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Abstract

Oscillation theory for canonical systems is developed. This is then applied to various topics related to semibounded systems and the essential spectrum. The correspondence between self-adjoint relations and self-adjoint operators coming from canonical systems is investigated. An upper bound on the number of solutions of a one-channel difference equation is obtained.

Chapter 1

Introduction

This dissertation is mainly concerned with the spectral theory of self-adjoint ordinary differential operators and their discrete analogues. The roots of this theory are in the nineteenth-century studies of Sturm-Liouville equations . In the early twentieth century, Weyl derived the spectral representation of arbitrary Sturm-Liouville operators. A spectral representation is a generalization, to a possibly infinite-dimensional setting, of the diagonalization of a self-adjoint matrix; for the operators being discussed here, this takes the form of a generalized Fourier transform. Next followed the mathematical foundation of quantum mechanics, in which observable quantities of a physical system are modeled by self-adjoint operators and the possible results of observation are determined by spectral properties of the corresponding operator. For example, the time-independent Schrödinger equation in one dimension, which is a kind of Sturm-Liouville equation, is used to model the energy of a particle restricted to a line, and the set of possible energy levels of the particle is the spectrum of the Schrödinger operator. Schrödinger operators are one of the most researched objects in spectral theory. Also important are the discrete analogues of Sturm-Liouville operators, which are called Jacobi operators.

All of the above equations, and others such as one-dimensional Dirac equations and Krein strings, can be explicitly rewritten as canonical systems. A canonical system is a differential equation of the form $Jf'(x) = -zH(x)f(x)$ where $x \in (a,b) \subset \mathbb{R}$, $z \in \mathbb{C}$, $f : (a,b) \to \mathbb{C}^2$, $H(x) \in M_2(\mathbb{R})$ is positive semidefinite and locally integrable, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In the rest of the introduction and most of this dissertation, I assume that $(a,b) = (0,\infty)$, $\operatorname{tr}(H) = 1$, and that the boundary condition $f_2(0) = 0$ has been imposed. I sometimes use the coefficient function H to denote the canonical system. A deep result of de Branges [11, Theorem 5.1] states that there is a one-to-one correspondence between such canonical systems and analytic functions $m : \mathbb{C}^+ \to \overline{\mathbb{C}^+}$, which are known as (generalized) Herglotz functions. Here $\overline{\mathbb{C}^+}$ denotes the closure of the open upper half plane in the extended complex plane. As evidence of the importance of Herglotz functions, and hence canonical systems, in spectral theory, all of the above operators have their spectral data encoded in Herglotz functions built out of certain solutions of the corresponding equation. Also, if A is any self-adjoint operator on a Hilbert space $\mathcal{H}$, then $z \mapsto \langle f, (A-z)^{-1}f\rangle$ is a Herglotz function for every $f \in \mathcal{H}$. The theorem of de Branges can be seen thus as a very general result in inverse spectral theory, in the sense that if arbitrary spectral data is presented in the form of a Herglotz function, then the theorem gives the existence of a unique canonical system with that spectral data.

One of the earliest results in the spectral theory of differential operators is Sturm's oscillation theory, which relates zeros of solutions to a Sturm-Liouville equation to spectral properties. Oscillation theory has been developed and applied in several contexts [2, 3, 4, 9, 14, 17, 18, 19, 22]. I, with Christian Remling, established a version of oscillation theory for canonical systems. Let E be the projection-valued measure corresponding to the canonical system H. In this context, $\dim E(s,t)$, when finite, is equal to the number of eigenvalues in (s,t), and $\dim E(s,t)$ is infinite when the spectrum in (s,t) has an accumulation point. The set of accumulation points of the spectrum is the essential spectrum, σ_{ess}.

Theorem 1.1 ([12]). *Assume it is not the case that $H(x)$ is eventually a constant, rank-one matrix. Let $u(x;t)$ be the solution $Ju'(x) = -tH(x)u(x)$, $t \in \mathbb{R}$, such that $u(0;t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Write u in polar coordinates as $u(x;t) = r(x;t)\begin{pmatrix} \cos\theta(x;t) \\ \sin\theta(x;t) \end{pmatrix}$ with $r > 0$, $\theta(0;t) = 0$, and θ continuous. Then, if $-\infty < s < t < \infty$, $\dim E(s,t) = \lim_{x\to\infty}\lfloor \frac{1}{\pi}(\theta(x;t) - \theta(x;s))\rfloor$.*

Chapter 3 covers the basics of oscillation theory for canonical systems.

This version of oscillation theory has been the key tool in deriving several results. Remling and I first applied oscillation theory to characterizing semibounded canonical systems. A result due to Winkler states that a canonical system with semibounded spectrum has determinant zero throughout [24]. This is fascinating because it means that that if the determinant of a semibounded canonical system is perturbed to be positive on an arbitrarily small set, then, as long as the set has positive measure, the spectrum becomes unbounded above and below. Below, a simple, oscillation-theoretic proof of this is obtained that shows, more generally, that if $\dim E(-t,c) = o(t)$ as $t \to \infty$, then the determinant is identically zero. This claim follows from Theorem 4.1.

The key result involving semibounded canonical systems is that the negative spectrum of a canonical system consists of exactly $N \geq 0$ points if and only if, for every x, $H(x)$ can be written as the projection onto $(\cos\varphi(x), \sin\varphi(x))^t$ for some decreasing function $\varphi(x)$ with $\varphi(0+) \in (-\pi/2, \pi/2]$ and

$$-(N-1)\pi - \frac{\pi}{2} > \varphi(\infty) \geq -N\pi - \frac{\pi}{2}$$

such that φ has no jumps of size $\geq \pi$ (with one exceptional case when H is identically the projection on $(1,0)^t$). See Theorem 4.3 below. This theorem had already been obtained by Winkler and Woracek using unrelated methods [24, 26], but the oscillation-theoretic proof seems simpler and yields more information; namely, it shows that $\varphi(x) = \theta(x, -\infty) + \pi/2$. These and other closely related results are derived in Chapter 4.

New results obtained via oscillation theory are estimates for the bottom of the essential spectrum of a nonnegative canonical system in terms of the asymptotics of φ and, as a result, a criterion for such a canonical system to have purely discrete spectrum.

Theorem 1.2 ([12]). *Suppose H is a canonical system with nonnegative spectrum, and write $H(x)$ as the projection onto $\begin{pmatrix} \cos\phi(x) \\ \sin\phi(x) \end{pmatrix}$, as above. Let $A = \limsup_{x\to\infty} x(\varphi(x) - \varphi(\infty))$ and $B = \liminf_{x\to\infty} x(\varphi(x) - \varphi(\infty))$. Then $\frac{1}{4A} \leq \min \sigma_{ess} \leq \frac{1}{A}$ and $\min \sigma_{ess} \leq \frac{1}{4B}$. The spectrum is purely discrete if and only if $A = B = 0$.*

See Theorems 5.1, 5.2, and 5.4. Much of the proof is fueled by the insight that the differential equation satisfied by θ for $t < \min \sigma_{ess}$ is asymptotically comparable to a well-known Riccati equation that can be solved explicitly.

This is then used to obtain a criterion for a Schrödinger operator, assuming it is bounded below, to have purely discrete spectrum in terms of a certain solution to the Schrödinger equation. See [12] and Theorem 5.5 below. This follows because a

Schrödinger equation can be rewritten as a canonical system using solutions to the equation at zero energy.

Soon after our initial investigations, Romanov and Woracek obtained new results about canonical systems with purely discrete spectrum. Among other things, they proved that the spectrum of a canonical system corresponding to $H = \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix}$ is purely discrete if and only if

$$\lim_{x\to\infty} x \int_x^\infty h_1(y)dy = 0,$$

under the assumption that $h_1 \in L^1(0,\infty)$. Notably, this limit does not depend on the off-diagonal elements of H and, hence, their result implies, assuming $h_1 \in L^1(0,\infty)$, that the canonical system H has purely discrete spectrum if and only if the spectrum of the canonical system $H_d = \begin{pmatrix} h_1 & 0 \\ 0 & h_3 \end{pmatrix}$ is purely discrete. Their arguments rely on applying tools from abstract operator theory to integral operators [15].

A simple oscillation-theoretic argument shows that if H_d has purely discrete spectrum, then so does H. This argument actually proves the stronger claim that $\dim E(0,t)$ and $\dim E(-t,0)$ are bounded above by $\dim E_d(0,2t)+1$ for any $t>0$, and this in turn implies that $M(H_d)/2 \leq M(H)$ where $M(H) = \min\{|t| : t \in \sigma_{ess}(H)\}$. Under certain assumptions on H, $\dim E_d(0,t)$ is bounded above by a spectral projection corresponding to H. For more information on these results, see Propositions 6.1 and 6.2.

The fact that H_d has purely discrete spectrum when H does and $h_1 \in L^1(0,\infty)$ is far less trivial to prove than the converse. However, oscillation theory yields a stronger result that allows the bottoms of their essential spectra to be compared in the following way.

Theorem 1.3 ([13]). *Assume* $h_1 \in L^1(0,\infty)$. *Let* $M(H) = \min\{|t| : t \in \sigma_{ess}(H)\}$. *Let* $H_d = \begin{pmatrix} h_1 & 0 \\ 0 & h_3 \end{pmatrix}$ *if* $H = \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix}$. *Then* $\frac{1}{2}M(H_d) \le M(H) \le \frac{2}{3-\sqrt{5}}M(H_d)$.

This and other related results can be found in Chapter 6. Note Theorem 6.7, which implies that the constant in the upper bound can be replaced with 1 when H has finite negative spectrum.

The last main chapter, Chapter 7, concerns the development of results like Theorem 1.2 for diagonal systems. Once again a crucial insight for the development is the comparison of the differential equation for θ with a Riccati equation. An interesting connection to Schrödinger operators was discovered during this research.

Theorem 1.4 ([13]). *Assume* $h_1 \in L^1(0,\infty)$. *Let* $\mathcal{L}(t)$ *be a self-adjoint Schrödinger operator on* $(0,\infty)$ *with potential* $-t^2 h_1$ *and any boundary condition at* 0. *Then*

$$M(H_d) = \sup\{t \ge 0 : \textit{the spectrum of } \mathcal{L}(t) \textit{ in } (-\infty, 0) \textit{ is finite}\}.$$

The form of the relation between $M(H_d)$ and the asymptotics of φ is quite similar to that in Theorem 1.2.

Theorem 1.5 ([13]). *Assume* $h_1 \in L^1(0,\infty)$. *Let*

$$A = \limsup_{x\to\infty} x \int_x^\infty h_1(y)dy$$

and

$$B = \liminf_{x\to\infty} x \int_x^\infty h_1(y)dy.$$

Then $\frac{1}{2\sqrt{A}} \le M(H_d) \le \frac{1}{\sqrt{A}}$ *and* $M(H_d) \le \frac{1}{2\sqrt{B}}$.

Diagonal canonical systems can be transformed into nonnegative canonical systems as in [7, 12]. It follows that bounds on $M(H_d)$ could also be obtained via Theorem 1.2. In general, these are different than the ones in Theorem 1.5. This topic is discussed after the proof of Theorem 7.3.

Various transformations can also be done to a canonical system that allow the above results to be applied to systems where it is not necessarily the case that $h_1 \in L^1(0, \infty)$. This is discussed at the end of Chapter 7.

An immediate consequence of Theorems 1.3 and 1.5 is the discreteness criterion of Romanov and Woracek mentioned earlier. The value of the development here lies in the information gained about the bottom of the essential spectrum and in the different perspective, from the point of view of oscillation theory.

Corollary 1.6 ([15, 13]). *The spectrum of a canonical system corresponding to* $H = \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix}$ *with* $h_1 \in L^1(0, \infty)$ *is purely discrete if and only if* $\lim_{x\to\infty} x \int_x^\infty h_1(y)dy = 0$.

Chapter 8 addresses the question of whether the standard procedure of obtaining self-adjoint operators from canonical systems produces all possible self-adjoint operators associated with the differential equation. A large part of this problem is to give a precise formulation of the question. The question is formulated in Chapter 8 as an abstract problem involving self-adjoint operators and relations, not necessarily those coming from a canonical system. An affirmative answer to that formulation of the question is obtained.

Chapter 9 concerns one-channel operators. One-channel operators are a higher-dimensional generalization of Jacobi operators that were introduced recently by Sadel in connection with random systems from mathematical physics [16]. They are self-adjoint operators corresponding to difference equations of the form

$$zu_n = A_n u_{n+1} + A_{n-1}^* u_{n-1} + B_n u_n$$

with $A_n, B_n \in \mathbb{R}^{d\times d}$, $\operatorname{rk} A_n = 1$, and $B_n^* = B_n$. Jacobi operators correspond to the case $d = 1$. Many of their spectral properties are determined by two-dimensional transfer matrices, which suggests that they have a rich spectral theory analogous to classical one-dimensional operators. A very basic issue is whether the difference equation has the right number, $2d$, of linearly independent solutions. An example is provided where there are fewer than $2d$ solutions. It is then shown that, in general, there are at most $2d$ solutions for $z \notin \mathbb{R}$. For classical Jacobi difference equations, the fact that the solution space is 2-dimensional follows easily from basic algebra. The proof of the upper bound $2d$ for $d > 1$ is not such an elementary argument.

Chapter 2

Preliminary definitions, results, and notation

The purpose of this chapter is to establish notation and to recall some basic definitions and results. Most of the results referred to can be found in [11].

A *canonical system* is a differential equation of the form

$$Ju'(x) = -zH(x)u(x), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Here, $x \in (a,b) \subset \mathbb{R}$, $z \in \mathbb{C}$, $H \in L^1_{loc}(a,b)$, $H(x) \in \mathbb{R}^{2\times 2}$, $H(x) \geq 0$, and $H(x) \neq 0$ for almost every x. A function $u : (a,b) \to \mathbb{C}^2$ is called a solution of (2.1) if u is (locally) absolutely continuous and (2.1) holds for almost every $x \in (a,b)$. Given any $c \in (a,b)$ and $v \in \mathbb{C}^2$, there is a unique solution of (2.1) such that $u(c) = v$.

Let V be the seminormed vector space of all Borel measurable $f : (a,b) \to \mathbb{C}^2$

such that $\int_a^b f(x)^* H(x) f(x)\,dx < \infty$ with the seminorm

$$\|f\| = \left(\int_a^b f(x)^* H(x) f(x)\,dx \right)^{1/2}.$$

Here $f(x)^* = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}^* = (\overline{f_1(x)}, \overline{f_2(x)})$. Let $N = \{f \in V : \|f\| = 0\}$. Define $L^2_H(a,b) = V/N$. $L^2_H(a,b)$ is a Hilbert space.

Suppose that H is integrable near a. Then solutions of (2.1) have continuous extensions to $[a,b)$. Suppose $f_0 : (a,b) \to \mathbb{C}^2$ is absolutely continuous, $f_0 \in L^2_H(a,b)$, and for some $g \in L^2_H(a,b)$, $Jf_0'(x) = -H(x)g(x)$ for almost every $x \in (a,b)$. Then f_0 has a continuous extension to $[a,b)$. Analogous results holds if H is integrable near b.

Given a Hilbert space $\mathcal{H}$, a *relation* R is a linear subspace of $\mathcal{H} \oplus \mathcal{H}$. The *adjoint* of a relation R is defined to be $R^* = \{(f,g) \in \mathcal{H} \oplus \mathcal{H} : \langle f, k \rangle = \langle g, h \rangle$ for all $(h,k) \in R\}$, where $\langle f, k \rangle$ denotes the inner product in $\mathcal{H}$. R is called *self-adjoint* if $R = R^*$. In particular, an operator is self-adjoint if and only if its graph is a self-adjoint relation.

The following relations coming from canonical systems will be considered. Given H, define

$$\begin{aligned} \mathcal{T} = \{(f,g) \in L^2_H(a,b) \oplus L^2_H(a,b) : f \text{ has an AC representative} \\ f_0 \text{ such that } Jf_0'(x) = -Hg(x) \text{ for a.e. } x \in (a,b)\}. \end{aligned}$$

For $\beta \in \mathbb{R}$, define $e_\beta = (\cos\beta, \sin\beta)^t$, and then let e_β^* denote the conjugate transpose, as above. If $H \in L^1(a,b)$, then

$$\begin{aligned}\mathcal{S}^{(\beta)} = \{(f,g) \in \mathcal{T} : f &\text{ has an AC representative } f_0 \text{ such that}\\ &Jf_0'(x) = -Hg(x) \text{ for a.e. } x \in (a,b) \text{ and}\\ &e_\beta^* Jf_0(b) = 0 = e_0^* Jf_0(a)\}\end{aligned}$$

defines a self-adjoint relation. If H is integrable near a, but not near b, then a self-adjoint relation is defined by

$$\begin{aligned}\mathcal{S} = \{(f,g) \in \mathcal{T} : f &\text{ has an AC representative } f_0 \text{ such that}\\ &Jf_0'(x) = -Hg(x) \text{ for a.e. } x \in (a,b) \text{ and } e_0^* Jf_0(a) = 0\}.\end{aligned}$$

If H is not integrable near a or b, then

$$\begin{aligned}\mathcal{S} = \{(f,g) \in \mathcal{T} : f &\text{ has an AC representative } f_0 \text{ such that}\\ &Jf_0'(x) = -Hg(x) \text{ for a.e. } x \in (a,b)\}\end{aligned}$$

is a self-adjoint relation. These claims requires some work to prove; see Chapter 2 of [11].

A self-adjoint operator can be obtained from a self-adjoint relation by the following procedure. Define the *multi-valued part* of a relation R to be

$$R(0) = \{g \in \mathcal{H} : (0,g) \in R\}.$$

Given a self-adjoint relation R, set $\mathcal{H}_1 = R(0)^\perp$ and $S = \{(f,g) \in R : f,g \in \mathcal{H}_1\}$. Then S is the graph of a self-adjoint operator on the Hilbert space $\mathcal{H}_1$.

The spectral properties of the self-adjoint operators associated with a canonical system are encoded in Herglotz functions. Let $\mathbb{C}^+$ denote the open upper half plane. A *Herglotz function* is an analytic function $m : \mathbb{C}^+ \to \overline{\mathbb{C}^+}$, where the closure is taken in the extended complex plane. If $H \in L^1(a,b)$ and $\beta \in \mathbb{R}$, then a Herglotz function $m^{(\beta)}(z)$ is defined by $m^{(\beta)}(z) = f_1(a,z)/f_2(a,z)$ where $f(x,z)$ is the solution of (2.1) with the value $f(b,z) = e_\beta$. If H is integrable near a, but not near b, then for any $z \notin \mathbb{R}$, there is a unique $L^2(a,b)$ solution $f(x,z)$ of (2.1). In this situation, $m(z) = f_1(a,z)/f_2(a,z)$ defines a Herglotz function. If H is not integrable near a or b and $c \in (a,b)$, then two Herglotz functions are defined by $m_\pm(z) = \pm f_1^{(\pm)}(c,z)/f_2^{(\pm)}(c,z)$ where $f^{(-)}(x,z)$ is the solution of (2.1) in $L^2(a,c)$ and $f^{(+)}(x,z)$ is the solution of (2.1) in $L^2(c,b)$.

