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OSCILLATION PHENOMENA FOR LINEAR

DIFFERENTIAL SYSTEMS IN A B^*-ALGEBRA

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OSCILLATION PHENOMENA FOR LINEAR

DIFFERENTIAL SYSTEMS IN A B^*-ALGEBRA

1. Introduction. In recent years many results have been established concerning properties of solutions of the self-adjoint linear matrix differential system

$$U' = A(t)U + B(t)V, \quad V' = C(t)U - A^*(t)V, \tag{1.1}$$

and the associated Riccati differential equation

$$W' + WA(t) + A^*(t)W + WB(t)W - C(t) = 0, \tag{1.2}$$

where $A(t)$, $B(t)$, and $C(t)$ are matrix valued functions. Many results concerning disconjugacy and oscillation phenomena, and Sturmian-type comparison theorems, are presented in books by Hartman [2] and Reid [11]. More recently, Hille [5] has considered nonoscillation properties for the self-adjoint linear differential system where the functions $A(t)$, $B(t)$, $C(t)$, $U(t)$, and $V(t)$ assume their values in a B^*-algebra.

The principal objective of this paper is to extend results for the matrix differential systems (1.1) and (1.2) to corresponding differential equations wherein the coefficient functions and solutions assume their values in a B^*-algebra. In Sections 2 and 3 we examine elementary properties of B^*-algebras and solutions of the linear differential system and the Riccati differential equation in a B^*-algebra. In Section 4 several preliminary results concerning nonoscillation of the linear differential system on a compact interval are presented, and the relationship of nonoscillation properties to the existence of solutions of the two-point boundary value problem is discussed. In Section 5 we consider disconjugacy properties of solutions of the linear system and their relation to a particular hermitian form on the B^*-algebra, which is a generalization of the Dirichlet functional frequently employed in the study of matrix differential systems. The hermitian form is also employed in Section 6 to establish necessary and sufficient conditions for the linear differential system to be nonoscillatory on a compact interval. Also in this section several Sturmian-type comparison theorems are proved. In Section 7 we consider cases wherein solutions of the nonlinear Riccati differential equation exist on an infinite interval, and establish bounds on the growth of solutions. Necessary and sufficient conditions for the linear system to be nonoscillatory for large t are presented in Section 8. Finally, in Section 9 we consider sufficient conditions for the linear differential system to be oscillatory for large t.

2. Properties of a B^*-algebra. In this section we define a B^*-algebra, and consider several elementary properties which are required in the proofs within this paper.

A Banach space is a normed linear vector space over a scalar field, which is complete in the metric determined by its norm. A Banach algebra is a Banach space with an associative multiplication defined, and such that the inequality $\|xy\| \leq \|x\| \; \|y\|$ holds for all elements x,y in the space. A Banach algebra is said to be unital if there exists an element e such that xe = ex = x for each element x in the algebra, and $\|e\| = 1$.

Furthermore, in a unital Banach algebra, an element x is said to be <u>non-singular</u>, or <u>regular</u>, if there exists an element x^{-1} in the algebra such that $xx^{-1} = x^{-1}x = e$; if an element fails to be nonsingular, it is said to be <u>singular</u>. If $\mathcal{B}$ is a unital Banach algebra over the complex scalar field Γ and x is an element of $\mathcal{B}$, the <u>spectrum</u> of x is defined to be

$$\sigma(x) = \{\lambda \in \Gamma \mid \lambda e - x \text{ is singular}\}.$$

A complex unital Banach algebra $\mathcal{B}$ is said to be a <u>B^*-algebra</u> if it has an involutory operation $(\ )^*$ possessing the following properties:

(i) <u>For each</u> $x \in \mathcal{B}$ <u>there exists a unique</u> $x^* \in \mathcal{B}$, <u>and</u> $(x^*)^* = x$.

(ii) $(x + y)^* = x^* + y^*$.

(iii) $(\alpha x)^* = \bar{\alpha}x^*$, <u>where</u> $\bar{\alpha}$ <u>is the complex conjugate of an element</u> α <u>in the complex scalar field</u>.

(iv) $(xy)^* = y^*x^*$.

(v) $\|x^*x\| = \|x\|^2$.

An element $x \in \mathcal{B}$ is said to be <u>symmetric</u>, or <u>hermitian</u>, if $x = x^*$. Furthermore, we require that the following additional properties hold:

(vi) <u>Each symmetric element has a real spectrum</u>.

(vii) <u>The collection of symmetric elements with non-negative real spectra forms a positive cone</u>, i.e., <u>the set is closed under addition, multiplication by positive scalars, and passage to the limit</u>.

(viii) <u>Each element of the form</u> x^*x <u>has a non-negative real spectrum</u>.

For convenience, the notation $x > 0$, $[x \geq 0]$, will be employed whenever x is symmetric and has a positive, [non-negative], spectrum; in this case, x is said to be a positive, [non-negative], element. It is clear that x is a nonsingular element whenever $x > 0$. Furthermore, the notation $x > y$, $[x \geq y]$, will be used whenever x and y are symmetric and $x - y > 0$, $[x - y \geq 0]$. It is easily verified that the relation $x > y$, $[x \geq y]$, determines a partial ordering on a B^*-algebra.

The definition given above for a B^*-algebra is that employed by Hille [5; p. 110]. One example of a B^*-algebra is the algebra of n by n matrices with complex entries. For this B^*-algebra the involution operation is defined so that A^* denotes the conjugate transpose of a matrix A. Another example of a B^*-algebra is the following, which is encountered frequently in functional analysis. If $\mathcal{H}$ is a complex Hilbert space, then the algebra of all bounded linear transformations on $\mathcal{H}$ is a B^*-algebra. In this case, if A is a bounded linear operator, then A^* represents the adjoint transformation. In fact, Richart [12; p. 244] has shown that every B^*-algebra is isometrically *-isomorphic to an algebra of bounded linear transformations over a complex Hilbert space.

The following theorem concerns the existence of nonsingular elements in a B^*-algebra $\mathcal{B}$ and the continuity of the inverse operation.

THEOREM 2.1. _If_ $x_o \in \mathcal{B}$ _is nonsingular, and_ $x \in \mathcal{B}$ _is such that_

$$\|x - x_o\| < 1/\|x_o^{-1}\|,$$

then x _is singular. Furthermore, in this case_

$$\|x^{-1} - x_o^{-1}\| \leq \|x_o^{-1}\|^2\|x - x_o\|/(1 - \|x_o^{-1}\|\,\|x - x_o\|),$$

so that the inverse operation is continuous.

The first statement in the theorem is proved in Hille [5; p. 107] and implies that the set of nonsingular elements in $\mathcal{B}$ is an open set. The method of proof is similar to a proof given by Taylor [14; p. 164] to show the existence of inverses of transformations on a Banach space. The second statement in the theorem may be proved by the method given by Taylor.

The following two theorems concerning the spectrum of an element are presented in Hille [5]; Theorem 2.2 is evident from the discussion on pages 108-112, and Theorem 2.3 is found on page 486.

THEOREM 2.2. _If_ x _is a symmetric element of_ $\mathcal{B}$, _then_

$$\|x\| = \sup_{\lambda \in \sigma(x)} |\lambda|.$$

THEOREM 2.3. _The unit element_ e _is positive. The inverse of a positive element_ x _is positive, and_

$$\sigma(x^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma(x)\}.$$

Furthermore, if $\sigma(x) \subset [\alpha,\beta]$, $\alpha > 0$, _then_

$$\alpha e \le x \le \beta e, \quad \text{and} \quad \beta^{-1} e \le x^{-1} \le \alpha^{-1} e.$$

Moreover, any integral power of a positive element x _is positive. In particular, if_ $\sigma(x) \subset [\alpha,\beta]$, $\alpha > 0$, _then_

$$\alpha^2 e \le x^2 \le \beta^2 e.$$

Theorems 2.2 may be used to prove the following important result.

THEOREM 2.4. _If_ x _is a symmetric element of_ $\mathcal{B}$, _then_

$$-\|x\| e \le x \le \|x\| e.$$

Furthermore, if x_1, x_2 _are symmetric elements of_ $\mathcal{B}$ _such that_ $0 \le x_1 \le x_2$, _then_ $\|x_1\| \le \|x_2\|$.

If x is symmetric, then $|\lambda| \le \|x\|$ for each $\lambda \in \sigma(x)$. Therefore we must have $\sigma(x) \subset [-\|x\|,\|x\|]$, and hence $-\|x\|e \le x \le \|x\|e$. If $0 \le x_1 \le x_2$, then from the first part of the theorem we have $x_2 \le \|x_2\|e$, and hence

$0 \leq x_1 \leq \|x_2\|e$. This implies that $\sigma(x_1) \subset [0,\|x_2\|]$ and consequently, by Theorem 2.2, we have $\|x_1\| \leq \|x_2\|$.

The next theorem demonstrates that certain elements of a B^*-algebra have square roots.

THEOREM 2.5. Let $b \in \mathcal{B}$ be such that $b > 0$, $[b \geq 0]$. Then there exists an element $m \in \mathcal{B}$ such that $m > 0$, $[m \geq 0]$, and $b = m^2$.

Hille [5; p. 486] proves the above theorem for the case where $b \geq 0$, by expressing m in terms of an integral in the complex plane. For the case where $b \geq 0$, Rickart [12; p. 183, and p. 231] employs a power series expansion used in complex variable theory to obtain a non-negative square root of b. The power series expansion is also employed by Reid [9] to obtain square roots of non-negative hermitian matrices; his method of proof is easily adapted to prove the above theorem.

The two following theorems are actually corollaries to Theorem 2.5.

THEOREM 2.6. Let $b \geq 0$ be an element of $\mathcal{B}$. If a is any element of $\mathcal{B}$, then $a^*ba \geq 0$.

Let $m \geq 0$ be the element in $\mathcal{B}$ such that $m^2 = b$. Since m is symmetric, we have $a^*ba = a^*mma = (ma)^*(ma)$. Therefore by property (viii) of a B^*-algebra, it follows that $a^*ba \geq 0$.

THEOREM 2.7. Let $b \geq 0$ be an element of $\mathcal{B}$. If a is an element of b such that $a^*ba = 0$, then $ba = 0$.

If m is the non-negative square root of b guaranteed by Theorem 2.5, then $a^*ba = (ma)^*(ma) = 0$. By property (v), we have $\|ma\| = 0$; consequently, $ma = 0$ and $ba = m(ma) = 0$.

Like many of the theorems of this section, the following result is a generalization of a theorem on transformations in Hilbert spaces.

THEOREM 2.8. <u>If</u> a <u>and</u> b <u>are elements of</u> $\mathcal{B}$ <u>such that</u> $0 < a \leq b$, <u>then</u> $0 < b^{-1} \leq a^{-1}$.

If $m > 0$ is the positive square root of b, then

$$e - m^{-1}am^{-1} = m^{-1}(b-a)m^{-1} \geq 0.$$

Therefore, $0 < m^{-1}am^{-1} \leq e$, and by Theorem 2.3 it follows that $e \leq (m^{-1}am^{-1})^{-1} = ma^{-1}m$. Furthermore, we have

$$a^{-1} - b^{-1} = a^{-1} - (m^2)^{-1} = m^{-1}(ma^{-1}m - e)m^{-1} \geq 0,$$

so that $0 < b^{-1} \leq a^{-1}$.

We will frequently require the use of integrals throughout this paper. The ordinary Riemann-type integral, such as discussed in Hille-Phillips [6; pp. 62-71] is sufficient for the methods in this paper. If $\pi = \{t_o = \alpha, t_1, \cdots, t_n = \beta]$ is a partition of the compact interval $[\alpha,\beta]$, and the values $\{\tau_i\}_{i=1}^{n}$ are chosen so that $t_{i-1} \leq \tau_i \leq t_i$ for $i = 1,2,\cdots,n$, we define

$$S(b;\pi:\alpha,\beta) = \textstyle\sum_{i=1}^{n} (t_i - t_{i-1})b(\tau_i),$$

where $b(t)$ is a $\mathcal{B}$-valued function on $[\alpha,\beta]$. Let $\|\tau\|$ denote the norm of the partition, defined as $\|\tau\| = \max\{t_i - t_{i-1} | i = 1,2,\cdots,n\}$. The function $b(t)$ is said to be <u>integrable</u> on $[\alpha,\beta]$ if $S(b;\pi:\alpha,\beta)$ tends to a limit in $\mathcal{B}$ as the norm of the partition approaches zero. The limit is denoted by $\int_\alpha^\beta b(t)dt$, and is called the integral of $b(t)$ on $[\alpha,\beta]$. It is easily shown that $b(t)$ is integrable on $[\alpha,\beta]$ whenever $b(t)$ is continuous or piecewise continuous.

If $b(t)$ is a continuous $\mathcal{B}$-valued function, and $\pi = \{t_o,t_1,\cdots,t_n\}$ is any partition of $[\alpha,\beta]$, then

$$\|S(b;\pi:\alpha,\beta)\| \leq \sum_{i=1}^{n} \|b(\tau_i)\| \ (t_i - t_{i-1}).$$

As the norm of π approaches zero, the quantity of the left member of this relation approaches $\|\int_\alpha^\beta b(t)dt\|$, whereas the right term tends to the Riemann integral of the continuous real-valued function $\|b(t)\|$. Consequently, we have the following result.

THEOREM 2.9. _If_ $b(t)$ _is a continuous_ $\mathcal{B}$_-valued function on_ $[\alpha,\beta]$, $\alpha < \beta$, _then_

$$\left\|\int_\alpha^\beta b(t)dt\right\| \leq \int_\alpha^\beta \|b(t)\|dt.$$

THEOREM 2.10. _Let_ $b(t)$ _be a continuous_ $\mathcal{B}$_-valued function on the interval_ $[\alpha,\beta]$, $\alpha < \beta$. _If_ $b(t) \geq 0$ _on_ $[\alpha,\beta]$ _then_

$$\int_\alpha^\beta b(t)dt \geq 0.$$

Furthermore, if $b(t) > 0$ _on_ $[\alpha,\beta]$, _then_

$$\int_\alpha^\beta b(t)dt > 0.$$

If $b(t) \geq 0$ on $[\alpha,\beta]$, then for each partition π we have

$$\sum_{i=1}^{n} (t_i - t_{i-1})b(\tau_i) \geq 0,$$

by property (vii) of a B^*-algebra. Therefore, passing to the limit as $\|\pi\|$ approaches 0, we find that $\int_\alpha^\beta b(t)dt \geq 0$. If $b(t) > 0$ on $[\alpha,\beta]$, then $b^{-1}(t) \leq \|b^{-1}(t)\|e$. Furthermore, since $b(t)$ is continuous on $[\alpha,\beta]$, by Theorem 2.1 the inverse operation is continuous so that $\|b^{-1}(t)\|$ is a continuous real-valued function on $[\alpha,\beta]$. If λ is a positive real number such that $\|b^{-1}(t)\| \leq \lambda$ on $[\alpha,\beta]$, then $b^{-1}(t) \leq \lambda e$ and $0 < \lambda^{-1}e \leq b(t)$. Consequently, we find that

$$\int_{\alpha}^{\beta} b(t)dt \geq (\beta-\alpha)\lambda^{-1}e > 0,$$

and the theorem is proved.

We will frequently use the concept of a derivative. If a(t) is a $\mathcal{B}$-valued function on an open set $\mathcal{D}$ and $t_o \in \mathcal{D}$, the function a(t) is said to be <u>differentiable</u> at t_o provided

$$\lim_{h \to o} \frac{a(t_o+h) - a(t_o)}{h}$$

exists; as usual, the limit is denoted by $a'(t_o)$. In particular, if b(t) is a continuous function on $[\alpha,\beta]$ and a(t) is defined as equal to $\int_{\alpha}^{t} b(s)ds$ on $[\alpha,\beta]$, then for each $t \in (\alpha,\beta)$ the derivative a'(t) exists and is given by b(t); the above statement is, of course, a form of the fundamental theorem of integral calculus.

THEOREM 2.11. <u>Let</u> $b(t) \geq 0$ <u>be a continuous</u>, $\mathcal{B}$<u>-valued function on the interval</u> $[\alpha,\beta]$. <u>If</u> $\int_{\alpha}^{\beta} b(s)ds = 0$, <u>then</u> $b(t) = 0$ <u>on</u> $[\alpha,\beta]$.

If a(t) is defined to be the integral $\int_{\alpha}^{t} b(s)ds$, then clearly $a(t) = 0$ on $[\alpha,\beta]$, and consequently $a'(t) = 0$ on (α,β). Therefore, $b(t) = a'(t) = 0$ on (α,β), and by the continuity of b(t) we have $b(t) = 0$ on $[\alpha,\beta]$.

If $\mathcal{B}$ is a Banach algebra, it will have a nonempty subset of elements which are called completely continuous or compact. An element $c \in \mathcal{B}$ is said to be <u>compact</u> if for each bounded sequence $\{x_n\}$ in $\mathcal{B}$ the sequence $\{cx_n\}$ contains a convergent subsequence. If T_c is defined to be the bounded linear operator on the Banach algebra $\mathcal{B}$ such that $T_c(x) = cx$ for each $x \in \mathcal{B}$, then clearly c is a compact element of $\mathcal{B}$ if and only if the bounded linear operator T_c is a compact operator.

