ON THE UNBIASED ESTIMATE OF
INTRA-CLASS CORRELATION

By

JACK DUANE TESTERMAN

Bachelor of Arts

Oklahoma Agricultural and Mechanical College

Stillwater, Oklahoma

1955

Submitted to the faculty of the Graduate School of
the Oklahoma Agricultural and Mechanical
College in partial fulfillment of the
requirements for the degree of
MASTER OF SCIENCE
May, 1957

ON THE UNBIASED ESTIMATE OF INTRA-CLASS CORRELATION

Thesis Approved:

Franklin A. Graybill
Thesis Adviser

Carl E. Marshall

Robert MacVicar
Dean of the Graduate School

PREFACE

The purpose of this thesis is the examination of the intra-class correlation ratio, a statistic commonly used by geneticists in breeding programs. An unbiased estimate of this intra-class correlation ratio is considered in the first part of the thesis. Biased estimates are sometimes desirable because of simplicity of computations. In the second part of this thesis one such estimate is examined for bias.

I am deeply indebted to Dr. Franklin Graybill for suggesting this problem to me and for his helpful criticism given me in the preparation of this thesis. I would also like to acknowledge the aid and encouragement given me by Dr. Carl Marshall. I am grateful to Dr. Hamblen of the Computing Center, and to Gary Spears, also of the Computing Center, for their help on the computing project.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Let us consider the model

$$(1.1) \qquad y_{ij} = \mu + \alpha_i + e_{ij} \qquad \begin{array}{l} i = 1, 2, \ldots m + 1 \\ j = 1, 2, \ldots c \end{array}$$

where the e_{ij} and the α_i are uncorrelated, and where

$$e_{ij} \sim \text{NID}(0, \sigma^2)$$
$$\alpha_i \sim \text{NID}(0, \sigma_w^2)$$

We shall use the symbol E to denote mathematical expectation and the notation $e_{ij} \sim \text{NID}(0, \sigma^2)$ to mean the e_{ij}'s are distributed normally and independently with mean zero and variance σ^2. Also we will adhere to the following notation:

$$Y_{i.} = \sum_j y_{ij}, \quad Y_{.j} = \sum_j y_{ij}, \quad Y_{..} = \sum_{i,j} y_{ij} .$$

We can now make the following analysis of variance.

TABLE I

Source	d.f	MS	EMS
Total	$c(m+1) - 1$		
Between classes	m	$\frac{1}{m}\left[\sum_i \frac{Y_{i.}^2}{c} - \frac{Y_{..}^2}{c(m+1)}\right] = B$	$\sigma^2 + c\sigma_w^2$
Within classes	n	$\frac{1}{n}\left[\sum_{i,j} y_{ij}^2 - \sum_i \frac{Y_{i.}^2}{c}\right] = W$	σ^2

We are concerned with the problem of estimating the intra-class correlation ratio

$$(1.2) \qquad \rho^2 = \frac{\sigma_w^2}{\sigma^2 + \sigma_w^2}$$

where the variance components, σ^2 and σ_w^2 , are defined in Table I.

It follows from our analysis of variance that the sum of squares divided by the expected mean square is distributed as the Chi-Square distribution with degrees of freedom associated with the source of variation. Symbolically we have that

$$(1.3) \qquad \frac{nW}{\sigma^2} \sim \chi^2(n)$$

and

$$(1.4) \qquad \frac{mB}{\sigma^2 + c\sigma_w^2} \sim \chi^2(m) .$$

CHAPTER II

AN UNBIASED ESTIMATE

The problem we will investigate in this section is to find an unbiased estimator of ρ^2.

Theorem I. <u>In the one way classification as given by (1.1), an unbiased estimator of</u> ρ^2 <u>is</u>

$$(2.1) \qquad \sum_{i=0}^{\infty} (-1)^i \frac{1}{c^i} \left[\frac{1}{F_1} \left(\frac{B}{W}\right) - 1 \right]^i$$

<u>where</u> $(F_1)^i = F_i$.

Proof: We shall first express ρ^2 as a function of the ratio σ_w^2/σ^2 and proceed to find an unbiased estimate of this ratio.

It follows from (1.2) that

$$1 - \rho^2 = 1 - \frac{\sigma_w^2}{\sigma^2 + \sigma_w^2} = \frac{\sigma^2}{\sigma^2 + \sigma_w^2}$$

Dividing both numerator and denominator by σ^2 we obtain

$$1 - \rho^2 = \frac{1}{1 + \frac{\sigma_w^2}{\sigma^2}} = \frac{1}{1 + r} \qquad \text{where } r = \frac{\sigma_w^2}{\sigma^2} .$$

Now

$$1 - \rho^2 = (1 + r)^{-1} = \sum_{i=0}^{\infty} (-r)^i$$

with the condition that $r^2 < 1$. More explicitly we have

$$1 - \rho^2 = 1 - r + r^2 - r^3 + \ldots$$

and

$$(2.2) \qquad \rho^2 = r - r^2 + r^3 - r^4 + \ldots \quad .$$

This relationship expresses the ratio ρ^2 in terms of the ratio r. It is obvious that to find an unbiased estimator of ρ^2, we need only to find a function such that the expected value of this function equals the infinite series expressed above as ρ^2.

We shall obtain a function V_t such that $E(V_t) = r^t$, $t > 0$.

