A Transformation of the Catalan Numbers and Related Counted Sets

By

William Anderson Davis

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Stillwater, OK
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A TRANSFORMATION OF THE CATALAN

NUMBERS AND RELATED

COUNTED SETS

Thesis Approved:

Dr. Jeffrey Mermin

Thesis Advisor

Dr. Edward Richmond

Dr. Jay Schweig

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Name: William Anderson Davis

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Abstract:

Borel's triangle is a triangular array of numbers constructed by a transformation of the Catalan numbers. By identifying natural bijections between them, we discuss several sets counted by Borel's triangle, including full binary trees with a fixed number of "marked" branching vertices, ballot sequences in elections between three parties, and permutations with a given set of properties. We further introduce a strategy which can be used to identify more combinatorial interpretations of Borel's triangle.

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Chapter 1

Introduction

The Catalan numbers C_n given by $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$ are pervasive in the field of enumerative combinatorics and can be used to count the elements of many sets. In [1], Stanley devotes a chapter to an exercise asking the reader to show that the elements of 214 different sets are counted by C_n. The Catalan numbers can be defined by the above formula, by the quadratic recursion $C_{n+1} = \sum_{j=0}^{n} C_j C_{n-j}$, or by constructing the following triangular array.

Definition 1.0.1. *Catalan's triangle* is the triangular array $\{C_{n,k}\}$ defined where $0 \leq k \leq n$ given by

$$C_{n,k} = \begin{cases} 1 & ; k = 0 \\ C_{n-1,k} + C_{n,k-1} & ; 0 < k < n \\ C_{n,k-1} & ; k = n \end{cases}$$

Rows zero through four of Catalan's triangle are given below.

$$\begin{array}{lllll} 1 \\ 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 5 & 5 \\ 1 & 4 & 9 & 14 & 14 \end{array}$$

The right boundary of Catalan's triangle gives the classic Catalan numbers; that is to say, $C_n = C_{n,n}$.

Remark 1.0.2. Note that the nth Catalan number C_n is the sum of the entries in the previous row of Catalan's triangle. That is, $C_n = C_{n,n} = \sum_{k=0}^{n-1} C_{n-1,k}$.

In this way, each row of Catalan's triangle can be seen as a specific partition of the subsequent Catalan number. Using this notion, then for many sets for which it is known how they are counted by the Catalan numbers, we can identify some additional parameter which we will use to further classify these sets into subsets whose elements are counted by each row of Catalan's triangle.

Example 1.0.3. For example, it is known that there are $C_4 = 14$ different ways to triangulate a regular hexagon. We can further classify this using Catalan's triangle by seeing 1 of these triangulations has 3 diagonals connected to a fixed vertex, 3 of these triangulations have exactly 2 diagonal connected to the same fixed vertex, 5 of these triangulations have exactly 1 diagonal connected to the same fixed vertex, and 5 of these triangulations have 0 diagonals connected to the same fixed vertex.

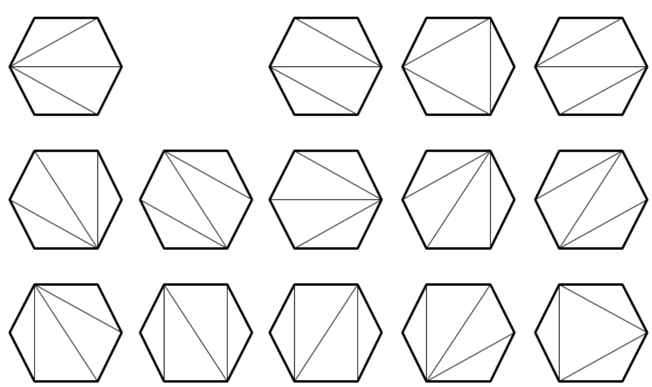

Figure 1.1: The 14 triangulations of a regular hexagon, arranged by the number of diagonals connected to the leftmost vertex

Definition 1.0.4. We construct a corresponding array $\{f_{n,k}\}$, known as *Borel's triangle*, by applying binomial coefficients to rows of Catalan's triangle in the following way:

$$f_{n,k} = \sum_{s=0}^{n} \binom{s}{k} C_{n,k}$$

Rows zero through four of Borel's triangle are given below.

1				
2	1			
5	6	2		
14	28	20	5	
42	120	135	70	14

In [2], Francisco, et al. give four interpretations of Borel's triangle and discuss sets counted by it including certain marked binary trees, pseudotriangulations, and Betti numbers of certain monomial ideals. It is then left as an open-ended problem to find and identify other set of objects counted by Borel's triangle. Here, we will discuss several of these sets counted by Borel's triangle and introduce a strategy utilizing the structure of Catalan's triangle to identify more of these sets.

As $f_{n,k}$ is defined in terms of the rows of Catalan's triangle, then for many sets whose elements are counted by the rows of Catalan's triangle, we can identify similar sets whose elements are counted by the rows of Borel's triangle. In particular, the entries in Borel's triangle are defined by applying certain binomial coefficients to the row's of Catalan's triangle, so the elements of sets counted by Borel's triangle can be constructed by "choosing" or "marking" a fixed number of a certain property of elements of sets counted by Catalan's triangle.

Using this notion of "marking", we introduce the following strategy for finding classes of objects counted by Borel's triangle. First, we begin with a set of objects which are counted by the classic Catalan numbers. Then, we attempt to identify an additional parameter or property which allows us to further classify this into subsets counted by the rows of Catalan's triangle. Next, we "mark" a certain fixed number of this property to create new sets whose elements are counted by Borel's triangle. Lastly, we aim to recontextualize these "markings" to find other sets which are counted by Borel's triangle. In [3], Cai and Yan have done similar work simultaneously and independently from ours, in which they discuss other marked Catalan structures and sets counted by Borel's triangle.

In Chapter 2, we will proceed to discuss several sets whose elements are counted by the Catalan numbers and their classification into subsets whose elements are counted by the rows of Catalan's triangle.

In Chapter 3, we will discuss several sets whose elements are counted by the rows of Borel's triangle, many of which are closely related to sets discussed in Chapter 2 whose elements now have some marked property.

Chapter 2

Structures Counted by Catalan's Triangle

2.1 Dyck Paths and Ballot Sequences

Definition 2.1.1. A *Dyck Path* of length $2n$ is a sequence of n *up* steps (labeled U) and n *down* steps (labeled D) for which each initial segment of the sequence has at least as many *up* steps as *down* steps.

A Dyck path can be visually represented as a "mountain range" with n upstrokes and n down strokes which never dips below the horizon. In this way, we refer to a *peak* as the point between an *up* step and a *down* step which immediately follows it, and the *height* of that peak is the number of *down* steps needed in order for the path to return to its original height.

Proposition 2.1.2. *The entry $C_{n,k}$ in Catalan's Triangle counts the number of Dyck paths of length $2(n+1)$ in which the height of the first peak is $n-k+1$.*

Proof. Let $X_{n,k}$ denote the number of Dyck paths of length $2(n+1)$ in which the height of the first peak is $n-k+1$. In order to show $C_{n,k}$ counts $X_{n,k}$, we will show $X_{n,k} = X_{n,k-1} + X_{n-1,k}$ and $X_{0,0} = 1$.

First, note that there is exactly one path of length 2 where the height of the first peak is 1 (namely, the path given by one *up* step followed by one *down* step), so $X_{0,0} = 1$.

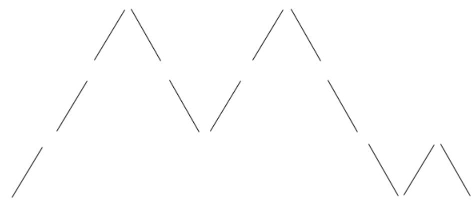

Figure 2.1: A Dyck path of length 12 and with a first peak of height 3 and ia final peak of height 1

Now, let A be a Dyck path of length $2(n+1)$ where the height of the first peak is $n-k+1 = h$. We can write $A = a_1, a_2, \ldots, a_{2n+2}$ as a sequence of *up* steps and *down* steps where each a_i is either U (denoting an *up* step) or D (denoting a *down* step). As the height of the first peak of A is h, then the first *down* step in A comes after h *up* steps. That is, $a_{h+1} = D$ and for $i \leq h$, $a_i = U$. Consider the two cases where $a_{h+2} = U$ and $a_{h+2} = D$.

