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LOCAL ESCAPE RATE DICHOTOMY FROM A PROBABILITY POINT OF VIEW

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Abstract

We will discuss a dichotomy pertaining to escape rates in dynamical systems. This dichotomy pertains to the limiting behavior of the escape rate as it is compared to the size of a shrinking hole (the local escape rate). In this case, it has been shown, with some robustness, that under certain mixing conditions on the system this limiting behavior is determined by the periodicity of of the set to which the hole shrinks. We will use a blocking argument to obtain error estimates for truncation of the limit described above. These will allow for the result that the double limit describing the local escape rate to be taken along different paths. Finally, we will discuss a result that ties the escape rate conditioned on being in the hole, to the usual escape rate.

Contents

Chapter 1

Introduction

In what follows, we will discuss the escape rate for dynamical systems. One should consider a particular subset of the phase space to be designated as a hole. In each iterate of the map, we will track the mass remaining in the system (not having entered the hole). We will call the average exponential rate of decay in that mass the escape rate. The escape rate has been generally discussed in [6, 10, 11, 21, 20, 22, 14, 30]. An intuitive property of the escape rate is its monotonicity. That is, if one hole contains another, then that hole has a larger escape rate (or at least it does not have a smaller escape rate). One can show readily that if the probability measure one considers to determine mass is invariant or non-singular, then the escape rate into a measure-zero set is zero.

A number of works, see for example [26, 25, 38, 15, 9], have recently discussed the asymptotics of the decay to zero of the escape rate as the size of the hole is shrunk to zero. In [18, 19, 16, 17] Freitas, Freitas, et al have investigated the connection to extreme value theory. Many related works on the extreme value theory have been compiled in [31]. The result in both cases is that the asymptotics are intimately tied to the mixing properties of the dynamics, as well as the asymptotics of the short return probabilities, both of which have been investigated by Abadi in several works [1, 4, 2].

In Chapter 2, we will discuss the escape rate, some basic results and first examples. In Chapter 3, we will discuss error estimates pertaining to local escape rates for periodic and non-periodic sets in the presence of high extremal index. In Chapter 4, we will discuss similar results obtained without error estimates for arbitrary extremal index. In Chapter 5, we will discuss the conditional escape rate and its connection with the usual escape rate. Finally in Chapter 6, we will discuss the application to extreme value theory and some examples.

Chapter 2

Escape Rate

In this chapter, we will begin by giving the basic setup of the escape rate, as well as some relevant definitions and examples. We will note that the primary computational tool of these examples is the transfer operator. That, however, will not end up being the case in the majority of the discussion that follows in subsequent chapters.

2.1 Escape Rate and its Basic Properties

Let X be a compact metric space and $T : X \to X$. Let μ be a Borel probability measure on X. We will use $\mathscr{B}$ to denote the Borel $\sigma-$algebra. We will assume throughout that μ is non-singular with respect to T, meaning $\mu(T^{-1}A) = 0$ if $\mu(A) = 0$. Let $H \in \mathscr{B}$.

Definition 2.1. *The* ***hitting time*** *to H is defined by*

$$\tau_H(x) = \inf\{j \geq 1 | T^j x \in H\}, \tag{2.1}$$

Definition 2.2. *The **escape rate** into H is defined by*

$$\rho(\mu, H, T) = \lim_{t\to\infty} -\frac{1}{t} \log \mu\{x|\tau_H(x) > t\}, \tag{2.2}$$

provided that the limit exists.

The arguments of ρ may be suppressed in the event that the context is clear.

Let us give some elementary properties of the escape rate. The first such result tells us that ρ is an invariant of metric conjugacy.

Proposition 2.3. *[11] Let $\pi : X \to Y$ be a metric conjugacy between $S : X \to X$ and $T : Y \to Y$ where X and Y are compact metric spaces. Let μ a Borel probability measure on X and ν a Borel probability measure on Y (so that $\pi_*\mu = \nu$ and $\pi T = S\pi$). Let $A = \pi^{-1}B$ for A,B measurable in X, Y respectively. Then $\rho(\mu, A, S) = \rho(\nu, B, T)$ provided that either limit exists.*

Proof. Let $x = \pi^{-1}y$. Then,

$$\begin{aligned}\tau_B(y) &= \inf\{j \geq 1|T^j y \in B\} \\ &= \inf\{j \geq 1|\pi S^j \pi^{-1} y \in B\} \\ &= \inf\{j \geq 1|S^j x \in A\} \\ &= \tau_A(x).\end{aligned}$$

Similarly $\mu\{\tau_A > t\} = \mu\pi^{-1}\{\tau_B > t\} = \nu\{\tau_B > t\}$. Taking logs, dividing by t, and then taking the limit completes the proof. □

Our second proposition tells us that if the size of the hole is negligible, then the escape rate is negligible. It also tells us that intuitively, if we expand a hole, its escape rate cannot decrease.

Proposition 2.4. *If $\mu(H) = 0$, then $\rho(\mu, H, T) = 0$. Furthermore, the escape rate is monotone with the hole. More specifically, if $H \subset K$, then $\rho(K) \geq \rho(H)$.*

Proof. We have $1 \geq \mu(\tau_H > t) = 1 - \mu(\tau_H \leq t) \geq 1 - \sum_{j=0}^{t} \mu(T^{-j}H) = 1$, where in sequence, we have used that μ is a probability measure, the sub-additivity, and nonsingularity of μ. So $\mu(\tau_H < t) = 1$ and thus, $\rho(\mu, A, T) = 0$. To show the monotonicity, we note that if $H \subset K$, then

$$\mu(\tau_K > t) \leq \mu(\tau_H > t), \tag{2.3}$$

and $f(x) = \frac{-\log(x)}{x}$ is decreasing.

□

The following proposition is gives some insight into the sensitivity of the choice of measure.

Proposition 2.5. *[14] If $\nu \ll \mu$ and $C^{-1} < \frac{d\nu}{d\mu} < C$ for some $C > 0$, then $\rho(\mu) = \rho(\nu)$.*

Proof. Suppose that $\nu \ll \mu$. Then $\forall B \in \mathscr{B}$, we have $\nu(B) = \int_B \frac{d\nu}{d\mu} d\mu$. Thus we have

$$C^{-1}\mu(B) \leq \nu(B) \leq C\mu(B).$$

Thus,

$$-\log \mu(B) + \log(C) \geq -\log \nu(B) \geq -\log \mu(B) - \log(C).$$

By setting $B = \{\tau_H < t\}$, and dividing throughout by t and taking the limit as $t \to \infty$, the proof is complete by squeezing theorem. □

2.2 Computation for Subshifts of Finite Type

In this section, we compute the escape rate for a basic example: subshifts of finite type. A similar computation is carried out in several works (see e.g. [6]).

Given a matrix A with entries in $\{0, 1\}$, we say a sequence, $\{s_k\}$, is admissible if, $A_{s_k s_{k+1}} = 1$ for all k. We denote the collection of **admissible sequences** by Σ_A^+.[1] A **subshift of finite type** is the left shift operator restricted to Σ_A^+. We will denote the subshift of finite type with associated matrix A by $\sigma_A : \Sigma_A^+ \to \Sigma_A^+$ or just σ.

Let $P = (P_{ij})$ be a row-stochastic matrix whose nonzero entries correspond to those of A. Also, let π a stochastic row-vector. The pair, (P, π) induces a measure on Σ_A^+ in the following way. We will denote by

$$[s_0...s_{n-1}] = \Big\{\{t_k\}|\forall 0 \leq i \leq n-1, t_i = s_i\Big\}$$

the **n-cylinders** of the subshifts of finite type. Let $\mathcal{F}^n$ be the sigma algebra generated by the n-cylinders. **The Markov measure induced by** (P, π)[2] of an n-cylinder is given by

$$\mu[s_0...s_{n-1}] = \pi_{s_0} \prod_{j=0}^{n-2} P_{s_j s_{j+1}}. \tag{2.4}$$

Suppose now that we would like to compute the escape rate into a 1-cylinder.

[1] The + references the fact that the sequences are indexed over $\mathbb{N}$ rather than $\mathbb{Z}$

[2] The pair (P, π) can be thought of as analogous to the initial distribution and transition matrix if the reader is familiar with Markov chains

Let us, for instance, choose $H = [0]$. Then we have[3]

$$\rho(\mu, H, \sigma) = -\lim_{t\to\infty} \frac{\log(\mu(\tau_H > t))}{t} \tag{2.5}$$

$$\mu(\tau_H > t) = \sum_{s_0 \dots s_t} \mu[s_0 \dots s_t] \prod_{j=0}^{t} \chi_{\{s_j \neq 0\}} \tag{2.6}$$

$$= \sum_{s_0 \dots s_t} \pi_{s_0} \chi_{s_0 \neq 0} \prod_{j=0}^{t-1} P_{s_j s_{j+1}} \chi_{\{s_{j+1} \neq 0\}} \tag{2.7}$$

$$= \tilde{\pi} \tilde{P}^t 1, \tag{2.8}$$

where $\tilde{\pi} = \pi \chi_{s_0 \neq 0}$, and $\tilde{P}$ is the matrix P_{ij} with the column corresponding to the hole changed to zero, and 1 is a column vector of ones. Evidently,

$$\rho = -\lim_{t\to\infty} \frac{\log(\tilde{\pi}\tilde{P}^t 1)}{t} \tag{2.9}$$

$$= -\log \lambda, \tag{2.10}$$

where λ is the largest eigenvalue of $\tilde{P}$, and the computation follows from Gelfand's formula for the spectral radius (cf. [28]).

2.3 Expanding Linear Markov Maps

In this section we use the results of the previous two to compute the escape rate into a Markov partition element for a piecewise-linear expanding Markov map. let $I = [0, 1]$ and $I_0, \dots, I_{n-1}$ be intervals so that $\mathcal{A} = \{I_j | j = 0, \dots, n-1\}$ be a partition of I. Let T be a map, $T : I \to I$ so that T is linear on each I_j and $|T|'_{I_j}| > 1$. We will also require that the image of each partition element is a

[3] Here, we denote by χ_A the indicator function on A and use a shorthand for describing sets common to probability, that is $\{s_j \neq 0\}$ in place of $\{\{s_k\} \in \Sigma_A^+ | s_j \neq 0\}$.

union of some of the other partition elements. We denote by $\mathcal{F}$, the $\sigma-$algebra generated by the partition elements, and by $\mathcal{F}^n$, the $\sigma-$algebra generated by its refinement via T. i.e. $\mathcal{F}^n = \sigma(\vee_{j=0}^{n-1} T^{-j}\mathcal{A})$. Note that by definition, $\mathcal{F}^n$ is the collection of unions of $n-$cylinders. We will assume, also that $\mathcal{A}$ is generating (cylinders shrink to points).

It has been shown in [5, 34] that I contains a full Lebesgue measure set on which T is conjugate to a subshift of finite type with associated Markov matrix $P_{ij} = \frac{\mu(T^{-1}I_j \cap I_i)}{\mu(I_i)}$. In light of the conjugacy invariance of the escape rate, holes that are chosen to correspond to the $1-$cylinders have an associated escape rate that can be computed using the same method of the previous section.

2.4 Survey of Results

Much of the discussion regarding escape rate rests on the work of Keller and Liverani regarding perturbation of transfer operators, see [30, 27]. In [7] Ulam's method is used to numerically approximate the escape rate for Lasota-Yorke maps. In [11], the problem of where to place a hole to achieve maximal escape rate for the doubling map and Markov hole is discussed. In the same paper, the problem of an asymptotics of shrinking the hole to measure zero are discussed in the same context, see the introduction for more work on this subject. In [9], the order of limits for this localized escape rate is relaxed, again using perturbation of transfer operator. In the same paper, there is some discussion of the connection between escape rates and the extreme value theory. For more discussion of the extreme value theory see the works of J. M. Freitas, and A.C.M. Freitas listed in the introduction, as well as the works of F. Yang, in particular [38]. In [23], the escape rate for products of expanding Markov maps is discussed. In [15], the escape rate for special flows is discussed.

Chapter 3

The Local Escape Rate Dichotomy for Points

3.1 Introduction

In this chapter, we will discuss the local escape rate dichotomy for neighborhoods shrinking to a single point. In particular, we will demonstrate the qualitative difference between periodic and non-periodic points that results from the periodicity of the point. We will also give tools for computation of error terms for the local escape rate in terms of this dichotomy that are not accessible for the case of general null sets.

3.2 Setup

As has been discussed in the previous chapter, if the hole defining the escape rate is a null set with respect to the reference measure, then the escape rate is zero, and decreases monotonically for nested sets. As such, it is natural to consider the asymptotics of a series of nested holes whose measure decreases to zero, normalized against their measure. To that end, we define the local escape rate as in the previous chapter.

Definition 3.1. *We define the **local escape rate** as*

$$\rho(\Lambda, \mu, T) = \lim_{n\to\infty} \frac{\rho(U_n, \mu, T)}{\mu(U_n)} \tag{3.1}$$

where $\mu(\Lambda) = 0$, *and* $U_{n+1} \subset U_n$ *with* $\cap_{n=0}^{\infty} U_n = \Lambda$, *provided that the limit exists.*

Definition 3.2. *For a sequence of* U_n *shrinking to a point* x,

$$\vartheta(x, \{U_n\}) = \lim_{n\to\infty} \frac{\mu(T^{-p}U_n \cap U_n)}{\mu(U_n)} \tag{3.2}$$

provided that the limit exists. We will refer to the quantity ϑ *as the **extremal index** where* p *is the least period of* x, *a periodic point, and provided that the limit exists.*

For more on the extremal index, see appendix. Under some technical requirements, we will show that the local escape rate to a non-periodic null set is 1 and for a periodic point, it is the extremal index.

To achieve a robust result, it is insufficient to simply consider any sequence of measurable holes. To avoid pathologies (one can, for instance, imagine a sequence of holes shrinking to a point in a highly asymmetric way), we insist on some conditions that the nested sequence $\{U_n\}$ must satisfy.

Definition 3.3. *We define* U_n^j *to be the smallest union of* $j-$*cylinders so that* $U_n \subset T^{-(n-j)}U_n$ *in the case of left* ϕ*-mixing and* $U_n \subset U_n^j$ *in the case of right* ϕ*-mixing*

Definition 3.4. *Suppose $\mathcal{A}$ is a Markov partition. We say that U_n is an* ***adapted neighborhood system*** *if it satisfies the following.*

(1) $U_n \in \sigma(\mathcal{A}^n)$.

(2) $U_{n+1} \subset U_n$.

(3) $\cap_n U_n = \Lambda$.

