ON A THEORY OF DISTRIBUTIONS FOR
ULTRAHYPERBOLIC EQUATIONS

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PREFACE

During the past few years Professor E. W. Titt and his students have been concerned with the problem of developing a method for deriving integration formulae for all types of linear second order partial differential equations. In this work they have developed a theory of distributions for normal hyperbolic equations which promises to be valuable in obtaining general integration formulae. This paper is an extension of this theory of distributions to apply to ultrahyperbolic equations. Specifically this paper is an extension of portions of three papers by Professor Titt and others. These papers are listed in the bibliography as numbers 3, 5 and 6.

My thanks are due to Professors R. B. Deal and E. W. Titt who served as my advisers during the preparation of this paper and to Professor L. W. Johnson for his sound counsel and kind interest given me in all matters pertaining to my work. Indebtedness is also acknowledged to the Office of Naval Research and the National Science Foundation for financial support of this work.

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I. INTRODUCTION

This paper is concerned with the general linear second order partial differential equation with constant coefficients. If one utilizes the summation convention, as is done extensively in this paper, this equation and its homogeneous adjoint can be written

$$(1.1)\qquad \begin{aligned} &\text{(a)}\quad L(u) \equiv A^{ij}u_{ij} + B^{i}u_{i} + Cu = f(x^{i}) \\ &\text{(b)}\quad L^{*}(v) \equiv A^{ij}v_{ij} - B^{i}v_{i} + Cv = 0 \qquad (i, j = 1, \ldots, n) \end{aligned}$$

where a subscript i indicates differentiation with respect to the variable x^i. There is no restriction in assuming the matrix A^{ij} to be symmetric. In order to restrict the discussion to non-parabolic equations the matrix A^{ij} is assumed to be non-singular.

In order to classify the differential equation (1.1a) one considers the characteristic form $A^{ij}y_i y_j$. By a real non-singular linear transformation, $\eta_i = a_i^j y_j$, it is possible to resolve the characteristic form into the canonical form,

$$(1.2)\qquad A^{ij}y_i y_j = (\eta_1^2 + \eta_2^2 + \ldots + \eta_p^2 - \eta_{p+1}^2 - \eta_{p+2}^2 - \ldots - \eta_n^2).$$

For definiteness it is assumed in this paper that $p \leq n/2$. For convenience in writing, n-p is replaced by q so that in what follows p and q are subject to the restrictions

(1.3) $$p+q = n \text{ and } p \leq q.$$

The non-parabolic equations are classified according to the value of p. If $p = 0$ then L(u) is elliptic. If $p = 1$ then L(u) is normal hyperbolic. If $p \geq 2$ then L(u) is ultrahyperbolic.

This paper is part of an attempt to develop Green's [1] ideas into a method for treating all types of linear second order partial differential equations.[1] In any attempt to apply Green's ideas to linear partial differential equations difficulty with divergent integrals seems to be inherent. In treating the normal hyperbolic equation in three dimensions, Volterra [7] escapes this difficulty by first distributing the Green potential along a line. Partly because of the obscurity of the origin of Volterra's distributions, Hadamard [2] uses the Green potential as his integrating factor but he is forced to develop a calculus for dealing with the finite part of a divergent integral. The school of thought represented by this paper is related to the ideas of Volterra and Hadamard in that it incorporates ideas from both.

In treating the normal hyperbolic equation Professor E. W. Titt and his students have developed chains of distributions which include not only quantities with line discontinuities employed by the Volterra school but also quantities with finite jump discontinuities at a non-characteristic hypersurface. The origin of these distributions lies in a weighted non-Euclidean area of the hypersurface, the weight factor being a retarded potential distinct from the Green potential. The analytical treatment of this weighted area is facilitated by the

[1] Numbers in brackets refer to the bibliography at the end of the paper.

reduction of the (n-1)-tuple integral to a single integral. This single integral is in the nature of a transform of the original potential, the kernel of which varies in analytical form with the dimension. This single integral has the property that n-2 applications of a differential operator consisting of certain terms of the adjoint operator changes it into the first or second type Green potential depending on whether the number of dimensions is even or odd. This property enables one to set up a chain of distributions which links the Green potential to the transform of the retarded potential. In the derivation of integration formulae the starting point is the same as Hadamard's except that it is not necessary to invoke an abstract theory of the finite part. The procedure is to delete the cone from the region of integration with an approximating quadric and then to use integration by parts to prepare the equation for passage to the limit. The above-mentioned chain of distributions is used in the integration by parts. These distributions are obtained by differentiation processes and so are used in the opposite order in the integration.

The problem considered in this paper is the extension of this theory of distributions to apply to ultrahyperbolic equations. The completion of this program would lead to a theory applicable to all types of non-parabolic equations with constant coefficients.

II. VECTOR SYSTEMS FOR ARBITRARY SUBSPACES

To construct a theory of distributions applicable to ultrahyperbolic equations it is first necessary to extend certain portions of the paper [5]. Specifically it is necessary to define and develop some of the properties of vector systems for arbitrary subspaces. The dimensionality of the subspaces is dictated by the metric defined by the coefficients of the second order terms of the differential equation. The letters p and q are used to denote the dimensionality of these subspaces and are determined by the type of equation as in (1.2). After defining the basic system and developing some of its properties it is necessary to define a non-parabolic, non-degenerate vector system and to derive rules of indices and Pythagorean identities for this system.

A basic vector system deals with two sets of n vectors each in n dimensions. In conformity with the tensor notation one of the sets, consisting of covariant vectors, is designated by L_i^a and ℓ_i^α where $i = 1, \ldots, n$; $a = 1, \ldots, p$ and $\alpha = 1, \ldots, q$. The other set, consisting of contravariant vectors, is designated by L_a^i and ℓ_α^i. In working with these vectors the indices i, j, k, m have a range $1, \ldots, n$; the indices a, b, c have a range $1, \ldots, p$; and the indices α, β, γ have a range $1, \ldots, q$. The integers p and q are subject to the restrictions (1.3). When and only when one of these indices appears in a term as a subscript and a superscript it is understood that this index is summed over the appropriate range.

A set of 2n vectors L_i^a, ℓ_i^α and L_a^i, ℓ_α^i constitutes a basic vector system when the vectors satisfy the conditions given in (2.1).

(2.1)

(a) $L_a^i L_i^b = \delta_a^b$

(b) $\ell_\alpha^i \ell_i^\beta = \delta_\alpha^\beta$

(c) $L_a^i \ell_i^\alpha = 0$

(d) $L_i^a \ell_\alpha^i = 0$

(e) Either $\begin{vmatrix} L_i^1 \\ \vdots \\ L_i^p \\ \ell_i^1 \\ \vdots \\ \ell_i^q \end{vmatrix}$ or $\begin{vmatrix} L_1^i \\ \vdots \\ L_p^i \\ \ell_1^i \\ \vdots \\ \ell_q^i \end{vmatrix}$ has the value unity.

In the above column notation for determinants the index i indicates the column. This definition is actually just a renaming of the basic algebra defined in [5].

As a consequence of the conditions (2.1) one has that both determinants in (2.1e) are unity as is demonstrated below.

(2.2)
$$\begin{vmatrix} L_i^1 \\ \vdots \\ L_i^p \\ \ell_i^1 \\ \vdots \\ \ell_i^q \end{vmatrix} \begin{vmatrix} L_1^i \\ \vdots \\ L_p^i \\ \ell_1^i \\ \vdots \\ \ell_q^i \end{vmatrix} = \begin{vmatrix} L_i^1 L_1^i & \ldots & L_i^1 L_p^i & L_i^1 \ell_1^i & \ldots & L_i^1 \ell_q^i \\ \vdots & & \vdots & \vdots & & \vdots \\ L_i^p L_1^i & \ldots & L_i^p L_p^i & L_i^p \ell_1^i & \ldots & L_i^p \ell_q^i \\ \ell_i^1 L_1^i & \ldots & \ell_i^1 L_p^i & \ell_i^1 \ell_1^i & \ldots & \ell_i^1 \ell_q^i \\ \vdots & & \vdots & \vdots & & \vdots \\ \ell_i^q L_1^i & \ldots & \ell_i^q L_p^i & \ell_i^q \ell_1^i & \ldots & \ell_i^q \ell_q^i \end{vmatrix} = \begin{vmatrix} 1 \ldots 0 & 0 \ldots 0 \\ \vdots & \vdots \\ 0 \ldots 1 & 0 \ldots 0 \\ 0 \ldots 0 & 1 \ldots 0 \\ \vdots & \vdots \\ 0 \ldots 0 & 0 \ldots 1 \end{vmatrix} = 1$$

Since the product of the two determinants is unity and one of them is unity by hypothesis, the value of the other must also be unity.

Another property of a basic system is that each of the covariant vectors is the cross product vector for n-1 contravariant vectors and similarly for the contravariant vectors.

$$(2.3)\quad L^a_i = \begin{vmatrix} L^j_1 \\ \vdots \\ L^j_{a-1} \\ \delta^j_i \\ L^j_{a+1} \\ \vdots \\ L^j_p \\ \ell^j_1 \\ \vdots \\ \ell^j_q \end{vmatrix}; \quad \ell^\alpha_i = \begin{vmatrix} L^j_1 \\ \vdots \\ L^j_p \\ \ell^j_1 \\ \vdots \\ \ell^j_{\alpha-1} \\ \delta^j_i \\ \ell^j_{\alpha+1} \\ \vdots \\ \ell^j_q \end{vmatrix}; \quad L_a^i = \begin{vmatrix} L^1_j \\ \vdots \\ L^{a-1}_j \\ \delta^i_j \\ L^{a+1}_j \\ \vdots \\ L^p_j \\ \ell^1_j \\ \vdots \\ \ell^q_j \end{vmatrix}; \quad \ell_\alpha^i = \begin{vmatrix} L^1_j \\ \vdots \\ L^p_j \\ \ell^1_j \\ \vdots \\ \ell^{\alpha-1}_j \\ \delta^i_j \\ \ell^{\alpha+1}_j \\ \vdots \\ \ell^q_j \end{vmatrix}.$$

The proof of (2.3) is exactly the same as the proof of the corresponding theorem in [5].

The column rule of indices completes the properties of the basic system.

$$(2.4)\qquad L^a_i L^j_a + \ell^\alpha_i \ell^j_\alpha = \delta^j_i.$$

To establish this, one notices that by (2.3) L^j_a is the cofactor of L^a_j in the determinant of the covariant vectors and ℓ^j_α is the cofactor of

ℓ^{a}_{j} in the same determinant. Hence $L^{a}_{i}L^{j}_{a} + \ell^{\alpha}_{i}\ell^{j}_{\alpha}$ is the sum of the products of the elements of the i-th column by the cofactors of the corresponding elements of the j-th column in this determinant. Since the value of this determinant is one, (2.4) follows.

In a restricted vector system one deals with a symmetric matrix A^{ij} as well as a basic system of 2n vectors. If the matrix A^{ij} is non-singular one can form the matrix A_{ij} of normalized cofactors so that

$$A^{jk}A_{ij} = \delta^{k}_{i}. \tag{2.6}$$

The matrix A^{ij} determines two positive integers p and q as in (1.2) and (1.3). Consider any set of p vectors M^{a}_{i}. If this set of vectors satisfies the condition

$$|A^{ij}M^{a}_{i}M^{b}_{j}| \neq 0 \tag{2.7}$$

then it is said to be a non-degenerate set.

A non-parabolic, non-degenerate vector system is a restricted vector system in which A^{ij} is non-singular, the L^{a}_{i} are non-degenerate and condition (2.8) is satisfied.

$$A^{ij}L^{a}_{j} = g^{ab}L^{i}_{b} \tag{2.8}$$

where g^{ab} is defined by

$$g^{ab} = A^{ij}L^{a}_{i}L^{b}_{j}. \tag{2.9}$$

The rules of indices for a non-parabolic, non-degenerate system are derived next. Since the matrix g^{ab} is non-singular, its inverse g_{ab} exists and satisfies the condition

$$g^{bc}g_{ab} = \delta_a^c. \tag{2.10}$$

Multiplying (2.10) by $A_{ik}L_c^i$ and using (2.8) and (2.6) one obtains the rule for lowering indices on L_a^i.

$$A_{ik}L_a^i = g_{ab}L_k^b. \tag{2.11}$$

Multiplying (2.11) by L_c^k and using (2.1a) one obtains

$$g_{ab} = A_{ij}L_a^iL_b^j. \tag{2.12}$$

In order to complete the rules of indices one defines the quantities $h^{\alpha\beta}$ and $h_{\alpha\beta}$ as follows:

$$\text{(a)}\quad h^{\alpha\beta} = A^{ij}\ell_i^\alpha\ell_j^\beta \qquad\qquad \text{(b)}\quad h_{\alpha\beta} = A_{ij}\ell_\alpha^i\ell_\beta^j. \tag{2.13}$$

If one multiplies (2.13a) by (2.13b), sums on a Greek index and then uses (2.4), (2.6), (2.8), (2.11) and (2.1) there results

$$h^{\alpha\beta}h_{\beta\gamma} = \delta_\gamma^\alpha. \tag{2.14}$$

Introducing the factor ℓ_β^k or ℓ_k^β into (2.13a) or (2.13b) respectively and using (2.4), (2.8), (2.11) and (2.1) one obtains

$$(2.15) \qquad \text{(a)} \quad A^{ik}\ell^{\alpha}_{i} = h^{\alpha\beta}\ell^{k}_{\beta}$$

$$\text{or} \quad \text{(b)} \quad A_{ik}\ell^{i}_{\alpha} = h_{\alpha\beta}\ell^{\beta}_{k}.$$

The relations (2.15), (2.11) and (2.8) constitute the A rules of indices for a non-parabolic, non-degenerate system.