It is known from (1.3) and (1.4) that the ratio

$$(2.3) \qquad u = \frac{\dfrac{B}{\sigma^2 + c\sigma_w^2}}{\dfrac{W}{\sigma^2}} \sim F(m,\ n)$$

where $F(m, n)$ denotes the F distribution with m degrees of freedom in the numerator and n degrees of freedom in the denominator. It follows that

$$E(u) = E\left[\frac{\sigma^2}{\sigma^2 + c\sigma_w^2} \ \frac{B}{W}\right] = \frac{\sigma^2}{\sigma^2 + c\sigma_w^2} E\left(\frac{B}{W}\right) = F_1$$

and

$$\frac{1}{F_1} E\left(\frac{B}{W}\right) = \frac{\sigma^2 + c\sigma_w^2}{\sigma^2} = 1 + cr \ ,$$

where F_1 is the first moment of the F distribution and equals $m/(m-2)$. From this equality we obtain a function V_1 such that

$$E\left\{\frac{1}{c}\left[\frac{1}{F_1}\left(\frac{B}{W}\right) - 1\right]\right\} = E(V_1) = r.$$

Thus we have obtained a function V_1 such that the expected value of V_1 is equal to r.

Now we shall examine a function, u^t, such that the expected value of u^t is equal to the t^{th} moment, F_t, of the F distribution about the origin, i.e.,

$$F_t = \frac{\Gamma(\frac{m+2t}{2})\ \Gamma(\frac{n-2t}{2})\ (\frac{n}{m})^t}{\Gamma(\frac{m}{2})\ \Gamma(\frac{n}{2})} \quad , \quad t < \frac{n}{2} \ .$$

Obviously

$$E\left[\frac{\sigma^2}{\sigma^2 + c\sigma_w^2}\ \frac{B}{W}\right]^t = F_t$$

and, therefore,

$$\frac{1}{F_t} E\left(\frac{B}{W}\right)^t = (1 + cr)^t \ .$$

Now let us examine this equality:

for $t = 1$:

$$\frac{1}{F_1} E\left(\frac{B}{W}\right) = (1 + cr)$$

so

$$E(V_1) = E\left\{\frac{1}{c}\left[\frac{1}{F_1}\left(\frac{B}{W}\right) - 1\right]\right\} = r \ .$$

for $t = 2$:

$$\frac{1}{F_2} E\left(\frac{B}{W}\right)^2 = (1 + cr)^2 = 1 + 2cr + c^2r^2$$

so

$$E(V_2) = E\left\{\frac{1}{c^2}\left[\frac{1}{F_2}\left(\frac{B}{W}\right)^2 - 2\frac{1}{F_1}\left(\frac{B}{W}\right) + 1\right]\right\} = r^2 \ .$$

for $t = 3$:

$$\frac{1}{F_3} E\left(\frac{B}{W}\right)^3 = (1 + cr)^3 = 1 + 3cr + 3c^2r^2 + c^3r^3$$

so

$$E(V_3) = E\left\{\frac{1}{c^3}\left[\frac{1}{F_3}\left(\frac{B}{W}\right)^3 - 3\frac{1}{F_2}\left(\frac{B}{W}\right)^2 + 3\frac{1}{F_1}\left(\frac{B}{W}\right) - 1\right]\right\} = r^3$$

For $t = 4$:

$$\frac{1}{F_4} E\left(\frac{B}{W}\right)^4 = (1 + cr)^4 = 1 + 4cr + 6c^2r^2 + 4c^3r^3 + c^4r^4$$

so

$$E(V_4) = E\left\{\frac{1}{c^4}\left[\frac{1}{F_4}\left(\frac{B}{W}\right)^4 - 4\frac{1}{F_3}\left(\frac{B}{W}\right)^3 + 6\frac{1}{F_2}\left(\frac{B}{W}\right)^2 - 4\frac{1}{F_1}\left(\frac{B}{W}\right) + 1\right]\right\} = r^4$$

For $t = 5$:

$$\frac{1}{F_5} E\left(\frac{B}{W}\right)^5 = (1 + cr)^5 = 1 + 5cr + 10c^2r^2 + 10c^3r^3 + 5c^4r^4 + c^5r^5$$

so

$$E(V_5) = E\left\{\frac{1}{c^5}\left[\frac{1}{F_5}\left(\frac{B}{W}\right)^5 - 5\frac{1}{F_4}\left(\frac{B}{W}\right)^4 + 10\frac{1}{F_3}\left(\frac{B}{W}\right)^3 - 10\frac{1}{F_2}\left(\frac{B}{W}\right)^2 + 5\frac{1}{F_1}\left(\frac{B}{W}\right) - 1\right]\right\}$$
$$= r^5$$

Upon examination of the binomial $\frac{1}{c^t}(a + b)^t$ we find that with $a = \frac{1}{F_1}\left(\frac{B}{W}\right)$ and $b = -1$ we can expand the binomial for different values of t and obtain our functions V_i, $i = 1, 2, 3, \cdots$. It follows that

$$E(V_t) = E\left\{\frac{1}{c^t}\left[\frac{1}{F_1}\left(\frac{B}{W}\right) - 1\right]^t\right\} = r^t .$$

It is necessary to define what we mean by $\left(\frac{1}{F_1}\right)^t$. In this thesis, we shall use the subscript on the F to act as an exponent in our expansion. Thus, $\left(\frac{1}{F_1}\right)^t = \frac{1}{F_t}$.

We now have a function V_t which is an unbiased estimate of r^t and, therefore, we can obtain an unbiased estimate for ρ^2.

CHAPTER III

A COMMON BIASED ESTIMATE

For purposes of simplicity, the quantity

$$\hat{\rho}^2 = \frac{B - W}{B + (c - 1)W} \tag{3.1}$$

is commonly used as an estimator of ρ^2. This is a consistent estimator of ρ^2. We will examine this estimator for bias. First we will find the expected value of $\hat{\rho}^2$.