Any Herglotz function other than the constant $m(z) = \infty$ can be written uniquely in the form

$$m(z) = a + bz + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\rho(t)$$

with $a \in \mathbb{R}$, $b \geq 0$, and ρ a positive Borel measure on $\mathbb{R}$ (possibly $\rho = 0$) with $\int \frac{d\rho(t)}{1+t^2} < \infty$ [21]. If $m(z) = \infty$ (Herglotz functions that take on values in $\mathbb{R} \cup \{\infty\}$ are always constant), then we associate with m the data $a = \infty$, $b = 0$, and the zero measure $\rho = 0$. If $H \in L^1(a,b)$ and $\beta \in \mathbb{R}$, then the measure $\rho^{(\beta)}$ associated with $m^{(\beta)}$ is discrete, and the self-adjoint operator $S^{(\beta)}$ corresponding to the relation $\mathcal{S}^{(\beta)}$ is unitarily equivalent to multiplication by the variable in $L^2(\rho^{(\beta)})$. If H is integrable near a, but not near b, then the self-adjoint operator S corresponding to the relation $\mathcal{S}$ is unitarily equivalent to multiplication by the variable in $L^2(\rho)$, where ρ is the measure associated with the m function defined above. If H is not integrable near a or b, then a spectral representation of the corresponding self-adjoint operator S

can be built using $m_\pm$, but the fine details of this will not be used here.

The discrete spectrum of a canonical system consists of isolated, simple eigenvalues. The essential spectrum of a canonical system, denoted by $\sigma_{ess}(H)$, is the set of accumulation points of the spectrum. Note that our notation $\sigma_{ess}(H)$ refers to the essential spectrum of the corresponding self-adjoint operator $\mathcal{S}$ or $\mathcal{S}^{(\beta)}$, and more proper notation might be $\sigma_{ess}(\mathcal{S})$ or $\sigma_{ess}(\mathcal{S}^{(\beta)})$, but our notation is more commonly used in the context of canonical systems. We similarly use the notation $\sigma(H)$ to denote the spectrum of the corresponding self-adjoint operator. We define

$$M(H) = \min\{|t| : t \in \sigma_{ess}(H)\},$$

the bottom of the essential spectrum in absolute value.

A fundamental result of de Branges states that the map $H \mapsto m$ sets up a one-to-one correspondence between the set of all Herglotz functions and the set of all canonical systems H on $(a,b) = (0,\infty)$ with $\operatorname{tr} H(x) \equiv 1$. The correspondence $H \mapsto m$ is also a homeomorphism with respect to certain natural topologies. The relevant spaces are in fact compact metric spaces, but the exact metrics are not used below. What is used is the fact that a sequence $m_n(z)$ of Herglotz functions converges locally uniformly to $m(z)$ if and only if $\int_0^\infty f(x)^* H_n(x) f(x)\, dx$ converges to $\int_0^\infty f(x)^* H(x) f(x)\, dx$ for all continuous functions $f : (0,\infty) \to \mathbb{C}^2$ with compact support.

An important notion is that of a singular interval of a canonical system. If $\operatorname{tr} H(x) \equiv 1$, a *singular interval* is a maximal open interval $I \subset (a,b)$ such that

$H(x) = P_\alpha$ for almost every $x \in I$. Here P_α is the projection

$$P_\alpha = \begin{pmatrix} \cos^2 \alpha & \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha & \sin^2 \alpha \end{pmatrix}$$

and α is a constant called the *type* of the singular interval. A point that is not in any singular interval is called a *regular point*.

Below, unless otherwise mentioned, it is assumed that H is trace-normed, i.e. $\operatorname{tr} H(x) \equiv 1$, and that the basic interval (a, b) starts at $a = 0$. The primary focus is on half-line systems, $(a, b) = (0, \infty)$, with the boundary condition $u_2(0) = 0$, the one used in the above definition of $\mathcal{S}$. An arbitrary trace-normed H can be written in the form

$$H(x) = \begin{pmatrix} \cos^2 \varphi(x) & g(x) \sin \varphi(x) \cos \varphi(x) \\ g(x) \sin \varphi(x) \cos \varphi(x) & \sin^2 \varphi(x) \end{pmatrix}$$

with Borel measurable $g : (a, b) \to [0, 1]$ and $\varphi : (a, b) \to \mathbb{R}$. The set of trace-normed H on $(0, \infty)$ with nonnegative spectrum is denoted by $\mathcal{C}_+$.

Chapter 3

Oscillation theory

Given a real, non-trivial solution u of (2.1) for $z = t \in \mathbb{R}$, write u in polar coordinates as $u = Re_\theta$, with $R(x) > 0$, $\theta(x)$ continuous, and $e_\theta = (\cos\theta, \sin\theta)^t$. It follows that θ is absolutely continuous and solves the *Prüfer equation*

$$\theta'(x) = te^*_{\theta(x)}H(x)e_{\theta(x)}. \tag{3.1}$$

Proposition 3.1 ([12])**.** *Let $\theta(x;t)$ be a solution for $x > 0$ of (3.1) with initial value $\theta(0;t) = \alpha$, where α does not depend on t. Then $\theta(x;t)$ is increasing as a function of $t \in \mathbb{R}$. Then as a function of $x > 0$, $\theta(x;t)$ is increasing if $t \geq 0$ and decreasing if $t \leq 0$. As a function of $t \in \mathbb{R}$, $\theta(x;t)$ is increasing, and $t \mapsto \theta(x;t)$ is strictly increasing for $x > 0$ except when $(0,x)$ is contained in a singular interval of type $\alpha + \pi/2$.*

Proof. The monotonicity of $\theta(x;t)$ as a function of x is immediate. The monotonicity as a function of t can be seen in one of two ways. First, $t \mapsto \theta(x;t)$ is increasing by the comparison principle for first order ordinary differential equations. Suppose $t \mapsto \theta(L;t)$ is constant for $t \in (a,b)$ and fixed $L > 0$. Then the problem on $(0, L)$ with boundary condition $\beta \equiv \theta(L; a) \bmod \pi$ at $x = L$ would have an eigenvalue at each $t \in (a, b)$ unless $H(x)u(x;t) = 0$ on $(0, L)$. The condition that $H(x)u(x;t) = 0$ on $(0, L)$ implies u is constant on $(0, L)$, since u solves the canonical system, and thus $H(x) = P_{\alpha+\pi/2}$ on $(0, L)$. Now, problems on bounded intervals have purely discrete spectrum, so there is a contradiction unless $H(x) = P_{\alpha+\pi/2}$ on $(0, L)$. The stated properties of $t \mapsto \theta(x;t)$ also follow from the fact that $\frac{d}{dt}\theta(x;t) = \int_0^x u(y;t)^* H(y) u(y;t)\, dy / R(x;t)^2$. To derive this formula, calculate that $-\frac{d}{dx}(u(x;t)^* J \frac{d}{dt} u(x;t)) = u(x;t)^* H(x) u(x;t)$ from the differential equation and directly that $-u(x;t)^* J \frac{d}{dt} u(x;t) = R(x;t)^2 \frac{d}{dt}\theta(x;t)$, and then use the fact that $\frac{d}{dt}\theta(0;t) = 0$, which holds because $\theta(0;t)$ is constant. $\square$

The following lemma is the basis for oscillation theory applied to canonical systems. Here $E_L^{(\beta)}$ is the spectral projection for the self-adjoint operator $S^{(\beta)}$ coming from the canonical system on the bounded interval $(0, L)$ with boundary conditions $u_2(0) = 0 = e_\beta^* J u(L)$.

Lemma 3.2 ([12]). *Let $\theta(x;t)$ be the solution of (3.1) with $\theta(0;t) = 0$. Then*

$$\dim E_L^{(\beta)}[s, t) = \left\lceil \frac{1}{\pi}\left(\theta(L;t) - \beta\right) \right\rceil - \left\lceil \frac{1}{\pi}\left(\theta(L;s) - \beta\right) \right\rceil .$$

Proof. If $(0, L)$ is a singular interval of type $\pi/2$, then the spectrum is empty, due to the operator acting on a 0-dimensional space, and $\theta(L;t) = 0$ for every t. Suppose $(0, L)$ is not a singular interval of type $\pi/2$. Then λ is an eigenvalue if and only if $\theta(L;\lambda) \equiv \beta \bmod \pi$. To see this first note that if λ is an eigenvalue, then $\theta(L;\lambda) \equiv \beta \bmod \pi$ by the choice of the boundary condition. Conversely, if $\theta(L;\lambda) \equiv \beta \bmod \pi$, then the corresponding solution of the canonical system lies in the domain of the self-adjoint relation and is thus an eigenvector of the operator (note that this solution does not represent 0 in the Hilbert space because of the assumption that $(0, L)$ is not a singular interval of type $\pi/2$). By the monotonicity and continuity of θ, the function $G(\lambda) = \left\lceil \frac{1}{\pi} (\theta(L;\lambda) - \beta) \right\rceil - \left\lceil \frac{1}{\pi} (\theta(L;s) - \beta) \right\rceil$ has the value $G(s) = 0$, jumps by one at each $\lambda \in [s, t)$ such that $\theta(L;\lambda) \equiv \beta \bmod \pi$, and is increasing, piecewise-constant, and left-continuous. So, $\left\lceil \frac{1}{\pi} (\theta(L;t) - \beta) \right\rceil - \left\lceil \frac{1}{\pi} (\theta(L;s) - \beta) \right\rceil$ is equal to the number of eigenvalues in $[s, t)$. □

The next step is to relate the spectral projections of the half-line operators with the solutions of (3.1). Here a half-line operator refers to a canonical system on $(0, \infty)$. Many of the steps in the following proof are motivated by Weidmann's proof of the corresponding classical result about Sturm-Liouville operators [22, Chapter 14].

Theorem 3.3 ([12])**.** *Suppose that* $(0, \infty)$ *does not end with a singular half line* (L, ∞)*. Let* E *denote the spectral projection of the half-line operator, and let* $\theta(x;t)$ *be the solution of* (3.1) *with* $\theta(0;t) = 0$*. Then*

$$\dim E(s, t) = \lim_{L \to \infty} \left\lfloor \frac{1}{\pi} (\theta(L;t) - \theta(L;s)) \right\rfloor . \tag{3.2}$$

Proof. Let

$$F(L) = \frac{1}{\pi} (\theta(L;t) - \theta(L;s)) .$$

It suffices to prove the following two inequalities:

$$\lfloor F(L) \rfloor \leq \dim E(s, t) \quad \text{for all } L > 0; \tag{3.3}$$

$$\dim E(s, t) \leq \liminf_{L \to \infty} \lceil F(L) \rceil - 1. \tag{3.4}$$

To show that these inequalities imply (3.2), assume they hold and note first that since then

$$\begin{aligned}\dim E(s,t) &\le \liminf_{L\to\infty} \lceil F(L)\rceil - 1\\ &\le \limsup_{L\to\infty} \lceil F(L)\rceil - 1\\ &\le \sup_{L>0} \lceil F(L)\rceil - 1\\ &\le \sup_{L>0} \lfloor F(L)\rfloor \le \dim E(s,t),\end{aligned}$$

equality holds everywhere. So, $\lim_{L\to\infty}\lceil F(L)\rceil$ exists in $\mathbb{Z}\cup\{\infty\}$. If $F(L)\notin\mathbb{Z}$ eventually, then it is clear that $\lim_{L\to\infty}\lfloor F(L)\rfloor$ exists. Suppose there exists $L_n \nearrow \infty$ with $F(L_n)\in\mathbb{Z}$. If $\lim_{L\to\infty}\lceil F(L)\rceil$ were finite, then $F(L_n)$ would equal that limit eventually, which would lead to a contradiction with (3.3) and (3.4). If $\lim_{L\to\infty}\lceil F(L)\rceil=\infty$, then clearly $\lim_{L\to\infty}\lfloor F(L)\rfloor=\infty$. In any case, $\lim_{L\to\infty}\lfloor F(L)\rfloor$ exists. It follows that $\dim E(s,t)=\lim_{L\to\infty}\lfloor F(L)\rfloor$, assuming (3.3) and (3.4).

First, we prove (3.3), by looking at the problem on $[0,L]$ with the boundary condition at L chosen so that t is an eigenvalue. So, given $L>0$, define $\beta\in[0,\pi)$ by writing $\theta(L;t)=n\pi+\beta$, $n\in\mathbb{Z}$. Then t is an eigenvalue of the problem with boundary condition $e_\beta^* Ju(L)=0$ as long as $(0,L)$ is not a singular interval of type $\pi/2$. If $H\equiv P_{e_2}$ on $(0,L)$, then $F(L)=0$ and (3.3) is immediate. Assume $(0,L)$ is not a singular interval of type $\pi/2$. By Lemma 3.2,

$$\dim E_L^{(\beta)}[s,t] = 1+n-\left\lceil \frac{1}{\pi}\left(\theta(L;s)-\beta\right)\right\rceil = \lfloor F(L)\rfloor + 1.$$

Suppose that (3.3) does not hold. Then

$$\dim\mathcal{M}\ge 2,\quad \mathcal{M}=R(E_L^{(\beta)}[s,t])\ominus R(E(s,t)). \tag{3.5}$$

Here, $R(E_L^{(\beta)}) \subset L^2_H(0,L)$ is identified with a subspace of $L^2_H(0,\infty)$ by extending elements of $R(E_L^{(\beta)})$ by the zero function on (L,∞), and, similarly, the self-adjoint relation $\mathcal{S}_L^{(\beta)}$ is identified with a relation on $L^2_H(0,\infty)$.

Since the elements of $R(E_L^{(\beta)}[s,t])$ are linear combinations of eigenvectors of the operator $S_L^{(\beta)}$, they are contained in $D(\mathcal{S}_L^{(\beta)})$, the domain of the self-adjoint relation. If (f,g) is any element of $\mathcal{S}_L^{(\beta)}$, then the standard, absolutely continuous representative $f(x)$ of $f \in L^2_H(0,L)$, determined by (f,g) as in [11, Lemma 2.1], satisfies the boundary condition $e_\beta^* Jf(L) = 0$. So, by (3.5), there is a non-zero element $f \in \mathcal{M}$ with $f(L) = 0$. This element, identified as above with an element of $L^2_H(0,\infty)$, lies in $D(\mathcal{S})$, the domain of the self-adjoint relation for the problem on the half line $(0,\infty)$.

Let $c = (s+t)/2$ and $g = S_L^{(\beta)} f$, the image of f under the operator $S_L^{(\beta)}$. Since $f = E_L^{(\beta)}[s,t]f$, it follows from functional calculus for $S_L^{(\beta)}$ that $\|g - cf\| \leq (t-s)/2\|f\|$. On the other hand, (f,g) is identified with an element of the self-adjoint relation $\mathcal{S}$ on the half line, after extending both functions by zero for $x > L$. So, $g = Sf + h$ for some $h \in \mathcal{S}(0)$, the multi-valued part of $\mathcal{S}$. Thus, since $f, Sf \in \overline{D(\mathcal{S})} = \mathcal{S}(0)^\perp$, $\|g - cf\|^2 \geq \|(S-c)f\|^2$. Using the functional calculus for S and the fact that $E(s,t)f = 0$, it follows that $\|(S-c)f\|^2 \geq \left(\frac{t-s}{2}\right)^2 \|f\|^2$. Hence, $\|g - cf\| = (t-s)/2\|f\|$.

Thus, $0 = \langle f, (S-c)^2 f\rangle - \langle f, (\frac{t-s}{2})^2 f\rangle = \langle f, (S-t)(S-s)f\rangle$. Using the functional calculus and the assumption that $f \in \mathcal{M}$, $0 = \langle f, (S-t)(S-s)f\rangle$ is possible only if $E((s,t)^c)f = 0$. Thus, f can be written as $f = u_s + u_t$ with u_λ solving $Ju'_\lambda = -\lambda H u_\lambda$ and $u_\lambda \in R(E\{\lambda\})$. The function $f(x) = u_s(x) + u_t(x)$ is absolutely continuous, satisfies $Jf' = -Hk$, with $k = su_s + tu_t \in L^2_H$, and has L^2_H norm 0 on (L,∞). So, by [11, Lemma 2.26], $f(c) = 0$ at all regular points $c > L$. Since (L,∞)

is assumed not to be contained in a singular half line, there are such regular points $c > L$. Fix a regular $c > L$. Since $u_s(c) = -u_t(c)$, u_s and u_t satisfy the same boundary condition at $x = c$. Thus, u_s and u_t are orthogonal on (c, ∞) either because one of them represents 0 in L^2_H or because they are eigenfunctions corresponding to different eigenvalues. Since $\|f\|_{L^2_H(c,\infty)} = 0$ and $f = u_s + u_t$ in $L^2_H(0, \infty)$, both u_s and u_t have zero norm on (c, ∞). However, a non-zero solution has zero norm on (c, ∞) only if (c, ∞) is contained in a singular half line. One of the solutions is non-zero because f was taken to be non-zero. So, there is a contradiction with the assumption that (L, ∞) is not contained in a singular interval. Hence, (3.3) holds.

The proof of (3.4) involves taking a sequence of endpoints L_n with $\lceil F(L_n) \rceil$ converging to $\liminf \lceil F(L) \rceil$ and then imposing boundary conditions at those endpoints that make s an eigenvalue. If $\liminf \lceil F(L) \rceil = \infty$, there is nothing to prove. So, assume $\liminf \lceil F(L) \rceil$ is finite, and take a sequence $L_n \nearrow \infty$ with $\lim_{n\to\infty} \lceil F(L_n) \rceil = \liminf \lceil F(L) \rceil$. Since the terms in the sequence and the limit are integers, we can assume $\lceil F(L_n) \rceil = \liminf \lceil F(L) \rceil$. Define $\beta_n \in [0, \pi)$ by writing $\theta(L_n; s) = N_n \pi + \beta_n$. We can focus on large enough n such that $H(x)$ is not identically equal to P_{e_2} on $(0, L_n)$. Then the problems on $[0, L_n]$ with boundary conditions $e^*_{\beta_n} J u(L_n) = 0$ have eigenvalue s. These boundary value problems can be identified with the the half line problems corresponding to

$$H_n(x) = \begin{cases} H(x) & x < L_n \\ P_{\beta_n + \pi/2} & x > L_n \end{cases}.$$

These half-line canonical systems obviously converge to H in the sense introduced in Chapter 2. Hence, their m functions converge locally uniformly to the m function of H. This implies that their spectral measures converge in the weak $*$ sense to the spectral measure corresponding to H. By Lemma 3.2 and the choice of β_n,

$$\dim E_n(s,t) = \lceil F(L_n) \rceil - 1 = \liminf \lceil F(L) \rceil - 1.$$

Hence, $\dim E(s,t) \leq \liminf_{L\to\infty} \lceil F(L) \rceil - 1$. $\square$

Note that when H does end with a singular half line $(0,L)$ of type $\beta + \pi/2$, then the problem can be identified with the problem on the bounded interval $(0,L)$ with the boundary condition $e_\beta^* J u(L) = 0$, as mentioned in the above proof. More specifically, there is an obviously defined unitary equivalence between S and $S_L^{(\beta)}$. In fact, L can be replaced with any $\tilde{L} \geq L$. Note that if $(0,\infty)$ is a singular interval itself, then all the spectral properties are trivial: $m(z)$ is a constant and the spectrum is empty. This trivial problem can be identified with the problem on $(0,L)$, for any $L \in (0,\infty)$, with the boundary condition just mentioned.