In order to derive Pythagorean identities for the non-parabolic, non-degenerate system the following definitions are introduced.

$$(2.16) \qquad \text{(a)} \quad G^{ij} = g^{ab}L^{i}_{a}L^{j}_{b}; \qquad \text{(b)} \quad G_{ij} = g_{ab}L^{a}_{i}L^{b}_{j};$$

$$\text{(c)} \quad H^{ij} = h^{\alpha\beta}\ell^{i}_{\alpha}\ell^{j}_{\beta}; \qquad \text{(d)} \quad H_{ij} = h_{\alpha\beta}\ell^{\alpha}_{i}\ell^{\beta}_{j}.$$

Starting from the definition of H_{ij} and using various rules of indices as follows

$$(2.17) \qquad \begin{aligned} H_{ij} &= h_{\alpha\beta}\ell^{\alpha}_{i}\ell^{\beta}_{j} \\ &= A_{ik}\ell^{k}_{\beta}\ell^{\beta}_{j} \\ &= A_{ik}(\delta^{k}_{j}-L^{k}_{a}L^{a}_{j}) \\ &= A_{ij}-A_{ik}L^{k}_{a}L^{a}_{j} \\ &= A_{ij}-g_{ab}L^{b}_{i}L^{a}_{j} \\ &= A_{ij}-G_{ij} \end{aligned}$$

one finds

$$A_{ij} = H_{ij} + G_{ij}. \tag{2.18}$$

In a similar fashion one obtains

$$A^{ij} = H^{ij} + G^{ij}. \tag{2.19}$$

Introducing the factor $X_i X_j$ into (2.19) and the factor $X^i X^j$ into (2.18) one obtains the Pythagorean identities,

$$\begin{aligned} &\text{(a)} \quad A^{ij} X_i X_j = H^{ij} X_i X_j + G^{ij} X_i X_j \text{ and} \\ &\text{(b)} \quad A_{ij} X^i X^j = H_{ij} X^i X^j + G_{ij} X^i X^j. \end{aligned} \tag{2.20}$$

The H and G rules of indices given below follow directly from the definitions (2.16) and the basic conditions (2.1).

$$\begin{aligned} &\text{(a)} \quad H^{ij} L^a_j = 0; &&\text{(b)} \quad H_{ij} L^j_a = 0; \\ &\text{(c)} \quad H^{ij} \ell^\alpha_j = h^{\alpha\beta} \ell^j_\beta; &&\text{(d)} \quad H_{ij} \ell^j_\alpha = h_{\alpha\beta} \ell^\beta_i; \\ &\text{(e)} \quad G^{ij} \ell^\alpha_j = 0; &&\text{(f)} \quad G_{ij} \ell^j_\alpha = 0; \\ &\text{(g)} \quad G^{ij} L^a_j = g^{ab} L^i_b; &&\text{(h)} \quad G_{ij} L^j_a = g_{ab} L^b_i. \end{aligned} \tag{2.21}$$

By calculations similar to the following,

$$\begin{aligned} A^{ij} H_{ki} &= A^{ij} h_{\alpha\beta} \ell^\alpha_k \ell^\beta_i \\ &= h_{\alpha\beta} h^{\beta\gamma} \ell^\alpha_k \ell^j_\gamma \\ &= \ell^\gamma_k \ell^j_\gamma \end{aligned} \tag{2.22}$$

(2.22) continued

$$= \delta^j_k - L^j_a L^a_k$$

one obtains the following relations between A and H and A and G.

(2.23)

$$\text{(a)} \quad A^{ij}H_{ki} = A_{ki}H^{ij} = \delta^j_k - L^j_a L^a_k$$

$$\text{(b)} \quad A^{ij}G_{ki} = A_{ki}G^{ij} = \delta^j_k - \ell^j_\alpha \ell^\alpha_k$$

$$\text{(c)} \quad A^{ij}H_{ij} = q$$

$$\text{(d)} \quad A^{ij}G_{ij} = p$$

Using (2.16d), (2.15a) and (2.21f) one obtains the following relation between A, H, and G.

(2.24)

$$A^{ij}H_{ik}G_{jm} = 0$$

This completes the properties of a non-parabolic, non-degenerate vector system as far as the needs of this paper are concerned.

To show that it is possible to construct a non-parabolic, non-degenerate vector system when given the matrix A^{ij} one could consider the following example. Since A^{ij} is a non-singular symmetric matrix there exists a proper orthogonal matrix C^j_i where i indicates row and j indicates column such that

(2.25)

$$A^{ij}C^k_i C^m_j = \rho^k \delta^{km}$$

where ρ^k are the characteristic roots of A^{ij}. For the vectors L^a_i one could choose any p columns of the matrix C^j_i and for the vectors ℓ^α_i one could choose the remaining q columns of C^j_i.

$$(2.26)\quad \begin{aligned} &(a)\quad L^a_i = C^{j_a}_i \text{ where } j_a \text{ assumes any p values from 1 to n,} \\ &(b)\quad \ell^\alpha_i = C^{j_\alpha}_i \text{ where } j_\alpha \neq j_a. \end{aligned}$$

Then

$$(2.27)\quad \begin{aligned} &(a)\quad A^{ij}L^a_iL^b_j \neq 0 \text{ when } a = b \text{ and} \\ &\qquad\qquad = 0 \text{ when } a \neq b, \\ &(b)\quad A^{ij}\ell^\alpha_i\ell^\beta_j \neq 0 \text{ when } \alpha = \beta \text{ and} \\ &\qquad\qquad = 0 \text{ when } \alpha \neq \beta, \\ &(c)\quad A^{ij}L^a_i\ell^\alpha_j = 0. \end{aligned}$$

Having the covariant vectors one could determine the matrices g^{ab} and $h^{\alpha\beta}$ and then determine the matrices g_{ab} and $h_{\alpha\beta}$ as their inverses. Then the contravariant vectors could be defined by the following relations,

$$(2.28)\quad \begin{aligned} &(a)\quad L^i_a = g_{ab}A^{ij}L^b_j \text{ and} \\ &(b)\quad \ell^i_\alpha = h_{\alpha\beta}A^{ij}\ell^\beta_j. \end{aligned}$$

To show that this system of 2n vectors is a non-parabolic, non-degenerate system one must show that the basic conditions (2.1) are satisfied, that g^{ab} is non-singular and that (2.8) is satisfied. With this choice of L^a_i and ℓ^α_i condition (2.1e) is satisfied since C^j_i is a proper orthogonal matrix. Further the matrices g^{ab} and $h^{\alpha\beta}$ are non-singular so one can determine their inverses g_{ab} and $h_{\alpha\beta}$. Having these matrices one can determine the L^i_a and ℓ^i_α by (2.28). With the L^i_a and ℓ^i_α determined in this fashion the first four of conditions (2.1) are satisfied as is demonstrated below.

(2.29)

$$\text{(a)}\quad L_a^i L_i^b = g_{ac} A^{ij} L_j^c L_i^b = g_{ac} g^{cb} = \delta_a^b.$$

$$\text{(b)}\quad \ell_\alpha^i \ell_i^\beta = h_{\alpha\gamma} A^{ij} \ell_j^\gamma \ell_i^\beta = h_{\alpha\gamma} h^{\gamma\beta} = \delta_\alpha^\beta.$$

$$\text{(c)}\quad L_a^i \ell_i^\alpha = g_{ab} A^{ij} L_j^b \ell_i^\alpha = 0.$$

$$\text{(d)}\quad L_i^a \ell_\alpha^i = h_{\alpha\beta} A^{ij} L_i^a \ell_j^\beta = 0.$$

The last two are zero by (2.27c). Condition (2.8) is satisfied as is demonstrated below.

(2.30)
$$g^{ab} L_b^i = g^{ab} g_{bc} A^{ij} L_j^c = \delta_c^a A^{ij} L_j^c = A^{ij} L_j^a.$$

As a consequence of this particular example one sees that it is possible to choose the vector system so that the quadratic forms $g^{ab} y_a y_b$ and $h^{\alpha\beta} \eta_\alpha \eta_\beta$ are definite. This could be accomplished by choosing the L_i^a as the columns of C_i^j corresponding to the p positive characteristic roots of A^{ij} and choosing for the ℓ_i^α the columns of C_i^j corresponding to the q negative characteristic roots of A^{ij}.

III. POTENTIALS

In general, potentials are solutions of the adjoint equation (1.1b) which are functions of differences $X^i = x^i - \xi^i$ where ξ^i is any point interior to the region where the value of u is desired. In order to construct the potentials desired here, one selects a non-parabolic, non-degenerate vector system using the matrix A^{ij} of the coefficients of the second order terms in the differential equation (1.1a). The vector system chosen is such that the matrix $h_{\alpha\beta}$ is given by

$$(3.1) \qquad h_{\alpha\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

To obtain a vector system in which condition (3.1) is satisfied one could proceed in the following manner. Since the matrix A^{ij} is non-singular one could determine its inverse A_{ij}. From the matrix A_{ij} one could determine an orthogonal matrix C^i_j where i indicates row and j indicates column such that

$$(3.2) \qquad A_{ij} C^i_k C^j_m = \rho_k \delta_{km}$$

where the ρ_k are the characteristic roots of A_{ij}. Now p of these characteristic roots are positive and q of them are negative. Consider any one of the positive roots and any q-1 of the negative roots. Denote the positive root by ρ_{k_1} and the q-1 negative roots by $\rho_{k_2}, \rho_{k_3}, \ldots, \rho_{k_q}$. Define the quantities D^i_α as follows:

(3.3)
$$D^i_1 = \frac{C^i_{k_1}}{\sqrt{\rho_{k_1}}}$$
$$D^i_\alpha = \frac{C^i_{k_\alpha}}{\sqrt{-\rho_{k_\alpha}}} \qquad \text{for } 1 < \alpha \leq q.$$

Then

(3.4)
$$A_{ij}D^i_\alpha D^j_\beta = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix},$$

but the product

(3.5)
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \ldots & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \ldots & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

is equal to the matrix desired for $h_{\alpha\beta}$ in (3.1). Hence one could define the ℓ^i_α as follows,

$$(3.6)\qquad \ell^i_\alpha = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} D^i_1 \\ D^i_2 \\ D^i_3 \\ \vdots \\ D^i_q \end{bmatrix} = \begin{bmatrix} \frac{C^i_{k_1}}{\sqrt{2\rho_{k_1}}} - \frac{C^i_{k_2}}{\sqrt{-2\rho_{k_2}}} \\ \frac{C^i_{k_1}}{\sqrt{2\rho_{k_1}}} + \frac{C^i_{k_2}}{\sqrt{-2\rho_{k_2}}} \\ \frac{C^i_{k_3}}{\sqrt{-\rho_{k_3}}} \\ \vdots \\ \frac{C^i_{k_q}}{\sqrt{-\rho_{k_q}}} \end{bmatrix}$$

and condition (3.1) would be satisfied. The L^i_a could be chosen as the remaining p columns of the matrix C^i_j after adjusting the magnitude to make the determinant of the L^i_a and ℓ^i_α equal to one. The covariant vectors could be defined in terms of these contravariant vectors as follows:

$$(3.7)\qquad \begin{aligned} L^a_i &= g^{ab}A_{ij}L^j_b \\ \ell^\alpha_i &= h^{\alpha\beta}A_{ij}\ell^j_\beta. \end{aligned}$$

The proof that a vector system chosen in this fashion is a non-parabolic, non-degenerate system is similar to the proof given in section II that the example given there is a non-parabolic, non-degenerate system.

The quantities R, S, and T, related to the terms of the Pythagorean identity (2.20b), are defined by

$$R = \sqrt{j A_{ij} X^i X^j}$$

(3.8)
$$S = \sqrt{\hat{j} H_{ij} X^i X^j}$$

$$T = \sqrt{\bar{j}\, G_{ij} X^i X^j}$$

where j, $\hat{j}$ and $\bar{j}$ are subject to $j^2 = \hat{j}^2 = \bar{j}^2 = 1$ and are chosen to make R, S and T real. The value of j depends on the position of the point x^i relative to the characteristic cone with vertex at ξ^i. In a similar fashion the values of $\hat{j}$ and $\bar{j}$ depend on the position of the point x^i relative to the cylinders $H_{ij} X^i X^j = 0$ and $G_{ij} X^i X^j = 0$. In this paper the values of x^i considered are such that $j = \hat{j} = -\bar{j} = 1$ so that the Pythagorean identity (2.20b) becomes

(3.9)
$$R^2 = S^2 - T^2.$$

The Green potentials are those solutions of (1.1b) which are exponentially damped functions of R alone. The retarded potentials are solutions of (1.1b) which are exponentially damped functions of R and S. Since the potentials are functions of differences and are exponentially damped one tries for a solution of (1.1b) of the form

(3.10)
$$v = e^{-\alpha_i X^i}\,\Phi(X^i).$$

For a function of the form (3.10) equation (1.1b) becomes

(3.11)
$$e^{-\alpha_i X^i}\left[A^{ij}(\alpha_i \alpha_j \Phi - \alpha_j \Phi_i - \alpha_i \Phi_j + \Phi_{ij}) - B^i(-\alpha_i \Phi + \Phi_i) + C\Phi\right] = 0.$$

Equation (3.11) can be written

$$(3.12) \qquad A^{ij}\Phi_{ij} + \Phi_i(-2A^{ij}\alpha_j - B^i) + \Phi(A^{ij}\alpha_i\alpha_j + B^i\alpha_i + C) = 0.$$

In order to simplify (3.12) one defines constants α_i and δ as follows:

$$(3.13) \qquad \begin{aligned} \alpha_i &= -\frac{1}{2} A_{ij}B^j, \\ \delta^2 &= J(A^{ij}\alpha_i\alpha_j + B^i\alpha_i + C) = J(-\frac{1}{4} A_{ij}B^iB^j + C). \end{aligned}$$

The symbol J is subject to $J^2 = 1$ and is chosen to make δ real. With these constants (3.12) becomes

$$(3.14) \qquad A^{ij}\Phi_{ij} + J\delta^2\Phi = 0.$$

For a function $\Phi(R, S, T)$ equation (3.14) becomes

$$(3.15) \qquad \begin{aligned} &\Phi_{RR}A^{ij}R_iR_j + \Phi_{SS}A^{ij}S_iS_j + \Phi_{TT}A^{ij}T_iT_j + 2\Phi_{RS}A^{ij}R_iS_j + 2\Phi_{RT}A^{ij}R_iT_j \\ &+ 2\Phi_{ST}A^{ij}S_iT_j + \Phi_R A^{ij}R_{ij} + \Phi_S A^{ij}S_{ij} + \Phi_T A^{ij}T_{ij} + J\delta^2\Phi = 0. \end{aligned}$$

Using the definitions of R, S and T and various rules of indices from section II one computes the sums appearing in (3.15). Differentiating the expressions for R^2, S^2 and T^2 from (3.8) with respect to x^i one obtains

$$(3.16) \qquad \begin{aligned} &\text{(a)} \quad RR_i = A_{ij}X^j \\ &\text{(b)} \quad SS_i = H_{ij}X^j \\ &\text{(c)} \quad TT_i = -G_{ij}X^j. \end{aligned}$$

Using (3.16) and rules of indices one obtains

(3.17)

$$\text{(a)}\quad A^{ij}R_iR_j = R^{-2}A^{ij}A_{ik}X^kA_{jm}X^m = 1 \qquad (2.6)$$

$$\text{(b)}\quad A^{ij}S_iS_j = S^{-2}A^{ij}H_{ik}X^kH_{jm}X^m = 1 \qquad (2.23a \text{ and } 2.21a)$$

$$\text{(c)}\quad A^{ij}T_iT_j = T^{-2}A^{ij}G_{ik}X^kG_{jm}X^m = -1 \qquad (2.23b \text{ and } 2.21b)$$

$$\text{(d)}\quad A^{ij}R_iS_j = (RS)^{-1}A^{ij}A_{ik}X^kH_{jm}X^m = \frac{S}{R} \qquad (2.6)$$

$$\text{(e)}\quad A^{ij}R_iT_j = -(RT)^{-1}A^{ij}A_{ik}X^kG_{jm}X^m = -\frac{T}{R} \qquad (2.6)$$

$$\text{(f)}\quad A^{ij}S_iT_j = -(ST)^{-1}A^{ij}H_{ik}X^kG_{jm}X^m = 0. \qquad (2.24)$$

To compute the sums involving second derivatives one uses the results on first derivatives. Differentiating the relations (3.16) with respect to x^j one obtains the following:

(3.18)

$$\text{(a)}\quad RR_{ij} + R_iR_j = A_{ij}$$

$$\text{(b)}\quad SS_{ij} + S_iS_j = H_{ij}$$

$$\text{(c)}\quad TT_{ij} + T_iT_j = -G_{ij}.$$

Using the relations in (3.18), (3.17) and rules of indices one obtains the following results for second derivatives.

