UNIVERSITY OF CENTRAL OKLAHOMA

Existence Results for a Class of Even-Order Boundary Value Problems

A THESIS

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

MASTER OF SCIENCE

in the field of

APPLIED MATHEMATICAL SCIENCE

by

Daniel Brumley

EDMOND, OKLAHOMA

May 2018

THE UNIVERSITY OF CENTRAL OKLAHOMA GRADUATE SCHOOL

SUPERVISORY COMMITTEE APPROVAL

of a thesis submitted by

Daniel Brumley

This thesis has been read by each member of the following supervisory committee and by majority vote has been found to be satisfactory.

5/3/18 — Britney Hopkins

Britney Hopkins, Ph.D., Chairperson

5-3-18

Tyler Cook, Ph.D.

05/03/2018

Michael Fulkerson, Ph.D.

ABSTRACT

Existence Results for a Class of Even-Order

Boundary Value Problems

Daniel Brumley

Advisor: Britney Hopkins, Ph.D.

This thesis focuses on establishing the existence of positive solutions to even-order boundary value problems of the form

$$
\begin{aligned}
&u^{(2n)}(t) = \lambda h\left(t, u(t), u''(t), \dots, u^{(2n-2)}(t)\right), \\
&\alpha_{i+1}u^{(2i)}(0) - \gamma_{i+1}u^{(2i)}(0) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \dots, n-1, \\
&\beta_{i+1}u^{(2i+1)}(1) - \delta_{i+1}u^{(2i+1)}(1) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \dots, n-1,
\end{aligned}
$$

where $n \geq 2,\ h : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,\infty) \to (-1)^n[0,\infty)$ is continuous, and $\lambda > 0$. In particular, for $i = 0, 1, \dots, n-1$, we let $\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}, \delta_{i+1} > 0$ and consider in tandem the cases where either $a_{i+1} > 0$ or $a_{i+1} < 0$.

Similar problems have been considered by other authors. What distinguishes this work is the method employed: Beginning with a transformation of the problem into a system of second-order differential equations satisfying homogeneous boundary conditions, the

work culminates in successive applications of the Guo-Krasnosel'skii Fixed Point Theorem giving at least three positive solutions. This method is based on the work of Marcos, Lorca, and Ubilla, who developed the technique to establish existence results for a class of fourth-order boundary value problems; however, the general framework under which this thesis operates is more accurately attributed to Hopkins, who extended the method to even-order problems on both continuous and discrete domains. Future work could seek to further demonstrate the broad applicability of the method by considering more challenging boundary conditions, by generalizing the results to time scales, or by utilizing different fixed point theorems in order to characterize solutions further.

Acknowledgements

This thesis would not have been possible were it not for the support of some truly amazing people.

For starters, I would like to thank the Department of Mathematics and Statistics at the University of Central Oklahoma for supporting me with my work over the past few years. I am especially grateful for the many opportunities the department gave me to present my work across the country. Also, though not directly tied into my thesis, I would nevertheless like to take this moment to express my gratitude to the department for allowing me to work as a Supplemental Instructor during my undergraduate years. The experience was truly life changing.

Of course, I am also indebted to those individuals who make up the heart of the Department, namely, my professors. If you told me ten years ago I would be in the position I am now, I would never have believed you. But you all sparked my passion for mathematics, and, for that, I am truly grateful. In particular, I would like to thank Drs. Tyler Cook, Michael Fulkerson, and Britney Hopkins for taking the time out of their busy schedules to serve on my thesis committee, as well as Drs. Kristi Karber and Thomas Milligan, who, along with the preceding trio of professors, exposed me to mathematical research and encouraged me to pursue doctoral studies. I would also like to give a shout-out to Dr. Brittany Bannish for helping me out on multiple occasions this past year—I owe you a lot!

One of these professors is so awesome that I feel obligated to thank her twice. I owe practically all of my academic successes over the past few years to my advisor, Dr. Britney Hopkins. Thank you for always being willing to take a chance on me.

There are many people outside of the classroom who have supported me along the way as well. There could be little doubt who is at the foundation of this support: my family. In particular, I will forever be indebted to my grandmother, “Guy,” who, when things took an unexpected turn for the worse a couple of years ago, took me in—oh, and also my mom and my dog! My parents also deserve singling out: My dad for encouraging me to always question and not fall in with the crowd, for inculcating me with a strong will, and for showing me that sometimes the best things in life require stepping over the fence; my mom for keeping me grounded and never allowing me to take myself too seriously, for loving me unconditionally, and for being the example of strength and warmth I will always strive toward.

Finally, this section would not be complete without mention of my friends. I will always be grateful for the times I got to goof around Edmond with my Caribbean sister from another mister, Jacintha Lawrence (a.k.a J-Dizzle, J-Dawg, Jelly Bean, Larry). I am also grateful for the unwavering loyalty and support of my “strongest and most independentest” of friends, Hussain Alasafra—I have received more from you than I could ever hope to give back! Last but certainly not least, I owe a great many thanks to my friend, my fellow life adventurer, my confidante, Christina Regier—it will be interesting to see where these next few years take us!

Table of Contents

CHAPTER 1

Introduction

This thesis focuses on establishing the existence of positive solutions to even-order boundary value problems of the form

$$u^{(2n)}(t) = \lambda h\left(t, u(t), u''(t), \ldots, u^{(2n-2)}(t)\right), \tag{1.1}$$

$$\alpha_{i+1}u^{(2i)}(0) - \gamma_{i+1}u^{(2i)}(0) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{1.2}$$

$$\beta_{i+1}u^{(2i+1)}(1) - \delta_{i+1}u^{(2i+1)}(1) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{1.3}$$

where $n \geq 2$, $h : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,\infty) \to (-1)^n[0,\infty)$ is continuous, and $\lambda > 0$. In particular, for $i = 0, 1, \ldots, n-1$, we require $\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}, \delta_{i+1} > 0$ and consider in tandem the cases (i) $a_{i+1} > 0$ and (ii) $a_{i+1} < 0$. We refer to the system (1.1)–(1.3) coupled with (i) as the *positive case*; the system (1.1)–(1.3) paired with (ii) we call the *negative case*.

The method that we use to demonstrate existence of multiple positive solutions begins with a transformation of the problem into a system of second-order differential equations satisfying homogeneous boundary conditions. An operator T is then defined in such a way that the fixed points of T (if any) correspond to positive solutions to the system. To establish the existence of fixed points of T, a sequence of lemmas is constructed that lead to contraction and expansion estimates of T over nested subsets of a cone. This allows for successive applications of the Guo-Krasnoselskii Fixed Point Theorem giving multiple fixed points of the operator T and, hence, multiple positive solutions to the system of boundary value problems. The multiplicity of positive solutions to (1.1)–(1.3) follows by corollary.

The rest of this chapter is devoted to a brief survey of the relevant history surrounding our particular problem and boundary value problems more generally. Definitions and results needed for the remainder of the work will also be presented. In Chapter 2, we prove the existence of positive solutions to systems of the form (1.1)–(1.3) in the special case where $n = 2$. The purpose of studying this lower-order problem is to provide a concrete setting in which to motivate the method above. Special attention is given to the differences between the positive and negative cases. Utilizing the insights of the preceding chapter, we turn to establishing the existence results for the general, even-order problem in Chapter 3. We close with a discussion considering possible avenues for future work in Chapter 4.

1.1. History

From a practical standpoint, the study of multiple solutions to boundary value problems is important to the modeling of various physical phenomena. For instance, Cohen [**10**] studied the multiplicity of solutions to the boundary value problem

$$\beta u'' - u' + f(u) = 0, \qquad 0 \le t \le 1,$$

$$u'(0) - \alpha u(0) = 0, u'(1) = 0,$$

which occurs in the modeling of a certain chemical reactor. Argawal addressed uniqueness issues to boundary value problems of the form

$$u^{(4)} = f(t, u, u', u'', u'''), \qquad a \leq t \leq b$$

$$u(a) = A, u'(a) = B, u(b) = C, u''(b) = D,$$

motivated by problems arising in beam analysis in [**1**]. Additional examples may be found in [**2**], [**3**], [**11**], [**12**], [**14**], [**20**], [**21**], [**22**] and the references therein.

A number of diverse methods exist to establish the existence of multiple solutions to boundary value problems. A particularly fruitful approach hinges on transforming a higher order problem into a system of second-order differential equations of the form $u''(t) = f(t, u(t))$ satisfying homogeneous boundary conditions and observing that solutions to this problem are just fixed points of the operator

$$Tu = \int_0^1 G(t, s) f(s, u(s)) ds,$$

where G is the Green's function corresponding to the specified homogeneous boundary conditions. As a result, various fixed point theorems have been utilized or proposed to address existence and uniqueness issues, such as the Leggett Williams Fixed Point Theorem [**19**] and its many offshoots ([**4**], [**5**], [**6**]).

One of the more important fixed point theorems to arise in the past sixty years in the study of solutions to boundary value problems is attributed to Krasnosel'skii. In [**18**], he established a fixed point result for operators acting on cones (see Definition 1.2.1). An extension was later formulated by Guo in [**13**] that held for less restrictive open subsets within a cone. This more general result is known as the Guo-Krasnosel'skii Fixed Point Theorem (see Theorem 1.2.3).

By utilizing the Guo-Krasnosel'skii Fixed Point Theorem, Marcos, Lorca, and Ubilla [**17**] demonstrated the existence of at least three positive solutions to the boundary value problem

$$u^{(4)} = \lambda h(t, u, u''), \qquad t \in (0, 1),$$

$$u(0) = u''(0) = 0, u(1) = a, u''(1) = -b.$$

Their method of proof was essentially the approach outlined above. Hopkins later expanded upon this work in [**15**], [**16**] by generalizing the system above to arbitrary order and considering analogous problems on both continuous and discrete domains. Brumley, Hopkins, et al. investigated further the existence of solutions to several classes of boundary value problems in [**7**], [**9**], including the positive case of (1.1)–(1.3) when $n = 2$ in [**8**]. This thesis is an outgrowth of these investigations into the multiplicity of solutions of even-order boundary value problems.

1.2. Definitions and Theorems

We now state the major definitions and theorems that will be used throughout this work. As noted above, the Guo-Krasnosel'skii Fixed Point Theorem gives fixed points within open subsets of a cone. This motivates the following definition:

Definition 1.2.1. *A nonempty, closed, convex subset C of a Banach space X is called a* ***cone*** *if* *(i) for all* $x \in C$ *and* $\lambda > 0$*, we have* $\lambda x \in C$*;*

(ii) if $x \in C$ *and* $-x \in C$*, then* $x = 0$*.*

We will restrict our attention to a particular class of operators, namely, completely continuous operators, which are defined as follows:

Definition 1.2.2. *Let* X *and* Y *be Banach spaces and* $\{x_n\}$ *a weakly convergent sequence in* X*. A bounded linear operator* $T : X \to Y$ *is called* ***completely continuous*** *if* $\{Tx_n\}$ *is norm-convergent in* Y*.*

With these preliminary definitions out of the way, we now state the fixed point result upon which all of our later work rests.

Theorem 1.2.3 (Guo-Krasnosel'skii Fixed Point Theorem)**.** *Let* $(X, \|\cdot\|)$ *be a Banach space, and let* $C \subset X$ *be a cone. Suppose* Ω_1, Ω_2 *are open subsets of* X *satisfying* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$*. If* $T : C \cap (\overline{\Omega}_2 - \Omega_1) \to C$ *is a completely continuous operator such that either*

(i) $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_2$ *or*

(ii) $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_2$*,*

then T *has a fixed point in* $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$*.*

We will refer to cases (i) and (ii) of the Guo-Krasnosel'skii Fixed Point Theorem as the *compression* (or *contraction*) and *expansion* forms of the theorem, respectively.

We conclude this section with a lemma that will be of tremendous use in establishing the various contraction and expansion estimates in the next three chapters. A proof is supplied in [**15**].