(4) If Λ is nonperiodic, then there exists $\gamma', C > 0$ and $K \in (0,1]$ such that $\mu(U_n^j) \leq Cj^{-\gamma'}$ for all $j \leq Kn$. If Λ is periodic, then there exists $C, \gamma' > 0$ and $K \in (0,1]$ so that $\mu(U_{n,u}^j) \leq Cj^{-\gamma'}$ for $j \leq K(n+pu)$ where p is the minimal period.

(5) If Λ is periodic with minimal period p, then there exists $J(n) \in (0,1)$ so that $J(n)n \to \infty$, and $\mu(\cap_{j=0}^k T^{-i_j} U_n) = (1 + \mathcal{O}^*(r_n))\mu(U_{n,\frac{i_k}{p}})$.

3.3 Preliminaries

We will begin by outlining some useful estimates and computations surrounding the hitting time. Suppose that μ is an invariant measure under the action of T. Then consider the set

$$\{\tau_U \leq k\} \tag{3.3}$$

We have

$$\mu\{\tau_U \leq k\} = \mu(\cup_{j=1}^k T^{-j} U) \tag{3.4}$$

$$\leq \sum_{j=1}^k \mu(T^{-j} U) \tag{3.5}$$

$$= \sum_{j=1}^k \mu(U) \tag{3.6}$$

$$= k\mu(U), \tag{3.7}$$

so we obtain $\mu\{\tau_U \leq k\} \leq k\mu(U)$ giving a coarse upper estimate for the distribution of hitting times.

Consider also,

$$\{\tau_U \circ T^k \in A \subset \mathbb{R}\} \tag{3.8}$$

$$= \{\inf\{j \geq 1 | T^j(T^k(x))\} \in A\} \tag{3.9}$$

$$= T^{-k}\{\tau_U \in A\}. \tag{3.10}$$

Collecting these, we have shown the following propositions.

Proposition 3.5. *Let μ be a $T-$invariant measure. Then we have*

$$\mu(\tau_U \leq k) \leq k\mu(U) \tag{3.11}$$

Proposition 3.6. $\{\tau_U \circ T^k \in A\} = T^{-k}\{\tau_U \in A\}$

Throughout the text, we will frequently make use of these two results without explicit reference.

3.4 The blocking argument

In this section, we introduce a blocking argument for $\phi-$mixing systems with respect to a particular measurable partition. $\phi-$mixing systems are systems for which the iterates of the map are asymptotically independent (mixing), where the rate function, ϕ is given tied to sets in particular $\sigma-$algebras generated by the dynamics acting on the partition. The blocking argument gives a bound on the error penalty that arises from the difference between asymptotic independence and actual independence with respect to the measure. We define these concepts more precisely as follows.

Let T a measurable map on $(\Omega, \mathcal{F}, \mu)$, with, and $\mathcal{A}$ a generating partition of Ω. We will denote by $\mathcal{A}^n$, the n-th refinement of $\mathcal{A}$ through T, $\vee_{k=0}^{n-1} T^{-k}\mathcal{A}$. We will denote by $\sigma(\mathcal{A}^n)$ the $\sigma-$algebra generated by $\mathcal{A}^n$.

Definition 3.7. *We say that the measure μ is **left** $\phi-$**mixing** with respect to the map, T and the partition, $\mathcal{A}$, if*

$$\left|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)\right| \leq \phi(k)\mu(A) \tag{3.12}$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\cup_j \mathcal{A}^j)$.

*We say that the measure μ is **right** $\phi-$**mixing** with respect to the map, T if*

$$\left|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)\right| \leq \phi(k)\mu(B) \tag{3.13}$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\cup_j \mathcal{A}^j)$.

The following lemma allows us to leverage the mixing to break the tail probability of the hitting time into two blocks and then applying the mixing to an artificial gap (Δ below) between the blocks.

Lemma 3.8. *[4] Suppose that T, μ are left $\phi-$mixing with respect to the partition $\mathcal{A}$. Then for $U \in \mathcal{A}^n$, $s, t > 0$, and $\Delta < \frac{t}{2}$,*

$$|\mu(\tau_U > s+t) - \mu(\tau_U > s)\mu(\tau_U > t)| \leq 2\mu(\tau_U > s - \Delta)(\phi(\Delta - n) + \Delta\mu(U)).$$

Proof. We have for $\Delta < \frac{t}{2}$,

$$\begin{aligned}
&|\mu(\tau_U > s+t) - \mu(\tau_U > s)\mu(\tau_U > t)| \\
&\leq \left|\mu(\tau_U > s+t) - \mu(\tau_U > s, \tau_U \circ T^{s+\Delta} > t-\Delta)\right| \\
&+ \left|\mu(\tau_U > s, \tau_U \circ T^{s+\Delta} > t-\Delta) - \mu(\tau_U > t-\Delta)\mu(\tau_U > s)\right| \\
&+ |\mu(\tau_U > s)\mu(\tau_U > t-\Delta) - \mu(\tau_U > s)\mu(\tau_U > t)|
\end{aligned}$$

We also have

$$\begin{aligned}
&\left|\mu(\tau_U > s+t) - \mu(\tau_U > s, \tau_U \circ T^{s+\Delta} > t-\Delta)\right| \\
&\leq |\mu(\tau_U > s, \tau_U \circ T^{s} \leq \Delta)| \\
&\leq \mu(\tau_U > s-\Delta)(\Delta\mu(U) + \phi(\Delta - n)),
\end{aligned}$$

using the $\phi-$mixing assumption. Also,

$$\begin{aligned}
&\left|\mu(\tau_U > s, \tau_U \circ T^{s+\Delta} > t-\Delta) - \mu(\tau_U > t-\Delta)\mu(\tau_U > s)\right| \\
&\leq \mu(\tau_U > s-\Delta)\phi(\Delta - n).
\end{aligned}$$

Finally,

$$\begin{aligned}
&|\mu(\tau_U > s)\mu(\tau_U > t-\Delta) - \mu(\tau_U > s)\mu(\tau_U > t)| \\
&\leq \mu(\tau_U > s-\Delta)\Delta\mu(U)
\end{aligned}$$

Collecting these estimates completes the proof. □

In the following lemma, we apply this result iteratively, using an induction argument, to split the tail further into more blocks so that we can find a multiplicative error.

Lemma 3.9. *Let $s > 0$ and $\Delta < \frac{s}{2}$. Define $q = \lfloor \frac{s}{\Delta} \rfloor$ and $\eta = \frac{q}{q+1}$. Then*

$$(\mu(\tau_U > s) + \delta^\eta)^{k+a(q)} \leq \mu(\tau_U > ks) \leq (\mu(\tau_U > s) + \delta^\eta)^{k-2} \tag{3.14}$$

where $\delta = 2(\Delta\mu(U) + \phi(\Delta - n))$ and for some $a(q) > 0$ and $k \geq 2 - q^{-1}$ an integer multiple of q^{-1}.

Proof. Fix any choice of $s, \Delta < \frac{s}{2}$ and $k \in [2 - q^{-1}, 3]$. Next, choose $a(q)$ sufficiently large so that

$$(\mu(\tau_U > s) - \delta^\eta)^{2-q^{-1}+a(q)} \leq \mu(\tau_U > 3s).$$

Note that for $k \leq k'$ we have

$$\begin{aligned}(\mu(\tau_U > s) - \delta^\eta)^{k+a(q)} &\leq (\mu(\tau_U > s) - \delta^\eta)^{2-q^{-1}+a(q)} \\ &\leq \mu(\tau_U > 3s) \\ &\leq \mu(\tau_U > ks)\end{aligned}$$

$\mu(\tau_U > k's) \leq \mu(\tau_U > ks)$. So for $k \in [2 - q^{-1}, 3]$

$$\begin{aligned}\mu(\tau_U > ks) &\leq \mu(\tau_U > s)\mu(\tau_U > (k-1)s) + \delta\mu(\tau_U > (k-1-q^{-1})s) \\ &\leq \mu(\tau_U > s)^2 + \delta\mu(\tau_U > s) \\ &= \mu(\tau_U > s)(\mu(\tau_U > s) + \delta) \\ &\leq (\mu(\tau_U > s) + \delta^\eta)^{k-2}.\end{aligned}$$

Thus we have

$$\mu(\tau_U > ks) \leq (\mu(\tau_U > s) + \delta^\eta)^{k-2} \tag{3.15}$$

for $k \in [2 - q^{-1}, 3]$, because $\mu(\tau_U > s) \leq \mu(\tau_U > s) + \delta^\eta)$, and $\mu(\tau_U > s) \leq 1$.

We claim the above bound holds for $k > 3$. We will proceed by induction on k:

$$\begin{aligned}
\mu(\tau_U > ks) &\leq \mu(\tau_U > s)\mu(\tau_U > (k-1)s) + \delta\mu(\tau_U > (k - 1 - q^{-1})s) \\
&\leq \mu(\tau_U > s)(\mu(\tau_U > s) + \delta^\eta)^{k-3} + \delta(\mu(\tau_U > s) + \delta^\eta)^{k-3-q^{-1}} \\
&= (\mu(\tau_U > s) + \delta^\eta)^{k-3-q^{-1}}[\mu(\tau_U > s)(\mu(\tau_U > s) + \delta)^{q^{-1}} + \delta] \\
&\leq (\mu(\tau_U > s) + \delta^\eta)^{k-2}.
\end{aligned}$$

We justify the last inequality as follows. By definition of η, we have $\delta = \delta^\eta \delta^{\frac{\eta}{q}} \leq \delta^\eta(\mu(\tau_U > s) + \delta^\eta)^{q^{-1}}$. Consider the bracketed term in the third line.

$$\begin{aligned}
&\mu(\tau_U > s)(\mu(\tau_U > s) + \delta)^{q^{-1}} + \delta \\
&\qquad \leq \mu(\tau_U > s)(\mu(\tau_U > s) + \delta)^{q^{-1}} + \delta^\eta(\mu(\tau_U > s) + \delta^\eta)^{q^{-1}} \\
&\qquad = \mu(\tau_U > s) + \delta^\eta)^{1+q^{-1}}.
\end{aligned}$$

By induction this completes the proof. □

3.5 Main Results

We will now state and prove the main results of this chapter, outlined as follows. First, we give results that determines the limiting behavior of an adapted neighborhood system shrinking for both non-periodic and periodic points. Next we discuss some immediate corollaries. Finally we will state and prove the main lemmas before returning to the proofs of the main theorems.

Theorem 3.10. *Let μ left $\phi-$mixing measure with respect to a generating partition $\mathcal{A}$. Suppose that $U \in \sigma(\mathcal{A}^n)$ for some n. Then for $t \sim ks$, $s = q\Delta$, and $\eta = \frac{q}{q+1}$*

$$\left|\frac{\log(\mu(\tau_U > t))}{t\mu(U)} - 1\right| \lesssim \inf_{s,\Delta<\frac{s}{2}} \frac{1}{k} + \frac{C\pi(U)^{-\theta+1}}{1 + C\pi(U)^{-\theta+1}} + s\mu(U) + \frac{\delta^\eta}{s\mu(U)}.$$

Theorem 3.11. *Let $U \in \sigma(\mathcal{A}^n)$ containing a periodic point with period p. Assume*

$$\left|Q_{\ell+1} - \sum_{u=\ell}^{\frac{Jn}{p}} \binom{u-1}{\ell-1}\mu(U_u)\right| \leq E_1(n, J, \ell), \tag{3.16}$$

$$|\mu(U_u) - \mu(U)\vartheta^u| \leq E_2(u). \tag{3.17}$$

Then we have for $t \sim ks$ and $J \in (0, 1)$,

$$\left|\frac{\log(\mu(\tau_U > t))}{t\mu(U_n)} - (1-\vartheta)\right|$$
$$\lesssim \inf_{s,\Delta<\frac{s}{2}} \frac{1}{s\mu(U)} \sum_{\ell=0}^{\frac{Jn}{p}} (-1)^\ell E(\ell) + \left(\frac{\vartheta}{\vartheta-1}\right)^s + R + \frac{\delta^\eta}{s\mu(U)},$$

where $R \lesssim \phi^2(\frac{Jn}{4}) + s^2\mu(U^{\frac{Jn}{4}})$, and

$$E(\ell) := sE_1 + \frac{Jn}{p}E_2 + s\mu(U)\left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{Jn}{p}+1}.$$

The local escape rate is defined using two limits. First, we take the limit as the cutoff for the hitting time (t) approaches infinity. Then we take the limit as the size of the hole ($\mu(U)$) shrinks to zero. The following allows us to relax this and considering different paths that $t, \mu(U)$ could take to $\infty, 0$ respectively.

Definition 3.12. *Let* $\rho_\alpha(U_n) = \lim_{n\to\infty} -\frac{1}{t\mu(U_n)} \log \mu(\tau_{U_n} > t)$, *where* $\lambda > 0$. *and* $t = \lambda \mu(U_n)^{-\alpha}$.

Using the error estimates proved in this section, We will show that this adjusted version of the escape rate follows the same dichotomy as the original formulation (definition 3.1).

Corollary 3.13. *Let* U_n *be an adapted neighborhood system for which* $\vartheta < \frac{1}{2}$. *Suppose that*

$$\sum_{\ell=0}^{\frac{Jn}{p}} \frac{E_1(n,\ell,J)}{\mu(U_n)} \to 0$$
$$\sum_{\ell=0}^{\frac{Jn}{p}} \frac{E_2(n,\ell,J)}{\mu(U_n)} \to 0.$$

If $\xi_1^n \lesssim \mu(U_n) \lesssim \xi_2^n$ *for* $0 < \xi_1 < \xi_2 < 1$ *or* $n^{-\gamma_1} \lesssim \mu(U_n) \lesssim n^{-\gamma_2}$ *and* $\frac{\alpha}{\gamma_1} < 1$ *then in either case*

$$\rho_\alpha(U_n) \to \begin{cases} 1 \text{ under the assumptions of 3.10} \\ 1 - \vartheta \text{ under the assumptions of 3.11} \end{cases}.$$

We will now state and prove the main lemma for non-periodic points. We note that this proof does not require the blocking argument and generally applies in the case of null sets as well. We will adjust the proof minimally when discussing the null-set case for the purposes of keeping that section somewhat self-contained.

Lemma 3.14. *Suppose that $U \in \sigma(\mathcal{A}^n)$ for some n. Then we have that*

$$\left|\frac{\mu(\tau_U \leq s)}{s\mu(U)} - 1\right| \leq \frac{C(\pi(U)^{-\theta+1} + s\mu(U))}{1 + C(\pi(U)^{-\theta+1} + s\mu(U))},$$

where $\theta = \min\{p, \gamma\}$ and some $C > 0$.

Proof. First, we find an upper bound for the desired quantity. By definition of the hitting time we have $\mu(\tau_U \leq s) = \mu(\cup_{j=1}^{s} T^{-j}U) \leq s\mu(U)$ Thus we have that $\frac{\mu(\tau_U \leq s)}{s\mu(U_n)} \leq 1$ for all n.