Chapter 4

Semibounded canonical systems and related topics

An interesting result of Winkler states that $\det H(x) \equiv 0$ for a nonnegative canonical system [24]. Oscillation theory leads to a simpler proof of this result, and the proof immediately gives the following generalization.

Theorem 4.1. *Suppose* $\dim E(-t, 0) = o(t)$ *as* $t \to \infty$*. Then* $\det H(x) = 0$ *for almost every* $x \in (0, \infty)$*.*

Proof. Suppose first that H does not end with a singular half line. By Theorem 3.3 and the Prüfer equation (3.1),

$$\lim_{t\to\infty} \int_0^\infty e^*_{\theta(x;-t)} H(x) e_{\theta(x;-t)} \, dx = 0.$$

Suppose there is a set $A \subset (0, \infty)$ with $|A| > 0$ and $\det H(x) > 0$ on A. Then, since

$e_\theta^* H(x) e_\theta \geq \det H(x)$ for every θ,

$$t \int_0^\infty e^*_{\theta(x;-t)} H(x) e_{\theta(x;-t)} \, dx \geq t \int_0^\infty \det H(x) \, dx \geq t \int_A \det H(x) \, dx$$

for every $t > 0$, which clearly contradicts the fact that the above limit is 0.

Now suppose that H ends with a singular interval (L, ∞). So, $\det H(x) = 0$ on (L, ∞). By Lemma 3.2 and the Prüfer equation,

$$\lim_{t \to \infty} \int_0^L e^*_{\theta(x;-t)} H(x) e_{\theta(x;-t)} \, dx = 0.$$

Suppose there is a set $A \subset (0, L)$ with $|A| > 0$ and $\det H(x) > 0$ on A. As above,

$$t \int_0^L e^*_{\theta(x;-t)} H(x) e_{\theta(x;-t)} \, dx \geq t \int_A \det H(x) \, dx$$

for every $t > 0$, which leads to a contradiction. $\square$

The above theorem also holds if $\dim E(-t, c) = o(t)$, where $c \in \mathbb{R}$ is arbitrary. To see this, one can apply a determinant-preserving transformation, as in (7.7) below, to H whose only effect on $m(z)$ is to shift the spectral measure so that $c \to 0$ [23]. In particular, it follows that $\det H(x) \equiv 0$ for semibounded canonical systems.

Lemma 4.2. *Let $\theta(x; t)$ be the solution of* (3.1) *with initial value $\theta(0; t) = 0$. The negative spectrum $\sigma(H) \cap (-\infty, 0)$ consists of exactly $N \geq 0$ points if and only if*

$$-N\pi \geq \theta(x; -t) > -(N+1)\pi \quad \textit{for all sufficiently large } x, t > 0. \tag{4.1}$$

Proof. The condition that $\sigma(H) \cap (-\infty, 0)$ consists of $N \geq 0$ points is equivalent to

$$\dim E(-t, 0) = N \quad \text{for all sufficiently large } t > 0, \tag{4.2}$$

and we have oscillation-theoretic formulas for $\dim E(-t, 0)$ coming from Lemma 3.2 and Theorem 3.3. Since $\theta(x; 0) = 0$, the claim follows immediately from Theorem 3.3 if H does not end with a singular half line.

If $(0, \infty)$ is a singular interval itself, then the spectrum is empty and $\theta(x; -t) > -\pi$ for all $x, t > 0$ by direct calculation. Suppose that $(0, \infty)$ ends with a singular interval (L, ∞), $L > 0$, of type $\beta + \pi/2$, say, with $0 \leq \beta < \pi$. As mentioned above, this problem can be identified with the problem on $(0, x)$ with boundary condition β at any $x \geq L$. By Lemma 3.2, (6.1) is equivalent, in this situation, to $-N\pi + \beta \geq \theta(x; -t) > -(N + 1)\pi + \beta$ for all sufficiently large $x, t > 0$. The inequality $\theta(x; -t) > -(N+1)\pi + \beta$ obviously implies the second inequality in (6.4). The first inequality in (6.4) also follows from the conditions from the lemma. To see this, note that we can take x large enough so that $(0, x)$ is not a singular interval of type $\pi/2$. Then $\theta(x; -t)$ is strictly decreasing as a function of t. So, for sufficiently large t, $-N\pi + \beta > \theta(x; -t)$. Fix such a t. Then since $\lim_{x\to\infty} \theta(x; -t)$ exists, it must be $-(N + 1)\pi + \beta$. Thus, there exist $x, t > 0$ such that $\theta(x; -t) < -N\pi$. By monotonicity, this inequality holds for all sufficiently large $x, t > 0$. Conversely, suppose (6.4) holds. Then obviously $-N\pi + \beta > \theta(x; -t)$ for all sufficiently large $x, t > 0$. This implies that there are at least N points in the negative spectrum. Assume for the sake of contradiction that there are more than N points in $\sigma(H) \cap (-\infty, 0)$. So, there is some $-s < 0$ with $-(N+1)\pi < \theta(x; -s) \leq -(N+1)\pi + \beta$ for all $x \geq L$. By (6.4) and monotonicity, $\lim_{x\to\infty} \theta(x; -t)$ exists for every $t > 0$, and it is clear from (3.1), that this limit is $\equiv \beta \bmod \pi$. Since there is (negative) spectrum, we can assume that, for sufficiently large x, $(0, x)$ is not a singular interval of type $\pi/2$ and hence $\theta(x; -t)$ is strictly decreasing as a function of t for such x. Thus, there are $t > s$ and x_0 with $\theta(x_0; -t) < -(N+1)\pi + \beta$. So, $\lim_{x\to\infty} \theta(x; -t) \leq -(N+2)\pi + \beta$, but this contradicts (6.4). $\square$

The following result, due to Winkler and Woracek [24, 26], gives a complete characterization of canonical systems with finite negative spectrum. Winkler proved this using unrelated methods, and once again, the oscillation theoretic proof seems simpler. The proof here also gives more information than the original proof. See (4.3) below.

Theorem 4.3 ([24, 26])**.** *Suppose $(0, \infty)$ is not a singular interval of type $\pi/2$. Then the negative spectrum $\sigma(H) \cap (-\infty, 0)$ consists of exactly $N \geq 0$ points if and only if $H(x) = P_{\varphi(x)}$ for some decreasing function $\varphi(x)$ with $\varphi(0+) \in (-\pi/2, \pi/2]$ and*

$$-(N-1)\pi - \frac{\pi}{2} > \varphi(\infty) \geq -N\pi - \frac{\pi}{2}$$

such that φ has no jumps of size $\geq \pi$.

Proof. Suppose that the negative spectrum consists of exactly N points. By Theorem 4.1, $\det H(x) = 0$. Thus, because $\operatorname{tr} H(x) = 1$, $H(x) = P_{\varphi(x)}$ for some function $\varphi(x)$. It will be shown that φ can be taken as

$$\varphi(x) = \varphi_0(x), \quad \varphi_0(x) = \lim_{t\to\infty} \theta(x; -t) + \frac{\pi}{2}, \tag{4.3}$$

using the assumption that (6.4) holds. It is easy to see that the monotonicity of $\theta(x; -t)$ in both arguments and the bounds (6.4) imply that the limit exists, that φ_0 is decreasing, $\varphi_0(0+) \leq \pi/2$, and $-(N-1)\pi - \pi/2 \geq \varphi_0(\infty) \geq -N\pi - \pi/2$. Except for the trivial case when $(0,\infty)$ is a singular interval of type $\pi/2$, (6.4) implies that $-N\pi > \theta(x;-t)$ for all sufficiently large $x, t > 0$. So, assume now that $(0,\infty)$ is not a singular interval of type $\pi/2$. This ensures that $-(N-1)\pi - \pi/2 > \varphi_0(\infty)$. We next establish (4.3). If $N = 0$, this is enough to show that H has the required form. If $N > 0$, then the only things left to establish are that φ does not have jumps of size $\geq \pi$ and that $\varphi(0+) > -\pi/2$.

Since $H(x) = P_{\varphi(x)}$, (3.1) can be rewritten as

$$\theta' = -t\sin^2(\theta - \psi(x)), \quad \psi(x) = \varphi(x) - \frac{\pi}{2}. \tag{4.4}$$

So, for all $L, t > 0$,

$$\int_0^L \sin^2\left(\theta(x;-t) - \varphi(x) + \frac{\pi}{2}\right) dx = -\frac{\theta(L;-t)}{t} < \frac{(N+1)\pi}{t}.$$

Hence, $\int_0^\infty \sin^2\left(\theta(x;-t) - \varphi(x) + \frac{\pi}{2}\right) dx \leq \frac{(N+1)\pi}{t}$ for every $t > 0$. Thus, by Fatou's lemma,

$$\liminf_{t\to\infty} \sin^2\left(\theta(x;-t) - \varphi(x) + \frac{\pi}{2}\right) = 0$$

for almost every $x > 0$. This implies $\varphi_0(x) \equiv \varphi(x) \bmod \pi$ almost everywhere, and, since $P_{\alpha+n\pi} = P_\alpha$, this proves (4.3). The remainder of this direction of the proof will be finished after starting on the converse direction.

Now suppose that $H(x) = P_{\varphi(x)}$ with $\varphi(x)$ as described in the theorem, and let $\psi(x) = \varphi(x) - \pi/2$. To show that there at most N points in the negative spectrum, the second inequality in (6.4) will be deduced. By the proof of Lemma 4.2, this will imply that H has at most N points in the negative spectrum.

Suppose first that $\psi(0+) < 0$. Since $\psi(x) \geq -(N+1)\pi$ for all x, it suffices to show that $\theta(x;-t) > \psi(x)$ for all $x, t > 0$. Fix $t > 0$, and let $\theta(x) = \theta(x;-t)$.

Suppose that

$$L := \sup\{b > 0 : \theta(x) > \psi(x-) \text{ on } 0 < x < b\}$$

is finite. So, $\theta(L) = \psi(L-)$. Take $a \in (0, L)$ such that $\theta(x) - \psi(L-) \leq \pi/2$ for all $x \in (a, L)$. Then

$$\sin^2(\theta(x) - \psi(x)) \leq \sin^2(\theta(x) - \psi(L-)) \leq (\theta(x) - \psi(L-))^2$$

for all $x \in (a, L)$. Now, consider the solution θ_1 of

$$\theta_1' = -t(\theta_1 - \psi(L-))^2, \quad \theta_1(a) = \theta(a) > \psi(L-).$$

By the comparison principle, $\theta(x) \geq \theta_1(x)$ for all $x \in (a, L)$. So, since $\theta(L) = \psi(L-)$, $\lim_{x\to L} \theta_1(x) \leq \psi(L-)$, but this contradicts the elementary fact that θ_1 does not reach $\psi(L-)$ in finite time. Hence, $L = \infty$, which means that $\theta(x) > \psi(x-)$ for all $x > 0$.

If $\psi(x) = 0$ on some initial interval $(0, L)$, then the problem on (L, ∞) with the boundary condition $u_2(L) = 0$ has the same spectral measure as the problem on $(0, \infty)$. So, to count the points in the negative spectrum, we can assume that $\psi(x) < 0$ for all $x > 0$. Let

$$H_n(x) = \begin{cases} H(x) & x > 1/n \\ P_{\varphi(1/n+)} & x < 1/n \end{cases}.$$

These converge to H in the metric referred to earlier, and hence their spectral measures converge in the weak $*$ sense. So, if each H_n has at most N points in the negative spectrum, then it also follows for H. The upper bound on the number of points in the negative spectrum of H_n follows from the above case.

Everything claimed about the case $N = 0$ has now been proved. The other cases follow by induction. Suppose the equivalence stated in the theorem is true for all $M < N$. Suppose that H has the form described for N. Then N has at most N points in the negative spectrum. Suppose it had $M < N$ points in the negative spectrum. Then by the induction hypothesis it could be written in the form required for some $M < N$, a contradiction to φ not having jumps of size $\geq \pi$. Conversely, suppose H has exactly N points in the negative spectrum. Then by the earlier arguments, the only things left to establish are that φ does not have jumps of size $\geq \pi$ and that $\varphi(0+) > -\pi/2$. If $\varphi(0+) \leq -\pi/2$, then by suitably adding multiples of π to φ, H can be written in the form corresponding to some $M < N$, which would imply that H has only M points in the negative spectrum by the induction hypothesis. Suppose φ has a jump of size $\geq \pi$. Then multiples of π could be suitably added to portions of φ to remove all those jumps so that then H would written in the form for some $M < N$. By the induction hypothesis, this would imply that H has only M points in the negative spectrum, a contradiction. □

The above theorem naturally leads to the question of what can be said if φ is still decreasing, but $\varphi(\infty)$ is not necessarily finite, or if the spectrum is just assumed to be bounded below. The following theorem gives an answer to these questions.

Corollary 4.4 ([12])**.** *(a) H can be written in the form $H(x) = P_{\varphi(x)}$ with $\varphi(x)$ decreasing and $\varphi(0+) < \infty$ if and only if the self-adjoint problems on $[0, L]$ have finite negative spectrum for all $L > 0$.*

(b) If $\sigma(H) \subset [c, \infty)$ for some $c \in \mathbb{R}$, then there is a decreasing function $\varphi(x)$ with $\varphi(0+) < \infty$ such that $H(x) = P_{\varphi(x)}$.

Proof. Suppose that $H(x) = P_{\varphi(x)}$ with $\varphi(x)$ decreasing and $\varphi(0+) < \infty$. The problem on $[0, L]$ with boundary condition β at L can be identified with the problem on $(0, \infty)$ with the same coefficient function on $(0, L)$ and then a singular half line of type $\beta + \pi/2$. Since adding multiples of π to β leaves the projection unchanged, one can take $\beta + \pi/2$ to be less than $\varphi(L)$. So, this problem has finite negative spectrum by Theorem 4.3.

Conversely, suppose that the problems on $[0, L]$ have finite negative spectrum for every $L > 0$. Since the boundary condition can be implemented by a singular half life, Theorem 4.3 implies that $H(x) = P_{\varphi_n(x)}$ on $(0, n)$ for some decreasing function φ_n. Since adding multiples of π can be added to φ_n without changing the projection, it is clear that one can define a decreasing function φ such that $H(x) = P_{\varphi(x)}$ for all $x > 0$.

Suppose that $\sigma(H) \subset [c, \infty)$. If H ends with a singular half line, then, since the spectrum is purely discrete, H has finite negative spectrum, which is the case addressed by Theorem 4.3. Suppose that H does not end with a singular half line. Then, from the proof of Theorem 3.3, $\lfloor \frac{1}{\pi} (\theta(L; c) - \theta(L; s)) \rfloor \leq \dim E(s, c) = 0$ for every $s < c$ and $L > 0$. Hence,

$$\dim E_L^{(\beta)}[s, c) = \left\lceil \frac{1}{\pi} (\theta(L; c) - \beta) \right\rceil - \left\lceil \frac{1}{\pi} (\theta(L; s) - \beta) \right\rceil \leq 1$$

for every $s < c$, $L > 0$, and $\beta \in [0, \pi)$. So, the problems on $[0, L]$ have finite negative spectrum for every $L > 0$. Then the claim follows from part (a). □

Winkler and Woracek used the theory of Krein strings to deduce the following result [26]. The proof here achieves this result by using oscillation theory for canonical systems, without appealing to Krein strings,. Note that if $H \in \mathcal{C}_+$, then $m(z)$ can be holomorphically continued to $\mathbb{C} \setminus [0, \infty)$. Assuming $H(x) \not\equiv P_{\pi/2}$, $m(t)$ is real-valued and increasing for $t \in (-\infty, 0)$.

Theorem 4.5 ([12, 26])**.** *Let $H \in \mathcal{C}_+$, and suppose that $(0, \infty)$ is not a singular interval of type $\pi/2$. Take φ as in Theorem 4.3 so that $H(x) = P_{\varphi(x)}$. Then*

$$\tan \varphi(0+) = -m(-\infty), \quad \tan \varphi(\infty) = -m(0-).$$

Proof. Suppose that it is not the case that $H(x) \equiv P_{e_2}$, and let $\psi(x) = \varphi(x) - \pi/2$. Define $\alpha(-t) \in (-\pi, 0)$ by $m(-t) = \cot \alpha(-t)$ for $t > 0$. The theorem claims that $\psi(0+) = \alpha(-\infty)$ and $\psi(\infty) = \alpha(0-)$.

Theorem 3.3 implies that the solutions of (3.1) at $-t < 0$ with initial value $\theta(0) = 0$ stay above $-\pi$ because $\dim E(-t, 0) = 0$. A version of this for the solution with initial value $\alpha(-t)$ is needed.