Lemma 1.2.4. *Suppose* $u : [0, 1] \subseteq \mathbb{R} \to \mathbb{R}$ *is nonnegative, concave, and continuous. Then, for all* $\alpha, \beta \in (0, 1)$ *with* $\alpha < \beta$*, we have*

$$\inf_{t \in [\alpha,\beta]} u(t) \geq \alpha\,(1 - \beta)\,\|u\|_{\infty},$$

where $\|u\|_{\infty} = \sup_{t \in [0,1]} u(t)$.

CHAPTER 2

The Fourth-Order Problem

In this chapter, we prove the existence of positive solutions to systems of the form (1.1)–(1.3) in the special case when $n = 2$. That is, we consider a boundary value problem of the form

$$u^{(4)}(t) = \lambda h\left(t, u(t), u''(t)\right), \tag{2.1}$$

$$\alpha_1 u(0) - \gamma_1 u(1) = \beta_1 u'(0) - \delta_1 u'(1) = -a_1, \tag{2.2}$$

$$\alpha_2 u''(0) - \gamma_2 u''(1) = \beta_2 u'''(0) - \delta_2 u'''(1) = a_2, \tag{2.3}$$

where $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is nonnegative and continuous, $\lambda > 0$, and $\alpha_i, \beta_i, \gamma_i, \delta_i > 0$ for $i = 1, 2$. Like their even-order counterparts, the fourth-order systems differ only with respect to the constraints placed on the parameters a_i. In Section 2.1, we establish existence results for (2.1)–(2.3) in the negative case, that is, when $a_i < 0$ for $i = 1, 2$. We prove analogous results for the positive case ($a_i > 0$) in Section 2.2.

We emphasize that the purpose of studying the lower-order cases is to provide a concrete setting in which to motivate the framework that, in the end, leads to the establishment of the general even-order results. In particular, we draw attention in this chapter to how changes in the parameters lead to modifications of the underlying hypotheses and the core sequence of four lemmas leading up to the main existence results.

2.1. Negative Case

In this section, we the establish existence of at least three positive solutions to systems of differential equations of the form (2.1)–(2.3) when $a_i < 0$ for $i = 1, 2$. In line with [**15**], [**16**], [**17**], Section 2.1.1 begins with a series of substitutions and transformations that converts the problem into a system of second-order differential equations satisfying homogeneous boundary conditions. We then define a cone C and an operator T in such a way that the fixed points of T (over C) correspond to positive solutions to a more generalized version of the second-order homogeneous system just obtained. In Section 2.1.2, a sequence of four lemmas is constructed that leads to various expansion and compression estimates of T over nested subsets of the cone C. This allows for a triple application of the Guo-Krasnosel'skii Fixed Point Theorem in Section 2.1.3 giving the existence of at least three fixed points of T and, hence, at least three positive solutions to the general system. The existence of multiple positive solutions to the original system is then established by corollary.

2.1.1. Preliminaries

Consider the boundary value problem

$$u^{(4)}(t) = \lambda h\left(t, u(t), u''(t)\right), \tag{2.4}$$

$$\alpha_1 u(0) - \gamma_1 u(1) = \beta_1 u'(0) - \delta_1 u'(1) = -a_1, \tag{2.5}$$

$$\alpha_2 u''(0) - \gamma_2 u''(1) = \beta_2 u'''(0) - \delta_2 u'''(1) = a_2, \tag{2.6}$$

where $h : [0,1]\times[0,\infty)\times(-\infty,0] \to [0,\infty)$ is nonnegative and continuous, $\lambda, \alpha_i, \beta_i, \gamma_i, \delta_i > 0$, and $a_i < 0$ for $i = 1, 2$.

Let $u_1 = u$, $u_2 = -u_1''$, $g(t, u_1, u_2) = u_2$, and $h(t, u_1, -u_2) = f(t, u_1, u_2)$. Then (2.4)–(2.6) becomes

$$-u_2''(t) = \lambda f\left(t, u_1, u_2\right), \tag{2.7}$$

$$-u_1''(t) = g\left(t, u_1, u_2\right), \tag{2.8}$$

$$\alpha_1 u_1(0) - \gamma_1 u_1(1) = \beta_1 u_1'(0) - \delta_1 u_1'(1) = -a_1, \tag{2.9}$$

$$\alpha_2 u_2(0) - \gamma_2 u_2(1) = \beta_2 u_2'(0) - \delta_2 u_2'(1) = -a_2. \tag{2.10}$$

Notice that the choice of substitutions combined with the sign changing properties of h implies that f and g are nonnegative, from which it follows that u_1 and u_2 are concave. These facts will be important later as the cone we define will be equipped with concavity and nonnegativity conditions.

We now proceed to transform the above system into one satisfying homogeneous boundary conditions. To accomplish this, let $\overline{u}_1'(t) \equiv u_1'(t) - \frac{a_1}{\delta_1}t$ and observe that this satisfies $\beta_1\overline{u}_1'(0) - \delta_1\overline{u}_1'(1) = 0$. Integrating both sides of the former equation with respect to t, we obtain $\overline{u}_1(t) = u_1(t) - \frac{a_1}{2\delta_1}t^2 + k$ for some $k \in \mathbb{R}$. We would like to choose k so that the remaining boundary condition $\alpha_1\overline{u}_1(0)-\gamma_1\overline{u}_1(1)=0$ is satisfied. That is, we need

$$\begin{aligned}0=\alpha_1\overline{u}_1(0)-\gamma_1\overline{u}_1(1)&=\alpha_1\left[u_1(0)+k\right]-\gamma_1\left[u_1(1)-\frac{a_1}{2\delta_1}+k\right]\\&=\left[\alpha_1u_1\left(0\right)-\gamma_1u_1\left(1\right)\right]+\frac{a_1\gamma_1}{2\delta_1}+\left(\alpha_1-\gamma_1\right)k\\&=-a_1+\frac{a_1\gamma_1}{2\delta_1}+\left(\alpha_1-\gamma_1\right)k,\end{aligned}$$

and so we must have $k=\frac{a_1(2\delta_1-\gamma_1)}{2\delta_1(\alpha_1-\gamma_1)}$. Thus, by setting $\overline{u}_1(t)=u_1(t)-\frac{a_1}{2\delta_1}t^2+\frac{a_1(2\delta_1-\gamma_1)}{2\delta_1(\alpha_1-\gamma_1)}$, we obtain a function that simultaneously satisfies the homogeneous boundary conditions $\alpha_1\overline{u}_1(0)-\gamma_1\overline{u}_1(1)=\beta_1\overline{u}_1'(0)-\delta_1\overline{u}_1'(1)=0$. A similar argument can be used to obtain the function $\overline{u}_2(t)\equiv u_2(t)-\frac{a_2}{2\delta_2}t^2+\frac{a_2(2\delta_2-\gamma_2)}{2\delta_2(\alpha_2-\gamma_2)}$ satisfying $\alpha_2\overline{u}_2(0)-\gamma_2\overline{u}_2(1)=\beta_2\overline{u}_2'(0)-\delta_2\overline{u}_2'(1)=0$.

We can therefore transform (2.7)–(2.10) into the equivalent homogeneous system

$$-u_2''(t)=\lambda f(t,u_1(t)+Q_1t^2+R_1,u_2(t)+Q_2t^2+R_2), \tag{2.11}$$

$$-u_1''(t)=g(t,u_1(t)+Q_1t^2+R_1,u_2(t)+Q_2t^2+R_2), \tag{2.12}$$

$$\alpha_iu_i(0)-\gamma_iu_i(1)=\beta_iu_i'(0)-\delta_iu_i'(1)=0, \tag{2.13}$$

where $Q_i=\frac{a_i}{2\delta_i}$ and $R_i=-\frac{a_i(2\delta_i-\gamma_i)}{2\delta_i(\alpha_i-\gamma_i)}$ for $i=1,2$. After integrating twice, solutions to (2.11)–(2.13) can be shown to be of the form

$$\begin{aligned}u_2(t)&=\lambda\int_0^1G_2(t,s)f(s,u_1(s)+Q_1s^2+R_1,u_2(s)+Q_2s^2+R_2)ds\\u_1(t)&=\int_0^1G_1(t,s)g(s,u_1(s)+Q_1s^2+R_1,u_2(s)+Q_2s^2+R_2)ds,\end{aligned}$$

where $G_i(t,s)$ is the Green's function

$$G_i(t,s) = \frac{1}{M_i N_i} \begin{cases} \delta_i N_i t + \gamma_i M_i s + \gamma_i \beta_i, & 0 \le t \le s \le 1, \\ \beta_i N_i t + \alpha_i M_i s + \gamma_i \beta_i, & 0 \le s \le t \le 1, \end{cases}$$

and $M_i = \delta_i - \beta_i$, $N_i = \alpha_i - \gamma_i$ for $i = 1, 2$.

Recall that we are ultimately interested in establishing the existence of *positive* solutions to (2.4)–(2.6). Of course, solutions to this system correspond to solutions of (2.11)–(2.13) given the one-one relationship established above. Yet, it is clear from the above definitions that, without additional constraints on the parameters $\alpha_i, \beta_i, \delta_i, \gamma_i$, the Green's functions are unrestricted in sign and, in turn, so too are any solutions of (2.4)–(2.6)—assuming such solutions even exist. With this in mind, we now make the assumption that $\alpha_i < \gamma_i$ and $\delta_i < \beta_i$ for $i = 1, 2$ so that

$$\delta_i N_i t + \gamma_i M_i s + \gamma_i \beta_i \ge \delta_i N_i + \gamma_i M_i + \gamma_i \beta_i = \alpha_i \delta_i > 0$$

and

$$\beta_i N_i t + \alpha_i M_i s + \gamma_i \beta_i \ge \beta_i N_i + \alpha_i M_i + \gamma_i \beta_i = \alpha_i \delta_i > 0,$$

from which it follows that $G_i(t,s)$ is positive. As a result, solutions to (2.4)–(2.6), provided they exist, would have to be positive as well.

With the general form of positive solutions to the homogeneous system known, we now proceed to set up the cone and operator that will be utilized in the main existence results. Let $(X, \|\cdot\|)$ be the Banach space $X = C([0,1];\mathbb{R}) \times C([0,1];\mathbb{R})$ endowed with the norm

$$\|(u_1, u_2)\| = \|u_1\|_\infty + \|u_2\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Define $C \subset X$ to be the cone

$$C = \{(u_1, u_2) \in X \mid u_i \text{ is nonnegative and concave;}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2\},$$

and let Ω_ρ denote the open set $\Omega_\rho = \{(u_1, u_2) \in X : \|(u_1, u_2)\| < \rho\}$. Finally, we define $T : X \to X$ to be the operator

$$T(u_1, u_2) = (A_1(u_1, u_2), A_2(u_1, u_2))$$

with

$$A_2(u_1, u_2)(t) = \lambda \int_0^1 G_2(t, s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds$$

and

$$A_1(u_1, u_2)(t) = \int_0^1 G_1(t, s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds,$$

where G_1, G_2 are defined as above, $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$, and f, g are assumed to satisfy the following hypothesis:

(A0) $f, g : [0, 1] \times [0, \infty)^2 \to [0, \infty)$ are continuous functions that are nondecreasing in their last two variables.

By design, the fixed points of T (over C), if any, are solutions to a system that is similar to (2.11)–(2.13) in form but in which the only constraints on Q_1, Q_2 and R_1, R_2

are nonpositivity and nonnegativity, respectively. To avoid any confusion, we refer to this more general system as (2.11*)–(2.13*).

Now, notice that the addition of (A0) introduces constraints on the functions f, g and the constants Q_1, Q_2, R_1, R_2 in (2.11*)–(2.13*) that may or may not hold for their counterparts in (2.11)–(2.13). As a result, it is possible the natural correspondence between the two systems will be compromised, unless we appropriately constrain the function h and parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ of (2.4)–(2.6). Incidentally, the continuity and nonnegativity properties of the functions f, g of (2.11)–(2.13) follow directly from the continuity and sign changing properties of h coupled with the choice of substitutions and transformations made earlier. The nondecreasing properties cannot be similarly deduced, so we make the following assumption on (2.4)–(2.6): h is nondecreasing in its second variable and nonincreasing in its third.