Case 1 $(a_{h+2} = U)$:Since $a_{h+2} = U$, then the first *down* step in A is immediately followed by an *up* step. Consider transforming A by swapping a_{h+1} and a_{h+2}. This gives us a sequence of the same length in which the first *down* step comes after $h+1$ *up* steps. In this way, we can uniquely associate A with a Dyck path of length $2(n+1)$ where the height of the first peak is $n-(k-1)+1$.

Case 2 $(a_{h+2} = D)$: Since $a_{h+2} = D$, then the first *down* step in A is immediately followed by another *down* step. Consider transforming A by removing a_h and a_{h+1}. This gives us a sequence of length $2n$ where the first *down* step comes after $h-1$ *up* steps. In this way, we can uniquely associate A with a Dyck path of length $2n$ where the height of the first peak is $n-k+1$.

Since each Dyck path of length $2(n+1)$ in which the height of the first peak is $n-k+1$ can be made uniquely by modifying either a Dyck path of length $2(n+1)$ in which the height of the first peak is $n-(k-1)+1$ or a Dyck path of length $2n$ in which the height of the first peak is $n-k+1$, then $X_{n,k} = X_{n,k-1} + X_{n-1,k}$. Therefore, $C_{n,k}$ counts $X_{n,k}$. □

Proposition 2.1.3. *The number of Dyck paths of length $2(n+1)$ in which the height of the first peak is $n-k+1$ is in natural bijection with the number of Dyck paths of length $2(n+1)$ in which the height of the final peak is $n-k+1$.*

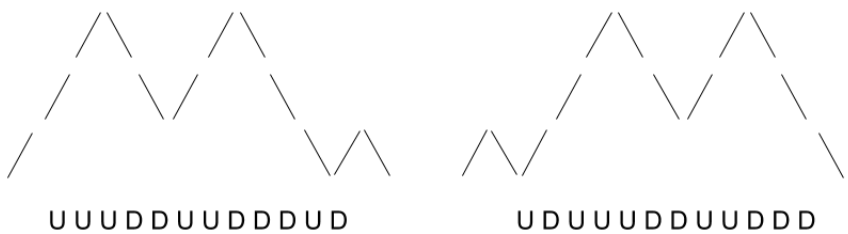

Figure 2.2: A Dyck path of length 12 and with a first peak of height 3 and its corresponding path of length 12 with a final peak of height 3

Proof. Let A be a Dyck path of length $2(n+1)$ where the height of the first peak is $n-k+1$. As before, write $A = a_1, a_2, \ldots, a_{2n+2}$ as a sequence of *up* steps and *down* steps where each a_i is either U or D. Define a new sequence $B = b_1, b_2, \ldots, b_{2n+2}$ by $b_i = -a_{2n+2-i+1}$ where negation in this sense changes an *up* step to a *down* step and vice versa.

This creates a new sequence of the same length by reversing and negating our original sequence; however, this is visually represented by a horizontal reflection of the diagram given by the original sequence. Hence, the height of the final peak in our new sequence B is the same as the height of the first peak in our original sequence A. □

Proposition 2.1.4. *The number of Dyck paths of length $2(n+1)$ in which the height of the first peak is $n-k+1$ is in natural bijection with the number of ballot sequences of elections between two parties each receiving $n+1$ votes in which the first party to receive a vote receives* *exactly k votes after the second party's first vote, and the second party never holds a majority of the votes at any stage in the count.*

Proof. Let A be a Dyck path of length $2(n+1)$ where the height of the first peak is $n-k+1$. As before, write $A = a_1, a_2, \ldots, a_{2n+2}$ as a sequence of *up* steps and *down* steps where each a_i is either U or D. We can view this sequence as a ballot sequence of an election between two parties in which each *up* step is a vote for Party X and each *down* step is a vote for Party Y.

As A has length $2(n+1)$ with an equal number of *up* and *down* steps, then each party receives exactly $n+1$ votes, and since the height of the first peak of A is $n-k+1$, then the first $n-k+1$ steps are all *up* steps, so the first $n-k+1$ votes are all for Party X with Party Y receiving the $(n-k)$th vote. Party X then receives exactly k more votes after Party Y receives its first vote. As A is a valid Dyck path, then at no point in the sequence are there more *down* steps than *up* steps (i.e. the path never dips below the height on which it began), so at no stage in the count are there more votes for Party Y than for Party X. This gives us our ballot sequence with the desired properties. □

2.2 Pattern Avoiding Permutations

Definition 2.2.1. Consider a permutation σ. We say σ has a 321 *pattern* if there exist $l < m < n$ such that $\sigma(n) < \sigma(m) < \sigma(l)$. Further, we say this pattern is *consecutive* if $m = l+1$ and $n = l+2$. We similarly define a 231 *pattern* if there exist $l < m < n$ such that $\sigma(n) < \sigma(l) < \sigma(m)$. We say a permutation is 321-*avoiding* if it contains no 321 patterns

Proposition 2.2.2. *The number of Dyck paths of length $2(n+1)$ in which the height of the first peak is $n-k+1$ is in natural bijection with the number of 321-avoiding, fixed-point-free involutions, σ, on $[2(n+1)]$ where $\sigma(1) = n-k+2$.*

Proof. Let A be a Dyck path of length $2(n+1)$ where the height of the first peak is $n-k+1 = h$, and write $A = a_1, a_2, \ldots, a_{2n+2}$ as a sequence of *up* steps and *down* steps where each a_i is either U or D. As the height of the first peak of A is h, then a_{h+1} is the first *down* step in the sequence. We will define a permutation σ by transposing the ith instance of an *up* step in the sequence with the ith instance of *down* step in the sequence. That is, if a_r the

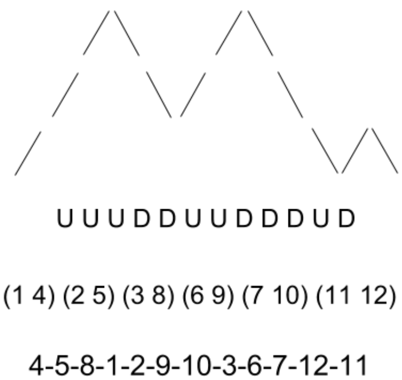

Figure 2.3: A Dyck path of length 12 and with a first peak of height 3 and its corresponding 321-avoiding fixed-point-free involution on [12] containing the transposition (1 4)

ith instance of an *up* step, and a_s is the ith instance of a *down* step, then σ contains the transposition $(r\ \ s)$. Clearly, this defines σ as a product of $n+1$ disjoint transpositions (i.e. a fixed-point-free involution on $[2(n+1)]$), so we need to show that σ avoids the pattern 321.

Suppose for contradiction that σ contains a 321 pattern (that is, there exist $x < y < z$ such that $\sigma(x) > \sigma(y) > \sigma(z)$). As $x < y$ and $\sigma(x) > \sigma(y)$, it must be true that a_x is an *up* step while a_y is a *down* step (if a_x and a_y were both *up* steps or both *down* steps, then this would require $\sigma(x) < \sigma(y)$). In the same way, as $y < z$ and $\sigma(y) > \sigma(z)$, then it must be true that a_y is an *up* step while a_z is a *down* step. However, this requires that a_y be both an *up* step and a *down* step, and this is a contradiction.