Therefore results pertaining to compact operators on a Banach space may be translated to obtain results for compact elements of a Banach algebra. Results found in Taylor [14; pp. 274-281] and Narici-Bachman [8; pp. 286-295] enable us to establish the following theorem.

THEOREM 2.12. *Let $\mathcal{B}$ be a Banach algebra. Then the following properties are valid:*

(a) *If c_1 and c_2 are compact elements of $\mathcal{B}$ and α,β are scalars, then $\alpha c_1 + \beta c_2$ is compact.*

(b) *If c is compact and $a \in \mathcal{B}$, then ac and ca are compact.*

(c) *If $\{c_n\}$ is a sequence of compact elements converging to an element $c \in \mathcal{B}$, then c is compact.*

(d) *If b is nonsingular, and c is compact, then the element $b + c$ is such that either $b + c$ is nonsingular, or there exists an $x \neq 0$ such that $(b+c)x = 0$.*

As a consequence of Theorem 2.12, we are able to prove the following result which will be used in Section 5.

COROLLARY. *Let a(t) and c(t) be continuous $\mathcal{B}$-valued functions on the interval $[\alpha,\beta]$. If c(t) is compact for each $t \in [\alpha,\beta]$, then the integrals*

$$\int_\alpha^\beta a(t)c(t)dt, \quad \text{and} \quad \int_\alpha^\beta c(t)a(t)dt$$

are compact elements of $\mathcal{B}$.

3. *Basic properties of solutions of the linear differential system and the Riccati equation.* Let I be a real interval of the form (α_o,∞), where $\alpha_o \geq -\infty$, and let a(t), b(t), and c(t) be continuous $\mathcal{B}$-valued

functions on I such that $b(t)$ and $c(t)$ are symmetric for each $t \in I$. This section will be concerned with properties of solutions of the self-adjoint differential system

$$(3.1)\qquad \begin{aligned} L_1[u,v](t) &\equiv -v' + c(t)u - a^*(t)v = 0, \\ L_2[u,v](t) &\equiv u' - a(t)u - b(t)v = 0, \end{aligned}$$

for $t \in I$. We shall also consider properties of solutions of the associated Riccati differential equation

$$(3.2)\qquad K[w](t) \equiv w' + wa(t) + a^*(t)w + wb(t)w - c(t) = 0,$$

on subintervals of I.

A pair of $\mathfrak{B}$-valued functions $(u(t),v(t))$ is said to be a solution of system (3.1) if both $u(t)$ and $v(t)$ are continuously differentiable on the interval I, and satisfy the differential equations (3.1) for each value $t \in I$. Similarly, a $\mathfrak{B}$-valued function $w(t)$ is said to be a solution of the Riccati equation (3.2) on a subinterval I_o of I if $w(t)$ is continuously differentiable on I_o and equation (3.2) is satisfied for each $t \in I_o$.

As a special case of system (3.1), we also consider the linear second-order differential equation

$$(3.3)\qquad [r(t)u' + q(t)u]' - [q^*(t)u' + p(t)u] = 0, \qquad t \in I.$$

It is assumed that $r(t)$, $q(t)$, and $p(t)$ are continuous $\mathfrak{B}$-valued functions on I, $r(t)$ and $p(t)$ are symmetric, and $r(t) > 0$ on I. A continuously differentiable function $u(t)$ is said to be a solution of (3.3) if $r(t)u'(t) + q(t)u(t)$ is also continuously differentiable on I, and equation (3.3) is satisfied for each $t \in I$. If $u(t)$ is a solution of (3.3), and $v(t)$ is defined as $r(t)u'(t) + q(t)u(t)$ on I, then it may be easily verified that (u,v) is a solution of system (3.1) under the identification $a(t) = -r^{-1}(t)q(t)$, $b(t) = r^{-1}(t)$, and $c(t) = p(t) - q^*(t)r^{-1}(t)q(t)$. Since $r(t)$ and $p(t)$ are symmetric, the corresponding functions $b(t)$ and $p(t)$ are also symmetric as required in system (3.1); in addition, we have $b(t) = r^{-1}(t) > 0$ on I. Finally, a particular condition is said to hold for equation (3.3) if and only if the condition holds for the system written in the form of system (3.1); therefore, all the results of this paper given for system (3.1) also apply to any equation of the form (3.3).

The following theorem shows that solutions to the initial value problem exist, and are unique for system (3.1).

THEOREM 3.1. *If* u_o, $v_o \in \mathcal{B}$ *and* $\tau \in I$, *then there exists a unique solution* $(u(t),v(t))$ *of system* (3.1) *on* I *such that* $u(\tau) = u_o$ *and* $v(t) = v_o$.

It may be verified readily that $(u(t),v(t))$ is a solution of (3.1) determined by the initial values $u(\tau) = u_o$, $v(\tau) = v_o$, if and only if the integral equations

$$(3.4)\qquad \begin{aligned} u(t) &= u_o + \int_\tau^t [a(s)u(s) + b(s)v(s)]ds, \\ v(t) &= v_o + \int_\tau^t [c(s)u(s) - a^*(s)v(s)]ds, \end{aligned}$$

are satisfied for each $t \in I$. Moreover, to show that solutions of (3.4) exist and are unique on I, it suffices to prove the existence and uniqueness of solutions on each compact subinterval $[\alpha,\beta]$ of I which contains τ. The classical method of successive approximations may be employed to show the existence of a solution $(u(t),v(t))$ of (3.4) on an interval $[\alpha,\beta]$. The uniqueness of solutions of (3.4) on an interval $[\alpha,\beta]$ may be demonstrated easily by the use of the well-known Gronwall inequality; see, for example, Reid [11; p. 13]. It is also to be noted that Theorem 3.1 is an immediate consequence of a result given by Bourbaki [1; p. 27] for linear differential equations on the Banach space $\mathcal{B} \times \mathcal{B}$ where $\|(x,y)\|$ is defined as $\|x\| + \|y\|$ for elements $x,y \in \mathcal{B}$.

In the following theorem, we show that solutions of the initial value problem are continuous with respect to the function $c(t)$.

THEOREM 3.2. *Let* $u_o, v_o \in \mathcal{B}$ *and* $\tau \in [\alpha,\beta]$. *Let* $(u(t),v(t))$ *be the unique solution of system* (3.1) *satisfying* $u(\tau) = u_o$, $v(\tau) = v_o$. *If* ε *is any positive real number, then there exists a positive number* δ *such that the solution* $(\hat{u}(t),\hat{v}(t))$ *of the system*

$$\begin{aligned} \hat{u}' &= a(t)\hat{u} + b(t)\hat{v}, \\ \hat{v}' &= \hat{c}(t)\hat{u} - a^*(t)\hat{v}, \qquad t \in [\alpha,\beta], \\ \hat{u}(\tau) &= u_o, \quad \hat{v}(\tau) = v_o, \end{aligned} \tag{3.5}$$

satisfies $\|\hat{u}(t) - u(t)\| < \varepsilon$ *on* $[\alpha,\beta]$ *whenever* $\|\hat{c}(t) - c(t)\| < \delta$ *on* $[\alpha,\beta]$.

With the aid of Gronwall's inequality, it can be shown that for $t \in [\alpha,\beta]$ we have

$$\|\hat{u}(t) - u(t)\| + \|\hat{v}(t) - v(t)\| \le \max_{s \in [\alpha,\beta]}\{\|\hat{c}(s) - c(s)\| \; \|u(s)\|\}\exp\{(\beta-\alpha)k\}$$

where k is a real number such that

$$\|a(s)\| + \|c(s)\| + \|\hat{c}(s) - c(s)\| \le k, \text{ and}$$
$$\|a(s)\| + \|b(s)\| \le k$$

for $s \in [\alpha,\beta]$. The proof of Theorem 3.2 is now immediate from the above inequality. Moreover, it is to be noted that solutions of system (3.1) on an interval $[\alpha,\beta]$ are continuous with respect to the initial data τ, u_o, and v_o, and the functions $a(t)$, $b(t)$, and $c(t)$; again, the result is established by the use of Gronwall's inequality.

If $(u_1(t),v_1(t))$ and $(u_2(t),v_2(t))$ are solutions of system (3.1), it is easily verified that $(u_1^*(t)v_2(t) - v_1^*(t)u_2(t))' = 0$ on I. Therefore, we have the following result.

THEOREM 3.3. *If* $(u_1(t),v_1(t))$ *and* $(u_2(t),v_2(t))$ *are solutions of system* (3.1) *on* I, *then* $u_1^*(t)v_2(t) - v_1^*(t)u_2(t) \equiv$ *constant on* I.

Two solutions (u_1,v_1) and (u_2,v_2) of system (3.1) are said to be *mutually conjoined*, or simply *conjoined*, if $u_1^*(t)v_2(t) - v_1^*(t)u_2(t) \equiv 0$ on I. Furthermore, a solution (u,v) of system (3.1) is said to be *self-conjoined* whenever $u^*(t)v(t) \equiv v^*(t)u(t)$ on I.

The following theorem establishes the relationship of solutions of the linear differential system (3.1) with solutions of the Riccati differential system (3.2). A similar theorem for matrix differential equations may be found in Reid [11; p. 101].

THEOREM 3.4. *Suppose* $(u(t),v(t))$ *is a solution of system* (3.1) *on* I *such that* $u(t)$ *is nonsingular on a subinterval* I_o *of* I. *Then* $w(t) = v(t)u^{-1}(t)$ *is a solution of the Riccati equation* (3.2) *on* I_o. *Conversely, if* $w(t)$ *is a solution of* (3.2) *on a subinterval* I_o *of* I, *then there exists a solution* $(u(t),v(t))$ *of system* (3.1) *on* I *such that* $u(t)$ *is nonsingular on* I_o *and* $w(t) = v(t)u^{-1}(t)$. *In each case,*

$$u^*(t)[w(t) - w^*(t)]u(t) = u^*(t)v(t) - v^*(t)u(t), \text{ for } t \in I_o,$$

so that $w(t)$ *is a symmetric solution if and only if* (u,v) *is self-conjoined.*

If (u,v) is a solution of (3.1) on I, and u(t) is nonsingular on I_o, then $u^{-1}(t)$ is continuously differentiable on I_o and $[u^{-1}(t)]' = -u^{-1}(t)u'(t)u^{-1}(t)$. It can be verified directly that $w(t) = v(t)u^{-1}(t)$ satisfies equation (3.2) on I_o.

Suppose $w(t)$ is a solution of (3.2) on I_o. Let $\tau \in I_o$, and let $u(t)$ be the unique solution of the initial value problem $u'(t) = h(t)u(t)$, $t \in I_o$, and $u(\tau) = e$, where $h(t) = a(t) + b(t)w(t)$. The function $u(t)$ is nonsingular on I_o, with its inverse determined by the solution of the system $[u^{-1}(t)]' = -u^{-1}(t)h(t)$, $u^{-1}(\tau) = e$. One now defines $v(t) = w(t)u(t)$ on I_o; substitution of (u,v) into the differential system (3.1) shows that $(u(t),v(t))$ is indeed a solution on I_o with $u(t)$ nonsingular. The domain of definition of the functions $u(t)$ and $v(t)$ can be extended to I in a manner such that $(u(t),v(t))$ is a solution of system (3.1) on I. The last statement of the theorem follows from the equation $w(t) = v(t)u^{-1}(t)$.

The previous theorem can be used to prove the local existence and uniqueness of solutions to the Riccati differential equation.

THEOREM 3.5. _Let_ $w_o \in \mathcal{B}$ _and_ $\tau \in I$. _Then there exists a positive real number_ δ _and a_ $\mathcal{B}$_-valued function_ $w(t)$ _defined on_ $(\tau-\delta,\tau+\delta)$ _such that_ $w(t)$ _is the unique solution of the Riccati equation_ (3.2) _on the interval_ $(\tau-\delta,\tau+\delta)$ _satisfying_ $w(\tau) = w_o$.

Let $(u(t),v(t))$ be the solution of system (3.1) on I such that $u(\tau) = e$, $v(\tau) = w_o$. Since $u(\tau)$ is nonsingular, there exists a $\delta > 0$ such that $u(t)$ is nonsingular on $(\tau-\delta,\tau+\delta)$. By Theorem 3.4, we have that $w(t) = v(t)u^{-1}(t)$ is a solution of that Riccati equation on $(\tau-\delta,\tau+\delta)$ which satisfies $w(\tau) = w_o$. To show uniqueness, suppose that $\hat{w}(t)$ is a solution of the Riccati equation on a subinterval I_o of $(\tau-\delta,\tau+\delta)$ such that $\hat{w}(\tau) = w_o$. By Theorem 3.4, there exists a solution $(\hat{u},\hat{v})$ of system (3.1) such that $\hat{w}(t) = \hat{v}(t)\hat{u}^{-1}(t)$. Furthermore, we have $\hat{v}(\tau)\hat{u}(\tau) = w_o$, so that $\hat{v}(\tau) = w_o\hat{u}(\tau) = v(\tau)\hat{u}(\tau)$. However, by the uniqueness of solutions of system (3.1) it follows that $\hat{u}(t) = u(t)\hat{u}(\tau)$ and $\hat{v}(t) = v(t)\hat{u}(\tau)$ on the interval I_o. Therefore, we find that $\hat{w}(t) = \hat{v}(t)\hat{u}^{-1}(t) = v(t)u^{-1}(t) = w(t)$ on I_o, and hence solutions of the Riccati equation are locally unique.

The following lemma is used to establish the nonsingularity of $u(t)$ at certain points in I and will be employed in Theorem 3.6.

LEMMA 3.1. *Let $u(t)$ be a continuous $\mathcal{B}$-valued function on the finite interval $[\alpha,\beta]$ such that $u(t)$ is nonsingular on $[\alpha,\beta)$. Then $u(t)$ is nonsingular at $t = \beta$ if and only if $\|u^{-1}(t)\|$ is bounded on $[\alpha,\beta)$.*

If $u(\beta)$ is nonsingular, then $u^{-1}(t)$ and $\|u^{-1}(t)\|$ are continuous functions on the compact interval $[\alpha,\beta]$, and therefore $\|u^{-1}(t)\|$ is bounded on $[\alpha,\beta)$. Conversely, suppose that $\|u^{-1}(t)\|$ is bounded by a positive real number κ on $[\alpha,\beta)$. By the continuity of $u(t)$, there exists a $\tau \in [\alpha,\beta)$ such that $\|u(\tau) - u(\beta)\| < \kappa^{-1}$, and consequently

$$\|u(\tau) - u(\beta)\| < 1/\|u^{-1}(\tau)\|.$$

By Theorem 2.1 we conclude that $u(\beta)$ is nonsingular, and the lemma is proved.

The following result is established readily, and will be employed in Section 6.

LEMMA 3.2. *Let $u(t)$ be a continuous function on $[\alpha,\beta]$ such that $u(t)$ is nonsingular on $[\alpha,\beta)$. If $u(\beta)$ is singular, then for each $\varepsilon > 0$ there exists an $x \in \mathcal{B}$ with $\|x\| = 1$ and such that $\|u(\beta)x\| < \varepsilon$.*

Since $\|u^{-1}(t)\|$ is unbounded on $[\alpha,\beta)$, and $u(t)$ is continuous on $[\alpha,\beta]$, there exists a $\tau \in [\alpha,\beta)$ such that $\|u^{-1}(\tau)\| > 2/\varepsilon$ and $\|u(\tau) - u(\beta)\| < \varepsilon/2$. If x is defined as $u^{-1}(\tau)/\|u^{-1}(\tau)\|$, then clearly $\|x\| = 1$ and

$$\|u(\tau)x\| = \|u(\tau)u^{-1}(\tau)\|/\|u^{-1}(\tau)\| = 1/\|u^{-1}(\tau)\| < \varepsilon/2.$$

Furthermore, we have

$$\|u(\tau)x - u(\beta)x\| \leq \|u(\tau) - u(\beta)\| \; \|x\| < \varepsilon/2,$$

and hence it follows from the triangle inequality that $\|u(\beta)x\| < \varepsilon$.

The following result concerns the extendability of solutions of the Riccati equation.

THEOREM 3.6. _Let_ $w(t)$ _be a solution of the Riccati equation_ (3.2) _on a finite interval_ $[\alpha,\beta)$. _If_ $\|w(t)\|$ _is bounded on_ $[\alpha,\beta)$, _then there exists a positive real number_ δ _such that_ $w(t)$ _can be extended to the interval_ $[\alpha,\beta+\delta)$.