Next, we find a lower bound. Consider the function that counts the number of hits up to time s. $N = \sum_{j=1}^{s} 1_U \circ T^j$. We have $\int N d\mu = \sum_{j=1}^{s} \mu(T^{-j}U) = s\mu(U)$. We also have by Hölder's inequality,

$$\int N d\mu^2 = \int N 1_{N\geq 1} d\mu^2 \leq \int N^2 d\mu \int 1_{N \geq 1} d\mu = \int N^2 d\mu \mu(\tau_U \leq s), \quad (3.18)$$

and thus $\frac{s\mu(U)}{\int N^2 d\mu} \leq \frac{\mu(\tau_U \leq s)}{s\mu(U)}$.

Finally we wish to show that this lower bound approaches 1 as $n \to \infty$. As such we must consider the second moment of N:

$$\begin{aligned}
\int N^2 d\mu &= \int \sum_{j=1}^{s} \sum_{k=1}^{s} 1_{U_n} \circ T^j \cdot 1_{U_n} \circ T^k d\mu \\
&= \sum_{j=1,k=1}^{s} \int 1_{T^{-j}U \cap T^{-k}U} d\mu \\
&= \sum_{j=k}^{s} \mu(T^{-j}U \cap T^{-k}U) + 2\sum_{j>k}^{s} \mu(T^{-j}U \cap T^{-k}U) \\
&= s\mu(U) + 2\sum_{k=1}^{s} (s-k)\mu(T^{-k}U \cap U).
\end{aligned}$$

We then split the values of k to use $\phi-$mixing. First, if $1 \leq k < \pi(U)$ then

$\mu(T^{-k}U \cap U) = 0$. by definition. Second, if $k > 2n$ then we use the $\phi-$mixing directly

$$\mu(T^{-k}U \cap U) \leq \mu(U)(\phi(k-n) + \mu(U)) \leq \mu(U)(\phi(k/2) + \mu(U)).$$

For the remaining values of k (i.e.$\pi(U) \leq k \leq 2n$), we use the assumption on U. Let $U^{[k/2]}$ be the smallest element of $\sigma(\mathcal{A}^{[k/2]})$ such that $T^{n-[k/2]}U \subset U^{[k/2]}$. Then we have

$$\mu(U \cap T^{-k}U) \leq \mu(U \cap T^{-(n+k-[k/2])}U^{[k/2]}) \leq \mu(U)(\phi(k/2) + \mu(U^{[k/2]})).$$

Collecting these terms we have, continuing our calculation from before:

$$\begin{aligned}
&2\sum_{k=1}^{s}(s-k)\mu(T^{-k}U \cap U) \\
&\leq 2s\sum_{k=1}^{s}\mu(T^{-k}U \cap U) \\
&\leq 2s\mu(U)\left(\sum_{k=\pi(U)}^{2n}\mu(U^{[\frac{k}{2}]}) + \sum_{k=2n+1}^{s}\mu(U) + \sum_{k=\pi(U)}^{s}\phi(\frac{k}{2})\right) \\
&\leq 2s\mu(U)(C_2\pi(U)^{-\gamma+1} + s\mu(U) + C_3\pi(U)^{-p+1}).
\end{aligned}$$

Thus,

$$\frac{1}{1 + C\pi(U)^{-\gamma+1} + 2s\mu(U) + D\pi(U)^{-p+1}} \leq \frac{\mu(\tau_U \leq s)}{s\mu(U)} \leq 1,$$

and finally

$$\begin{aligned}
&\left|\frac{\mu(\tau_U \leq s)}{s\mu(U)} - 1\right| \\
&\leq 1 - \frac{1}{1 + C\pi(U)^{-\gamma+1} + 2s\mu(U) + D\pi(U)^{-p+1}} \\
&\leq \frac{C(\pi(U)^{-\theta+1} + s\mu(U))}{1 + C(\pi(U)^{-\theta+1} + s\mu(U))}.
\end{aligned}$$

The final inequality follows from calculus on the function $\frac{x}{1+x}$.

□

As an immediate consequence, we have the following.

Corollary 3.15. *Let $U_n \in \sigma(\mathcal{A}^n)$ be a sequence of nested sets so that $\cap_n U_n = \Lambda$. Let s to vary with n so that $s(n)\mu(U_n) \to 0$, and $\pi(U_n) \nearrow \infty$, then we have*

$$\frac{\mu(\tau_{U_n} \leq s)}{s\mu(U_n)} \to 1.$$

Remark 3.16. *In the above proof, and lemmas π can be replaced by π_{ess} because $\pi_{ess} \leq \pi$*

Lemma 3.17. *Let $U \in \sigma(\mathcal{A}^n)$ containing a periodic point. Suppose that*

$$\left|Q_{\ell+1} - \sum_{u=\ell}^{\frac{Jn}{p}} \binom{u-1}{\ell-1}\mu(U_u)\right| \leq E_1(n, J, \ell), \tag{3.19}$$

$$|\mu(U_u) - \mu(U)\vartheta^u| \leq E_2(u). \tag{3.20}$$

Then we have

$$\left|\frac{\mu(\tau_U \leq s)}{s\mu(U)} - (1-\vartheta)\right| \leq \frac{1}{s\mu(U)} \sum_{\ell=0}^{\frac{Jn}{p}} (-1)^\ell E(\ell) + \left(\frac{\vartheta}{\vartheta - 1}\right)^s (1-\vartheta) + R,$$

where $R \lesssim \phi^2(\frac{Jn}{4}) + s^2\mu(U^{\frac{Jn}{4}})$, *and*

$$E(\ell) := sE_1 + \frac{Jn}{p}E_2 + s\mu(U)\left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{Jn}{p}+1}.$$

Proof. First, we apply inclusion-exclusion, showing

$$\mu(\tau_U \leq s) = \mu(\cup_{j=1}^{s} T^{-j}U) = \sum_{j=1}^{s} \mu(T^{-j}U) + \sum_{\ell}^{s-1} (-1)^n M_{\ell+1}, \tag{3.21}$$

where $M_{\ell+1}$ is the measure of all points which hit U exactly $\ell+1$ times before s. Next, we split $M_{\ell+1}$ into a principal, $P_{\ell+1}$, and remainder part, $R_{\ell+1}$.

If a sequence of consecutive hits are within Jn of each other, we say they are **clustered**. Points at least Jn apart are said to be in **different clusters**.

The principal part measures points which only have a single cluster of short returns to the set. The remainder are those that have multiple clusters. We will use the invariance of the measure to move the starting position of all such sequences of hits to 0. This will contribute a multiple of at most s. We denote by $G_{\ell+1}(s)$, the collection of hit arrangements that only have one cluster and have their first hit at 0.

Consider an element of $G_{\ell+1}(s)$ that ends at $i_\ell = up$, so that we only hit on multiples of a fixed period p. There are precisely $\binom{u-1}{\ell-1}$ configurations of hits with the above requirements.

If the probability of an overlap is ϑ, we have that the probability of having

u overlap and $\ell + 1$ hits is given by

$$\binom{u-1}{\ell-1}\vartheta^u = \left(\frac{\vartheta}{1-\vartheta}\right)^\ell \binom{u-1}{\ell-1}\vartheta^{u-\ell}(1-\vartheta)^\ell$$
$$= \left(\frac{\vartheta}{1-\vartheta}\right)^\ell \binom{k+\ell-1}{\ell-1}\vartheta^k(1-\vartheta)^\ell, \tag{3.22}$$

where $k = u - \ell$. We can recognize, in the last term the negative binomial distribution. The possible values of k are $0 \leq k \leq \frac{Jn}{p} - \ell$. Summing (3.22) yields

$$\left(\frac{\vartheta}{1-\vartheta}\right)^\ell \sum_{k=0}^{\infty} \binom{k+\ell-1}{\ell-1}\vartheta^k(1-\vartheta)^\ell = \left(\frac{\vartheta}{1-\vartheta}\right)^\ell \tag{3.23}$$

Contributing an error for the tail sum. Taking the alternating sum, we retrieve

$$\sum_{\ell=0}^{\infty}(-1)^\ell \left(\frac{\vartheta}{1-\vartheta}\right)^\ell = 1 - \vartheta. \tag{3.24}$$

The bulk of the argument consists of using the mixing to argue that the contribution of sequences of hits that do not meet the above description becomes negligible in the limit.

By (3.19), we have for $Q_{\ell+1}(s) := \sum_{\vec{i} \in G_{\ell+1}(s)} \mu(C_{\vec{i}})$ where $C_{\vec{i}}$ is the set of points that hit U at the times prescribed by $\vec{i}$. We have

$$\left| P_{\ell+1} - s\mu(U)\left(\frac{\vartheta}{1-\vartheta}\right)^\ell \right| \leq I + II + III + IV, \tag{3.25}$$

where

$$\begin{aligned}
I &= |P_{\ell+1} - sQ_{\ell+1}| \leq \ell Q_{\ell+1} \\
II &= \left| sQ_{\ell+1} - s\sum_{u=\ell}^{\frac{Jn}{p}} \binom{u-1}{\ell-1} \mu(U_u) \right| \leq sE_1 \\
III &= \left| s\sum_{u=\ell}^{\frac{Jn}{p}} \binom{u-1}{\ell-1} \mu(U_u) - s\mu(U) \sum_{u=\ell}^{\frac{Jn}{p}} \binom{u-1}{\ell-1} \vartheta^u \right| \leq \frac{Jn}{p} E_2 \\
IV &= \left| s\sum_{u=\ell}^{\frac{Jn}{p}} \binom{u-1}{\ell-1} \mu(U_u) - s\mu(U) \sum_{u=\ell}^{\infty} \binom{u-1}{\ell-1} \vartheta^u \right| \leq s\mu(U) \left(\frac{\vartheta}{1-\vartheta} \right)^{\frac{Jn}{p}-\ell}.
\end{aligned}$$

Collecting the above estimates yields

$$\begin{aligned}
&\left| P_{\ell+1} - s\mu(U) \left(\frac{\vartheta}{1-\vartheta} \right)^{\ell} \right| \\
&\leq \ell Q_{\ell+1} + sE_1 + \frac{Jn}{p} E_2 + s\mu(U) \left(\frac{\vartheta}{1-\vartheta} \right)^{\frac{Jn}{p}-\ell+1} \\
&=: E'(\ell).
\end{aligned}$$

By repeating the estimates II, III, IV with $\ell Q_{\ell+1}$ instead of $sQ_\ell + 1$ we can absorb the $\ell Q_{\ell+1}$ into the other terms at the cost of a constant multiple. Thus we have

$$\begin{aligned}
\left| P_{\ell+1} - s\mu(U) \left(\frac{\vartheta}{1-\vartheta} \right)^{\ell} \right| &\lesssim sE_1 + \frac{Jn}{p} E_2 + s\mu(U) \left(\frac{\vartheta}{1-\vartheta} \right)^{\frac{Jn}{p}-\ell+1} \\
&=: E(\ell).
\end{aligned}$$

Thus, to conclude our argument, we have

$$\begin{aligned}\mu(\tau_U \leq s) &= s\mu(U) + \sum_{\ell=0}^{s-1}(-1)^{\ell}P_{\ell+1} + \sum_{\ell=0}^{s-1}(-1)^{\ell}R_{\ell+1} \\ &= s\mu(U) + \sum_{\ell=0}^{\frac{Jn}{m}}(-1)^{\ell}P_{\ell+1} + \sum_{\ell=0}^{s-1}(-1)^{\ell}R_{\ell+1},\end{aligned}$$

noting that for $\ell + 1 \geq \frac{Jn}{p}$, $P_{\ell+1}$ is empty. Thus, we have shown that

$$\left| s\mu(U) + \sum_{\ell=1}^{\frac{Jn}{p}}(-1)^{\ell}P_{\ell+1} - s\mu(U)\sum_{\ell=0}^{\frac{Jn}{p}}(-1)^{\ell}\left(\frac{\vartheta}{1-\vartheta}\right)^{\ell}\right| \leq \sum_{\ell=0}^{\frac{Jn}{p}}(-1)^{\ell}E(\ell),$$

noting that we have absorbed the $s\mu(U)$ starting index. Next, we have

$$\left| s\mu(U)\sum_{\ell=0}^{\frac{Jn}{p}}(-1)^{\ell}\left(\frac{\vartheta}{1-\vartheta}\right)^{\ell} - s\mu(U)(1-\vartheta)\right| \leq s\mu(U)\left(\frac{\vartheta}{\vartheta-1}\right)^{\frac{Jn}{p}}(1-\vartheta).$$

Thus, finally we have

$$\begin{aligned}|\mu(\tau_U \leq s) - s\mu(U)(1-\vartheta)| \leq \sum_{\ell=0}^{\frac{Jn}{p}}(-1)^{\ell}E(\ell) + s\mu(U)\left(\frac{\vartheta}{\vartheta-1}\right)^{\frac{Jn}{p}}(1-\vartheta) \\ + s\mu(U)R,\end{aligned}$$

where $R \leq C(\phi^2(\frac{Jn}{4}) + s^2\mu(U_n^{\frac{Jn}{4}}))$ is found by using mixing on the terms with multiple clusters (here, ϕ^2 is the second tail sum of ϕ). Dividing through by $s\mu(U)$ completes the our sketch of the proof. The remaining details on the remainder terms can be found in [26, Lemma 3], however we will give an idea for how the mixing is used in the appendix. □

We now complete the proof of the two main theorems.

Proof of Theorems 3.9 and 3.10. We first consider $\mu(\tau_U > t)$.Allowing s, Δ to be given with $\Delta < s$. We use the division algorithm on t by s to write $t = ks+r$ where $0 \leq r < s$. We will also define q by $s = q\Delta$ and assume that q is a natural number. If $t = ks$, then we have, by Lemma 3,

$$\mu(\tau_U > ks) \leq (\mu(\tau_U > s) + \delta^\eta)^{k-2}$$

Taking logs and dividing by $ks\mu(U)$, we get

$$\begin{aligned}
\frac{1}{ks\mu(U)}\log(\mu(\tau_U > ks)) &= \frac{k-2}{ks\mu(U)}\log(\mu(\tau_U > s) + \delta^\eta) \\
&= \frac{k-2}{ks\mu(U)}\log(1-\mu(\tau_U \leq s) + \delta^\eta) \\
&= (1-\frac{2}{k})\left(\frac{\mu(\tau_U \leq s)}{s\mu(U)} - \frac{\delta^\eta}{s\mu(U)} + ...\right) \\
\left|\frac{1}{ks\mu(U)}\log(\mu(\tau_U > ks)) - 1\right| &\lesssim \frac{1}{k} + \frac{C\pi(U)^{-\theta+1}}{1+C\pi(U)^{-\theta+1}} + \frac{\delta^\eta}{s(n)\mu(U)} + s\mu(U))
\end{aligned}$$

Where in the third equality we have suppressed the much smaller higher order terms in the expansion. Similarly we have

$$\begin{aligned}
\frac{k-2}{ks\mu(U)}\log(1-\mu(\tau_U \leq s) + \delta^\eta) = 1 - \vartheta + \sum_{\ell=0}^{s-1}(-1)^\ell E + \left(\frac{\vartheta}{\vartheta-1}\right)^s (1-\vartheta) \\
+ R - \frac{\delta^\eta}{s\mu(U)}
\end{aligned}$$

Where the terms E, R are as defined in 3.14. Thus we find that

$$\left|\frac{\log(\mu(\tau_U > ks))}{ks\mu(U)} - (1-\vartheta)\right| \lesssim \frac{1}{k} + \sum_{\ell=0}^{s-1}(-1)^\ell E(\ell) + \left(\frac{\vartheta}{\vartheta-1}\right)^s + R + \frac{\delta^\eta}{s\mu(U)}$$

$\square$

We now prove the main corollary.