To see that this is invertible, let σ be a 321-avoiding fixed-point-free involution on $[2(n+1)]$ containing the transposition $(1\ \ h+1)$. As σ is the product of $n+1$ disjoint transpositions, we can uniquely construct a Dyck path $A = a_1, a_2, \ldots, a_{2n+2}$ in the following way: If $(i\ \ j)$ is a transposition in σ where $i < j$, then define $a_i = U$ and $a_j = D$. As each *down* step is paired with an *up* step preceding it, then this gives a valid Dyck path in that there are never more *down* steps than *up* steps at any stage in the sequence, and as σ contains the transposition $(1\ \ h+1)$, then a_{h+1} is the first *down* step in the sequence, and the height of the first peak of A is h. $\square$

2.3 Triangulations, Parenthesizations, and Full Binary Trees

Proposition 2.3.1. *The number of Dyck paths of length* $2(n+1)$ *in which the height of the final peak is* $n-k+1$ *is in natural bijection with the number of triangulations of a regular polygon with* $n+3$ *sides where* k *of the diagonals do not contain a given fixed vertex.*

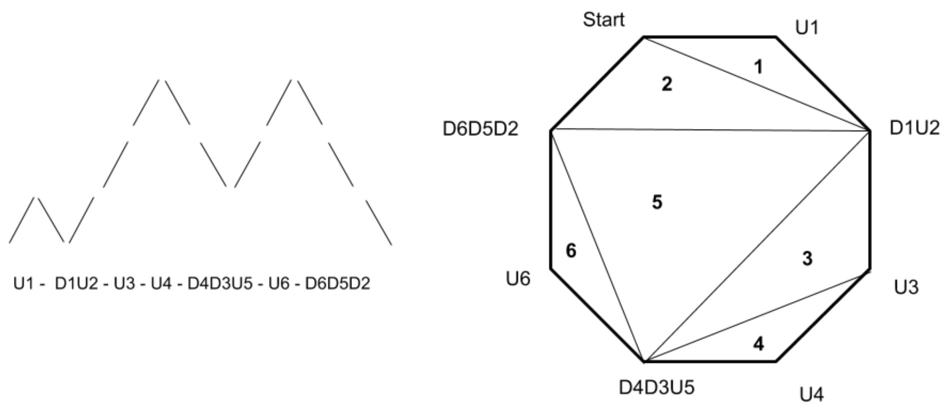

Figure 2.4: A Dyck path of length 12 and its corresponding triangulation of a regular octagon

Proof. Let A be a Dyck path of length $2(n+1)$ where the height of the final peak is $n-k+1$. Note that since the length of the path is $2(n+1)$, then there are $n+1$ total *down* steps, so there are k down steps which are not included in the final descent. We will show a method of mapping A to a triangulation of a regular $(n+2)$-sided polygon.

First, we want to pair together certain *up* and *down* steps of A in the following way: we label the *up* steps in increasing order, then starting with the first *down* step, we will pair each *down* step, with the most recent unpaired *up* step. For example, the sequence $UDUUUDDUUDDD$ becomes the paired sequence $U_1D_1U_2U_3U_4D_4D_3U_5U_6D_6D_5D_2$ where each U_i is paired with D_i.

Now, we want to divide this paired sequence into $(n+2)$ segments in the following way: the first segment will simply be U_1 and each subsequent segment will be the sequence of steps beginning immediately after the end of the last segment and ending with the next *up* step (except for the final segment, in which case the final $(n-k+1)$ *down* steps comprise the final segment). For example, the paired sequence $U_1D_1U_2U_3U_4D_4D_3U_5U_6D_6D_5D_2$ becomes the segmented sequence $U_1 - D_1U_2 - U_3 - U_4 - D_4D_3U_5 - U_6 - D_6D_5D_2$.

Now, we will assign this segmented sequence to the vertices of an $(n+3)$ sided polygon, and use that assignment to construct a triangulation. First, label one vertex of the polygon as v_0, and continue clockwise around the polygon labeling the vertices $v_1, v_2, \ldots, v_{n+2}$. Next, we assign the first segment of our segmented sequence to v_1, the second segment of our sequence to v_2, and so on. Now, to triangulate the polygon, we draw diagonals in the following way: treating each *down* step in the order in which they occur in the original sequence, draw a diagonal from the vertex assigned to D_i to the lowest-numbered vertex connected to the vertex assigned to U_i.

In our example, we connect v_2 to v_0 as v_0 is the lowest numbered vertex connected to v_1. Then, we connect v_5 to v_3 as v_3 is the lowest numbered vertex connected to v_4 and connect v_5 to v_2 as v_2 is the lowest numbered vertex connected to v_2. Finally, we connect v_7 to v_5 as v_5 is the lowest numbered vertex connected to v_6 and connect v_7 to v_2 as v_2 is the lowest numbered vertex connected to v_5 (note that the last *down* step, D_2 would have us connect v_7, to v_0, since v_2 is connected to v_0, but as this is the final *down* step, v_7 and v_0 are already connected by a side of the polygon).

Note that since there were k *down* steps which were not included in the final descent, there were k *down* steps which were not included in the final sequence, so there are k diagonals in the triangulation of the polygon which are not connected to the final vertex v_{n+2}. Since we can uniquely define a paired sequence from a Dyck path, can uniquely define a segmented sequence from a paired sequence, and can uniquely define a triangulation of an $(n+3)$-sided polygon from a segmented sequence, then we can uniquely define such a triangulation from a Dyck path.

To show this invertible, we will show how to uniquely define a Dyck path from a triangulation of a $(n+3)$-sided polygon. Begin with a triangulation of a $(n+3)$-sided polygon in which k of the vertices are not connected to a given vertex v_{n+2} and label the vertices $v_0, v_1, \ldots, v_{n+2}$ in a clockwise manner as before. This triangulation divides the polygon into $n+1$ triangles, so we will create a segmented sequence in the following manner: for each of these $n+1$ triangles, and the three vertices v_i, v_j, v_l comprising it (with $i < j < l$), add an *up* step to the jth segment and a *down* step to the lth segment of the segmented sequence.

As each vertex can only be the second-highest-numbered vertex for at most one triangle (and for exactly one triangle for all vertices except v_0 and v_{n+2}), then each of the first $n+1$ segments contains exactly one *up* step, so make the *up* step the last step in each segment. This gives us $n+2$ segments, the first $n+1$ of which end in an *up* step. By concatenating these segments together, this gives us a new sequence which precisely defines a Dyck path. Further, as k of the diagonals are not connected to v_{n+2}, then then the final segment of the segmented sequence contains $(n+1-k)$ *down* steps, so the height of the final peak of the Dyck path is $(n+1-k)$. □

Proposition 2.3.2. *The number of triangulations of a regular polygon with $n+3$ sides where k of the diagonals do not contain a given fixed vertex is in natural bijection with the number of parenthesizations of $n+2$ elements where k sets of the parentheses do not contain a right parenthesis on the far right of all of the elements.*

Proof. Consider a regular $(n+3)$-sided polygon whose vertices are labaled in clockwise order around it $v_0, v_1, v_2, \ldots, v_{n+2}$. Now consider some triangulation that polygon where k of the diagonals are not connected to the vertex v_{n+2}.

Now consider the currently unparenthesized product $x_1 \cdot x_2 \cdot \ \ldots \ \cdot x_{n+1}$. We will add parentheses to the product in the following way: if v_i is connected to v_j by a diagonal where $i < j$, add a set of parentheses to the product with the left parenthesis between the elements x_i and x_{i+1} (or on the far left of the product if $i = 0$) and a right parenthesis between the elements x_j and x_{j+1} (or on the far right of the product if $j = n+2$). To ensure this is a valid parenthesization, then in each space between two adjacent elements, place the right parentheses to the left of all left parentheses within that space.

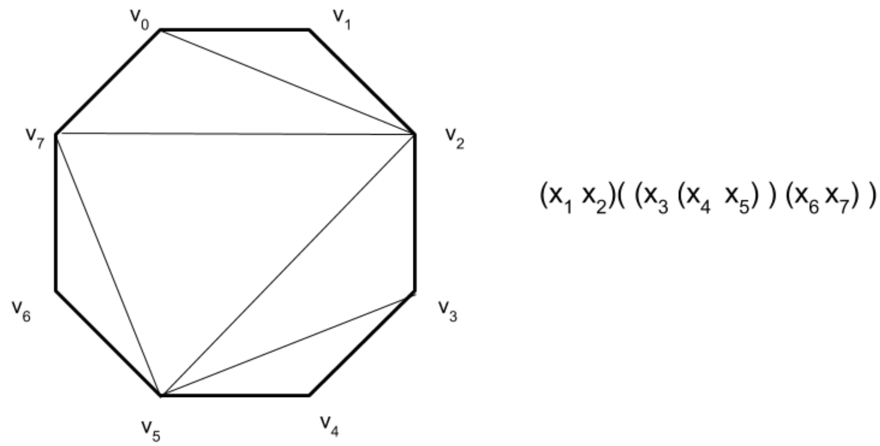

Figure 2.5: A triangulation of a regular octagon and its corresponding parenthesization of the product of 7 elements

This gives us a valid parenthesization of the product; as no two diagonals cross each other in the triangulation, this leads to a well-defined notion of pairs of parentheses in our product. Further, as there are k diagonals which are not connected to v_{n+2}, then there are k right parentheses in the parenthesized product which are not immediately to the right of x_{n+2}. That is, there are k right parentheses which are not at the far right of all elements in the product.