(3.2)

$$E(\hat{\rho}^2) = E\left[\frac{B - W}{B + (c - 1)W}\right] = E\left[1 - c\frac{W}{B + (c - 1)W}\right] = 1 - cE\left[\frac{W}{B + (c - 1)W}\right]$$

since $\frac{B - W}{B + (c - 1)W} = 1 - \frac{cW}{B + (c - 1)W}$. If we divide numerator and denominator by B we get

$$E(\hat{\rho}^2) = 1 - cE\left[\frac{\frac{W}{B}}{1 + (c - 1)\frac{W}{B}}\right] . \tag{3.3}$$

Let us examine the distribution of $\frac{W}{B} = X$. Using the same assumptions as in (2.3) we know that

$$\frac{\frac{W}{\sigma^2}}{\frac{B}{\sigma^2 + c\sigma_w^2}} \sim F(n, m) .$$

Let $K = \frac{\sigma^2 + c\sigma_w^2}{\sigma^2}$. Then $y = KX \sim F(n, m)$. Hence the expected value of $\hat{\rho}^2$ is determined by first finding $E\left[\frac{X}{1 + (c - 1)X}\right]$.

$$E\left[\frac{X}{1+(c-1)X}\right] = \int_0^\infty \frac{X}{1+(c-1)X} \; \frac{\Gamma(\frac{m+n}{2})\,(\frac{n}{m})^{n/2}(KX)^{(n-2)/2}d(KX)}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})\,(1+\frac{n}{m}KX)^{(n+m)/2}}$$

$$= \int_0^\infty \frac{\frac{y}{K}}{1+(c-1)\frac{y}{K}} \; \frac{\Gamma(\frac{m+n}{2})\,(\frac{n}{m})^{n/2}\,y^{(n-2)/2}\,dy}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})\,(1+\frac{n}{m}y)^{(n+m)/2}} \; .$$

The substitution $y = \frac{mz}{n(1-z)}$ transforms this integral to

$$E\left[\frac{X}{1+(c-1)X}\right] = \int_0^1 \frac{\frac{mz}{Kn(1-z)}}{1+\frac{c-1}{K}\frac{mz}{n(1-z)}} \; \frac{\Gamma(\frac{m+n}{2})\,z^{(n/2-1)}(1-z)^{(m/2-1}}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})}$$

$$= \int_0^1 \frac{\frac{m}{Kn}}{1+(\frac{c-1}{K}\,\frac{m}{n}-1)z} \; \frac{\Gamma(\frac{m+n}{2})\,z^{(n/2-1)}(1-z)^{(m/2-1)}}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})}$$

$$(3.4) \qquad = \frac{m}{Kn}\,\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} \int_0^1 \frac{1}{(1-w^2z)}\, z^{n/2}(1-z)^{(m/2-1)}dz$$

where

$$-w^2 = \frac{c-1}{K}\,\frac{m}{n} - 1$$

and

$$-1 < -w^2 < 0 \; .$$

Now using the identity

$$1 - z = \left[(1-w^2z) + (w^2-1)z\right]$$

we obtain

$$E\left[\frac{X}{1+(c-1)X}\right] = \frac{m}{Kn}\,\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} \int_0^1 (1-w^2z)^{-1}\,z^{n/2}\Big[(1-w^2z) + (w^2 -$$

$$(1-z)^{(m/2-2)}dz$$

$$= \frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{n}{2})\,\Gamma(\frac{m}{2})} \left[\int_0^1 z^{n/2}(1-z)^{(m/2-2)}dz + (w^2-1)\int_0^1 (1-w^2z)^{-1} z^{(n/2+1)}(1-z)^{(m/2-2)}dz \right]$$

$$= \frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{n}{2})\,\Gamma(\frac{m}{2})} \left[B(\tfrac{n}{2}+1, \tfrac{m}{2}-1) + (w^2-1)\int_0^1 z^{(n/2+1)}(1-z)^{(m/2-3)} + (w^2-1)^2 \int_0^1 (1-w^2z)^{-1} z^{(n/2+2)}(1-z)^{(m/2-3)}dz \right]$$

$$= \frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} \left[B(\tfrac{n}{2}+1, \tfrac{m}{2}-1) + (w^2-1)\,B(\tfrac{n}{2}+2, \tfrac{m}{2}-2) + (w^2-1)^2 \int_0^1 z^{(n/2+2)}(1-z)^{(m/2-4)}dz + (w^2-1)^3 \int_0^1 (1-w^2z)^{-1} z^{(n/2+3)}(1-z)^{(m/2-4)}dz \right]$$

$$= \frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} \left[B(\tfrac{n}{2}+1, \tfrac{m}{2}-1) + (w^2-1)\,B(\tfrac{n}{2}+2, \tfrac{m}{2}-2) + (w^2-1)^2\,B(\tfrac{n}{2}+3, \tfrac{m}{2}-3) + (w^2-1)^3 \int_0^1 (1-w^2z)^{-1} z^{(n/2+3)}(1-z)^{(m/2-4)}\,dz \right]$$

(3.5)

$$= \frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} \left[\sum_{i=1}^{m/2-1} (w^2-1)^{i-1}\,B(\tfrac{n}{2}+i, \tfrac{m}{2}-i) + (w^2-1)^{m/2} \int_0^1 (1-w^2z)^{-1} z^{(\frac{m+n}{2}-1)}\,dz \right]$$

where $B(m, n) = \int_0^1 z^{m-1}(1-z)^{n-1}\,dz$, $m, n > 0$, is the Beta function. Let us examine the quantity $(1 - w^2 z)^{-1}$. Since $0 \leq z \leq 1$ and $-1 < -w^2 < 0$ we shall expand $(1 - w^2 z)^{-1}$ into an infinite series, i.e.,

$$(3.6) \qquad (1 - w^2 z)^{-1} = \sum_{i=0}^{\infty} (w^2 z)^i .$$

The integral

$$\int_0^1 (1 - w^2 z)^{-1} z^{(\frac{m+n}{2} - 1)} dz$$

can now be written as

$$\int_0^1 \sum_{i=0}^{\infty} (w^2 z)^i \; z^{(\frac{m+n}{2} - 1)} dz .$$

Interchanging the summation and integral signs we obtain

$$\sum_{i=0}^{\infty} (w^2)^i \int_0^1 z^{(\frac{m+n}{2} + i - 1)} dz = \sum_{i=0}^{\infty} (w^2)^i \frac{1}{\frac{m+n}{2} + i} .$$

