \frac{M^{r+1} N_{r+1} N^r}{N^{r+1}}$$

Collecting coefficients of $(Y'^{r}_{..} - Y^{r+1}_{..})(Y'^{s}_{..} - Y^{s+1}_{..})$, $r < s$, we have

$$- \frac{M^{r+1} N_{r+1} N^r N_{s+1}}{N^{r+1} N^{s+1}} + \frac{M^r N_r M^{r+1} N_{r+1} N_{s+1}}{M^r N^{r+1} N^{s+1}} + \dots + \frac{M^1 N_1 M^{r+1} N_{r+1} N_{s+1}}{M^1 N^{r+1} N^{s+1}}$$

$$- \frac{M^{r+1} N_{r+1} N^r M^{s+1} N_{s+1}}{N^{r+1} M^{r+1} N^{s+1}} + \frac{M^r N_r N_{r+1} M^{s+1} N_{s+1}}{N^{r+1} M^r N^{s+1}} + \dots$$

$$+ \frac{M^1 N_1 N_{r+1} M^{s+1} N_{s+1}}{N^{r+1} M^1 N^{s+1}} .$$

Combining all but the first term of each part gives

$$- \frac{M^{r+1} N_{r+1} N^r N_{s+1}}{N^{r+1} N^{s+1}} + \frac{N^r M^{r+1} N_{r+1} N_{s+1}}{N^{r+1} N^{s+1}} - \frac{N_{r+1} N^r M^{s+1} N_{s+1}}{N^{r+1} N^{s+1}}$$

$$+ \frac{N^r N_{r+1} M^{s+1} N_{s+1}}{N^{r+1} N^{s+1}} = 0.$$

Now since these two general terms are the only possible ones involved in the second summation of the expression for

$$\sum_{j=1}^{N} Q_j \widehat{\tau}_j ,$$

we have

$$\sum_{j=1}^{N} Q_j \widehat{\tau}_j = \sum_{p=1}^{k} \left[M^p \sum_{j=N^{p-1}+1}^{N^p} (Y^p_{.j} - Y^p_{..})^2 \right] + \frac{M^1 N_1 N^0}{N^1} (Y'^{0}_{..} - Y^1_{..})^2$$

$$+ \sum_{p=1}^{k-1} \frac{M^{p+1} N_{p+1} N^{p}}{N^{p+1}} (Y'^{p}_{..} - Y^{p+1}_{..})^2$$

$$= \sum_{\ell=1}^{k} q_{\ell}^{3} + \sum_{\ell=1}^{k-1} q_{\ell}^{4} ,$$

since $N^0 = 0$.

Now by subtraction the Error S. S. must be

$$\sum_{\ell=1}^{k} q_{\ell}^{1} + \sum_{\ell=1}^{k-1} q_{\ell}^{2} .$$

Also since the degrees of freedom for error and treatments eliminating blocks in Table 3.1 are the same as q and p of (2.2), then we have W = V. Thus we have shown that the test function given in the last section can be derived by the method of least squares.

MEANS AND STANDARD ERRORS

We will now derive the best, linear, unbiased estimates of $\tau_s - \tau_t$ and the standard errors of these estimates.

Theorem:

$$\hat{\tau}^p_s = Y^p_{.s} - Y^p_{..} - \frac{N^{p-1}}{N^p}(Y'^{p-1}_{..} - Y^p_{..}) - \sum_{\ell = p+1}^{k} \frac{N_\ell}{N^\ell}(Y^\ell_{..} - Y'^{\ell-1}_{..}) ,$$

$$s = N^{p-1} + 1,\ N^{p-1} + 2,\ \ldots ,\ N^p ,\ p = 1,\ 2,\ \ldots ,\ k$$

is the best, linear, unbiased estimate of $\tau_s - \tau_.$, and therefore $\hat{\tau}_s - \hat{\tau}_t$ is the best, linear, unbiased estimate of $\tau_s - \tau_t$.

Proof: Since $\hat{\tau}_s$ was found by the method of least squares using the linear restriction $\tau_. = 0$, this is an obvious result of the extended Markoff Theorem as proved by David and Neyman (1938).