More subtle are the assumptions needed on the parameters. We note that all of our subsequent results will take place in the cone C defined above, where the functions u_1 and u_2 are assumed nonnegative. Combining this with the fact that $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$ and $f, g : [0,1] \times [0,\infty)^2 \to [0, \infty)$, we must therefore have

$$0 \leq \min_{\substack{u_i \in C, \\ s \in [0,1]}} \left\{ u_i(s) + Q_i s^2 + R_i \right\} = Q_i + R_i$$

for $i = 1, 2$. In the transformed system (2.11)–(2.13), this amounts to

$$0 \leq Q_i + R_i = \frac{a_i \left(\alpha_i - 2\delta_i\right)}{2\delta_i \left(\alpha_i - \gamma_i\right)},$$

from which we get the requirement in (2.4)–(2.6) that $2\delta_i \leq \alpha_i$ for $i = 1, 2$.

We close this section with a lemma that establishes the cone preserving and completely continuous properties of the operator T, both of which are required in order to apply the Guo-Krasnosel'skii Fixed Point Theorem. We note the following bounds, which will be useful not only in the immediate proof but also in the proofs of the lemmas in the next section:

$$\max_{t\in[0,1]} \int_0^1 |G_i(t,s)|\, ds = \frac{\gamma_i\,(\delta_i+\beta_i)}{2M_iN_i}, \qquad i=1,2, \tag{2.14}$$

and

$$\max_{t\in[0,1]} \int_0^1 \left|\frac{\partial}{\partial t}G_i(t,s)\right| ds = \left|\frac{\beta_i}{M_i}\right| = \frac{-\beta_i}{M_i}, \qquad i=1,2. \tag{2.15}$$

Lemma 2.1.1. *Suppose* (A0) *holds. Then $T : X \to X$ is a completely continuous operator such that $T(C) \subseteq C$.*

Proof. We begin by showing $T(C) \subseteq C$. Let $(u_1,u_2) \in C$. That $T(u_1,u_2)$ is nonnegative follows immediately from the fact $f, g \geq 0$ and $\lambda, G_1, G_2 > 0$. Moreover, since

$$A_1''(u_1,u_2)(t) = -g(t, u_1(t)+Q_1t^2+R_1, u_2(t)+Q_2t^2+R_2) \leq 0$$

and

$$A_2''(u_1,u_2)''(t) = -\lambda f(s, u_1(t)+Q_1t^2+R_1, u_2(t)+Q_2t^2+R_2) \leq 0$$

for all $t \in [0,1]$, we have that A_1 and A_2 are concave. To show $T(u_1, u_2)$ satisfies the boundary conditions of the cone, we consider A_1 and A_2 individually. Note first

$$\begin{aligned}
\alpha_1 A_1(0) - \gamma_1 A_1(1) &= \alpha_1 \int_0^1 G_1(0,s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds - \\
&\quad \gamma_1 \int_0^1 G_1(1,s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&= \int_0^1 \frac{\alpha_1 (\gamma_1 M_1 s + \gamma_1 \beta_1)}{M_1 N_1} g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds - \\
&\quad \int_0^1 \frac{\gamma_1 (\beta_1 N_1 + \alpha_1 M_1 s + \gamma_1 \beta_1)}{M_1 N_1} g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&= 0.
\end{aligned}$$

Next, because G_1 is continuously differentiable on each of the regions $s < t$ and $t < s$, we are justified in differentiating under the integral sign on each region to obtain

$$\begin{aligned}
\beta_1 A_1'(0) - \delta_1 A_1'(1) &= \beta_1 \int_0^1 \frac{\partial}{\partial t} G_1(t,s) \Big|_{t=0} g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds - \\
&\quad \delta_1 \int_0^1 \frac{\partial}{\partial t} G_1(t,s) \Big|_{t=1} g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&= \beta_1 \int_0^1 \frac{\delta_1 N_1}{M_1 N_1} g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds - \\
&\quad \delta_1 \int_0^1 \frac{\beta_1 N_1}{M_1 N_1} g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&= 0.
\end{aligned}$$

The proofs for A_2 proceed similarly. Thus, T is cone preserving.

To prove T is a completely continuous operator, it is enough to show that each $A_i : X \to C([0,1]; \mathbb{R})$ is completely continuous. Because the A_i are linear operators between Banach spaces, it suffices to demonstrate compactness of the A_i. So, define

$B_r = \{(u,v) \in X \mid \|(u,v)\| \leq r\}$. Since g is continuous, there exists $k_r > 0$ such that

$$\left|g(t, u(t) + Q_1 t^2 + R_1, v(t) + Q_2 t^2 + R_2)\right| < k_r$$

for all $(u,v) \in B_r$ and $t \in [0,1]$. Let $\{(u_n, v_n)\}$ be a sequence in B_r. Then, for all n and all $t \in [0,1]$,

$$\begin{aligned}
|A_1(u_n, v_n)(t)| &\leq \int_0^1 |G_1(t,s) g(t, u_n(s) + Q_1 s^2 + R_1, v_n(s) + Q_2 s^2 + R_2)| ds \\
&< k_r \int_0^1 |G_1(t,s)| \, ds \\
&\leq k_r \frac{\gamma_1 (\delta_1 + \beta_1)}{2 M_1 N_1},
\end{aligned}$$

where the last inequality follows from (2.14). Thus, $\{A_1(u_n, v_n)\}$ is uniformly bounded.

Now, let $\epsilon > 0$, and set $\eta = \frac{-\epsilon M_1}{k_r \delta_1} > 0$. Then, for all n and for all $t_1, t_2 \in [0,1]$ satisfying $0 < t_2 - t_1 < \eta$, we can apply the Mean Value Theorem to obtain $\tau \in (t_1, t_2)$ such that

$$\begin{aligned}
|A_1(u_n, v_n)(t_2) - A_1(u_n, v_n)(t_1)| &= \left| \frac{d}{dt} A_1(u_n, v_n)(\tau) \right| |t_2 - t_1| \\
&< \eta \int_0^1 \left| \frac{\partial}{\partial t} G_1(\tau, s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) \right| ds \\
&< \eta k_r \int_0^1 \left| \frac{\partial}{\partial t} G_1(\tau, s) \right| ds.
\end{aligned}$$

Combining the above result with the bound in (2.15) gives $|A_1(u_n, v_n)(t_2) - A_1(u_n, v_n)(t_1)| < \epsilon$. Thus, $\{A_1(u_n, v_n)\}$ is also uniformly equicontinuous. By the Arzelà-Ascoli Theorem, it follows that $\{A_1(u_n, v_n)\}$ has a convergent subsequence. Therefore, A_1 is compact and therefore completely continuous. A similar argument can be applied to A_2. Thus, T is a completely continuous operator. $\square$

2.1.2. Lemmas

In this section, we prove a sequence of four lemmas that give various compression and expansion estimates for the operator T. Each estimate will hold on the boundary of a particular subset of the cone C. By strategically nesting these subsets, we will be able combine the estimates to apply one of the two forms of the Guo-Krasnosel'skii Fixed Point Theorem to obtain a fixed point for T. As previously noted, this, in turn, will yield a positive solution to the general system (2.11*)–(2.13*). For reference, we restate this system and its current constraints:

$$-u_2''(t) = \lambda f(t, u_1(t) + Q_1t^2 + R_1, u_2(t) + Q_2t^2 + R_2), \tag{2.11*}$$

$$-u_1''(t) = g(t, u_1(t) + Q_1t^2 + R_1, u_2(t) + Q_2t^2 + R_2), \tag{2.12*}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0, \tag{2.13*}$$

where $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$ such that $Q_i + R_i \geq 0$ for $i = 1, 2$, f and g are assumed to satisfy hypothesis (A0), $0 < \delta_i < \beta_i$, and $2\delta_i \leq \alpha_i < \gamma_i$ for $i = 1, 2$.

We now move to the first two lemmas, each of which gives expansion estimates on the operator T. The following hypothesis will be needed in the proofs of both lemmas:

(A1) There exists $\alpha, \beta \in (0, 1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0, \infty)^2$ with $x_1 + x_2 \neq 0$, there exists $\kappa > 0$ such that $f(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.

As mentioned in the preceding section, we must be sure that any assumptions we place on (2.11*)–(2.13*) are reflected in the original system (2.4)–(2.6). In the case of (A1), the corresponding hypothesis on h is obvious: There must exist $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0,\infty) \times (-\infty, 0]$ with $x_1 - x_2 \neq 0$, there exists $\kappa > 0$ such that $h(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.

Lemma 2.1.2. *Suppose* (A0) *and* (A1) *hold, and let* $\rho^* > 0$. *Then there exists* Λ *such that, for every* $\lambda \geq \Lambda$ *and* $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$, *we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and let $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$. Assume α and β are as in (A1) and set $r = \alpha(1 - \beta)$. Define

$$K = \inf\left\{ \frac{f(t, rc_1, rc_2)}{r(c_1 + c_2)} : c_1, c_2 \geq 0, c_1 + c_2 = p^*, t \in [\alpha, \beta] \right\}.$$

The existence of a positive K follows from assumption (A1).

Now set $\Lambda \geq \left[Kr \int_\alpha^\beta G_2(1,s)ds \right]^{-1}$. Utilizing Lemma 1.2.4, we know that

$$u_i(t) + Q_i t^2 + R_i \geq \inf_{t \in [\alpha, \beta]} u_i(t) \geq r\|u_i\|_\infty$$

for $i = 1, 2$. Pairing this with the nondecreasing properties of f, it follows that

$$\begin{aligned}
\|T(u_1, u_2)\| &\geq \|A_2(u_1, u_2)\|_\infty \\
&\geq \lambda \int_0^1 G_2(1, s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&\geq \lambda \int_\alpha^\beta G_2(1, s) f(s, r\|u_1\|_\infty, r\|u_2\|_\infty) ds \\
&= \lambda r \|(u_1, u_2)\| \int_\alpha^\beta G_2(1, s) \frac{f(s, r\|u_1\|_\infty, r\|u_2\|_\infty)}{r\|(u_1, u_2)\|} ds \\
&\geq \lambda K r \|(u_1, u_2)\| \int_\alpha^\beta G_2(1, s) ds \\
&\geq \Lambda K r \|(u_1, u_2)\| \int_\alpha^\beta G_2(1, s) ds \\
&\geq \|(u_1, u_2)\|
\end{aligned}$$

for $\lambda \geq \Lambda$, which completes the proof. $\square$

Lemma 2.1.3. *Fix $\Lambda > 0$, and suppose* (A0) *and* (A1) *hold. Then, for every $\lambda \geq \Lambda$ and $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$, there exists positive $\rho_1 = \rho_1(\Lambda, Q_1, Q_2, R_1, R_2)$ such that, for every $\rho \in (0, \rho_1]$, we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_\rho$.