Let κ be a positive real number such that $\|w(t)a(t) + a^*(t)w(t) + w(t)b(t)w(t) - c(t)\| \leq \kappa$ on the interval $[\alpha,\beta)$. Since $w(t)$ is a solution of the Riccati equation (3.2), it follows that

$$w(t) - w(s) = \int_t^s (wa + a^*w + wbw - c)d\tau,$$

and consequently $\|w(t) - w(s)\| \leq |t-s|\kappa$ for s and t values on $[\alpha,\beta)$. Therefore the limit of $w(t)$ as t approaches β exists, and will be denoted by w_o. Theorem 3.5 guarantees that there is a $\delta > 0$, and a solution $w(t)$ of the Riccati equation on $(\beta-\delta,\beta+\delta)$, satisfying $\hat{w}(\beta) = w_o$. Defining $w(t)$ to be $\hat{w}(t)$ on the interval $[\beta,\beta+\delta)$, we find that $w(t)$ is a solution on the interval $[\alpha,\beta+\delta)$.

The final theorem of this section is similar to a result of Hayden and Howard [3], where the elements are endomorphisms on a Banach space.

THEOREM 3.7. _Let_ $(u(t),v(t))$ _be a solution of system_ (3.1) _on_ I _such that_ $u(t)$ _is nonsingular on a finite subinterval_ $[\alpha,\beta)$ _of_ I. _Then_ $\|u^{-1}(t)\|$ _is bounded on_ $[\alpha,\beta)$ _if and only if_ $\|v(t)u^{-1}(t)\|$ _is bounded on_ $[\alpha,\beta)$.

If $\|u^{-1}(t)\|$ is bounded on $[\alpha,\beta)$, then clearly $\|v(t)u^{-1}(t)\|$ is bounded as a result of the inequality $\|v(t)u^{-1}(t)\| \le \|v(t)\|\ \|u^{-1}(t)\|$. Conversely, if $\|v(t)u^{-1}(t)\|$ is bounded on $[\alpha,\beta)$, the function $w(t)$ defined by $w(t) = v(t)u^{-1}(t)$ is a solution of the Riccati equation (3.2) on $[\alpha,\beta)$, and $\|w(t)\|$ is bounded on this interval. By Theorem 3.6, $w(t)$ can be extended as a solution of the Riccati equation to an interval $[\alpha,\beta+\delta)$ where $\delta > 0$, and Theorem 3.4 guarantees the existence of a solution $(\hat{u},\hat{v})$ of system (3.1) such that $\hat{u}(t)$ is nonsingular on $[\alpha,\beta+\delta)$ and $w(t) = \hat{v}(t)\hat{u}^{-1}(t)$ on the interval. As in the proof of Theorem 3.5, it can be shown that $u(t) = \hat{u}(t)\hat{u}^{-1}(\alpha)u(\alpha)$ and $v(t) = \hat{v}(t)\hat{u}^{-1}(\alpha)u(\alpha)$ on $[\alpha,\beta)$. Furthermore, we have

$$\|u^{-1}(t)\| \le \|u^{-1}(\alpha)\hat{u}(\alpha)\|\ \|\hat{u}^{-1}(t)\|, \text{ for } t \in [\alpha,\beta). \tag{3.6}$$

However, the function $\hat{u}(t)$ is nonsingular on $[\alpha,\beta]$, so that $\|\hat{u}^{-1}(t)\|$ is bounded on $[\alpha,\beta)$. From equation (3.6) it then follows that $\|u^{-1}(t)\|$ is bounded on $[\alpha,\beta)$ and the theorem is proved.

4. Preliminary nonoscillation theorems for the linear differential system, and the two-point boundary value problem. In this section we examine necessary and sufficient conditions for the linear differential system (3.1) to be nonoscillatory on a compact interval $[\alpha,\beta]$. These preliminary results will be employed in the proofs of the nonoscillation theorems presented in Section 6. Theorems 4.6 - 4.8 relate nonoscillation properties of the linear differential system to the existence and uniqueness of solutions of the two-point boundary value problem. As in Section 3, we require that the coefficient functions $a(t)$, $b(t)$, and $c(t)$ of system (3.1) are continuous $\mathscr{B}$-valued functions, and that $b(t)$ and

and $c(t)$ are symmetric for each $t \in I$.

The linear differential system (3.1) is said to be <u>nonoscillatory</u> on a compact interval $[\alpha,\beta]$ if there exists a self-conjoined solution (u,v) of system (3.1) with $u(t)$ nonsingular on $[\alpha,\beta]$. In this terminology, Theorem 3.4 is a nonoscillation theorem. It states that system (3.1) is nonoscillatory on $[\alpha,\beta]$ if and only if there exists a symmetric solution $w(t)$ of the Riccati equation.

The following lemma is essential in proving the nonoscillation theorems in this section. The result may be found in a more general form in Reid [11; p. 308]. Although Reid establishes the result for linear matrix differential systems, the method of proof is the same for the B^*-algebra case.

LEMMA 4.1. *Let $(u(t),v(t))$ be a self-conjoined solution of system (3.1) on I such that $u(t)$ is nonsingular on a subinterval I_o of I. If $\tau \in I_o$, then the unique solution $(u_\tau(t),v_\tau(t))$ of system (3.1) satisfying $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$, is given by*

$$(4.1)\qquad \begin{aligned} u_\tau(t) &= u(t)\phi(t,\tau;u)u^*(\tau), \\ v_\tau(t) &= v(t)\phi(t,\tau;u)u^*(\tau) + u^{*-1}(t)u^*(\tau), \end{aligned} \qquad t \in I_o,$$

where

$$\phi(t,\tau;u) = \int_\tau^t u^{-1}(s)b(s)u^{*-1}(s)ds.$$

THEOREM 4.1. *Suppose that system (3.1) is nonoscillatory on a finite interval $[\alpha,\beta]$, and let (u_α,v_α) be the solution of system (3.1) satisfying $u_\alpha(\alpha) = 0$, $v_\alpha(\alpha) = e$. Then for $t \in (\alpha,\beta]$ the function $u_\alpha(t)$ is nonsingular if and only if $\phi(t,\alpha;u)$ is nonsingular. In particular, if $b(t) > 0$ for each $t \in [\alpha,\beta]$, then $u_\alpha(t)$ is nonsingular on $(\alpha,\beta]$.*

By hypothesis, there exists a self-conjoined solution (u,v) of system (3.1) with $u(t)$ nonsingular on the interval $I_o = [\alpha,\beta]$. We now use Lemma 4.1 with $\tau = \alpha$. Since $u(t)$ is nonsingular on $[\alpha,\beta]$ and $u^*(\tau)$ is nonsingular, it follows from equations (4.1) that $u_\tau(t)$ is nonsingular if and only if $\phi(t,\alpha;u)$ is nonsingular. In the case that $b(t) > 0$ on $[\alpha,\beta]$, we have that $u^{-1}(s)b(s)u^{*-1}(s) > 0$ on $[\alpha,\beta]$, and therefore $\phi(t,\alpha;u) > 0$ for $t \in (\alpha,\beta]$. Hence, $\phi(t,\alpha;u)$ and $u_\alpha(t)$ are nonsingular for $t \in (\alpha,\beta]$.

The following result is analogous to Theorem 4.1, and will be used in the proofs of Theorems 4.3 and 4.4.

THEOREM 4.2. _Suppose that system_ (3.1) _is nonoscillatory on a finite interval_ $[\alpha,\beta]$, _and let_ (u_β,v_β) _be the solution of system_ (3.1) _satisfying_ $u_\beta(\beta) = 0$, $v_\beta(\beta) = e$. _Then for_ $t \in [\alpha,\beta)$, _the function_ $u_\beta(t)$ _is nonsingular if and only if_ $\phi(\beta,t;u)$ _is nonsingular. In particular, if_ $b(t) > 0$ _for each_ $t \in [\alpha,\beta]$, _then_ $u_\beta(t)$ _is nonsingular on_ $[\alpha,\beta)$.

The following theorem presents an interesting property of solutions of the linear differential system whenever $b(t) > 0$ on I.

THEOREM 4.3. _Suppose that we have_ $b(t) > 0$ _on_ I _and_ $\tau \in I$, _and let_ (u_τ,v_τ) _be the solution of system_ (3.1) _such that_ $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$. _Then there exists a_ $\delta > 0$ _such that_ $u_\tau(t)$ _is nonsingular on_ $(\tau-\delta,\tau) \cup (\tau,\tau+\delta)$.

Let (u,v) be the solution of system (3.1) such that $u(\tau) = e$, $v(\tau) = 0$; clearly this solution is self-conjoined since $u^*(\tau)v(\tau) - v^*(\tau)u(\tau) = 0$. Furthermore, the function $u(t)$ is nonsingular at $t = \tau$ so that there exists a $\delta > 0$ such that $u(t)$ is nonsingular on $[\tau-\delta,\tau+\delta]$. Therefore, system (3.1) is nonoscillatory on each of the intervals $[\tau-\delta,\tau]$ and $[\tau,\tau+\delta]$. It follows from Theorems 4.1 and 4.2 that $u_\tau(t)$ is nonsingular on the intervals $[\tau-\delta,\tau)$ and $(\tau,\tau+\delta]$, which is the desired conclusion.

The following result is employed in Section 6, wherein nonoscillation properties of the linear differential system are studied in further detail.

THEOREM 4.4. *Suppose that* $b(t) \geq 0$ *on* $[\alpha,\beta]$, *and let* (u_α,v_α) *be the solution of system* (3.1) *satisfying* $u_\alpha(\alpha) = 0$, $v_\alpha(\alpha) = e$. *If* $u_\alpha(t)$ *is nonsingular on* $(\alpha,\beta]$, *then system* (3.1) *is nonoscillatory on the interval* $[\alpha,\beta]$.

Let ε be any positive real number, and define new functions $\hat{a}(t)$, $\hat{b}(t)$, and $\hat{c}(t)$ on the interval $[\alpha,\beta+\varepsilon]$ as follows:

$$\hat{a}(t) = \begin{cases} a(t) & \text{on } [\alpha,\beta], \\ a(\beta) & \text{on } [\beta,\beta+\varepsilon], \end{cases}$$

$$\hat{b}(t) = \begin{cases} b(t) & \text{on } [\alpha,\beta], \\ b(\beta) + (t-\beta)e & \text{on } [\beta,\beta+\varepsilon], \end{cases} \tag{4.2}$$

$$\hat{c}(t) = \begin{cases} c(t) & \text{on } [\alpha,\beta], \\ c(\beta) & \text{on } [\beta,\beta+\varepsilon]. \end{cases}$$

Let $(u(t),v(t))$ be the extension of (u_α,v_α) to the interval $[\alpha,\beta+\varepsilon]$, which is a solution of the differential system

$$\begin{aligned} u' &= \hat{a}(t)u + \hat{b}(t)v, \\ v' &= \hat{c}(t)u - \hat{a}^*(t)v, \end{aligned} \qquad t \in [\alpha,\beta+\varepsilon]. \tag{4.3}$$

Since $u(\beta) = u_\alpha(\beta)$ is nonsingular, there exists a positive number $\delta < \varepsilon$ such that $u(t)$ is nonsingular on the interval $(\alpha,\tau]$, where $\tau = \beta + \delta$. If (u_τ,v_τ) is the solution of system (4.3) satisfying $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$, then for $t \in (\alpha,\tau]$ we have

$$u_\tau(t) = -u(t)\left[\int_t^\tau u^{-1}(s)\hat{b}(s)u^{*-1}(s)ds\right]u^*(\tau). \tag{4.4}$$

Clearly, we have that $u^{-1}(s)\hat{b}(s)u^{*-1}(s) \geq 0$ on $(\alpha,\tau]$, and moreover, since $\hat{b}(s) > 0$ for $s \in (\beta,\tau]$, we have $u^{-1}(s)\hat{b}(s)u^{*-1}(s) > 0$ on $(\beta,\tau]$. Therefore,

$$\int_t^\tau u^{-1}(s)\hat{b}(s)u^{*-1}(s)ds > 0 \text{ for } t \in [\beta,\tau).$$

If $t \in (\alpha,\beta]$, we also have

$$\int_t^\tau u^{-1}(s)\hat{b}(s)u^{*-1}(s)ds \geq \int_\beta^\tau u^{-1}(s)\hat{b}(s)u^{*-1}(s)ds > 0.$$

Therefore, since $u^*(\tau)$ is nonsingular, and $u(t)$ and $\int_t^\tau u^{-1}(s)\hat{b}(s)u^{*-1}(s)ds$ are nonsingular for $t \in (\alpha,\tau)$, it follows from equation (4.4) that $u_\tau(t)$ is nonsingular on (α,τ). Furthermore, by Theorem 3.3, the function $u_\tau^*(t)v(t) - v_\tau^*(t)u(t)$ is constant on $[\alpha,\tau]$. Evaluating this function at $t = \alpha$ and $t = \tau$, we find that $u_\tau^*(\alpha) = -u(\tau)$; since $u(\tau)$ is nonsingular, $u_\tau(\alpha)$ is also nonsingular. Therefore (u_τ,v_τ) is a self-conjoined solution of system (4.3) on $[\alpha,\tau]$ with $u_\tau(t)$ nonsingular on $[\alpha,\tau)$. If (u_τ,v_τ) is restricted to the interval $[\alpha,\beta]$, then clearly it is a self-conjoined solution of system (3.1) with $u_\tau(t)$ nonsingular on $[\alpha,\beta]$, and therefore system (3.1) is nonoscillatory on $[\alpha,\beta]$.

It should be noted that the converse of Theorem 4.4 is in general not true. For example, if $a(t) = b(t) = c(t) = 0$ on I, then $(u(t) \equiv e, v(t) \equiv 0)$ is a self-conjoined solution with $u(t)$ nonsingular on I; however, the solution (u_α,v_α) satisfying $u_\alpha(\alpha) = 0$, $v_\alpha(\alpha) = e$ is given by $u_\alpha(t) \equiv 0$ on I. A partial converse of Theorem 4.4 is given in Theorem 4.1 under the restriction that $b(t) > 0$ on the interval $[\alpha,\beta]$. To obtain another partial converse of Theorem 4.4, we will consider linear differential systems which satisfy the following hypothesis on an interval $[\alpha,\beta]$.

(H) For each point $\tau \in [\alpha,\beta]$, the solution (u_τ,v_τ) of system (3.1) satisfying $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$, is such that there exists a $\delta > 0$ so that $u_\tau(t)$ is nonsingular on $\{[\tau-\delta,\tau) \cup (\tau,\tau+\delta]\} \cap [\alpha,\beta]$.

If $b(t) > 0$ on $[\alpha,\beta]$, Theorem 4.3 guarantees that condition (H) holds for the linear differential system. The following result shows that the converse of Theorem 4.4 holds whenever hypothesis (H) holds.

THEOREM 4.5. Let $b(t) \geq 0$ on $[\alpha,\beta]$ and suppose that property (H) holds on the interval $[\alpha,\beta]$. Furthermore, suppose that system (3.1) is nonoscillatory on $[\alpha,\beta]$. If $\tau \in [\alpha,\beta]$ and (u_τ,v_τ) is the solution of system (3.1) satisfying $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$, then $u_\tau(t)$ is nonsingular on $[\alpha,\tau) \cup (\tau,\beta]$.

We prove that for $\tau < t \leq \beta$, the function $u_\tau(t)$ is nonsingular; a similar argument holds for $t < \tau$. Let (u,v) be a self-conjoined solution of system (3.1) with $u(t)$ nonsingular on $[\alpha,\beta]$. By Theorem 4.1 it follows that $u_\tau(t)$ is nonsingular if and only if $\phi(t,\tau;u)$ is nonsingular. Since $b(t) \geq 0$ on $[\alpha,\beta]$, we have that $u^{-1}(s)b(s)u^{*-1}(s) \geq 0$ and therefore $\phi(t,\tau;u) \geq 0$ for $t \geq \tau$. By property (H) we have that $u_\tau(t)$ is nonsingular on an interval of the form $(\tau,\tau+\delta]$, and consequently $\phi(t,\tau;u) > 0$ on this interval. Furthermore, for $t \in (\tau+\delta,\beta]$ it follows that

$$\phi(t,\tau;u) \geq \phi(\tau+\delta,\tau;u) > 0.$$

Hence $\phi(t,\tau;u)$ is nonsingular for $t \in (\tau,\beta]$, and $u_\tau(t)$ is also nonsingular on this interval.

The final theorems of this section concern the two-point boundary value problem and its relationship with nonoscillation properties on a compact interval. Solutions of the two-point boundary value problem are said to exist and be unique on arbitrary subintervals of $[\alpha,\beta]$ if for each pair of distinct points τ_1, τ_2 of $[\alpha,\beta]$, and for arbitrary elements u_1, u_2 of the B^*-algebra, there exists a unique solution (u,v) of the system

$$
(4.5) \qquad \begin{aligned} L_1[u,v](t) &= 0, & L_2[u,v](t) &= 0, \\ u(\tau_1) &= u_1, & u(\tau_2) &= u_2. \end{aligned}
$$

The principal result concerning solutions of the two-point boundary value problem is given in the following theorem.

THEOREM 4.6. For distinct τ_1, τ_2 belonging to $[\alpha,\beta]$, and arbitrary u_1, $u_2 \in \mathcal{B}$, there exists a unique solution of the two-point boundary value problem (4.5) if and only if for each $\tau \in [\alpha,\beta]$, the solution (u_τ,v_τ) of system (3.1) satisfying $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$ is such that $u_\tau(t)$ is nonsingular for t distinct from τ on the interval $[\alpha,\beta]$.