Using this result in (3.5) we have

$$E\left[\frac{X}{1 + (c-1)X}\right] = \frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} \left[\sum_{i=1}^{m/2 - 1} (w^2 - 1)^{i-1} B(\frac{n}{2} + i, \frac{m}{2} - i) \right.$$

$$\left. + (w^2 - 1)^{m/2} \sum_{i=0}^{\infty} (w^2)^i \frac{1}{\frac{m+n}{2} + i} \right] .$$

Examination of the last summation in the brackets yield a result which can be computed.

$$\sum_{i=0}^{\infty} \frac{(w^2)^i}{\frac{m+n}{2} + i} = \sum_{j=\frac{m+n}{2}}^{\infty} \frac{(w^2)^{(j - \frac{m+n}{2})}}{j} \qquad \text{if } j = \frac{m+n}{2} + i$$

$$= (w^2)^{-\frac{m+n}{2}} \sum_{j=\frac{m+n}{2}}^{\infty} \frac{(w^2)^j}{j}$$

$$= (w^2)^{-\frac{m+n}{2}} \left[\sum_{j=1}^{\infty} \frac{(w^2)^j}{j} - \sum_{j=1}^{\frac{m+n}{2}-1} \frac{(w^2)^j}{j} \right]$$

$$= (w^2)^{-\frac{m+n}{2}} \left[-\ln(1-w^2) - \sum_{j=1}^{\frac{m+n}{2}-1} \frac{(w^2)^j}{j} \right] .$$

So (3.7)

$$E\left[\frac{X}{1+(c-1)X}\right] = \frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} \left[\sum_{i=1}^{\frac{m}{2}-1} (w^2-1)^{i-1} B(\tfrac{n}{2}+i, \tfrac{m}{2}-i) \right.$$

$$\left. + (w^2-1)^{m/2} (w^2)^{-\frac{m+n}{2}} \left\{ -\ln(1-w^2) - \sum_{j=1}^{\frac{m+n}{2}-1} \frac{(w^2)^j}{j} \right\} \right]$$

is a form of the expected value which is convenient for using a desk calculator. Now substituting in (3. 3) we can obtain the expected value of $\hat{\rho}^2$.

A program for the calculation of the expected value of $\hat{\rho}^2$ was written for use on the IBM 650 Data Processing Computer. The formula used for computations was

$$E(\rho^2) = 1 - cE\left[\frac{X}{1+(c-1)X}\right]$$

$$= 1 - c\,\frac{m}{Kn} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \int_0^1 (1-w^2 z)^{-1} z^{n/2} (1-z)^{\frac{m}{2}-1} dz$$

where

$$\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\,\Gamma(\frac{n}{2})} = \frac{1}{B(\frac{m}{2}, \frac{n}{2})} = \frac{1}{\int_0^1 z^{\frac{m}{2}-1}(1-z)^{\frac{n}{2}-1}\,dz} .$$

A subroutine of Simpson's rule was used for the two integrations with increments of .01 from 0 to 1.

The values obtained by this method were very close to the values obtained by the results of computing (3.7) for different values of m, n, and ρ^2. The results are shown in the tables in the appendix. A few of the results are plotted on the graphs in the appendix to show the bias incurred by using the estimate

$$\frac{B - W}{B + (c - 1)W} .$$

CHAPTER IV

HERITABILITY RATIO

Geneticists often use the statistic

$$h^2 = \frac{4\sigma_w^2}{\sigma^2 + \sigma_w^2} \tag{4.1}$$

called the heritability ratio, which is based on the model defined by (1.1). We may now use the results of Theorem I to obtain an unbiased estimate of h^2. We shall now examine some of the properties of h^2.

The ratio h^2 has the limits

$$0 \leq h^2 \leq 1 \quad . \tag{4.2}$$

Substituting $4\rho^2$ for h^2 we have

$$0 \leq 4\rho^2 \leq 1$$

or

$$0 \leq \rho^2 \leq 1/4 \tag{4.3}$$

By definition

$$0 \leq \frac{\sigma_w^2}{\sigma^2 + \sigma_w^2} \leq 1/4$$

and dividing both numerator and denominator by σ^2 we obtain

$$0 \leq \frac{r}{1 + r} \leq 1/4 \quad .$$

Now solving for r we have

$$0 \leq r \leq 1/3 \quad . \tag{4.4}$$

From (2.2) we have

$$\rho^2 = r - r^2 + r^3 - r^4 + \cdots \quad .$$

We can estimate ρ^2 by one or more of the terms on the right-hand side of the equation and approximate the bias by using (4.4). For example, the bias is less than $.111\cdots$ when we use r as an estimator of ρ^2. If we estimate ρ^2 by $r - r^2$ the bias is less than $.037\cdots$.

In many cases the limits on h^2 are

(4.5) $$0 \leqq h^2 \leqq 1/2 \ .$$

In this case the corresponding limits on r will be

(4.6) $$0 \leqq r \leqq 1/7 \ .$$

If we now approximate ρ^2 by r our bias would be less than $.0204\cdots$. If we approximate ρ^2 by $r - r^2$ our bias would be less than $.00291\cdots$.

Thus we can estimate h^2 to any desired accuracy by using more terms with higher powers of r.