Lemma 4.6 ([12])**.** *Suppose $H \in \mathcal{C}_+$, $t > 0$, and let $\theta(x; -t)$ be the solution of (3.1) with initial value $\theta(0; -t) = \alpha(-t)$. Then $\theta(x; -t) \geq -\pi$ for all $x \geq 0$.*

Proof. Let f be the solution $Jf' = tHf$ with initial value $e_{\alpha(-t)}$. So, $f(x) = R(x)e_{\theta(x;-t)}$ for some function $R(x) > 0$. Since $e_{\alpha(-t)}$ is a multiple of $(m(-t), 1)^t$, $f \in L^2_H(0, \infty)$. Suppose there is an $L > 0$ such that $\theta(L; -t) = -\pi$. Then f can and will be redefined by multiplying it by a constant so that $f(L) = e_1 = (1, 0)^t$. Define

$$f_L(x) = \begin{cases} e_1 & x < L \\ f(x) & x > L \end{cases}$$

and

$$g_L(x) = \begin{cases} 0 & x < L \\ -tf(x) & x > L \end{cases}.$$

Then (f_L, g_L) is clearly in $\mathcal{S}$, the self-adjoint relation on $(0, \infty)$. Let S be the self-adjoint operator. Since $g_L = Sf_L + h$ for some $h \in \mathcal{S}(0)$ and $f_L \in D(\mathcal{S}) \subset \mathcal{S}(0)^{\perp}$,

$$\langle f_L, g_L \rangle = \langle f_L, Sf_L \rangle.$$

Thus, since $H \in \mathcal{C}_+$, $\langle f_L, g_L \rangle = \langle f_L, Sf_L \rangle \geq 0$. From the above definitions,

$$\langle f_L, g_L \rangle = -t \int_L^{\infty} f^*(x) H(x) f(x)\, dx \leq 0.$$

Hence, $\int_L^{\infty} f^*(x) H(x) f(x)\, dx = 0$, which implies $Hf = 0$ almost everywhere on (L, ∞). So, since f is a solution, $f(x) = e_1$ and thus $\theta(x; -t) = -\pi$ for all $x \geq L$. $\square$

Note that $\alpha(-t) = \theta(0; -t)$ is increasing as a function of $t > 0$ because $m(-t)$ is decreasing, $m(-t) = \cot \alpha(-t)$, and $\alpha(-t) \in (-\pi, 0)$. This implies that $\theta(x; -t)$ is increasing as a function of $t > 0$ since, by Lemma 4.6 and its proof, $\cot \theta(x; -t) = m_x(-t)$ is the m function of the problem on (x, ∞) and $\theta(x; -t) \in [-\pi, 0)$ (if $m_x(-t) = \infty$ for some $t > 0$, then $\theta(x; -t) \equiv -\pi$). Also, note that $\alpha(-\infty) = -\pi$ is impossible because this would imply $\alpha(-t) \leq -\pi$ for $t > 0$, which contradicts the choice of $\alpha \in (-\pi, 0)$.

For the proof of Theorem 4.5, first suppose that $\alpha(-\infty) < \psi(0+)$. Fix $\delta \in (0, \min(\alpha(-\infty) + \pi, \psi(0+) - \alpha(-\infty)))$. Then take an $a > 0$ such that $\psi(x) \in (\alpha(-\infty) + \delta, \psi(0+)]$ for all $x \in (0, a)$. So, for all $x \in (0, a)$, $\psi(x) - \alpha(-\infty) > \delta$. Now, since $\alpha(-t) = \theta(0; -t)$ is increasing as a function of $t > 0$, $\psi(x) - \theta(0; -t) > \delta$ for all $x \in (0, a)$ and $t > 0$. Then, for all $x \in (0, a)$ and $t > 0$, $\psi(x) - \theta(x; -t) > \delta$ since $\theta(x; -t)$ is decreasing as a function of x. Now fix an $s > 0$ such that $\alpha(-\infty) - \theta(0; -s) < \frac{1}{2}(\alpha(-\infty) - \delta + \pi)$. Take a $0 < b \leq a$ such that $\theta(0; -s) - \theta(x; -s) < \frac{1}{2}(\alpha(-\infty) - \delta + \pi)$ for all $x \in (0, b)$. Then for all $x \in (0, b)$, $\theta(x; -s) > \delta - \pi$.

Since $\theta(x;-t)$ is increasing as a function of $t>0$, this implies that $\theta(x;-t)>\delta-\pi$ for all $t\geq s$ and $x\in(0,b)$. So, $\sin^2(\psi(x)-\theta(x;-t))\geq\sin^2(\delta)$ for all $t\geq s$ and $x\in(0,b)$. Hence, $\theta'(x;-t)\leq -t\sin^2\delta$ for all $t\geq s$ and $x\in(0,b)$. Thus, there exist $t\geq s$ and $x\in(0,b)$ such that $\theta(x;-t)<-\pi$, which contradicts Lemma 4.6. So, $\alpha(-\infty)\geq\psi(0+)$.

Now suppose that $\alpha(-\infty)>\psi(0+)$. Note that $\psi(0+)>-\pi$ by our assumption that $(0,\infty)$ is not a singular interval of type $\pi/2$. Let $\delta=\min(\frac{1}{2}(\psi(0+)+\pi),\frac{1}{4}(\alpha(-\infty)-\psi(0+)))$. Take $a>0$ such that $\psi(x)\in(\delta-\pi,\psi(0+)]$ for all $x\in(0,a)$. Fix an $s>0$ such that $\alpha(-\infty)-\theta(0;-s)<\frac{1}{2}(\alpha(-\infty)-\psi(0+))$. Now take a $b\in(0,a]$ such that $\theta(0;-s)-\theta(x;-s)<\delta$ for all $x\in(0,b)$. Then $\theta(x;-s)-\psi(0+)>\delta$ for all $x\in(0,b)$. Hence, $\theta(x;-t)-\psi(x)>\delta$ for all $x\in(0,b)$ and $t\geq s$. Also, $\theta(x;-t)-\psi(x)<\pi-\delta$ for $x\in(0,b)$ since $\psi(x)>\delta-\pi$ on $(0,a)$. Thus, $\sin^2(\psi(x)-\theta(x;-t))\geq\sin^2(\delta)$ for all $t\geq s$ and $x\in(0,b)$. Hence, $\theta'(x;-t)\leq -t\sin^2\delta$ for all $t\geq s$ and $x\in(0,b)$. Thus, there exist $t\geq s$ and $x\in(0,b)$ such that $\theta(x;-t)<-\pi$, which contradicts Lemma 4.6. So, $\alpha(-\infty)\leq\psi(0+)$.

To show that $\psi(\infty)\leq\alpha(0-)$, suppose it were not the case. Take a $t>0$ such that $\psi(\infty)>\theta(0;-t)$. Now $\theta(x;-t)$ is a decreasing function of x with values in $(0,-\pi]$. By (4.4), $\lim_{x\to\infty}\theta(x;-t)\equiv\psi(\infty) \bmod \pi$. So, $\lim_{x\to\infty}\theta(x;-t)\leq\psi(\infty)-\pi<-\pi$, due to the assumption that $\psi(x)$ is not identically 0, but this contradicts Lemma 4.6.

At last, suppose that $\psi(\infty)<\alpha(0-)$. We can assume $m(z)$ is not a constant because the theorem is well-known and trivial in that case. Let $\gamma=-\pi-\alpha(0-)$,

$$R_\gamma=\begin{pmatrix}\cos\gamma & -\sin\gamma\\ \sin\gamma & \cos\gamma\end{pmatrix},$$

and $H_\gamma(x) = R_\gamma H(x) R_{-\gamma}$. By direct calculation $H_\gamma(x) = P_{\psi(x)+\frac{\pi}{2}+\gamma}$. It is well-known that $m_\gamma = R_\gamma m$ [11, Theorem 3.20]. So, $\alpha_\gamma = \alpha + \gamma$. Now,

$$m_\gamma(z) = \frac{m(z)\cos\gamma - \sin\gamma}{m(z)\sin\gamma + \cos\gamma},$$

and $m(-t)\sin\gamma + \cos\gamma = m(-t)\sin\alpha(0-) - \cos\alpha(0-)$ could only be 0 if $m(-t) = \cot\alpha(0-) = \lim_{s\to 0^+} m(-s)$, which would contradict $m(-t)$ being strictly decreasing. So, m_γ is also holomorphic on an open set containing $(-\infty, 0)$. Thus, $H_\gamma \in \mathcal{C}_+$. As above, $\lim_{x\to\infty}\theta_\gamma(x; -t) \equiv \psi(\infty) + \gamma \mod \pi$. Now, $\theta_\gamma(0; -t) = \alpha(-t) + \gamma \geq -\pi$, and $\theta_\gamma(0; -t) < \psi(\infty) + \gamma + \pi$. So, $\lim_{x\to\infty}\theta_\gamma(x; -t) \leq \psi(\infty) + \gamma < -\pi$, which contradicts Lemma 4.6. (Alternatively, one could note that, although we see that $H_\gamma \in \mathcal{C}_+$ by looking at m_γ, $P_{\psi(x)+\frac{\pi}{2}+\gamma}$ obviously cannot be rewritten in the form required by Theorem 4.3). $\square$

A characterization of whole-line canonical systems with nonnegative spectrum can be obtained by applying Theorems 4.3 and 4.5.

Theorem 4.7 ([12])**.** *The spectrum of the whole-line canonical system with coefficient function $H(x)$, $x \in \mathbb{R}$, is nonnegative if and only if $H(x) = P_{\varphi(x)}$ for some decreasing function $\varphi(x)$ with $\varphi(-\infty) - \varphi(\infty) \leq \pi$.*

Proof. Assume that $(-\infty, \infty)$ is not one or two singular intervals, in which case the theorem is trivial. Suppose that $\sigma(H) \subset [0, \infty)$. Since the essential spectrum of the whole-line problem is the union of the essential spectra of the half-line problems on $(-\infty, 0)$ and $(0, \infty)$, the two half-line m functions $m_\pm$ are meromorphic on an open set containing $(-\infty, 0)$. Since the whole line problem has no negative eigenvalues, there are no $t > 0$ such that $m_+(-t) = -m_-(-t)$ or $m_+(-t) = m_-(-t) = \infty$ (on the Riemann sphere). Due to the Herglotz representations of $m_\pm$, $m_\pm(t)$ are increasing on each subinterval of $(-\infty, 0)$ that avoids the poles. I claim that $m_\pm$ together have at most one pole. Suppose there are $0 < s < t$ with $m_+(-s) = m_-(-t) = \infty$ and no poles in between. Then since $m_\pm$ are increasing and continuous on $(-t, -s)$, there is a $-u \in (-t, -s)$ with $m_+(-u) = -m_-(-u)$, but this was ruled out. Similarly if there were $0 < s < t$ with $m_+(-s) = m_+(-t) = \infty$ and no poles in between, then there would be a $-u \in (-t, -s)$ with $m_+(-u) = -m_-(-u)$. So, by applying Theorem 4.3 to both half lines and possibly adding a multiple of π to one of the function obtained, $H(x) = P_{\varphi(x)}$ for some decreasing function $\varphi(x)$. If φ has any jumps of size $\geq \pi$, then we appropriately add multiples of π to make the jumps less than π.

Suppose for the sake of contradiction that $\varphi(-\infty) - \varphi(\infty) > \pi$. Then there exists a real number γ such that $\varphi_\gamma = \varphi + \gamma$ has range contained in an interval containing $[-\pi/2, \pi/2]$. Consider $H_\gamma(x) = R_\gamma H(x) R_{-\gamma} = P_{\varphi_\gamma}$, where R_γ is the rotation matrix given as in the proof of Theorem 4.5. H_γ has the same spectrum as H because the two whole-line operators are unitarily equivalent [11, Theorem 7.2]. Since φ_γ has no jumps of size $\geq \pi$, there exists an $a \in \mathbb{R}$ such that $-\pi/2 < \varphi_\gamma(a+) \leq \varphi_\gamma(a-) < \pi/2$. So, since $\varphi(-\infty) - \varphi(\infty) > \pi$, the problems on $(-\infty, a)$ and (a, ∞) both have negative spectrum by Theorem 4.3, which must be discrete since the essential spectrum of the whole-line problem is the union of the essential spectra of the half-line problems. This contradicts the earlier observation that at most one of the half-line problems has a negative eigenvalue.

The converse also follows by splitting the whole line into half lines. Choose φ so that $\varphi(\infty) \in [-\pi/2, \pi/2)$. Take the $a \in \mathbb{R}$ where $\varphi(x) \geq \pi/2$ for $x < a$ and $\varphi(x) < \pi/2$ for $x > a$ if there is such an a, and if there is no such a, then let $a = 0$. Then the problems on $(-\infty, a)$ and (a, ∞) are as in Theorem 4.3. Refer to their m functions by m_- and m_+, respectively . So, by Theorem 4.5, $\tan\varphi(a+) = -m_+(-\infty)$, $\tan\varphi(\infty) = -m_+(0-)$, $\tan\varphi(a-) = m_-(-\infty)$, and $\tan\varphi(-\infty) = m_-(0-)$. Suppose $\varphi(-\infty) > \pi/2$. Then $\tan\varphi(-\infty) \leq \tan\varphi(\infty)$. So, $m_-(t) \leq m_-(0-) = \tan\varphi(-\infty) \leq \tan\varphi(\infty) = -m_+(0-) \leq -m_+(t)$ for $t < 0$, and since either $m_-(t)$ or $m_+(t)$ is strictly increasing, either the first or last of these inequalities is strict. It follows that the whole-line problem has no negative eigenvalues. There is no negative essential spectrum because neither half-line problem has it. Now, suppose that $\varphi(-\infty) \leq \pi/2$. Then $m_-(t) \geq m_-(-\infty) = \tan\varphi(a-) \geq \tan\varphi(a+) = -m_+(-\infty) \geq -m_+(t)$ for $t < 0$. Hence, by the same argument as above, the whole-line problem has no negative spectrum. $\square$

Chapter 5

The essential spectrum of nonnegative canonical systems

The bottom of the essential spectrum of an $H \in \mathcal{C}_+$ is controlled by the asymptotics of φ. The precise statements are the contents of the next two theorems.

Theorem 5.1 ([12])**.** *Let $H \in \mathcal{C}_+$. Take φ as in Theorem 4.3 so that $H(x) = P_{\varphi(x)}$ (or if $(0,\infty)$ is a singular interval of type $\pi/2$, let $\varphi(x) \equiv \pi/2$), and let*

$$A = \limsup_{x\to\infty} x(\varphi(x) - \varphi(\infty))$$

Then

$$\frac{1}{4A} \leq M(H) \leq \frac{1}{A}.$$

Proof. Let $T = M(H)$. Since $H \in \mathcal{C}_+$, this definition of T is equivalent to the condition that $\dim E(0,t) < \infty$ for every $0 < t < T$ and $\dim E(0,t) = \infty$ for every $t > T$. By Lemma 3.2 or Theorem 3.3 and Theorem 4.3, this is equivalent to the oscillation-theoretic conditions

$$\lim_{x\to\infty}\theta(x;t)<\infty \quad (0<t<T); \qquad \lim_{x\to\infty}\theta(x;t)=\infty \quad (t>T) \tag{5.1}$$

where $\theta(x;t)$ is the solution of (4.4) with $\theta(0;t)=0$.

As noticed earlier, $\lim_{x\to\infty}\theta(x;t)$ is either infinite or congruent to $\psi(\infty)$ mod π. Take ψ with range in $[0,\pi]$. Assume that $\psi(\infty)=0$. It is sufficient to prove the claim in this case because, by applying a rotation, which leaves the essential spectrum and A unchanged, one can obtain a new $H\in\mathcal{C}_+$ with a ψ that satisfies $\psi(\infty)=0$.

For the first inequality from Theorem 5.1, we can assume $A<\infty$ since there is nothing to do otherwise. Take any $C>A$. Then fix an $a>0$ such that $0\le\psi(x)\le C/x$ for all $x\ge a$. Let $0<t<1/(4C)$. To prove the first inequality in the theorem, it suffices to show that there is an angle $\theta_0<0$ such that the solution $\theta(x)$ of

$$\theta'=t\sin^2(\theta-\psi(x)) \tag{5.2}$$

with initial value $\theta(a)=\theta_0$ satisfies $\theta(x)\le 0$ for all $x>a$. To see that this implies the first inequality, note first that since one can find a negative integer n such that $\theta(a;t)+n\pi\le\theta_0$, and then $\theta(x;t)+n\pi\le\theta(x)$ for $x>a$ by the comparison principle, $\lim_{x\to\infty}\theta(x;t)=\infty$ is impossible . This then means that $t\le T$ for every $0<t<1/(4C)$ and every $C>A$, which obviously implies the first inequality.

Let $b=at/2$, $D=Ct$, and

$$p_\pm=\frac{1}{2}\left(1\pm\sqrt{1-4D}\right).$$

Define

$$\alpha(x) = -\frac{1}{b\xi}\frac{p_+\xi^d - p_-}{\xi^d - 1}, \quad \xi = \frac{x}{b} > 1, \quad d = p_+ - p_- = \sqrt{1-4D}.$$

By direct calculation, $\alpha(x)$ solves

$$\alpha' = \alpha^2 + \frac{D}{x^2} \tag{5.3}$$

for $x > b$. I claim that $\alpha(x) < -D/x$ for all $x > b$. Note that $\alpha(x) \to -\infty$ as $x \to b^+$. So, $\alpha(x) < -D/x$ for all $x > b$ near b. It thus suffices to show that $\alpha(x) = -D/x$ has no solution with $x > b$. Let $y = \xi^d$. The claim follows because

$$\frac{p_+ y - p_-}{y-1} = D, \quad 0 < D < 1/4,$$

has no solutions $y > 1$ by basic algebra.

Let $\theta_1(x) = \alpha(x) + D/x$. So, $\theta_1(x) < 0$ for all $x > b$. By direct calculation, $\theta_1(x)$ solves $\theta_1' = \left(\theta_1 - \frac{D}{x}\right)^2$. Let $\theta_0 = \theta_1(at)$. Then by a change of variables, the solution $\theta_2(x)$ of $\theta_2' = t(\theta_2 - C/x)^2$ with $\theta_2(a) = \theta_0$ stays below 0.

For $\theta \le 0$ and $x \ge a$, $t\sin^2(\theta - \psi(x)) \le t(\theta - C/x)^2$ since $0 \le \psi(x) \le C/x$ for all $x \ge a$. So, the solution $\theta(x)$ of

$$\theta' = t\sin^2(\theta - \psi(x)), \tag{5.4}$$

with initial value $\theta(a) = \theta_0$ satisfies $\theta(x) \le \theta_2(x) < 0$ for $x > a$.

Now, for the upper bound $M(H) \le 1/A$, we can obviously assume $A > 0$. Take $0 < C < A$, $t > 1/C$, and $0 < \epsilon < 1 - 1/(Ct)$. Suppose for the sake of contradiction that $\lim_{x\to\infty}\theta(x;t) < \infty$. Then there exists a negative integer

n such that $\lim_{x\to\infty} \theta(x;t) + n\pi = 0$. So, $\theta(x;t) + n\pi \leq 0$ for all $x > 0$. Let $\theta(x) = \theta(x;t) + n\pi$. Using the assumption $\psi(\infty) = 0 = \theta(\infty)$, take $a > 0$ such that

$$\sin^2(\theta(x) - \psi(x)) \geq (1-\epsilon)\left(\theta(x) - \psi(x)\right)^2$$

for all $x \geq a$. Then, by the definition of A, the monotonicity of ψ, and the choice of ϵ, there exists $b > a$ such that $\psi(x) \geq C/b$ for $x \leq b$ and $(C - \frac{1}{t(1-\epsilon)})b - a > 0$. Thus, since $\theta(x) \leq 0$,

$$\sin^2(\theta(x) - \psi(x)) \geq (1-\epsilon)\left(\theta(x) - \frac{C}{b}\right)^2$$

for all $x \in [a, b]$. Let $\theta_0 = \theta(a)$.