Suppose that $u_\tau(t)$ is nonsingular for distinct points τ,t of the interval $[\alpha,\beta]$. If τ_1, τ_2 are distinct points of $[\alpha,\beta]$ and u_1, u_2 are elements of $\mathcal{B}$, then it may be easily verified that

$$
\begin{aligned}
u(t) &= u_{\tau_1}(t)u_{\tau_1}^{-1}(\tau_2)u_2 + u_{\tau_2}(t)u_{\tau_2}^{-1}(\tau_1)u_1, \\
v(t) &= v_{\tau_1}(t)u_{\tau_1}^{-1}(\tau_2)u_2 + v_{\tau_2}(t)u_{\tau_2}^{-1}(\tau_1)u_1,
\end{aligned}
$$

is a solution of system (3.1) satisfying $u(\tau_1) = u_1$ and $u(\tau_2) = u_2$. To show the uniqueness of the solution suppose that $(\hat{u},\hat{v})$ also satisfies $\hat{u}(\tau_1) = u_1$ and $\hat{u}(\tau_2) = u_2$. Then the function $u(t) - \hat{u}(t)$ vanishes at both $t = \tau_1$ and $t = \tau_2$. Therefore, by the uniqueness of solutions of the initial value problem we must have

$$
\begin{aligned}
u(t) - \hat{u}(t) &= u_{\tau_1}(t)[v(\tau_1) - \hat{v}(\tau_1)], \\
v(t) - \hat{v}(t) &= v_{\tau_1}(t)[v(\tau_1) - \hat{v}(\tau_1)],
\end{aligned} \tag{4.6}
$$

for $t \in [\alpha,\beta]$. Since $u(\tau_2) - \hat{u}(\tau_2) = 0$, and $u_{\tau_1}(\tau_2)$ is nonsingular, it follows from equations (4.6) that $v(\tau_1) = \hat{v}(\tau_1)$. Consequently, we have that $u(t) \equiv \hat{u}(t)$ and $v(t) \equiv \hat{v}(t)$ and the solution of (4.5) is unique.

Conversely, suppose that solutions of the two-point boundary value problem (4.5) exist and are unique. Let τ_1, τ_2 be distinct points of the interval $[\alpha,\beta]$. Then there exists a solution (u,v) of system (3.1) such that $u(\tau_1) = 0$ and $u(\tau_2) = e$. Again by the uniqueness of solutions to the initial value problem, we have that

$$
\left.
\begin{aligned}
u(t) &= u_{\tau_1}(t)v(\tau_1), \\
v(t) &= v_{\tau_1}(t)v(\tau_1),
\end{aligned}
\right\} \quad t \in [\alpha,\beta],
$$

and consequently $u_{\tau_1}(\tau_2)v(\tau_1) = e$. To show that $u_{\tau_1}(t)$ is nonsingular at $t = \tau_2$, it suffices to show that $v(\tau_1)u_{\tau_1}(\tau_2) = e$. Now consider the solution $(\hat{u},\hat{v})$ of system (3.1) defined by

$$
\begin{aligned}
\hat{u}(t) &= u_{\tau_1}(t)[v(\tau_1)u_{\tau_1}(\tau_2) - e], \\
\hat{v}(t) &= v_{\tau_1}(t)[v(\tau_1)u_{\tau_1}(\tau_2) - e].
\end{aligned}
$$

It is easily seen that $\hat{u}(\tau_1) = 0 = \hat{u}(\tau_2)$. However, the solution $(u_o(t) \equiv 0, v_o(t) \equiv 0)$ also satisfies the conditions $u_o(\tau_1) = 0 = u_o(\tau_2)$. Therefore, by the uniqueness of solutions of the two-point boundary value problem, it follows that $\hat{u}(t) \equiv 0$ and $\hat{v}(t) \equiv 0$. In particular, for $t = \tau_1$

we find that $0 = \hat{v}(\tau_1) = e[v(\tau_1)u_{\tau_1}(\tau_2) - e]$, so that $v(\tau_1)u_{\tau_1}(\tau_2) = e$, and consequently $u_{\tau_1}(t)$ is nonsingular at $t = \tau_2$.

The following results follow directly from Theorems 4.4 and 4.5 by employing the criteria established in Theorem 4.6.

THEOREM 4.7. _Let_ $b(t) \geq 0$ _on_ $[\alpha,\beta]$, _and suppose that solutions of system_ (4.5) _exist and are unique for distinct_ $\tau_1, \tau_2 \in [\alpha,\beta]$ _and_ $u_1, u_2 \in \mathcal{B}$. _Then system_ (3.1) _is nonoscillatory on the interval_ $[\alpha,\beta]$.

THEOREM 4.8. _Let_ $b(t) \geq 0$ _on_ $[\alpha,\beta]$, _and suppose that property_ (H) _holds on_ $[\alpha,\beta]$. _If system_ (3.1) _is nonoscillatory on_ $[\alpha,\beta]$, _then there exists a unique solution of system_ (4.5) _for distinct_ $\tau_1,\tau_2 \in [\alpha,\beta]$ _and_ $u_1,u_2 \in \mathcal{B}$.

5. _Disconjugacy properties of the linear differential system and an associated hermitian form._ This section is concerned with a property of solutions of system (3.1) termed _disconjugacy_. Two distinct points t_1, t_2 of the interval I are said to be _(mutually) conjugate_ with respect to system (3.1) provided there exists a solution $(u(t),v(t))$ of (3.1) such that $u(t) \not\equiv 0$ on $[t_1,t_2]$, while $u(t_1) = 0 = u(t_2)$. System (3.1) is said to be _disconjugate_ on an interval $[\alpha,\beta]$ provided no two distinct points of $[\alpha,\beta]$ are conjugate. The first results of this section concern properties of a hermitian form associated with system (3.1); the relationship of the hermitian form to disconjugacy and nonoscillation properties is considered in Theorems 5.1 - 5.4. In Section 6 this hermitian form is employed to establish necessary and sufficient conditions for the linear differential system (3.1) to be nonoscillatory. For the case of matrix differential equations, results similar to Lemmas 5.1 - 5.3 may be found in Reid [11; pp. 322-325].

A $\mathcal{B}$-valued function $\xi(t)$ is said to be <u>piecewise continuous</u> on an interval $[\alpha,\beta]$ if $\xi(t)$ is continuous on $[\alpha,\beta]$ except for at most a finite number of points and the right-hand and left-hand (deleted) limits exist at the points of discontinuity. A $\mathcal{B}$-valued function $\eta(t)$ is said to be <u>piecewise smooth</u> on $[\alpha,\beta]$ if $\eta(t)$ is continuous, while its derivative $\eta'(t)$ exists on $[\alpha,\beta]$ except for at most a finite number of points and is piecewise continuous. The set $D[\alpha,\beta]$ is defined as the collection of $\mathcal{B}$-valued functions $\eta(t)$ which are piecewise smooth on $[\alpha,\beta]$ and such that there exists a piecewise continuous $\mathcal{B}$-valued function $\xi(t)$ satisfying the differential equation

$$\eta'(t) - a(t)\eta(t) = b(t)\xi(t)$$

whenever $\eta'(t)$ exists on $[\alpha,\beta]$. The notation $\eta \in D[\alpha,\beta]:\xi$ is used to indicate that $\eta(t)$ is associated in this manner with a function $\xi(t)$. Furthermore, the set $D_o[\alpha,\beta]$ is defined to be the collection of $\eta \in D[\alpha,\beta]$ such that $\eta(\alpha) = 0 = \eta(\beta)$.

If $\eta_1 \in D[\alpha,\beta]:\xi_1$ and $\eta_2 \in D[\alpha,\beta]:\xi_2$, we define $J[\eta_1:\xi_1,\eta_2:\xi_2;\alpha,\beta]$ to be

$$\int_\alpha^\beta [\xi_2^*(s)b(s)\xi_1(s) + \eta_2^*(s)c(s)\eta_1(s)]ds.$$

The function J is clearly a mapping from $D[\alpha,\beta] \times D[\alpha,\beta]$ into the B^*-algebra $\mathcal{B}$. If $b(t)$ is singular at points in $[\alpha,\beta]$, then the functions $\xi_j(t)$ associated with $\eta_j(t) \in D[\alpha,\beta]$, $j = 1,2$, may not be uniquely determined. However, it may be verified readily that the value of J is independent of the particular function $\xi_j(t)$. Therefore, we employ the simpler notation

$$J[\eta_1,\eta_2;\alpha,\beta] = \int_\alpha^\beta [\xi_2^*(s)b(s)\xi_1(s) + \eta_2^*(s)c(s)\eta_1(s)]ds.$$

For the linear differential system (3.1), it is required that $b(t)$ and $c(t)$ be continuous symmetric functions on I. Consequently, several elementary properties of the mapping J may be established readily. In particular, if $\eta_j \in D[\alpha,\beta]$ for $j = 1,2,3$, then

(i) $J[\eta_1,\eta_2;\alpha,\beta] = (J[\eta_2,\eta_1;\alpha,\beta])^*$,

(ii) $J[\lambda\eta_1,\eta_2;\alpha,\beta] = \lambda J[\eta_1,\eta_2;\alpha,\beta]$, where λ is a complex number,

(iii) $J[\eta_1+\eta_2,\eta_3;\alpha,\beta] = J[\eta_1,\eta_3;\alpha,\beta] + J[\eta_2,\eta_3;\alpha,\beta]$.

As a result of the above properties it is easily seen that J is a hermitian form on the set $D[\alpha,\beta]$.

If $\eta \in D[\alpha,\beta]$, for convenience we define

$$J[\eta;\alpha,\beta] = J[\eta,\eta;\alpha,\beta].$$

From property (i) above it follows that for each $\eta \in D[\alpha,\beta]$ we have that $J[\eta;\alpha,\beta]$ is a symmetric element of the B^*-algebra $\mathcal{B}$. In the event that $J[\eta;\alpha,\beta]$ has a non-negative spectrum, we can write $J[\eta;\alpha,\beta] \geq 0$, as defined in Section 2. In particular, in Theorems 5.1, 5.2 and 5.4 of this section we will be interested in the cases wherein the following property holds.

(H_1) $J[\eta;\alpha,\beta] \geq 0$, <u>for each</u> $\eta \in D_o[\alpha,\beta]$ <u>and</u> $J[\eta;\alpha,\beta] = 0$ <u>only if</u> $\eta(t) \equiv 0$.

The following lemmas establish several important properties of the hermitian form J which will be required in the proofs of theorems in this section and in Section 6. The proof of Lemma 5.1 is immediate from the definition of the hermitian from J.

LEMMA 5.1. <u>Let</u> $\eta_j \in D[\alpha,\beta]:\xi_j$ <u>for</u> $j = 1,2$. <u>If</u> $\xi_1(t)$ <u>is continuously</u>

differentiable on $[\alpha,\beta]$, then

$$J[\eta_1,\eta_2;\alpha,\beta] = \eta_2^*\xi_1\Big|_\alpha^\beta + \int_\alpha^\beta \eta_2^*(s)L_1[\eta_1,\xi_1](s)ds.$$

The following lemma shows that if system (3.1) is not disconjugate on $[\alpha,\beta]$ then $J[\eta;\alpha,\beta]$ will vanish for some non-identically vanishing $\eta \in D_o[\alpha,\beta]$.

LEMMA 5.2. Suppose there are distinct points $t_1 < t_2$ on $[\alpha,\beta]$ which are conjugate with respect to system (3.1). Then there exists an $\eta \in D_o[\alpha,\beta]$ such that $\eta(t) \not\equiv 0$ on $[t_1,t_2]$ and $J[\eta;\alpha,\beta] = 0$.

Suppose that $t_1 < t_2$ and (u,v) is a solution of system (3.1) such that $u(t) \not\equiv 0$ on $[t_1,t_2]$, while $u(t_1) = 0 = u(t_2)$. Define $\eta(t)$ as equal to $u(t)$ on $[t_1,t_2]$, and to 0 elsewhere on $[\alpha,\beta]$; correspondingly, let $\xi(t)$ equal $v(t)$ on $[t_1,t_2]$ and 0 otherwise. Then $\eta \in D_o[\alpha,\beta]:\xi$ with $\eta(t) \not\equiv 0$, and by Lemma 5.1 we have

$$J[\eta;\alpha,\beta] = J[\eta;t_1,t_2] = \eta^*\xi\Big|_{t_1}^{t_2} = 0.$$

If $u(t)$, $v(t)$, $\eta(t)$, $\xi(t)$ and $h(t)$ are differentiable functions such that $\eta(t) = u(t)h(t)$, then we have the following identity:

$$\begin{aligned}\xi^*b\xi + \eta^*c\eta &= (\xi-vh)^*b(\xi-vh)\\ &\quad - (vh)^*L_2[\eta,\xi] - L_2[\eta,\xi]^*(vh)\\ &\quad + h^*(v^*L_2[u,v] + u^*L_1[u,v])h\\ &\quad - h^*(u^*v - v^*u)h' + (h^*u^*vh)'.\end{aligned} \tag{5.1}$$

For the case of matrix differential systems, (see, for example, Reid [11; p. 325]), the result of the following lemma is that of the so-called Legendre or Clebsch transformation of $J[\eta;\alpha,\beta]$. The result is proved easily with the aid of (5.1).

LEMMA 5.3. *Let* (u,v) *be a self-conjoined solution of system* (3.1) *on* $[\alpha,\beta]$ *and* $\eta \in D[\alpha,\beta]:\xi$. *If there exists a piecewise smooth function* $h(t)$ *such that* $\eta(t) = u(t)h(t)$, *then*

$$(5.2) \qquad J[\eta;\alpha,\beta] = \eta^* vh\Big|_\alpha^\beta + \int_\alpha^\beta (\xi - vh)^* b(\xi - vh)ds.$$

We are now able to establish relations between properties of the hermitian form J and disconjugacy and nonoscillation properties of the linear differential system (3.1). As an immediate consequence of Lemma 5.2, we have the following result.

THEOREM 5.1. *If the hermitian form* $J[\eta;\alpha,\beta]$ *satisfies condition* (H_1) *on an interval* $[\alpha,\beta]$ *then system* (3.1) *is disconjugate on this interval.*

The following result provides a partial converse to Theorem 5.1.

THEOREM 5.2. *Let* $b(t) \geq 0$ *on* $[\alpha,\beta]$, *and suppose that system* (3.1) *is nonoscillatory on the interval* $[\alpha,\beta]$. *Then the hermitian form* $J[\eta;\alpha,\beta]$ *satisfies condition* (H_1).

Let (u,v) be a self-conjoined solution of system (3.1) with $u(t)$ nonsingular on the interval $[\alpha,\beta]$. If $\eta \in D_0[\alpha,\beta]$, define $h(t) = u^{-1}(t)\eta(t)$ on $[\alpha,\beta]$. Therefore $h(\alpha) = 0 = h(\beta)$, and by Lemma 5.3 it follows that

$$(5.3) \qquad J[\eta;\alpha,\beta] = \int_\alpha^\beta (\xi - vh)^* b(\xi - vh)ds.$$

However, $b(s) \geq 0$ on $[\alpha,\beta]$, so that by Theorem 2.10 we have that $J[\eta;\alpha,\beta] \geq 0$. If $\eta \in D_0[\alpha,\beta]$ is such that $J[\eta;\alpha,\beta] = 0$, then $\int_\alpha^\beta (\xi - vh)^* b(\xi - vh)ds = 0$ and by Theorems 2.7 and 2.11 it follows that

$b(t)(\xi(t) - v(t)h(t)) = 0$ on $[\alpha,\beta]$. Consequently, we have

$$uh' + u'h - auh = \eta' - a\eta = b\xi = bvh,$$

and therefore $h' = (u^{-1}au + u^{-1}bv - u^{-1}u')h$. Since $h(\alpha) = 0$, it follows that $h(t) \equiv 0$ on $[\alpha,\beta]$, and therefore $\eta(t) = u(t)h(t) \equiv 0$ on $[\alpha,\beta]$.

For the finite dimensional matrix case it has been shown, (see [11; p. 337]), that the following conditions are equivalent whenever $b(t) \geq 0$ on $[\alpha,\beta]$:

(i) the linear differential system (3.1) is nonoscillatory on $[\alpha,\beta]$;

(ii) the hermitian form J satisfies property (H_1) on $[\alpha,\beta]$;

(iii) system (3.1) is disconjugate on the interval $[\alpha,\beta]$.

The following example, given by Heimes [4], illustrates that these conditions are not equivalent for the B^*-algebra case, even under the restriction that $b(t) = e > 0$.