(3.19)

$$\text{(a)}\quad A^{ij}R_{ij} = \frac{n-1}{R}$$

$$\text{(b)}\quad A^{ij}S_{ij} = \frac{q-1}{R}$$

(3.19) continued

$$\text{(c)} \quad A^{ij}T_{ij} = -\frac{p-1}{R}.$$

If one substitutes the results from (3.17) and (3.19) into equation (3.15) it becomes

(3.20)

$$\Phi_{RR} + \Phi_{SS} - \Phi_{TT} + 2\frac{S}{R}\Phi_{RS} + 2\frac{T}{R}\Phi_{RT} + \frac{n-1}{R}\Phi_R + \frac{q-1}{S}\Phi_S - \frac{p-1}{T}\Phi_T + J\delta^2\Phi = 0.$$

In the solutions designated as the Green potentials and the retarded potentials, the function Φ has the form

(3.21) $$\Phi = S^{\mu}F(R).$$

Since Φ does not involve T explicitly Φ_{TT}, Φ_{RT} and Φ_T do not occur when one uses this form for Φ in (3.20). Substituting the value for Φ given in (3.21) into (3.20) one obtains

(3.22) $$S^{\mu}F'' + \mu(\mu-1)S^{\mu-2}F + 2\frac{S^{\mu}}{R}\mu F' + \frac{n-1}{R}S^{\mu}F' + (q-1)\mu S^{\mu-2}F + J\delta^2 F = 0$$

where the primes indicate differentiation with respect to R. Using the second and fifth terms of the left member of (3.22) to determine μ one has the following equation for μ.

(3.23) $$\mu(\mu-1) + (q-1)\mu = 0$$

from which $\mu = 0$ or $\mu = 2-q$.

After using the Φ_S and Φ_{SS} terms to determine μ one is left with an ordinary differential equation for F as follows:

$$F'' + \frac{2\mu + n - 1}{R} F' + J\delta^2 F = 0. \tag{3.24}$$

In treating the differential equation,

$$F'' + \frac{k}{R} F' + J\delta^2 F = 0, \tag{3.25}$$

if one makes the substitution

$$F(R) = (\delta R)^{(1-k)/2} Z(\delta R) = \rho^{(1-k)/2} Z(\rho) \tag{3.26}$$

then (3.25) becomes

$$\rho^{-(k+1)/2} \left\{ \rho Z'' + Z' + Z\left[J\rho - \left(\frac{1-k}{2}\right)^2 \frac{1}{\rho} \right] \right\} = 0 \tag{3.27}$$

where the primes indicate differentiation with respect to ρ. From equation (3.27) one sees that Z is a Bessel function of order $(1-k)/2$. Hence $F(R) = (\delta R)^{(1-k)/2} Z_{(1-k)/2}(\delta R)$. The Bessel function is oscillating or non-oscillating according as $J = 1$ or $J = -1$ where J is determined in (3.13).

Comparing equations (3.24) and (3.25) one sees that $k = 2\mu + n - 1$. Corresponding to each of the two values of μ from (3.23) one has a value of k. If $\mu = 0$ then $k = n-1$ and $(1-k)/2 = (2-n)/2$. If $\mu = 2-q$ then $k = n-2q+3$ and $(1-k)/2 = (2q-n-2)/2$. For these values of k one has

$$\text{(3.28)} \quad \begin{aligned} &\text{(a)} \quad F(R) = (\delta R)^{-(n-2)/2} Z_{(n-2)/2}(\delta R) \\ &\text{(b)} \quad F(R) = (\delta R)^{(2q-n-2)/2} Z_{(2q-n-2)/2}(\delta R). \end{aligned}$$

The Green potentials are obtained by using (3.28a), which corresponds to $\mu = 0$, in (3.10). The retarded potentials are obtained by using (3.28b), which corresponds to $\mu = 2-q$, in (3.10). With these values of F and μ (3.10) becomes

$$\text{(3.29)} \quad \begin{aligned} &\text{(a)} \quad v = e^{-\alpha_i X^i} \frac{Z_{(n-2)/2}(\delta R)}{(\delta R)^{(n-2)/2}}, && \text{Green;} \\ &\text{(b)} \quad v = e^{-\alpha_i X^i} \frac{(\delta R)^{(2q-n-2)/2} Z_{(2q-n-2)/2}(\delta R)}{S^{q-2}}, && \text{retarded.} \end{aligned}$$

The Green potentials are classified as first or second type according as the Bessel function used is first or second type. When m is an odd integer the symbol $Y_{m/2}$ means $J_{-m/2}$. The function Φ for the Green potentials for the oscillating case is given below.

$$\begin{array}{lll} & \text{First type} & \text{Second type} \\ \text{(3.30)} & \left(\frac{1}{\delta R}\right)^{(n-2)/2} J_{(n-2)/2}(\delta R), & \left(\frac{\delta}{R}\right)^{(n-2)/2} Y_{(n-2)/2}(\delta R). \end{array}$$

The retarded potentials from (3.29b) are classified as first or second type retarded potentials in the same manner as the Green potentials except for the first three values of n for each value of p. The function Φ for the retarded potentials for the oscillating case is given in (3.31).

$$\begin{array}{lll} & & \text{First type} \\ & n = 2p & S^{-(q-2)}\left(\frac{\delta}{R}\right) Y_1(\delta R) \\ \text{(3.31)} & & \\ & n = 2p+1 & S^{-(q-2)} \left(\frac{\delta}{R}\right)^{1/2} Y_{1/2}(\delta R) \end{array}$$

(3.31) continued

First type (continued)

$n = 2p+2 \qquad S^{-(q-2)}\left[Y_0(\delta R) - \ln \delta J_0(\delta R)\right]$

$n \geq 2p+3 \qquad S^{-(q-2)}\left(\frac{R}{\delta}\right)^{(2q-n-2)/2} J_{(2q-n-2)/2}(\delta R)$

Second type

$n = 2p \qquad S^{-(q-2)}\left(\frac{1}{\delta R}\right) J_1(\delta R)$

$n = 2p+1 \qquad S^{-(q-2)}\left(\frac{1}{\delta R}\right)^{1/2} J_{1/2}(\delta R)$

$n = 2p+2 \qquad S^{-(q-2)} J_0(\delta R)$

$n \geq 2p+3 \qquad S^{-(q-2)}(\delta R)^{(2q-n-2)/2} Y_{(2q-n-2)/2}(\delta R)$

When $q = n-p = 2$, which is the case for normal hyperbolic $n = 3$ and ultrahyperbolic $n = 4$, $p = 2$, the two solutions in (3.29) are the same. For the desired results in this paper for $n = 4$, $p = 2$ the second type retarded potential is used; however, it may be interesting to note that for this case if one looks for a solution of the form $\Phi = F(R) \ln S + G(R)$ as is done in [5] one obtains $\Phi = \frac{1}{R^2} \ln \frac{R^2}{S}$ when $\delta = 0$. This is similar to the quantity $\frac{1}{R} \ln \frac{R^2}{S}$ used by Volterra for $n = 3$; normal hyperbolic and region of integration exterior to the cone.

The following list of values of Φ for first type retarded potentials with $\delta = 0$ for small values of n and p may be of interest.

n	p = 1	p = 2	p = 3	p = 4	p = 5
2	$\frac{S}{R^2}$				
3	$\frac{1}{R} \ln \frac{R^2}{S}$				
4	$\frac{\ln R}{S}$	$\frac{1}{R^2} \ln \frac{R^2}{S}$			
5	$\frac{R}{S^2}$	$\frac{1}{RS}$			
6	$\frac{R^2}{S^3}$	$\frac{\ln R}{S^2}$	$\frac{1}{R^2 S}$		
7	$\frac{R^3}{S^4}$	$\frac{R}{S^3}$	$\frac{1}{RS^2}$		
8	$\frac{R^4}{S^5}$	$\frac{R^2}{S^4}$	$\frac{\ln R}{S^3}$	$\frac{1}{R^2 S^2}$	
9	$\frac{R^5}{S^6}$	$\frac{R^3}{S^5}$	$\frac{R}{S^4}$	$\frac{1}{RS^3}$	
10	$\frac{R^6}{S^7}$	$\frac{R^4}{S^6}$	$\frac{R^2}{S^5}$	$\frac{\ln R}{S^4}$	$\frac{1}{R^2 S^3}$

IV. DISTRIBUTION OF POTENTIALS

This section is concerned with the problem of integrating a retarded potential over a portion of a q-dimensional subspace. The q-dimensional subspace used is the subspace whose equations are

$$L^a_i(\bar{x}^i - \xi^i) = 0 \tag{4.1}$$

where the L^a_i are from the non-parabolic, non-degenerate vector system used in the construction of the potential. The $\bar{x}^i$ are the current coordinates. The portion of this q-dimensional subspace used as the region of integration is that part of (4.1) cut out by the two cones

$$(4.2)\qquad \begin{aligned} &\text{(a)}\quad A_{ij}(\bar{x}^i - \xi^i)(\bar{x}^j - \xi^j) = 0 \text{ and} \\ &\text{(b)}\quad A_{ij}(\bar{x}^i - x^i)(\bar{x}^j - x^j) = 0. \end{aligned}$$

This region of integration is designated by Q_q.

Except for the cases $n = 2p$ and $n = 2p+2$ the potential which is distributed over the q-dimensional subspace is the first type retarded potential without exponential damping. In these exceptional cases the potential which is distributed over the q-dimensional subspace is the second type retarded potential without exponential damping. In all cases the potentials are given by

$$\Phi = S^{-(q-2)}\left(\frac{R}{\delta}\right)^{\frac{2q-n-2}{2}} J_{\frac{2q-n-2}{2}}(\delta R). \tag{4.3}$$

For the case $n = 2p$ one must multiply the second type retarded potential given in (3. 31) by $-\delta^2$ to put it in the form given in (4.3)

The q-tuple integral is a weighted non-Euclidean area of the region Q_q; the weight factor being the potential (4.3). In the integrand the variables $x^i - \xi^i$ in Φ are replaced by $x^i - \bar{x}^i$ so that the weighted area is given by

$$\bar{I} = \iint_{Q_q} \cdots \int \Phi(x^i - \bar{x}^i) dA. \tag{4.4}$$

In order to explain the parametric representation used in the region Q_q the quantities

(4.5)

$$\text{(a)} \quad \lambda^\alpha = \ell^\alpha_i (x^i - \bar{x}^i),$$

$$\text{(b)} \quad \mu = \sqrt{H_{ij}(x^i - \bar{x}^i)(x^j - \bar{x}^j)} \qquad \text{and}$$

$$\text{(c)} \quad t^\alpha = \ell^\alpha_i (x^i - \xi^i)$$

are introduced. From (2.16d) $H_{ij} = h_{\alpha\beta}\ell^\alpha_i \ell^\beta_j$, hence $H_{ij}(x^i - \bar{x}^i)(x^j - \bar{x}^j)$ $= h_{\alpha\beta}\ell^\alpha_i(x^i - \bar{x}^i)\ell^\beta_j(x^j - \bar{x}^j)$. If one uses (3.1) this becomes

$$\mu^2 = 2\lambda_1\lambda_2 - \lambda_3^2 - \lambda_4^2 - \cdots - \lambda_q^2 \tag{4.6}$$

where for convenience in writing, subscripts have been used on the λ's. This practice of using subscripts is followed in the succeeding work on both the λ's and the t's. It can be shown that the λ^α suffice for a parametric representation in Q_q, however, for parameters in the q-dimensional subspace the quantities $\lambda_2, \lambda_3, \ldots, \lambda_q$ and μ are used.

Since the region of integration is bounded by the cones (4.2) it is necessary to convert the equations of these loci into conditions on the parameters. If one uses (2.20b), (2.16b), (4.5) and (3.8) the two equations (4.2) yield

$$(4.7)\quad \begin{aligned}&\text{(a)}\quad \mu^2+S^2-2t_1\lambda_2-2t_2\lambda_1+2t_3\lambda_3+2t_4\lambda_4+\ldots+2t_q\lambda_q=0 \quad\text{and}\\ &\text{(b)}\quad \mu=T.\end{aligned}$$

Using (4.6) to eliminate λ_1 from (4.7a) and then multiplying the result by $t_2\lambda_2$ one has

$$(4.8)\quad \begin{aligned}-t_2^2\lambda_3^2-t_2^2\lambda_4^2-\ldots-t_2^2\lambda_q^2+2t_2t_3\lambda_2\lambda_3+2t_2t_4\lambda_2\lambda_4+\ldots+2t_2t_q\lambda_2\lambda_q\\ +(\mu^2+S^2)t_2\lambda_2-t_2^2\mu^2-2t_1t_2\lambda_2^2=0.\end{aligned}$$

If one completes the squares in the terms of the first line of (4.8) and makes use of the relation $S^2=2t_1t_2-t_3^2-t_4^2-\ldots-t_q^2$ equation (4.8) becomes

$$(4.9)\quad \begin{aligned}-S^2\lambda_2^2+t_2(\mu^2+S^2)\lambda_2-t_2^2\mu^2-(t_2\lambda_3-\lambda_2t_3)^2\\ -(t_2\lambda_4-\lambda_2t_4)^2-\ldots-(t_2\lambda_q-\lambda_2t_q)^2=0.\end{aligned}$$

Thus the region of integration is bounded by (4.9) and $\mu=T$.