To show this is invertible, consider a parenthesization of the product of $n+2$ elements $x_1 \cdot x_2 \cdot \ \dots \ \cdot x_{n+2}$ where k sets of these parentheses do not contain a right parenthesis on the far right of the product. We can uniquely define a triangulation of a $(n+3)$-sided polygon by first pairing up each set of parentheses in the parenthesization in the natural way. Then, for each pair of parentheses in which the left parenthesis occurs between elements x_i and x_{i+1} (or on the far left of the product if $i = 0$) and the right parenthesis occurs between the elements x_j and x_{j+1} (or on the far right of the product if $j = n+2$), then we will connect the vertices v_i and v_j in the triangulation.

As we paired the parentheses in the natural way, none of these constructed diagonals will cross each other, and as k of these right parentheses did not occur to the right of x_{n+2}, then k of these diagonals are not connected to v_{n+2}. □

Let T be a binary tree. Recall that T has a unique vertex known as the *root* vertex, v_r. For two vertices v_1 and v_2, if there is an edge between v_1 and v_2 and the path from v_r to v_2 passes through v_1, then we say v_2 is a *child* of v_1, and v_1 is the *parent* of v_2. If the edge connecting a parent and child has a positive slope, we say that the child is a *left child* of the parent and that the edge is a *left edge.* Alternatively if the edge has a negative slope, we say that the child is a *right child* of the parent and that the edge is a *right edge.*

A *leaf* vertex (or *leaf node*) is a vertex with no children. A *branching* vertex (or *branching node*) is a vertex with exactly two children. Note that the *root* vertex is the unique vertex which does not have a parent. A *full* binary tree is a binary tree in which every vertex is either a branching vertex or a leaf vertex.

Definition 2.3.3. Let T be a full binary tree. The *rightmost leaf* of T is the leaf found by starting at the root vertex and taking only right edges until reaching a leaf node. The *rightmost branch* is the path from the root vertex to the rightmost leaf.

Remark 2.3.4. Note that for a full binary tree T, T must have an odd number of vertices and must have one more leaf vertex than branching vertices. That is, T has $2n + 1$ vertices for some positive whole number n; n of these vertices must be branching vertices, and the other $n + 1$ must be leaf vertices.

This fact may be shown inductively; a full binary tree with 1 vertex is simply a single leaf vertex. The only way to add vertices to the tree while maintaining its fullness is to add 2 children to a leaf vertex, turning that leaf vertex into a branch vertex and adding 2 additional leaf vertices. Hence this increases both the number of leaf vertices and the number of branching vertices by 1.

Proposition 2.3.5. *The number of parenthesizations of $n + 2$ elements where k sets of the parentheses do not contain a right parenthesis on the far right of all of the elements is in natural bijection with the number of full binary trees with $2n + 3$ vertices with k branching vertices not contained in the rightmost branch.*

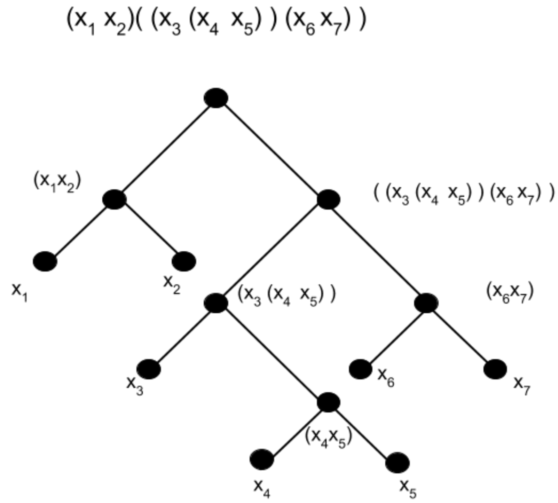

Figure 2.6: A parenthesization of the product of 7 elements and its corresponding full binary tree with 13 vertices

Proof. Consider a parenthesization of a product of $n+2$ elements $x_1 \cdot x_2 \cdot \ \ldots \ \cdot x_{n+2}$ in which there are k right parentheses not on the far right of the product. Note that our parenthesization has n pairs of parentheses, leading to $n+1$ instances of multiplication. We will construct a binary tree of $n+1$ branching vertices and $n+2$ leaf vertices where each instance of multiplication corresponds to a branching vertex and the left and right factors of that instance of multiplication correspond to the left and right children of that vertex.

First, assign the parenthesized product to the root vertex of the tree. By convention, the parenthesized product does not have a pair of parentheses surrounding it, so the product is of the form $A \cdot B$ where both A and B are either a single element or are themselves a parenthesized product. We create a right and left child vertex from the root node, assign A to the left child and B to the right child.

We continue the above process for all other instances of multiplication of the form $(A \cdot B)$ where $(A \cdot B)$ is assigned to a branching vertex, A is assigned to the left child of that vertex and B is assigned to the right child of that vertex. If at any point in the process a single element is assigned to a vertex, then that vertex is a leaf vertex.

As our parenthesization contained $n+1$ instances of multiplication, then there are $n+1$ branching vertices in the newly constructed tree, and as our product contained $n+2$ elements, then there are $n+2$ leaf vertices in the tree. Further, as the parenthesization contains k right parentheses which are not at the far right of the product, then there are k instances of multiplication which do not involve the element x_{n+2} and hence, there are k branching vertices which are not contained in the rightmost path.

To see this is invertible, begin with a full binary tree T which has $2n+3$ vertices where k of the branching vertices are not included in the rightmost branch. As we can see from Remark 2.3.4, T has $n+2$ leaf vertices and $n+1$ branching vertices. We will assign parenthesized products to each vertex in the following way. First, for each leaf vertex, assign the element x to that vertex. Then, for each branching vertex, aside from the root vertex, where A is the parenthesized product assigned to its left child and B is the parenthesized product assigned to its right child, assign the product $(A \cdot B)$ to that vertex. Finally, for the root vertex, assign the product $A \cdot B$ where A is the product assigned to the left child of the root vertex and B is the product assigned to the right child of the root vertex.

As each branching vertex corresponds to an instance of multiplication and each leaf vertex corresponds to an identical copy of the element x, then the parenthesized product corresponding to the root vertex includes $n+1$ instances of multiplication and $n+2$ copies of the element x. We can then number these elements from left to right $x_1, \ldots, x_{n+2}$. As k of the branching vertices are not contained in the rightmost branch, then k of the instances of multiplication do not involve the element x_{n+2}, so there are k right parentheses which are not on the far right of the product.

Thus, the parenthesized product corresponding to the root node of the tree is exactly a parenthesization of a product of $n+2$ identical elements where k of the right parentheses are not on the far right of the product. □

Theorem 2.3.6. *Let $n \geq 1$ and $k \leq n-1$. Then the entry in Catalan's triangle $C_{n,k}$ counts* *each of the following sets, and there exist natural bijections between them:*

(i) The number of Dyck paths of length $2(n+1)$ *in which the height of the first peak is* $n-k+1$.

(ii) The number of Dyck paths of length $2(n+1)$ *in which the height of the final peak is* $n-k+1$.