Theorem: The variance of the estimate of $\tau^p_s - \tau^p_t$ is $\frac{2\sigma^2}{M^p}$ if $s \neq t$, and $s, t = N^{p-1} + 1,\ N^{p-1} + 2,\ \ldots ,\ N^p$ for $p = 1,\ 2,\ \ldots ,\ k$. The variance of the estimate of $\tau^p_s - \tau^r_t$ is

$$\sigma^2 \left[\frac{N^p - 1}{M^p N^p} + \frac{N^{r-1} + 1}{M^r N^{r-1}} + \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{M^\ell N^\ell N^{\ell-1}} \right] ,$$

for $s = N^{p-1} + 1,\ N^{p-1} + 2,\ \ldots ,\ N^p$, $t = N^{r-1} + 1,\ N^{r-1} + 2,\ \ldots ,\ N^r$, $p = 1,\ 2,\ \ldots ,\ k-1$, $r = p+1,\ p+2,\ \ldots ,\ k$.

Proof:

$$\hat{\tau}^p_s - \hat{\tau}^p_t = Y^p_{.s} - Y^p_{.t} ,\quad s \neq t \text{ and } s, t = N^{p-1} + 1,\ N^{p-1} + 2,\ \ldots ,\ N^p,$$

$$p = 1,\ 2,\ \ldots ,\ k.$$

And

$$\mathrm{Var}\,(\hat{\tau}_s^p - \hat{\tau}_t^p) = E\left[\frac{\sum_{i=1}^{M^p} e_{is}}{M^p} - \frac{\sum_{i=1}^{M^p} e_{it}}{M^p}\right]^2$$

$$= \frac{\sigma^2}{M^p} + \frac{\sigma^2}{M^p} = \frac{2\sigma^2}{M^p}\,, \qquad p = 1, 2, \ldots, k.$$

Now

$$\hat{\tau}_s^p - \hat{\tau}_t^r = Y_{.s}^p - Y_{..}^p - \frac{N^{p-1}}{N^p}(Y'^{p-1}_{..} - Y_{..}^p) - \sum_{\ell=p+1}^{k} \frac{N_\ell}{N^\ell}(Y_{..}^\ell - Y'^{\ell-1}_{..})$$

$$- Y_{.t}^r + Y_{..}^r + \frac{N^{r-1}}{N^r}(Y'^{r-1}_{..} - Y_{..}^r) + \sum_{\ell=r+1}^{k} \frac{N_\ell}{N^\ell}(Y_{..}^\ell - Y'^{\ell-1}_{..})$$

$$= Y_{.s}^p - \frac{N^{p-1}Y'^{p-1}_{..} + N_p Y_{..}^p}{N^p} - \sum_{\ell=p+1}^{r} \frac{N_\ell}{N^\ell}(Y_{..}^\ell - Y'^{\ell-1}_{..})$$

$$- Y_{.t}^r + \frac{N^{r-1}Y'^{r-1}_{..} + N_r Y_{..}^r}{N^r}$$

$$= Y_{.s}^p - \frac{N^{p-1}Y'^{p-1}_{..} + N_p Y_{..}^p}{N^p} - \sum_{\ell=p+1}^{r-1} \frac{N_\ell}{N^\ell}(Y_{..}^\ell - Y'^{\ell-1}_{..})$$

$$- Y_{.t}^r + \frac{N^{r-1}Y'^{r-1}_{..} + N_r Y_{..}^r}{N^r} - \frac{N_r}{N^r}(Y_{..}^r - Y'^{r-1}_{..})$$

$$= Y_{.s}^p - \frac{N^{p-1}Y'^{p-1}_{..} + N_p Y_{..}^p}{N^p} - \sum_{\ell=p+1}^{r-1} \frac{N_\ell}{N^\ell}(Y_{..}^\ell - Y'^{\ell-1}_{..})$$

$$- Y^{r}_{.t} + Y'^{r-1}_{..} \; .$$

And

$$\mathrm{Var}(\tilde{\tau}^{p}_{s} - \tilde{\tau}^{r}_{t}) = E\left\{ \frac{\sum_{i=1}^{M^p} e_{is}}{M^p} - \frac{\sum_{i=1}^{M^p}\sum_{j=1}^{N^p} e_{ij}}{M^p N^p} - \sum_{\ell=p+1}^{r-1} \frac{N_\ell}{N^\ell}\left[\frac{\sum_{i=1}^{M^\ell}\sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij}}{M^\ell N_\ell} \right.\right.$$