In order to express (4.4) as a repeated integral it is necessary to determine the limits of integration for the parameters. The limits for λ_q are found by solving (4.9) for λ_q obtaining

$$(4.10)\quad \lambda_q = \frac{1}{t_2}\left\{\lambda_2 t_q \pm \left[-S^2\lambda_2^2 + t_2(\mu^2+S^2)\lambda_2 - t_2^2\mu^2 - (t_2\lambda_3-\lambda_2 t_3)^2 - (t_2\lambda_4-\lambda_2 t_4)^2 - \ldots - (t_2\lambda_{q-1}-\lambda_2 t_{q-1})^2\right]^{\frac{1}{2}}\right\}.$$

To determine the limits for λ_{q-1} one equates the discriminant in (4.10) to zero and solves for λ_{q-1}. This process is repeated to determine the limits for λ_r from the limits for λ_{r+1}. One obtains

$$\text{(a)}\quad \lambda_r = \frac{1}{t_2}\left\{\lambda_2 t_r \pm \left[-S^2\lambda_2^2 + t_2(\mu^2+S^2)\lambda_2 - t_2^2\mu^2 - (t_2\lambda_3-\lambda_2 t_3)^2 - (t_2\lambda_4-\lambda_2 t_4)^2 - \ldots - (t_2\lambda_{r-1}-\lambda_2 t_{r-1})^2\right]^{\frac{1}{2}}\right\} \text{ for } r > 3,$$

$$(4.11)\quad \text{(b)}\quad \lambda_3 = \frac{1}{t_2}\left\{\lambda_2 t_3 \pm \left[-S^2\lambda_2^2 + t_2(\mu^2+S^2)\lambda_2 - t_2^2\mu^2\right]^{\frac{1}{2}}\right\} \quad \text{and}$$

$$\text{(c)}\quad \lambda_2 = t_2\frac{\mu^2}{S^2} \text{ and } \lambda_2 = t_2.$$

The other limit for μ is obtained from the intersection of $\lambda_2 = t_2\frac{\mu^2}{S^2}$ and $\lambda_2 = t_2$. This yields $\mu = S$, hence the limits for μ are

$$(4.12)\qquad \mu = T \text{ and } \mu = S.$$

The non-Euclidean element of q-dimensional area is given by

$$(4.13)\qquad dA = \sqrt{\pm\left|h_{\alpha\beta}\right|}\;\left|h_\beta^\alpha\right|\; d\nu^1 d\nu^2 \ldots d\nu^q$$

where the ν^α are the parameters and $h_\beta^\alpha = \ell_i^\alpha \frac{\partial \overline{x}^i}{\partial \nu^\beta}$ [4]. Using (4.5a) and (4.6) one finds

(4.14) $$\left|h^{\alpha}_{\beta}\right| = \begin{vmatrix} \frac{\partial\lambda_1}{\partial\lambda_2} & \frac{\partial\lambda_1}{\partial\lambda_3} & \cdots & \frac{\partial\lambda_1}{\partial\lambda_q} & \frac{\mu}{\lambda_2} \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & -1 & 0 \end{vmatrix} = \frac{\mu}{\lambda_2}.$$

From (3.1) one sees that $\left|h_{\alpha\beta}\right|$ is ± 1, hence (4.13) is

(4.15) $$dA = \frac{\mu}{\lambda_2} d\lambda_q d\lambda_{q-1} \ldots d\lambda_3 d\lambda_2 d\mu.$$

Using (4.5b), (3.8), (2.16b) and (4.1) the potential Φ becomes

(4.16) $$\Phi(x^i - \bar{x}^i) = \mu^{-(q-2)} \left(\frac{\sqrt{\mu^2 - T^2}}{\delta}\right)^{\frac{2q-n-2}{2}} J_{\frac{2q-n-2}{2}}(\delta\sqrt{\mu^2 - T^2})$$
$$= \Phi(\mu, T).$$

Collecting the results from (4.10), (4.11), (4.12), (4.15) and (4.16) the integral (4.4) becomes

(4.17) $$\mathbb{I} = \int_T^S \int_{t_2 \frac{\mu^2}{S^2}}^{t_2} \int_{\bar{\lambda}_3}^{\bar{\bar{\lambda}}_3} \cdots \int_{\bar{\lambda}_q}^{\bar{\bar{\lambda}}_q} \Phi(\mu, T) \frac{\mu}{\lambda_2} d\lambda_q \ldots d\lambda_3 d\lambda_2 d\mu$$

where $\bar{\lambda}_r$ and $\bar{\bar{\lambda}}_r$ are used to denote the two branches of the surfaces (4.10) and (4.11a).

The analytical treatment of the integral (4.17) is facilitated by reducing it to a single integral. This is done in the succeeding work.

If one performs the λ_q integration (4.17) becomes

$$(4.18)\quad \mathbb{T} = 2\int\int\ldots\int\frac{\Phi\mu}{\lambda_2 t_2}\Big[-S^2\lambda_2^2+t_2(\mu^2+S^2)\lambda_2-t_2^2\mu^2-(t_2\lambda_3-\lambda_2 t_3)^2 -(t_2\lambda_4-\lambda_2 t_4)^2-\ldots-(t_2\lambda_{q-1}-\lambda_2 t_{q-1})^2\Big]^{\frac{1}{2}}d\lambda_{q-1}\ldots d\lambda_2 d\mu.$$

If one performs the λ_{q-1} integration (4.18) becomes

$$(4.19)\quad \mathbb{T} = \frac{\pi}{1!}\int\int\ldots\int\frac{\Phi\mu}{\lambda_2 t_2^2}\Big[-S^2\lambda_2^2+t_2(\mu^2+S^2)\lambda_2-t_2^2\mu^2-(t_2\lambda_3-\lambda_2 t_3)^2 -(t_2\lambda_4-\lambda_2 t_4)^2-\ldots-(t_2\lambda_{q-2}-\lambda_2 t_{q-2})^2\Big]\,d\lambda_{q-2}\ldots d\lambda_2 d\mu.$$

With each integration one loses the last term in the brackets, increases the exponent of the factor in the brackets by $\frac{1}{2}$, increases the exponent of t_2 in the denominator by 1, and multiplies the integral by a constant factor. After r integrations one has for $q-r \geq 3$

$$(4.20)\quad \mathbb{T} = C\int\int\ldots\int\frac{\Phi\mu}{\lambda_2 t_2^r}\Big[-S^2\lambda_2^2+t_2(\mu^2+S^2)\lambda_2-t_2^2\mu^2-(t_2\lambda_3-\lambda_2 t_3)^2 -(t_2\lambda_4-\lambda_2 t_4)^2-\ldots-(t_2\lambda_{q-r}-\lambda_2 t_{q-r})^2\Big]^{\frac{r}{2}}d\lambda_{q-r}\ldots d\lambda_2 d\mu$$

where $C = \dfrac{2^r\left[\frac{r-1}{2}\right]!\,\pi^{\frac{r-1}{2}}}{r!}$ when r is odd and $C = \dfrac{\pi^{\frac{r}{2}}}{\left[\frac{r}{2}\right]!}$ when r is even.

After q-2 integrations one has

$$(4.21)\quad \mathbb{T} = C\int_T^S\int_{t_2\frac{\mu^2}{S^2}}^{t_2}\frac{\Phi\mu}{\lambda_2 t_2^{q-2}}\Big[-S^2\lambda_2^2+t_2(\mu^2+S^2)\lambda_2-t_2^2\mu^2\Big]^{\frac{q-2}{2}}d\lambda_2 d\mu$$

where C is given by the expressions following (4.20) with $r = q-2$.

When q is odd one has after the λ_2 integration

$$(4.22)\quad \mathrm{I} = \frac{2^{q-2}\left[\frac{q-3}{2}\right]!\;\pi^{\frac{q-1}{2}}}{(q-2)!}\int_T^S \Phi\mu\Bigg\{(-1)^{\frac{q+1}{2}}\mu^{q-2}+\frac{S^2+\mu^2}{2S}\Bigg[(-1)^{\frac{q-1}{2}}\mu^{q-3}$$

$$+(-1)^{\frac{q-3}{2}}\mu^{q-5}\left(\frac{S^2-\mu^2}{2S}\right)^2\frac{1}{2}+(-1)^{\frac{q-5}{2}}\mu^{q-7}\left(\frac{S^2-\mu^2}{2S}\right)^4\frac{3.1}{4.2}+\dots$$

$$+\mu^2\left(\frac{S^2-\mu^2}{2S}\right)^{q-5}\frac{(q-6)(q-8)\dots3.1}{(q-5)(q-7)\dots4.2}-\left(\frac{S^2-\mu^2}{2S}\right)^{q-3}\frac{(q-4)(q-6)\dots3.1}{(q-3)(q-5)\dots4.2}\Bigg]\Bigg\}d\mu.$$

When q is even one has after the λ_2 integration

$$(4.23)\quad \mathrm{I} = \frac{2\pi^{\frac{q-2}{2}}}{\left[\frac{q-2}{2}\right]!}\int_T^S \Phi\mu\Bigg\{(-1)^{\frac{q-2}{2}}\mu^{q-2}\ln\frac{S}{\mu}+\frac{S^2+\mu^2}{2S}\Bigg[(-1)^{\frac{q-4}{2}}\mu^{q-4}\left(\frac{S^2-\mu^2}{2S}\right)$$

$$+(-1)^{\frac{q-6}{2}}\mu^{q-6}\left(\frac{S^2-\mu^2}{2S}\right)^3\frac{2}{3}+(-1)^{\frac{q-8}{2}}\mu^{q-8}\left(\frac{S^2-\mu^2}{2S}\right)^5\frac{4.2}{5.3}+\dots$$

$$-\mu^2\left(\frac{S^2-\mu^2}{2S}\right)^{q-5}\frac{(q-6)(q-8)\dots4.2}{(q-5)(q-7)\dots5.3}+\left(\frac{S^2-\mu^2}{2S}\right)^{q-3}\frac{(q-4)(q-6)\dots4.2}{(q-3)(q-5)\dots5.3}\Bigg]\Bigg\}d\mu.$$

In both (4.22) and (4.23) the agreement is that the sum in the braces terminates with the term in which the exponent of μ is zero. Equations (4.22) and (4.23) express the integral (4.4) as a single integral which is the desired result in this section.

V. THE KERNEL

Equations (4.22) and (4.23) express the integral (4.4) as a single integral which is a transform of the potential Φ. The kernel of the transform is given below in (5.1). The sum in the braces terminates as before with the term in which the exponent of μ is zero. The constant C' is the same as the constant before the integral in the corresponding case.

(a) q odd

$$K = C'\mu\Bigg\{(-1)^{\frac{q+1}{2}}\mu^{q-2}+\frac{S^2+\mu^2}{2S}\Bigg[(-1)^{\frac{q-1}{2}}\mu^{q-3}+(-1)^{\frac{q-3}{2}}\mu^{q-5}\left(\frac{S^2-\mu^2}{2S}\right)^2\frac{1}{2}$$

$$+(-1)^{\frac{q-5}{2}}\mu^{q-7}\left(\frac{S^2-\mu^2}{2S}\right)^4\frac{3.1}{4.2}+\ldots+\mu^2\left(\frac{S^2-\mu^2}{2S}\right)^{q-5}\frac{(q-6)(q-8)\ldots 3.1}{(q-5)(q-7)\ldots 4.2}$$

(5.1)

$$-\left(\frac{S^2-\mu^2}{2S}\right)^{q-3}\frac{(q-4)(q-6)\ldots 3.1}{(q-3)(q-5)\ldots 4.2}\Bigg]\Bigg\}$$

(b) q even

$$K = C'\mu\Bigg\{(-1)^{\frac{q-2}{2}}\mu^{q-2}\ln\frac{S}{\mu}+\frac{S^2+\mu^2}{2S}\Bigg[(-1)^{\frac{q-4}{2}}\mu^{q-4}\left(\frac{S^2-\mu^2}{2S}\right)$$

$$+(-1)^{\frac{q-6}{2}}\mu^{q-6}\left(\frac{S^2-\mu^2}{2S}\right)^3\frac{2}{3}+(-1)^{\frac{q-8}{2}}\mu^{q-8}\left(\frac{S^2-\mu^2}{2S}\right)^5\frac{4.2}{5.3}+\ldots$$

$$-\mu^2\left(\frac{S^2-\mu^2}{2S}\right)^{q-5}\frac{(q-6)(q-8)\ldots 4.2}{(q-5)(q-7)\ldots 5.3}+\left(\frac{S^2-\mu^2}{2S}\right)^{q-3}\frac{(q-4)(q-6)\ldots 4.2}{(q-3)(q-5)\ldots 5.3}\Bigg]\Bigg\}$$

In order to obtain a more compact form for the kernel, one first makes the transformation given below.

(5.2) $$S = e^t, \qquad \mu = e^a,$$

(5.2) continued

$$\frac{S^2+\mu^2}{2S\mu} = \cosh(t-a), \qquad \frac{S^2-\mu^2}{2S\mu} = \sinh(t-a).$$

After making this substitution in (5.1), if one differentiates the resulting expressions with respect to t, one finds

(5.3)

$$K_t = \frac{-2\pi^{\frac{q-1}{2}}}{\left[\frac{q-3}{2}\right]!} e^{(q-1)a} \sinh^{q-2}(t-a) \quad \text{for } q \text{ odd and}$$

$$K_t = \frac{2^{q-2}\left[\frac{q-4}{2}\right]!\,\pi^{\frac{q-2}{2}}}{(q-3)!} e^{(q-1)a} \sinh^{q-2}(t-a) \quad \text{for } q \text{ even.}$$

Transforming back to the original variables S and μ and using the fact that $K_S = K_t t_S = \frac{K_t}{S}$ one obtains

(5.4)
$$K_S = C'' \frac{\mu}{S}\left(\frac{S^2-\mu^2}{S}\right)^{q-2}$$

where $C'' = \frac{-\pi^{\frac{q-1}{2}}}{2^{q-3}\left[\frac{q-3}{2}\right]!}$ when q is odd and $C'' = \frac{\left[\frac{q-4}{2}\right]!\,\pi^{\frac{q-2}{2}}}{(q-3)!}$ when q is even. Solving the differential equation (5.4) subject to the condition $K = 0$ when $S = \mu$ one has the following expression for K after discarding a constant factor.

(5.5)
$$K(S,\mu) = \int_{\mu}^{S} \frac{\mu}{\theta}\left(\frac{\theta^2-\mu^2}{\theta}\right)^{q-2} d\theta.$$

The chain of distributions developed in this paper is based on the integral

(5.6)
$$I = e^{-\alpha_i X^i} \int_T^S K(S,\mu)\, \Phi(\mu, T)\, d\mu$$

where $K(S, \mu)$ is given by (5.5) and $\Phi(\mu, T)$ is given by (4.16). This integral is essentially the integral $\overline{I}$ of (4.22) and (4.23) multiplied by the exponential factor $e^{-\alpha_i X^i}$. For such an exponentially damped function of S and T the adjoint operator takes the form

$$(5.7) \quad L^*I = L^*\left[e^{-\alpha_i X^i}\overline{I}(S, T)\right]$$

$$= \left[A^{ij}(D_i + \alpha_i)(D_j + \alpha_j) + J\delta^2\right]\left[e^{-\alpha_i X^i}\overline{I}(S, T)\right]$$

$$= e^{-\alpha_i X^i}\left[D_{SS} + \frac{q-1}{S}D_S - D_{TT} - \frac{p-1}{T}D_T + J\delta^2\right]\overline{I}(S, T).$$

The operator Δ used in the construction of the chain of distributions consists of certain terms of the adjoint operator as follows:

$$(5.8) \qquad \Delta e^{-\alpha_i X^i}F(S) = H^{ij}(D_i + \alpha_i)(D_j + \alpha_j)\left[e^{-\alpha_i X^i}F(S)\right]$$

$$= e^{-\alpha_i X^i}\left[D_{SS} + \frac{q-1}{S}D_S\right]F(S)$$

$$= e^{-\alpha_i X^i}\overline{\Delta}F(S)$$

where $\overline{\Delta} = D_{SS} + \frac{q-1}{S}D_S$.