Proof of Corollary 3.12. Let $s = \mu(U_n)^{-\alpha+\epsilon}$ for some $\epsilon > 0$. Then we have $\frac{t}{s} = \lambda\mu(U_n)^{-\epsilon} = k \to \infty$ as $n \to \infty$, as described in the definition of $\rho_\alpha(U_n)$. Consider

$$s\mu(U_n) = \mu(U_n)^{1-\alpha+\epsilon} \to 0.$$

Next, let $\Delta = s^\beta$ for $\beta \in (0, 1)$. Assume that $\phi(j) \lesssim j^{-p}$, and that $n^{-\gamma_1} \lesssim \mu(U_n) \lesssim n^{-\gamma_2}$. Then we have

$$\frac{\delta^\eta}{s\mu(U_n)} \lesssim \frac{\Delta^\eta\mu(U_n)^\eta}{s\mu(U_n)} + \frac{\phi(\Delta - n)^\eta}{s\mu(U_n)}$$

$$\frac{\Delta^\eta\mu(U_n)^\eta}{s\mu(U_n)} = \mu(U_n)^{(\alpha-\epsilon)(1-\beta\eta)+\eta-1} \to 0 \tag{3.26}$$

and

$$\frac{\phi(\Delta - n)^\eta}{s\mu(U_n)} \lesssim n^{-\gamma_1(\eta\beta p(\alpha-\epsilon)+1-\alpha+\epsilon)} \to 0.$$

If we consider the right-hand side of (3.26) we note that for n sufficiently large the

exponent approaches $\alpha(1 - \beta) > 0$.

Now take $\alpha > 1$, and let $a = \frac{1}{\alpha}$, and $s = \mu(U_n)^{-a}$. Then $k = \frac{t}{s} = \lambda\mu(U_n)^{-(\alpha-a)} \to \infty$. We also have $s\mu(U_n) = \mu(U_n)^{1-a} \to 0$. If we take $\Delta = s^\beta$ for some $\beta \in (0, 1)$, then we have

$$\frac{\Delta^\eta\mu(U_n)^\eta}{s\mu(U_n)} = \mu(U_n)^{a(1-\beta\eta)+\eta-1} \to 0,$$

since for n sufficiently large we have $|\eta-1|$ small and $a(1-\beta\eta) \sim a(1-\beta) > 0$.

Given the assumptions on ϕ and $\mu(U_n)$, we have

$$\frac{\phi(\Delta - n)^\eta}{s\mu(U_n)} \lesssim n^{-\gamma_1(a\beta p\eta + a - 1)}.$$

Thus we have $\frac{\phi(\Delta-n)^\eta}{s\mu(U_n)} \to 0$ provided that $\beta > \frac{\alpha-1}{p} \vee \frac{\alpha}{\gamma_1}$ ($\beta > \frac{\alpha}{\gamma_1}$ is required for $\Delta - n \sim \Delta$).

We now consider the case where $\mu(U_n)$ are decreasing exponentially fast, that is, we assume that $\xi_1^n \lesssim \mu(U_n) \lesssim \xi_2^n$ with $0 < \xi_1 < \xi_2 < 1$. For this case, we choose $s = \xi_2^{-\chi n}$ for some $\chi \in (0,1)$, and again $\Delta = s^\beta$ for some $\beta \in (0,1)$ Then $s \to \infty$, and $s\mu(U_n) \to 0$.

$$\frac{\phi(\Delta - n)^\eta}{s\mu(U_n))} \lesssim \xi_2^{n\left(\chi(1-p\beta) - \frac{\log \xi_2}{\log \xi_1}\right)},$$

which implies that $\frac{\phi(\Delta-n)^\eta}{s\mu(U_n))} \to 0$ if β is sufficiently close to 0 and χ is sufficiently close to 1.

Similarly,

$$\frac{\Delta^\eta \mu(U)^\eta}{s\mu(U)} \lesssim \xi_2^{n\left(\chi(1-\eta\beta) + \eta - \frac{\log \xi_2}{\log \xi_1}\right)} \to 0,$$

since for n sufficiently large, $\eta > \frac{\log \xi_2}{\log \xi_1}$.

We now consider the term $\frac{1}{s\mu(U)} \sum_{\ell=0}^{\frac{Jn}{p}} (-1)^\ell E(\ell)$. We have

$$E(\ell) = sE_1 + \frac{Jn}{p} E_2 + s\mu(U) \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{Jn}{p}+1}$$

$$\frac{E(\ell)}{s\mu(U)} = \frac{E_1}{\mu(U)} + \frac{Jn}{ps} \frac{E_2}{\mu(U)} + \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{Jn}{p}+1}.$$

Since $\frac{\ell}{s}, \frac{Jn}{ps} \leq 1$. Assuming that $\sum_{\ell=0}^{\frac{Jn}{p}} \frac{E_1(n,\ell,J)}{\mu(U)} \to 0$, and $\sum_{\ell=0}^{\frac{Jn}{p}} \frac{E_2(n,\ell,J)}{\mu(U)} \to 0$ yields the desired convergence. □

Chapter 4

Escape Rate Dichotomy for General Null Sets

4.1 Introduction

In this chapter, we will discuss the local escape rate dichotomy as it applies to general null-sets, not just points. Some adjustments will be made to the framework of the discussion in light of the fact that we will no longer be looking to show the error estimates discussed in Chapter 3.

4.2 Preliminaries

We begin by introducing some notations and definitions.

Definition 4.1. *We define the* ***higher order hitting times*** *recursively. Let* $\tau_U^0 = 0$ *and* $\tau_U^1 = \tau_U$ *as previously defined, then*

$$\tau_U^\ell = \tau_U^{\ell-1} + \tau_U \circ T^{\tau_U^{\ell-1}}. \tag{4.1}$$

Definition 4.2. *We define the parameters* $\hat{\alpha}_\ell(K) = \lim_{n\to\infty} \mu_{U_n}(\tau_{U_n}^{\ell-1} \le K)$, *and the parameters*[1]

$$\hat{\alpha}_\ell = \lim_{K\to\infty} \hat{\alpha}_\ell(K). \tag{4.2}$$

Remark 4.3. *We note here that* $\hat{\alpha}_\ell(K)$ *is monotonically increasing in* K, *and bounded above by* 1. *so provided the* $\hat{\alpha}_\ell(K)$ *exist, the* $\hat{\alpha}_\ell$ *exist. Similarly, the* $\hat{\alpha}_\ell(K)$ *are decreasing in* ℓ *and thus so are the* $\hat{\alpha}_\ell$.

Definition 4.4. *We define the level sets* p_i^ℓ *by*

$$p_i^\ell = \lim_{n\to\infty} \mu_{U_n}(\tau_{U_n}^{\ell-1} = i) \tag{4.3}$$

Definition 4.5. *We define the level sets of the hitting times after a specific cutoff in the following way. First, we let*

$$\alpha_\ell(K) = \lim_{n\to\infty} \mu_{U_n}(\tau_{U_n}^{\ell-1} \le K < \tau_{U_n}^{\ell}), \tag{4.4}$$

and

$$\alpha_\ell = \lim_{K\to\infty} \alpha_\ell. \tag{4.5}$$

Definition 4.6. *A nested sequence* $\{U_n \in \mathcal{A}^{\kappa_n}\}$ *is called a* ***good neighborhood system****, if:*

1. $\kappa_n \nearrow \infty$ *and* $\kappa_n \mu(U_n)^\varepsilon \to 0$ *for some* $\varepsilon \in (0,1)$;
2. *there exists* $C > 0$ *and* $p' > 1$ *such that* $\mu(U_n^j) \le \mu(U_n) + Cj^{-p'}$ *for all* $j < \kappa_n$.

[1] μ_{U_n} refers to the measure μ conditioned on U_n

4.3 Gibbs-Markov Systems

A map $T : \mathbf{M} \to \mathbf{M}$ is called *Markov* if there is a countable measurable partition $\mathcal{A}$ on $\mathbf{M}$ with $\mu(A) > 0$ for all $A \in \mathcal{A}$, such that for all $A \in \mathcal{A}$, $T(A)$ is injective and can be written as a union of elements in $\mathcal{A}$. Write $\mathcal{A}^n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{A}$ as before, it is also assumed that $\mathcal{A}$ is (one-sided) generating.

Fix any $\lambda \in (0,1)$ and define the metric d_λ on $\mathbf{M}$ by $d_\lambda(x,y) = \lambda^{s(x,y)}$, where $s(x,y)$ is the largest positive integer n such that x, y lie in the same n-cylinder. Define the Jacobian $g = JT^{-1} = \frac{d\mu}{d\mu \circ T}$ and $g_k = g \cdot g \circ T \cdots g \circ T^{k-1}$.

The map T is called *Gibbs-Markov* if it preserves the measure μ, and also satisfies the following two assumptions:
(i) The big image property: there exists $C > 0$ such that $\mu(T(A)) > C$ for all $A \in \mathcal{A}$.
(ii) Distortion: $\log g|_A$ is Lipschitz for all $A \in \mathcal{A}$.

In view of (i) and (ii), there exists a constant $D > 1$ such that for all x, y in the same n-cylinder, we have the following distortion bound:

$$\left| \frac{g_n(x)}{g_n(y)} - 1 \right| \le D d_\lambda(T^n x, T^n y),$$

and the Gibbs property:

$$D^{-1} \le \frac{\mu(A_n(x))}{g_n(x)} \le D.$$

For an excellent survey of results relating to limit laws for Gibbs-Markov systems, we refer the reader to the dissertation of X. Zhang [41].

Remark 4.7. *In light of in [32, Lemma 2.4(b)], Gibbs-Markov systems are exponentially left ϕ-mixing, that is, $\phi(k) \lesssim \eta^k$ for some $\eta \in (0,1)$. In fact*

$|\mu(A \cap (T^{-(n+k)}B) - \mu(A)\mu(B)| \leq \eta^k \mu(A)\sqrt{\mu(B)}$, *but Gibbs-Markov systems need not be* $\psi-$*mixing or both left and right* $\phi-$*mixing.*

4.4 Main results

Theorem 4.8. *Assume that* $T : \mathbf{M} \to \mathbf{M}$ *preserves a probability measure* μ *that is both left and right* ϕ*-mixing with* $\phi(k) \leq Ck^{-p}$ *for some* $C > 0$ *and* $p > 1$*, and* $\{U_n\}$ *is a good neighborhood system such that* $\{\hat{\alpha}_\ell\}$ *exists, and satisfies* $\sum_\ell \ell\hat{\alpha}_\ell < \infty$*.*

Then α_1 *exists, and the localized escape rate at* Λ *exists and satisfies*

$$\rho(\Lambda, \{U_n\}) = \alpha_1.$$

For Gibbs-Markov systems, the same result is true:

Theorem 4.9. *Assume that* $T : \mathbf{M} \to \mathbf{M}$ *is a Gibbs-Markov system with respect to the partition* $\mathcal{A}$*. Let* $\{U_n\}$ *be a good neighborhood system such that* $\{\hat{\alpha}_\ell\}$ *exists, and satisfies* $\sum_\ell \ell\hat{\alpha}_\ell < \infty$*.*

Then α_1 *exists. Furthermore, the localized escape rate at* Λ *exists and satisfies*

$$\rho(\Lambda, \{U_n\}) = \alpha_1.$$

Both of these theorems follow immediately from the corresponding lemmas in the previous sections.

Corollary 4.10. *Let* $(\mathbf{M}, T, \mathcal{B}, \mu)$ *be a measure preserving system. Assume that* $\{U_n\}$ *is a good neighborhood system with* $\pi_{\text{ess}}(U_n) \to \infty$*, and* $(T, \mu, \mathcal{A})$ *satisfies one of the following two assumptions:*

1. *either μ is both left and right ϕ-mixing with $\phi(k) \leq Ck^{-p}$ for some $p > 1$;*

2. *or T is Gibbs-Markov;*

then the localized escape rate at Λ exists and satisfies

$$\rho(\Lambda, \{U_n\}) = 1.$$

As an immediate corollary, we have:

Corollary 4.11. *The conclusion of Corollary 4.10 holds if the assumption: "$\pi_{\mathrm{ess}}(U_n) \to \infty$" is replaced by the following assumptions:*

1. *T is continuous, $\Lambda = \cap_n U_n = \cap_n \overline{U}_n$;*

2. *Λ intersects every forward orbit at most once, that is, for every $x \in \Lambda$ we have $\Lambda \cap \{T^k(x) : k \geq 1\} = \emptyset$.*

4.5 Proofs of Main Results

The next lemma establishes the relation between the escape rate and the probability of short entries:

Lemma 4.12. *Assume that μ is left ϕ-mixing for the partition $\mathcal{A}$, with $\phi(k) \leq Ck^{-p}$ for some $p > 0$. Let $\{U_n \in \sigma(\mathcal{A}^{\kappa_n})\}$ be a nested sequence of sets for some $\kappa_n \nearrow \infty$. Furthermore, assume that there exists $\varepsilon \in (0,1)$, such that $\kappa_n \mu(U_n)^\varepsilon \to 0$.*

Then we have

$$\rho(\Lambda) = \lim_{n\to\infty} \frac{\mu(\tau_{U_n} \leq s_n)}{s_n \mu(U_n)}, \tag{4.6}$$

where $s_n = \lfloor \mu(U_n)^{-(1-a)} \rfloor$ for any fixed $a > 0$ small enough.