Proof. By (A1) and the nondecreasing property of f, there exists $\alpha, \beta \in (0, 1)$ with $\alpha < \beta$ and $\kappa > 0$ such that, for all $(u_1, u_2) \in C$ and all $t \in [\alpha, \beta]$, we have

$$f(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2) \geq f(t, Q_1 \beta^2 + R_1, Q_2 \beta^2 + R_2) > \kappa.$$

Take $\rho_1 = \Lambda\kappa \int_\alpha^\beta G_2(1,s)ds$. Then, for all $(u_1,u_2) \in C \cap \partial\Omega_\rho$ where $\rho \in (0,\rho_1]$, we have

$$\begin{aligned}
\|T(u_1,u_2)\| &\geq \|A_2(u_1,u_2)\|_\infty \\
&\geq \lambda \int_0^1 G_2(1,s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&\geq \lambda \int_\alpha^\beta G_2(1,s) f(s, Q_1\beta^2 + R_1, Q_2\beta^2 + R_2) ds \\
&> \lambda\kappa \int_\alpha^\beta G_2(1,s) ds \\
&= \lambda\kappa \|(u_1,u_2)\| \int_\alpha^\beta \frac{G_2(1,s)}{\|(u_1,u_2)\|} ds \\
&\geq \frac{\rho_1}{\rho} \|(u_1,u_2)\| \\
&\geq \|(u_1,u_2)\|.
\end{aligned}$$

□

So far, we have found subsets $C \cap \partial\Omega_{\rho^*}$ and $C \cap \partial\Omega_{\rho_1}$ on which

$$\|T(u_1,u_2)\| \geq \|(u_1,u_2)\|.$$

Note that we have no explicit relationship between ρ^* and ρ_1 at the moment. However, suppose we were to find $\rho_2 \in (0,\rho^*)$ such that

$$\|T(u_1,u_2)\| \leq \|(u_1,u_2)\|$$

for all $(u_1, u_2) \in C \cap \partial\Omega_{\rho_2}$. The existence of such a ρ_2 combined with the fact that Lemma 2 holds for *all* positive $\rho \leq \rho_1$, would entail $\rho_1 < \rho_2 < \rho^*$. As a result, the Guo-Krosnoselskii Fixed Point Theorem would be satisfied twice: Once over $C \cap (\overline{\Omega}_{\rho^*} - \Omega_{\rho_2})$ via the expansion form of the theorem; a second time over $C \cap (\overline{\Omega}_{\rho_2} - \Omega_{\rho_1})$ by the compression form of the theorem. This is the stacking method alluded to earlier.

We find exactly such a ρ_2 in Lemma 2.1.4. To do so, we assume the following hypotheses hold:

(A2) Let $z = x_1 + x_2$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0, 1]$.

(A4) For all $\zeta \in \left(1, \frac{2M_1N_1}{\gamma_1(\delta_1+\beta_1)}\right)$, there exists $q > 0$ such that, for all $(\overline{x}_1, \overline{x}_2) \in [0, \infty)^2$ with $0 < \overline{x}_1 + \overline{x}_2 < q$, we have $g\left(t, \overline{x}_1, \overline{x}_2\right) \leq \zeta\left(\overline{x}_1 + \overline{x}_2\right)$ for each $t \in [0, 1]$.

It is clear from the statement of (A2) what its parallel should be in the original system; however, gleaning the corresponding hypothesis from (A4) is less straightforward. In the proof below, we utilize (A4) to obtain the inequality

$$g(t, \|u_1\|_\infty + R_1, \|u_2\|_\infty + R_2) \leq \zeta(\|u_1\|_\infty + \|u_2\|_\infty + R_1 + R_2),$$

where u_1, u_2 are appropriately chosen. Recall that in transforming (2.4)–(2.6) into (2.11)–(2.13), the function g ultimately takes the form of a projection onto its last coordinate. Thus, we can recast the above inequality in terms of the the original system as

$$\|u_2\|_\infty + R_2 \leq \zeta(\|u_1\|_\infty + \|u_2\|_\infty + R_1 + R_2).$$

To ensure the above bound can be obtained in (2.4)–(2.6), we assume $a_i, \alpha_i, \beta_i, \gamma_i, \delta_i$ satisfy $0 < R_2 < R_1$ and $\frac{2M_1N_1}{\gamma_1(\delta_1+\beta_1)} > 1$ where R_i, M_i, N_i are defined as above for $i = 1, 2$. It should be noted that the first constraint serves a dual purpose in that it also prevents any conflict in the definition of $\overline{\zeta}$ in the proof below.

As a final aside, it is worth mentioning that (A4) is stated more strongly than needed to prove Lemma 2.1.4 for the general system (2.11*)–(2.13*). It would suffice in this more general case to require only the existence of $\zeta \in \left(0, \frac{2M_1N_1}{\gamma_1(\delta_1+\beta_1)}\right)$; the specification that $\zeta > 1$ is needed for the corresponding assumption on the original system only. However, we have opted for the current form of (A4) in order to maintain a certain continuity between the hypotheses of the two systems.

Lemma 2.1.4. *Suppose* (A0), (A2), *and* (A4) *hold, and let $\rho^* > 0$ be fixed. Then given $\lambda > 0$, there exists $\rho_2 \in (0, \rho^*)$ and $\overline{\zeta} > 0$ such that for every $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$ with $R_1 + R_2 < \overline{\zeta}$, we have*

$$\|T(u_1, u_2)\| \leq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon < \frac{M_2N_2}{\gamma_2(\delta_2+\beta_2)}$. From (A2), there exists $\overline{\rho}_2 \in (0, \rho^*)$ such that for $x_1 + x_2 = \overline{\rho}_2$ with $(x_1, x_2) \in [0, \infty)^2$ and for $R_1 + R_2 < \overline{\rho}_2$, we have

$$f(t, x_1 + R_1, x_2 + R_2) \leq \epsilon\left[(x_1 + R_1) + (x_2 + R_2)\right]$$

for $t \in [0,1]$. Also, pick $\zeta \in \left(1, \frac{2M_1N_1}{\gamma_1(\delta_1+\beta_1)}\right)$. Then, by (A4), there exists $q > 0$ such that for $(x_1 + R_1, x_2 + R_2) \in [0,\infty)^2$ with $x_1 + R_1 + x_2 + R_2 < q$ we have

$$g(t, x_1 + R_1, x_2 + R_2) \leq \zeta \left[(x_1 + R_1) + (x_2 + R_2)\right]$$

for $t \in [0,1]$.

Let $0 < \rho_2 < \min(q/2, \overline{\rho}_2)$. Take $(u_1, u_2) \in C \cap \partial\Omega_{\rho_2}$, and $R_1 + R_2 \leq \rho_2$. Then, by (A0) and above, we have

$$\begin{aligned}
A_2(u_1, u_2) &= \lambda \int_0^1 G_2(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&\leq \lambda \int_0^1 G_2(t,s) f(s, \|u_1\|_\infty + R_1, \|u_2\|_\infty + R_2) ds \\
&\leq \lambda\epsilon \left[\|(u_1, u_2)\| + R_1 + R_2\right] \int_0^1 G_2(t,s) ds \\
&\leq 2\lambda\epsilon \|(u_1, u_2)\| \int_0^1 G_2(t,s) ds \\
&\leq \lambda\epsilon \frac{\gamma_2\left(\delta_2 + \beta_2\right)}{M_2 N_2} \|(u_1, u_2)\|
\end{aligned}$$

for $t \in [0,1]$.

Note that $\|u_1\|_\infty + \|u_2\|_\infty + R_1 + R_2 \leq 2\rho_2 < q$, so it follows

$$g(t, \|u_1\|_\infty + R_1, \|u_2\|_\infty + R_2) \leq \zeta(\|u_1\|_\infty + \|u_2\|_\infty + R_1 + R_2).$$

Pick $\zeta' < 1$, and suppose $R_1 + R_2 < \zeta' \rho_2$. Let $\overline{\zeta} = \zeta' \rho_2$. We have, by (A0) and above,

$$\begin{aligned}
A_1(u_1, u_2) &= \int_0^1 G_1(t,s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&\leq \zeta \left[\|(u_1, u_2)\| + R_1 + R_2 \right] \int_0^1 G_1(t,s) ds \\
&\leq \zeta \left(1 + \zeta'\right) \|(u_1, u_2)\| \int_0^1 G_1(t,s) ds \\
&\leq \zeta \left(1 + \zeta'\right) \frac{\gamma_1 \left(\delta_1 + \beta_1\right)}{2 M_1 N_1} \|(u_1, u_2)\|
\end{aligned}$$

for $t \in [0,1]$.

Thus,

$$\|T(u_1, u_2)\| \leq \left[\zeta \left(1 + \zeta'\right) \frac{\gamma_1 \left(\delta_1 + \beta_1\right)}{2 M_1 N_1} + \lambda \epsilon \frac{\gamma_2 \left(\delta_2 + \beta_2\right)}{M_2 N_2} \right] \|(u_1, u_2)\|$$

for $(u_1, u_2) \in C \cap \Omega_{\rho_2}$ and $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$ with $R_1 + R_2 < \overline{\zeta}$. Picking ϵ and ζ' small enough gives the desired result. $\square$

A third fixed point can be obtained by establishing a compression estimate for T on $C \cap \partial\Omega_{\rho_3}$, where $\rho_3 > \rho^*$, and then combining this estimate with the one secured by Lemma 2.1.2. Notice that, in this case, we would apply the compression form of the Guo-Krasnosel'skii Fixed Point Theorem on the set $C \cap (\overline{\Omega}_{\rho_3} - \Omega_{\rho^*})$.

This is the purpose of Lemma 2.1.5. The proof of this will require the addition of hypotheses (A3) and (A5) below. Note that the parallel of (A5) in (2.4)–(2.6) is guaranteed to be satisfied as a result of the constraints given prior to Lemma 2.1.4. The corresponding hypothesis to (A3) is evident from the definitions.

(A3) Let $z = x_1 + x_2$. Then

$$\lim_{z\to\infty} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0, 1]$.

(A5) For all $\theta \in \left(1, \frac{2M_1N_1}{\gamma_1(\delta_1+\beta_1)}\right)$, there exists $r > 0$ such that, for all $(\overline{x}_1, \overline{x}_2) \in [0, \infty)^2$ with $\overline{x}_1 + \overline{x}_2 > r$, we have $g(t, \overline{x}_1, \overline{x}_2) \leq \theta(\overline{x}_1 + \overline{x}_2)$ for each $t \in [0, 1]$.

Lemma 2.1.5. *Let $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$, and suppose $R_1 + R_2 < \overline{\zeta}$, where $\overline{\zeta} > 0$ is given. Suppose further that assumptions* (A0), (A3), *and* (A5) *hold. Then, for every $\lambda > 0$, there exists $\rho_3 = \rho_3(\overline{\zeta}, \lambda)$ such that for every $\rho \geq \rho_3$, we have*

$$\|T(u_1, u_2)\| \leq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_\rho$.

Proof. Let $\overline{\zeta} > 0$ be given, and let $R_1 + R_2 < \overline{\zeta}$. Pick $\theta \in \left(1, \frac{2M_1N_1}{\gamma_1(\delta_1+\beta_1)}\right)$. Then, by (A5), there exists $r > 0$ such that for all $(x_1+R_1, x_2+R_2) \in [0, \infty)^2$ with $x_1+R_1+x_2+R_2 > r$, we have $g(t, x_1 + R_1, x_2 + R_2) \leq \theta(x_1 + R_1 + x_2 + R_2)$ for $t \in [0, 1]$. Let $\epsilon > 0$ and pick q_1 large enough so that $q_1 + (R_1 + R_2) > r$ and $\epsilon > \frac{\theta\overline{\zeta}}{q_1}$. Then, for $(u_1, u_2) \in C \cap \partial\Omega_{q_1}$,

$$\begin{aligned} g(t, \|u_1\|_\infty + R_1, \|u_2\|_\infty + R_2) &\leq \theta\left[(\|u_1\|_\infty + R_1) + (\|u_2\|_\infty + R_2)\right] \\ &= \theta(\|u_1\|_\infty + \|u_2\|_\infty) + \theta(R_1 + R_2) \\ &\leq \theta(\|u_1\|_\infty + \|u_2\|_\infty) + \epsilon(\|u_1\|_\infty + \|u_2\|_\infty) \\ &= (\epsilon + \theta)\|(u_1, u_2)\|. \end{aligned}$$