Consider the solution $\theta_1(x)$ of

$$\theta_1' = (1-\epsilon)\left(\theta_1 - \frac{C}{b}\right)^2, \quad \theta_1(at) = \theta_0.$$

To solve this explicitly, let $\alpha = \theta_1 - C/b$. So, $\alpha' = (1-\epsilon)\alpha^2$, and the solution of this with the right initial value is

$$\alpha(x) = \frac{\theta_0 - C/b}{1 - (\theta_0 - C/b)(1-\epsilon)(x - at)}.$$

So, for $x > at$

$$\theta_1(x) = \frac{C}{b} + \frac{\theta_0 - C/b}{1 - (\theta_0 - C/b)(1-\epsilon)(x-at)} \geq \frac{C}{b} - \frac{1}{(1-\epsilon)(x-at)}.$$

By a change of variable, the solution $\theta_2(x)$ of

$$\theta_2' = t(1-\epsilon)\left(\theta_2 - \frac{C}{b}\right)^2, \quad \theta_2(a) = \theta_0$$

satisfies

$$\theta_2(x) \geq \frac{C}{b} - \frac{1}{(1-\epsilon)(xt - at)}$$

for all $x > a$. By the choice of b, $\frac{C}{b} - \frac{1}{(1-\epsilon)(bt-at)} > 0$. So, $\theta_2(b) > 0$, but by the comparison principle $\theta(b) \geq \theta_2(b)$. Thus, there is a contradiction with the assumption that $\theta(x) = \theta(x;t) + n\pi \leq 0$ for all $x > 0$. $\square$

Theorem 5.2 ([12]). *Suppose that $H \in \mathcal{C}_+$, and take φ as above. Let*

$$B = \liminf_{x\to\infty} x(\varphi(x) - \varphi(\infty)).$$

Then $M(H) \leq 1/(4B)$.

Proof. As above, we can assume that $\psi(\infty) = 0$. For this inequality, we can obviously assume $B > 0$. Take any $0 < C < B$ and $t > 1/(4C)$. Take $\epsilon > 0$ such that $\epsilon < 1 - 1/(4Ct)$. Suppose for the sake of contradiction that $\lim_{x\to\infty}\theta(x;t) < \infty$. Then there exists a negative integer n such that $\lim_{x\to\infty}\theta(x;t) + n\pi = 0$. So, $\theta(x;t) + n\pi \leq 0$ for all $x > 0$. Let $\theta(x) = \theta(x;t) + n\pi$. Using the assumption $\psi(\infty) = 0 = \theta(\infty)$, take $a_0 > 0$ such that

$$\sin^2(\theta(x) - \psi(x)) \geq (1-\epsilon)\left(\theta(x) - \psi(x)\right)^2$$

for all $x \geq a_0$. Take $a \geq a_0$ such that $\psi(x) \geq C/x$ for all $x \geq a$. Thus, since $\theta(x) \leq 0$

for all $x > 0$,

$$\sin^2(\theta(x) - \psi(x)) \geq (1 - \epsilon) \left(\theta(x) - \frac{C}{x} \right)^2$$

for all $x \geq a$.

So, the solution $\theta_2(x)$ of $\theta_2' = t(1-\epsilon)(\theta_2 - C/x)^2$ with $\theta_2(a) = \theta(a)$ exists for $x > a$. Let $D = (1 - \epsilon)Ct$. By a change of variable, the solution $\theta_1(x)$ of $\theta_1' = \left(\theta_1 - \frac{D}{x}\right)^2$ with $\theta_1(at(1 - \epsilon)) = \theta(a)$ exists for $x > at(1 - \epsilon)$. Let $\alpha(x) = \theta_1(x) - D/x$ and $u(x) = \exp(-\alpha(at(1 - \epsilon)) - \int_{at(1-\epsilon)}^{x} \alpha(y)\, dy)$. Then $u(x)$ is a zero-free solution, for $x > at(1 - \epsilon)$, of the Schrödinger equation

$$- u'' - \frac{D}{x^2} u = 0. \tag{5.5}$$

However, this contradicts the well-know fact that this Schrödinger equation is oscillatory for $D > 1/4$. $\square$

Since $\sigma_{ess}(H) = \emptyset$ if and only if $M(H) = \infty$, Theorem 5.1 immediately gives a precise description of the $H \in \mathcal{C}_+$ with purely discrete spectrum.

Theorem 5.3 ([12])**.** *Let $H \in \mathcal{C}_+$. Take φ as in Theorem 4.3 so that $H(x) = P_{\varphi(x)}$ (or if $(0, \infty)$ is a singular interval of type $\pi/2$, let $\varphi(x) \equiv \pi/2$). Then $\sigma_{ess}(H) = \emptyset$ if and only if*

$$\varphi(x) - \varphi(\infty) = o(1/x) \quad \text{as } x \to \infty.$$

Theorem 5.1 also tells exactly when $0 \in \sigma_{ess}(H)$ for $H \in \mathcal{C}_+$.

Theorem 5.4 ([12])**.** *Let $H \in \mathcal{C}_+$. Take φ as in Theorem 4.3 so that $H(x) = P_{\varphi(x)}$ (or if $(0, \infty)$ is a singular interval of type $\pi/2$, let $\varphi(x) \equiv \pi/2$). Then 0 is in the* *essential spectrum of H if and only if*

$$\limsup_{x\to\infty} x(\varphi(x) - \varphi(\infty)) = \infty.$$

A one-dimensional Schrödinger operator can be rewritten as a canonical system using a well-known transformation of writing it as a first-order system and then doing a variation of constants about $z = 0$. Thus, Theorem 5.3 gives a criterion for a semibounded Schrödinger operator to have purely discrete spectrum.

Theorem 5.5 ([12]). *Let $\mathcal{L} = -d^2/dx^2 + V(x)$ be a Schrödinger operator on $(0, \infty)$ that is bounded below. Let $E_0 < \min \sigma(\mathcal{L})$. Suppose $g(x)$ is a solution of $-y'' + Vy = E_0 y$ with $g \notin L^2(0, \infty)$. Then $\sigma_{ess}(\mathcal{L}) = \emptyset$ if and only if*

$$\lim_{x\to\infty} \int_0^x g^2(t)\, dt \int_x^\infty \frac{dt}{g^2(t)} = 0.$$

Proof. Consider the self-adjoint Schrödinger operator $\mathcal{A}$ corresponding to $-d^2/dx^2 + V(x) - E_0$ with the same boundary condition at 0 (the existence of a non-L^2 solution implies the limit point case holds at infinity). Notice that $0 < \min \sigma(\mathcal{A})$. One can take certain solutions p, q, determined by the boundary condition of $\mathcal{A}$, to the equation $-y'' + (V - E_0)y = 0$ such that the canonical system

$$H_0(x) = \begin{pmatrix} p^2 & pq \\ pq & q^2 \end{pmatrix}$$

has the same m function as $\mathcal{A}$. Obtain a trace-normed H from H_0 by changing the variable to

$$X = \int_0^x \left(p^2(t) + q^2(t)\right)\, dt. \tag{5.6}$$

Thus, $H = P_\varphi$ with $\cot\varphi = p/q$. Keep in mind that p and q each have at most one zero by oscillation theory for Schrödinger operators. So, by Theorem 4.3, either $\lim_{x\to\infty} p/q$ or $\lim_{x\to\infty} q/p$ exists in $\mathbb{R}$. Suppose $M = \lim_{x\to\infty} p/q \in \mathbb{R}$. Let $f = p - Mq$. So, $f/q \to 0$. Since $0 < \min\sigma(\mathcal{A})$, there is a nontrivial L^2 solution of $-y'' + (V - E_0)y = 0$, and it is not hard to see that this solution must be (a multiple of) f.

By expanding $\cot\varphi = p/q$ in a Taylor series about $\varphi(\infty)$, one sees that $X(\varphi(X) - \varphi(\infty)) \to 0$ if and only if $Xf/q \to 0$. So, by Theorem 5.3, $\sigma_{ess} = \emptyset$ if and only if $Xf/q \to 0$. By constancy of the Wronskian $W = f'q - fq' = q^2(f/q)'$,

$$\frac{f(x)}{q(x)} = -W \int_x^\infty \frac{dt}{q^2(t)}, \tag{5.7}$$

and, again by constancy of a Wronskian,

$$p(x) = q(x)\left(M - \int_x^\infty \frac{dt}{q^2(t)}\right). \tag{5.8}$$

So, $\int_0^x p^2(t)\,dt \le C\int_0^x q^2(t)\,dt$. Hence, by using (5.6) and (5.8) to evaluate Xf/q, one sees that $\sigma_{ess} = \emptyset$ if and only if

$$\lim_{x\to\infty} \int_0^x q^2(t)\,dt \int_x^\infty \frac{dt}{q^2(t)} = 0.$$

Since f and q span the solution space and $f \in L^2$, g/q goes to a nonzero constant at infinity. This gives the criterion in the Theorem, assuming $\lim_{x\to\infty} p/q \in \mathbb{R}$. If If $\lim_{x\to\infty} p/q$ is not finite, then $\varphi(\infty) = 0$. In this case, swap every p and q in the above arguments (without changing H_0 and and the relation between p/q and φ), replace M with 0, and expand $\tan\varphi$ about $\varphi(\infty) = 0$. Note that $f = q$ in this situation. □

Chapter 6

Comparison results

We write an arbitrary trace-normed canonical system H on $(0, \infty)$ in the form

$$H(x) = \begin{pmatrix} \cos^2 \varphi(x) & g(x) \sin \varphi(x) \cos \varphi(x) \\ g(x) \sin \varphi(x) \cos \varphi(x) & \sin^2 \varphi(x) \end{pmatrix}$$

with a Borel measurable $g : (0, \infty) \to [0, 1]$. The goal in this chapter is to compare the spectra of H and

$$H_d(x) = \begin{pmatrix} \cos^2 \varphi(x) & 0 \\ 0 & \sin^2 \varphi(x) \end{pmatrix}.$$

Below $\psi = \varphi - \pi/2$.

The dimensions of the spectral projections $E(0, t)$ for a half-line canonical system are always bounded above $E_d(0, 2t) + 1$ where E_d is the spectral projection for the corresponding diagonal system. If $\sup_{x \in (0,\infty)} g(x) < 1$, then the dimensions of the spectral projections $E_d(0, s)$ are bounded above by $\dim E(0, (1 - \sup g(x))^{-1} s) + 1$. The next two propositions contain more precise statements along these lines.

Proposition 6.1. *Suppose that H does not end with a singular interval. Then*

$$\dim E(-t,0), \dim E(0,t) \leq \dim E_d(0,2t),$$
$$\dim E_d(-(1-\sup g(x))t, 0) \leq \dim E(-t,0), \ \ \textit{and}$$
$$\dim E_d(0,(1-\sup g(x))t) \leq \dim E(0,t)$$

for all $t > 0$.

Proof. Denote the right-hand side of (3.1) by $tf(H)$. So,

$$tf(H) = t\left(\sin^2\psi\cos^2\theta + \cos^2\psi\sin^2\theta - 2g\sin\psi\cos\psi\sin\theta\cos\theta\right). \tag{6.1}$$

Obviously, since $f(H_d) = \sin^2\psi\cos^2\theta + \cos^2\psi\sin^2\theta$,

$$f(H) \leq 2f(H_d). \tag{6.2}$$

Let E and E_d denote the spectral projections for H and H_d, respectively. Suppose H does not end with a singular half line. Then by the comparison principle and Theorem 3.3, $\dim E(0,t) \leq \dim E_d(0,2t)$ and $\dim E(-t,0) \leq \dim E_d(-2t,0)$ for $t > 0$.

Let $t > 0$. An easy calculation shows that

$$tH(x) - (1-\sup g(x))tH_d(x) \geq 0. \tag{6.3}$$

Hence, by the comparison principle, $\theta_d(x;(1-\sup g(y))t) \leq \theta(x;t)$ and $\theta_d(x;-(1-\sup g(y))t) \geq \theta(x;-t)$ for all $x, t > 0$.

Since H does not end with a singular half line, Theorem 3.3 and the above inequalities imply that $\dim E_d(0,(1-\sup g(x))t) \leq \dim E(0,t)$ and $\dim E_d(-(1-\sup g(x))t,0) \leq \dim E(-t,0)$ for all $t>0$. $\square$

Proposition 6.2. *Suppose that H ends a singular interval (L,∞) of type $\beta-\pi/2$ with $\beta\in[0,\pi)$. If $\beta-\pi/2\neq 0,-\pi/2$, then*

$$\dim E(-t,0), \dim E(0,t) \leq \dim E_d(0,2t)+1,$$
$$\dim E_d(-(1-\sup g(x))t,0) \leq \dim E(-t,0)+1, \text{ and}$$
$$\dim E_d(0,(1-\sup g(x))t) \leq \dim E(0,t)+1$$

for all $t>0$. If $\beta-\pi/2=0$ or $\beta-\pi/2=0$, then

$$\dim E(-t,0) \leq \dim E_d(0,-2t)+1,$$
$$\dim E(0,t) \leq \dim E_d(0,2t),$$
$$\dim E_d(-(1-\sup g(x))t,0) \leq \dim E(-t,0)+1, \text{ and}$$
$$\dim E_d(0,(1-\sup g(x))t) \leq \dim E(0,t)$$

for all $t>0$.

Proof. For the first, fourth, and fifth inequalities, we use (6.2). Suppose H ends with a singular half line (L,∞) of type $\beta-\pi/2\neq 0,-\pi/2$, $\beta\in[0,\pi)$. Then H_d does not end with a singular half line. Let $t>0$. 0 is not an eigenvalue for H, due to the choice of β, and hence $\dim E(0,t)=\lceil\frac{1}{\pi}(\theta(L;t)-\beta)\rceil$ by Lemma 3.2. Note that $\lceil\frac{1}{\pi}(\theta(L;t)-\beta)\rceil \leq \lfloor\frac{1}{\pi}\theta(L;t)\rfloor+1$, and by the comparison principle and (6.2), $\lfloor\frac{1}{\pi}\theta(L;t)\rfloor \leq \lfloor\frac{1}{\pi}\theta_d(L;2t)\rfloor$. Now, by Theorem 3.3, $\dim E_d(0,2t)=\lim_{x\to\infty}\left\lfloor\frac{1}{\pi}\theta_d(x;2t)\right\rfloor$. So, by the monotonicity of θ,

$$\dim E(0,t) \le \dim E_d(0,2t) + 1.$$

Turning to negative spectrum,

$$\dim E(-t,0) \le \dim E[-t,0) = -\lceil \frac{1}{\pi}(\theta(L;-t) - \beta)\rceil$$

by Lemma 3.2. Note that $-\lceil \frac{1}{\pi}(\theta(L;-t) - \beta)\rceil \le -\lceil \frac{1}{\pi}\theta(L;-t)\rceil + 1$, and by the comparison principle $-\lceil \frac{1}{\pi}\theta(L;-t)\rceil \le -\lceil \frac{1}{\pi}\theta_d(L;-2t)\rceil$. By the monotonicity of θ,

$$\dim E(-t,0) \le \dim E_d(-2t,0) + 1.$$

Suppose H ends with a singular half line (L,∞) of type $\beta - \pi/2 = 0$ or $\beta - \pi/2 = -\pi/2$. So, H_d ends with the same singular half line. Let $t > 0$. Now, 0 is an eigenvalue for H if and only if it is an eigenvalue for H_d. Suppose 0 is not an eigenvalue. Then

$$\dim E(0,t) = \dim E[0,t) = \lceil \frac{1}{\pi}(\theta(L;t) - \beta)\rceil,$$

and similarly $\dim E_d(0,2t) = \lceil \frac{1}{\pi}(\theta_d(L;t) - \beta)\rceil$. So, by the comparison principle and (6.2), $\dim E(0,t) \le \dim E_d(0,2t)$. Suppose 0 is an eigenvalue. Then

$$\dim E(0,t) = \dim E[0,t) - 1 = \lceil \frac{1}{\pi}(\theta(L;t) - \beta)\rceil - 1,$$

and similarly $\dim E_d(0,2t) = \lceil \frac{1}{\pi}(\theta_d(L;t) - \beta)\rceil - 1$. Then by the comparison principle,

$\dim E(0,t) \le \dim E_d(0,2t)$. For negative spectrum,

$$\dim E(-t,0) \le \dim E[-t,0) = -\lceil \frac{1}{\pi}(\theta(L;-t) - \beta) \rceil,$$

and similarly $\dim E_d(-2t,0) \ge -\lceil \frac{1}{\pi}(\theta_d(L;-t) - \beta) \rceil - 1$. By the comparison principle, $-\lceil \frac{1}{\pi}(\theta(L;-t) - \beta) \rceil \le -\lceil \frac{1}{\pi}(\theta_d(L;-2t) - \beta) \rceil$. So, $\dim E(-t,0) \le \dim E_d(-2t,0) + 1$.

For the other inequalities, we use (6.3). Suppose H ends with a singular half line (L,∞) of type $\beta - \pi/2 \ne 0, -\pi/2$, $\beta \in [0,\pi)$. H_d then does not end with a singular half line. Let $t > 0$. Due to the choice of β, 0 is not an eigenvalue for H, and hence $\dim E(0,t) = \lceil \frac{1}{\pi}(\theta(L;t) - \beta) \rceil$ by Lemma 3.2. For any $x \ge L$, the problems on $(0,L)$ and $(0,x)$ with the boundary condition β at the right endpoints have the same spectra. So, $\dim E(0,t) = \lceil \frac{1}{\pi}(\theta(x;t) - \beta) \rceil$ for any $x \ge L$. Note that $\lceil \frac{1}{\pi}(\theta(x;t) - \beta) \rceil \ge \lfloor \frac{1}{\pi}\theta(x;t) \rfloor - 1$ for all $x \ge L$. By the comparison principle and (6.3), $\lfloor \frac{1}{\pi}\theta(x;t) \rfloor \ge \lfloor \frac{1}{\pi}\theta_d(x;(1-\sup g(y))t) \rfloor$. Now, by Theorem 3.3,

$$\dim E_d(0,(1-\sup g(y))t) = \lim_{x\to\infty} \left\lfloor \frac{1}{\pi}\theta(x;(1-\sup g(y))t) \right\rfloor.$$

So,

$$\dim E_d(0,(1-\sup g(x))t) \le \dim E(0,t) + 1.$$

For negative spectrum, note that

$$\dim E(-t,0) \ge \dim E[-t,0) - 1 = -\lceil \frac{1}{\pi}(\theta(x;-t) - \beta) \rceil - 1$$

for any $x \ge L$ by Lemma 3.2. Also note that $-\lceil \frac{1}{\pi}(\theta(x;-t) - \beta) \rceil \ge -\lceil \frac{1}{\pi}\theta(x;-t) \rceil$.