Remark 4.13. *At first glance, the right-hand side of* (3.1) *is similar to the definition of the local escape rate given above. However, since $s_n \ll \mu(U_n)^{-1}$ (where the latter is the average return time given by Kac's formula), $\mu(\tau_{U_n} \leq s_n)$ concerns the probability of short entries to U.*

Proof. Let $\{s_n\}, \{\Delta_n\}$ be increasing sequences of positive integers with $\Delta_n < s_n/2$, whose choice will be specified later. Write $q_n = \lfloor s_n/\Delta_n \rfloor$, $\eta_n = \frac{q_n}{q_n+1}$ and $\delta_n = 2(\Delta_n \mu(U_n) + \phi(\Delta_n - \kappa_n))$ as before. By Lemma 3.9 we get

$$\frac{1}{ks_n}|\log \mu(\tau_{U_n} > ks_n)| = \frac{k-2}{ks_n}|\log\left(\mu(\tau_{U_n} > s_n) + \delta_n^{\eta_n}\right)|.$$

Take limit as $k \to \infty$ and note that $\mu(\tau_{U_n} > s_n) = 1 - \mu(\tau_{U_n} \leq s_n)$, we obtain

$$\begin{aligned}\rho(U_n) =& \lim_{k\to\infty} \frac{1}{ks_n}|\log \mu(\tau_{U_n} > ks_n)| \\ =& \frac{1}{s_n}\big(\mu(\tau_{U_n} \leq s_n) + o(s_n\mu(U_n)) + \mathcal{O}(\delta_n^{\eta_n})\big).\end{aligned} \tag{4.7}$$

Here we used the trivial estimate

$$\mu(\tau_{U_n} \leq s_n) \leq \mu(\cup_{1\leq k\leq s_n} T^{-k}(U_n)) \leq s_n\mu(U_n).$$

Divide (4.7) by $\mu(U_n)$ and let $n \to \infty$, we obtain

$$\rho(\Lambda) = \lim_{n\to\infty}\left(\frac{\mu(\tau_{U_n} \leq s_n)}{s_n\mu(U_n)} + \frac{(\delta_n)^{\eta_n}}{s_n\mu(U_n)}\right). \tag{4.8}$$

It remains to show that the second term converges to zero for some proper choice of $\{s_n\}$ and Δ_n. For this purpose, we fix some $a \in (0,1), b \in (\varepsilon, 1)$ and choose $s_n = \lfloor \mu(U_n)^{-(1-a)} \rfloor$, and $\Delta_n = \lfloor \mu(U_n)^{-b} \rfloor \gg \kappa_n = o(\mu(U_n)^{-\varepsilon})$. Then we have ($a_n \lesssim b_n$ means there exists constant C such that $a_n \le C \cdot b_n$):

$$\begin{aligned}\frac{(\delta_n)^{\eta_n}}{s_n\mu(U_n)} \lesssim & \frac{(\Delta_n)^{\eta_n}\mu(U_n)^{\eta_n}}{s_n\mu(U_n)} + \frac{\phi(\Delta_n - \kappa_n)^{\eta_n}}{s_n\mu(U_n)} \\ \le & \Delta_n\mu(U_n)^{\eta_n - a} + (\Delta_n)^{-p\eta_n}\mu(U_n)^{-a} \\ \le & \mu(U_n)^{\eta_n - a - b} + \mu(U_n)^{bp\eta_n - a}.\end{aligned}$$

In order for both terms to go to zero, we need:

1. $1 - a > b$, which guarantees that $s_n \gg \Delta_n$, so $q_n \to \infty$ and consequently $\eta_n \nearrow 1$; then the first term will go to zero;
2. $bp > a$, so that the second term goes to zero.

Both requirements are satisfied if we take any $b \in (\varepsilon, 1)$, then choose $0 < a < \min\{1 - b, bp\}$. Combine this with (4.8) we conclude that

$$\rho(\Lambda) = \lim_{n\to\infty} \frac{\mu(\tau_{U_n} \le s_n)}{s_n\mu(U_n)},$$

as desired. □

4.5.1 Systems that are left and right mixing

Lemma 4.14. *Let μ be both left and right ϕ-mixing for the partition $\mathcal{A}$, with $\phi(k) \le Ck^{-m}$ for some $m > 1$. Assume that $\{U_n\}$ is a good neighborhood system, such that $\hat{\alpha}_\ell(K)$ exists for K large enough, and $\sum_\ell \ell\hat{\alpha}_\ell < \infty$. Then we have*

$$\lim_{n\to\infty} \frac{\mu(\tau_{U_n} \le s_n)}{s_n\mu(U_n)} = \alpha_1$$

for any increasing sequence $\{s_n\}$ for which $s_n\mu(U_n) \to 0$ as $n \to \infty$.

Proof. For an given integer s, write $Z_n^s = \sum_{j=1}^{s} \mathbb{I}_{U_n} \circ T^j$ which counts the number of entries to U_n before time s. Let K be a large integer, then by [25] Lemma 3 for every $\varepsilon > 0$ one has $\mu(\tau_{U_n} \le K) = \alpha_1 K \mu(U_n)(1 + \mathcal{O}^*(\varepsilon))$, where the notation $\mathcal{O}^*$ means that the implied constant is one (i.e. $x = \mathcal{O}^*(\varepsilon)$ if $|x| < \varepsilon$). For simplicity, assume $r = s_n/K$ is an integer and put

$$V_q = \{Z_n^K \circ T^{qK} \ge 1\},$$

$q = 0, 1, \dots, r-1$, and

$$D_q = \{V_q, Z_n^{(r-q-1)K} \circ T^{(q+1)K} = 0\}.$$

Then

$$\{Z_n^{s_n} \ge 1\} = \bigcup_{q=0}^{r-1} D_q$$

is a disjoint union. Let us now estimate

$$\begin{aligned}
\mu(Z_n^{(r-q-1)K} \circ T^{(q+1)K} &\ge 1, V_q) \\
&\le \mu(Z_n^{(r-q-1)K-2\sqrt{K}} \circ T^{(q+1)K+2\sqrt{K}} \ge 1, V_q) + 2\sqrt{K}\mu(U_n) \\
&\le 2\sqrt{K}\mu(U_n) + \sum_{i=(q+1)K+2\kappa_n}^{s_n} \mu(T^{-i}U_n \cap V_q) \\
&\qquad + \sum_{j=qK}^{(q+1)K-1} \sum_{i=(q+1)K+2\sqrt{K}}^{(q+1)K+2\kappa_n} \mu(T^{-j}U_n \cap T^{-i}U_n).
\end{aligned}$$

Using the left ϕ-mixing property, the first sum above can be bounded by

$$\mathrm{II} \le \sum_{i=\kappa_n}^{s_n-\kappa_n} \mu(V_q)(\mu(U_n) + \phi(i)).$$

For the second sum, we use right ϕ-mixing to get (and recall that U_n^j is the outer-approximation of U_n by j-cylinders):

$$\begin{aligned}
\mathrm{III} \leq & \sum_{j=qK}^{(q+1)K-1} \sum_{i=(q+1)K+2\sqrt{K}}^{(q+1)K+2\kappa_n} \mu(U_n \cap T^{-(i-j)}U_n) \\
\leq & K \sum_{j=2\sqrt{K}}^{2\kappa_n} \mu(U_n^{j/2} \cap T^{-j}U_n) \\
\leq & K \sum_{j=2\sqrt{K}}^{2\kappa_n} \mu(U_n)(\mu(U_n^{j/2}) + \phi(j/2)) \\
= & \mathcal{O}(1)\mu(V_q) \sum_{j=2\sqrt{K}}^{2\kappa_n} (\mu(U_n^{j/2}) + \phi(j/2)),
\end{aligned}$$

where the last equality follows from

$$\mu(V_q) = \mu(\tau_{U_n} \leq K) = \alpha_1 K \mu(U_n)(1 + \mathcal{O}^*(\varepsilon)).$$

Combine the previous estimates, we get

$$\begin{aligned}
&\mu(Z_n^{(r-q-1)K} \circ T^{(q+1)K} \geq 1, V_q) \\
&\quad \leq \mu(Z_n^{(r-q-1)K-2\sqrt{K}} \circ T^{(q+1)K+2\sqrt{K}} \geq 1, V_q) + 2\sqrt{K}\mu(U_n) \\
&\quad \leq 2\sqrt{K}\mu(U_n) + \mu(V_q) \sum_{i=\kappa_n}^{s_n-\kappa_n} (\mu(U_n) + \phi(i)) \\
&\qquad\qquad + \mu(V_q)\mathcal{O}(1) \sum_{j=\sqrt{K}}^{\kappa_n} (\mu(U_n^{(j)}) + \phi(j)) \\
&\quad \leq \mu(V_q)F,
\end{aligned}$$

where

$$F = \frac{2}{\sqrt{K}} + s_n\mu(U_n) + \mathcal{O}(1)(\phi^1(\sqrt{K}) + \sum_{j=\sqrt{K}}^{\kappa_n} \mu(U_n^{(i)}))$$

and $\phi^1(u) = \sum_{i=u}^{\infty} \phi(i)$ is the tail-sum of ϕ.

If n is large enough so that $\max\{s_n\mu(U_n), \kappa_n\mu(U_n), \phi^1(\kappa_n)\} < \varepsilon$ then

$$\begin{aligned} F &\leq 2\varepsilon + \frac{2}{\sqrt{K}} + \mathcal{O}(1)\left(\phi^1(\sqrt{K}) + \kappa_n\mu(U_n) + \sum_{i=\sqrt{K}}^{\kappa_n} i^{-p'}\right) \\ &\lesssim \varepsilon + \frac{1}{\sqrt{K}} + \phi^1(\sqrt{K}) + K^{-\frac{p'-1}{2}}, \end{aligned}$$

where we used the assumption that $\mu(U_n^{(i)}) \leq \mu(U_n) + Ci^{-m'}$ for some $m' > 1$. Consequently

$$\mu(D_q) = \mu(V_q) - \mu(V_q, Z_n^{(r-q-1)K} \circ T^{(q+1)K} \geq 1) = \mu(V_q)(1 + \mathcal{O}^*(F)),$$

and since $\{Z_n^{qK} \geq 1, V_q\} = V_q$ and $\mu(V_q) = \mu(V_0)$ we get

$$\mu(Z_n^{s_n} \geq 1) = \sum_{q=0}^{r-1} \mu(D_q) = r\mu(V_0)(1 + \mathcal{O}^*(F)).$$

Since by [25] Lemma 3 $\mu(V_0) = \alpha_1 K\mu(U_n)(1 + \mathcal{O}^*(\varepsilon))$ we obtain

$$\mu(\tau_{U_n} \leq s_n) = r\mu(V_0)(1 + \mathcal{O}^*(F)) = \alpha_1 s_n\mu(U_n)(1 + \mathcal{O}^*(\varepsilon + F)).$$

The statement of the lemma now follows if we let $\varepsilon \to 0$ and $K \to \infty$. □

4.5.2 Main lemma for Gibbs-Markov systems

Lemma 4.15. *Let $(T, \mu, \mathcal{A})$ be a Gibbs-Markov system. Assume that $\{U_n\}$ is a good neighborhood system, such that $\hat{\alpha}_\ell(K)$ exists for K large enough, and*

$\sum_\ell \ell\hat{\alpha}_\ell < \infty$. *Then we have*

$$\lim_{n\to\infty} \frac{\mathbb{P}(\tau_{U_n} \le s_n)}{s_n\mu(U_n)} = \alpha_1$$

for any increasing sequence $\{s_n\}$ *for which* $s_n\mu(U_n) \to 0$ *as* $n \to \infty$.

Proof. Recall that Gibbs-Markov systems are left ϕ-mixing with exponential rate. The proof follows the lines of Lemma 4.14 with only one modification: the term is now estimated as:

$$\begin{aligned} \mathrm{III} \le & \sum_{j=qK}^{(q+1)K-1} \sum_{i=(p+1)K+2\sqrt{K}}^{(q+1)K+2\kappa_n} \mu(U_n \cap T^{-(i-j)}U_n) \\ \le & K \sum_{j=2\sqrt{K}}^{2\kappa_n} \mu(U_n \cap T^{-j}U_n), \end{aligned}$$

Each term in the summation can be bounded by:

$$\begin{aligned} \mu(U_n \cap T^{-j}U_n) \le & \sum_{A\in\mathcal{C}_j(U_n)} \mu(T^{-j}U_n \cap A) \\ = & \sum_{A\in\mathcal{C}_j(U_n)} \frac{\mu(T^{-j}U_n \cap A)}{\mu(A)}\mu(A) \\ \lesssim & \sum_{A\in\mathcal{C}_j(U_n)} \frac{\mu(T^j(T^{-j}U_n \cap A))}{\mu(T^jA)}\mu(A) \\ \lesssim & \sum_{A\in\mathcal{C}_j(U_n)} \mu(U_n)\mu(A) \\ = & \mu(U_n)\mu\left(\bigcup_{A\in\mathcal{C}_j(U_n)} A\right) = \mu(U_n)\mu(U_n^j), \end{aligned}$$

where the third and forth inequality follow from the distortion and the big image property of Gibbs-Markov systems.

Then we have

$$\mathrm{III} \le K\mu(U_n) \sum_{j=2\sqrt{K}}^{\kappa_n} \mu(U_n^j) = \mathcal{O}(1)\mu(V_p) \sum_{j=2\sqrt{K}}^{2\kappa_n} \mu(U_n^j),$$

and the rest of the proof is identical to Lemma 4.14. □

4.5.3 Some remarks on the extremal index

In the classic literature (for example, [19] and [17]), the extremal index is defined as

$$\theta = \lim_{n\to\infty} \mu_{U_n}(\tau_{U_n} > K_n), \tag{4.9}$$

where $K_n \to \infty$ is some increasing sequence of integers. It is shown in [38, Proposition 5.4] that under the assumption of Theorem 4.9 we have

$$\alpha_1 = \theta.$$

It is also straight forward to check that the proof of Lemma 4.14 and **??** remain true with α_1 replaced by θ. We state this as the following proposition:

Proposition 4.16. *Assume that one of the following assumptions holds:*

1. *either μ is both left and right ϕ-mixing with $\phi(k) \lesssim k^{-p}$, $p > 1$;*

2. *or $(T, \mu, \mathcal{A})$ is a Gibbs-Markov system.*

Let θ be the extremal index defined by (4.9) *for some sequence $\{K_n\}$. Then for any good neighborhood system $\{U_n\}$ and any increasing sequence $\{s_n\}$ with $s_n\mu(U_n) \to 0$ and $s_n/K_n \to \infty$, we have*

$$\lim_{n\to\infty} \frac{\mu(\tau_{U_n} \le s_n)}{s_n\mu(U_n)} = \theta.$$

Furthermore, the local escape rate at $\Lambda = \cap_n U_n$ *exists and satisfies*

$$\rho(\Lambda) = \theta.$$

Theorem 4.17. *Assume that* $T : \mathbf{M} \to \mathbf{M}$ *preserves a measure* μ *that is left* ϕ*-mixing with* $\phi(k) \leq Ck^p$ *for some* $C > 0$ *and* $p > 1$*, and* $\{U_n \in \mathcal{A}^{\kappa_n}\}$ *is a nested sequence of sets with* $\kappa_n \mu(U_n)^\varepsilon \to 0$ *for some* $\varepsilon \in (0, 1)$.