Before proceeding to the discussion of the chain of distributions it is necessary to develop some of the properties of the kernel K. First it should be pointed out that except for a constant factor the kernel K reduces to the kernel given by Kainen [3] when q is replaced by n-1 which is the case for the normal hyperbolic equation. The kernel K satisfies the conditions

(5.9) $\bar{\Delta}K = 2(q-2)\frac{\mu}{S^{q-2}}(S^2-\mu^2)^{q-3}$ and $K = K_S = 0$ when $\mu = S$ for $q > 2$;

$\bar{\Delta}K = 0$ and $K = 0$, $K_S = 1$ when $\mu = S$ for $q = 2$.

If one replaces q by n-1 in (5.9) there results

$$\bar{\Delta}K = 2(n-3)\frac{\mu}{S^{n-3}}(S^2-\mu^2)^{n-4} \tag{5.10}$$

and this is the same as the expression given by Kainen except for the factor 2(n-3).

In order to compute higher orders of $\bar{\Delta}^m K$ one makes the exponential change of variable given by (5.2). The operator $\bar{\Delta}$ becomes

$$D_{SS}+\frac{q-1}{S}D_S = e^{-2t}\left[D_{tt}+(q-2)D_t\right] \tag{5.11}$$

and $\bar{\Delta}K$ in (5.9) for $q > 2$ becomes

$$\bar{\Delta}K = 2(q-2)(2e^a)^{q-3}e^{-(t-a)}\sinh^{q-3}(t-a). \tag{5.12}$$

For reference the following rules for operators are given.

(5.13)
(a) $(D\pm\gamma)\sinh^\gamma(t-a) = \gamma e^{\pm(t-a)}\sinh^{\gamma-1}(t-a)$,

(b) $(D^2-c^2)\sinh^\gamma(t-a) = (\gamma^2-c^2)\sinh^\gamma(t-a)+\gamma(\gamma-1)\sinh^{\gamma-2}(t-a)$.

Using (5.13a) and well known rules for D operators one calculates $\bar{\Delta}^2K$ as follows:

$$\bar{\Delta}^2K = e^{-2t}D(D+q-2)2(q-2)(2e^a)^{q-3}e^{-(t-a)}\sinh^{q-3}(t-a) \tag{5.14}$$

(5.14) continued

$$= 2(q-2)(2e^a)^{q-3}e^{-3t+a}(D-1)(D+q-3)\sinh^{q-3}(t-a)$$

$$= 2(q-2)(2e^a)^{q-3}e^{-3t+a}(D-1)(q-3)e^{t-a}\sinh^{q-4}(t-a)$$

$$= 2(q-2)(q-3)(2e^a)^{q-3}e^{-2t}D\sinh^{q-4}(t-a).$$

In a similar fashion one calculates

$$\bar{\Delta}^3K = 2(q-2)(q-3)(q-4)(2e^a)^{q-3}e^{-3t-a}(D-1)(D+1)\sinh^{q-5}(t-a),$$

(5.15)

$$\bar{\Delta}^4K = 2(q-2)(q-3)(q-4)(q-5)(2e^a)^{q-3}e^{-4t-2a}(D-2)D(D+2)\sinh^{q-6}(t-a).$$

By an induction process one obtains the following expression for $\bar{\Delta}^mK$.

(5.16)

$$\bar{\Delta}^mK = 2(q-2)(q-3)\ldots(q-1-m)(2e^a)^{q-3}e^{-mt-(m-2)a}f(D)\sinh^{q-2-m}(t-a)$$

where $f(D) = D[D^2-2^2][D^2-4^2]\ldots[D^2-(m-2)^2]$ for m even and greater than 2 and $f(D) = [D^2-1^2][D^2-3^2]\ldots[D^2-(m-2)^2]$ for m odd and greater than 1.

One observes in (5.16) that when $m = q-1$ the expression for $\bar{\Delta}^mK$ has a zero factor and so is zero. One can sharpen this statement when q is even; in this case $\bar{\Delta}^{q/2}K = 0$. In verifying this, one notices that when $m = q/2$ (5.16) contains the factor

$$(5.17)\qquad \left[D^2-\left(\frac{q-4}{2}\right)^2\right]\sinh^{\frac{q-4}{2}}(t-a).$$

If one uses formula (5.13b) to evaluate (5.17) one has $\gamma = c$ in the formula, hence the effect of performing the operation indicated in (5.17) is simply to bring in a constant factor and decrease the exponent of sinh(t-a) by 2. After performing this operation one has the factor

$$(5.18)\qquad \left[D^2-\left(\frac{q-8}{2}\right)^2\right]\sinh^{\frac{q-8}{2}}(t-a)$$

in (5.16). This is the same type of operation as (5.17), hence successive applications of (5.13b) lead either to the factor $D\left[\left(\frac{q-4}{2}\right)!\right]$ when q/2 is even or to the factor $\left[D^2-1^2\right]\sinh(t-a)$ when q/2 is odd. In view of the preceding argument one has the following results:

$$(5.19)\qquad \begin{aligned} &\bar{\Delta}^m K = 0 \text{ for } m \geq q-1 \text{ when } q \text{ is odd and} \\ &\bar{\Delta}^m K = 0 \text{ for } m \geq q/2 \text{ when } q \text{ is even.} \end{aligned}$$

If one collects factors in (5.16) it becomes

$$(5.20)\qquad \bar{\Delta}^m K = \frac{2^{q-2}(q-2)!}{(q-2-m)!}\, e^{(q-1-m)a} e^{-mt} f(D) \sinh^{q-2-m}(t-a)$$

where f(D) is given following (5.16).

To obtain another form of $\bar{\Delta}^m K$ one makes use of the following differential expressions for Legendre and Tschebyscheff polynomials given in [6].

(5.21)

$$P_\mu^\lambda(\cosh t) = \frac{(2\lambda)!}{2^\lambda \mu!(\mu-\lambda)!\sinh^{2\lambda} t} D(D^2-2^2)\ldots\left[D^2-(\mu-\lambda-1)^2\right]\sinh^{\mu+\lambda} t$$

$$= \frac{(-1)^\lambda 2^\lambda \lambda!}{(\mu-\lambda)!(2\lambda)!} D(D^2-2^2)\ldots\left[D^2-(\mu+\lambda-1)^2\right]\sinh^{\mu-\lambda} t,$$

$$(\mu-\lambda = 3,\ 5, \ldots);$$

$$P_\mu^\lambda(\cosh t) = \frac{(2\lambda)!}{2^\lambda \lambda!(\mu-\lambda)!\sinh^{2\lambda} t}(D^2-1^2)(D^2-3^2)\ldots\left[D^2-(\mu-\lambda-1)^2\right]\sinh^{\mu-\lambda} t.$$

(5.21) continued

$$= \frac{(-1)^{\lambda} 2^{\lambda} \lambda!}{(\mu-\lambda)!(2\lambda)!}(D^2-1^2)(D^2-3^2)\dots\left[D^2-(\mu+\lambda-1)^2\right]\sinh^{\mu-\lambda}t,$$

$$(\mu-\lambda = 2, 4, \dots);$$

$$P_\mu^{\mu-1}(\cosh t) = \frac{(2\mu-1)!}{2^{\mu-1}(\mu-1)!}\cosh t; \qquad P_\mu^{\mu}(\cosh t) = \frac{(2\mu-1)!}{2^{\mu-1}(\mu-1)!}.$$

$$T_\mu^\lambda(\cosh t) = \frac{\mu 2^{\lambda-1}(\lambda-1)!}{(\mu-\lambda)!\sinh^{2\lambda-1}t}D(D^2-2^2)\dots\left[D^2-(\mu-\lambda-1)^2\right]\sinh^{\mu+\lambda-1}t,$$

$$(\mu-\lambda = 3, 5, \dots), \ \lambda \neq 0;$$

$$T_\mu^\lambda(\cosh t) = \frac{\mu 2^{\lambda-1}(\lambda-1)!}{(\mu-\lambda)!\sinh^{2\lambda-1}t}(D^2-1^2)(D^2-3^2)\dots\left[D^2-(\mu-\lambda-1)^2\right]\sinh^{\mu+\lambda-1}t,$$

$$(\mu-\lambda = 2, 4, \dots), \ \lambda \neq 0;$$

$$T_\mu^{\mu-1}(\cosh t) = 2^{\mu-1}\mu!\cosh t; \qquad T_\mu^{\mu}(\cosh t) = 2^{\mu-1}\mu!.$$

In (5.21) P_μ^λ and T_μ^λ indicate derived polynomials, for instance P_μ^λ is the λ-th derivative of P_μ.

If one uses (5.21) equation (5.20 becomes

(5.22)

(a) $$\bar{\Delta}^m K = C_{a}\frac{\mu}{S^{q-2}}(2S\mu)^{q-2-m}\left(\frac{S^2-\mu^2}{2S\mu}\right)^{q-1-2m} P_{\frac{q-3}{2}}^{\frac{q-1-2m}{2}}\left(\frac{S^2+\mu^2}{2S\mu}\right)$$

$$1 \leq m \leq (q-1)/2, \ q \text{ odd};$$

(b) $$\bar{\Delta}^m K = C_{b}\frac{\mu}{S^{q-2}}(2S\mu)^{q-2-m} P_{\frac{q-3}{2}}^{\frac{2m-q+1}{2}}\left(\frac{S^2+\mu^2}{2S\mu}\right)$$

$$(q-1)/2 \leq m \leq q-2, \ q \text{ odd};$$

(c) $$\bar{\Delta}^m K = C_{c}\frac{\mu}{S^{q-2}}(2S\mu)^{q-2-m}\left(\frac{S^2-\mu^2}{2S\mu}\right)^{q-1-2m} T_{\frac{q-2}{2}}^{\frac{q-2m}{2}}\left(\frac{S^2+\mu^2}{2S\mu}\right)$$

$$1 \leq m \leq (q-2)/2, \ q \text{ even};$$

(5.22) continued

$$\text{where}\quad C_a = \frac{2^{\frac{q-1}{2}}(m-1)!(q-2)!\left[\frac{q-1-2m}{2}\right]!}{(q-2-m)!(q-1-2m)!},$$

$$C_b = \frac{(-1)^{\frac{2m-q-1}{2}}\, 2^{\frac{q-1}{2}}(2m-q+1)!(q-2)!}{\left[\frac{2m-q+1}{2}\right]!}\quad\text{and}$$

$$C_c = \frac{2^{\frac{4m-q+4}{2}}(m-1)!(q-3)!}{(q-2-m)!\left[\frac{q-2-2m}{2}\right]!}.$$

Another form of $\overline{\Delta}^m K$ is obtained by replacing the hyperbolic function in (5.20) by exponentials and then expanding by the binomial theorem. The result is given below in (5.23).

(5.23)

$$\overline{\Delta}^m K = \frac{2^{q-2}(q-2)!}{(q-2-m)!}\, e^{(q-1-m)a} e^{-mt} f(D) \sum_{i=0}^{q-2-m} \binom{q-2-m}{i} \frac{(-e^{2a})^{q-2-m-i}\,(e^{2t})^i}{(2e^t e^a)^{q-2-m}}$$

If one arranges f(D) in the form (D-m+2)(D-m+4)...(D+m-4)(D+m-2) and performs the differential operations in (5.23) one has

(5.24)

$$\overline{\Delta}^m K = \frac{2^m (q-2)!}{(q-2-m)!} e^a e^{-(q-2)t} (D-q+4)(D-q+6)\ldots$$

$$(D-q+2m-2)(D-q+2m) \sum_{i=0}^{q-2-m} \binom{q-2-m}{i} (-e^{2a})^{q-2-m-i} (e^{2t})^i$$

$$= \frac{2^m (q-2)!}{(q-2-m)!} e^a e^{-(q-2)t} \sum_{i=0}^{q-2-m} \binom{q-2-m}{i} (2i-q+4)(2i-q+6)\ldots$$

$$(2i+2m-q-2)(2i+2m-q)(-e^{2a})^{q-2-m-i}(e^{2t})^i$$

$$= \frac{2^m (q-2)!}{(q-2-m)!} \sum_{i=0}^{q-2-m} \binom{q-2-m}{i} (-1)^{q-2-m-i} (2i-q+4)(2i-q+6)\ldots$$

$$(2i+2m-q-2)(2i+2m-q)\, e^{(2q-3-2m-2i)a} e^{(2i-q+2)t}.$$

VI. THE CHAIN OF DISTRIBUTIONS

The chain of distributions is based on the integral (5.6) which is repeated here.

$$(6.1) \qquad I(S, T) = e^{-\alpha_i X^i} \int_T^S K(S, \mu)\, \Phi(\mu, T)\, d\mu.$$

In the construction of the chain of distributions one uses the operator Δ given in (5.7). The first link in the chain of distributions is the integral (6.1). The second link in the chain is obtained by applying the operator Δ to the first link. The third link is $\Delta^2 I$ and the r-th link is $\Delta^{r-1} I$. The last link is $\Delta^{q-1} I$.

For the case $n = 2p+1$, the integral (6.1) is improper; however, the differentiation processes used here can be justified in this case. For the case $q = 2$, the kernel is such that $K_S(S, S) = 1$ and so is not included in the general case. For $q = 2$ the chain of distributions consists of the two links given below in (6.2).

$$(6.2) \qquad \begin{aligned} I &= e^{-\alpha_i X^i} \int_T^S \mu \ln \frac{S}{\mu} \left(\frac{\delta}{\sqrt{\mu^2 - T^2}} \right) J_1(\delta \sqrt{\mu^2 - T^2}\,)\, d\mu \\ \Delta I &= e^{-\alpha_i X^i} \left(\frac{\delta}{R} \right) J_1(\delta R) \end{aligned}$$

If one multiplies the expression for ΔI in (6.2) by $\frac{1}{\delta^2}$ then it becomes $e^{-\alpha_i X^i} \left(\frac{1}{\delta R} \right) J_1(\delta R)$ which is, for this case, the first type Green potential given in (3.30) since for $q = 2$, n must be 4 for ultrahyperbolic equations.