Then by the comparison principle,

$$\dim E_d(-(1-\sup g(x))t, 0) \leq \dim E(-t, 0) + 1.$$

Suppose H ends with a singular half line (L, ∞) of type $\beta - \pi/2 = 0$ or $\beta - \pi/2 = -\pi/2$, which forces H_d to end with the same singular half line. Let $t > 0$. Again, 0 is an eigenvalue for H if and only if it is an eigenvalue for H_d. Suppose 0 is not an eigenvalue. Then

$$\dim E(0, t) = \dim E[0, t) = \lceil \frac{1}{\pi}(\theta(L; t) - \beta) \rceil,$$

and similarly $\dim E_d(0, (1-\sup g(x))t) = \lceil \frac{1}{\pi}(\theta_d(L; (1-\sup g(x))t) - \beta) \rceil$. So, the comparison principle implies that $\dim E_d(0, (1-\sup g(x))t) \leq \dim E(0, t)$. Suppose 0 is an eigenvalue. Then

$$\dim E(0, t) = \dim E[0, t) - 1 = \lceil \frac{1}{\pi}(\theta(L; t) - \beta) \rceil - 1,$$

and $\dim E_d(0, (1-\sup g(x))t) = \lceil \frac{1}{\pi}(\theta_d(L; (1-\sup g(x))t) - \beta) \rceil - 1$. Then $\dim E_d(0, (1-\sup g(x))t) \leq \dim E(0, t)$ by the comparison principle. For negative spectrum,

$$\dim E(-t, 0) \geq \dim E[-t, 0) - 1 = -\lceil \frac{1}{\pi}(\theta(L; -t) - \beta) \rceil - 1,$$

and $\dim E_d(-(1-\sup g(x))t, 0) \leq -\lceil \frac{1}{\pi}(\theta_d(L; -(1-\sup g(x))t) - \beta) \rceil$. The comparison principle implies that $-\lceil \frac{1}{\pi}\theta(L; -t) \rceil \geq -\lceil \frac{1}{\pi}\theta_d(L; -(1-\sup g(x))t) \rceil$. So, $\dim E_d(-(1-\sup g(x))t, 0) \leq \dim E(-t, 0) + 1$. □

Corollary 6.3. *It is always the case that $M(H_d)/2 \leq M(H)$. .*

Proof. This follows immediately from the preceding propositions. □

This corollary and the preceding propositions raise the question of whether $M(H)$ is bounded above by some constant times $M(H_d)$. This is not always the case, as easy examples show. Note that as soon as $\sin\psi$ and $\cos\psi$ are both not in L^2, $M(H_d) = 0$. However, the arguments for the above inequalities involving $\sup g(x)$ suggest the following result.

Theorem 6.4. *Suppose $G = \limsup_{x\to\infty} g(x) < 1$. Then $M(H) \leq M(H_d)/(1-G)$, and if $\sigma_{ess}(H) \cap (-\infty, 0] = \emptyset$ or $\sigma_{ess}(H) \cap [0, \infty) = \emptyset$, then $\sigma_{ess}(H) = \emptyset$.*

Proof. Let $0 < \eta < 1 - G$. Take $a > 0$ such that $g(x) \leq G + \eta$ for all $x \geq a$. Let $t \in (0, M(H))$. Once again it is easy to check that

$$tH(x) - (1 - (G + \eta))tH_d(x) \geq 0$$

for all $x \geq a$. It follows from the comparison principle that $\theta_d(x; (1-(G+\eta))t)$ has a finite limit as $x \to \infty$. Hence, $(1 - (G + \eta))t \leq M(H_d)$ for all $0 < \eta < 1 - G$ and $t \in (0, M(H))$. Hence,

$$M(H) \leq \frac{1}{1-G} M(H_d).$$

Now, suppose that $\sigma_{ess}(H) \cap (-\infty, 0] = \emptyset$. Suppose for the sake of contradiction that $\sigma_{ess}(H) \neq \emptyset$, so that $M(H) < \infty$. Take a $t > M(H)$. So, $2t > M(H_d)$, and hence, by the symmetry of the spectrum of diagonal systems, $\lim_{x\to\infty} \theta_d(x; -2t) = -\infty$. Take η and a as above. Take any s with $(1 - (G + \eta))s \geq t$. Hence, $\lim_{x\to\infty} \theta_d(x; -2(1-(G+\eta))s) = -\infty$. Then, since $-2sH(x)+2(1-(G+\eta))sH_d(x) \leq 0$ for all $x \geq a$, the comparison principle implies that $\lim_{x\to\infty} \theta(x; -2s) = -\infty$. This contradicts the assumption that $\sigma_{ess}(H) \cap (-\infty, 0] = \emptyset$. The claim that $\sigma_{ess}(H) \cap [0, \infty) = \emptyset$ implies $\sigma_{ess}(H) = \emptyset$ follows from a similar argument. □

Another case when $M(H)$ is bounded above by a constant times $M(H_d)$ is when $\sin\psi \in L^2(0,\infty)$. The proof of this works with no alteration if it is instead assumed that $\lim_{x\to\infty}\psi(x) = 0$. That case is trivial if $\sin\psi \notin L^2(0,\infty)$, but it is included for completeness.

Theorem 6.5. *Assume that* $\sin\psi \in L^2(0,\infty)$ *or* $\lim_{x\to\infty}\psi(x) = 0$. *Let* $G = \limsup_{x\to\infty} g(x)$. *If* $G < \sqrt{3}/2$, *then* $M(H) \le 2M(H_d)$. *If* $G \ge \sqrt{3}/2$, *then*

$$M(H) \le \frac{2}{3-\sqrt{1+4G^2}} M(H_d).$$

If $\lim_{x\to\infty}\psi(x) = 0$ *and* $\sin\psi \notin L^2(0,\infty)$, *then* $M(H) = M(H_d) = 0$.

Proof. Let $\eta \in (0, \sqrt{2}-1)$ be arbitrary, and define

$$c = \begin{cases} \left(3-\sqrt{1+4(G+\eta)^2}\right)/2 & G \ge \frac{\sqrt{3}}{2} \\ 1/2 & G < \frac{\sqrt{3}}{2} \end{cases}.$$

It suffices to prove that $M(H) \le \frac{1}{c}M(H_d)$ for every $\eta \in (0, \sqrt{2}-1)$ (this choice of η ensures that $c > 0$). Take $a > 0$ such that, for all $x \ge a$, $|g(x)| \le G+\eta$ if $G \ge \frac{\sqrt{3}}{2}$ or $|g(x)| \le \frac{\sqrt{3}}{2}$ if $G < \frac{\sqrt{3}}{2}$. If $M(H) = 0$, there is nothing to be done, so suppose $M(H) > 0$. Let $t \in (0, M(H))$. So, due to the assumptions of ψ, there exist integers m, n such that $\theta_+(x) = \theta(x;t) + m\pi$ and $\theta_-(x) = \theta(x;-t) + n\pi$ satisfy $\lim_{x\to\infty}\theta_\pm(x) = 0$. Note that $\theta_+(x) - \theta_-(x) \le 0$ for all $x > 0$. To show that $t \le \frac{1}{c}M(H_d)$, it suffices to show that, for every $0 < \epsilon < c$, there is a $b \ge a$ such

$$(\theta_+ - \theta_-)' \ge t(c-\epsilon)(\sin^2\psi\cos^2(\theta_+ - \theta_-) + \cos^2\psi\sin^2(\theta_+ - \theta_-)) \tag{6.4}$$

at all $x \ge b$. To see this, suppose $t > \frac{1}{c}M(H_d)$. Take $\epsilon > 0$ such that $t(c-\epsilon) >$

$M(H_d)$. Take $b \geq a$ such that (6.4) holds at all $x \geq b$. Since $t(c-\epsilon) > M(H_d)$ and the spectrum of H_d is symmetric about 0, $\lim_{x\to\infty} \theta_d(x; t(c-\epsilon)) = \infty$. Take a negative integer k such that $\theta_d(b; t(c-\epsilon)) + k\pi \leq \theta_+(b) - \theta_-(b)$. . This would imply, by (6.4) and the comparison principle, that $\theta_d(x; t(c-\epsilon)) + k\pi \leq \theta_+(x) - \theta_-(x) \leq 0$ for all $x \geq b$, contradicting the assumption that $\lim_{x\to\infty} \theta_d(x; t(c-\epsilon)) = \infty$. Note that the right-hand side above is obtained by taking the right-hand side of the Prüfer equation for H_d and replacing θ_d with $\theta_+ - \theta_-$.

So, fix $\epsilon > 0$. Let $F = (\theta_+ - \theta_-)' / t$. Note that

$$\begin{aligned} F = \sin^2\psi(\cos^2\theta_+ + \cos^2\theta_-) + \cos^2\psi(\sin^2\theta_+ + \sin^2\theta_-) \\ -g\sin\psi\cos\psi(\sin 2\theta_+ + \sin 2\theta_-)\ . \end{aligned}$$

Using the fact that $\lim_{x\to\infty} \theta_\pm(x) = 0$, take $b \geq a$ such that

$$F \geq \left(1 - \frac{\epsilon}{c}\right)\left[2\sin^2\psi + (\theta_+^2 + \theta_-^2)\cos^2\psi - 2\left|g(\theta_+ + \theta_-)\sin\psi\cos\psi\right|\right]$$

on $[b, \infty)$. Let

$$K = \begin{cases} G + \eta & G \geq \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & G < \frac{\sqrt{3}}{2} \end{cases}.$$

Then, by the choice of a, the right hand side is bounded below by

$$\left(1 - \frac{\epsilon}{c}\right)\left[2\sin^2\psi + (\theta_+^2 + \theta_-^2)\cos^2\psi - 2\left|K(\theta_+ + \theta_-)\sin\psi\cos\psi\right|\right]$$

on $[b, \infty)$.

Now, the inequality

$$\sin^2\psi\cos^2(\theta_+ - \theta_-) + \cos^2\psi\sin^2(\theta_+ - \theta_-) \le \sin^2\psi + (\theta_+ - \theta_-)^2\cos^2\psi$$

always holds. So, to prove (6.4), it suffices to show that $c(\sin^2\psi + (\theta_+ - \theta_-)^2\cos^2\psi)$ is bounded above by $2\sin^2\psi + (\theta_+^2 + \theta_-^2)\cos^2\psi - 2\,|K(\theta_+ + \theta_-)\sin\psi\cos\psi|$ on $[b,\infty)$.

This last bound follows easily if $\cos\psi = 0$ from the fact that $c < 2$. Assume $\cos\psi \ne 0$, and let $T = \tan\psi$. Then the desired bound is equivalent to

$$(2-c)T^2 - 2K|T|\;|\theta_+ + \theta_-| + \theta_+^2 + \theta_-^2 - c(\theta_+ - \theta_-)^2 \ge 0$$

on $[b,\infty)$. By basic calculus, the left-hand side is at a minimum when $|T| = K|\theta_+ + \theta_-|/(2-c)$. Thus, it suffices to prove that

$$-\frac{K^2(\theta_+ + \theta_-)^2}{2-c} + \theta_+^2 + \theta_-^2 - c(\theta_+ - \theta_-)^2 \ge 0 \tag{6.5}$$

on $[b,\infty)$.

Suppose there is an $x_0 > 0$ such that $\theta_-(x_0) = 0$. Then $\theta_-(x) = 0$ for all $x \ge x_0$, and it follows from the Prüfer equation that $\sin\psi(x) = 0$ for all $x \ge x_0$. In this situation, $M(H) = M(H_d) = \infty$ because the spectra are purely discrete.

So now suppose that $\theta_-(x_0) > 0$ for all $x \ge b$. Let $q = -\theta_+/\theta_-$. Then (6.5) is equivalent to

$$-\frac{K^2(-q+1)^2}{2-c} + q^2 + 1 - c(q+1)^2 \ge 0,$$

and this is equivalent to

$$(c^2 - 3c + 2 - K^2)q^2 + 2(c^2 - 2c + K^2)q + c^2 - 3c + 2 - K^2 \ge 0.$$

By the choice of c, $c^2 - 3c + 2 - K^2 = 0$. Also, because $K^2 \geq 3/4$,

$$c^2 - 2c + K^2 = c^2 - 3c + 2 - K^2 + c - 2 + 2K^2 = \frac{3 - \sqrt{1 + 4K^2}}{2} - 2 + 2K^2 \geq 0.$$

So, since $q \geq 0$, the desired inequality follows.

Suppose $\lim_{x\to\infty} \psi(x) = 0$ and $\sin\psi \notin L^2(0,\infty)$. Then also $\cos\psi \notin L^2(0,\infty)$. Hence, there is no $0 \neq v \in \mathbb{R}^2$ with $\int_0^\infty v^* H_d(x) v\, dx < \infty$. So, $0 \in \sigma_{ess}(H_d)$. Then by the inequality in the theorem, $M(H) = M(H_d) = 0$. $\square$

Theorem 6.6 ([13]). *Assume that* $\sin\psi \in L^2(0,\infty)$. *Then*

$$\frac{1}{2} M(H_d) \leq M(H) \leq \frac{2}{3 - \sqrt{5}} M(H_d).$$

Proof. The inequalities are immediate from Corollary 6.3 and Theorem 6.5. $\square$

An immediate consequence of this is that, when $\sin\psi \in L^2(0,\infty)$, H has purely discrete spectrum if and only if H_d has purely discrete spectrum. This was noticed earlier by Romanov and Woracek using different methods [15].

Example 6.1. The lower bound in Theorem 6.6 is sharp. Consider a Schrödinger equation

$$-y'' + \frac{1}{4} y = zy \tag{6.6}$$

with constant potential $V(x) \equiv 1/4$. Using the transformation, which is discussed in in [11, Section 1.3], alluded to earlier, (6.6) can be rewritten as a canonical system. Given an absolutely continuous y, let $Y = (y', y)^t$. Then y is a solution of (6.6) if and only if Y is a solution of

$$Y' = \begin{pmatrix} 0 & 1/4 - z \\ 1 & 0 \end{pmatrix} Y. \tag{6.7}$$

Let $p = e^{x/2}$ and $q = e^{-x/2}$, two independent solutions of (6.6). Define

$$T_0 = \begin{pmatrix} p' & q' \\ p & q \end{pmatrix}.$$

Then T_0 solves (6.7) for $z = 0$ and $\det T_0(x) = 1$.

Given Y, let $u = T_0^{-1} Y$. By direct calculation, Y is a solution of (6.7) if and only if u is a solution of (2.1) with

$$H(x) = \begin{pmatrix} p^2 & pq \\ pq & q^2 \end{pmatrix} = \begin{pmatrix} e^x & 1 \\ 1 & e^{-x} \end{pmatrix}.$$

Note that

$$H_d(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$$

has determinant 1 and absolutely continuous entries. Hence, the diagonal system can be rewritten as a Dirac equation

$$Jv' = \begin{pmatrix} 0 & W(x) \\ W(x) & 0 \end{pmatrix} v - zv,$$

where $W(x) = (e^x)'/(2e^x) = 1/2$ [11, Section 6.4].

The bottom of the essential spectrum for a Schrödinger or Dirac operator with a constant potential is well-known. It follows that $M(H) = 1/4$ and $M(H_d) = 1/2$ [13]. In fact, one could compute $M(H)$ and $M(H_d)$ directly using Theorems 5.1, 5.2, 7.2, and 7.3 (all the limits involved exist and can be computed with basic calculus).

It is not known whether the constant in the upper bound of Theorem 6.6 is sharp. Obviously, Theorems 6.4 and 6.5 give better constants in some situations. Another result along these lines is the following theorem.

Theorem 6.7. *Assume that either $\psi(x) \in [-\pi, -\pi/2]$ eventually or $\psi(x) \in [-\pi/2, 0]$ eventually. Let $\gamma \in \{0, -\pi/2, -\pi\}$ and suppose that either $\gamma = \lim_{x\to\infty} \psi(x)$ or $\sin(\psi(x) - \gamma) \in L^2(0, \infty)$. Then*

$$M(H) \leq M(H_d).$$

If $\psi(x) \in [-\pi/2, 0]$ eventually and $\gamma = 0$ or $\psi(x) \in [-\pi, -\pi/2]$ eventually and $\gamma = -\pi/2$, then $\sigma_{ess}(H) \cap (-\infty, 0] = \emptyset$ implies $\sigma_{ess}(H) = \emptyset$. If $\psi(x) \in [-\pi, -\pi/2]$ eventually and $\gamma = -\pi$ or $\psi(x) \in [-\pi/2, 0]$ eventually and $\gamma = -\pi/2$, then $\sigma_{ess}(H) \cap [0, \infty) = \emptyset$ implies $\sigma_{ess}(H) = \emptyset$.

Proof. First suppose that $\psi(x) \in [-\pi/2, 0]$ eventually and $\gamma = 0$ (or $\psi(x) \in [-\pi, -\pi/2]$ eventually and $\gamma = -\pi/2$). Let $t \in (0, M(H))$. So, $\theta(x; -t)$ converges to a finite limit as $x \to \infty$, and due to the assumptions on ψ, this limit must be $\equiv \gamma$ mod π. So, we can take an $a > 0$ such that $\sin 2\theta(x; -t) \geq 0$ (or ≤ 0) for all $x \geq a$. Now take a $b \geq a$ such that $\psi(x) \in [-\pi/2, 0]$ (or $\psi(x) \in [-\pi, -\pi/2]$) for all $x \geq b$. Note that

$$\sin^2\psi \cos^2\theta + \cos^2\psi \sin^2\theta - \frac{g}{2}\sin 2\psi \sin 2\theta \geq \sin^2\psi \cos^2\theta + \cos^2\psi \sin^2\theta \qquad (6.8)$$

when $\psi \in [-\pi/2, 0]$ and $\sin 2\theta \geq 0$ (or $\psi \in [-\pi, -\pi/2]$ and $\sin 2\theta \leq 0$). Take an integer m such that $\theta_d(b;-t)+m\pi \geq \theta(b;-t)$. Then, by the comparison principle, $\theta_d(x;-t)+m\pi \geq \theta(x;-t)$ for all $x \geq b$. Thus, $t \leq M(H_d)$.

Now suppose that $\sigma_{ess}(H) \neq \emptyset$. Hence, $M(H) < \infty$. Take a $t > M(H)$. So, by Corollary 6.3 and the symmetry of the spectrum of diagonal systems, $\theta_d(x,-2t) \to -\infty$ as $x \to \infty$. The above inequalities and the comparison principle then imply that $\lim_{x\to\infty} \theta(x,-2t) = -\infty$. Thus, $\sigma_{ess}(H) \cap (-\infty, 0] \neq \emptyset$.