Assume that θ *defined by* (4.9) *exists for some sequence* $\{K_n\}$ *with* $K_n > (\kappa_n)^2$*. Then the localized escape rate at* Λ *exists and satisfies*

$$\rho(\Lambda) = \theta.$$

Chapter 5

Conditional Escape Rate

5.1 Introduction

In this brief chapter, we discuss the conditional escape rate in reference to the usual escape rate. We will show using only basic tools that if either the conditional or usual escape rate exists, then the other exists, and they are equal.

5.2 Preliminaries

Let $T : \mathbf{M} \to \mathbf{M}$ and μ a T-invariant measure with associated $\sigma-$algebra, $\mathcal{F}$. The following lemma establishes a relationship between the hitting times and the return times.

Lemma 5.1. *For any set $U \subset \mathbf{M}$ with $\mu(U) > 0$, let $A_k := \{x \in \mathbf{M} | \tau_U \geq k\}$, and $B_k := \{x \in U | \tau_U \geq k\} = A_k \cap U$. Then we have*

$$\mu_U(A_k)\mu(U) = \mu(B_k) = \mu(A_k) - \mu(A_{k+1}) \tag{5.1}$$

Proof. By definition we have $A_{k+1} \subset A_k$. Thus, we compute

$$\begin{aligned}
\mu(A_{k+1}) &= \mu(\cap_{j=1}^{k} T^{-j} U^c) \\
&= \mu(T^{-1}(\cap_{j=0}^{k-1} T^{-j} U^c)) \\
&= \mu(U^c \cap_{j=1}^{k-1} T^{-j} U^c) \\
&= \mu(U^c \cap A_k) \\
&= \mu(A_k) - \mu(U \cap A_k) \\
&= \mu(A_k) - \mu(B_k),
\end{aligned}$$

where the third equality follows from the invariance of μ. □

Next, we establish the exponential rate for a sequence with of telescoping terms based on the original sequence it was built from.

Lemma 5.2. *Suppose that* $\lim_{n\to\infty} -\frac{\log(a_n)}{n} = \vartheta > 0$. *Let* $b_n = (a_n - a_{n+1})$. *Suppose, also, that* b_n *is monotonically decreasing. Then we have*

$$\lim_{n\to\infty} -\frac{\log(b_n)}{n} = \vartheta$$

as well.

Remark 5.3. *Note that there are counterexamples for which the statement of the lemma does not hold when* b_n *is not monotonically decreasing.*

Proof. Fix $\epsilon > 0$. Then we have for n sufficiently large,

$$a_n \in \left(e^{-n(\vartheta+\epsilon)}, e^{-n(\vartheta-\epsilon)}\right) \tag{5.2}$$

$$a_{n+1} \in \left(e^{-(n+1)(\vartheta+\epsilon)}, e^{-(n+1)(\vartheta-\epsilon)}\right). \tag{5.3}$$

Then by (5.2), and (5.3), we have

$$\begin{aligned}b_n = a_n - a_{n+1} &< e^{-n(\vartheta-\epsilon)} - e^{-(n+1)(\vartheta+\epsilon)} \\ &= e^{-n(\vartheta-\epsilon)}\left(1 - e^{-((2n+1)\epsilon+\vartheta)}\right).\end{aligned}$$

Thus we have

$$\begin{aligned}-\frac{\log(b_n)}{n} = -\frac{\log(a_n - a_{n+1})}{n} &\geq -\frac{\log(e^{-n(\vartheta-\epsilon)}\left(1 - e^{-((2n+1)\epsilon+\vartheta)}\right))}{n} \\ &= \vartheta - \epsilon - \frac{\log\left(1 - e^{-((2n+1)\epsilon+\vartheta)}\right)}{n}.\end{aligned}$$

taking lim inf on both sides, we obtain

$$\liminf_{n\to\infty} -\frac{\log(b_n)}{n} \geq \vartheta - \epsilon,$$

for all $\epsilon > 0$. Sending $\epsilon \to 0$, we get $\liminf_{n\to\infty} -\frac{\log(b_n)}{n} \geq \vartheta$. Note that this bound on the lim inf does not require the monotonicity.

To bound the lim sup, we now proceed by contradiction. Assume that

$$\limsup_{n\to\infty} -\frac{\log(b_n)}{n} = \vartheta' > \vartheta.$$

Then there is a sub-sequence b_{n_k} so that for k sufficiently large, we have

$$b_{n_k} \in \left(e^{-n_k(\vartheta'+\epsilon)}, e^{-n_k(\vartheta'-\epsilon)}\right).$$

But then for n_k sufficiently large,

$$e^{-n_k(\vartheta+\epsilon)} - e^{-(2n_k)(\vartheta-\epsilon)} \leq \sum_{j=n_k}^{2n_k-1} b_j \leq (n_k)e^{-n_k(\vartheta'-\epsilon)}, \tag{5.4}$$

where the second inequality follows from the monotonicity of the b_j which yields a contradiction since we can choose $\epsilon > 0$ so that $\vartheta' - \epsilon > \vartheta + \epsilon$, and $2(\vartheta - \epsilon) > \vartheta + \epsilon$ and then the decay rate of the right-hand side in (5.4) is more than that of the left-hand side.

Thus we have

$$\limsup_{n\to\infty} -\frac{\log b_n}{n} \leq \vartheta \leq \liminf_{n\to\infty} -\frac{\log b_n}{n},$$

which completes the proof. □

By choosing $a_k = \mu(A_k)$, $b_k = \mu(B_k)$, and $C = \frac{1}{\mu(U)}$, we retrieve the following:

Proposition 5.4. $-\frac{1}{k}\log \mu(\tau_U \geq k) \to \vartheta$ *if and only if* $-\frac{1}{k}\log \mu_U(\tau_U \geq k) \to \vartheta$.

Definition 5.5. *We define the* ***escape rate*** *and the* ***conditional escape rate*** *respectively as follows.*

$$\rho(U) := \lim_{k\to\infty} -\frac{1}{k}\log(\mu(\tau_U > k)) \tag{5.5}$$

$$\rho_U(U) := \lim_{k\to\infty} -\frac{1}{k}\log(\mu_U(\tau_U > k)). \tag{5.6}$$

5.3 Main Result

Theorem 5.6. *For every* $U \in \mathcal{F}$ *so that* $\mu(U) > 0$, *assume that either* $\rho(U)$ *exists, or* $\rho_U(U)$ *exists. Then both exist and* $\rho(U) = \rho_U(U)$.

Remark 5.7. *Note that this theorem does not require any mixing assumptions. It is just a cosmetic restatement of the immediately preceding proposition in* *terms of escape rate.*

Chapter 6

Applications and Examples

6.1 Extreme value theory

Let M a manifold with $\phi : M \to \mathbb{R} \cup \{\infty\}$ continuous achieving its maximum on a null set, Λ. We then consider $T : M \to M$ and the dynamics of ϕ.

Definition 6.1. *Let $X_j = \phi \circ T^j$ for $j \in \mathbb{N}$. Let u_n a sequence of real numbers to be thought of as thresholds and let $M_t = \max_{0 \leq k \leq t}\{X_k\}$. Define the* ***threshold family*** *of open set*

$$U_n = \{\phi > u_n\} \tag{6.1}$$

.

Definition 6.2. *We will define the* ***exceedence rate*** *of ϕ along $\{u_n\}$ as*

$$\zeta(\{u_n\}) = \lim_{t\to\infty} -\frac{1}{t}\log \mu(M_t < u_n) \tag{6.2}$$

$$\zeta(\phi, \{u_n\}) = \lim_{n\to\infty} \frac{\zeta(\{u_n\})}{\mu(U_n)}. \tag{6.3}$$

Definition 6.3. *We also define the* ***inner and outer approximating sets***

for the sets U_n. For a sequence of real numbers r_n,

$$U_n^i = U_n \backslash \overline{\cup_{x\in\partial U_n} B_{r_n}(x)} \tag{6.4}$$

$$U_n^o = \cup_{x\in U_n} B_{r_n}(x). \tag{6.5}$$

Note that we have $\overline{U_n^i} \subset U_n \subset \overline{U_n} \subset U_n^o$.

Additionally, we will require the assumption that with respect to our measure, these are good approximations of the set U_n in the following sense.

Definition 6.4. *that there exists a sequence $r_n \to 0$ so that*

$$\mu(U_n^o \backslash U_n^i) = o(1)\mu(U_n). \tag{6.6}$$

where $o(1)$ refers to the limit in n. We say that U_n is ***well-approximated*** *if such a sequence, r_n exists*

Theorem 6.5. *Assume that*

1. *either μ is both left and right ϕ-mixing with $\phi(k) \le Ck^{-p}$, $p > 1$;*

2. *or $(T, \mu, \mathcal{A})$ is a Gibbs-Markov system.*

Let $\varphi : M \to \mathbb{R} \cup \{+\infty\}$ be a continuous function achieving its maximum on a measure zero set Λ. Let $\{u_n\}$ be a non-decreasing sequence of real numbers with $u_n \nearrow \sup\varphi$, such that the open sets U_n defined above are well-approximated and $\{\hat{\alpha}_\ell\}$ exists and satisfies $\sum_\ell \ell\hat{\alpha}_\ell < \infty$. Write κ_n for the smallest positive integer with $\operatorname{diam}\mathcal{A}^{\kappa_n} \le r_n$ where $\{r_n\}$ is the sequence given by assumption on U_n. We assume that:

1. *$\kappa_n\mu(U_n)^\varepsilon \to 0$ for some $\varepsilon \in (0, 1)$;*

2. U_n has small boundary: there exists $C>0$ and $p'>1$, such that $\mu\left(\bigcup_{A\in\mathcal{A}^j, A\cap B_{r_n}(\partial U_n)\neq\emptyset} A\right)\le Cj^{-p'}$ for all n and $j\le\kappa_n$.

Then the exceedance rate of φ along $\{u_n\}$ exists and satisfies

$$\zeta(\varphi,\{u_n\})=\alpha_1.$$

Proposition 6.6. *Let $\{U_n\}$, $\{V_n\}$ and $\{W_n\}$ be sequences of nested sets with $V_n\subset U_n\subset W_n$ for each n, and $\Lambda=\cap_n U_n=\cap_n V_n=\cap_n W_n$. Assume that*

$$\mu(W_n\setminus V_n)=o(1)\mu(V_n), \tag{6.7}$$

and $\rho(\Lambda,\{W_n\})=\rho(\Lambda,\{V_n\})=\alpha$.

Then we have

$$\rho(\Lambda,\{U_n\})=\alpha.$$

Proof. $V_n\subset U_n\subset W_n$ implies that $\tau_{W_n}\ge\tau_{V_n}\ge\tau_{U_n}$. Therefore we have

$$\rho(W_n)\ge\rho(U_n)\ge\rho(V_n).$$

On the other hand, (6.7) means that $\mu(W_n)/\mu(V_n)\to 1$. We thus obtain

$$\rho(\Lambda,\{W_n\})\ge\rho(\Lambda,\{U_n\})\ge\rho(\Lambda,\{V_n\}),$$

and the proposition follows from the squeeze theorem. □

Proof of Theorem 6.5. For the sequence $\{r_n\}$ given in Assumption 6.6, we write κ_n for the smallest integer such that $\operatorname{diam}\mathcal{A}^{\kappa_n} \le r_n$. Then consider

$$V_n = \cup_{A\in\mathcal{A}^{\kappa_n}, A\subset U_n} A, \qquad\qquad W_n = \cup_{A\in\mathcal{A}^{\kappa_n}, A\cap U_n\neq\emptyset} A.$$

Clearly we have $V_n \subset U_n \subset W_n$ for each n. Moreover, the choice of κ_n gives

$$U_n^i \subset V_n, \qquad\qquad W_n \subset U_n^o.$$

Combine this with (6.6), we have $\mu(W_n \setminus V_n) = o(1)\mu(V_n)$.

Let us write $\hat\alpha_\ell^*$, $* = U, V, W$ for $\hat\alpha_\ell$ defined using $\{U_n\}, \{V_n\}, \{W_n\}$, respectively. Then it is proven in [38, Lemma 5.6] that

$$\hat\alpha_\ell^V = \hat\alpha_\ell^U = \hat\alpha_\ell^W.$$

In particular, $\sum_\ell \ell\hat\alpha_\ell^U < \infty$ implies that the same holds for $\hat\alpha_\ell^*$, $* = V, W$, and the value of α_1 defined by $\{V_n\}, \{U_n\}, \{W_n\}$ are equal.

It remains to show that $\{V_n\}$ and $\{W_n\}$ are good neighborhood systems. (1) of Definition 4.6 holds due to (a) in Theorem 6.5. For (2) of Definition 4.6, observe that

$$\mu(V_n^j) = \mu\left(\bigcup_{A\in\mathcal{C}_j(V_n)} A\right) \le \mu(V_n) + \mu\left(\bigcup_{A\in\mathcal{A}^j, A\cap B_{r_n}(\partial U_n)\neq\emptyset} A\right) \le \mu(V_n) + Cj^{-p'},$$

thanks to (b) in Theorem 6.5. A similar argument shows that $\{W_n\}$ is also a good neighborhood system.