For the purpose of illustrating certain properties of the chain of distributions, the result of the process of successive applications of the operator Δ to the integral I is given below for two cases. For the potentials one is referred to (5.9). The kernel is obtained from (5.6) but it is arranged here for comparison with (5.1).

(6.3)

$$n = 7 \qquad K(S, \mu) = -\frac{16\mu}{3}\left\{-\mu^3 + \frac{S^2+\mu^2}{2S}\left[\mu^2 - \left(\frac{S^2-\mu^2}{2S}\right)^2 \frac{1}{2}\right]\right\}$$

$$p = 2$$

$$q = 5 \qquad \Phi(\mu, T) = \frac{1}{\mu^3}\left(\frac{\sqrt{\mu^2-T^2}}{\delta}\right)^{1/2} J_{1/2}(\delta\sqrt{\mu^2-T^2})$$

$$I = e^{-\alpha_i X^i}\int_T^S K(S,\mu)\,\Phi(\mu, T)\, d\mu$$

$$\Delta I = e^{-\alpha_i X^i}\int_T^S \overline{\Delta} K(S,\mu)\,\Phi(\mu, T)\, d\mu$$

$$\Delta^2 I = e^{-\alpha_i X^i}\int_T^S \overline{\Delta}^2 K(S,\mu)\,\Phi(\mu, T)\, d\mu$$

$$\Delta^3 I = e^{-\alpha_i X^i}\int_T^S \overline{\Delta}^3 K(S,\mu)\,\Phi(\mu, T)\, d\mu + 48\, e^{-\alpha_i X^i}\left[\frac{1}{S^2}\left(\frac{\delta}{R}\right)^{1/2} Y_{1/2}(\delta R) - \frac{1}{S^4}\left(\frac{R}{\delta}\right)^{1/2} J_{1/2}(\delta R)\right]$$

$$\Delta^4 I = 48\, e^{-\alpha_i X^i}\left(\frac{\delta}{R}\right)^{5/2} Y_{5/2}(\delta R)$$

(6.4)

$$n = 8 \qquad K(S, \mu) = 6\mu\left\{\mu^4 \ln\frac{S}{\mu} + \frac{S^2+\mu^2}{2S}\left[-\mu^2\left(\frac{S^2-\mu^2}{2S}\right) + \left(\frac{S^2-\mu^2}{2S}\right)^3 \frac{2}{3}\right]\right\}$$

$$p = 2$$

$$q = 6 \qquad \Phi(\mu, T) = \frac{1}{\mu^4}\left(\frac{\sqrt{\mu^2-T^2}}{\delta}\right) J_1(\delta\sqrt{\mu^2-T^2})$$

(6.4) continued

$$I = e^{-a_i X^i} \int_T^S K(S, \mu)\, \Phi(\mu, T) d\mu$$

$$\Delta I = e^{-a_i X^i} \int_T^S \overline{\Delta} K(S, \mu)\, \Phi(\mu, T) d\mu$$

$$\Delta^2 I = e^{-a_i X^i} \int_T^S \overline{\Delta}^2 K(S, \mu)\, \Phi(\mu, T) d\mu$$

$$\Delta^3 I = 384\ e^{-a_i X^i} \frac{1}{S^4} \left(\frac{R}{\delta}\right)^1 J_1(\delta R)$$

$$\Delta^4 I = 384\ e^{-a_i X^i} \left[- \frac{1}{S^2} \left(\frac{\delta}{R}\right)^1 J_1(\delta R) - \frac{2}{S^4} \left(\frac{R}{\delta}\right)^0 J_0(\delta R) \right]$$

$$\Delta^5 I = -384\ e^{-a_i X^i} \left(\frac{\delta}{R}\right)^3 J_3(\delta R)$$

In the two examples, (6.3) and (6.4), one should notice first that $q-1$ applications of the operator Δ to the integral I results in the first type Green potential for n even and the second type Green potential for n odd. Another property of the chain of distributions is that for q odd the first $(q+1)/2$ links of the chain consist of integrals only and that for q even the first $q/2$ links of the chain consist of integrals only. Another characteristic of the chain of distributions is that the integral disappears with the $(q/2)$-th application of the operator Δ when q is even but does not disappear until the last application of the operator when q is odd. This last property is a consequence of the property of the kernel given in (5.19).

The remainder of this paper is concerned with the problem of writing expressions for the links of the chain of distributions for the general case and with demonstrating that the q-th link is the Green potential which is the principal result of this paper.

For q odd the first $(q+1)/2$ links of the chain and for q even the first $q/2$ links of the chain are given by

$$\Delta^m I = e^{-\alpha_i X^i} \int_T^S \bar{\Delta}^m K(S, \mu)\,\Phi(\mu, T)\,d\mu \tag{6.5}$$

where $m \leq (q-1)/2$ when q is odd and greater than one, and where $m \leq (q-2)/2$ when q is even and greater than two. This is proved by mathematical induction. That (6.5) is true for $m = 1$ follows from the fact that $K(S, S) = K_S(S, S) = 0$. Assuming that (6.5) is true for $m = r-1$ one has the following expression for $\Delta^r I$.

$$\begin{aligned}\Delta^r I &= \Delta e^{-\alpha_i X^i} \int_T^S \bar{\Delta}^{r-1} K(S, \mu)\,\Phi(\mu, T)\,d\mu \\ &= e^{-\alpha_i X^i}\, \bar{\Delta} \int_T^S \bar{\Delta}^{r-1} K(S, \mu)\,\Phi(\mu, T)\,d\mu\end{aligned} \tag{6.6}$$

To determine the value of the right member of (6.6) one needs

$$\begin{aligned} & e^{-\alpha_i X^i} D_S \int_T^S \bar{\Delta}^{r-1} K(S, \mu)\,\Phi(\mu, T)\,d\mu \\ &= e^{-\alpha_i X^i} \int_T^S D_S \bar{\Delta}^{r-1} K(S, \mu)\,\Phi(\mu, T)\,d\mu + e^{-\alpha_i X^i} \left[\bar{\Delta}^{r-1} K(S, \mu)\,\Phi(\mu, T)\right]_{\mu = S} .\end{aligned} \tag{6.7}$$

That the last term on the right in (6.7) is zero for $r \leq (q-1)/2$ when q is odd and for $r \leq (q-2)/2$ when q is even follows from the fact that $\bar{\Delta}^{r-1} K$ contains $(S^2-\mu^2)^{q-2r+1}$ as a factor as is seen in (5.22a) and (5.22c). In addition to (6.7) one needs the value of the second derivative of the integral in (6.6).

$$(6.8) \quad e^{-\alpha_i X^i} D_{SS} \int_T^S \bar{\Delta}^{r-1} K(S, \mu)\, \Phi(\mu, T) d\mu$$

$$= e^{-\alpha_i X^i} \left\{ \int_T^S D_{SS} \bar{\Delta}^{r-1} K(S, \mu)\, \Phi(\mu, T) d\mu + \left[D_S \bar{\Delta}^{r-1} K(S, \mu)\, \Phi(\mu, T) \right]_{\mu = S} \right\}$$

That the last term in the right member of (6.8) is zero for $r \leq (q-1)/2$ when q is odd and for $r \leq (q-2)/2$ when q is even follows from the observation made following (6.7). For the values of r considered here, $q-2r+1 \geq 2$ and so $D_S[\bar{\Delta}^{r-1} K(S, \mu)]$ contains the factor $S^2 - \mu^2$ which is zero for $\mu = S$. In view of the preceding arguments there are no contributions from the limits of integration in the first or second derivatives of the integral. The operator $\bar{\Delta}$ is $D_{SS} + \frac{q-1}{S} D_S$, hence if the first derivative is multiplied by $(q-1)/S$ and added to the second derivative one has

$$(6.9) \qquad \Delta^r I = e^{-\alpha_i X^i} \int_T^S \bar{\Delta}^r K(S, \mu)\, \Phi(\mu, T) d\mu$$

where r is subject to the restrictions following (6.7). Therefore (6.5) is true.

In the consideration of $\Delta^m I$ for $m \geq (q+1)/2$ when q is odd and for $m \geq q/2$ when q is even the expressions are quite different for q even and q odd, consequently the cases are considered separately. The case of q even is presented first.

In preparation for the Δ operation on

$$(6.10)\qquad \Delta^{\frac{q-2}{2}} I = e^{-\alpha_i X^i} \int_T^S \overline{\Delta}^{\frac{q-2}{2}} K(S,\mu)\Phi(\mu, T)d\mu$$

one considers the following form of $\overline{\Delta}^{\frac{q-2}{2}} K$ obtained from (5.24) by replacing e^a by μ, e^t by S and m by $(q-2)/2$.

$$(6.11)\qquad \overline{\Delta}^{\frac{q-2}{2}} K = \frac{2^{\frac{q-2}{2}}(q-2)!}{\left[\frac{q-2}{2}\right]!} \sum_{i=0}^{\frac{q-2}{2}} \binom{\frac{q-2}{2}}{i} (-1)^{\frac{q-2-2i}{2}} (2i-q+4)(2i-q+6)\ldots$$

$$(2i-4)(2i-2)\mu^{q-1-2i} S^{2i-q+2}$$

Now $(2i-q+4)(2i-q+6)\ldots(2i-4)(2i-2) = 2^{\frac{q-4}{2}} (i-\frac{q-4}{2})(i-\frac{q-6}{2})\ldots(i-2)(i-1)$ so (6.11) can be written

$$(6.12)\qquad \overline{\Delta}^{\frac{q-2}{2}} K =$$

$$\frac{2^{q-3}(q-2)!}{\left[\frac{q-2}{2}\right]!} \sum_{i=0}^{\frac{q-2}{2}} \binom{\frac{q-2}{2}}{i} (-1)^{\frac{q-2-2i}{2}} (i-\frac{q-4}{2})(i-\frac{q-6}{2})\ldots(i-2)(i-1)\mu^{q-1-2i} S^{2i-q+2}.$$

From (6.12) it is seen that the only terms in the sum which are not zero are the terms for $i = 0$ and $i = \frac{q-2}{2}$. Therefore

$$(6.13)\qquad \overline{\Delta}^{\frac{q-2}{2}} K = \frac{2^{q-3}(q-2)!}{\left[\frac{q-2}{2}\right]!} \left\{ -\left[\frac{q-4}{2}\right]! \frac{\mu^{q-1}}{S^{q-2}} + \left[\frac{q-4}{2}\right]! \right\}$$

$$= 2^{q-2}(q-3)! \left[-\frac{\mu^{q-1}}{S^{q-2}} + \mu \right]$$

$$= 2^{q-2}(q-3)! \frac{\mu}{S^{q-2}} (S^{q-2} - \mu^{q-2}).$$

Using the results from (6.13) and performing the Δ operation on (6.10) one has

$$(6.14)\quad \Delta^{\frac{q}{2}} I = e^{-\alpha_i X^i} 2^{q-2} (q-2)! \frac{1}{S^{q-2}} \left(\frac{R}{\delta}\right)^{\frac{2q-n-2}{2}} J_{\frac{2q-n-2}{2}}(\delta R).$$

The integral disappears in this step because $\bar{\Delta}^{\frac{q}{2}} K(S, \mu) = 0$ when q is even by (5.19).

In the following work with higher orders of $\Delta^m I$ the following notation is used.

$$(6.15)\qquad F^r = \left(\frac{R}{\delta}\right)^r J_r(\delta R)$$

The r on the left is simply a symbol for identification and does not represent an exponent. In performing further Δ operations on (6.14) one needs the following rules for differentiation.

$$(6.16)\quad \begin{aligned} &(a)\quad \frac{1}{R} F^r_R = \left(\frac{R}{\delta}\right)^{r-1} J_{r-1}(\delta R) = F^{r-1} \\ &(b)\quad \bar{\Delta} \frac{F^r}{S^\gamma} = \frac{F^{r-2}}{S^{\gamma-2}} + (q-2\gamma)\frac{F^{r-1}}{S^\gamma} - \gamma(q-2-\gamma)\frac{F^r}{S^{\gamma+2}} \end{aligned}$$

Because of the repeated occurrence of the factor $e^{-\alpha_i X^i} 2^{q-2}(q-2)!$ in the following work, this factor is replaced by the symbol E for convenience in writing. If one introduces the notation of (6.15) into (6.14) and then applies the Δ operator to the result using (6.16) one has

$$(6.17)\quad \Delta^{\frac{q+2}{2}} I = E\left[\frac{F^{\frac{2q-n-6}{2}}}{S^{q-4}} - \frac{(q-4)F^{\frac{2q-n-4}{2}}}{S^{q-2}}\right],$$

(6.17) continued

$$\Delta^{\frac{q+4}{2}} I = E\left[\frac{F^{\frac{2q-n-10}{2}}}{S^{q-6}} - 2\frac{(q-6)F^{\frac{2q-n-8}{2}}}{S^{q-4}} + \frac{(q-6)(q-4)F^{\frac{2q-n-6}{2}}}{S^{q-2}}\right] \text{ and}$$

$$\Delta^{\frac{q+6}{2}} I = E\left[\frac{F^{\frac{2q-n-14}{2}}}{S^{q-8}} - 3\frac{(q-8)F^{\frac{2q-n-12}{2}}}{S^{q-6}} + 3\frac{(q-8)(q-6)F^{\frac{2q-n-10}{2}}}{S^{q-4}} - \frac{(q-8)(q-6)(q-4)F^{\frac{2q-n-8}{2}}}{S^{q-2}}\right].$$

The foregoing leads to the general expression for $\Delta^{\frac{q+2k}{2}} I$.

(6.18)

$$\Delta^{\frac{q+2k}{2}} I = E\sum_{i=0}^{k}\binom{k}{i}(-1)^i \frac{(q-2k-4)(q-2k-2)\dots(q-2k+2i-4)F^{\frac{2q-n-4k+2i-2}{2}}}{(q-2k-4)\,S^{q-2k+2i-2}}$$

To prove (6.18) one again uses mathematical induction. Assuming that (6.18) is true for $k = r-1$ one has

(6.19)

$$\Delta^{\frac{q+2(r-1)}{2}} I = E\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i \frac{(q-2r-2)(q-2r)\dots(q-2r+2i-2)F^{\frac{2q-n-4r+2i+2}{2}}}{(q-2r-2)\,S^{q-2r+2i}},$$

Applying the operator Δ to (6.19) and using (6.16) one has

(6.20)

$$\Delta^{\frac{q+2r}{2}} I = E\left[\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i \frac{(q-2r-2)(q-2r)\dots(q-2r+2i-2)F^{\frac{2q-n-4r+2i-2}{2}}}{(q-2r-2)\,S^{q-2r+2i-2}}\right.$$

$$\left.+\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^{i+1}(q-4r+4i)\frac{(q-2r-2)(q-2r)\dots(q-2r+2i-2)F^{\frac{2q-n-4r+2i}{2}}}{(q-2r-2)\,S^{q-2r+2i}}\right.$$

(6.20) continued

$$+\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i(q-2r+2i)(2-2r+2i)\frac{(q-2r-2)(q-2r)\dots}{(q-2r-2)}$$

$$\left.\frac{(q-2r+2i-2)F^{\frac{2q-n-4r+2i+2}{2}}}{S^{q-2r+2i+2}}\right].$$

In combining like terms in the right member of (6.20) one notices that there is no term to combine with the first term of the first sum and that the second term of the first sum combines with the first term of the second sum. These first two terms of $\Delta^{(q+2r)/2}I$ are

$$(6.21)\qquad E\left[\frac{F^{\frac{2q-n-4r-2}{2}}}{S^{q-2r-2}} - r\frac{(q-2r-2)F^{\frac{2q-n-4r}{2}}}{S^{q-2r}}\right].$$

In general the k-th term of the first sum combines with the (k-1)-th term of the second sum and the (k-2)-th term of the third sum. To obtain the next r-2 terms of $\Delta^{(q+2r)/2}I$ one considers the following terms of (6.20).