(iii) The number of ballot sequences of elections between two parties each receiving $n+1$ *votes in which the first party to receive a vote receives exactly* k *votes after the second party's first vote, and the second party never holds a majority of the votes at any stage in the count.*

(iv) The number of 321*-avoiding, fixed-point-free involutions,* σ*, on* $[2(n+1)]$ *where* $\sigma(1) = n-k+2$

(v) The number of triangulations of a regular polygon with $n+3$ *sides where* k *of the diagonals do not contain a given fixed vertex.*

(vi) The number of parenthesizations of $n+2$ *elements where* k *sets of the parentheses do not contain a right parenthesis on the far right of all of the elements.*

(vii) The number of full binary trees with $2n+3$ *vertices with* k *branching vertices not contained in the rightmost branch.*

Proof. By Proposition 2.1.2, item (i) is counted by $C_{n,k}$. By Proposition 2.1.3, items (i) and (ii) are in bijection with each other, and by Proposition 2.1.4, items (i) and (iii) are in bijection with each other. By Proposition 2.2.2, items (ii) and (iv) are in bijection, and Propositions 2.3.1, 2.3.2, and **??** show that items (ii), (v), (vi), and (vii) are all in bijection. Therefore, all of the listed items are in bijection with each other and are all counted by the entry in Catalan's triangle $C_{n,k}$. □

Chapter 3

Structures Counted by Borel's Triangle

3.1 Marked Dyck Paths and Three-Party Ballot Sequences

Proposition 3.1.1. *The entry in Borel's triangle $f_{n,k}$ counts the number of marked Dyck paths of length $2(n+1)$ with k marked non-initial up steps.*

Proof. Let A be a Dyck path of length $2(n+1)$ with s non-initial *up* steps, k of which are marked. As shown in Proposition 2.1.2, the total number of possible Dyck paths of length $2(n+1)$ with s non-initial *up* steps is $C_{n,s}$, and the number of possible ways to mark k of these s non-initial *up* steps is $\binom{s}{k}$. Therefore, the total number of possible ways to mark k non-initial *up* steps in a Dyck path of length $2(n+1)$ is the sum over all possible s,

$$\sum_{s=0}^{n} \binom{s}{k} C_{n,k}$$

which is exactly our definition of $f_{n,k}$. □

Proposition 3.1.2. *The number of marked Dyck paths of length $2(n+1)$ with k marked non-initial up steps is in natural bijection with the number of marked Dyck paths of length $2(n+1)$ with k marked non-terminal down steps.*

Proof. Let A be a Dyck path of length $2(n+1)$ with k marked non-initial *up* steps. Write $A = a_1, a_2, \ldots, a_{2n+2}$ as a sequence of *up* steps and *down* steps where each a_i is either an unmarked *up* step, U, a marked *up* step, U^*, or an unmarked *down* step, D. Following the

Figure 3.1: A Dyck path of length 12 with 2 marked non-initial *up* steps and its corresponding path of length 12 with 2 marked non-terminal *down* steps

reasoning of the proof of Proposition 2.1.3, we define a new sequence $B = b_1, b_2, \ldots, b_{2n+2}$ by $b_i = -a_{2n+2-i+1}$ where negation in this sense changes an unmarked *up* step to an unmarked *down* step, a marked *up* step to a marked *down* step, and a *down* step to an *up* step.

As before, this creates a new sequence of the same length. As none of the marked *up* steps of A were part of the first peak, none of the marked *down* steps of B are part of the last peak, so B is a sequence of the same length as A with k marked non-terminal *down* steps. □

Proposition 3.1.3. *The number of marked Dyck paths of length* $2(n+1)$ *with* k *marked non-initial up steps is in natural bijection with the number of marked Dyck paths of length* $2(n+k+1)$ *with* k *marked non-initial* $up - up - down$ *patterns.*

Proof. Let A be a Dyck path of length $2(n+1)$ with k marked non-initial *up* steps. As before, write $A = a_1, a_2, \ldots, a_{2n+2}$ as a sequence of *up* steps and *down* steps where each a_i is either an *up* step, U or a *down* step, D and a marked *up* step or marked *down* step is denoted by U^* or D^* respectively. Define a new sequence B by inserting a marked *up* step and a marked *down* step immediately following each marked *up* step of A.

This increases the length of the sequence by 2 for each marked *up* step in A, so our new sequence B is a sequence of length $2(n+k+1)$, and since A has k non-initial *up* steps, then B has k marked $up - up - down$ patterns following the first peak.

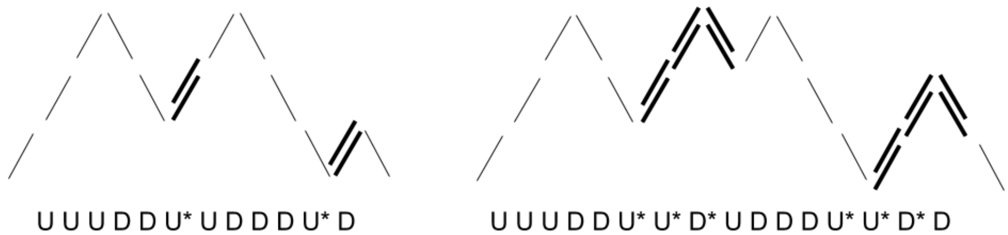

Figure 3.2: A Dyck path of length 12 with 2 marked non-initial *up* steps and its corresponding path of length 16 with 2 marked non-initial $up - up - down$ patterns

To show this is invertible, let B be a Dyck path of length $2(n + k + 1)$ with k marked $up - up - down$ patterns following the first peak. By removing the second *up* step and the *down* step in each of these patterns, we decrease the length of the sequence by $2k$ and are left with a sequence of length $2(n + 1)$ with k marked *up* steps following the first peak. □

Proposition 3.1.4. *The number of marked Dyck paths of length $2(n + 1)$ with k marked non-initial up steps is in natural bijection with the number of ballot sequences of elections between three parties in which the total number of votes cast is $2(n + 1)$, the second party to receive a vote receives exactly half the votes, the third party to receive a vote receives k votes, and the second party never holds a majority of the votes at any stage in the count.*

Proof. Let A be a Dyck path of length $2(n+1)$ with k marked non-initial *up* steps. As before, write $A = a_1, a_2, \ldots, a_{2n+2}$ as a sequence of *up* steps and *down* steps where each a_i is either an unmarked *up* step, U, a marked *up* step, U^*, or an unmarked *down* step, D. We can view this sequence as a ballot sequence between three parties in which each unmarked *up* step is a vote for Party X, each *down* step is a vote for Party Y, and each marked *up* step is a vote for Party Z.

As A has length $2(n + 1)$, and has an equal number of *up* and *down* steps, then Party Y receives exactly $(n + 1)$ votes, and since A has exactly k marked *up* steps, then Party Z receives exactly k votes and party X receives the remaining $(n - k + 1)$ votes. As the first step must be an unmarked *up* step and the first marked *up* step must come after the first *down* step, then Party X must be the first party to receive a vote and Party Y must be the second party to receive a vote. As A is a valid Dyck path, then at no point in the sequence are there more *down* steps than *up* steps (i.e. the path never dips below the height on which it began), so at no stage in the count does Party Y obtain a majority of the votes. This gives us our ballot sequence with the desired properties. □

3.2 Marked Triangulations, Parenthesizations, and Full Binary Trees

Proposition 3.2.1. *The number of marked Dyck paths of length* $2(n+1)$ *with* k *marked non-terminal down steps is in natural bijection with the number of triangulations of a regular polygon with* $n+3$ *sides with* k *marked diagonals, each not containing a given fixed vertex.*

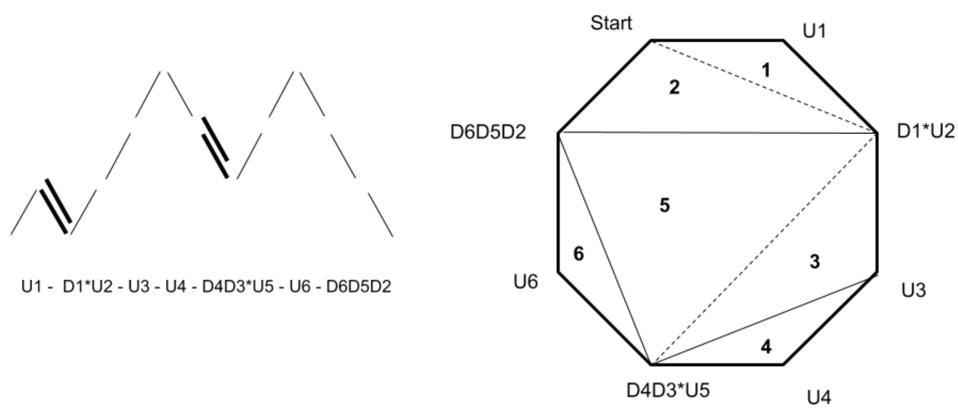

Figure 3.3: A Dyck path of length 12 with 2 marked non-terminal *down* steps and its corresponding triangulation of a regular octagon with 2 marked diagonals not connected to a given vertex

Proof. Let A be a Dyck path of length $2(n+1)$ with k marked non-terminal *down* steps. Following the methods outlined in the proof of Proposition 2.3.1, we want to create a segmented paired sequence from A which contains $n+2$ segments where each segment but the final segment ends in an *up* step.