Applying the above with the nondecreasing property of g, we see that, for $t \in [0,1]$,

$$\begin{aligned}
A_1(u_1,u_2) &= \int_0^1 G_1(t,s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&\leq \int_0^1 G_1(t,s) g(s, \|u_1\|_\infty + R_1, \|u_2\|_\infty + R_2) ds \\
&\leq (\epsilon + \theta) \|(u_1,u_2)\| \int_0^1 G_1(t,s) ds \\
&\leq (\epsilon + \theta) \frac{\gamma_1 (\delta_1 + \beta_1)}{2 M_1 N_1} \|(u_1,u_2)\|.
\end{aligned}$$

Now, let $\eta > 0$. Then by (A0) and (A3), there exists $q_2 > 0$ such that for every $(x_1,x_2) \in [0,\infty)^2$ with $x_1 + x_2 \geq q_2$, we have

$$f(t, x_1 + R_1, x_2 + R_2) \leq \eta \left[(x_1 + R_1) + (x_2 + R_2) \right].$$

Let $q_3 = \max\left\{\overline{\zeta}, q_2\right\}$ and $(u_1,u_2) \in C \cap \partial\Omega_{q_3}$. Recalling $R_1 + R_2 < \overline{\zeta}$, it follows

$$\begin{aligned}
f(t, \|u_1\|_\infty + R_1, \|u_2\|_\infty + R_2) &\leq \eta(\|u_1\|_\infty + \|u_2\|_\infty) + \overline{\zeta}\eta \\
&\leq 2\eta \|(u_1,u_2)\|.
\end{aligned}$$

Combining this with the nondecreasing properties of f, we have

$$\begin{aligned}
A_2(u_1,u_2) &\leq 2\eta\lambda \|(u_1,u_2)\| \int_0^1 G_2(t,s) ds \\
&\leq \eta\lambda \frac{\gamma_2 (\delta_2 + \beta_2)}{M_2 N_2} \|(u_1,u_2)\|.
\end{aligned}$$

Take $\rho_3 = \max\{q_1, q_3\}$, and let $\rho \geq \rho_3$. Then, given $(u_1, u_2) \in C \cap \partial\Omega_\rho$, we have

$$\|T(u_1, u_2)\| \leq \left[(\epsilon + \theta)\frac{\gamma_1(\delta_1 + \beta_1)}{2M_1N_1} + \eta\lambda\frac{\gamma_2(\delta_2 + \beta_2)}{M_2N_2}\right]\|(u_1, u_2)\|.$$

Picking ϵ and η small enough gives $\|T(u_1, u_2)\| \leq \|(u_1, u_2)\|$ as needed.

□

2.1.3. Main Results

We are now in a position to establish the main existence results of this section. Our first result combines the lemmas above to show that the general system (2.11*)–(2.13*) contains at least three positive solutions.

Theorem 2.1.6. *Suppose hypotheses* (A0)–(A5) *are satisfied for functions f and g. Suppose additionally that $0 < \delta_i < \beta_i$ and $2\delta_i \leq \alpha_i < \gamma_i$ for $i = 1, 2$. Then there exists $\Lambda > 0$ such that, given $\lambda \geq \Lambda$, there exists $\overline{\zeta} > 0$ such that for every $a_1, a_2 < 0$ satisfying*

$$\frac{a_1(2\delta_1 - \gamma_1)}{2\delta_1(\gamma_1 - \alpha_1)} + \frac{a_2(2\delta_2 - \gamma_2)}{2\delta_i(\gamma_2 - \alpha_2)} < \overline{\zeta}$$

and every $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$ satisfying $R_1 + R_2 < \overline{\zeta}$, the system (2.11*)–(2.13*) *has at least three positive solutions.*

Proof. Suppose that f and g satisfy hypotheses (A0)–(A5) and that $0 < \delta_i < \beta_i$ and $2\delta_i \leq \alpha_i < \gamma_i$ for $i = 1, 2$. Let $\rho^* > 0$ be fixed. By Lemma 2.1.2, there exists $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $a_1, a_2 < 0$, we have

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$.

Fix $\lambda \geq \Lambda$. By Lemmas 2.1.3 through 2.1.3, there exists $\overline{\zeta} > 0$ and $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that for every $a_1, a_2 < 0$ satisfying

$$-\frac{a_1\left(2\delta_1 - \gamma_1\right)}{2\delta_1\left(\alpha_1 - \gamma_1\right)} - \frac{a_i\left(2\delta_i - \gamma_i\right)}{2\delta_i\left(\alpha_i - \gamma_i\right)} < \overline{\zeta}$$

and every $(Q_1, Q_2, R_1, R_2) \in (-\infty, 0]^2 \times [0, \infty)^2$ satisfying $R_1 + R_2 < \overline{\zeta}$, we have

$$\begin{aligned}
\|T(u_1, u_2)\| &\geq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_1}, \\
\|T(u_1, u_2)\| &\leq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_2}, \\
\|T(u_1, u_2)\| &\leq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_3}.
\end{aligned}$$

After applying the Guo-Krasnosel'skii Fixed Point Theorem three times, we get that there exist three positive solutions (u_1, u_2), (v_1, v_2),$(w_1, w_2) \in C$ such that

$$\rho_1 < \|(u_1, u_2)\| < \rho_2 < \|(v_1, v_2)\| < \rho^* < \|(w_1, w_2)\| < \rho_3.$$

$\square$

By taking advantage of the one-one correspondence between (2.4)–(2.6) and (2.11)–(2.13) and noting that the latter system is of the form (2.11*)–(2.13*), we obtain the following corollary. Note that the addition of constraints (2.16) and (2.17) below ensures that the assumptions of (A4) and (A5) are satisfied. These new constraints allow for the removal of any hypotheses corresponding to (A4) and (A5).

Corollary 2.1.7. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times [0,\infty) \times (-\infty,0] \to [0,\infty)$*:*

(A0)* *h is continuous, nondecreasing in its second variable, and nonincreasing in its third variable.*

(A1)* *There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0,\infty) \times (-\infty, 0]$ with $x_1 - x_2 \neq 0$, there exists $\kappa > 0$ such that $h(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.*

(A2)* *Let $z = x_1 - x_2 > 0$. Then*

$$\lim_{z \to 0^+} \frac{h(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

(A3)* *Let $z = x_1 - x_2 > 0$. Then*

$$\lim_{z \to \infty} \frac{h(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

Suppose also that $0 < \delta_i < \beta_i$ and $2\delta_i \leq \alpha_i < \gamma_i$ for $i = 1, 2$. Then there exists $\Lambda > 0$ such that, given $\lambda \geq \Lambda$ there exists $\overline{\zeta} > 0$ such that, for every $a_1, a_2 < 0$ that satisfies the properties that after setting $Q_i = \frac{a_i}{2\delta_i}$, $R_i = -\frac{a_i(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$, $M_i = \delta_i - \beta_i$, and $N_i = \alpha_i - \gamma_i$ for $i = 1, 2$ we have $0 < R_1 + R_2 < \overline{\zeta}$,

$$0 < R_2 < R_1, \tag{2.16}$$

and

$$\frac{2M_1 N_1}{\gamma_1 (\delta_1 + \beta_1)} > 1, \tag{2.17}$$

then the system (2.4)–(2.6) *has at least three positive solutions.*

2.2. Positive Case

We now turn our attention to boundary value problems of the form (2.1)–(2.3) when $a_i > 0$ for $i = 1, 2$. The structure of this section is similar to the previous one, however, the presentation will be more compact given the significant overlap between the two cases—a full treatment can be found in [**8**]. To reiterate, the main purpose of this chapter is to highlight the differences between the two cases, and so it is such details that we wish to bring to the forefront in this section.

2.2.1. Preliminaries

Consider the boundary value problem

$$u^{(4)}(t) = \lambda h\left(t, u(t), u''(t)\right), \tag{2.18}$$

$$\alpha_1 u(0) - \gamma_1 u(1) = \beta_1 u'(0) - \delta_1 u'(1) = -a_1, \tag{2.19}$$

$$\alpha_2 u''(0) - \gamma_2 u''(1) = \beta_2 u'''(0) - \delta_2 u'''(1) = a_2, \tag{2.20}$$

where $h : [0, 1] \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$ is nonnegative and continuous, $\lambda, \alpha_i, \beta_i, \gamma_i, \delta_i > 0$, and $a_i > 0$ for $i = 1, 2$.

Because (2.18)–(2.20) is identical to (2.4)–(2.6), excepting a change in sign of the parameters a_1 and a_2, we can use the transformations of the preceding section to obtain

$$-u_2''(t) = \lambda f(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \tag{2.21}$$

$$-u_1''(t) = g(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \tag{2.22}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0, \tag{2.23}$$

where $Q_i = \frac{a_i}{2\delta_i}$ and $R_i = -\frac{a_i(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ for $i = 1, 2$. Notice that $Q_i > 0$, whereas the sign of R_i is currently undetermined. Solutions to this system are of the form

$$u_2(t) = \lambda \int_0^1 G_2(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds$$
$$u_1(t) = \int_0^1 G_1(t,s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds,$$

where $G_i(t,s)$ is the Green's function

$$G_i(t,s) = \frac{1}{M_i N_i} \begin{cases} \delta_i N_i t + \gamma_i M_i s + \gamma_i \beta_i, & 0 \le t \le s \le 1, \\ \beta_i N_i t + \alpha_i M_i s + \gamma_i \beta_i, & 0 \le s \le t \le 1, \end{cases}$$

and $M_i = \delta_i - \beta_i$, $N_i = \alpha_i - \gamma_i$ for $i = 1, 2$. By making the assumptions that $\alpha_i > \gamma_i$ and $\delta_i > \beta_i$ for $i = 1, 2$, we guarantee that the solutions (2.18)–(2.20) are positive. Note that the inequalities on the parameters have flipped in comparison to the negative case.

For convenience, we note the following bounds:

$$\max_{t \in [0,1]} \int_0^1 G_i(t,s)\, ds = \frac{\alpha_i (\delta_i + \beta_i)}{2 M_i N_i}, i = 1, 2, \tag{2.24}$$

and

$$\max_{t\in[0,1]} \int_0^1 \frac{\partial}{\partial t} G_k(t,s)\, ds = \frac{\delta_k}{M_k}, i = 1, 2. \tag{2.25}$$

It is interesting to point out the symmetry in the bounds above with those given in (2.14)–(2.15). Given the key role the corresponding inequalities played in the proofs above, one might anticipate that these differences will lead to points of divergence in what follows. This hunch would be well justified—see (2.27) in Corollary 2.2.7 for instance.

We now proceed to set up the cone and operator that will be utilized in the main existence results. Let $(X, \|\cdot\|)$ be the Banach space $X = C([0,1];\mathbb{R}) \times C([0,1];\mathbb{R})$ endowed with the norm

$$\|(u_1, u_2)\| = \|u_1\|_\infty + \|u_2\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Define $C \subset X$ to be the cone

$$C = \{(u_1, u_2) \in X \mid u_i \text{ is nonnegative and concave;}$$
$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2\},$$

and let Ω_ρ denote the open set $\Omega_\rho = \{(u_1, u_2) \in X : \|(u_1, u_2)\| < \rho\}$. Finally, we define $T : X \to X$ to be the operator

$$T(u_1, u_2) = (A_1(u_1, u_2), A_2(u_1, u_2))$$

with

$$A_2(u_1, u_2)(t) = \lambda \int_0^1 G_2(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds$$

and

$$A_1(u_1, u_2)(t) = \int_0^1 G_1(t,s)g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2)ds,$$

where G_1, G_2 are defined as above, $(Q_1, Q_2, R_1, R_2) \in [0,\infty)^4$, and f, g are assumed to satisfy the following hypothesis:

(B0) $f, g : [0,1] \times [0,\infty)^2 \to [0,\infty)$ are continuous functions that are nondecreasing in their last two variables.

The operator T (over C) was constructed so that its fixed points are solutions to a system that is similar to (2.21)–(2.23) in form but in which the only constraints on Q_1, Q_2, R_1, R_2 are nonnegativity. We adopt the same strategy as we did in the negative case and refer to this more general system as (2.21*)–(2.23*).