CHAPTER V

CONCLUSIONS

This thesis gives an unbiased estimator for the intra-class correlation ratio. This estimator, stated in Theorem I, is in the form of an infinite series. A few terms of this series will, in most problems, yield the desired accuracy needed. For many problems in genetics the first, or the first and second, terms will suffice. In all cases an approximation to the bias can be computed easily.

The estimator examined in Chapter III may be used to estimate the intra-class correlation ratio. The graphs in the appendix exhibit the amount of bias that occurs for different degrees of freedom. The estimator very closely approximates ρ^2 when the degrees of freedom are large. The amount of bias increases as the degrees of freedom decrease. For small degrees of freedom, the estimator given in Theorem I may be used to obtain more accuracy in the experiment. The tables and graphs in the appendix are for guidance in determining which estimator should be selected for use in experiments.

APPENDIX

TABLE II

Expected values of $\hat{\rho}^2$ for $m = 1$ and selected values of ρ^2 and c.

c \ ρ^2	0.1	0.3	0.5	0.7	0.9
2	- 0.025	0.101	0.242	0.415	0.681
4	0.069	0.207	0.348	0.512	0.753
6	0.093	0.240	0.385	0.550	0.785
8	0.105	0.259	0.408	0.575	0.806
10	0.112	0.272	0.426	0.595	0.822

TABLE III

Expected values of $\hat{\rho}^2$ for $m = 2$ and selected values of ρ^2 and c.

c \ ρ^2	0.1	0.3	0.5	0.7	0.9
2	0.017	0.163	0.322	0.509	0.765
4	0.067	0.219	0.375	0.551	0.789
6	0.080	0.237	0.393	0.568	0.801
8	0.087	0.246	0.404	0.579	0.810
10	0.091	0.253	0.413	0.588	0.819

TABLE IV

Expected values of $\hat{\rho}^2$ for $m = 4$ and selected values of ρ^2 and c.

c \ ρ^2	0.1	0.3	0.5	0.7	0.9
2	0.061	0.228	0.405	0.601	0.840
4	0.082	0.252	0.426	0.616	0.844
6	0.087	0.259	0.433	0.620	0.846
8	0.089	0.263	0.436	0.622	0.847
10	0.091	0.265	0.438	0.624	0.848

TABLE V

Expected values of $\hat{\rho}^2$ for $m = 6$ and selected values of ρ^2 and c.

c \ ρ^2	0.1	0.3	0.5	0.7	0.9
2	0.076	0.252	0.436	0.636	0.864
4	0.088	0.268	0.449	0.644	0.866
6	0.092	0.272	0.454	0.646	0.866
8	0.093	0.275	0.456	0.648	0.866
10	0.094	0.276	0.457	0.648	0.867

TABLE VI

Expected values of $\hat{\rho}^2$ for $m = 8$ and selected values of ρ^2 and c.

c \ ρ^2	0.1	0.3	0.5	0.7	0.9
2	0.083	0.265	0.452	0.652	0.874
4	0.092	0.276	0.462	0.658	0.875
6	0.094	0.279	0.465	0.659	0.876
8	0.095	0.281	0.466	0.661	0.876
10	0.096	0.282	0.467	0.661	0.876