Again as before, we assign this segmented to the vertices of a $(n + 3)$-sided polygon by labeling one vertex as v_0, labeling the remaining vertices $v_1, v_2, \ldots, v_{n+2}$ in clockwise order, and assigning the ith segment of the sequence to the vertex labeled v_i.

Now, to triangulate the polygon, we draw diagonals using the same process as before: treating each *down* step in the order in which they occur in the original sequence, draw a diagonal from the vertex assigned to D_i to the lowest-numbered vertex connected to the vertex assigned to U_i; however, in this case we add the further stipulation that if D_i is a marked step, then the diagonal drawn from the vertex assigned to D_i to the lowest-numbered vertex connected to the vertex assigned to U_i is a marked diagonal.

Note that since none of the marked *down* steps of A were part of the final descent, then none of the marked *down* steps are assigned to the vertex v_{n+2}, so none of the marked diagonals in the triangulation are connected to the vertex v_{n+2}. In this way, we've constructed a triangulation of a $(n+3)$-sided polygon with k marked diagonals, none of which are connected to the vertex v_{n+2}.

To show this invertible, begin with a triangulation of a $(n + 3)$-sided polygon containing k marked diagonals each not connected to a vertex v_{n+2}. We use the methods outlined in the proof of Proposition 2.3.1 to uniquely define a Dyck path from a triangulation of a $(n + 3)$-sided polygon. Then, for each *down* step in the path corresponding to a marked diagonal in the triangulation, we will mark that *down* step. Since the triangulation had k marked diagonals, this path will have k marked *down* steps, and as each marked diagonal is not connected to the vertex v_{n+2}, then each marked *down* step is not part of the final descent. □

Proposition 3.2.2. *The number of triangulations of a regular polygon with $n + 3$ sides with k marked diagonals, each not containing a given fixed vertex, is in natural bijection with the number of parenthesizations of $n + 2$ elements with k marked sets of parentheses, none of which contain a right parenthesis on the far right of all of the elements.*

Proof. Consider some triangulation of a regular $(n + 3)$-sided polygon containing k marked

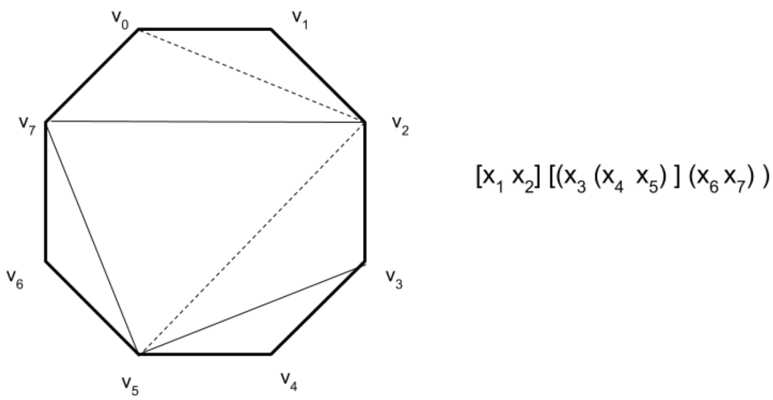

Figure 3.4: A triangulation of a regular octagon with 2 marked diagonals not connected to a given vertex and its corresponding parenthesization of the product of 7 elements with 2 marked pairs of parentheses

diagonals which are not connected to some vertex v_{n+2}. First, label one vertex of the polygon as v_0, and continue clockwise around the polygon labeling the vertices $v_1, v_2, \ldots, v_{n+1}$. Without loss of generality, suppose v_{n+2} is the given vertex where the marked diagonals in the triangulation are not connected to v_{n+2}.

As in the proof of Proposition 2.3.2, consider the currently unparenthesized product $x_1 \cdot x_2 \cdot \ldots \cdot x_{n+1}$. We will add parentheses to the product in the following way: if v_i is connected to v_j by a diagonal where $i < j$, add a set of parentheses to the product with the left parenthesis between the elements x_i and x_{i+1} (or on the far left of the product if $i = 0$) and a right parenthesis between the elements x_j and x_{j+1} (or on the far right of the product if $j = n+2$). To ensure this is a valid parenthesization, then in each space between two adjacent elements, place the right parentheses to the left of all left parentheses within that space. Further, if the diagonal connecting v_i and v_j is marked, then the corresponding pair of parentheses is a marked pair of parentheses (here denoted as a pair of square brackets).

Again, this gives us a valid parenthesization of the product with a well-defined notion of pairs of parentheses. Further, as the triangulation contains k marked diagonals—none of which are connected to v_{n+2}—the parenthesization of the product contains k marked pairs of parentheses—none of which contain a parenthesis immediately to the right of x_{n+2}.

To see this is invertible, begin with a parenthesization of the product of $n+2$ elements in which k pairs of these parentheses are marked, and none of the marked pairs of parentheses include a right parenthesis on the far right of the product. We construct a triangulation of a $(n+3)$-sided polygon using the methods outlined in the proof of Proposition 2.3.2; however, we further stipulate that if a pair of parentheses is marked, then its corresponding diagonal is marked as well.

Since this parenthesization had k pairs of marked parentheses, the corresponding triangulation has k marked diagonals, and as none of the pairs of marked parentheses contain a right parenthesis at the far right of the product, none of the marked diagonals are connected to the vertex v_{n+2}. □

Proposition 3.2.3. *The number of parenthesizations of* $n+2$ *elements with* k *marked sets of parentheses, none of which contain a right parenthesis on the far right of all of the elements, is in natural bijection with the number of binary trees with* $2n+3$ *unmarked vertices,* k *of which are marked vertices not contained in the rightmost branch.*

Proof. Consider a parenthesization of a product of $n+2$ elements $x_1 \cdot x_2 \cdot \ \ldots \ \cdot x_{n+2}$ with k pairs of marked parentheses—none of which contain a right parenthesis at the far right of the product. Note that our parenthesization has n pairs of parentheses, leading to $n+1$ instances of multiplication. Following the methods outlined in the proof of Proposition 2.3.5 we will construct a binary tree of $n+1$ branching vertices and $n+2$ leaf vertices where each instance of multiplication corresponds to a branching vertex and the left and right factors of that instance of multiplication correspond to the left and right children of that vertex, now with the added condition that any vertex constructed by an instance of multiplication with a marked pair of parentheses will be a marked vertex.

First, assign the parenthesized product to the root vertex of the tree. By convention, the parenthesized product does not have a pair of parentheses surrounding it, so the product is of the form $A \cdot B$ where both A and B are either a single element or are themselves a

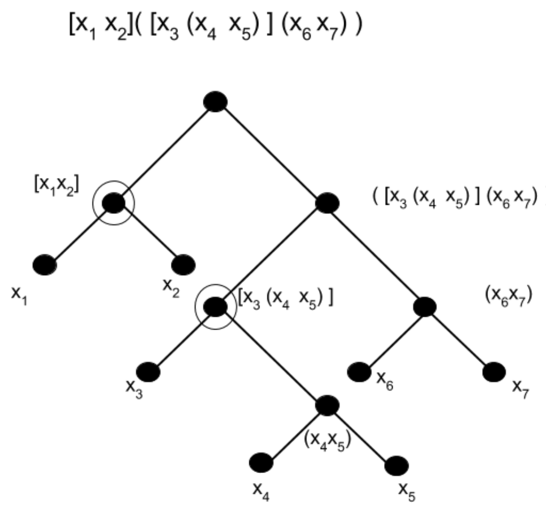

Figure 3.5: A parenthesization of the product of 7 elements with 2 marked pairs of parentheses and its corresponding full binary tree with 13 vertices and 2 marked branching vertices

parenthesized product. We create a right and left child vertex from the root node, assign A to the left child and B to the right child. Note that if either A or B is a product surrounded by a marked pair of parentheses, then the created vertex to which it is assigned is a marked vertex.