Furthermore, the addition of (B0) brings with it the same problems that we faced in the preceding section. We can address the nondecreasing property by making the same constraint on the function h as we did in the negative case, that is, we require that h is nondecreasing in its second variable and nonincreasing in its third variable. Our solution to handling the parameters is slightly different, but the approach is the same as above. We note that our later work will take place in the cone C defined above, where the functions u_1 and u_2 are assumed nonnegative, and that we have $(Q_1, Q_2, R_1, R_2) \in [0,\infty)^4$ and $f, g : [0,1] \times [0,\infty)^2 \to [0,\infty)$. As a result, this leads to

$$0 \leq \min_{\substack{u_i \in C, \\ s \in [0,1]}} \left\{ u_i(s) + Q_i s^2 + R_i \right\} = R_i$$

for $i = 1, 2$. Observe that Q_i does not appear on the right-hand side, due to its nonnegativity. This was not so in the negative case where Q_i was assumed nonpositive. Since $R_i = -\frac{a_i(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ in (2.21)–(2.23), this leads to the requirement $2\delta_i \leq \gamma_i$ for $i = 1, 2$ in (2.18)–(2.20). Notice that this leads to an altogether different inequality than the one obtained in the negative case.

We end this section with the following preliminary lemma, which establishes the cone preserving and completely continuous properties of T. The proof, which we omit here, follows a standard Arzelà-Ascoli argument as seen in Section 2.1.

Lemma 2.2.1. *Suppose* (B0) *holds. Then* $T : X \to X$ *is a completely continuous operator such that* $T(C) \subseteq C$.

2.2.2. Lemmas

We now establish the sequence of four lemmas that will allow us to obtain the fixed points of the operator T. For reference, we restate the system (2.21*)–(2.23*) and its current constraints:

$$-u_2''(t) = \lambda f(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \tag{2.21*}$$

$$-u_1''(t) = g(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \tag{2.22*}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0, \tag{2.23*}$$

where $(Q_1, Q_2, R_1, R_2) \in \times[0, \infty)^4$, f and g are assumed to satisfy hypothesis (B0), and $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ for $i = 1, 2$.

The first two lemmas give expansion estimates for T and depend on hypothesis (B1) below. The corresponding hypothesis on h can be stated as follows: There must exist $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0, \infty) \times (-\infty, 0]$ with $x_1 + x_2 \neq 0$, there exists $\kappa > 0$ such that $h(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.

(B1) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0, \infty)^2$ with $x_1 + x_2 \neq 0$, there exists $\kappa > 0$ such that $f(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.

We state these lemmas without proof as their proofs can be modified from the ones given for Lemmas 2.1.2 and Lemma 2.1.3 above. The key difference between the two cases has to do with how we utilize the nondecreasing properties of f. In the proof of Lemma 2.1.3, for instance, we appealed to (A0) and (A1) to obtain

$$f(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2) \geq f(t, Q_1 \beta^2 + R_1, Q_2 \beta^2 + R_2) > \kappa$$

for all $(u_1, u_2) \in C$ and all $t \in [\alpha, \beta]$, where α, β are given as in (A1). This inequality holds because of the nonpositivity of Q_1 and Q_2 combined with the fact that $Q_i + R_i \geq 0$ for $i = 1, 2$. In the present case, we have Q_1 and Q_2 nonnegative, so the same argument will not work. Instead, we argue that

$$f(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2) \geq f(t, Q_1 \alpha^2 + R_1, Q_2 \alpha^2 + R_2) > \kappa$$

making a similar appeal to the appropriate hypotheses but this time utilizing the nonnegativity of Q_1 and Q_2, as well as the sum $Q_i + R_i$.

Lemma 2.2.2. *Suppose* (B0) *and* (B1) *hold, and let* $\rho^* > 0$*. Then there exists* Λ *such that, for every* $\lambda \geq \Lambda$ *and* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$*, we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$*.*

Lemma 2.2.3. *Fix* $\Lambda > 0$*, and suppose* (B0) *and* (B1) *hold. Then, for every* $\lambda \geq \Lambda$ *and* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$*, there exists* $\rho_1 = \rho_1(\Lambda, Q_1, Q_2, R_1, R_2)$ *such that, for every positive* $\rho \leq \rho_1$*, we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho}$*.*

The motivation for Lemma 2.2.4 is precisely the same as that given for Lemma 2.1.4 in the preceding section. We want to find $\rho_2 \in (0, \rho^*)$ for which we get a compression estimate for T on $C \cap \partial\Omega_{\rho_2}$. This then allows for a double application of the Guo-Krosnoselskii Fixed Point Theorem: Once over $C \cap (\overline{\Omega}_{\rho^*} - \Omega_{\rho_2})$ via the expansion form of the theorem, and a second time over $C \cap (\overline{\Omega}_{\rho_2} - \Omega_{\rho_1})$ by the compression form of the theorem.

We assume the following hypotheses hold:

(B2) Let $z = x_1 + x_2$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0, 1]$.

(B4) For all $\zeta \in \left(1, \frac{2M_1N_1}{\alpha_1(\delta_1+\beta_1)}\right)$, there exists $q > 0$ such that, for all $(\overline{x}_1, \overline{x}_2) \in [0, \infty)^2$ with $0 < \overline{x}_1 + \overline{x}_2 < q$, we have $g\left(t, \overline{x}_1, \overline{x}_2\right) \leq \zeta\left(\overline{x}_1 + \overline{x}_2\right)$ for each $t \in [0, 1]$.

The hypothesis corresponding to (B2) in the original system is clear, and the one corresponding to (B4) follows from the same rationale given in the negative case. In particular, we assume $a_i, \alpha_i, \beta_i, \gamma_i, \delta_i$ satisfy $0 < Q_2 + R_2 < Q_1 + R_1$ and $\frac{2M_1N_1}{\alpha_1(\delta_1+\beta_1)} > 1$ where Q_i, R_i, M_i, N_i are defined as above for $i = 1, 2$. Note that, in contrast to the negative case, the first inequality involves Q_1 and Q_2, not just R_1 and R_2.

With these details out of the way, we now state Lemma 2.2.4. Again, we omit the proof as it is similar to the one given for Lemma 2.1.4.

Lemma 2.2.4. *Suppose* (B0), (B2), *and* (B4) *hold, and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$, *there exists* $\rho_2 \in (0, \rho^*)$ *and* $\overline{\zeta} > 0$ *such that for every* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$ *with* $Q_1 + Q_2 + R_1 + R_2 < \overline{\zeta}$, *we have*

$$\|T(u_1, u_2)\| \leq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho_2}$.

In Lemma 2.2.5 below, we establish a compression estimate for T on $C \cap \partial\Omega_{\rho_3}$, where $\rho_3 > \rho^*$. This allows for a third and final application of the Guo-Krasnosel'skii Fixed Point Theorem on the set $C \cap (\overline{\Omega}_{\rho_3} - \Omega_{\rho^*})$. The proof requires hypotheses (B3) and (B5). Note that the bounds given prior to Lemma 2.2.4 will ensure any corresponding hypothesis in (2.18)–(2.20) is satisfied, and the corresponding hypothesis to (B3) follows with only slight changes. The proof itself is a straightforward modification of the one given for Lemma 2.1.5; we omit the details here.

(B3) Let $z = x_1 + x_2$. Then

$$\lim_{z\to\infty} \frac{f(t,x_1,x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

(B5) For all $\theta \in \left(1, \frac{2M_1N_1}{\gamma_1(\delta_1+\beta_1)}\right)$, there exists $r > 0$ such that, for all $(\overline{x}_1, \overline{x}_2) \in [0,\infty)^2$ with $\overline{x}_1 + \overline{x}_2 > r$, we have $g(t, \overline{x}_1, \overline{x}_2) \leq \theta(\overline{x}_1 + \overline{x}_2)$ for each $t \in [0,1]$.

Lemma 2.2.5. *Suppose $Q_1 + Q_2 + R_1 + R_2 < \overline{\zeta}$, where $\overline{\zeta} > 0$ is given. Suppose further that assumptions* (B0), (B3), *and* (B5) *hold. Then, for every $\lambda > 0$, there exists $\rho_3 = \rho_3(\overline{\zeta}, \lambda)$ such that for every $\rho \geq \rho_3$, we have*

$$\|T(u_1,u_2)\| \leq \|(u_1,u_2)\|$$

for each $(u_1,u_2) \in C \cap \partial\Omega_\rho$.

2.2.3. Main Results

We now establish the main existence results of this section. Theorem 2.2.6 below combines the work above to show that the general system (2.21*)–(2.23*) contains at least three positive solutions.

Theorem 2.2.6. *Suppose hypotheses* (B0)–(B5) *are satisfied for functions f and g. Suppose additionally that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ for $i = 1,2$. Then there exists $\Lambda > 0$ such that, given $\lambda \geq \Lambda$, there exists $\overline{\zeta} > 0$ such that, for every $a_1, a_2 > 0$ satisfying*

$$\frac{a_1}{2\delta_1}\left[1 - \frac{2\delta_1 - \gamma_1}{\alpha_1 - \gamma_1}\right] + \frac{a_2}{2\delta_2}\left[1 - \frac{2\delta_2 - \gamma_2}{\alpha_2 - \gamma_2}\right] < \overline{\zeta}$$

and every $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$ *satisfying* $Q_1 + R_1 + Q_2 + R_2 < \overline{\zeta}$, *the system* (7)–(9) *has at least three positive solutions.*

Proof. Suppose f and g satisfy hypotheses (Z0)–(Z5) and that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ for $i = 1, 2$. Let $\rho^* > 0$ be fixed. By Lemma 2.2.2, there exists $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $a_1, a_2 > 0$,

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$.

Fix $\lambda \geq \Lambda$. By Lemmas 2.2.3 through 2.2.5, there exists $\overline{\zeta} > 0$ and $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that, for every $a_1, a_2 > 0$ satisfying

$$\frac{a_1}{2\delta_1}\left[1 - \frac{2\delta_1 - \gamma_1}{\alpha_1 - \gamma_1}\right] + \frac{a_2}{2\delta_2}\left[1 - \frac{2\delta_2 - \gamma_2}{\alpha_2 - \gamma_2}\right] < \overline{\zeta}$$

and every $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$ satisfying $Q_1 + R_1 + Q_2 + R_2 < \overline{\zeta}$, we have

$$\begin{aligned}
\|T(u_1, u_2)\| &\geq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_1}, \\
\|T(u_1, u_2)\| &\leq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_2}, \\
\|T(u_1, u_2)\| &\leq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_3}.
\end{aligned}$$

After applying the Guo-Krasnosel'skii Fixed Point Theorem three times, we get that there exist three positive solutions (u_1, u_2), (v_1, v_2),$(w_1, w_2) \in C$ such that

$$\rho_1 < \|(u_1, u_2)\| < \rho_2 < \|(v_1, v_2)\| < \rho^* < \|(w_1, w_2)\| < \rho_3.$$

$\square$

Utilizing the one-one correspondence between (2.18)–(2.20) and (2.21)–(2.23) and noting that the latter system is of the form (2.21*)–(2.23*), we obtain the following corollary. Note that the addition of constraints (2.26) and (2.27) ensure that the assumptions of (B4) and (B5) are satisfied, and, as a result, we drop any explicit hypotheses corresponding to (B4) and (B5).