For the other case, suppose that either $\psi(x) \in [-\pi,-\pi/2]$ eventually and $\gamma = -\pi$ (or $\psi(x) \in [-\pi/2, 0]$ eventually and $\gamma = -\pi/2$). Let $t \in (0, M(H))$. As above, $\lim_{x\to\infty} \theta(x;t)$ exists and must be $\equiv \gamma \bmod \pi$. So, now we can take an $a > 0$ such that $\sin 2\theta(x;t) \leq 0$ (or ≥ 0) for all $x \geq a$. Now take a $b \geq a$ such that $\psi(x) \in [-\pi,-\pi/2]$(or $[0,\pi/2]$) for all $x \geq b$. Note that (6.8) holds when $\psi \in [-\pi,-\pi/2]$ and $\sin 2\theta \leq 0$ (or $\psi \in [-\pi/2, 0]$ and $\sin 2\theta \geq 0$). Take an integer m such that $\theta_d(b;t)+m\pi \leq \theta(b;t)$. Then, by the comparison principle, $\theta_d(x;t)+m\pi \leq \theta(x;t)$ for all $x \geq b$. Thus, $t \leq M(H_d)$. The other claim follows from an argument very similar to the previous case. □

Corollary 6.8. *Let H have finite negative spectrum and take φ as is Theorem 4.3. If $\varphi(\infty) \equiv 0$ mod $\pi/2$, then $M(H) \leq M(H_d)$.*

Proof. Due to the monotonicity of φ, this follows immediately from Theorem 6.7. □

Chapter 7

The essential spectrum of canonical systems

In this chapter, the bottom of the essential spectrum of a diagonal system is studied. Some comments are given at the end about how the theorems here and in the previous section could be applied to systems that do not satisfy the running assumption that $\sin\psi \in L^2(0,\infty)$ or to measuring the distance of an eigenvalue from the essential spectrum.

For small θ, $t(\cos^2\psi \sin^2\theta + \sin^2\psi \cos^2\theta)$, the right-hand side of the Prüfer equation for the diagonal system, is comparable to $t(\theta^2 + \sin^2\psi)$, and the differential equation $\theta' = t(\theta^2 + \sin^2\psi)$ can be rewritten as a Schrödinger equation $-u'' - t^2(\sin^2\psi)u = 0$. If $\sin\psi \in L^2(0,\infty)$, then the potentials $-t^2(\sin^2\psi) \in L^1(0,\infty)$. So, the spectra in $(-\infty, 0)$ of the corresponding Schrödinger operators are purely discrete and bounded below. Also, note that the limit point case holds for these Schrödinger equations since the potentials are bounded. So, only a boundary condition at 0 is needed to obtain a self-adjoint operator, and there is the possibility that the spectrum in $(-\infty, 0)$ accumulates at 0. By Sturm's comparison theorem, the set of

$t \geq 0$ such that $-u'' - t^2(\sin^2\psi)u = 0$ is non-oscillatory is either $\{0\}$ or an interval. The following theorem states that the value of t where these equations switch from non-oscillatory to oscillatory is $M(H_d)$.

Theorem 7.1 ([13])**.** *Suppose* $\sin\psi \in L^2(0,\infty)$*. Let* $\mathcal{L}(t)$ *be a self-adjoint Schrödinger operator on* $L^2(0,\infty)$ *with any boundary conditions at* 0 *that is generated by*

$$-\frac{d^2}{dx^2} - t^2\sin^2\psi(x),$$

and let

$$S = \sup\{t \geq 0 : \mathcal{L}(t) \text{ has finite negative spectrum}\}.$$

Then $M(H_d) = S$*.*

Proof. To show that $M(H_d) \leq S$, assume $M(H_d) > 0$ since there is nothing to prove if $M(H_d) = 0$. Take $t \in (0, M(H_d))$ and then $\epsilon > 0$ such that $t < (1-\epsilon)M(H_d)$. Using the assumption that $\sin\psi \in L^2$ take a solution θ of

$$\theta' = \frac{t}{1-\epsilon}(\cos^2\psi\sin^2\theta + \sin^2\psi\cos^2\theta)$$

such that $\theta(x) \leq 0$ for all $x > 0$ and $\lim_{x\to\infty}\theta(x) = 0$. Take $a > 0$ such that

$$\begin{aligned}\cos^2\psi\sin^2\theta + \sin^2\psi\cos^2\theta &= \sin^2\theta + \cos 2\theta\sin^2\psi \\ &\geq (1-\epsilon)(\theta^2 + \sin^2\psi)\end{aligned}$$

on $[a,\infty)$. So, by the comparison principle, the solution θ_1 of

$$\theta_1' = t(\theta_1^2 + \sin^2\psi), \qquad \theta_1(a) = \theta(a)$$

also has the property that $\theta_1(x) \leq 0$ for all $x \geq a$. In particular, $\theta_1(x)$ exists as a solution for all $x > a$. Set

$$u(x) = \exp\left(-t\int_a^x \theta_1(s)\,ds\right).$$

for $x > a$. Then $u(x)$ is a solution of the Schrödinger equation, $-u''-t^2(\sin^2\psi)u = 0$. Clearly, $u(x) > 0$ for all $x > a$. Hence, by classical oscillation theory for Schrödinger operators, $\mathcal{L}(t)$ has finite negative spectrum. Thus, $t \leq S$ for all $t \in (0, M(H_d))$.

To show the opposite inequality, let $t \in (0, S)$. By classical oscillation theory, which implies that all solutions of $-u'' - t^2(\sin^2\psi)u = 0$ have finitely many zeros, one can take an $a > 0$ such that some solution of $-u'' - t^2(\sin^2\psi)u = 0$ has no zeros in $[a, \infty)$. Also by classical oscillation theory, it follows that each solution of $-u'' - t^2(\sin^2\psi)u = 0$ has at most one zero in $[a, \infty)$. Let u be the solution with $u(a+1) = u'(a+1) = 1$. Suppose $u(x) > 0$ for every $x \in [a, a+1]$. Then $u'' = -t^2(\sin^2\psi)u \leq 0$ on $[a, a+1]$. Thus, $u'(x) \geq u'(a+1) = 1$ for all $x \in [a, a+1]$. So, $u(a) = 1 - \int_a^{a+1} u'(y)\,dx \leq 0$. So, u has a zero in $[a, a+1)$, but this contradicts the assumption that $u > 0$ on $[a, a+1]$. So, there is some $x \in [a, a+1)$ with $u(x) \leq 0$. Since $u(a+1) > 0$, this implies that u has a zero in $[a, a+1)$. Hence, since u has no zeros in $[a+1, \infty)$, $u > 0$ on $[a+1, \infty)$. So, $u'' \leq 0$ on $[a+1, \infty)$. It cannot be the case that u' is negative somewhere in $[a+1, \infty)$ because u' is decreasing on that interval, so if it were negative, this would force u to reach 0 eventually. Suppose there is an $x_0 > a+1$ such that $u'(x_0) = 0$. Then, since u' is decreasing and never negative on $[a+1, \infty)$, $u'(x) = 0$ for all $x > x_0$. Hence, since $u'' = -t^2(\sin^2\psi)u$ and $u > 0$ on $[a+1, \infty)$, $\sin^2\psi \equiv 0$ on (x_0, ∞). If $\sin^2\psi \equiv 0$ eventually, then H_d has purely discrete spectrum since it ends with a singular half line, and $L(t)$ has finite negative spectrum for every t since solutions of $-u'' = 0$ obviously do not have infinitely many zeros. Thus, $M(H_d) = S = \infty$ when $\sin^2\psi \equiv 0$ eventually, and when this is not the case, u' cannot have a zero in $[a+1,\infty)$. So, $u, u' > 0$ on $[a+1,\infty)$.

Now, $\theta_1' = t(\theta_1^2 + \sin^2\psi)$ is solved, following the same transformation as in the first step, by $\theta_1 = -u'/(tu)$. Due to the above observations about u, this solution satisfies $\theta_1(x) = -u'(x)/(tu(x)) < 0$ for all $x \geq a+1$. Since $\cos^2\psi\sin^2\theta + \sin^2\psi\cos^2\theta \leq (\theta^2 + \sin^2\psi)$, the comparison principle implies that (3.1) for the diagonal system has a solution that stays below 0. As argued earlier, this implies that $\theta_d(x;t)$ has a finite limit as $x \to \infty$. So, $t \leq M(H_d)$ for every $0 < t < S$. The corresponding inequality for negative t follows from this case because the spectrum of H_d is symmetric about 0. $\square$

We now prove analogs of Theorems 5.1 and 5.2 for diagonal systems.

Theorem 7.2 ([13]). *Suppose* $\sin\psi \in L^2(0,\infty)$, *and set*

$$A = \limsup_{x\to\infty} x \int_x^\infty \sin^2\psi(t)\,dt.$$

Then

$$\frac{1}{2\sqrt{A}} \leq M(H_d) \leq \frac{1}{\sqrt{A}}.$$

Proof. For the first inequality, we can obviously assume $A < \infty$. Take any $C > A$. Let $0 < t < 1/(2\sqrt{C})$. To prove the first inequality in the theorem, it suffices to show that there is an $a > 0$ and a $\theta_0 \leq 0$ such that the solution $\theta(x)$ of

$$\theta' = t(\cos^2\psi\sin^2\theta + \sin^2\psi\cos^2\theta) \tag{7.1}$$

with initial value $\theta(a) = \theta_0$ satisfies $\theta(x) \leq 0$ for all $x > a$. To see that this implies the first inequality, note first that since one can find a negative integer n such that $\theta(a;t)+n\pi \le \theta_0$, and then $\theta(x;t)+n\pi \le \theta(x)$ for $x > a$ by the comparison principle, $\lim_{x\to\infty} \theta(x;t) = \infty$ is impossible . So, this leads to the conclusion that $t \le M(H_d)$ for every $0 < t < 1/(2\sqrt{C})$ and every $C > A$, which obviously implies the first inequality.

As show in the proof of Theorem 5.1, since $t^2C < 1/4$, there exists an $a_1 > 0$ such that

$$\alpha_1' = \left(\alpha_1 - \frac{t^2C}{x}\right)^2 \tag{7.2}$$

has a solution $\alpha_1(x)$ satisfying $\alpha_1(x) \le 0$ on (a_1, ∞). Take an $a > a_1$ such that $W(x) = \int_x^\infty \sin^2\psi(s)\,ds \le C/x$ for all $x \ge a$. Then

$$\left(\alpha_1(x) - \frac{t^2C}{x}\right)^2 \ge \left(\alpha_1(x) - t^2W(x)\right)^2$$

for all $x \ge a$. So, by the comparison principle, the initial value problem $\alpha' = (\alpha - t^2W)^2$, $\alpha(a) = \alpha_1(a)$, has a solution $\alpha(x) \le 0$ on (a, ∞). Let $\theta_1(x) = \frac{1}{t}(\alpha(x) - t^2W(x))$. Then $\theta_1' = t(\theta_1^2 + \sin^2\psi)$. Note that $\theta_1(x) \le 0$ for all $x \ge a$, and as noted earlier,

$$\cos^2\psi \sin^2\theta + \sin^2\psi\cos^2\theta \le (\theta^2 + \sin^2\psi).$$

So, by the comparison principle, the solution $\theta(x)$ of

$$\theta' = t(\cos^2\psi\sin^2\theta + \sin^2\psi\cos^2\theta) \tag{7.3}$$

with initial value $\theta(a) = \theta_2(a)$ satisfies $\theta(x) \le 0$ for all $x > a$.

For the second inequality, we use Theorem 7.1, $M(H_d) = S$. Take $t \in (0, S)$ and $C < A$. So, by classical oscillation theory, one can take $a \ge 1$ large enough so that the Schrödinger operator $L = -d^2/dx^2 - t^2 \sin^2 \psi(x)$ on $L^2(a,\infty)$ with Dirichlet boundary conditions $u(a) = 0$ has no negative spectrum. By the definition of A, one can take a sequence $a < b_n \nearrow \infty$ such that $\int_{b_n}^{\infty} \sin^2 \psi \, dx \geq C/b_n$.

Let

$$u_n(x) = \begin{cases} \frac{x-a}{b_n-a} & a < x < b_n \\ 1 & b_n < x < b_n + 1 \\ \frac{b_n^2+1-x}{b_n^2-b_n} & b_n + 1 < x < b_n^2 + 1 \\ 0 & x > b_n^2 + 1 \end{cases},$$

which is obviously in H^1. So, since L is nonnegative, L's quadratic form $Q(u) = \int_a^{\infty} \left(|u'(x)|^2 - t^2 (\sin^2 \psi(x)) \, |u(x)|^2 \right) dx$ satisfies $Q(u_n) \geq 0$ for all n. By direct calculation and the choice of b_n,

$$Q(u_n) \leq \frac{1}{b_n - a} - \frac{t^2 C}{b_n} + \frac{1}{b_n^2 - b_n}.$$

Hence,

$$\frac{1}{b_n - a} - \frac{t^2 C}{b_n} + \frac{1}{b_n^2 - b_n} \geq 0.$$

By rearranging this inequality, one obtains that

$$t^2 C \leq 1 + \frac{a}{b_n - a} + \frac{1}{b_n - 1}.$$

Since the right-hand side converges to 1 as $n \to \infty$, $t^2 C \leq 1$. So, $t \leq 1/\sqrt{C}$ for every $t \in (0, S) = (0, M(H_d))$ and $C < A$. □

Theorem 7.3 ([13]). *Suppose* $\sin \psi \in L^2(0,\infty)$, *and set*

$$B = \liminf_{x \to \infty} x \int_x^{\infty} \sin^2 \psi(t) \, dt.$$

Then

$$M(H_d) \leq \frac{1}{2\sqrt{B}}.$$

Proof. We use the identity $M(H_d) = S$ from Theorem 7.1. Fix $t \in (0, S)$. Let

$$W(x) = \int_x^\infty \sin^2 \psi(s)\, ds.$$

Take $a_0 > 0$ such that $t^2 W(a_0) < 1$. Fix $C < B$. Take $a_1 \geq a_0$ such that $W(x) \geq C/x$ for all $x \geq a_1$. As argued in the proof of Theorem 7.1, the solution of $-u'' - t^2(\sin^2 \psi)u = 0$ with initial values $u(a) = u'(a) = 1$ will stay above 0 for $x \geq a$ for sufficiently large a. Fix such an $a \geq a_1$ and u.

Let $\alpha = u'/u - t^2 W$. So, α solves

$$\alpha' = -\left(\alpha + t^2 W\right)^2, \tag{7.4}$$

and $\alpha(a) > 0$. In fact, $\alpha(x) \geq 0$ for all $x \geq a$. Suppose this is not the case, say $\alpha(b) = \alpha_0 < 0$ with $b > a$. Note that α is decreasing. Thus, since $W(x) \to 0$ as $x \to \infty$, we can take $c \geq b$ such that $t^2 W(c) \leq -\alpha(c)/2$. So, since W and α are decreasing, $t^2 W(x) \leq -\alpha(x)/2$ for all $x \geq c$. Hence,

$$-(\alpha(x) + t^2 W(x)) \leq -\frac{1}{4}\, \alpha(x)^2$$

for all $x \geq c$. Now, the solution $\alpha_1(x) = 4\alpha(c)/\left(\alpha(c)(x-c)+4\right)$ of $\alpha_1' = -\frac{1}{4}\alpha_1^2$, $\alpha_1(c) = \alpha(c)$, goes to $-\infty$ as $x \to c - 4/\alpha(c)$, and this forces the same to be true of $\alpha(x)$ by the comparison principle. However, this contradicts the fact that $\alpha(x)$ is continuous on (a, ∞).

So, since $\alpha(x) \geq 0$ and $W(x) \geq C/x$ for all $x \geq a$,

$$-\left(\alpha + t^2 W\right)^2 \leq -(\alpha + t^2 C/x)^2$$

on $[a, \infty)$. Let α_2 solve

$$\alpha_2' = -\left(\alpha_2 + \frac{t^2 C}{x}\right)^2, \qquad \alpha_2(a) = \alpha(a). \tag{7.5}$$

Thus, $\alpha_2(x) \geq 0$ for all $x \geq a$, and in particular $\alpha_2(x)$ exists as a solution on (a, ∞). Let $\beta(x) = \alpha_2(x) + t^2 C/x$. Define

$$y(x) = \exp\left(\int_a^x \beta(s)\, ds\right).$$

Then y is a zero-free solution of the Schrödinger equation

$$-y'' - \frac{t^2 C}{x^2}\, y = 0. \tag{7.6}$$

The solutions of this equation have infinitely many zeros if $t^2 C > 1/4$. Hence, $t^2 C \leq 1/4$. So, $t \leq 1/(2\sqrt{C})$ for all $t \in (0, S) = (0, M(H_d))$ and $C < B$. $\square$

Remark 7.1. One can obtain different bounds on $M(H_d)$ by transforming the diagonal system into a projection and then applying Theorem 5.1 or 5.2. For simplicity, suppose that $\sin\psi(T) \neq 0$ for all $T > 0$. Let $f(T) = \int_0^T \cos^2\psi(S)\, dS$, introduce a change of variable $x = g(T) = \int_0^T (1 + f(S)^2)\sin^2\psi(S)\, dS$, and define $\phi(x) = \arctan(-f(T))$.

Suppose $\lim_{T\to\infty} g(T) < \infty$. Combining this assumption with the assumption that $\sin\psi \in L^2(0, \infty)$, one obtains that $\int_0^\infty S^2 \sin^2\psi(S)\, dS$ is finite. Hence, as

$T \to \infty$,

$$T^2 \int_T^\infty \sin^2 \psi(S)\, dS \le \int_T^\infty S^2 \sin^2 \psi(S)\, dS \to 0.$$

So, H_d has purely discrete spectrum by Theorem 7.2 when $g(T)$ does not go to ∞, which settles that case.

Suppose $\lim_{T\to\infty} g(T) = \infty$. Hence, $P_{\phi(x)}$ is defined for $x \in (0, \infty)$. By construction, ϕ is decreasing with range in $(-\pi/2, 0]$ and $\phi(\infty) = -\pi/2$. Hence, by Theorem 5.1,

$$\frac{1}{4A} \le M(P_\phi) \le \frac{1}{A},$$

where $A = \limsup_{x\to\infty} x(\psi(x) + \pi/2)$.

Now, if $p(T)$ is a solution of $J\frac{d}{dT}p(T) = -zH_d(T)p(T)$, then

$$u(x) = \begin{pmatrix} z & -zf(T) \\ 0 & 1 \end{pmatrix}^{-1} p(T)$$

solves $J\frac{d}{dx}u(x) = -z^2 P_{\phi(x)}u(x)$. So, $M(H_d)^2 = M(P_\phi)$, and by Theorem 7.2,

$$\frac{1}{4\tilde{A}} \le M(H_d)^2 \le \frac{1}{\tilde{A}}$$

where $\tilde{A} = \limsup_{T\to\infty} T \int_T^\infty \sin^2 \psi(S)\, dS$.