We continue the above process for all other instances of multiplication of the form $(A \cdot B)$ where $(A \cdot B)$ is assigned to a branching vertex, A is assigned to the left child of that vertex and B is assigned to the right child of that vertex. If at any point in the process a single element is assigned to a vertex, then that vertex is a leaf vertex. Similarly, for all instances of multiplication of the form $[A \cdot B]$ where $[A \cdot B]$ is assigned to a marked branching vertex, A and B are assigned to the right and left children of that vertex, respectively.

As our parenthesization contained $n+1$ instances of multiplication, then there are $n+1$ branching vertices in the newly constructed tree, and as our product contained $n+2$ elements, then there are $n+2$ leaf vertices in the tree. Further, as the parenthesization contains k marked pairs of parentheses—none of which contain a right parenthesis which are not at the far right of the product—then there are k instances of multiplication surrounded by marked parentheses which do not involve the element x_{n+2} and hence, there are k marked branching vertices which are not contained in the rightmost path.

To see this is invertible, begin with a full binary tree T which has $2n + 3$ vertices, k of which are marked branching vertices not contained in the rightmost branch. As before, and highlighted in Remark 2.3.4, T has $n + 2$ leaf vertices and $n + 1$ branching vertices. Again, we will assign parenthesized products to each vertex using the methods outlined in the proof of Preposition 2.3.5, now with the added condition that products assigned to marked vertices will now be surrounded by a marked pair of parentheses.

First, for each leaf vertex, assign the element x to that vertex. Then for each unmarked branching vertex, aside from the root vertex, where A is the parenthesized product assigned to its left child and B is the parenthesized product assigned to its right child, assign the product $(A \cdot B)$ to that vertex, and for each marked branching vertex where A is the parenthesized product assigned to its left child and B is the parenthesized product assigned to its right child, assign the product $[A \cdot B]$ to that vertex. Finally, for the root vertex, assign the product $A \cdot B$ where A is the product assigned to the left child of the root vertex and B is the product assigned to the right child of the root vertex.

As each branching vertex corresponds to an instance of multiplication and each leaf vertex corresponds to an identical copy of the element x, then the parenthesized product corresponding to the root vertex includes $n + 1$ instances of multiplication and $n + 2$ copies of the element x. We can then number these elements from left to right $x_1, \ldots, x_{n+2}$. As the tree has k marked branching vertices, each not contained in the rightmost branch, then k of the instances of multiplication are given by a marked pair of parentheses. Further, none of the instances of multiplication given by a marked pair of parentheses involve the element x_{n+2}, so none of the marked pairs of parentheses contain a right parenthesis on the far right of the product.

Thus, the parenthesized product corresponding to the root node of the tree is exactly a parenthesization of a product of $n+2$ identical elements including k marked pairs of parentheses, none of which contain a right parenthesis on the far right of the product. $\square$

3.3 More Permutations

Lemma 3.3.1. *For any Dyck path and corresponding* 321*-avoiding fixed-point-free involution, an* $up - up - down$ *pattern in the Dyck path corresponds to a consecutive* 231 *pattern in the corresponding permutation.*

Proof. Let $A = a_1, a_2, \ldots, a_{2n+2}$ be a Dyck path of length $2(n+1)$ and σ be its corresponding 321-avoiding fixed-point-free involution. Further, let a_x, a_{x+1}, a_{x+2} be an $up - up - down$ pattern (that is, $a_x = a_{x+1} = U$ and $a_{x+2} = D$). As a_x and a_{x+1} are both *up* steps, then by our bijection relating Dyck paths to permutations, $\sigma(x) > x$ and $\sigma(x+1) > x+1$, and since a_{x+2} is a *down* step, then $\sigma(x+2) < x+2$.

Further, as a_x is an *up* step which precedes a_{x+1} then a_x is paired with a *down* step which precedes the *down* step paired with a_{x+1}, therefore $\sigma(x) < \sigma(x+1)$. As A is a valid Dyck path, we know a_{x+1} is not paired with a_{x+2} (if these two steps were paired, then a_x must be paired with some down step a_y where $y < x$ meaning at some stage in the path, namely the first y steps of the path, there are more *down* steps than *up* steps), so $\sigma(x+2) < x+1$. Additionally, a_x is not paired with a_{x+1} as they are both *up* steps, so $\sigma(x) > x+1$. Therefore, $\sigma(x+2) < \sigma(x) < \sigma(x+1)$ which is exactly a consecutive 231 pattern. $\square$

Remark 3.3.2. Consider permutations σ on $[2(n+k+1)]$ with the following properties:

(a) $\sigma = \tau\rho_1\rho_2\cdots\rho_k$ where τ is the product of $n+1-k$ disjoint transpositions, one of which is $(1\ a)$.

(b) For each $i \in \{1,\ldots,k\}$, $\rho_i = (x_i\ \ c_i\ \ x_i+1\ \ b_i)$ with $a < x_i < x_i+1 < b_i < c_i$.

(c) For each $i \in \{1,\ldots,k\}$, $\sigma(x_i+2) < x_i+2$.

(d) If we define $\rho'_i = (x_i\ \ b_i)(x_i+1\ \ c_i)$, then $\sigma' = \tau\rho'_1\rho'_2\ldots\rho'_k$ is a 321-avoiding fixed-point-free involution.

We will show first that such permutations with the described properties have exactly k consecutive 321 patterns. Then, we will show that these permutations are counted by

Borel's triangle by demonstrating a bijection between them and Dyck paths with marked $up-up-down$ patterns.

Claim 3.3.3. *The permutations described in Remark 3.3.2 have exactly* k *consecutive* 321 *patterns.*

Proof. Let $\sigma = \tau\rho_1\rho_2\ldots\rho_k$ be a permutation described in Remark 3.3.2 and consider each $\rho_i = (x_i \;\; c_i \;\; x_i+1 \;\; b_i)$ for $i \in \{1,\ldots,k\}$. By property (b), we have that $\sigma(1) < x_i < x_i + 1 < b_i < c_i$, therefore, b_i is at least $x_i + 2$. By property (c), we have that $\sigma(x_i + 2) < x_i + 2$, therefore $\sigma(x_i + 2) < x_i + 2 \leq b_i$. In any case, we have $\sigma(x_i + 2) < \sigma(x_i + 1) < \sigma(x_i)$ which is exactly a consecutive 321 pattern.

To show that there are exactly k of these patterns, suppose for contradiction that there is an additional consecutive 321 pattern where $\sigma(m+2) < \sigma(m+1) < \sigma(m)$ and $m \neq x_i$ for all i. If this 321 pattern intersects one of the k aforementioned 321 patterns, then this means σ actually contains a consecutive 4321 pattern. Further, if this 321 pattern does not intersect any of the k previously counted 321 patterns, then $\sigma(m) = \sigma'(m)$, $\sigma(m+1) = \sigma'(m+1)$, and $\sigma(m+2) = \sigma'(m+2)$. In either case, this contradicts that σ' is 321-avoiding, so the only consecutive 321 patterns in σ are exactly the ones corresponding to each ρ_i. □

Proposition 3.3.4. *The number of marked Dyck paths of length* $2(n+k+1)$ *with* k *marked non-initial* $up-up-down$ *patterns is in natural bijection with the number of permutations* σ *with the properties described in Remark 3.3.2.*

Proof. Let $A = a_1, a_2, \ldots, a_{2n+2k+2}$ be a marked Dyck path of length $2(n+k+1)$ with k marked non-initial $up-up-down$ patterns. Suppose the height of the first peak is $a-1$. Then, following the proof of Proposition 2.2.2, A has a corresponding 321-avoiding fixed-point-free involution σ' on $[2(n+k+1)]$ where σ' contains the transposition $(1 \;\; a)$. Further, by Lemma 3.3.1, as A has k marked $up-up-down$ patterns, then σ' has k marked consecutive 231 patterns. As these $up-up-down$ patterns in A are non-initial, then the corresponding consecutive 231 patterns $\sigma'(x), \sigma'(x+1), \sigma'(x+2)$ occur where $x > a$.