Corollary 2.2.7. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times [0,\infty) \times (-\infty,0] \to [0,\infty)$*:*

(B0)* *h is continuous, nondecreasing in its second variable, and nonincreasing in its third variable.*

(B1)* *There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0,\infty)^2$ with $x_1+x_2 \neq 0$, there exists $\kappa > 0$ such that $f(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.*

(B2)* *Let $z = x_1 + x_2 > 0$. Then*

$$\lim_{z \to 0^+} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

(B3)* *Let $z = x_1 + x_2 > 0$. Then*

$$\lim_{z \to \infty} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

Suppose also that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ for $i = 1, 2$. Then there exists $\Lambda > 0$ such that, given $\lambda \geq \Lambda$, there exists $\overline{\zeta} > 0$ such that, for every $a_1, a_2 > 0$ that satisfies the *properties that after setting* $Q_i = \frac{a}{2\delta_i}$, $R_i = -\frac{a(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$, $M_i = \delta_i - \beta_i$, *and* $N_i = \alpha_i - \gamma_i$ *for* $i = 1, 2$ *we have* $0 < Q_1 + Q_2 + R_1 + R_2 < \overline{\zeta}$,

$$0 < Q_2 + R_2 < Q_1 + R_1, \tag{2.26}$$

and

$$\frac{2M_1 N_1}{\alpha_1 (\delta_1 + \beta_1)} > 1, \tag{2.27}$$

then the system (2.18)–(2.20) *has at least three positive solutions.*

CHAPTER 3

The Even-Order Problem

In this chapter, we accomplish the primary goal of this thesis: establishing the existence of multiple positive solutions to boundary value problems of the form (1.1)–(1.3) when either $a_{i+1} > 0$ or $a_{i+1} < 0$ for $i = 1, 2$. In contrast to our work above, we present only the negative case and provide only the statements, not the proofs, of the various lemmas. Our approach here also differs in that we will state the required constraints on the system upfront. The eager reader can fill in the missing details by adjusting the proofs of Chapter 2 to account for the jump in order and, if necessary, noting the differences between the positive and negative cases.

3.1. Preliminaries

Consider the boundary value problem

$$u^{(2n)}(t) = \lambda h\left(t, u(t), u''(t), \ldots, u^{(2n-2)}(t)\right), \tag{3.1}$$

$$\alpha_{i+1}u^{(2i)}(0) - \gamma_{i+1}u^{(2i)}(0) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{3.2}$$

$$\beta_{i+1}u^{(2i+1)}(1) - \delta_{i+1}u^{(2i+1)}(1) = (-1)^{i+1}a_{i+1}, \qquad i = 0, 1, \ldots, n-1, \tag{3.3}$$

where $n \geq 2$, $h : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,\infty) \to (-1)^n[0,\infty)$ is continuous, and $\lambda > 0$. Moreover, we require $a_{i+1} < 0 < \delta_{i+1} < \beta_{i+1}$ and $2\delta_{i+1} \leq \alpha_{i+1} < \gamma_{i+1}$ for $i = 0, 1, \ldots, n-1$ such that, after setting $Q_{i+1} = \frac{a_{i+1}}{2\delta_{i+1}}$, $R_{i+1} = -\frac{a_{i+1}(2\delta_{i+1}-\gamma_{i+1})}{2\delta_{i+1}(\alpha_{i+1}-\gamma_{i+1})}$, $M_{i+1} = \delta_{i+1} - \beta_{i+1}$, and

$N_{i+1} = \alpha_{i+1} - \gamma_{i+1}$, we have $0 < R_{i+1} < \sum_{\substack{1\le j\le n-1,\\ j\ne i+1}} R_j$ for all $i = 1, 2, \ldots, n-1$ and $\frac{2M_{i+1}N_{i+1}}{\gamma_{i+1}(\delta_{i+1}+\beta_{i+1})} > 1$ for all $i = 0, 1, \ldots, n-1$.

For $t \in [0,1]$, we apply the substitutions

$$u_{i+1}(t) = (-1)^i u^{(2i)}(t), \qquad i = 0, 1, \ldots, n-1$$

$$u_{i+1}(t) = g_i(t, u_1, u_2, \ldots, u_n), \qquad i = 1, 2, \ldots, n-1$$

$$f\left(t, u_1, u_2, \ldots, u_n\right) = h\left(t, u_1, -u_2, \ldots, (-1)^{n+1} u_n\right),$$

which gives

$$-u_n''(t) = \lambda f\left(t, u_1, u_2, \ldots, u_n\right), \tag{3.4}$$

$$-u_i''(t) = g_i\left(t, u_1, u_2, \ldots, u_n\right), \qquad i = 1, 2, \ldots, n-1 \tag{3.5}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = -a_i, \qquad i = 1, 2, \ldots, n. \tag{3.6}$$

Notice that our particular choice of substitutions combined with the sign changing properties of h leave f and g_i nonnegative for all i. The concavity of u_j, $j = 1, 2, \ldots, n$, follows immediately from this observation. The above system can be transformed into the equivalent homogeneous system

$$-u_n''(t) = \lambda f(t, u_1(t) + Q_1 t^2 + R_1, \ldots, u_n(t) + Q_n t^2 + R_n), \tag{3.7}$$

$$-u_i''(t) = g_i(t, u_1(t) + Q_1 t^2 + R_1, \ldots, u_n(t) + Q_n t^2 + R_n), \qquad i = 1, 2, \ldots, n-1 \tag{3.8}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0, \qquad i = 1, 2, \ldots, n \tag{3.9}$$

where Q_j and R_j are as above for $j = 1, 2, \ldots, n$.

Solutions to (3.7)–(3.9) are of the form

$$u_n(t) = \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds$$
$$u_i(t) = \int_0^1 G_i(t,s) g_i(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds, \qquad i = 1, 2, \ldots, n-1,$$

where $G_j(t,s)$ denotes the Green's function

$$G_j(t,s) = \frac{1}{M_j N_j} \begin{cases} \delta_j N_j t + \gamma_j M_j s + \gamma_j \beta_j, & 0 \le t \le s \le 1, \\ \beta_j N_j t + \alpha_j M_j s + \gamma_j \beta_j, & 0 \le s \le t \le 1, \end{cases}$$

with M_j and N_j defined as above for $j = 1, 2, \ldots, n$. Note that G_j is positive due to the given constraints, and, therefore, solutions to (3.1)–(3.3) will be positive, provided they exist. We also note the following bounds, which will be of use in the proofs of the lemmas below:

$$\max_{t \in [0,1]} \int_0^1 G_i(t,s)\, ds = \frac{\gamma_i\,(\delta_i + \beta_i)}{2 M_i N_i}, \qquad i = 1, 2, \ldots, n, \tag{3.10}$$

and

$$\max_{t \in [0,1]} \int_0^1 \frac{\partial}{\partial t} G_i(t,s)\, ds = \left| \frac{\beta_i}{M_i} \right| = \frac{\beta_i}{\beta_i - \delta_i}, \qquad i = 1, 2, \ldots, n. \tag{3.11}$$

Now, let $(X, \|\cdot\|)$ be the Banach space $X = \prod_{i=1}^n C([0,1]; \mathbb{R})$ endowed with the norm

$$\|(u_1, \ldots, u_n)\| = \sum_{i=1}^n \|u_i\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Define $C \subset X$ to be the cone

$$C = \{(u_1, \ldots, u_n) \in X \mid u_i \text{ is nonnegative and concave;}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2, \ldots, n.\},$$

and let Ω_ρ denote the open set $\Omega_\rho = \{(u_1, \ldots, u_n) \in X : \|(u_1, \ldots, u_n)\| < \rho\}$. Finally, define $T : X \to X$ to be the operator

$$T(u_1, \ldots, u_n) = (A_1(u_1, \ldots, u_n), \ldots, A_n(u_1, \ldots, u_n))$$

with

$$A_n(u_1, \ldots, u_n)(t) = \lambda \int_0^1 G_n(t,s) f(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds$$

and

$$A_i(u_1, \ldots, u_n)(t) = \int_0^1 G_i(t,s) g_i(s, u_1(s) + Q_1 s^2 + R_1, \ldots, u_n(s) + Q_n s^2 + R_n) ds,$$

for $i = 1, 2, \ldots, n-1$ and where $G_1, \ldots, G_n$ are defined as above, $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in (-\infty, 0]^n \times [0, \infty)^n$ with $Q_j + R_j \geq 0$ for $j = 1, 2, \ldots, n$, and $f, g_1, \ldots, g_{n-1}$ are assumed to satisfy the following hypotheses:

(C0) For $i = 1, 2, \ldots, n-1$ the functions $f, g_i : [0,1] \times [0, \infty)^n \to [0, \infty)$ are continuous and nondecreasing in their last $2n$ variables.

(C1) There exists $\alpha, \beta \in (0,1)$, where $\alpha < \beta$, such that, given $(x_1, \ldots, x_n) \in [0, \infty)^n$ with $\sum_{i=1}^n x_i \neq 0$, there exists $\kappa > 0$ such that $f(t, x_1, \ldots, x_n) > \kappa$ for $t \in [\alpha, \beta]$.

(C2) Let $z = \sum_{i=1}^{n} x_i$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0, 1]$.

(C3) Let $z = \sum_{i=1}^{n} x_i$. Then

$$\lim_{z \to \infty} \frac{f(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0, 1]$.

(C4) For all $\zeta_i \in \left(1, \frac{2M_i N_i}{\gamma_i(\delta_i + \beta_i)}\right)$, there exists $q_i > 0$ such that, for all $(\overline{x}_1, \ldots, \overline{x}_n) \in [0, \infty)^n$ with $0 < \sum_{j=1}^{n} \overline{x}_j < q_i$, we have $g_i(t, \overline{x}_1, \ldots, \overline{x}_n) \leq \zeta_i \sum_{j=1}^{n} \overline{x}_j$ for each $t \in [0, 1]$ and $i = 1, 2, \ldots, n-1$.

(C5) For all $\theta_i \in \left(1, \frac{2M_i N_i}{\gamma_1(\delta_i + \beta_i)}\right)$, there exists $r_i > 0$ such that, for all $(\overline{x}_1, \ldots, \overline{x}_n) \in [0, \infty)^n$ with $\sum_{j=1}^{n} \overline{x}_j > r_i$, we have $g_i(t, \overline{x}_1, \ldots, \overline{x}_n) \leq \theta \sum_{j=1}^{n} \overline{x}_j$ for each $t \in [0, 1]$ and $i = 1, 2, \ldots, n-1$.

As a reminder, we note that the fixed points of T, if any, are solutions to a system that is similar to (3.7)–(3.9) in form but in which $Q_1, Q_2, \ldots, Q_n$ and $R_1, R_2, \ldots, R_n$ are not fixed; instead, they need only be nonpositive and nonnegative, respectively, with the additional restriction that $Q_j + R_j \geq 0$ for $j = 1, 2, \ldots, n$. We refer to this more general system as (3.7*)–(3.9*).

The following preliminary lemma establishes the cone preserving and completely continuous properties of the operator T, both of which are needed in order to apply the Guo-Krasnosel'skii Fixed Point Theorem.

Lemma 3.1.1. *Suppose* (C0) *holds. Then* $T : X \to X$ *is a completely continuous operator such that* $T(C) \subseteq C$.

3.2. Lemmas

We now state the sequence of four lemmas that will allow us to establish the fixed points of the operator T. As noted above, we omit the proofs of the lemmas as they are constructed similarly to those in the preceding chapter.

Lemma 3.2.1. *Suppose* (C0) *and* (C1) *hold, and let* $\rho^* > 0$*. Then there exists* Λ *such that, for every* $\lambda \geq \Lambda$ *and* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in (-\infty, 0]^n \times [0, \infty)^n$*, we have*

$$\|T(u_1, \ldots, u_n)\| \geq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho^*}$.

Lemma 3.2.2. *Fix* $\Lambda > 0$*, and suppose* (C0) *and* (C1) *hold. Then, for every* $\lambda \geq \Lambda$ *and* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in (-\infty, 0]^n \times [0, \infty)^n$*, there exists positive* $\rho_1 = \rho_1(\Lambda, Q_1, \ldots, Q_n, R_1, \ldots, R_n)$ *such that, for every* $\rho \in (0, \rho_1)$*, we have*

$$\|T(u_1, \ldots, u_n)\| \geq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_\rho$.

Lemma 3.2.3. *Suppose* (C0), (C2), *and* (C4) *hold, and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$*, there exists* $\rho_2 \in (0, \rho^*)$ *and* $\overline{\zeta} > 0$ *such that for every* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in$

$(-\infty, 0]^n \times [0, \infty)^n$ *with* $\sum_{i=1}^n R_i < \overline{\zeta}$, *we have*

$$\|T(u_1, \ldots, u_n)\| \leq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}$.