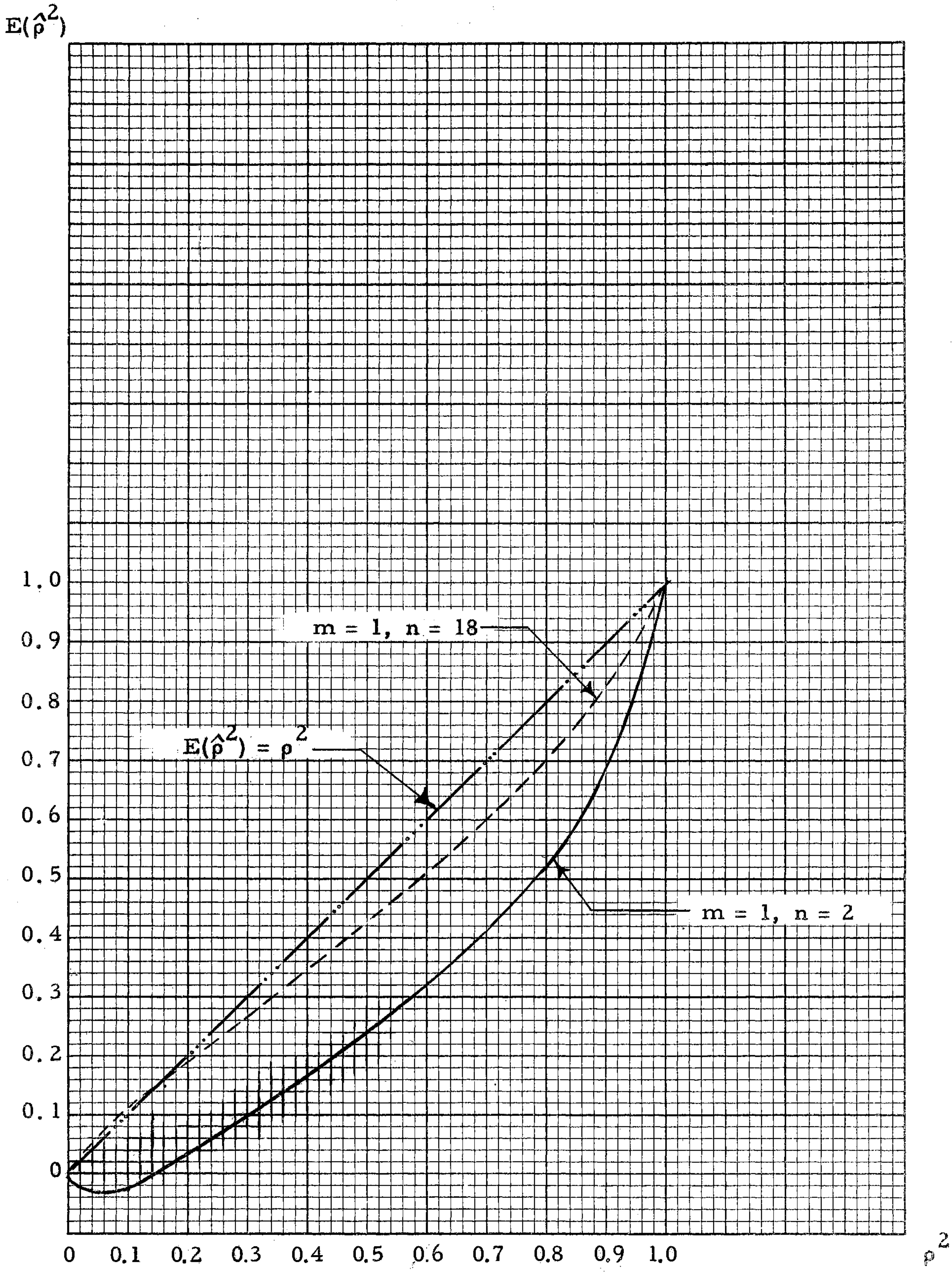

Fig. 1

Expected value of $\hat{\rho}^2$ for m = 1 and n = 2, 18

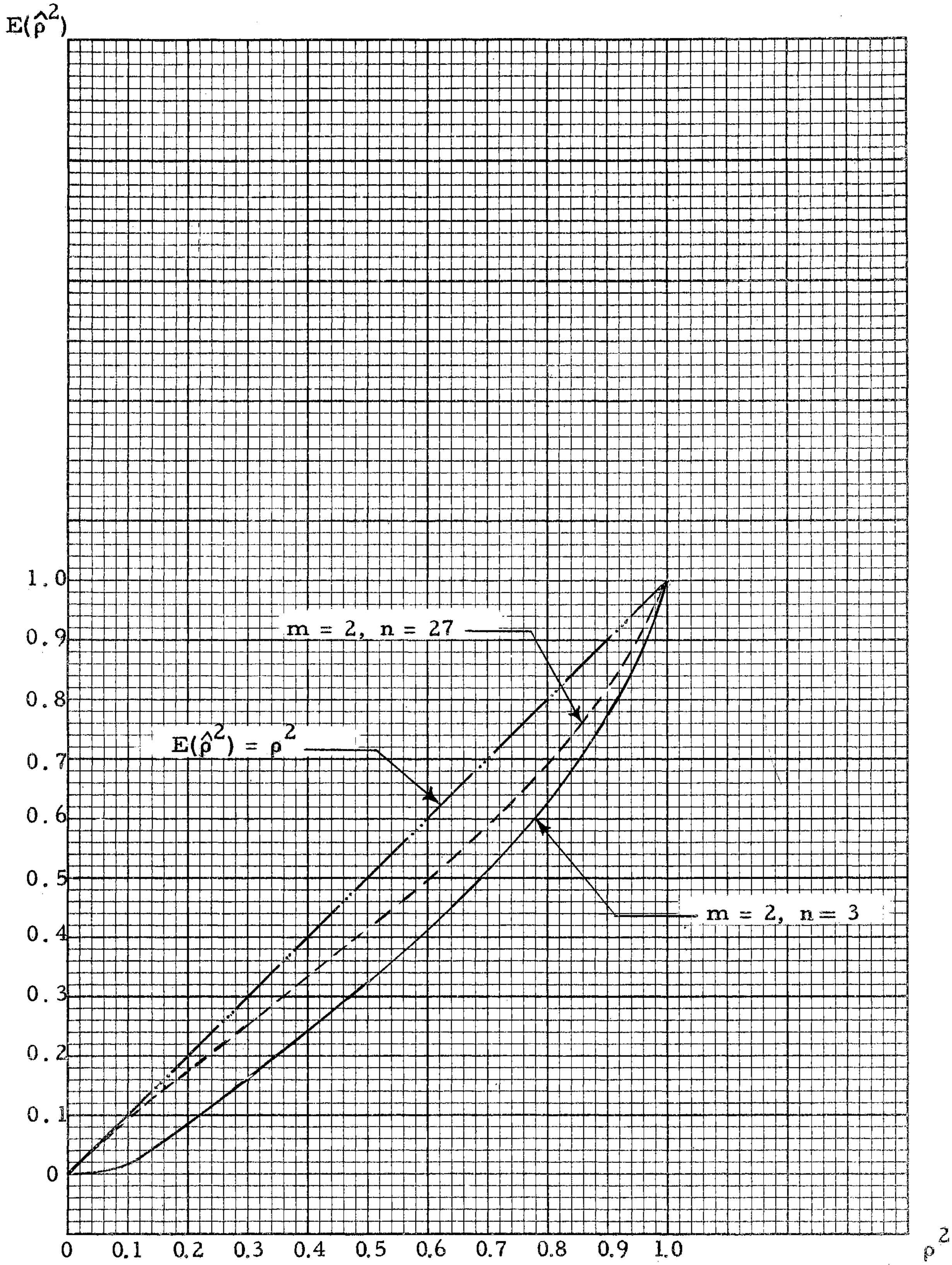

Fig. 2