Let $\mathcal{H}$ be the Hilbert space ℓ_2, and for $n \geq 1$ let $e_n = (0,\cdots,0,1,0,\cdots)$ be the usual complete orthonormal set. The B^*-algebra will be the collection of bounded linear operators on $\mathcal{H}$. Let $b(t) = e$ and $a(t) = 0$ on the interval $[0,1]$, and let $c(t)$ be the constant operator defined by $c(e_n) = -k_n^2 e_n$ where k_n is the real number $n\pi/(n+1)$. Since $v'(t) = u(t)$, system (3.1) may be written in the form

$$u''(t) = cu(t), \quad t \in [0,1].$$

If $\tau \in [0,1]$, then the solution $u_\tau(t)$ of this equation satisfying $u_\tau(\tau) = 0$, $u_\tau'(\tau) = e$ is defined by the relation

$$u_\tau(t)e_n = \{k_n^{-1}\sin[k_n(t-\tau)]\}e_n.$$

If $x = \Sigma\lambda_n e_n$ is an element of $\mathcal{H}$ such that $u_\tau(t)x = 0$ for some $t \neq \tau$ on $[0,1]$, then clearly $x = 0$; consequently the differential system is disconjugate on the interval [0,1].

The solution of the differential system satisfying $u_o(0) = 0$, $u_o'(0) = e$ is defined by

$$u_o(t)e_n = \{k_n^{-1}\sin(k_n t)\}e_n.$$

For $t = 1$, we find that

$$u_o(1)e_n = \{k_n^{-1}\sin(k_n)\}e_n.$$

The linear operator $u_o(1)$ is not onto for $\Sigma(1/n)e_n$ is not in its range; and consequently $u_o(1)$ is singular. Since $b(t) > 0$ on [0,1], it follows from Theorem 4.1 that the linear differential system fails to be nonoscillatory on the interval [0,1].

By Theorems 5.1 and 5.2, it follows that nonoscillation implies disconjugacy on an interval $[\alpha,\beta]$ provided that $b(t) \geq 0$. The preceding example shows that the concepts of nonoscillation and disconjugacy are not equivalent in the B^*-algebra case. However, the property which unifies the concepts of nonoscillation and disconjugacy in the matrix case is easily isolated. An element $u \in \mathcal{B}$ is said to have property (P) if either u is nonsingular or there exists a nonzero element $x \in \mathcal{B}$ such that $ux = 0$. We now have the following result relating disconjugacy and nonoscillation properties.

THEOREM 5.3. *Let* $b(t) \geq 0$ *on* $[\alpha,\beta]$, *and suppose that property* (H) *holds on* $[\alpha,\beta]$. *If system* (3.1) *is disconjugate on the interval* $[\alpha,\beta]$, *then this system is nonoscillatory on* $[\alpha,\beta]$ *if and only if for each* $\tau \in [\alpha,\beta]$ *the solution* (u_τ, v_τ) *of* (3.1) *satisfying* $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$, *is such that* $u_\tau(t)$ *has property* (P) *for each* $t \in [\alpha,\beta]$.

If system (3.1) is nonoscillatory on $[\alpha,\beta]$, it follows from Theorem

4.5 that $u_\tau(t)$ is nonsingular for $t \neq \tau$ on $[\alpha,\beta]$. Furthermore $u_\tau(\tau) = 0$ so that for any $x \in \mathcal{B}$, we have $u_\tau(\tau)x = 0$. Therefore $u_\tau(t)$ has property (P) for all $t \in [\alpha,\beta]$.

Conversely, suppose that system (3.1) is disconjugate on $[\alpha,\beta]$ and that $u_\tau(t)$ satisfies property (P) for $t \in [\alpha,\beta]$. In particular, consider the solution (u_α,v_α) satisfying $u_\alpha(\alpha) = 0$, $v_\alpha(\alpha) = e$. To show that system (3.1) is nonoscillatory on $[\alpha,\beta]$, it suffices by Theorem 4.4 to show that $u_\alpha(t)$ is nonsingular on $(\alpha,\beta]$. Suppose that there exists a $\tau \in (\alpha,\beta]$ such that $u_\alpha(\tau)$ is singular; then, by hypothesis, there exists a nonzero element $x \in \mathcal{B}$ such that $u_\alpha(\tau)x = 0$. Therefore, $(u_\alpha(t)x,v_\alpha(t)x)$ is a solution of system (3.1) such that $u_\alpha(\alpha)x = 0 = u_\alpha(\tau)x$. However, system (3.1) is disconjugate on $[\alpha,\beta]$, and it follows that $u_\alpha(t)x \equiv 0$ on $[\alpha,\tau]$. By property (H) we know that $u_\alpha(t)$ is nonsingular on an interval $(\alpha,\alpha+\delta)$, where δ is chosen so that $\alpha + \delta < \tau$. Since $u_\alpha(t)x \equiv 0$ on $(\alpha,\alpha+\delta)$, it follows that $x = 0$. Therefore we conclude that $u_\alpha(t)$ is nonsingular on $(\alpha,\beta]$, and hence system (3.1) is nonoscillatory on this interval.

The preceding theorem illustrates why the concepts of disconjugacy and nonoscillation are equivalent for matrix differential systems. That is, if u is any finite dimensional square matrix then either u is nonsingular or it has zero divisors so that property (P) holds for all such matrices. In the following result we exhibit one set of conditions where the concepts of nonoscillation and disconjugacy are equivalent for the B^*-algebra case.

THEOREM 5.4. *Let* $b(t) > 0$ *on* $[\alpha,\beta]$, *and suppose that* $a(t)$, $a^*(t)$, *and* $c(t)$ *are compact for each* $t \in [\alpha,\beta]$. *Then the following conditions are* equivalent:

(i) system (3.1) is nonoscillatory on the interval $[\alpha,\beta]$;

(ii) the hermitian form J satisfies condition (H_1) on the interval $[\alpha,\beta]$;

(iii) system (3.1) is disconjugate on the interval $[\alpha,\beta]$.

By Theorems 5.1 and 5.2, we know that (i) implies (ii), and that (ii) implies (iii). Therefore it suffices to show that (iii) implies (i). This result is obtained with the aid of Theorem 5.3. For $\tau \in [\alpha,\beta]$ let (u_τ, v_τ) be the solution of (3.1) satisfying $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$. We now show that $u_\tau(t)$ has property (P) for each $t \in [\alpha,\beta]$. For each $s \in [\alpha,\beta]$, define

$$k_1(s) = \int_\tau^s (cu_\tau - a^* v_\tau)dr,$$

so that $v_\tau(s) = e + k_1(s)$. Moreover, it follows that for $t \in [\alpha,\beta]$ we have

$$\begin{aligned} u_\tau(t) &= \int_\tau^t (au_\tau + bv_\tau)ds \\ &= \int_\tau^t bds + \int_\tau^t (bk_1 + au_\tau)ds. \end{aligned}$$

As the elements $a(s)$, $a^*(s)$, and $c(s)$ are compact for $s \in [\alpha,\beta]$, we have by the Corollary to Theorem 2.12 that $k_1(s)$ and $k(t) = \int_\tau^t (bk_1 + au_\tau)ds$ are also compact for each $t \in [\alpha,\beta]$; furthermore, for $t > \tau$, we have that $\int_\tau^t b(s)ds > 0$ so that $\int_\tau^t b(s)ds$ is nonsingular. Therefore, by Theorem 2.12 it follows that for $t > \tau$ the function $u_\tau(t)$ is either nonsingular, or there exists a nonzero element $x \in \mathcal{B}$ such that $u_\tau(t)x = 0$. Hence for $t > \tau$ we have that $u_\tau(t)$ has property (P). It can be shown in a similar manner that $u_\tau(t)$ has property (P) for $t \in [\alpha,\tau]$, and it then follows from Theorem 5.3 that system (3.1) is nonoscillatory on the interval $[\alpha,\beta]$.

6. <u>Nonoscillation and comparison theorems for the linear differential system on a compact interval.</u> In this section we use a generalization of a method employed in variational theory to establish necessary and sufficient conditions for system (3.1) to be nonoscillatory on a finite subinterval $[\alpha,\beta]$ of the interval $I = (\alpha_o,\infty)$. The use of variational techniques will also enable us to establish generalizations of certain Sturmian-type comparison theorems. In particular, Theorems 6.1 and 6.2 are generalizations of classical nonoscillation theorems for finite dimensional matrix differential systems; see, for example, Reid [11; p. 328].

We shall be concerned with the set of real numbers λ such that

$$\begin{aligned} J[\eta;\alpha,\beta] &\equiv \int_\alpha^\beta [\xi^* b\xi + \eta^* c\eta]ds \\ &\geq \lambda\int_\alpha^\beta \eta^*\eta ds, \quad \text{for} \quad \eta \in D_o[\alpha,\beta], \end{aligned} \tag{6.1}$$

where $J[\eta;\alpha,\beta]$ is the hermitian form defined in Section 5. The following result shows that this set is nonempty if $b(t) \geq 0$ on $[\alpha,\beta]$.

LEMMA 6.1. <u>If</u> $b(t) \geq 0$ <u>on</u> $[\alpha,\beta]$, <u>then there exists a real number</u> λ_o <u>such that</u> (6.1) <u>holds with</u> $\lambda = \lambda_o$.

Let $-\lambda_o$ be defined as the maximum of $\|c(t)\|$ on the interval $[\alpha,\beta]$. Then we have that $\lambda_o e \leq c(t) \leq -\lambda_o e$, and consequently $c(t) - \lambda_o e \geq 0$, on $[\alpha,\beta]$. Furthermore, since $b(t) \geq 0$ on $[\alpha,\beta]$, it follows that

$$\int_\alpha^\beta [\xi^* b\xi + \eta^*(c - \lambda_o e)\eta]ds \geq 0$$

for each $\eta \in D_o[\alpha,\beta]:\xi$, and therefore inequality (6.1) holds for $\lambda = \lambda_o$.

We will be particularly interested in those cases wherein there exists a positive real λ such that (6.1) holds. In fact, Theorems 6.1 and 6.2 establish that under certain hypotheses system (3.1) is nonoscillatory on

$[\alpha,\beta]$ if and only if there exists a positive real λ_o such that inequality (6.1) is satisfied for $\lambda = \lambda_o$.

THEOREM 6.1. <u>Let</u> $b(t) \geq 0$ <u>on</u> $[\alpha,\beta]$, <u>and suppose that system</u> (3.1) <u>is nonoscillatory on this interval. Then there exists a positive real</u> λ_o <u>such that</u> (6.1) <u>holds with</u> $\lambda = \lambda_o$.

Since (3.1) is nonoscillatory on $[\alpha,\beta]$, there exists a self-conjoined solution (u,v) of this system such that $u(t)$ is nonsingular on $[\alpha,\beta]$. Let M be a positive real number such that $\|u^{-1}(t)\| \leq M$ on $[\alpha,\beta]$. Now consider solutions of the linear differential system

$$(6.2) \qquad \begin{aligned} \hat{u}' &= a(t)\hat{u} + b(t)\hat{v}, \\ \hat{v}' &= [c(t) - \lambda e]\hat{u} - a^*(t)\hat{v}, \end{aligned} \qquad t \in [\alpha,\beta],$$

where λ is a real number. From Theorem 3.2 it follows that there exists a $\delta > 0$ such that for any λ with $|\lambda| < \delta$ the solution $(\hat{u},\hat{v})$ of system (6.2) satisfying $\hat{u}(\alpha) = u(\alpha)$, $\hat{v}(\alpha) = v(\alpha)$ has the property that $\|u(t) - \hat{u}(t)\| < 1/M$ on $[\alpha,\beta]$. Therefore we have that $\|u(t) - \hat{u}(t)\| < 1/\|u^{-1}(t)\|$ for all $t \in [\alpha,\beta]$, and by Theorem 2.1 it follows that $\hat{u}(t)$ is nonsingular on $[\alpha,\beta]$. In particular, this property holds for a positive real λ_o chosen so that $0 < \lambda_o < \delta$. Furthermore, the solution $(\hat{u},\hat{v})$ is self-conjoined, since $\hat{u}^*(\alpha)\hat{v}(\alpha) - \hat{v}^*(\alpha)\hat{u}(\alpha) = u^*(\alpha)v(\alpha) - v^*(\alpha)u(\alpha) = 0$.

The associated hermitian form for system (6.2) with $\lambda = \lambda_o$ is given by

$$J_{\lambda_o}[\eta;\alpha,\beta] = \int_\alpha^\beta [\xi^* b\xi + \eta^*(c - \lambda_o e)\eta]ds.$$

Since system (6.2) is nonoscillatory in $[\alpha,\beta]$ for $\lambda = \lambda_o$, it follows from Theorem 5.2 that

$$\int_\alpha^\beta [\xi^* b\xi + \eta^*(c - \lambda_o e)\eta]ds \geq 0, \quad \text{for} \quad \eta \in D_o[\alpha,\beta],$$

and therefore inequality (6.1) holds for the positive real number λ_o.

The following result shows that the converse of Theorem 6.1 is valid whenever condition (H), as introduced in Section 4, holds.

THEOREM 6.2. *Let* $b(t) \geq 0$ *on* $[\alpha,\beta]$ *and suppose that property* (H) *holds on the interval* $[\alpha,\beta]$. *If there exists a positive real number* λ_o *such that inequality* (6.1) *holds for* $\lambda = \lambda_o$, *then system* (3.1) *is non-oscillatory on the* interval $[\alpha,\beta]$.

To show that (3.1) is nonoscillatory on the interval $[\alpha,\beta]$, it is to be noted that by Theorem 4.4 it suffices to show that the solution (u_α,v_α) satisfying $u_\alpha(\alpha) = 0$, $v_\alpha(\alpha) = e$ is such that $u_\alpha(t)$ is nonsingular on $(\alpha,\beta]$. Now, by hypothesis, there exists a $\delta > 0$ such that $u_\alpha(t)$ is non-singular on $(\alpha,\alpha+\delta]$; suppose that (α,τ), where $\tau \leq b$, is the maximal interval on which $u_\alpha(t)$ is nonsingular. Let (u_τ,v_τ) be the solution of (3.1) satisfying $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$, and choose $s \in (\alpha,\tau)$ so that $u_\tau(t)$ is nonsingular on $[s,\tau)$. Suppose $x \in \mathcal{B}$ and define $\eta \in D_o[\alpha,\beta]:\xi$ as follows:

$$\eta(t) = \begin{cases} u_\alpha(t)x & \text{on } [\alpha,s], \\ u_\tau(t)u_\tau^{-1}(s)u_\alpha(s)x & \text{on } [s,\tau], \\ 0 & \text{on } [\tau,\beta], \end{cases}$$

(6.3)

$$\xi(t) = \begin{cases} v_\alpha(t)x & \text{on } [\alpha,s), \\ v_\tau(t)u_\tau^{-1}(s)u_\alpha(s)x & \text{on } (s,\tau), \\ 0 & \text{on } (\tau,\beta]. \end{cases}$$

Since (η,ξ) is a solution of system (3.1) on the intervals $[a,s)$ and (s,τ), it follows from Lemma 5.1 that

$$J[\eta;\alpha,s] = \eta^*(s)\xi(s^-) = x^*u_\alpha^*(s)v_\alpha(s)x, \tag{6.4}$$

and that

$$(6.5) \qquad J[\eta;s,\tau] = -\eta^*(s)\xi(s^+) = -x^*u_\alpha^*(s)v_\tau(s)u_\tau^{-1}(s)u_\alpha(s)x.$$

By Theorem 3.3, we know that $u_\alpha^*(t)v_\tau(t) - v_\alpha^*(t)u_\tau(t)$ is equal to the constant $u_\alpha^*(\tau)$ on $[\alpha,\tau]$. Consequently, we have the relation

$$(6.6) \qquad u_\alpha^*(s)v_\tau(s)u_\tau^{-1}(s)u_\alpha(s) = v_\alpha^*(s)u_\alpha(s) + u_\alpha^*(\tau)u_\tau^{-1}(s)u_\alpha(s).$$

From equations (6.4) - (6.6), we find that

$$J[\eta;\alpha,\beta] = x^*[u_\alpha^*(s)v_\alpha(s) - v_\alpha^*(s)u_\alpha(s) - u_\alpha^*(\tau)u_\tau^{-1}(s)u_\alpha(s)]x.$$

Since (u_α, v_α) is a self-conjoined solution of (3.1), it follows that $u_\alpha^*(s)v_\alpha(s) = v_\alpha^*(s)u_\alpha(s)$, and therefore we have

$$(6.7) \qquad J[\eta;\alpha,\beta] = -x^*u_\alpha^*(\tau)u_\tau^{-1}(s)u_\alpha(s)x,$$

for the $\eta(t)$ constructed in (6.3).

Since $u_\alpha(t)$ is nonsingular on $(\alpha,s]$, it follows that $u_\alpha^*(t)u_\alpha(t) > 0$ on $(\alpha,s]$ and hence $\int_\alpha^s u_\alpha^*u_\alpha ds > 0$. Therefore there exists a positive real number k such that

$$\int_\alpha^s u_\alpha^*u_\alpha ds \geq ke > 0.$$

Consequently, if x is any element of $\mathcal{B}$ with $\|x\| = 1$, then for $\eta(t)$ defined as in (6.3) we have

$$\int_\alpha^b \eta^*\eta ds \geq \int_\alpha^s \eta^*\eta ds = x^*\left[\int_\alpha^s u_\alpha^*u_\alpha ds\right]x \geq kx^*x \geq 0,$$

and therefore $\left\|\lambda_o\int_\alpha^\beta \eta^*\eta ds\right\| \geq \lambda_o k > 0$.