(6.22)

$$E\left[\sum_{i=2}^{r-1}\binom{r-1}{i}(-1)^i\frac{(q-2r-2)(q-2r)\dots(q-2r+2i-2)F^{\frac{2q-n-4r+2i-2}{2}}}{(q-2r-2)\,S^{q-2r+2i-2}}\right.$$

$$+\sum_{i=1}^{r-2}\binom{r-1}{i}(-1)^{i+1}(q-4r+4i)\frac{(q-2r-2)(q-2r)\dots(q-2r+2i-2)F^{\frac{2q-n-4r+2i}{2}}}{(q-2r-2)\,S^{q-2r+2i}}$$

$$\left.+\sum_{i=0}^{r-3}\binom{r-1}{i}(-1)^i(q-2r+2i)(2-2r+2i)\frac{(q-2r-2)(q-2r)\dots(q-2r+2i-2)F^{\frac{2q-n-4r+2i+2}{2}}}{(q-2r-2)\,S^{q-2r+2i+2}}\right]$$

The sums in (6.22) can be written

(6.23) $$E\sum_{i=2}^{r-1}(-1)^i\left[\binom{r-1}{i}\frac{q-2r+2i-2}{q-2r-2}+\binom{r-1}{i-1}\frac{q-4r+4i-4}{q-2r-2}+\binom{r-1}{i-2}\frac{-2-2r+2i}{q-2r-2}\right]\frac{(q-2r-4)(q-2r-2)\ldots(q-2r+2i-4)F^{\frac{2q-n-4r+2i-2}{2}}}{(q-2r-4)\,S^{q-2r+2i-2}}.$$

Considering the factor in (6.23) immediately following $(-1)^i$ one has

(6.24) $$\left[\binom{r-1}{i}\frac{q-2r+2i-2}{q-2r-2}+\binom{r-1}{i-1}\frac{q-4r+4i-4}{q-2r-2}+\binom{r-1}{i-2}\frac{-2-2r+2i}{q-2r-2}\right]$$

$$=\binom{r}{i}\left[\frac{(q-2r+2i-2)(r-i)}{(q-2r-2)r}+\frac{(q-4r+4i-4)i}{(q-2r-2)r}+\frac{(-2-2r+2i)(i-1)i}{(q-2r-2)(r-i+1)r}\right]$$

$$=\binom{r}{i}.$$

If one uses (6.24) then (6.23) can be written

(6.25) $$E\sum_{i=0}^{r-1}\binom{r}{i}(-1)^i\frac{(q-2r-4)(q-2r-2)\ldots(q-2r+2i-4)F^{\frac{2q-n-4r+2i-2}{2}}}{(q-2r-4)\,S^{q-2r+2i-2}}.$$

The last term of $\Delta^{(q+2r)/2}I$ is obtained by combining the term in the second sum in (6.20) for which $i = r-1$ and the term in the third sum in (6.20) for which $i = r-2$. This term is

(6.26) $$E(-1)^r\frac{(q-2r-4)(q-2r-2)\ldots(q-4)F^{\frac{2q-n-2r-2}{2}}}{(q-2r-4)\,S^{q-2}}.$$

That the term of the third sum in (6.20) for which $i = r-1$, is zero is seen by observing that the factor $(2-2r+2i)$ is zero for $i = r-1$. If one combines the results from (6.21), (6.25) and (6.26) one has the following expression for $\Delta^{(q+2r)/2}I$.

(6.27)

$$\Delta^{\frac{q+2r}{2}} I = E \sum_{i=0}^{r} \binom{r}{i} (-1)^i \frac{(q-2r-4)(q-2r-2)\ldots(q-2r+2i-4)}{(q-2r-4)} \frac{F^{\frac{2q-n-4r+2i-2}{2}}}{S^{q-2r+2i-2}}$$

In view of the preceding argument (6.18) is true.

If one sets $k = (q-2)/2$ in (6.18) only the first term in the sum contributes and one has after replacing E and F by their equivalents

$$(6.28) \qquad \Delta^{q-1} I = e^{-\alpha_i X^i} 2^{q-2}(q-2)! \left(\frac{R}{\delta}\right)^{-(n-2)/2} J_{-(n-2)/2}(\delta R).$$

Converting the negative index on the Bessel function into a positive index one has

(6.29)

$$\Delta^{q-1} I = 2^{q-2}(q-2)! e^{-\alpha_i X^i} \left(\frac{\delta}{R}\right)^{\frac{n-2}{2}} Y_{\frac{n-2}{2}}(\delta R), \text{ n odd;}$$

$$\Delta^{q-1} I = (-J)^{\frac{n-2}{2}} 2^{q-2}(q-2)! e^{-\alpha_i X^i} \left(\frac{\delta}{R}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(\delta R), \text{ n even.}$$

The symbol J used in (6.29) before the expression for the case of n even is the same as the symbol J determined in (3.13). The expressions in (6.29) are the final link in the chain of distributions for q even. One should notice that they are, except for a constant factor, the first type Green potentials for n even and the second type Green potentials when n is odd. This is the desired result.

The consideration of $\Delta^m I$ for $m \geq (q+1)/2$ for the case of q odd follows. As was pointed out previously, in this case the integral does not disappear until the last Δ operation because of the property of the kernel given in (5.19). Because of this, the treatment of $\Delta^m I$ is expedited by the determination of a formula for the $\bar{\Delta}$ operation applied to the integral at any stage. To obtain this formula one needs an expression for $\bar{\Delta}^m K(S, \mu)$ evaluated at $\mu = S$. If one sets $\mu = S$ in the formula (5.22b) it becomes

$$(6.30)\quad \left[\bar{\Delta}^m K(S,\mu)\right]_{\mu=S} = \frac{(-1)^{\frac{2m-q+1}{2}}\, 2^{\frac{3q-2m-5}{2}}}{\left[\frac{2m-q+1}{2}\right]!}\, \frac{(2m-q+1)!(q-2)!}{S^{2m-q+1}}\, P_{\frac{q-3}{2}}^{\frac{2m-q+1}{2}}(1).$$

By using certain recurrence relations for Legendre polynomials from [6] one has

$$(6.31)\quad P_\mu^\lambda(1) = \frac{(\mu+\lambda)!}{2^\lambda \lambda!(\mu-\lambda)!}.$$

Using (6.31) in (6.30) one has the desired formula for $\bar{\Delta}^m K$.

$$(6.32)\quad \left[\bar{\Delta}^m K(S,\mu)\right]_{\mu=S} = \frac{(-1)^{\frac{2m-q+1}{2}}}{\left(\left[\frac{2m-q+1}{2}\right]!\right)^2}\, \frac{2^{2q-2m-3}}{(q-m-2)!}\, \frac{(2m-q+1)!(q-2)!(m-1)!}{S^{2m-q+1}}.$$

To determine the formula for $\bar{\Delta}$ applied to the integral at any stage one needs the value of the derivative of the integral. This is

$$(6.33)\quad D_S \int_T^S \bar{\Delta}^m K(S,\mu)\Phi(\mu,T)d\mu = \int_T^S D_S \bar{\Delta}^m K(S,\mu)\Phi(\mu,T)d\mu + \left[\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)\right]_{\mu=S}.$$

In addition one needs the value of the second derivative of the integral. Preparatory to this computation one determines the value of $\left[D_S\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)\right]_{\mu=S}$ and $D_S\left\{\left[\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)\right]_{\mu=S}\right\}$. The first of these is found below using (5.22b).

$$(6.34)\qquad D_S \bar{\Delta}^m K(S,\mu) =$$

$$\frac{(-1)^{\frac{2m-q+1}{2}} 2^{\frac{q-1}{2}} (2m-q+1)!(q-2)!\mu(2\mu)^{q-2-m}}{\left[\frac{2m-q+1}{2}\right]!}\left[\frac{-m}{S^{m+1}} P_{\frac{q-3}{2}}^{\frac{2m-q+1}{2}}\left(\frac{S^2+\mu^2}{2S\mu}\right)\right.$$

$$\left. + \frac{1}{S^{m+1}} \frac{S^2-\mu^2}{2S\mu} P_{\frac{q-3}{2}}^{\frac{2m-q+3}{2}}\left(\frac{S^2+\mu^2}{2S\mu}\right)\right]$$

hence $\left[D_S \bar{\Delta}^m K(S,\mu)\,\Phi(\mu,T)\right]_{\mu=S} = -\frac{m}{S}\left[\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)\right]_{\mu=S}$.

If one uses (6.32) and the F notation introduced in (6.15) the second expression mentioned above is

$$(6.35)\qquad D_S\left\{\left[\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)\right]_{\mu=S}\right\} =$$

$$\frac{(-1)^{2m-q+1} 2^{2q-2m-3}(2m-q+1)!(q-2)!(m-1)!}{\left(\left[\frac{2m-q+1}{2}\right]!\right)^2 (q-m-2)!} D_S\left[\frac{F^{\frac{2q-n-2}{2}}}{S^{2m-1}}\right] =$$

$$\frac{(-1)^{2m-q+1} 2^{2q-2m-3}(2m-q+1)!(q-2)!(m-1)!}{\left(\left[\frac{2m-q+1}{2}\right]!\right)^2 (q-m-2)!}\left[\frac{(-2m+1)F^{\frac{2q-n-2}{2}}}{S^{2m}} + \frac{F^{\frac{2q-n-4}{2}}}{S^{2m-2}}\right]$$

$$= \frac{-2m+1}{S}\left[\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)\right]_{\mu=S} + \left[\bar{\Delta}^m K(S,\mu)\right]_{\mu=S} \frac{F^{\frac{2q-n-4}{2}}}{S^{q-3}} .$$

Using the results from (6.33), (6.34) and (6.35) one finds the following expression for the second derivative of the integral.

$$(6.36)\quad D_{SS}\int_T^S \bar{\Delta}^m K(S,\mu)\Phi(\mu,T)d\mu = \int_T^S D_{SS}\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)d\mu + \frac{-3m+1}{S}\left[\bar{\Delta}^m K(S,\mu)\Phi(\mu,T)\right]_{\mu=S} + \left[\bar{\Delta}^m K(S,\mu)\right]_{\mu=S}\frac{F^{\frac{2q-n-4}{2}}}{S^{q-3}}.$$

Multiplying (6.33) by the factor $(q-1)/S$ and adding (6.36) to it one has the formula for $\bar{\Delta}$ applied to the integral at any stage. In the formula given for this below use has been made of (6.32).

$$(6.37)\quad \bar{\Delta}\int_T^S \bar{\Delta}^m K(S,\mu)\Phi(\mu,T)d\mu = \int_T^S \bar{\Delta}^{m+1}K(S,\mu)\Phi(\mu,T)d\mu$$

$$+\frac{(-1)^{\frac{2m-q+1}{2}}\,2^{2q-2m-3}(2m-q+1)!(q-2)!(m-1)!}{\left(\left[\frac{2m-q+1}{2}\right]!\right)^2(q-m-2)!}\left[\frac{F^{\frac{2q-n-4}{2}}}{S^{2m-2}}-\frac{(3m-q)F^{\frac{2q-n-2}{2}}}{S^{2m}}\right]$$

Preparatory to writing a general expression for $\Delta^{(q+2k-1)/2}I$ a few cases are given below. These expressions are based on (6.5) with $m = (q-1)/2$ and are computed by using the differentiation formulae (6.16) and (6.37). The symbol E is again used for $2^{q-2}(q-2)!e^{-a_iX^i}$.

(6.38)

$$\Delta^{\frac{q+1}{2}}I = e^{-a_iX^i}\int_T^S \bar{\Delta}^{\frac{q+1}{2}}K(S,\mu)\Phi(\mu,T)d\mu + E\left[\frac{F^{\frac{2q-n-4}{2}}}{S^{q-3}}-\frac{1}{2}\frac{(q-3)F^{\frac{2q-n-2}{2}}}{S^{q-1}}\right],$$

$$\Delta^{\frac{q+3}{2}}I = e^{-a_iX^i}\int_T^S \bar{\Delta}^{\frac{q+3}{2}}K(S,\mu)\Phi(\mu,T)d\mu + E\left[\frac{F^{\frac{2q-n-8}{2}}}{S^{q-5}}-\frac{3}{2}\frac{(q-5)F^{\frac{2q-n-6}{2}}}{S^{q-3}}\right.$$

$$\left.+\frac{3}{8}\frac{(q-5)(q-3)F^{\frac{2q-n-4}{2}}}{S^{q-1}}+\frac{1}{16}\frac{(q-5)(q-3)(q-1)F^{\frac{2q-n-2}{2}}}{S^{q+1}}\right],$$

(6.38) continued

$$\Delta^{\frac{q+5}{2}} I = e^{-\alpha_i X^i} \int_T^S \bar{\Delta}^{\frac{q+5}{2}} K(S,\mu)\Phi(\mu,T)d\mu + E\left[\frac{F^{\frac{2q-n-12}{2}}}{S^{q-7}} - \frac{5}{2}\frac{(q-7)F^{\frac{2q-n-10}{2}}}{S^{q-5}}\right.$$

$$+ \frac{15}{8}\frac{(q-7)(q-5)F^{\frac{2q-n-8}{2}}}{S^{q-3}} - \frac{5}{16}\frac{(q-7)(q-5)(q-3)F^{\frac{2q-n-6}{2}}}{S^{q-1}}$$

$$\left. - \frac{5}{128}\frac{(q-7)(q-5)(q-3)(q-1)F^{\frac{2q-n-4}{2}}}{S^{q+1}} - \frac{3}{256}\frac{(q-7)(q-5)(q-3)(q-1)(q+1)F^{\frac{2q-n-2}{2}}}{S^{q+3}}\right].$$

One should recognize the binomial coefficients for fractional exponents in the preceding expressions. These expressions lead to the general expression for $\Delta^{(q+2k-1)/2} I$.