$$\left.\left. - \frac{\sum_{i=1}^{M^\ell}\sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^\ell N^{\ell-1}} \right] - \frac{\sum_{i=1}^{M^r} e_{it}}{M^r} + \frac{\sum_{i=1}^{M^r}\sum_{j=1}^{N^{r-1}} e_{ij}}{M^r N^{r-1}} \right\}^2$$

$$= E\left[\frac{\sum_{i=1}^{M^p} e_{is}}{M^p} \right]^2 + E\left[\frac{\sum_{i=1}^{M^p}\sum_{j=1}^{N^p} e_{ij}}{M^p N^p} \right]^2$$

$$+ \sum_{\ell=p+1}^{r-1} \frac{(N_\ell)^2}{(N^\ell)^2} E\left[\frac{\sum_{i=1}^{M^\ell}\sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij}}{M^\ell N_\ell} \right]^2$$

$$+ \sum_{\ell=p+1}^{r-1} \frac{(N_\ell)^2}{(N^\ell)^2} E\left[\frac{\sum_{i=1}^{M^\ell}\sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^\ell N^{\ell-1}} \right]^2 + E\left[\frac{\sum_{i=1}^{M^r} e_{it}}{M^r} \right]^2$$

$$+ E\left[\frac{\sum_{i=1}^{M^r}\sum_{j=1}^{N^{r-1}} e_{ij}}{M^r N^{r-1}} \right]^2 - \frac{2}{N^p} E\left[\frac{\sum_{i=1}^{M^p} e_{is}}{M^p} \right]^2$$

$$- 2E\left[\frac{\sum_{i=1}^{M^p} e_{is} \sum_{i=1}^{M^p}\sum_{j \neq s = 1}^{N^p} e_{ij}}{(M^p)^2 N^p} \right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=1}^{M^p} e_{is} \sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij}}{M^p M^\ell N_\ell} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{M^\ell N_\ell}{M^p N^\ell N^{\ell-1}} E \left[\frac{\sum_{i=1}^{M^\ell} e_{is}}{M^\ell} \right]^2$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=1}^{M^\ell} e_{is} \sum_{i=1}^{M^\ell} \sum_{j \neq s = 1}^{N^{\ell-1}} e_{ij}}{M^p M^\ell N^{\ell-1}} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=M^\ell+1}^{M^p} e_{is} \sum_{i=1}^{M^\ell} \sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^p M^\ell N^{\ell-1}} \right] - 2 E \left[\frac{\sum_{i=1}^{M^p} e_{is} \sum_{i=1}^{M^r} e_{it}}{M^p M^r} \right]$$

$$+ 2 \frac{M^r}{M^p N^{r-1}} E \left[\frac{\sum_{i=1}^{M^r} e_{is}}{M^r} \right]^2 + 2 E \left[\frac{\sum_{i=1}^{M^r} e_{is} \sum_{i=1}^{M^r} \sum_{j \neq s = 1}^{N^{r-1}} e_{ij}}{M^p M^r N^{r-1}} \right]$$

$$+ 2 E \left[\frac{\sum_{i=M^r+1}^{M^p} e_{is} \sum_{i=1}^{M^r} \sum_{j=1}^{N^{r-1}} e_{ij}}{M^p M^r N^{r-1}} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=1}^{M^p} \sum_{j=1}^{N^p} e_{ij} \sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij}}{M^p N^p M^\ell N_\ell} \right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{M^\ell N_\ell N^p}{M^p N^\ell N^{\ell-1}} E \left[\frac{\sum_{i=1}^{M^\ell} \sum_{j=1}^{N^p} e_{ij}}{M^\ell N^p} \right]^2$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=M^\ell+1}^{M^p} \sum_{j=1}^{N^p} e_{ij} \sum_{i=1}^{M^\ell} \sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^p N^p M^\ell N^{\ell-1}} \right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=1}^{M^\ell} \sum_{j=1}^{N^p} e_{ij} \sum_{i=1}^{M^\ell} \sum_{j=N^p+1}^{N^{\ell-1}} e_{ij}}{M^p N^p M^\ell N^{\ell-1}} \right]$$

$$+ 2 E \left[\frac{\sum_{i=1}^{M^p} \sum_{j=1}^{N^p} e_{ij} \sum_{i=1}^{M^r} e_{it}}{M^p N^p M^r} \right] - 2 \frac{M^r N^p}{M^p N^{r-1}} E \left[\frac{\sum_{i=1}^{M^r} \sum_{j=1}^{N^p} e_{ij}}{M^r N^p} \right]^2$$