Let $K = \|u_\tau^{-1}(s)u_\alpha(s)\|$. Since $u_\alpha(t)$ is singular at $t = \tau$, by Lemma 3.2 there exists an $x \in \mathcal{B}$ with $\|x\| = 1$ and such that $\|u_\alpha(\tau)x\| < \lambda_o k/K$. Consequently, for such an x it follows from (6.7) that $\|J[\eta;\alpha,\beta]\| < \lambda_o k$, whereas $\left\|\lambda_o\int_\alpha^\beta \eta^*\eta ds\right\| \geq \lambda_o k$. This clearly violates the condition that

$$J[\eta;\alpha,\beta] \geq \lambda_o \int_\alpha^\beta \eta^* \eta ds, \text{ for all } \eta \in D_o[\alpha,\beta].$$

Therefore it follows that $u_\alpha(t)$ is nonsingular on $(\alpha,\beta]$; accordingly, system (3.1) is nonoscillatory on the interval $[\alpha,\beta]$.

If λ_o is a real number such that inequality (6.1) holds, it follows trivially that for any $\lambda \leq \lambda_o$ the inequality remains valid. Therefore there exists a largest λ, denoted by λ_β, such that inequality (6.1) is valid. It is easily seen that if $D_o[\alpha,\beta]$ contains any element other than $\eta(t) \equiv 0$, then λ_β must be finite.

With the aid of Theorem 6.2, we are able to establish some non-oscillatory properties of the linear differential system (6.2).

THEOREM 6.3. <u>Let</u> $b(t) > 0$ <u>on</u> $[\alpha,\beta]$, <u>and suppose</u> λ_β <u>is the largest real number such that inequality</u> (6.1) <u>holds</u>. <u>Then for each</u> $\lambda < \lambda_\beta$ <u>the system</u> (6.2) <u>is nonoscillatory</u>.

If $\lambda < \lambda_\beta$, then clearly $\lambda_\beta - \lambda > 0$ and

$$\begin{aligned} J_\lambda[\eta;\alpha,\beta] &= \int_\alpha^\beta [\xi^* b \xi + \eta^*(c - \lambda e)\eta] ds \\ &\geq (\lambda_\beta - \lambda) \int_\alpha^\beta \eta^* \eta ds, \quad \text{for} \quad \eta \in D_o[\alpha,\beta]. \end{aligned}$$

From Theorem 6.2, it follows immediately that system (6.2) is nonoscillatory for the chosen λ. One would expect the conclusion of the theorem to remain valid under the weaker hypothesis of $b(t) \geq 0$ on $[\alpha,\beta]$, and property (H). However, property (H) is a property of system (6.2) with $\lambda = 0$, and we have not shown that the property is sustained for other values of λ. The requirement that $b(t) > 0$ on $[\alpha,\beta]$ assures that property (H) holds for system (6.2) for any real λ.

As in the discussion preceding Theorem 6.3, for each $t \in (\alpha,\infty) \subset I$, there exists a largest value λ_t such that

$$J[\eta;\alpha,t] \geq \lambda_t \int_\alpha^t \eta^*\eta ds, \quad \text{for} \quad \eta \in D_o[\alpha,t].$$

Clearly λ_t is a monotone, nonincreasing function of t. The following result shows that λ_t is continuous from the right.

THEOREM 6.4. _Let_ $b(t) > 0$ _on_ $[\alpha,\beta]$, _and let_ ε _be an arbitrary positive real number. Then there exists a_ $\delta > 0$ _such that_

$$J[\eta;\alpha,\beta+\delta] \geq (\lambda_\beta - \varepsilon)\int_\alpha^{\beta+\delta} \eta^*\eta ds$$

for $\eta \in D_o[\alpha,\beta+\delta]$; _in particular_, $\lambda_t \geq \lambda_\beta - \varepsilon$ _for_ $t \in [\beta,\beta+\delta]$.

Since $J[\eta;\alpha,\beta] \geq \lambda_\beta \int_\alpha^\beta \eta^*\eta ds$ on $D_o[\alpha,\beta]$, by Theorem 6.3 there exists a self-conjoined solution $(\hat{u},\hat{v})$ of system (6.2) with $\lambda = \lambda_\beta - \varepsilon$ such that $\hat{u}(t)$ is nonsingular on $[\alpha,\beta]$. Therefore $\hat{u}(t)$ must be nonsingular on a larger interval $[\alpha,\beta+\delta]$, where $\delta > 0$. By Theorem 5.2, we have that

$$\int_\alpha^{\beta+\delta} [\xi^*b\xi + \eta^*\{c - (\lambda_\beta - \varepsilon)e\}\eta]ds \geq 0$$

on $D_o[\alpha,\beta+\delta]$. Therefore,

$$J[\eta;\alpha,\beta+\delta] = \int_\alpha^{\beta+\delta} [\xi^*b\xi + \eta^*c\eta]ds \geq (\lambda_\beta - \varepsilon)\int_\alpha^{\beta+\delta} \eta^*\eta ds$$

for $\eta \in D_o[\alpha,\beta+\delta]$, and the theorem is proved.

In the remainder of this section we will establish several comparison theorems for linear differential systems by applying the results of Theorems 6.1 and 6.2. Our attention will be restricted to the case where $b(t) > 0$ on $[\alpha,\beta]$. In this case, if $\eta \in D_o[\alpha,\beta]:\xi$, then $\xi(t)$ is uniquely determined by the relation

$$\xi(t) = b^{-1}(t)[\eta'(t) - a(t)\eta(t)]$$

whenever $\eta'(t)$ exists. Therefore the set $D_o[\alpha,\beta]$ consists of all piecewise smooth functions $\eta(t)$ such that $\eta(\alpha) = 0 = \eta(\beta)$. Furthermore, for $\eta \in D_o[\alpha,\beta]$ we can write

$$J[\eta;\alpha,\beta] = \int_\alpha^\beta [(\eta' - a\eta)^* b^{-1}(\eta' - a\eta) + \eta^* c\eta]ds.$$

Consider the two linear differential systems

$$(6.8_j) \qquad \begin{aligned} u' &= au + b_j v, \\ v' &= c_j u - a^* v, \end{aligned}$$

for $j = 1,2$, where $b_1(t)$, $b_2(t)$, $c_1(t)$, and $c_2(t)$ are continuous symmetric $\mathcal{B}$-valued functions, and $a(t)$ is continuous. Corresponding to the two differential systems are the hermitian forms

$$J_j[\eta;\alpha,\beta] = \int_\alpha^\beta [(\eta' - a\eta)^* b_j^{-1}(\eta' - a\eta) + \eta^* c_j \eta]ds, \quad j = 1,2,$$

defined on the common domain $D_o[\alpha,\beta]$. We also define an additional hermitian form

$$\begin{aligned} J_{1,2}[\eta;\alpha,\beta] &= J_1[\eta;\alpha,\beta] - J_2[\eta;\alpha,\beta] \\ &= \int_\alpha^\beta [(\eta' - a\eta)^*(b_1^{-1} - b_2^{-1})(\eta' - a\eta) + \eta^*(c_1 - c_2)\eta]ds \end{aligned}$$

on the set $D_o[\alpha,\beta]$. As in the discussion preceding Theorem 6.3, there exist largest values λ_j, $j = 1,2$, such that

$$J_j[\eta;\alpha,\beta] \geq \lambda_j \int_\alpha^\beta \eta^*\eta ds, \quad \text{for} \quad \eta \in D_o[\alpha,\beta].$$

Furthermore, in the case where $b_1^{-1}(t) \geq b_2^{-1}(t) > 0$, (equivalently $b_2(t) \geq b_1(t) > 0$), on $[\alpha,\beta]$, it follows that there exists a largest real number $\lambda_{1,2}$ such that

$$J_{1,2}[\eta;\alpha,\beta] \geq \lambda_{1,2} \int_\alpha^\beta \eta^*\eta ds, \quad \text{for} \quad \eta \in D_o[\alpha,\beta].$$

We can now establish easily the following result.

THEOREM 6.5. *Let* $b_2(t) \geq b_1(t) > 0$ *on* $[\alpha,\beta]$. *If* λ_1, λ_2, *and* $\lambda_{1,2}$ *are the real numbers defined in the preceding discussion, then* $\lambda_1 \geq \lambda_2 + \lambda_{1,2}$. *In particular, if* $\lambda_2 + \lambda_{1,2} > 0$ *then system* (6.8_1) *is nonoscillatory on* $[\alpha,\beta]$.

Clearly,

$$\begin{aligned} J_1[\eta;\alpha,\beta] &= J_2[\eta;\alpha,\beta] + J_{1,2}[\eta;\alpha,\beta] \\ &\geq (\lambda_2 + \lambda_{1,2})\int_\alpha^\beta \eta^*\eta ds, \end{aligned}$$

for all $\eta \in D_o[\alpha,\beta]$, and by the definition of λ_1 it follows readily that $\lambda_1 \geq \lambda_2 + \lambda_{1,2}$.

Although the preceding result is seemingly trivial, it manifests the basic idea involved in several Sturmian-type comparison theorems. The following result is such a comparison theorem.

THEOREM 6.6. *Let* $b_2(t) \geq b_1(t) > 0$ *on* $[\alpha,\beta]$, *and suppose that* $J_{1,2}[\eta;\alpha,\beta] \geq 0$ *for each* $\eta \in D_o[\alpha,\beta]$. *If system* (6.8_2) *is nonoscillatory on* $[\alpha,\beta]$, *then system* (6.8_1) *is also nonoscillatory on* $[\alpha,\beta]$.

Since $J_{1,2}[\eta;\alpha,\beta] \geq 0$ on $D_o[\alpha,\beta]$, it follows that $\lambda_{1,2} \geq 0$. If system (6.8_2) is nonoscillatory on $[\alpha,\beta]$, then $\lambda_2 > 0$, and by the preceding theorem we have $\lambda_1 \geq \lambda_2 + \lambda_{1,2} > 0$. Therefore, by Theorem 6.2, system (6.8_1) must be nonoscillatory on $[\alpha,\beta]$.

If $c_1(t)$ and $c_2(t)$ are such that $c_1(t) \geq c_2(t)$, and $b_1^{-1}(t) \geq b_2^{-1}(t)$ on $[\alpha,\beta]$, then clearly $J_{1,2}[\eta;\alpha,\beta] \geq 0$ for $\eta \in D_o[\alpha,\beta]$. Therefore we have the following result.

COROLLARY. *Let* $b_2(t) \geq b_1(t) > 0$ *and* $c_1(t) \geq c_2(t)$ *for* $t \in [\alpha,\beta]$.

If system (6.8_2) *is nonoscillatory on* $[\alpha,\beta]$, *then system* (6.8_1) *is also nonoscillatory on this interval.*

The final theorem of this section concerns linear second-order differential equations of the form

$$(6.9) \qquad [r(t)u' + q(t)u]' - [q^*(t)u' + p(t)u] = 0,$$

as introduced in Section 3. In particular, we establish a comparison theorem for two differential equations of the form (6.9) where $q(t) \equiv 0$ on $[\alpha,\beta]$. For $j = 1,2$, consider the differential equations

$$(6.10_j) \qquad [r_j(t)u']' - p_j(t)u = 0, \qquad t \in [\alpha,\beta].$$

We now have the following result.

THEOREM 6.7. *Let* $r_1(t) \geq r_2(t) > 0$ *and* $p_1(t) \geq p_2(t)$ *on* $[\alpha,\beta]$. *If equation* (6.10_2) *is nonoscillatory on* $[\alpha,\beta]$, *then so is equation* (6.10_1).

In the corresponding linear differential systems we have $b_j(t) = r_j^{-1}(t)$, and $b_2(t) \geq b_1(t) > 0$; furthermore, $c_1(t) - c_2(t) = p_1(t) - p_2(t) \geq 0$. The conclusion of the theorem is now immediate from the Corollary to Theorem 6.6.

7. *Properties of solutions of the Riccati differential equation and resulting comparison theorems.* In Theorem 7.1 of this section there is established a result which guarantees the existence of solutions of the Riccati differential equation (3.2) on an infinite interval $[\alpha,\infty)$. Theorems 7.2 and 7.3 are comparison and existence theorems for (3.2), and are proved with the aid of Theorem 7.1.

It is to be noted that for Riccati matrix differential equations a result similar to that of the following Theorem 7.1 is to be found in Reid [10]. However, the method of proof used by Reid does not appear to be adaptable to the present general situation, and the method of proof given here differs greatly from that used in the matrix case.

THEOREM 7.1. *If* $b(t) \geq 0$ *and* $c(t) \geq 0$ *on* $[\alpha,\infty)$, *and* $q > 0$, $[q \geq 0]$, *then there exists a symmetric solution* $w(t)$ *of the Riccati equation* (3.2) *on the interval* $[\alpha,\infty)$ *which satisfies* $w(\alpha) = q$. *Furthermore*, $w(t) > 0$, $[w(t) \geq 0]$, *on* $[\alpha,\infty)$.

We first consider the case where $q > 0$. Let (u,v) be the solution of system (3.1) satisfying $u(\alpha) = e$, $v(\alpha) = q$. It follows from the method of proof of Theroem 3.5 that the solution $w(t)$ of (3.2) satisfying $w(\alpha) = q$ exists on an interval $[\alpha,\tau)$ if and only if $u(t)$ is nonsingular on $[\alpha,\tau)$, and that $w(t) = v(t)u^{-1}(t)$ on this interval. Suppose that $[\alpha,\tau)$ is the maximal right interval of existence of $w(t)$ and that $\tau < \infty$. Since (u,v) is a solution of system (3.1), by Lemma 5.1 it follows that

$$\int_\alpha^t (v^*bv + u^*cu)ds = u^*v\Big|_\alpha^t \quad \text{for} \quad t \in [\alpha,\tau).$$

Furthermore, since $b(t) \geq 0$ and $c(t) \geq 0$, we have that $0 \leq u^*(t)v(t) - q$ and therefore

$$(7.1) \qquad 0 < q \leq u^*(t)v(t), \quad \text{for} \quad t \in [\alpha,\tau).$$

Since $u(t)$ is nonsingular on $[\alpha,\tau)$, it follows from inequality (7.1) that

$$(7.2) \qquad 0 < u^{*-1}(t)qu^{-1}(t) \leq v(t)u^{-1}(t) = w(t),$$

and therefore $w(t)$ is nonsingular on $[\alpha,\tau)$. In addition we have that $v(t) = w(t)u(t)$ is also nonsingular on $[\alpha,\tau)$. The solution (u,v) of system (3.1) is self-conjoined, so that inequality (7.1) may be written also as $0 < q \leq v^*(t)u(t)$. By Theorem 2.8, it follows that $0 < u^{-1}(t)v^{*-1}(t) \leq q^{-1}$, and consequently

$$(7.3) \qquad 0 < w(t) = v(t)u^{-1}(t) \leq v(t)q^{-1}v^*(t), \text{ for } t \in [\alpha,\tau).$$

By inequality (7.3) we find that $\|w(t)\| \leq \|v(t)q^{-1}v^*(t)\|$, and therefore $\|w(t)\|$ is bounded on $[\alpha,\tau)$. It then follows from Theorem 3.6 that $w(t)$ can be extended to a larger interval $[\alpha,\tau+\delta)$ where $\delta > 0$, which contradicts the maximality of the interval $[\alpha,\tau)$. Hence the solution $w(t)$ exists on the infinite interval $[\alpha,\infty)$, and by (7.2) we have $w(t) > 0$ on this interval.

Now consider the case where $q \geq 0$, and suppose that the maximal right interval of existence of $w(t)$ is $[\alpha,\tau)$, where $\tau < \infty$; it follows that $u(t)$ is nonsingular on this interval. In this case we find that

$$(7.4) \qquad 0 \leq u^{*-1}(t)qu^{-1}(t) \leq v(t)u^{-1}(t) = w(t)$$

on $[\alpha,\tau)$. For $\varepsilon > 0$, let $w_\varepsilon(t)$ be the solution of the Riccati equation determined by the initial condition $w_\varepsilon(\alpha) = q + \varepsilon e > 0$. From the preceding result we have that $w_\varepsilon(t)$ exists on the entire interval $[\alpha,\infty)$. For $z(t)$ defined by $z(t) = w_\varepsilon(t) - w(t)$ on $[\alpha,\tau)$, it is to be noted that $z(t)$ satisfies $z(\alpha) = \varepsilon e > 0$, and

$$(7.5) \qquad z' + za_o(t) + a_o^*(t)z + zb(t)z - c_o(t) = 0$$

on $[\alpha,\tau)$, where $a_o(t) = a(t) + b(t)w(t)$ and $c_o(t) \equiv 0$. As a result of the first part of the theorem applied to (7.5), it follows that $z(t) > 0$ on $[\alpha,\tau)$. Consequently, we have that $0 \leq w(t) < w_\varepsilon(t)$ on $[\alpha,\tau)$. If $\tau < \infty$, then for $t \in [\alpha,\tau)$ we have $\|w(t)\| \leq \max_{s\in[\alpha,\tau]} \|w_\varepsilon(s)\| < \infty$ so that $\|w(t)\|$ is bounded on $[\alpha,\tau)$. Again, in view of Theorem 3.6, this contradicts the maximality of the interval $[\alpha,\tau)$. Therefore the solution $w(t)$ exists on the infinite interval $[\alpha,\infty)$, and $w(t) \geq 0$ on this interval.