$$(6.39) \quad \Delta^{\frac{q+2k-1}{2}} I = e^{-\alpha_i X^i} \int_T^S \bar{\Delta}^{\frac{q+2k-1}{2}} K(S,\mu)\Phi(\mu,T)d\mu$$

$$+ E\sum_{i=0}^{2k-1} \binom{\frac{2k-1}{2}}{i} (-1)^i \frac{(q-2k-3)(q-2k-1)\ldots(q-2k+2i-3)F^{\frac{2q-n-4k+2i}{2}}}{(q-2k-3)\, S^{q-2k+2i-1}}.$$

In proving (6.39) one uses mathematical induction. Assuming that (6.39) is true for $k = r-1$ one has

$$(6.40) \quad \Delta^{\frac{q+2r-3}{2}} I = e^{-\alpha_i X^i} \int_T^S \bar{\Delta}^{\frac{q+2r-3}{2}} K(S,\mu)\Phi(\mu,T)d\mu$$

$$+ E\sum_{i=0}^{2r-3} \binom{\frac{2r-3}{2}}{i} (-1)^i \frac{(q-2r-1)(q-2r+1)\ldots(q-2r+2i-1)F^{\frac{2q-n-4r+2i+4}{2}}}{(q-2r-1)\, S^{q-2r+2i+1}}.$$

Applying the operator Δ to (6.40) and then using (6.16) and (6.37) one has

$$(6.41)\quad \Delta^{\frac{q+2r-1}{2}} I = e^{-\alpha_i X^i} \int_T^S \bar{\Delta}^{\frac{q+2r-1}{2}} K(S,\mu)\,\Phi(\mu,T)\,d\mu$$

$$+ E \frac{(-1)^r (2r-2)!\left[\frac{q+2r-5}{2}\right]!}{\left[(r-1)!\right]^2 \left[\frac{q-2r-1}{2}\right]! 2^{2r-2}} \left\{ \frac{-F^{\frac{2q-n-4}{2}}}{S^{q+2r-5}} + \frac{\left[\frac{q+6r-9}{2}\right] F^{\frac{2q-n-2}{2}}}{S^{q+2r-3}} \right\}$$

$$+ E \left[\sum_{i=0}^{\frac{2r-3}{2}} \binom{\frac{2r-3}{2}}{i} (-1)^i \frac{(q-2r-1)(q-2r+1)\dots(q-2r+2i-1)}{(q-2r-1)} \frac{F^{\frac{2q-n-4r+2i}{2}}}{S^{q-2r+2i-1}} \right.$$

$$+ \sum_{i=0}^{\frac{2r-3}{2}} \binom{\frac{2r-3}{2}}{i} (-1)^{i+1} (q-4r+4i+2) \frac{(q-2r-1)(q-2r+1)\dots(q-2r+2i-1)}{(q-2r-1)} \frac{F^{\frac{2q-n-4r+2i+2}{2}}}{S^{q-2r+2i+1}}$$

$$+ \sum_{i=0}^{\frac{2r-3}{2}} \binom{\frac{2r-3}{2}}{i} (-1)^i (q-2r+2i+1)(3-2r+2i) \frac{(q-2r-1)(q-2r+1)\dots}{(q-2r-1)}$$

$$\left. \frac{(q-2r+2i-1) F^{\frac{2q-n-4r+2i+4}{2}}}{S^{q-2r+2i+3}} \right].$$

The combination of like terms in the right member of (6.41) takes place in a manner similar to the case of q even, except that the expression in braces immediately following the integral in (6.41) must be taken into consideration. These two terms contribute to the last two terms in the final result. If one considers the three sums in brackets one sees that there is no term to combine with the first term of the first sum and that the second term of the first sum and the first term of the second sum combine to form the second term of the final result. These first two terms of the sum in the expression for $\Delta^{(q+2r-1)/2} I$ are

$$(6.42)\qquad E\left[\frac{F^{\frac{2q-n-4r}{2}}}{S^{q-2r-1}} - \frac{2r-1}{2} \frac{(q-2r-1) F^{\frac{2q-n-4r+2}{2}}}{S^{q-2r+1}} \right].$$

In general the k-th term of the first sum combines with the (k-1)-th term of the second sum and the (k-2)-th term of the third sum. To obtain the next 2r-4 terms of the sum in the expression for $\Delta^{(q+2r-1)/2}I$ one considers the following terms of (6.41).

$$(6.43)\quad E\left[\sum_{i=2}^{2r-3}\binom{\frac{2r-3}{2}}{i}(-1)^i\,\frac{(q-2r-1)(q-2r+1)\ldots(q-2r+2i-1)}{(q-2r-1)}\,\frac{F^{\frac{2q-n-4r+2i}{2}}}{S^{q-2r+2i-1}}\right.$$

$$+\sum_{i=1}^{2r-4}\binom{\frac{2r-3}{2}}{i}(-1)^{i+1}(q-4r+4i+2)\,\frac{(q-2r-1)(q-2r+1)\ldots(q-2r+2i-1)}{(q-2r-1)}\,\frac{F^{\frac{2q-n-4r+2i+2}{2}}}{S^{q-2r+2i+1}}$$

$$+\sum_{i=0}^{2r-5}\binom{\frac{2r-3}{2}}{i}(-1)^i(q-2r+2i+1)(3-2r+2i)\,\frac{(q-2r-1)(q-2r+1)\ldots}{(q-2r-1)}$$

$$\left.\frac{(q-2r+2i-1)F^{\frac{2q-n-4r+2i+4}{2}}}{S^{q-2r+2i+3}}\right].$$

The sums in (6.43) can be written

$$(6.44)\quad E\left\{\sum_{i=2}^{2r-3}(-1)^i\left[\binom{\frac{2r-3}{2}}{i}\frac{q-2r+2i-1}{q-2r-1}+\binom{\frac{2r-3}{2}}{i-1}\frac{q-4r+4i-2}{q-2r-1}\right.\right.$$

$$\left.\left.+\binom{\frac{2r-3}{2}}{i-2}\frac{-2r+2i-1}{q-2r-1}\right]\frac{(q-2r-3)(q-2r-1)\ldots(q-2r+2i-3)}{(q-2r-3)}\,\frac{F^{\frac{2q-n-4r+2i}{2}}}{S^{q-2r+2i-1}}\right\}.$$

Considering the factor in (6.44) immediately following $(-1)^i$ one has

$$(6.45)\quad \binom{\frac{2r-3}{2}}{i}\frac{q-2r+2i-1}{q-2r-1}+\binom{\frac{2r-3}{2}}{i-1}\frac{q-4r+4i-2}{q-2r-1}+\binom{\frac{2r-3}{2}}{i-2}\frac{-2r+2i-1}{q-2r-1}$$

$$=\binom{\frac{2r-1}{2}}{i}\left[\frac{(q-2r+2i-1)\left[\frac{2r-2i-1}{2}\right]}{(q-2r-1)\left[\frac{2r-1}{2}\right]}+\frac{(q-4r+4i-2)i}{(q-2r-1)\left[\frac{2r-1}{2}\right]}+\frac{(-2r+2i-1)(i-1)i}{(q-2r-1)\left[\frac{2r-1}{2}\right]\left[\frac{2r-2i+1}{2}\right]}\right]$$

(6.45) continued

$$= \binom{\frac{2r-1}{2}}{i}.$$

If one uses (6.45) then (6.44) can be written

$$(6.46) \quad E\sum_{i=2}^{2r-3}(-1)^i\binom{\frac{2r-1}{2}}{i}\frac{(q-2r-3)(q-2r-1)\ldots(q-2r+2i-3)F^{\frac{2q-n-4r+2i}{2}}}{(q-2r-3)\qquad S^{q-2r+2i-1}}.$$

The next to last term of the sum in $\Delta^{(q+2r-1)/2}I$ comes from the first term in the braces immediately following the integral in (6.41), the term of the second sum in (6.41) for which i = 2r-3 and the term of the third sum in (6.41) for which i = 2r-4. These terms are

$$(6.47) \quad E\left[\frac{(-1)^{r-1}(2r-2)!\left[\frac{q+2r-5}{2}\right]!F^{\frac{2q-n-4}{2}}}{\left[(r-1)!\right]^2\left[\frac{q-2r-1}{2}\right]!2^{2r-2}S^{q+2r-5}}\right.$$

$$+\binom{\frac{2r-3}{2}}{2r-3}(-1)^{2r-2}(q+4r-10)\frac{(q-2r-1)(q-2r+1)\ldots(q+2r-7)F^{\frac{2q-n-4}{2}}}{(q-2r-1)\qquad S^{q+2r-5}}$$

$$\left.+\binom{\frac{2r-3}{2}}{2r-4}(-1)^{2r-4}(q+2r-7)(2r-5)\frac{(q-2r-1)(q-2r+1)\ldots(q+2r-9)F^{\frac{2q-n-4}{2}}}{(q-2r-1)\qquad S^{q+2r-5}}\right]$$

$$= E\binom{\frac{2r-1}{2}}{2r-2}\left[-\frac{(q+2r-5)(2r-3)}{(q-2r-1)(2r-1)}+\frac{(q+4r-10)(2r-2)}{(q-2r-1)\left[\frac{2r-1}{2}\right]}\right.$$

$$\left.+\frac{(2r-5)(2r-2)(2r-3)}{(q-2r-1)\left[\frac{2r-1}{2}\right]\left[\frac{-2r+5}{2}\right]}\right]\frac{(q-2r-3)(q-2r-1)\ldots(q+2r-7)F^{\frac{2q-n-4}{2}}}{(q-2r-3)\qquad S^{q+2r-5}}$$

$$= E\binom{\frac{2r-1}{2}}{2r-2}(-1)^{2r-2}\frac{(q-2r-3)(q-2r-1)\ldots(q+2r-7)F^{\frac{2q-n-4}{2}}}{(q-2r-3)\qquad S^{q+2r-5}}.$$

The last term of the sum in $\Delta^{(q+2r-1)/2}I$ comes from the second term in the braces immediately following the integral in (6.41) and the term of the third sum in (6.41) for which $i = 2r-3$. These terms are

$$(6.48)\qquad E\left[\frac{(-1)^r(2r-2)!\left[\frac{q+2r-5}{2}\right]!\left[\frac{q+6r-9}{2}\right]}{\left[(r-1)!\right]^2\left[\frac{q-2r-1}{2}\right]!2^{2r-2}}\frac{F^{\frac{2q-n-2}{2}}}{S^{q+2r-3}}\right.$$

$$\left.+\binom{\frac{2r-3}{2}}{2r-3}(-1)^{2r-3}(q+2r-5)(2r-3)\frac{(q-2r-1)(q-2r+1)\ldots(q+2r-7)F^{\frac{2q-n-2}{2}}}{(q-2r-1)\,S^{q+2r-3}}\right]$$

$$= E\binom{\frac{2r-1}{2}}{2r-1}(-1)^{2r-1}\left[\frac{q+6r-9}{q-2r-1}-\frac{8r-8}{q-2r-1}\right]\frac{(q-2r-3)(q-2r-1)\ldots(q+2r-5)F^{\frac{2q-n-2}{2}}}{(q-2r-3)\,S^{q+2r-3}}$$

$$= E\binom{\frac{2r-1}{2}}{2r-1}(-1)^{2r-1}\frac{(q-2r-3)(q-2r-1)\ldots(q+2r-5)F^{\frac{2q-n-2}{2}}}{(q-2r-3)\,S^{q+2r-3}}\,.$$

Combining the results from (6.42), (6.46), (6.47) and (6.48) one has the following expression for $\Delta^{(q+2r-1)/2}I$.

$$(6.49)\qquad \Delta^{\frac{q+2r-1}{2}}I = e^{-a_iX^i}\int_T^S \bar{\Delta}^{\frac{q+2r-1}{2}}K(S,\mu)\Phi(\mu,T)d\mu$$

$$+\ E\sum_{i=0}^{2r-1}(-1)^i\binom{\frac{2r-1}{2}}{i}\frac{(q-2r-3)(q-2r-1)\ldots(q-2r+2i-3)F^{\frac{2q-n-4r+2i}{2}}}{(q-2r-3)\,S^{q-2r+2i-1}}$$

In view of the preceding argument (6.39) is true.

If one sets $k = (q-1)/2$ in (6.39) the integral drops out because $\Delta^{q-1}K(S,\mu) = 0$, and with this value of k, only the first term in the sum contributes. Hence one has after replacing E and F by their equivalents

$$(6.50)\qquad \Delta^{q-1}I = 2^{q-2}(q-2)!e^{-a_iX^i}\left(\frac{R}{\delta}\right)^{-(n-2)/2}J_{-(n-2)/2}(\delta R)\,.$$

Converting the negative index on the Bessel function into a positive index one has

(6.51)
$$\Delta^{q-1} I = 2^{q-2}(q-2)!\, e^{-\alpha_i X^i}\left(\frac{\delta}{R}\right)^{\frac{n-2}{2}} Y_{\frac{n-2}{2}}(\delta R), \quad n \text{ odd};$$
$$\Delta^{q-1} I = (-J)^{\frac{n-2}{2}}\, 2^{q-2}(q-2)!\, e^{-\alpha_i X^i}\left(\frac{\delta}{R}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(\delta R), \quad n \text{ even}.$$

These expressions are of the same type as those given in (6.29). They are the final link in the chain of distributions for q odd. They are, except for a constant factor, the first type Green potentials for n even and the second type Green potentials for n odd.

Reviewing the results of this chapter one sees that the desired chain of distributions is given by the equations listed below.

q even	q odd
(6.1)	(6.1)
(6.5)	(6.5)
(6.14)	
(6.18)	(6.39)
(6.29)	(6.51)

VII. SUMMARY

In this paper certain aspects of a theory of distributions for ultrahyperbolic equations have been developed. First, a basic vector system was defined and some of the properties for such a vector system were developed. After the work on the basic vector system a non-parabolic, non-degenerate vector system was defined and some properties of this system were developed. Rules of indices and Pythagorean identities for a non-parabolic, non-degenerate system were derived.

A non-parabolic, non-degenerate vector system was chosen and certain solutions of the adjoint equation were determined. These solutions were classified as retarded potentials and Green potentials. A retarded potential was integrated over a portion of a q-dimensional subspace determined by the vector system chosen. The q-tuple integral was reduced to a single integral which is in the nature of a transform of the retarded potential. From this single integral the kernel of the transform was determined and some properties of the kernel were developed. From the single integral a chain of distributions was constructed by repeated applications of an operator Δ consisting of certain terms of the adjoint operator. Some properties of the chain of distributions were discussed; in particular it was shown that $q-1$ applications of the operator Δ reduces the single integral to either the first or second type Green potential according as the number of dimensions is even or odd. Thus the chain of distributions links the transform of the retarded potential to the Green potentials.

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