Expected value of $\hat{\rho}^2$ for m = 2 and n = 3, 27

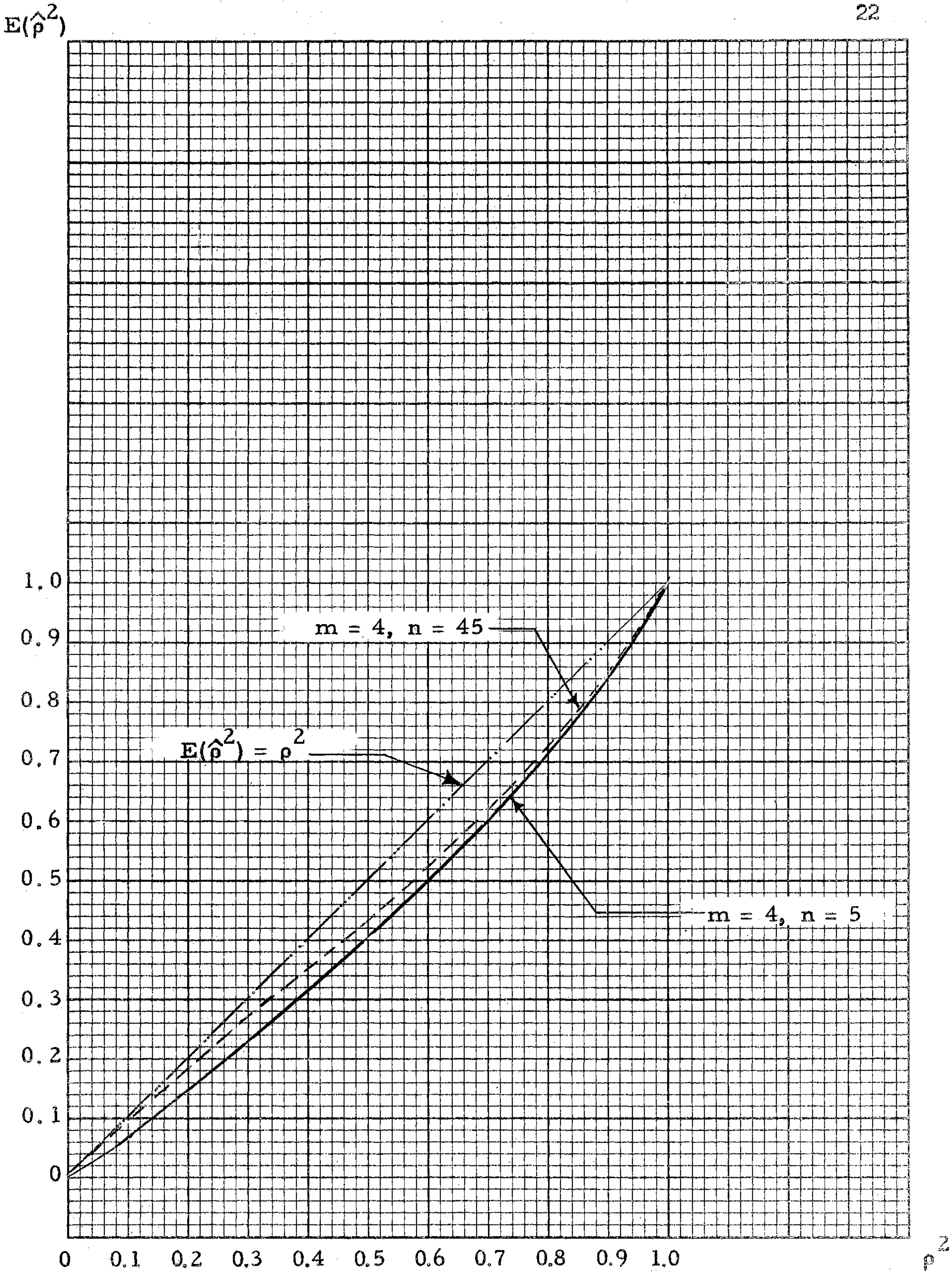

Fig. 3

Expected value of $\hat{\rho}^2$ for m = 4 and n = 5, 45

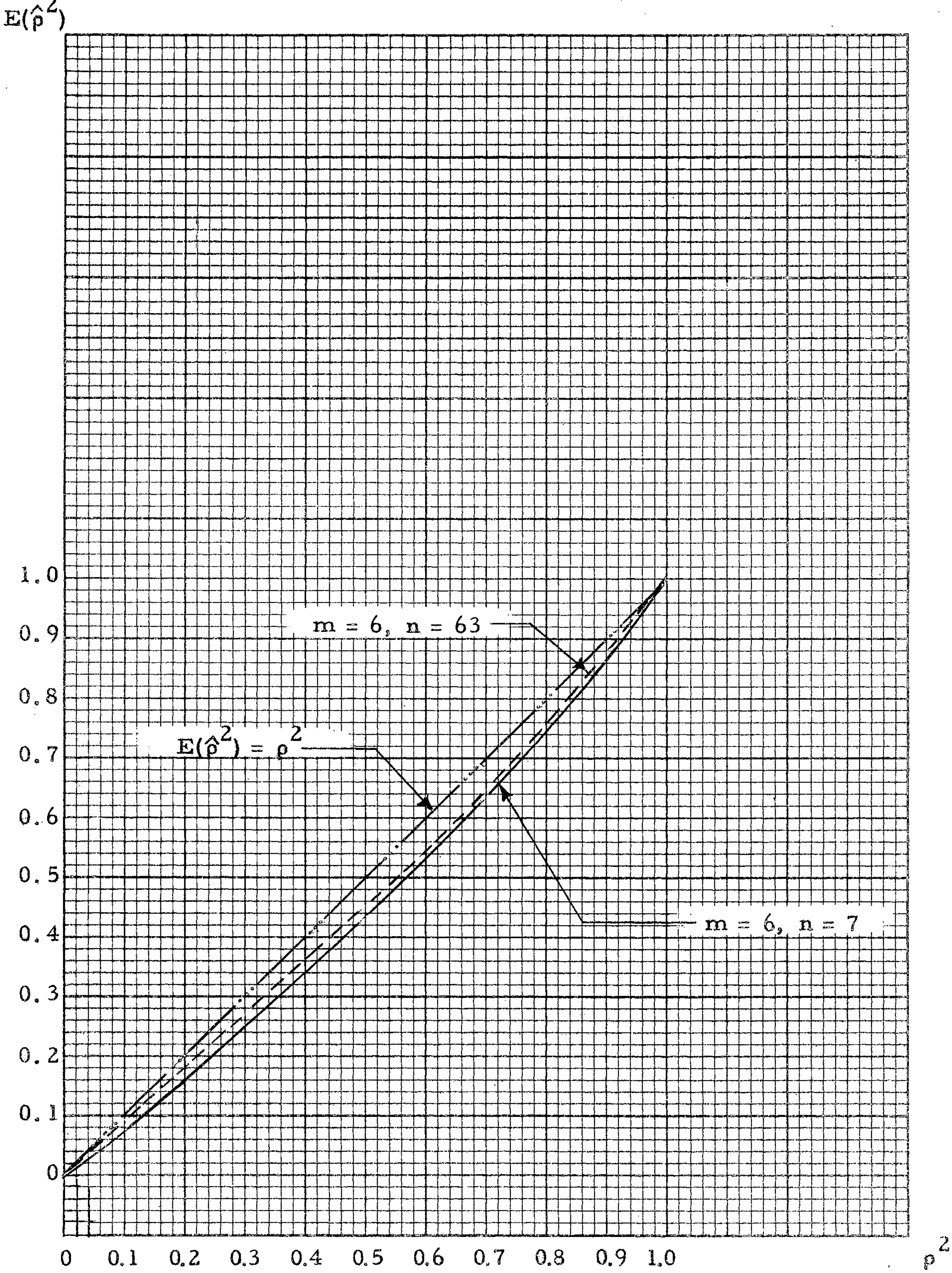