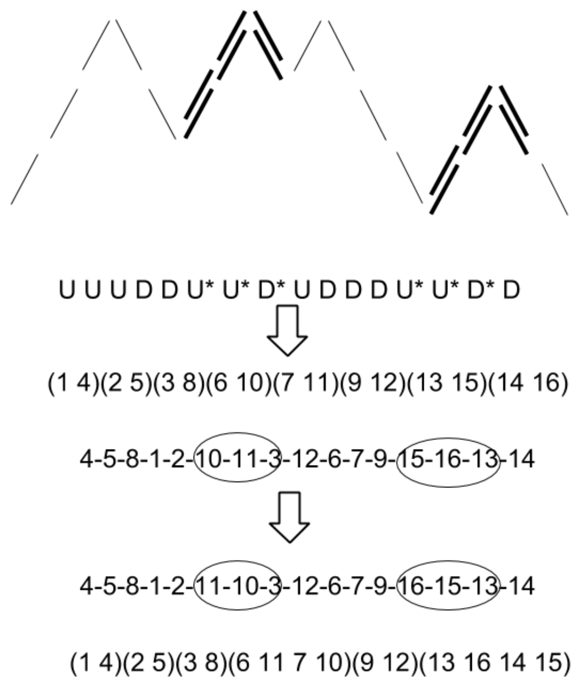

Figure 3.6: A Dyck path of length 16 with with 2 marked non-initial $up - up - down$ patterns and its corresponding permutation on [16] with 2 consecutive 321 patterns

Now for each of these marked consecutive 231 patterns, we will transpose the "2" and "3" in the pattern, creating a consecutive 321 pattern (that is to say, if the 231 pattern occurs in the positions $x_i, (x_i + 1), (x_i + 2)$ we will transpose the elements $\sigma'(x_i)$ and $\sigma'(x_i + 1)$). Therefore, we can define a new permutation σ in the following way. First, suppose the k marked consecutive 231 patterns of σ' occur in the positions $x_i, (x_i + 1), (x_i + 2)$ for $i \in \{1, \ldots, k\}$. Then, define $\sigma(x_i) = \sigma'(x_i + 1)$, $\sigma(x_i + 1) = \sigma'(x_i)$, for all such x_i, and $\sigma(m) = \sigma'(m)$ for all m which are not equal to any x_i or $x_i + 1$. This means that if σ' contains the transpositions $(x_i \;\; b_i)$ and $(x_i{+}1 \;\; c_i)$ then σ contains the 4-cycle $(x_i \;\; c_i \;\; x_i{+}1 \;\; b_i)$.

We claim that σ has the properties described in Remark 3.3.2. As σ' is a 321-avoiding fixed-point-free involution on $[2(n + k + 1)]$, then it is the product of $n + k + 1$ transpositions. For each of the k consecutive 231 patterns, the pattern in the positions $x_i, (x_i + 1), (x_i + 2)$

corresponds to σ' containing the transpositions $(x_i \;\; b_i)$ and $(x_i\!+\!1 \;\; c_i)$ and by our construction of σ, σ contains the 4-cycle $\rho_i = (x_i \;\; c_i \;\; x_i + 1 \;\; b_i)$ with $x_i < x_i + 1 < b_i < c_i$. Therefore for each of the k consecutive 231 patterns, our construction of σ takes two transpositions of σ' and combines them into a 4-cycle. Thus, σ is the product of k 4-cycles and $n\!+\!k\!+\!1\!-\!2k = n\!-\!k\!+\!1$ transpositions. Lastly, each of the 231 patterns of σ' occur in positions where $x_i > \sigma(1)$. Hence, σ has the desired properties.

To see this is invertible, begin with a permutation σ with the properties described in Remark 3.3.2. We know the corresponding σ' is a 321-avoiding fixed-point-free involution, so we claim that the Dyck path $A = a_1, a_2, \ldots, a_{2n+2k+2}$ corresponding to σ' has k marked non-initial $up\!-\!up\!-\!down$ patterns occurring in the positions $x_i, x_i\!+\!1$, and $x_i\!+\!2$ for each $i \in \{1, \ldots, k\}$.

For each x_i, we are given that $x_i < x_i + 1 < b_i < c_i$, so as $b_i < c_i$, we have $\sigma'(x_i) < \sigma'(x_i + 1)$. It remains to show that $\sigma'(x_i + 2) < \sigma'(x_i) = b_i$. The only elements for which $\sigma'(m) \neq \sigma(m)$ are where $m = x_j$ or $m = x_j + 1$ for some j. However, we know $x_i + 2$ cannot be either x_j or $x_j + 1$ as $\sigma(x_i + 2) < x_i + 2$. Therefore, $\sigma'(x_i + 2) = \sigma(x_i + 2) < x_i + 2 \leq b_i$. Hence, $\sigma'(x_i + 2) < \sigma'(x_i) < \sigma'(x_i + 1)$.

Therefore, as $\sigma'(x_i + 2) < \sigma'(x_i) < \sigma'(x_i + 1)$, then σ' contains a consecutive 231-pattern in positions $x_i, (x_i + 1), (x_i + 2)$ and by Lemma 3.3.1, then the corresponding Dyck path has a marked $up - up - down$ pattern in the same positions, and we will mark this pattern (in particular, $a_{x_i}, a_{x_i+1}, a_{x_i+2} = UUD$. Since σ' has k such marked 231 patterns, A has k such marked $up - up - down$ patterns. □

Theorem 3.3.5. *Let $n \geq 1$ and $k \leq n - 1$. Then the entry in Borel's triangle $f_{n,k}$ counts each of the following sets, and there exist natural bijections between them:*

(i) The number of marked Dyck paths of length $2(n+1)$ with k marked non-initial up steps.

(ii) The number of marked Dyck paths of length $2(n+1)$ with k marked non-terminal down steps.

(iii) The number of marked Dyck paths of length $2(n+k+1)$ with k marked non-initial $up - up - down$ patterns.

(iv) The number of ballot sequences of elections between three parties in which the total number of votes cast is $2(n+1)$, the second party to receive a vote receives exactly half the votes, the third party to receive a vote receives k votes, and the second party never holds a majority of the votes at any stage in the count.

(v) The number of triangulations of a regular polygon with $n+2$ sides with k marked diagonals, each not containing a given fixed vertex.

(vi) The number of parenthesizations of $n+1$ elements with k marked sets of parentheses, none of which contain a right parenthesis on the far right of all of the elements.

(vii) The number of full binary trees with $2n+3-k$ unmarked vertices and k marked branching vertices not contained in the rightmost branch.

(viii) Permutations σ on $[2(n+k+1)]$ with the properties described in Remark 3.3.2

Proof. By Proposition 3.1.1, item (i) is counted by $f_{n,k}$. By Proposition 3.1.2, items (i) and (ii) are in bijection with each other, by Proposition 3.1.3 items (i) and (iii) are in bijection with each other, and by Proposition 3.1.4, items (i) and (iv) are in bijection with each other. Propositions 3.2.1, 3.2.2, and 3.2.3 show that items (ii), (v), (vi), and (vii) are all in bijection. Lastly, Proposition 3.3.4 shows that items (iii) and (viii) are in bijection with each other. Therefore, all of the listed items are in bijection with each other and are all counted by the entry in Borel's triangle $f_{n,k}$. □

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VITA

William Anderson Davis

Candidate for the Degree of:

Master of Science

Thesis: A Transformation of the Catalan Numbers and Related Counted Sets

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for Bachelor of Science in Mathematics at Oklahoma State University, Stillwater Oklahoma in May, 2017.

Experience:

Employed by Oklahoma State Univeristy in the position of Teaching Assistant in Stillwater, Oklahoma from August 2017 to Present.

Professional Memberships:

Member of Pi Mu Epsilon as of February, 2017.