Now we can apply Theorem 4.8 or 4.9 on $\{V_n\}$ and $\{W_n\}$ to obtain

$$\rho(\Lambda, \{W_n\}) = \rho(\Lambda, \{V_n\}) = \alpha_1.$$

It then follows from Proposition 6.6 that $\rho(\Lambda, \{U_n\}) = \alpha_1$. This concludes the proof of Theorem 6.5. $\square$

Similar to Theorem 4.17, when the extremal index θ is defined as

$$\theta = \lim_{n\to\infty} \mu_{U_n}(\tau_{U_n} > K_n)$$

for some sequence $K_n > \kappa_n^2$, the conditions on the right ϕ-mixing and V_n^j can be dropped. We thus obtain the following version of Theorem 4.17 for open sets $\{U_n\}$:

Theorem 6.7. *Assume that $T : \mathbf{M} \to \mathbf{M}$ preserves a measure μ that is left ϕ-mixing with $\phi(k) \leq Ck^p$ for some $C > 0$ and $p > 1$.*

Let $\varphi : \mathbf{M} \to \mathbb{R} \cup \{+\infty\}$ be a continuous function achieving its maximum on a measure zero set Λ. Let $\{u_n\}$ be a non-decreasing sequence of real numbers with $u_n \nearrow \sup \varphi$, such that the open sets U_n defined by (6.1) satisfy Assumption 6.6. Write κ_n for the smallest positive integer with $\operatorname{diam} \mathcal{A}^{\kappa_n} \leq r_n$ where $\{r_n\}$ is the sequence in Assumption 6.6. We assume that:

1. *$\kappa_n \mu(U_n)^{\varepsilon} \to 0$ for some $\varepsilon \in (0, 1)$;*
2. *the extremal index θ defined by (4.9) exists for some sequence $K_n > (\kappa_n)^2$.*

Then the exceedance rate of φ along $\{u_n\}$ exists and satisfies

$$\zeta(\varphi, \{u_n\}) = \rho(\Lambda, \{U_n\}) = \theta.$$

6.2 Escape rate under inducing

In this section, we will state a general theorem for the local escape rate under inducing. For this purpose, we consider a measure preserving dynamical system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ with $\tilde{\mu}$ being a probability measure. Given a measurable function $R : \tilde{\Omega} \to \mathbb{Z}^+$ consider the space $\Omega = \tilde{\Omega} \times \mathbb{Z}^+ / \sim$ with the equivalence relation $\sim$ given by

$$(x, R(x)) \sim (\tilde{T}(x), 0).$$

Define *the (discrete-time) suspension map over* $\tilde{\Omega}$ *with roof function* R as the measurable map T on the space Ω acting by

$$T(x, j) = \begin{cases} (x, j+1) & \text{if } j < R(x) - 1, \\ (\tilde{T}x, 0) & \text{if } j = R(x) - 1. \end{cases}$$

We will call Ω a *tower over* $\tilde{\Omega}$ and refer to the set $\Omega_k := \{(x, k) : x \in \tilde{\Omega}, k < R(x)\}$ as the *kth floor* where $\tilde{\Omega}$ can be naturally identified with the 0th floor called *the base of the tower.*

For $0 \leq k < i$, set $\Omega_{k,i} = \{(x, k) : R(x) = i\}$. The map

$$\Pi : (x, k) \mapsto x$$

is naturally viewed as a projection from the tower Ω to the base $\tilde{\Omega}$ and for any given set $U \subset \Omega$ we will write

$$\tilde{U} = \Pi(U).$$

The measure $\tilde{\mu}$ can be lifted to a measure $\hat{\mu}$ on Ω by

$$\hat{\mu}(A) = \sum_{i=1}^{\infty} \sum_{k=0}^{i-1} \tilde{\mu}(\Pi(A \cap \Omega_{k,i})).$$

It is easy to verify that $\hat{\mu}$ is T-invariant and if $\tilde{\mu}(R) = \int R\, d\tilde{\mu} < \infty$ then $\hat{\mu}$ is a finite measure. In this case, the measure

$$\mu = \frac{\hat{\mu}}{\tilde{\mu}(R)}$$

is a T-invariant probability measure on Ω.

We write $\tilde{U} = \Pi(U) \subset \tilde{\Omega}$, $\tilde{\Lambda} = \cap_n \tilde{U}_n$ and define $\tilde{\rho}(\tilde{\Lambda}, \{\tilde{U}_n\})$ to be the localized escape rate at $\tilde{\Lambda}$ for the system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$. The following theorem relates the escaped rate of the base system with that of the suspension. A similar result is obtained for continuous suspensions under the assumption that R is bounded, see [15].

Theorem 6.8. *Let (Ω, T, μ) be a discrete-time suspension over an ergodic measure preserving system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ with a roof function R satisfying the following assumptions:*

1. R has exponential tail: *there exists $C, c > 0$ such that $\tilde{\mu}(R > n) \le Ce^{-cn}$;*

2. exponential large deviation estimate: *for every $\varepsilon > 0$ small, there exists $C_\varepsilon, c_\varepsilon > 0$ such that the set*

$$B_{\varepsilon,k} = \left\{ y \in \tilde{\Omega} : \left| \frac{1}{n} \sum_{j=0}^{n-1} R(\tilde{T}^j y_0) - \frac{1}{\mu(\Omega_0)} \right| > \varepsilon \text{ for some } n \ge k \right\},$$

   satisfies $\tilde{\mu}(B_{\varepsilon,k}) \le C_\varepsilon e^{-c_\varepsilon k}$.

Then for every nested sequence $\{U_n\}$, we have

$$\rho(\Lambda, \{U_n\}) = \tilde{\rho}(\tilde{\Lambda}, \{\tilde{U}_n\}).$$

Proof. The result of this theorem is in fact hidden in the proof of Theorem 4 of [26] and Theorem 3.2 (1) in [9]. We include the proof here for completeness.

For every $y = (x, m) \in \Omega$, we take $y_0 = x \in \tilde{\Omega}$. Then we have

$$\tau_{U_n}(y) = -m + \sum_{j=0}^{\tilde{\tau}_{\tilde{U}_n}(y_0)-1} R(\tilde{T}^j(y_0)), \tag{6.8}$$

where $\tilde{\tau}$ is the return times defined for the system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$. By the Birkhoff ergodic theorem on $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$, we see that

$$\frac{1}{n}\sum_{j=0}^{n-1} R(\tilde{T}^j y_0) \to \int_{\tilde{\Omega}} R(y)\, d\tilde{\mu}(y) = \frac{1}{\mu(\Omega_0)},$$

where we apply the Kac's formula on the last equality and use the fact that μ is the lift of $\tilde{\mu}$.

On the other hand, since the return time function R has exponential tail, we get, for each $\varepsilon > 0$ and t large enough,

$$\mu((x, m) : m > \varepsilon t) \lesssim e^{-c\varepsilon t}.$$

To simplify notation, we introduce the set (n is fixed)

$$A_t = \left\{ y = (x, m) : m < \varepsilon t, \sum_{j=0}^{\tilde{\tau}_{\tilde{U}_n}(y_0)-1} R(\tilde{T}^j(y_0)) > (1+\varepsilon)t \right\} \cap B^c_{\varepsilon,k}.$$

Combine (6.8) with the previous estimates on $B_{\varepsilon,k}$, for $k = t(1+\varepsilon)$ we get

$$\left|\mu(\tau_{\tilde{U}_n} > t) - \mu(A_t)\right| \lesssim e^{-c\varepsilon t} + e^{-c_\varepsilon (1+\varepsilon)t}. \tag{6.9}$$

Note that A_t contains the set

$$A_t^- = \left\{ y : m < \varepsilon t, \tilde{\tau}_{\tilde{U}_n}(y_0) > \frac{(1+\varepsilon)t}{\mu^{-1}(\Omega_0) - \varepsilon} \right\},$$

and is contained in

$$A_t^+ = \left\{ y : m < \varepsilon t, \tilde{\tau}_{\tilde{U}_n}(y_0) > \frac{(1+\varepsilon)t}{\mu^{-1}(\Omega_0) + \varepsilon} \right\}.$$

Now we are left to estimate $\mu(A_t^\pm)$. Since μ is the lift of $\tilde{\mu}$, we have

$$\begin{aligned}\mu(A_t^\pm) =& \frac{1}{\tilde{\mu}(R)} \sum_{j=0}^{\infty} \sum_{i=0}^{\min(\varepsilon t, R_j)} \tilde{\mu}(T^{-i} A_t^\pm \cap \Omega_{0,i}) \\ =& \mu(\Omega_0)(1 + \mathcal{O}(\varepsilon t))\tilde{\mu}(\tilde{A}_t^\pm),\end{aligned} \tag{6.10}$$

where

$$\tilde{A}_t^\pm = \left\{ y_0 \in \Omega_0 : \tilde{\tau}_{\tilde{U}_n}(y_0) > \frac{(1+\varepsilon)t}{\mu^{-1}(\Omega_0) \pm \varepsilon} \right\}.$$

Let $\alpha = \tilde{\rho}(\tilde{\Lambda}, \{\tilde{U}_n\})$. Then we have (recall that $\tilde{\mu}(\tilde{U}_n)\mu(\Omega_0) = \mu(U_n)$)

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t\mu(U_n)} |\log \tilde{\mu}(\tilde{A}_t^\pm)| = \alpha \frac{(1+\varepsilon)}{1 \pm \varepsilon\mu(\Omega_0)}.$$

By (6.10), we get that

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t\mu(U_n)} |\log \mu(\tilde{A}_t^\pm)| = \alpha \frac{(1+\varepsilon)}{1 \pm \varepsilon\mu(\Omega_0)}.$$

For each $\varepsilon > 0$ we can take n_0 large enough, such that for $n > n_0$:

$$\alpha \frac{1+\varepsilon}{1 \pm \varepsilon\mu(\Omega_0)} \mu(U_n) < \min\{c\varepsilon, c_\varepsilon(1+\varepsilon)\}.$$

It then follows that the right-hand-side of (6.9) is of order $o(\mu(A_t^{\pm}))$. We thus obtain

$$\begin{aligned} \rho(\Lambda, \{U_n\}) &= \lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t\mu(U_n)} |\log \mu(\tau_{U_n} > t)| \\ &\in \left(\alpha \frac{(1+\varepsilon)}{1+\varepsilon\mu(\Omega_0)}, \alpha \frac{(1+\varepsilon)}{1-\varepsilon\mu(\Omega_0)} \right) \end{aligned}$$

for every $\varepsilon > 0$. This shows that $\rho(\Lambda, \{U_n\}) = \alpha = \tilde{\rho}(\tilde{\Lambda}, \{\tilde{U}_n\})$. □

6.3 Young Towers

Young's towers, also known as the Gibbs-Markov-Young structure, is first introduced by Young in [39] and [40]. Young's tower can be viewed as a discrete time suspension over a Gibbs-Markov system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$, such that the roof function R (in this case, it is usually call the *return time function*) is integrable with respect to the measure $\tilde{\mu}$.

Theorem 6.9. *Assume that T is a C^2 map modeled by Young's tower, such that the return time function R has exponential tail: there exists $\lambda \in (0,1)$ such that*

$$\tilde{\mu}(R > n) \lesssim \lambda^n.$$

Let $\{U_n \subset \tilde{\Omega}\}$ be a nested sequence of sets satisfying the assumption of Theorem 4.9 in the cylinder case, or Theorem 6.5 in the open set case. Then *the localized escape rate at* $\Lambda = \cap_n U_n$ *exists and satisfies*

$$\rho(\Lambda, \{U_n\}) = \alpha_1.$$

Proof. Young towers can be interpreted as discrete-time suspension over Gibbs-Markov maps. The exponential tail of $\mu(R > n)$ implies the exponential large deviation estimate (see for example [9] Appendix B). Therefore Theorem 6.9 immediately follows from Theorem 4.9, Theorem 6.5 and Theorem 6.8. □

6.4 Examples

6.4.1 Periodic and non-periodic points dichotomy

First we consider the case where $\Lambda = \{x\}$ is a singleton, and $U_n = B_{\delta_n}(x)$ is a sequence of balls shrinking to x. Alternatively one could take $\varphi(y) = g(d(y, x))$ for some function $g(x) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ achieving its maximum at 0 (for example, $g(y) = -\log y$) and let $u_n \nearrow \infty$ be a sequence of threshold tending to infinity. Then $U_n = \{y : \varphi(y) > u_n\}$ is a sequence of balls with diameter shrinking to zero.

This situation has been dealt with in [9] for certain interval maps, and in [26] for maps that are polynomially ϕ-mixing. A dichotomy is obtained: when x is non-periodic the local escape rate is 1; when x is periodic then $\rho(x) = 1 - \theta$ where

$$\theta = \theta(x) = \lim_{n\to\infty} \frac{\mu(U_n \cap T^{-p} U_n)}{\mu(U_n)}, \tag{6.11}$$

where p is the period of x. When μ is an equilibrium state for some potential function $h(x)$ with zero pressure, one has $\theta = e^{S_p h(x)}$ where S_p is the Birkhoff sum. See [9].

Note that if x is non-periodic then one naturally deduces that $\pi(U_n) \nearrow \infty$ (see for example [26, Lemma 1]). When x is periodic, in [25, Section 8.3] it is shown that $\hat{\alpha}_\ell = \theta^{l-1}$ is a geometric distribution. In particular one has $\sum_\ell \ell\hat{\alpha}_\ell < \infty$ and $\alpha_1 = 1 - \theta$. This leads to the following theorem:

Theorem 6.10. *Assume that*

1. *either μ is both left and right ϕ-mixing with $\phi(k) \leq Ck^{-m}$, $m > 1$;*

2. *or $(T, \mu, \mathcal{A})$ is a Gibbs-Markov system.*

Assume that $0 < r_n < \delta_n$ satisfies

$$\mu(B_{\delta_n + r_n}(x) \setminus B_{\delta_n - r_n}(x)) = o(1)\mu(B_{\delta_n}(x)).$$

Write κ_n for the smallest positive integer with $\operatorname{diam} \mathcal{A}^{\kappa_n} \leq r_n$. We assume that:

1. *$\kappa_n \mu(U_n)^\varepsilon \to 0$ for some $\varepsilon \in (0,1)$;*

2. *U_n has small boundary: there exists $C > 0$ and $m' > 1$, such that $\mu\left(\bigcup_{A\in\mathcal{A}^j, A\cap B_{r_n}(\partial U_n)\neq\emptyset} A\right) \leq Cj^{-m'}$ for all n and $j \leq \kappa_n$.*

3. *when x is periodic with period p, θ defined by (6.11) exists.*

Then we have

$$\rho(\{x\}, \{B_{\delta_n}(x)\}) = \alpha_1 = \begin{cases} 1 & \text{if } x \text{ is non-periodic} \\ 1 - \theta & \text{if } x \text{ is periodic} \end{cases}.$$

This theorem improves [26, Theorem 2] by dropping the assumption $\theta < 1/2$. Also note that such results can be generalized to interval maps which can be modeled by Young's towers using Theorem 6.9.

6.4.2 Cantor sets for interval expanding maps

For simplicity, below we will only consider the Cantor ternary set. However the argument below can be adapted to a large family of dynamically-defined Cantor set discussed in [17] with only minor modification.

Consider the uniformly expanding map $T(x) = 3x \mod 1$ defined on the unit interval $[0, 1]$. We take Λ to be the ternary Cantor set on $[0, 1]$, and define recursively: $U_0 = [0, 1]$; U_{n+1} is obtained by removing the middle third of each connected component of U_n. Then we have $\cap_n U_n = \Lambda$.

Theorem 6.11. *For the uniformly expanding map $T(x) = 3x \mod 1$ on $[0, 1]$, the Cantor ternary set Λ and the nested sets $\{U_n\}$, we have*

$$\rho(\Lambda, \{U_n\}) = \frac{1}{3}.$$

Proof. Let $\mathcal{A} = \{[0, 1/3), [1/3, 2/3), [2/3, 1]\}$ be a Markov partition for T, with respect to which the Lebesgue measure μ is exponentially ψ-mixing[1]. Below we will verify the assumptions of Proposition 4.16.