Lemma 3.2.4. *Let* $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in (-\infty, 0]^n \times [0, \infty)^n$, *and suppose* $\sum_{i=1}^n R_i < \overline{\zeta}$, *where* $\overline{\zeta} > 0$ *is given. Suppose further that assumptions* (C0), (C3), *and* (C5) *hold. Then, for every* $\lambda > 0$, *there exists* $\rho_3 = \rho_3(\overline{\zeta}, \lambda)$ *such that for every* $\rho \geq \rho_3$, *we have*

$$\|T(u_1, \ldots, u_n)\| \leq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_\rho$.

3.3. Main Results

We now establish the main existence results of the chapter and, in fact, the thesis as a whole. Theorem 3.3.1 combines the work of the previous section to show that the general system (3.7*)–(3.9*) has at least three positive solutions.

Theorem 3.3.1. *Suppose hypotheses* (C0)–(C5) *are satisfied for functions* $f, g_1, g_2, \ldots, g_{n-1}$. *Suppose additionally that* $0 < \delta_i < \beta_i$ *and* $2\delta_i \leq \alpha_i < \gamma_i$ *for* $i = 1, 2, \ldots, n$. *Then there exists* $\Lambda > 0$ *such that, given* $\lambda \geq \Lambda$, *there exists* $\overline{\zeta} > 0$ *such that, for every* $a_1, a_2, \ldots, a_n < 0$ *satisfying*

$$\sum_{i=1}^{n} \frac{a_i\,(2\delta_i - \gamma_i)}{2\delta_i\,(\gamma_i - \alpha_i)} < \overline{\zeta}$$

and every $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in (-\infty, 0]^n \times [0, \infty)^n$ *satisfying* $\sum_{i=1}^n R_i < \overline{\zeta}$, *the system* (3.7*)–(3.9*) *has at least three positive solutions.*

Proof. Suppose $f, g_1, g_2, \ldots, g_{n-1}$ satisfy hypotheses (C0)–(C5) and that $0 < \delta_i < \beta_i$ and $2\delta_i \leq \alpha_i < \gamma_i$ for $i = 1, 2, \ldots, n$. Let $\rho^* > 0$ be fixed. By Lemma 3.2.1, there exists $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $a_1, a_2, \ldots, a_n < 0$, we have

$$\|T(u_1, \ldots, u_n)\| \geq \|(u_1, \ldots, u_n)\|$$

for each $(u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho^*}$.

Fix $\lambda \geq \Lambda$. By Lemma 3.2.2 through Lemma 3.2.4, there exists $\overline{\zeta} > 0$ and $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that, for $a_1, a_2, \ldots, a_n < 0$ satisfying

$$\sum_{i=1}^{n} \frac{a_i\,(2\delta_i - \gamma_i)}{2\delta_i\,(\gamma_i - \alpha_i)} < \overline{\zeta}$$

and $(Q_1, \ldots, Q_n, R_1, \ldots, R_n) \in (-\infty, 0]^n \times [0, \infty)^n$ satisfying $\sum_{i=1}^{n} R_i < \overline{\zeta}$, we have

$$\begin{aligned}
\|T(u_1, \ldots, u_n)\| &\geq \|(u_1, \ldots, u_n)\|, \text{ for } (u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_1}, \\
\|T(u_1, \ldots, u_n)\| &\leq \|(u_1, \ldots, u_n)\|, \text{ for } (u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_2}, \\
\|T(u_1, \ldots, u_n)\| &\leq \|(u_1, \ldots, u_n)\|, \text{ for } (u_1, \ldots, u_n) \in C \cap \partial\Omega_{\rho_3}.
\end{aligned}$$

After applying the Guo-Krasnosel'skii Fixed Point Theorem three times, we get that there exist three positive solutions $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_n)$,$(w_1, \ldots, w_n) \in C$ such that

$$\rho_1 < \|(u_1, \ldots, u_n)\| < \rho_2 < \|(v_1, \ldots, v_n)\| < \rho^* < \|(w_1, \ldots, w_n)\| < \rho_3.$$

□

Through the following corollary, we obtain the sought after existence results for (3.1)–(3.3). The result is an immediate consequence of the one-one correspondence between the two systems (3.1)–(3.3) and (3.7)–(3.9) and Theorem 3.3.1. Like the corresponding corollaries of Chapter 2, the addition of constraints (3.12) and (3.13) below allow for the removal of any hypotheses related to (C4) and (C5).

Corollary 3.3.2. *Suppose the following hypotheses are satisfied for a function* $h : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,\infty) \to (-1)^n[0,\infty)$*:*

(C0)* *h is continuous, nondecreasing in its $(2j)$th variables, and nonincreasing in its $(2j+1)$th variables for $j = 1, 2, \ldots, n$.*

(C1)* *There exists $\alpha, \beta \in (0,1)$, where $\alpha < \beta$, such that, given $(x_1, \ldots, x_n) \in \prod_{j=0}^{n-1}(-1)^j[0,\infty)$ with $\sum_{j=1}^{n} x_j \neq 0$, there exists $\kappa > 0$ such that $h(t, x_1, x_2, \ldots, x_n) > \kappa$ for $t \in [\alpha, \beta]$.*

(C2)* *Let $z = \sum_{i=1}^{n}(-1)^{i+1}x_i > 0$. Then*

$$\lim_{z \to 0^+} \frac{h(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0,1]$.

(C3)* *Let $z = \sum_{i=1}^{n}(-1)^{i+1}x_i > 0$. Then*

$$\lim_{z \to \infty} \frac{h(t, x_1, \ldots, x_n)}{z} = 0$$

uniformly for $t \in [0,1]$.

Suppose also that $0 < \delta_i < \beta_i$ and $2\delta_i \leq \alpha_i < \gamma_i$ for $i = 1, 2, \ldots, n$. Then there exists $\Lambda > 0$ such that, given $\lambda \geq \Lambda$ there exists $\overline{\zeta} > 0$ such that, for every $a_1, a_2, \ldots, a_n < 0$

for which, after setting $Q_i = \frac{a_i}{2\delta_i}$, $R_i = -\frac{a_i(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$, $M_i = \delta_i - \beta_i$, *and* $N_i = \alpha_i - \gamma_i$ *for* $i = 1, 2, \ldots, n$, *we have* $0 < \sum_{i=1}^{n} R_i < \overline{\zeta}$,

$$0 < R_i < \sum_{\substack{1 \le j \le n-1, \\ j \ne i}} R_j, \tag{3.12}$$

for all $i = 2, 3, \ldots, n$ *and*

$$\frac{2M_i N_i}{\gamma_i \left(\delta_i + \beta_i\right)} > 1, \tag{3.13}$$

for all $i = 1, \ldots, n$, *then the system* (3.1)–(3.3) *has at least three positive solutions.*

CHAPTER 4

Future Work

There are several possible avenues one could take with regard to future work. It is apparent that the method, which plays a central not only in this thesis but also the results in [**7**], [**9**], [**15**], [**16**], [**17**], affords a great deal of latitude in the classes of problems considered, and the most obvious path would be to continue applying the transformation technique to more challenging boundary value problems. For instance, one could investigate whether the method would be amenable to differential equations with multipoint boundary conditions.

Another possible direction would be to instead focus on ways of directly modifying the differential equation (1.1). The addition of odd-order derivatives is one possibility; discretizing the domain of h is another. Both have precedent in [**15**]. If one chooses to embark on the latter route, the question of whether analogous results might not also hold on time scales seems only natural to explore.

A final option worth thinking about would involve altering the method itself. The decision to use the infinity norm appears to be grounded more in tradition than in any inherent merit. There is no reason to think another norm could not accomplish the same job, and it interesting to consider the effects that such a change could have on the various estimates and bounds. A more far-reaching modification to the method would be to exchange the Guo-Krasnosel'skii Fixed Point Theorem for another fixed point theorem.

This would have the potential to not only affect the construction of the existence results but also the way solutions are characterized.

References

[1] Agarwal, R. On fourth order boundary value problems arising in beam analysis.

[2] Amann, H., and Moser, J. K. Existence of multiple solutions for nonlinear elliptic boundary value problems. *Indiana University Mathematics Journal 21*, 10 (1972), 925–935.

[3] Amundson, N. R., and Luss, D. Qualitative and quantitative observations on the tubular reactor. *The Canadian Journal of Chemical Engineering 46*, 6 (1968), 424–433.

[4] Avery, R. A generalization of the leggett-williams fixed point theorem. 9–14.

[5] Avery, R., Anderson, D., and Henderson, J. Some fixed point theorems of leggett-williams type. *Rocky Mountain J. Math. 41*, 2 (04 2011), 371–386.

[6] Avery, R., Henderson, J., and ORegan, D. Four functionals fixed point theorem. *Mathematical and Computer Modelling 48*, 7 (2008), 1081 – 1089.

[7] Bennett, O., Brumley, D., Hopkins, B., Karber, K., and Milligan, T. The multiplicity of solutions for a system of second-order differential equations. *Involve, a Journal of Mathematics 10*, 1 (2016), 77–87.

[8] Brumley, D., Fulkerson, M., Hopkins, B., and Karber, K. Existence of positive solutions for a class of fourth order boundary value problems. *International Journal of Differential Equations and Applications 15*, 2 (2016).

[9] Brumley, D., Hopkins, B., Karber, K., and Milligan, T. The existence of solutions for classes of even-order differential equations. *Advances in Dynamical Systems and Applications 11*, 1 (2016), 15–32.

[10] Cohen, D. S. Multiple stable solutions of nonlinear boundary value problems arising in chemical reactor theory. *SIAM Journal on Applied Mathematics 20*, 1 (1971), 1–13.

[11] Del Pino, M. A., and Manásevich, R. F. Existence for a fourth-order boundary value problem under a two-parameter nonresonance condition. *Proceedings of the American Mathematical Society 112*, 1 (1991), 81–86.

[12] Dunninger, D. Multiplicity of positive solutions for a nonlinear fourth order equation. In *Annales Polonici Mathematici* (2001), vol. 77, Instytut Matematyczny Polskiej Akademii Nauk, pp. 161–168.

[13] Guo, D., and Lakshmikantham, V. Nonlinear problems in abstract cones, vol. 5 of notes and reports in mathematics in science and engineering, 1988.

[14] Henderson, J., and Wang, H. Positive solutions for nonlinear eigenvalue problems. *Journal of Mathematical Analysis and Applications 208*, 1 (1997), 252–259.

[15] Hopkins, B. *Multiplicity of positive solutions of even-order nonhomogeneous boundary value problems*. Baylor University, 2009.

[16] Hopkins, B. Multiplicity of positive solutions for an even-order right focal boundary value problem. *Adv. Dyn. Syst. Appl 10*, 2 (2015), 189–200.

[17] J. Marcos, S. Lorca, P. U. Multiplicity of solutions for a class of nonhomogeneous fourth-order boundary value problems. *Applied Mathematics Letters 21*, 3 (2008), 279 – 286.

[18] Krasnoselskii, M. Fixed points of cone-compressing or cone-extending operators. 527–530.

[19] Leggett, R., and Williams, L. Multiple positive fixed points of nonlinear operators on ordered banach spaces.

[20] Markus, L., and Amundson, N. R. Nonlinear boundary-value problems arising in chemical reactor theory. *Journal of Differential Equations 4*, 1 (1968), 102–113.

[21] Parter, S. V. Solutions of a differential equation arising in chemical reactor processes. *SIAM Journal on Applied Mathematics 26*, 4 (1974), 687–716.

[22] Parter, S. V., Stein, M. L., and Stein, P. R. On the multiplicity of solutions of a differential equation arising in chemical reactor theory. *Studies in Applied Mathematics 54*, 4 (1975), 293–314.