$$- 2 E \left[\frac{\sum_{i=M^r+1}^{M^p} \sum_{j=1}^{N^p} e_{ij} \sum_{i=1}^{M^r} \sum_{j=1}^{N^{r-1}} e_{ij}}{M^p N^p M^r N^{r-1}} \right]$$

$$- 2 E \left[\frac{\sum_{i=1}^{M^r} \sum_{j=1}^{N^p} e_{ij} \sum_{i=1}^{M^r} \sum_{j=N^p+1}^{N^{r-1}} e_{ij}}{M^p N^p M^r N^{r-1}} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m}{N^\ell N^m} E \left[\frac{\sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^m} \sum_{j=N^{m-1}+1}^{N^m} e_{ij}}{M^\ell N_\ell M^m N_m} \right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \sum_{m = p+1}^{\ell} \frac{N_\ell N_m}{N^\ell N^m} E\left[\frac{\sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^m} \sum_{j=1}^{N^{m-1}} e_{ij}}{M^\ell N_\ell M^m N^{m-1}}\right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m M^m N_\ell}{N^\ell N^m M^\ell N^{m-1}} E\left[\frac{\sum_{i=1}^{M^m} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij}}{M^m N_\ell}\right]^2$$

$$- 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m}{N^\ell N^m} E\left[\frac{\sum_{i=M^m+1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^m} \sum_{j=1}^{N^{m-1}} e_{ij}}{M^\ell N_\ell M^m N^{m-1}}\right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m}{N^\ell N^m} E\left[\frac{\sum_{i=1}^{M^m} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^m} \sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^\ell N_\ell M^m N^{m-1}}\right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m}{N^\ell N^m} E\left[\frac{\sum_{i=1}^{M^m} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^m} \sum_{j=N^\ell+1}^{N^{m-1}} e_{ij}}{M^\ell N_\ell M^m N^{m-1}}\right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E\left[\frac{\sum_{i=1}^{M^\ell} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^r} e_{it}}{M^\ell N_\ell M^r}\right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell M^r N_\ell}{N^\ell M^\ell N^{r-1}} E\left[\frac{\sum_{i=1}^{M^r} \sum_{j=N^{\ell-1}+1}^{N^\ell} e_{ij}}{M^r N_\ell}\right]^2$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i = M^r+1}^{M^\ell} \sum_{j = N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^r} \sum_{j=1}^{N^{r-1}} e_{ij}}{M^\ell N_\ell M^r N^{r-1}} \right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i = 1}^{M^r} \sum_{j = N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^r} \sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^\ell N_\ell M^r N^{r-1}} \right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i = 1}^{M^r} \sum_{j = N^{\ell-1}+1}^{N^\ell} e_{ij} \sum_{i=1}^{M^r} \sum_{j=N^\ell+1}^{N^{r-1}} e_{ij}}{M^\ell N_\ell M^r N^{r-1}} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m M^m N^{\ell-1}}{N^\ell N^m M^\ell N^{m-1}} E \left[\frac{\sum_{i=1}^{M^m} \sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^m N^{\ell-1}} \right]^2$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m}{N^\ell N^m} E \left[\frac{\sum_{i = M^m+1}^{M^\ell} \sum_{j=1}^{N^{\ell-1}} e_{ij} \sum_{i=1}^{M^m} \sum_{j=1}^{N^{m-1}} e_{ij}}{M^\ell N^{\ell-1} M^m N^{m-1}} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m}{N^\ell N^m} E \left[\frac{\sum_{i=1}^{M^m} \sum_{j=1}^{N^{\ell-1}} e_{ij} \sum_{i=1}^{M^m} \sum_{j=N^{\ell-1}+1}^{N^{m-1}} e_{ij}}{M^\ell N^{\ell-1} M^m N^{m-1}} \right]$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=1}^{M^\ell} \sum_{j=1}^{N^{\ell-1}} e_{ij} \sum_{i=1}^{M^r} e_{it}}{M^\ell N^{\ell-1} M^r} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell M^r N^{\ell-1}}{N^\ell M^\ell N^{r-1}} E \left[\frac{\sum_{i=1}^{M^r} \sum_{j=1}^{N^{\ell-1}} e_{ij}}{M^r N^{\ell-1}} \right]^2$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=M^r+1}^{M^\ell} \sum_{j=1}^{N^{\ell-1}} e_{ij} \sum_{i=1}^{M^r} \sum_{j=1}^{N^{r-1}} e_{ij}}{M^\ell N^{\ell-1} M^r N^{r-1}} \right]$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{N^\ell} E \left[\frac{\sum_{i=1}^{M^r} \sum_{j=1}^{N^{\ell-1}} e_{ij} \sum_{i=1}^{M^r} \sum_{j=N^{\ell-1}+1}^{N^{r-1}} e_{ij}}{M^\ell N^{\ell-1} M^r N^{r-1}} \right]$$