The following result is obtained from inequalities (7.2) and (7.3), which were used in the proof of the preceding theorem.

COROLLARY. *Let* $b(t) \geq 0$ *and* $c(t) \geq 0$ *on the interval* $[\alpha,\infty)$, *and suppose that* $w(t)$ *is the solution of the Riccati differential equation* (3.2) *satisfying the initial condition* $w(\alpha) = q > 0$. *If* (u,v) *is the solution of the linear differential system* (3.1) *satisfying* $u(\alpha) = e$, $v(\alpha) = q$, *then*

$$0 < u^{*-1}(t)qu^{-1}(t) \leq w(t) \leq v(t)q^{-1}v^{*}(t) \quad \textit{for} \quad t \in [\alpha,\infty).$$

The following result is a comparison theorem, which establishes the existence of solutions to certain Riccati differential equations.

THEOREM 7.2. *Suppose that* $b(t) \geq 0$ *on* $[\alpha,\infty)$, *and suppose that* $w(t)$ *is a symmetric solution of the Riccati differential equation* (3.2) *on the interval* $[\alpha,\infty)$. *If* $c_1(t)$ *is a continuous symmetric* $\mathfrak{B}$*-valued function satisfying* $c_1(t) \geq c(t)$ *on the interval* $[\alpha,\infty)$, *and* $w_1(t)$ *is a solution of the Riccati differential equation*

$$K_1[w_1] \equiv w_1' + w_1a(t) + a^{*}(t)w_1 + w_1b(t)w_1 - c_1(t) = 0$$

with $w_1(\alpha) > w(\alpha)$, $[w_1(\alpha) \geq w(\alpha)]$, *then* $w_1(t)$ *exists on the entire interval* $[\alpha,\infty)$ *and* $w_1(t) > w(t)$, $[w_1(t) \geq w(t)]$, *throughout this interval.*

To prove the above theorem, we consider the auxiliary Riccati equation

$$K_2[w_2] \equiv w_2' + w_2a_2(t) + a_2^{*}(t)w_2 + w_2b(t)w_2 - c_2(t) = 0,$$

where $a_2(t) = a(t) + b(t)w(t)$ and $c_2(t) = c_1(t) - c(t) \geq 0$. Theorem 7.1 guarantees the existence of a solution $w_2(t)$ of $K_2[w_2] = 0$ on $[\alpha,\infty)$ satisfying the initial condition $w_2(\alpha) = w_1(\alpha) - w(\alpha) > 0$, $[w_2(\alpha) \geq 0]$; in addition, we have that $w_2(t) > 0$, $[w_2(t) \geq 0]$, on the interval $[\alpha,\infty)$. If $w_1(t)$ is defined as $w_1(t) = w(t) + w_2(t)$, it may be verified readily that $w_1(t)$ satisfies $K_1[w_1] = 0$ on $[\alpha,\infty)$. Furthermore, $w_1(t) - w(t) = w_2(t) > 0$ for $t \in [\alpha,\infty)$, so that $w_1(t) > w(t)$, $[w_1(t) \geq w(t)]$, on this interval.

Theorem 7.1 assures the existence of solutions of the Riccati equation (3.2) on $[\alpha,\infty)$ only when $b(t) \geq 0$ and $c(t) \geq 0$. However, Theorem 7.2 can be used to demonstrate the existence of solutions on $[\alpha,\infty)$, although the function $c(t)$ may not satisfy $c(t) \geq 0$. This is illustrated in the following example. Consider the Riccati differential equation

$$w' + w^2 + \frac{1}{4t^2}e = 0 \tag{7.6}$$

on $[1,\infty)$. It may be verified easily that $w(t) = (1/2t)e$ is a solution of (7.6) on $[1,\infty)$ satisfying $w(1) = (1/2)e$. If $c_1(t)$ is a continuous symmetric $\mathcal{B}$-valued function such that $c_1(t) \geq -(1/4t^2)e$ on $[1,\infty)$, then Theorem 7.2 implies that there exists a solution of the differential equation

$$w_1' + w_1^2 - c_1(t) = 0$$

on $[1,\infty)$ satisfying any initial condition of the form $w_1(1) > (1/2)e$, $[w_1(1) \geq (1/2)e]$. Furthermore, we have that $w_1(t) > (1/2t)e$, $[w_1(t) \geq (1/2t)e]$, for $t \geq 1$.

Corresponding to well-known results for the case of finite dimensional matrix differential equations, we have the following necessary and sufficient condition for the existence of solutions of the Riccati differential system (3.2).

THEOREM 7.3. <u>Let</u> $b(t) \geq 0$ <u>on</u> $[\alpha,\infty)$ <u>and let</u> q <u>be a symmetric element of</u> $\mathcal{B}$. <u>Then there exists on</u> $[\alpha,\infty)$ <u>a symmetric solution</u> $w(t)$ <u>of the Riccati equation</u> $K[w] = 0$ <u>satisfying the initial condition</u> $w(\alpha) = q$ <u>if</u>

<u>and only if there exists a continuously differentiable symmetric function</u> $w_o(t)$, $t \in [\alpha,\infty)$, <u>such that</u> $K[w_o](t) \leq 0$ <u>on</u> $[\alpha,\infty)$ <u>and</u> $w_o(\alpha) \leq q$.

It should be noted that the hypothesis of the theorem does not require that $q \geq 0$, or that $c(t) \geq 0$ on $[\alpha,\infty)$. Clearly if $w(t)$ is a solution of (3.2) satisfying $w(\alpha) = q$, then $K[w](t) \leq 0$ for $t \in [\alpha,\infty)$, and $w(\alpha) \leq q$. Conversely, suppose that there exists a continuously differentiable function $w_o(t)$ such that

$$w_o' + w_o a(t) + a^*(t)w_o + w_o b(t)w_o - c(t) = m(t) \leq 0$$

on $[\alpha,\infty)$. If $c_o(t)$ is defined by $c_o(t) = c(t) + m(t)$, then $c(t) \geq c_o(t)$ and $w_o(t)$ is a solution of

$$w_o' + w_o a(t) + a^*(t)w_o + w_o b(t)w_o - c_o(t) = 0$$

for $t \in [\alpha,\infty)$. An application of Theorem 7.2 guarantees the existence of a solution of the Riccati differential equation

$$K[w] \equiv w' + wa(t) + a^*(t)w + wb(t)w - c(t) = 0,$$

satisfying the initial condition $w(\alpha) = q \geq w_o(\alpha)$. In addition, it follows that $w(t) \geq w_o(t)$ on $[\alpha,\infty)$.

8. <u>Nonoscillation theorems for large</u> t. System (3.1) is said to be <u>nonoscillatory for large</u> t whenever there exists a self-conjoined solution (u,v) of this system on $[\alpha,\infty)$ such that $u(t)$ is nonsingular on an infinite interval $[\beta,\infty)$ with $\beta \geq \alpha$. In the previous section it was proved that (3.1) is nonoscillatory for large t whenever $b(t) \geq 0$ and $c(t) \geq 0$ on $[\alpha,\infty)$. Theorems 8.1 and 8.2 of this section involve the cases where $b(t) > 0$ and $c(t) \leq 0$ on $[\alpha,\infty)$. These theorems are simple generalizations of nonoscillation results for differential systems in a

B^*-algebra presented by Hille [5; pp. 487-490]. The comparison theorems of Section 6 are employed in Theorem 8.3 to prove that the linear differential system is nonoscillatory for large t under certain hypothesis.

In this section we consider the linear differential system

$$(8.1)\qquad \begin{aligned} u' &= g(t)v, \\ v' &= -f(t)u, \end{aligned} \qquad t \in [\alpha,\infty),$$

where f(t) and g(t) are continuous symmetric $\mathcal{B}$-valued functions. This is a special case of system (3.1) under the identification $a(t) \equiv 0$, $b(t) = g(t)$, and $c(t) = -f(t)$ on $[\alpha,\infty)$.

If $\tau \in [\alpha,\infty)$, let h(t) be a solution of the system

$$\begin{aligned} h'(t) &= a(t)h(t), \\ h(\tau) &= e, \end{aligned} \qquad t \in [\alpha,\infty),$$

where a(t) is the coefficient function in (3.1). If (u_o,v_o) is a solution of (3.1), then it may be verified readily that

$$u(t) = h^{-1}(t)u_o(t), \quad v(t) = h^*(t)v_o(t),$$

is a solution of (8.1), where f(t), g(t) are the continuous symmetric functions defined by $f(t) = -h^*(t)c(t)h(t)$, $g(t) = h^{-1}(t)b(t)h^{*-1}(t)$. Moreover, it can be shown that (u_o,v_o) is a self-conjoined solution of system (3.1) if and only if (u,v) is a self-conjoined solution of system (8.1). Consequently, the results of this section dealing with system (8.1) may also be applied to system (3.1) after the appropriate transformation has been made.

We now have the following result for system (8.1).

THEOREM 8.1. _Let_ $f(t) \geq 0$ _and suppose that_ $0 < g(t) \leq e$ _on the interval_ $[\alpha,\infty)$ _where_ $\alpha > 0$. _If for each_ $t \in [\alpha,\infty)$ _the integral_

$$h(t) = t\int_t^\infty f(s)ds$$

exists _and_ $h(t) \leq (1/4)e$, _then_ (8.1) _is_ _nonoscillatory_ _for_ _large_ t.

The method of proof is similar to that used by Hille in the case where $g(t) \equiv e$ on $[\alpha,\infty)$. Define a sequence of $\mathcal{B}$-valued functions on $[\alpha,\infty)$ as follows:

$$\text{(8.2)} \qquad \begin{aligned} z_1(t) &= h(t), \\ z_{n+1}(t) &= t\int_t^\infty \left[s^{-2}z_n(s)g(s)z_n(s)\right]ds + h(t), \quad n = 1,2,3,\cdots. \end{aligned}$$

Let $\sigma_1 = 1/4$ and for $n \geq 1$ define σ_{n+1} by the recursive relation $\sigma_{n+1} = \sigma_n^2 + \sigma_1$. It can be shown by induction that $\{\sigma_n\}$ is a monotone non-decreasing sequence and that

$$\|z_n(t)\| \leq \sigma_n,$$

$$\|z_n(t) - z_{n-1}(t)\| \leq \sigma_n - \sigma_{n-1},$$

$$\|z_n(t)g(t)z_n(t) - z_{n-1}(t)g(t)z_{n-1}(t)\| \leq \sigma_{n+1} - \sigma_n,$$

for $n \geq 1$ and $t \in [\alpha,\infty)$. Furthermore, if m, n are positive integers with $m > n$, then $\|z_m(t) - z_n(t)\| \leq \sigma_m - \sigma_n$. It is established easily that the sequence $\{\sigma_n\}$ converges to $\sigma = 1/2$. Therefore, for each $t \in [\alpha,\infty)$ the sequence $\{z_n(t)\}$ is a Cauchy sequence, and consequently has a limit $z(t)$. Indeed, on $[\alpha,\infty)$ the sequence $\{z_n(t)\}$ is bounded and converges uniformly to $z(t)$ on this interval. By (8.2) it then follows that

$$\text{(8.3)} \qquad z(t) = t\int_t^\infty [s^{-2}z(s)g(s)z(s)]ds + h(t)$$

on $[\alpha,\infty)$. If k is the constant element defined by

$$k = \int_\alpha^\infty [s^{-2}z(s)g(s)z(s)]ds + \int_\alpha^\infty f(s)ds,$$

the equation (8.3) can be written as

$$z(t) = tk - t\int_\alpha^t [s^{-2}z(s)g(s)z(s)]ds - t\int_\alpha^t f(s)ds$$

for $t \in [\alpha,\infty)$. If $w(t)$ is defined as $t^{-1}z(t)$ on $[\alpha,\infty)$, then we have that

$$w(t) = k - \int_\alpha^t [w(s)g(s)w(s)]ds - \int_\alpha^t f(s)ds,$$

and therefore $w(t)$ satisfies the associated Riccati differential equation

$$w'(t) + w(t)g(t)w(t) + f(t) = 0$$

on $[\alpha,\infty)$ and $w(\alpha) = k$. Furthermore, the solution $w(t)$ is symmetric since each of the functions $f(t)$, $h(t)$, $z_n(t)$ and $z(t)$ is symmetric. By Theorem 3.4, there exists a self-conjoined solution (u,v) of system (8.1) with $u(t)$ nonsingular on $[\alpha,\infty)$, and hence system (8.1) is nonoscillatory for large t.

In the following nonoscillation theorem we employ the concept of limit inferior and limit superior of a symmetric $\mathcal{B}$-valued function $h(t)$, with norm $\|h(t)\|$ bounded on $[\alpha,\infty)$. If $h(t)$ is such a function, then $\liminf h(t)$ is defined as $h_* e$, where

$$h_* = \sup\{\lambda \text{ a real number} \mid \text{there exists a } \tau \in [\alpha,\infty) \text{ such that } \lambda e \le h(t) \text{ on } [\tau,\infty)\}.$$

Similarly, $\limsup h(t)$ is defined as $h^* e$, where

$$h^* = \inf\{\lambda \text{ a real number} \mid \text{there exists a } \tau \in [\alpha,\infty) \text{ such that } h(t) \le \lambda e \text{ on } [\tau,\infty)\}.$$

According to the above definitions, if

$$\limsup h(t) = h^* e, \text{ and}$$
$$\liminf h(t) = h_* e,$$

then for each $\varepsilon > 0$, there exists a $\tau \in [\alpha,\infty)$ such that

$$(h_* - \varepsilon)e \le h(t) \le (h^* + \varepsilon)e \text{ for } t \in [\tau,\infty).$$

Furthermore, the inequality $\liminf h(t) \leq (1/4)e$ means that if λ is any real number such that $\lambda e \leq h(t)$ on a subinterval $[\tau,\infty)$ of $[\alpha,\infty)$, then $\lambda \leq 1/4$.

The following theorem is presented by Hille [5; p. 487] for the case where $g(t) \equiv e$ on $[\alpha,\infty)$. The method of proof is similar to the proof of Hille's result.