Notice that since $-\tan\phi(g(T)) = f(T)$ and

$$\cos^2 \phi(g(T)) = \frac{1}{1 + \tan^2 \phi(g(T))} = \frac{1}{1 + f^2(T)},$$

$$\frac{d}{dT}\left[\int_0^{g(T)} \cos^2 \phi(y)\, dy - \tan\phi(g(T))\right] = 1.$$

So, since $\int_0^{g(0)} \cos^2 \phi(y)\, dy - \tan \phi(g(0)) = 0$ as well,

$$T = \int_0^{g(T)} \cos^2 \phi(y)\, dy - \tan \phi(g(T)).$$

Also, since $\frac{d}{dT} \int_0^{g(T)} \cos^2 \phi(y)\, dy = \sin^2 \psi(T)$ and $\int_0^{g(0)} \cos^2 \phi(y)\, dy = 0$,

$$\int_0^{g(T)} \cos^2 \phi(y)\, dy = \int_0^T \sin^2 \psi(S)\, dS.$$

Thus, since $\cos \phi \in L^2(0, \infty)$,

$$\limsup_{T\to\infty} T \int_T^\infty \sin^2 \psi(S)\, dS = \limsup_{x\to\infty} - \tan \phi(x) \int_x^\infty \cos^2 \phi(y)\, dy.$$

So, since $\phi(\infty) = -\pi/2$, one quantity that we can use in the estimation of $M(H_d)^2 = M(P_\phi)$ via Theorem 7.2 is

$$\limsup_{x\to\infty} \frac{1}{\phi(x) + \pi/2} \int_x^\infty (\phi(y) + \pi/2)^2\, dy,$$

which is not always the same as $A = \limsup_{x\to\infty} x(\psi(x) + \pi/2)$, the quantity from Theorem 5.1.

Theorem 6.6 allows one to compare $M(H)$ and $M(H_d)$, and Theorems 7.3 and 7.2 then provide estimates of these quantites, all under the assumption that $\sin \psi \in L^2(0, \infty)$. These theorems can be applied in several ways without that assumption. Suppose first that $0 \notin \sigma_{ess}(H)$. Then there is a nontrivial $L^2(0, \infty)$ solution of the canonical system (2.1) at $z = 0$ [11, Theorem 3.8(b)]. Hence, there is a unit vector $v \in \mathbb{R}^2$ such that $\int_0^\infty v * H(x) v\, dx < \infty$. Let R be the rotation matrix whose first column is v. Then $R^* H R$ is a trace normed canonical system with the same essential spectrum that has its upper left entry in $L^1(0, \infty)$. So, $M(H) = M(R^* H R)$, and the above theorems apply to R^*HR [13, 15].

Another way to apply the above theorems in the investigation of $M(H)$, without the assumption that $\sin\psi \in L^2(0,\infty)$, is to transform H so that $m(z)$ is changed only by the addition of a point mass at 0 to the spectral measure. This transformation can be found in [25]. Since 0 is then an eigenvalue, the upper left entry of the new canonical system is in $L^1(0,\infty)$. The essential spectrum is unchanged, so $M(H)$ could be computed using the new system that the theorems do apply to.

Finally, Theorems 6.6, 7.3, and 7.2 can be used to estimate the distance of an arbitrary eigenvalue of a canonical system to the essential spectrum. Let t be an eigenvalue of the canonical system H, and let u and v be solutions of (2.1) with $u(0) = (1,0)^t$ and $v(0) = (0,1)^t$. The m function of the canonical system

$$\tilde{H} = \begin{pmatrix} u^*Hu & u^*Hv \\ u^*Hv & v^*Hv \end{pmatrix} \tag{7.7}$$

differs from the original only by a shift of the spectral measure so that $t \to 0$ [23]. So, the distance of t to $\sigma_{ess}(H)$ is equal to $M(\tilde{H})$. Since u is an eigenfunction for the eigenvalue t, $u^*Hu \in L^1(0,\infty)$. Hence, $M(\tilde{H})$ can be compared to $M(\tilde{H}_d)$ using Theorem 6.6, and $M(\tilde{H}_d)$ could be estimated using Theorems 7.3 and 7.2, after changing the variable to make the trace 1.

Chapter 8

Correspondence between self-adjoint operators and relations

Self-adjoint operators corresponding to canonical systems are obtained by a general procedure. A maximal relation is defined, the corresponding minimal operator is computed, and self-adjoint relations are obtained in between them. The self-adjoint relations then give self-adjoint operators as indicated in Chapter 2. One might question whether all possible self-adjoint operators associated with the canonical system are obtained in this way.

Of course, the question needs to be made precise to give a satisfactory answer to it. So, suppose the norm is agreed to be the one used in the definition of L^2_H. Then the possible Hilbert spaces the operator acts on are the Hilbert spaces inside L^2_H. Suppose it is agreed that the self-adjoint operator must be defined at least on everything in the domain of the minimal relation (this assumption is perhaps the most objectionable). Finally, suppose it is agreed that the graph of the operator must be a subset of the maximal relation, so that the operator acts in a way that the canonical system suggests. Then it does follow that the operator is one of those obtained by the above procedure starting with relations. The next theorem is an abstract generalization of this result.

Theorem 8.1. *Suppose $\mathcal{H}_1 \subset \mathcal{H}$ are Hilbert spaces. Let T_0 be a closed symmetric operator with a self-adjoint operator extension S on $\mathcal{H}_1$. Let R_0 be a closed symmetric relation on $\mathcal{H}$ such that $D(R_0) \subset D(T_0)$ and $T_0^{*_1} \subset R_0^*$ where $T_0^{*_1}$ is the adjoint, in $\mathcal{H}_1$, of T_0, considered as a relation. Then there exists a self-adjoint relation R such that $R_0 \subset R \subset R_0^*$ and $R \ominus \{0\} \times R(0) = S$.*

Proof. Let $R = S + \{0\} \times (D(S))^\perp$, where S is being considered as a relation and the sum refers to the sum of relations. Note that $R(0) = (D(S))^\perp$. So, to show that $R \ominus \{0\} \times R(0) = S$, it suffices to show that S and $\{0\} \times (D(S))^\perp$ are orthogonal. Let $f \in D(S)$ and $g \in D(S)^\perp$. Then, since $Sf \in \overline{D(S)} = D(S)^{\perp\perp}$, $\langle f, 0 \rangle + \langle Sf, g \rangle = 0$.

Next, we show that R is closed. Since S and $\{0\} \times (D(S))^\perp$ are orthogonal, it suffices to show that S and $\{0\} \times (D(S))^\perp$ are closed. S is closed because, being self-adjoint in $\mathcal{H}_1$, it is closed in $\mathcal{H}_1$, and $\mathcal{H}_1$ is closed in $\mathcal{H}$. $\{0\} \times (D(S))^\perp$ is closed because $D(S)^\perp$ is closed.

Now, we check the claim that $R \subset R_0^*$. Take any element of R and write it as $(f, Sf + g)$ with $f \in D(S)$ and $g \in D(S)^\perp$. So, since $S \subset T_0^{*_1} \subset R_0^*$, $(f, Sf) \in R_0^*$. Since $D(R_0) \subset D(T_0)$,

$$D(S)^\perp \subset D(T_0)^\perp \subset D(R_0)^\perp = R_0^*(0).$$

Thus, $(0, g) \in R_0^*$. So, $(f, Sf + g) \in R_0^*$.

To show that R is symmetric, let $f, h \in D(S)$ and $g, k \in D(S)^\perp$. So, since S is self-adjoint,

$$\langle f, Sh + k \rangle = \langle f, Sh \rangle = \langle Sf, h \rangle = \langle Sf + g, h \rangle.$$

Hence, for any $(f, Sf+g) \in R$, $(f, Sf+g) \in R^*$.

Finally it suffices to show that the deficiency indices of R are 0. Since R is closed and symmetric, this will imply that R is self-adjoint. This will then imply that $R_0 \subset R$ since we already know $R \subset R_0^*$ and, by assumption, R_0 is closed. The deficiency indices of R are the dimensions of the sets

$$\{Sf + g \pm if : f \in D(S), g \in D(S)^{\perp}\}^{\perp}.$$

Suppose $k \in \{Sf+g \pm if : f \in D(S), g \in D(S)^{\perp}\}^{\perp}$. Write $k = k_1 + k_2$ with $k_1 \in \mathcal{H}_1$ and $k_2 \in \mathcal{H}_1^{\perp}$. Let $f \in D(S)$ and $g \in D(S)^{\perp}$ be arbitrary. So,

$$0 = \langle k, Sf + g \pm if \rangle = \langle k_1, Sf + g \pm if \rangle + \langle k_2, Sf + g \pm if \rangle.$$

Now, $\langle k_1, Sf + g \pm if \rangle = \langle k_1, Sf \pm if \rangle$ since $k_1 \in \mathcal{H}_1 = D(S)^{\perp\perp}$. Since $f, Sf \in \overline{D(S)} = D(S)^{\perp\perp}$ and $k_2 \in \mathcal{H}_1^{\perp} = \overline{D(S)}^{\perp} = D(S)^{\perp}$, $\langle k_2, Sf + g \pm if \rangle = \langle k_2, g \rangle$. Hence,

$$0 = \langle k_1, Sf \pm if \rangle + \langle k_2, g \rangle$$

for all $f \in D(S)$ and $g \in D(S)^{\perp}$. Letting $g = 0$ and $f \in D(S)$ be arbitrary, this implies that $k_1 = 0$ since S, being a self-adjoint operator on $\mathcal{H}_1$, has deficiency indices 0. Letting $f = 0$ and $g \in D(S)^{\perp} = \mathcal{H}_1^{\perp}$ be arbitrary, the above equation implies that $k_2 \in \mathcal{H}_1^{\perp}$ is 0. So, the only element of $\{Sf + g \pm if : f \in D(S), g \in D(S)^{\perp}\}^{\perp}$ is 0, and the claim follows. $\square$

Chapter 9

One-channel operators

One-channel operators are a generalization of Jacobi operators to a vector-valued setting [16]. The starting point is a difference equation of the form

$$zu_n = A_n u_{n+1} + A^*_{n-1} u_{n-1} + B_n u_n$$

where $A_n, B_n \in \mathbb{R}^{d\times d}$, $\operatorname{rk} A_n = 1$, $B_n = B_n^*$, $u_n \in \mathbb{C}^d$. Note that Jacobi difference equations correspond to $d = 1$. These share with Jacobi operators the nice property of having spectral data encoded in two-dimensional transfer matrices, under certain assumptions. Another common way of generalizing Jacobi operators to a vector-valued setting would be to replace the assumption $\operatorname{rk} A_n = 1$ with the assumption that $A_n \in \mathrm{GL}_d(\mathbb{R})$. An important fact about Jacobi difference equations is that their solution spaces are two-dimensional. This is very easily proved by basic algebra for Jacobi equations. The same proof shows that the difference equation above with $A_n \in \mathrm{GL}_d(\mathbb{R})$ has a $2d$-dimensional solution space. The argument is recalled in the proof below.

Now, for one-channel equations it is not the case that the solution space is always

$2d$-dimensional. One can easily deduce that for $B_n \equiv 0$ and A_n defined by

$$\begin{cases} P_1 & n \text{ odd} \\ P_2 & n \text{ even} \end{cases},$$

where P_1 and P_2 are the projections on $(1,0)^t$ and $(0,1)^t$, respectively, the solution space is 0-dimensional. Suppose $z \neq 0$ and u_n, $n \in \mathbb{Z}$, is a solution. Then $zu_0 = P_2u_1 + P_1u_{-1}$ and $zu_1 = P_1u_2 + P_2u_0$. Thus, $zP_2u_0 = P_2u_1$ and $zP_2u_1 = P_2u_0$. Hence, $P_2u_0 = zP_2u_1 = z^2P_2u_0$. So, $P_2u_0 = 0$. Thus, $zu_1 = P_1u_2 + P_2u_0 = P_1u_2$. So, u_1 is in the range of P_1. Since $zu_2 = P_2u_3 + P_1u_1$, $zP_1u_2 = P_1u_1$. Thus, $P_1u_1 = zP_1u_2 = z^2u_1 = z^2P_1u_1$. So, $u_1 = P_1u_1 = 0$. Obviously, this argument can be generalized to show that $u_n = 0$ for all $n \in \mathbb{Z}$.

However, it is the case that the solution space of a one-channel equation is at most $2d$-dimensional for $z \notin \mathbb{R}$. This algebraic fact seems to escape an elementary proof. The proof here uses the known fact [8], whose proof we recall for convenience, that the deficiency indices of the corresponding minimal operator are at most $2d$.

Theorem 9.1. *Let $I = \mathbb{Z}$ or $I = \mathbb{N}$. Let $A_n, B_n \in \mathbb{R}^{d\times d}$ and $B_n = B_n^*$ for all $n \in I$. Consider a difference equation of the form*

$$zu_n = A_nu_{n+1} + A_{n-1}^*u_{n-1} + B_nu_n \ (n \in I) \tag{9.1}$$

where $z \in \mathbb{C}$, $u = (u_n)_{n\in I}$ is a sequence of vectors in $\mathbb{C}^d$, and if $I = N$, $u_0 = 0$. If $z \notin \mathbb{R}$, then there are at most $2d$ linearly independent solutions of (9.1).

Proof. First consider whole-line equations, $I = \mathbb{Z}$. Suppose that A_j is invertible for every $j \in \mathbb{Z}$. Then the dimension of the solution space of

$$zu_n = A_n u_{n+1} + A_{n-1}^* u_{n-1} + B_n u_n \tag{9.2}$$

is exactly $2d$. To see this it suffices to show that for any $v, w \in \mathbb{C}^d$ there exists a unique solution (u_j) of (9.2) with $u_0 = v$ and $u_1 = w$. This is trivial because the equation (9.2) has a unique solution u_{n+1} given u_{n-1} and u_n and a unique solution u_{n-1} given u_n and u_{n+1}.

Now suppose that it is not necessarily the case that every A_j is invertible. There are countably many eigenvalues of the A_j collectively. So, take a $\lambda \in \mathbb{R}$ that is not an eigenvalue of any A_j. Then $A_j - \lambda$ is invertible for every $j \in \mathbb{Z}$. Let τ be defined by

$$(\tau u)_n = A_n u_{n+1} + A_{n-1}^* u_{n-1} + B_n u_n$$

on sequences $u = (u_j)$. Define the minimal closed operator T to be the closure of the operator T_0 defined by $T_0 u = \tau u$ for $u \in D(T_0) = \{u \in \ell_2(\mathbb{C}_d) : u \text{ has finite support}\}$. It is known that T is symmetric and that its adjoint is the operator with the domain $\{u \in \ell_2(\mathbb{C}_d)) : \tau u \in \ell_2(\mathbb{C}_d))\}$ defined by $T^* u = \tau u$ for $u \in D(T^*)$. Consider the minimal closed operator $\tilde{T}$ corresponding to the coefficients $\tilde{A}_n = A_n - \lambda$ and $\tilde{B}_n = B_n$. Then $T - \tilde{T}$ is a bounded symmetric operator (defined on a dense subset of $\ell_2(\mathbb{C}_d)$). Hence, the deficiency indices of T and $\tilde{T}$ are the same. The deficiency indices of $\tilde{T}$ are at most $2d$ since the equation

$$zu_n = \tilde{A}_n u_{n+1} + \tilde{A}_{n-1}^* u_{n-1} + B_n u_n$$

has $2d$ linearly independent solutions. Thus, for all $z \notin \mathbb{R}$, the equation (9.2) has at most $2d$ linearly independent $\ell_2(\mathbb{C}_d)$ solutions.

Let $z \notin \mathbb{R}$, and suppose that (9.2) has $2d+1$ linearly independent solutions $u^{(1)}, ..., u^{(2d+1)}$. Take N so that the $u^{(k)}|_{[-N,N]}$ are linearly independent. Let $(a_j, b_j)_{j\in\mathbb{Z}}$ be limit circle Jacobi coefficients with $a_j \neq 0$ for all j. Define

$$C_n = \begin{cases} A_n & n \in [-N-1, N+1) \\ diag(a_n, ...a_n) & n \in [-N-1, N+1)^c \end{cases}$$

and

$$D_n = \begin{cases} B_n & |n| \leq N+1 \\ diag(b_n, ...b_n) & |n| > N+1 \end{cases}.$$

Let $v_n^{(k)} = u_n^{(k)}$ for $|n| \leq N+1$. Define $v_{N+2}^{(k)} = C_{N+1}^{-1}((z - D_{N+1})v_{N+1}^{(k)} - C_N^* v_N^{(k)})$ and $v_{-N-2}^{(k)} = C_{-N-2}^{*-1}((z - D_{-N-1})v_{-N-1}^{(k)} - C_{-N-1}v_{-N}^{(k)})$. Define $v_n^{(k)}$ for $n \geq N+3$ by solving, for $n \geq N+2$,

$$zv_n = C_n v_{n+1} + C_{n-1}^* v_{n-1} + D_n v_n$$

with the initial conditions $v_{N+1}^{(k)}, v_{N+2}^{(k)}$ given. Define $v_n^{(k)}$ for $n \leq -N-3$ by solving, for $n \leq -N-2$,

$$zv_n = C_n v_{n+1} + C_{n-1}^* v_{n-1} + D_n v_n$$

with the initial conditions $v_{-N-1}^{(k)}, v_{-N-2}^{(k)}$ given. Then $v_n^{(k)}$ solves

$$zv_n = C_n v_{n+1} + C_{n-1}^* v_{n-1} + D_n v_n$$

for all $n \in \mathbb{Z}$. The $2d+1$ solutions $v_n^{(k)}$ are linearly independent because they agree with $u_n^{(k)}$ on $[-N, N]$. Since the Jacobi coefficients (a_j, b_j) are limit circle, and each coordinate sequence of $v_n^{(k)}$ solves the corresponding Jacobi equation at large n, $v_n^{(k)} \in \ell_2(\mathbb{C}^d)$. This contradicts the earlier conclusion that

$$zv_n = C_n v_{n+1} + C_{n-1}^* v_{n-1} + D_n v_n$$

has at most $2d$ linearly independent $\ell_2(\mathbb{C}_d)$ solutions for $z \notin \mathbb{R}$.

The half line case is a trivial consequence of the whole line case. Suppose there were $2d+1$ linearly independent solutions of a half line equation for $z \notin \mathbb{R}$. Expand the equation to the other half line by choosing any B_n and any invertible A_n for the other half line. The original solutions could then be extended to solutions of the whole line equation, but this would contradict the whole line problem having at most $2d$ linearly independent solutions. $\square$

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