$$- 2 E \left[\frac{\sum_{i=1}^{M^r} e_{it} \sum_{i=1}^{M^r} \sum_{j=1}^{N^{r-1}} e_{ij}}{(M^r)^2 N^{r-1}} \right] .$$

Omitting the terms whose expected value is zero, we have

$$\mathrm{Var}\,(\hat{\tau}_s^p - \hat{\tau}_t^r) = \sigma^2 \left[\frac{1}{M^p} + \frac{1}{M^p N^p} + \sum_{\ell = p+1}^{r-1} \frac{(N_\ell)^2}{(N^\ell)^2 M^\ell N_\ell} \right.$$

$$+ \sum_{\ell = p+1}^{r-1} \frac{(N_\ell)^2}{(N^\ell)^2 M^\ell N^{\ell-1}} + \frac{1}{M^r} + \frac{1}{M^r N^{r-1}}$$

$$- \frac{2}{N^p M^p} + 2 \sum_{\ell = p+1}^{r-1} \frac{M^\ell N_\ell}{M^p N^\ell N^{\ell-1} M^\ell}$$

$$+ 2 \frac{M^r}{M^p N^{r-1} M^r} - 2 \sum_{\ell = p+1}^{r-1} \frac{M^\ell N_\ell N^p}{M^p N^\ell N^{\ell-1} M^\ell N^p}$$

$$- 2 \frac{M^r N^p}{M^p N^{r-1} M^r N^p}$$

$$- 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m M^m N_\ell}{N^\ell N^m M^\ell N^{m-1} M^m N_\ell}$$

$$- 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell M^r N_\ell}{N^\ell M^\ell N^{r-1} M^r N_\ell}$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \sum_{m = \ell+1}^{r-1} \frac{N_\ell N_m M^m N^{\ell-1}}{N^\ell N^m M^\ell N^{m-1} M^m N^{\ell-1}}$$

$$+ 2 \sum_{\ell = p+1}^{r-1} \frac{N_\ell M^r N^{\ell-1}}{N^\ell M^\ell N^{r-1} M^r N^{\ell-1}} \Bigg]$$

$$= \sigma^2 \Bigg[\frac{1}{M^p} - \frac{1}{M^p N^p} + \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{(N^\ell)^2 M^\ell}$$

$$+ \sum_{\ell = p+1}^{r-1} \frac{(N_\ell)^2}{(N^\ell)^2 M^\ell N^{\ell-1}} + \frac{1}{M^r} + \frac{1}{M^r N^{r-1}} \Bigg]$$

$$= \sigma^2 \left[\frac{N^p - 1}{M^p N^p} + \frac{N^{r-1} + 1}{M^r N^{r-1}} + \sum_{\ell = p+1}^{r-1} \frac{N_\ell (N^{\ell-1} + N_\ell)}{(N^\ell)^2 M^\ell N^{\ell-1}} \right]$$

$$= \sigma^2 \left[\frac{N^p - 1}{M^p N^p} + \frac{N^{r-1} + 1}{M^r N^{r-1}} + \sum_{\ell = p+1}^{r-1} \frac{N_\ell}{M^\ell N^\ell N^{\ell-1}} \right]$$

$p = 1, 2, \ldots, k-1$, $r = p+1, p+2, \ldots, k$.

We know from the Markoff Theorem that the error mean square is an estimate of σ^2. We may then use this estimate in the variances we have just derived for $\hat{\tau}_j - \hat{\tau}_{j'}$, and thus set confidence intervals on the estimates of treatment differences.

CONCLUSIONS

The randomized block designs that have been proposed to date have been for blocks containing equal numbers of experimental units. This has been due, primarily at least, to the ease of computation in this case. However, the method proposed in this paper lends itself nicely to computation, and therefore extends the scope of the randomized block designs to include the realm of blocks containing different numbers of experimental units.

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