THEOREM 8.2. _Let_ $g(t) \geq e$ _and_ $f(t) \geq 0$ _on_ $[\alpha,\infty)$, _where_ $\alpha > 0$. _Suppose that there exists a self-conjoined solution_ (u,v) _of system_ (8.1) _with both_ $u(t)$ _and_ $v(t)$ _nonsingular on an infinite interval_ $[\beta,\infty)$, _where_ $\beta \geq \alpha$. _Then for each_ $t \in [\alpha,\infty)$ _the improper integral_

$$\int_t^\infty f(s)ds$$

exists and

$$\left.\begin{matrix}\limsup \\ \liminf\end{matrix}\right\}_t \int_t^\infty f(s)ds \leq \begin{cases} e \\ (1/4)e. \end{cases}$$

Since (u,v) is a self-conjoined solution of (8.1) with $u(t)$ nonsingular on $[\beta,\infty)$, it follows that $w(t) = v(t)u^{-1}(t)$ is a symmetric solution of the Riccati differential equation

$$w' + wg(t)w + f(t) = 0, \quad \text{for} \quad t \in [\beta,\infty). \tag{8.4}$$

Furthermore, since by hypothesis we have that $v(t)$ is nonsingular on $[\beta,\infty)$, it follows that $w^{-1}(t) = u(t)v^{-1}(t)$ exists on $[\beta,\infty)$ and satisfies

$$(w^{-1})' = w^{-1}f(t)w^{-1} + g(t).$$

Therefore,

$$w^{-1}(t) = w^{-1}(\beta) + \int_\beta^t g(s)ds + \int_\beta^t [w^{-1}(s)f(s)w^{-1}(s)]ds$$

on $[\beta,\infty)$. Moreover, since $g(t) \geq e$ and $f(t) \geq 0$ on $[\beta,\infty)$, it follows that

$$w^{-1}(t) \geq w^{-1}(\beta) + (t-\beta)e$$

on this interval. If τ is defined by $\tau = \beta + \|w^{-1}(\beta)\|$, then

$$w^{-1}(\beta) + (\tau - \beta)e \geq 0.$$

Consequently, for $t > \tau$ it follows that

$$\begin{aligned} w^{-1}(t) &\geq w^{-1}(\beta) + (t-\beta)e \\ &\geq w^{-1}(\beta) + (\tau-\beta)e + (t-\tau)e \\ &\geq (t-\tau)e > 0. \end{aligned}$$

Therefore, we have that

$$0 < w(t) \leq (t-\tau)^{-1}e \quad \text{for} \quad t \in (\tau,\infty)$$

and hence

$$\lim_{t\to\infty} w(t) = 0. \tag{8.5}$$

Furthermore, we have

$$0 < tw(t) \leq t(t-\tau)^{-1}e$$

on (τ,∞) so that

$$\limsup tw(t) \leq e. \tag{8.6}$$

If $\beta \leq t < \gamma$, then by equation (8.4) it follows that

$$w(t) - w(\gamma) = \int_t^\gamma f(s)ds + \int_t^\gamma [w(s)g(s)w(s)]ds, \tag{8.7}$$

and consequently

$$w(t) - w(\gamma) \geq \int_t^\gamma f(s)ds \geq 0, \quad \text{for } \gamma > t. \tag{8.8}$$

If for fixed t the function $z(\gamma)$ is defined by $z(\gamma) = \int_t^\gamma f(s)ds$, then by inequality (8.8) it follows that

$$w(\gamma_1) - w(\gamma_2) \geq z(\gamma_2) - z(\gamma_1) \geq 0,$$

and consequently $\|z(\gamma_2) - z(\gamma_1)\| \le \|w(\gamma_1) - w(\gamma_2)\|$ whenever $t \le \gamma_1 \le \gamma_2$. However, in view of (8.5) we have that $w(\gamma_1)$ and $w(\gamma_2)$ converge to 0 as γ_1 and γ_2 become infinite, so that $z(\gamma)$ has a limit as γ becomes infinite; that is, the improper integral $\int_t^\infty f(s)ds = \lim_{\gamma\to\infty}\int_t^\gamma f(s)ds$ exists. Similarly, the improper integral $\int_t^\infty [w(s)g(s)w(s)]ds$ exists for each $t \in [\beta,\infty)$. Since $w(\gamma)$ approaches 0 as γ becomes infinite, it follows from equation (8.7) that

$$w(t) = \int_t^\infty f(s)ds + \int_t^\infty [w(s)g(s)w(s)]ds \text{ for } t \in [\beta,\infty).$$

If we define $h(t) = t\int_t^\infty f(s)ds$, then the above equation can be written as

$$tw(t) = t\int_t^\infty [w(s)g(s)w(s)]ds + h(t). \tag{8.9}$$

Now let h^*, h_*, u^*, and u_* be real numbers such that

$$\left.\begin{matrix}\limsup\\ \liminf\end{matrix}\right\} h(t) = \begin{cases} h^* e, \\ h_* e, \end{cases}$$

$$\left.\begin{matrix}\limsup\\ \liminf\end{matrix}\right\} tw(t) = \begin{cases} u^* e, \\ u_* e. \end{cases}$$

In view of (8.6), we know that $u^* \le 1$, and it follows from equation (8.9) that $h^* \le u^* \le 1$. Let $\varepsilon > 0$, and choose $\tau \ge \beta$ so that for $s \in [\tau,\infty)$

$$(u_* - \varepsilon)e \le sw(s) \le (u^* + \varepsilon)e,$$

$$(h_* - \varepsilon)e \le h(s) \le (h^* + \varepsilon)e,$$

where a negative lower bound is replaced by 0. Therefore for $s \ge \tau$, we have that

$$s^2 w(s)w(s) \ge (u_* - \varepsilon)^2 e.$$

Furthermore, for $s \geq \tau$ we have $w(s)g(s)w(s) \geq w(s)w(s)$ so that

$$t \int_t^\infty [w(s)g(s)w(s)]ds \geq t\int_t^\infty s^{-2}(u_* - \varepsilon)^2 e \, ds$$

$$\geq (u_* - \varepsilon)^2 e.$$

Hence from equation (8.9) it follows that

$$tw(t) \geq (u_* - \varepsilon)^2 e + (h_* - \varepsilon)^2 e$$

for $t > \tau$, and therefore

$$\lim\inf tw(t) \geq (u_* - \varepsilon)^2 e + (h_* - \varepsilon)^2 e.$$

However, $\lim\inf tw(t) = u_* e$ so that $u_* \geq (u_* - \varepsilon)^2 + (h_* - \varepsilon)^2$. Since ε was arbitrarily chosen, we have $u_* \geq u_*^2 + h_*^2$. This inequality implies that $(u_* - 1/2)^2 \leq 1/4 - h_*$; therefore, we have $h_* \leq 1/4$ and the theorem is proved.

The following theorem is a generalization of a result presented by Reid [9] for finite dimensional matrix differential systems.

THEOREM 8.3. <u>Let</u> $g(t) > 0$ <u>on</u> $[\alpha,\infty)$ <u>and suppose that there exists a continuous positive real-valued function</u> $r(t)$ <u>such that</u> $f(t) \leq r(t)e$, $g(t) \leq r(t)e$, <u>and the improper Riemann integral</u> $\int_\alpha^\infty r(s)ds$ <u>exists</u>. <u>Then system</u> (8.1) <u>is nonoscillatory for large</u> t.

Let $\tau \in [\alpha,\infty)$ be chosen so that $\int_\tau^\infty r(s)ds < \pi$. If (u_τ, v_τ) is the solution of (8.1) determined by the initial conditions $u_\tau(\tau) = 0$, $v_\tau(\tau) = e$, we will show that $u_\tau(t)$ is nonsingular on (τ,∞). Let $\beta \in (\tau,\infty)$ and consider the system

$$\begin{aligned} u'(t) &= r(t)v(t), \\ v'(t) &= -r(t)u(t), \end{aligned} \tag{8.10}$$

on $[\tau,\beta]$. Clearly,

$$u(t) = \sin(-\int_t^{\infty} r(s)ds)e,$$

$$v(t) = \cos(-\int_t^{\infty} r(s)ds)e$$

is a self-conjoined solution of system (8.10). Moreover,

$$0 < \int_t^{\infty} r(s)ds < \pi$$

for $t \in [\tau,\beta]$, so that $u(t)$ is nonsingular on $[\tau,\beta]$, and hence system (8.10) is nonoscillatory on $[\tau,\beta]$. Furthermore, we know that $0 < g(t) \leq r(t)e$ and $f(t) \leq r(t)e$ so by the Corollary to Theorem 6.6 it follows that (8.1) is nonoscillatory on $[\tau,\beta]$. Since $g(t) > 0$ on $[\tau,\beta]$, Theorem 4.1 guarantees that $u_\tau(t)$ is nonsingular on $[\tau,\beta]$. However, β was arbitrarily chosen, so that $u_\tau(t)$ must be nonsingular on (τ,∞), and hence system (8.1) is nonoscillatory for large t.

9. Comparison and oscillation theorems for large t. Hayden and Howard establish several oscillation theorems for differential systems of matrices and endomorphisms on a Banach space in [7] and [3]. In this section we present corresponding results for differential systems in a B^*-algebra.

The first result is a comparison theorem, implicitly used by Hayden and Howard in their proofs.

THEOREM 9.1. *Let* $g(t)$ *be a symmetric solution of the Riccati differential equation*

$$g'(t) = [g(t) + h(t)]f(t)[g(t) + h(t)] \tag{9.1}$$

on the interval $[\alpha,\infty)$, *where* $g(t)$, $h(t)$, *and* $f(t)$ *are continuous symmetric* $\mathcal{B}$*-valued functions. Suppose that* k *is a positive real number, and* $\phi(t)$

is a positive real-valued continuous function such that $h(t) \geq ke$, *and* $f(t) \geq \phi(t)e$ *on* $[\alpha,\infty)$. *If* ρ_o *is a real number such that* $\rho_o + k > 0$ *and* $g(\alpha) > \rho_o e$, *then the solution of the scalar differential system*

$$(9.2)\qquad \begin{aligned} \rho'(t) &= \phi(t)[\rho(t) + k]^2, \\ \rho(\alpha) &= \rho_o, \end{aligned}$$

exists on $[\alpha,\infty)$ *and is such that*

$$g(t) > \rho(t)e \quad \textit{for} \quad t \in [\alpha,\infty).$$

Since $g(t)$ and $\rho(t)$ are continuous functions, and $g(\alpha) - \rho(\alpha)e > 0$, there exists a maximal subinterval $[\alpha,\tau)$ of $[\alpha,\infty)$ with $g(t) > \rho(t)e$ on $[\alpha,\tau)$. We shall proceed to show that if $\tau < \infty$ then $g(\tau) > \rho(\tau)e$, in which case an extension argument yields a contradiction to the maximality of the interval $[\alpha,\tau)$. Since $g(t) > \rho(t)e$ and $h(t) \geq ke$ on $[\alpha,\tau)$, we have $g(t) + h(t) \geq [\rho(t) + k]e$. Furthermore, the solution $\rho(t)$ of system (9.2) is nondecreasing, so that $\rho(t) + k \geq \rho(\alpha) + k > 0$. Therefore, by Theorem 2.3, it follows that

$$[g(t) + h(t)]^2 \geq [\rho(t) + k]^2 e \quad \text{on} \quad [\alpha,\tau).$$

Moreover, $f(t) \geq \phi(t)e > 0$, so that for $t \in [\alpha,\tau)$ we have

$$\begin{aligned} [g(t) + h(t)]f(t)[g(t) + h(t)] &\geq \phi(t)[g(t) + h(t)]^2 \\ &\geq \phi(t)[\rho(t) + k]^2 e. \end{aligned}$$

Hence for each $t \in [\alpha,\tau)$ we have that $g'(t) \geq \rho'(t)e$, and consequently,

$$g(\tau) - g(\alpha) = \int_\alpha^\tau g'(s)ds \geq \int_\alpha^\tau \rho'(s)ds = [\rho(\tau) - \rho(\alpha)]e.$$

Therefore, it follows that

$$g(\tau) - \rho(\tau)e \geq g(\alpha) - \rho(\alpha)e > 0,$$

so that $g(\tau) > \rho(\tau)e$.

It may be verified easily that the solution of the scalar differential system (9.2) is given by

$$\rho(t) = [(\rho_o + k)^{-1} - \int_\alpha^t \phi(s)ds]^{-1} - k \tag{9.3}$$

on the interval where $\int_\alpha^t \phi(s)ds < (\rho_o + k)^{-1}$. Therefore, if there exists a t_o such that $\int_\alpha^{t_o} \phi(s)ds = (\rho_o + k)^{-1}$, then the solution $\rho(t)$ of system (9.2) becomes unbounded as t approaches t_o; consequently, the maximal right-hand interval of existence of the solution g(t) of (9.1) cannot exceed $[\alpha, t_o)$.

We shall say that system (3.1) is <u>oscillatory for large</u> t if this system fails to be nonoscillatory for large t. In this connection it is to be noted that the term "oscillatory" has been used in different contexts for even the case of finite dimensional matrix differential equations and for scalar differential equations of higher order. The choice of terminology employed here is related to the presence of singularities in u(t) for an arbitrary self-conjoined solution (u(t),v(t)) of the linear differential system on an infinite interval.

In Theorem 9.2 and its corollaries we will be interested in functions satisfying the following condition. <u>A symmetric $\mathcal{B}$-valued function</u> n(t) <u>is said to have property</u> (D) <u>on</u> $[\gamma,\infty)$ <u>if for each real number</u> k <u>there exists a</u> $\tau \in [\gamma,\infty)$ <u>such that</u> $n(t) \geq ke$ <u>on</u> $[\tau,\infty)$. It is to be noted that this condition might be expressed as $\lim\inf n(t) = (+\infty)e$.

THEOREM 9.2. <u>Let</u> $b(t) > 0$, <u>and suppose that</u> a(t) <u>is a continuous symmetric function which commutes with</u> b(t) <u>for each</u> $t \in [\gamma,\infty)$. <u>Furthermore, suppose that there exists a positive real-valued differentiable</u>

function $\lambda(t)$ *such that for*

$$(9.4) \qquad m(t) = [(1/2)\lambda'(t)e - \lambda(t)a(t)]b^{-1}(t)$$

and

$$(9.5) \qquad n(t) = -\int_{\gamma}^{t} [\lambda(s)c(s) + m(s)\lambda^{-1}(s)b(s)m(s)]ds + m(t),$$

the function $n(t)$ *has property* (D). *Finally, suppose that there exists a positive continuous real-valued function* $\phi(t)$ *such that*

$$(9.6) \qquad \lambda^{-1}(t)b(t) \geq \phi(t)e, \quad \text{for} \quad t \in [\gamma,\infty),$$

and

$$(9.7) \qquad \int_{\gamma}^{t} \phi(s)ds \to \infty \quad \text{as} \quad t \to \infty.$$

Then system (3.1) *is oscillatory for large* t.

Suppose that there exists a self-conjoined solution (u,v) of (3.1) such that $u(t)$ is nonsingular on $[\tau,\infty)$ where $\tau \geq \gamma$. It then follows that $w(t)$ defined as $w(t) = v(t)u^{-1}(t)$ satisfies the Riccati differential equation

$$(9.8) \qquad w' + wa + aw + wbw - c = 0$$

on the interval $[\tau,\infty)$. If $q(t)$ is defined as $q(t) = \lambda(t)w(t)$ on $[\tau,\infty)$, then it follows from equation (9.8) that

$$(9.9) \qquad \lambda^{-1}q' = \lambda'\lambda^{-2}q - \lambda^{-1}qa - \lambda^{-1}aq + \lambda^{-2}qbq - c.$$

Equation (9.9) can be written as

$$\lambda^{-1}q' = [q + m]\lambda^{-2}b[q + m] - c - m\lambda^{-2}bm,$$

where $m(t)$ is defined in (9.4), and hence

$$(9.10) \qquad q' = [q + m]\lambda^{-1}b[q + m] - \lambda c - m\lambda^{-1}bm$$

on $[\tau,\infty)$. If $g(t)$ is defined on $[\tau,\infty)$ as

$$g(t) = q(t) + \int_{\gamma}^{t} [\lambda c + m\lambda^{-1}bm]ds,$$

then the derivative $g'(t)$ exists and

$$g' = q' + \lambda c + m\lambda^{-1}bm.$$

It follows that equation (9.10) can be written as

$$g' = [g + n]\lambda^{-1}b[g + n], \tag{9.11}$$

where $n(t)$ is defined in (9.5).

Let ρ_o be a real number such that $g(\tau) > \rho_o e$; it is clear from equation (9.11) that $g(t)$ is nondecreasing, so that $g(t) > \rho_o e$ for each $t \in [\tau,\infty)$. Now let k be a positive real number such that $k + \rho_o > 0$. Since $n(t)$ satisfies property (D), there exists an $\alpha \in [\tau,\infty)$ such that $n(t) \geq ke$ for $t \geq \alpha$; moreover, for this α we have that $g(\alpha) > \rho_o e$. By hypothesis there exists a function $\phi(t)$ such that $\lambda^{-1}(t)b(t) \geq \phi(t)e > 0$. With the aid of Theorem 9.1 we have that $g(t) > \rho(t)e$ on $[\alpha,\infty)$, where $\rho(t)$ is defined in equation (9.3). However, by (9.7) there exists a $t_o \in [\alpha,\infty)$ such that $\int_{\alpha}^{t_o} \phi(s)ds = (\rho_o + k)^{-1}$. Consequently, as t approaches t_o the function $\rho(t)$ becomes infinite. This contradicts the possibility that $g(t)$ exists on $[\alpha,\infty)$; furthermore, neither $q(t)$ nor $w(t) = v(t)u^{-1}(t)$ can exist on the infinite interval $[\alpha,\infty)$. Therefore system (3.1) must be oscillatory for large t.

The above result corresponds to Theorem 5 in [7]; the following two corollaries correspond to Theorem 3 and Theorem 2, respectively, in the same article.

COROLLARY 1. Let $b(t) > 0$ and $a(t) \equiv 0$ on $[\gamma,\infty)$. Suppose that there exists a positive real-valued differentiable function $\lambda(t)$ such that

$$n(t) = -\int_{\gamma}^{t} [\lambda(s)c(s) + (1/4)(\lambda'(s))^2\lambda^{-1}(s)b^{-1}(s)]ds + (1/2)\lambda'(t)b^{-1}(t)$$

has property (D) on $[\gamma,\infty)$. Furthermore, suppose that there exists a positive continuous real-valued function $\phi(t)$ such that relations (9.6) and (9.7) hold. Then system (3.1) is oscillatory for large t.

The following result is a special case of Corollary 1, where $b(t) \equiv e$ and $\phi(t)$ may be defined as $\lambda^{-1}(t)$.

COROLLARY 2. Suppose that there exists a positive real-valued differentiable function $\lambda(t)$ such that

$$n(t) = -\int_{\gamma}^{t} [\lambda(s)c(s) + (1/4)(\lambda'(s))^2\lambda^{-1}(s)e]ds + (1/2)\lambda'(t)e$$

has property (D), and that

$$\int_{\gamma}^{t} \lambda^{-1}(s) \to \infty \text{ as } t \to \infty.$$

Then the linear differential system

$$u''(t) + c(t)u(t) = 0$$

is oscillatory for large t on the interval $[\gamma,\infty)$.

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