It is easy to see that $U_n \in \mathcal{A}^n$, i.e., $\kappa_n = n$. On the other hand, $\mu(U_n) = 2^n/3^n$ which shows that item (1) of Definition 4.6 is satisfied for any $\varepsilon \in (0, 1)$.

[1] $\psi-$mixing is a stronger assumption than both left and right $\phi-$mixing. See Appendix B

For item (2), note that $U_n^j = U^j$ which implies that

$$\mu(U_n^j) \leq \mu(U_n) + \mu(U_j) = \mu(U_n) + \left(\frac{2}{3}\right)^j.$$

We conclude that $\{U_n\}$ is a good neighborhood system.

The extremal index of Λ is studied in the recent work [17]. It is proven in [17, Theorem 3.3] that θ defined by (4.9) with $\kappa_n = n$ exists and satisfies

$$\theta = 1/3.$$

By Proposition 4.16 we conclude that $\rho(\Lambda, \{U_n\}) = 1/3$. □

6.4.3 Submanifolds of Anosov maps

In this section we consider the case where Λ is a submanifold for some Anosov map T. More importantly, we will show how our results can be applied to those cases where the extremal index θ is defined using time cut-off K_n that depends on U_n (see (4.9)).

Let $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ be an Anosov system on $\mathbb{T}^2$ and μ be the Lebesgue measure. It is well known that μ is exponentially ψ-mixing with respect to its Markov partition $\mathcal{A}$. Also denote by $\lambda > 1$ the eigenvalue of T. Following [13] we take Λ to be a line segment with finite length $l(\Lambda)$. We will lift Λ to $\hat{\Lambda} \subset \mathbb{R}^2$ and parametrize $\hat{\Lambda}$ by $p_1 + tv$ for some $p_1 \in \mathbb{R}^2$ and $t \in [0, l(\Lambda)]$. Write p_2 for the other end point of $\hat{\Lambda}$, that is, $p_2 = p_1 + l(\Lambda)v$.

Consider the function $\varphi_\Lambda(y) = -\log d(x, \Lambda)$ which achieves its maximum $(+\infty)$ on Λ. Write $v^*, * = s, u$ for the unit vector along the stable and unstable direction respectively. Then we have:

Theorem 6.12. *For the sequence* $\{u_n = \log n\}$*,*

1. *if* Λ *is not aligned with the stable direction* v^s *or the unstable direction* v^u *then* $\zeta(\varphi_\Lambda, \{u_n\}) = 1$*;*

2. *if* Λ *is aligned with the unstable direction but* $\{p_1 + tv^u, t \in \mathbb{R}\}$ *has no periodic point, then* $\zeta(\varphi_\Lambda, \{u_n\}) = 1$*;*

3. *if* Λ *is aligned with the stable direction but* $\{p_1 + tv^s, t \in \mathbb{R}\}$ *has no periodic point, then* $\zeta(\varphi_\Lambda, \{u_n\}) = 1$*;*

4. Λ *is aligned with* $v^*, * = s, u$ *and* L *contains a periodic point with prime period* q*, then* $\zeta(\varphi_\Lambda, \{u_n\}) = 1 - \lambda^{-q}$*;*

5. Λ *is aligned with the unstable direction* v^u*,* Λ *has no periodic points but* $\{p_1+tv^u, t \in \mathbb{R}\}$ *contains a periodic point of prime period* q*; if* $\Lambda \cap T^{-q}\Lambda = \emptyset$ *then* $\zeta(\varphi_\Lambda, \{u_n\}) = 1$*; if* $\Lambda \cap T^{-q}\Lambda \neq \emptyset$ *then* $\zeta(\varphi_\Lambda, \{u_n\}) = (1-\lambda^{-q})\frac{|p_2|}{l(\Lambda)}$*;*

6. Λ *is aligned with the stable direction* v^u*,* Λ *has no periodic points but* $\{p_1+tv^u, t \in \mathbb{R}\}$ *contains a periodic point of prime period* q*; if* $\Lambda \cap T^{-q}\Lambda = \emptyset$ *then* $\zeta(\varphi_\Lambda, \{u_n\}) = 1$*; if* $\Lambda \cap T^{-q}\Lambda \neq \emptyset$ *then* $\zeta(\varphi_\Lambda, \{u_n\}) = (1-\lambda^{-q})\frac{|p_2|}{l(\Lambda)}$*;*

Proof. We will only prove case (1), in which we will need the result of [13, Theorem 2.1 (1)]. The other cases use similar arguments and correspond to case (2) to (6) of [13, Theorem 2.1].

Below we verify the assumptions of Theorem 6.5.

Put $\delta_n = e^{-u_n}$. Then we see that $U_n = \{y : \varphi_\Lambda(y) > u_n\} = B_{\delta_n}(\Lambda)$. Since μ is the Lebesgue measure, it is straight forward to verify that Assumption 1 is satisfied with $r_n = (\delta_n)^2 = e^{-2u_n}$.

By the hyperbolicity of T, there exists $C > 0$ such that $\operatorname{diam} \mathcal{A}^n < C\lambda^{-n}$. This invites us to take

$$\kappa_n = \lfloor \frac{\ln C + 2u_n}{\ln \lambda} \rfloor + 1 = \mathcal{O}(\log n)$$

which guarantees that $\operatorname{diam} \mathcal{A}^{\kappa_n} < r_n$. On the other hand, $\mu(U_n) \lesssim e^{-u_n} l(\Lambda) = \mathcal{O}(1/n)$, so item (i) of Theorem 6.7 is satisfied for any $\varepsilon \in (0, 1)$.

We are left with the extremal index θ defined by (4.9). For this purpose we choose $K_n = (\log n)^5 \gg \kappa_n^2$. Now we estimate:

$$\begin{aligned} \mu_{U_n}(\tau_{U_n} \leq K_n) \leq & \frac{1}{\mu(U_n)} \sum_{j=1}^{(\log n)^5} \mu(U_n \cap T^{-j} U_n) \\ \lesssim & n \sum_{j=1}^{(\log n)^5} \mu(U_n \cap T^{-j} U_n) \\ = & o(1) \end{aligned}$$

where the last inequality follows from [13, Section 3.3, page 16]. This shows that

$$\theta = \lim_n \mu_{U_n}(\tau_{U_n} > K_n) = 1 - \lim_n \mu_{U_n}(\tau_{U_n} \leq K_n) = 1,$$

finishing the proof of (ii) of Theorem 6.7. We conclude that

$$\zeta(\varphi, \{\log n\}) = \theta = 1.$$

□

Appendix A

Extremal Index

For a comprehensive review of materials relating to the extremal index, see [33]. Conveniently, this survey also includes examples constructed from dynamical systems which are more in line with the discussion herein.

Definition A.1. *Let X_i a stationary process. One can show under certain conditions that $n\mu(X < u_n) \to \tau \iff \mu(M_n \leq u_n) \to e^{-\theta\tau}$ where M_n is the maximum process corresponding to X_i. In this case, we refer to θ as the* ***extremal index****.*

We will not discuss the role of the specific conditions required but instead reference condition $D(u_n)$ of [29] and note that this is a weaker assumption than ϕ-mixing. The necessary convergence to compound Poisson law for ϕ-mixing dynamical systems is discussed in [24, 25]

The extremal index in a process is generally intuitively viewed as referencing the reciprocal of the average size of clusters of occurrences of some rare event, although this is not always the case (see [36]). In Remark 2 of [25] a sufficient condition is given for equating the extremal index with the limiting reciprocal of the cluster size.

Despite its importance in the extreme value theory, there seems to be limited literature on estimation of the extremal index, see however [37]. In [12] some estimators are discussed based on a formula due to O'Brien (see [31] equation 3.2.4).

Appendix B

Mixing with Rate Functions

Mixing is a type of asymptotic independence. In the context of stochastic processes, this typically involves a sequence of random variables adapted to a filtration. Events describing random variables, measurable with respect to far apart sigma algebras, are then closer to being independent. That closeness is captured in a precise way by the rate function.

In the context of dynamical systems, we begin with a map, $T : X \to X$ and a partition $\mathcal{A}$ of X. As in the preliminaries we define the refinement

$$\mathcal{A} \vee \mathcal{B} := \{A \cap B | A \in \mathcal{A}, B \in \mathcal{B}\}$$

and the partition into $n-$cylinders by $\vee_{j=0}^{n-1} T^{-j}\mathcal{A}$, noting that the preimage of a partition is a partition.

Definition B.1. *We say that a measure, μ is $\alpha-$**mixing** with respect to T if*

$$\left|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)\right| \leq \alpha(k) \tag{B.1}$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\cup_j \mathcal{A}^j)$.

Early use of $\alpha-$mixing can be seen in a version of the central limit theorem proven by Rosenblatt [35]. Convergence to exponential distribution for hitting times to $n-$strings in $\alpha-$mixing processes is also shown by Abadi in [3]. We will briefly summarize how the mixing is used in the next appendix.

Definition B.2. *We say that a measure* μ *is* ***left*** $\phi-$***mixing*** *with respect to the map* T *and the partition* $\mathcal{A}$ *if*

$$\left|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)\right| \leq \phi(k)\mu(A) \tag{B.2}$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\cup_j \mathcal{A}^j)$.

We say that the measure μ *is* ***right*** $\phi-$***mixing*** *with respect to the map* T *and the partition* $\mathcal{A}$ *if*

$$\left|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)\right| \leq \phi(k)\mu(B) \tag{B.3}$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\cup_j \mathcal{A}^j)$.

Definition B.3. *We say that a measure is* $\psi-$***mixing*** *with respect to the map* T *and the partition* $\mathcal{A}$ *if*

$$\left|\mu(A \cap T^{-n-k}B) - \mu(A)\mu(B)\right| \leq \phi(k)\mu(A)\mu(B) \tag{B.4}$$

for all $A \in \sigma(\mathcal{A}^n)$, $B \in \sigma(\cup_j \mathcal{A}^j)$.

We first note that because μ is a probability measure, $\phi-$mixing is stronger than $\alpha-$mixing estimates regarding $\alpha-$mixing systems also apply to those that are left or right $\phi-$mixing.

Proposition B.4. *Let μ a T-invariant probability space and suppose that $\mathcal{A}$ is a partition such that T is left $\phi-$mixing with respect to $\mathcal{A}$. Suppose $A_{n_k} \in \sigma(\mathcal{A}^{n_k})$. Then we have*

$$\mu(\cap_{k=0}^{M} T^{-j_k} A_{n_k}) \leq \mu(A_{n_0}) \prod_{k=1}^{M} (\phi(\Delta_k) + \mu(A_{n_k}))$$

where $\Delta_k := j_k - j_{k-1} - n_{k-1} > 0$ is assumed and the sequences n_k, j_k are increasing.

Proof. We proceed by induction on M. Supposing that M=1, we have

$$\mu(A_{n_0} \cap T^{-j_1} A_{n_1}) \leq \mu(A_{n_0})\mu(A_{n_1}) + \mu(A_{n_0})\phi(j_1 - n_0) = \mu(A_0)(\phi(\Delta_1) + \mu(A_{n_1}))$$

where we have assumed without loss of generality by invariance that $j_0 = 0$. Now we have

$$\begin{aligned} \mu(\cap_{k=0}^{M} T^{-j_k} A_{n_k}) &= \mu(\cap_{k=0}^{M-1} T^{-j_k} A_{n_k} \cap T^{-j_M} A_{n_M}) \\ &\leq \mu(\cap_{k=0}^{M-1} T^{-j_k} A_{n_k})(\phi(\Delta_M) + \mu(A_{n_M})) \end{aligned}$$

by $\phi-$mixing (noting that $\cap_{k=0}^{M-1} T^{-j_k} A_{n_k} \in \sigma(\mathcal{A}^{n_k + j_k})$). Thus, by induction the result is proved □

The above bound gives detailed control in bounding dynamically defined sets like cylinders in the following way. Consider a cylinder, $C_n = \cap_{j=0}^{n-1} T^{-j} A_{a_j}$ where $A_{a_j} \in \mathcal{A}$. For clarity of exposition, we will denote C_n by $[a_0, ..., a_{n-1}]$. to union over all cylinders containing a particular substring, we will use asterisks.

For example,

$$[a_0, *_1, a_2] = \bigcup_{A_{a_1} \in \mathcal{A}} [a_0, a_1, a_2] = A_{a_0} \cap T^{-2} A_{a_2}. \tag{B.5}$$

Clearly deleting entries as such increases the measure, i.e.

$$\mu([a_0, *_1, a_2]) \geq \mu([a_0, a_1, a_2]).$$

and the mixing can then be used to estimate

$$\mu(A_{a_0} \cap T^{-2} A_{a_2}) \leq \mu(A_{a_0}) \left(\mu(A_{a_2}) + \phi(1)\right).$$

Detailed knowledge of ϕ, or the maximum measure of the sub-strings of a certain length cylinder allow for increasingly sharp bounds achieved by making informed choices about how to block off entries and introduce gaps.

Appendix C

Hitting Times Estimates

In this chapter we will summarize how mixing is used to show hitting times estimates. We will follow, more or less, the method used by Abadi in [3]. For a map, T and a parition $\mathcal{A}$ as in the previous section, we define the hitting time.

Definition C.1. *The hitting time of* $T : X \to X$ *to* $A \subset X$ *measurable is given by*

$$\tau_A(x) = \inf\{j \geq 1 | T^j(x) \in A\}. \tag{C.1}$$

Let $k > s$. Then, let $k = qs + r$ with $0 \leq r < s$. We have for an invariant measure, μ

$$\begin{aligned} \mu(\tau_A > js) &= \mu(\{\tau_A > (j-1)s\} \cap \{\tau_A \circ T^{(j-1)s} > s\}) \\ &\leq \mu(\{\tau_A > (j-1)s\} \cap \{\tau_A \circ T^{(j-1)s+2n} > s - 2n\}) \end{aligned}$$

and then we have, using the invariance, for $n \ll s$

$$\left|\mu(\tau_A > js) - \mu(\{\tau_A > (j-1)s\} \cap \{\tau_A \circ T^{(j-1)s+2n} > s-2n\})\right| \leq 2n\mu(A). \tag{C.2}$$

if μ is $\alpha-$mixing with respect to $T, \mathcal{A}$ then

$$|\mu(\tau_A > js) - \mu(\tau_A > s-2n)\mu(\tau_A > (j-1)s| \leq \alpha(n) + 2n\mu(A). \tag{C.3}$$

Applying this iteratively for $1 \leq j \leq q$, and then using standard arguments, Abadi was able to prove that the normalized limiting distribution for hitting time to $n-$cylinders is exponential. He, and others have made extensive use of this type of argument, which appears to be due to Bernstein [8].

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