Fig. 4

Expected value of $\hat{\rho}^2$ for m = 6 and n = 7, 63

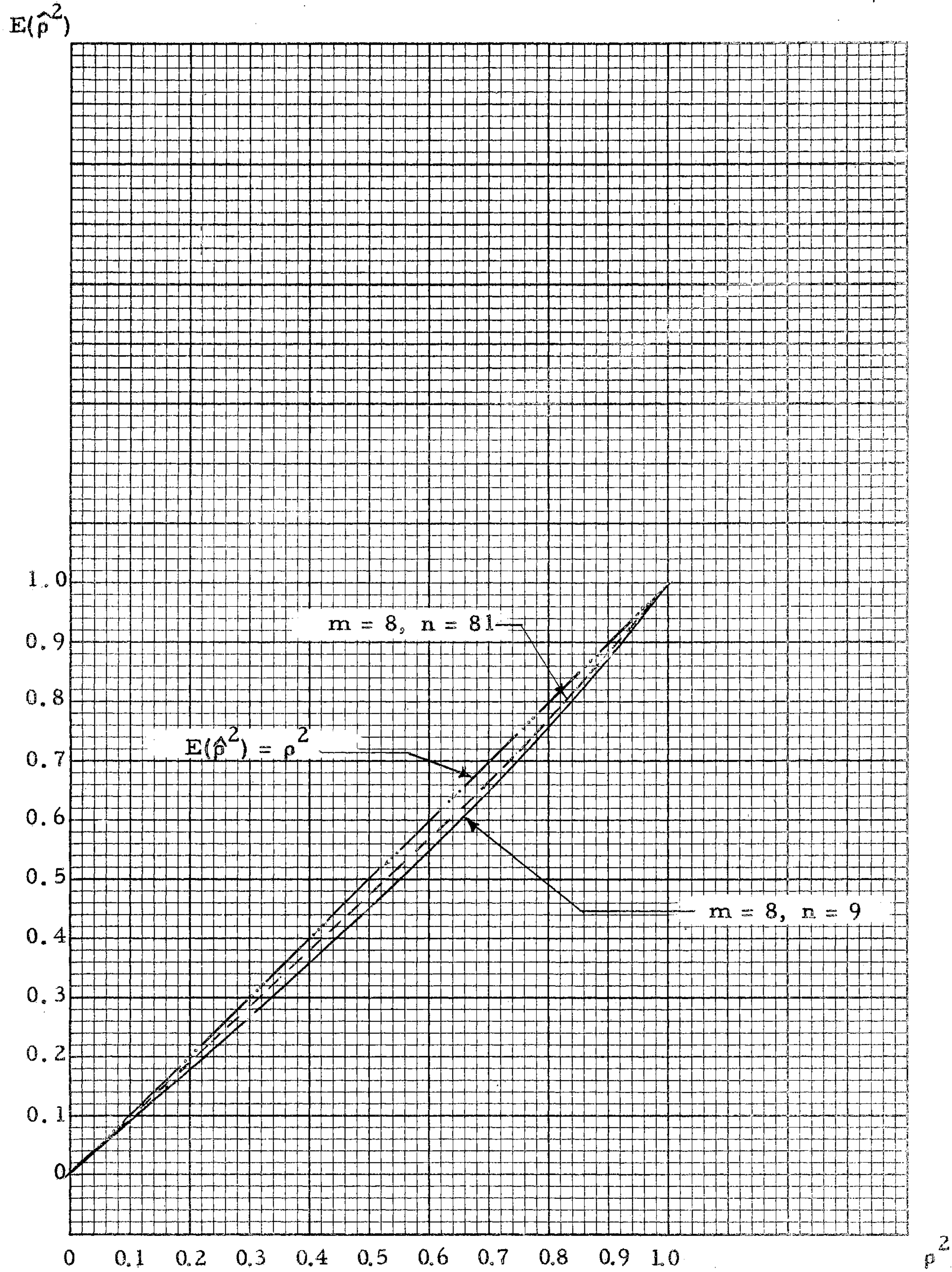

Fig. 5

Expected value of $\hat{\rho}^2$ for m = 8 and n = 9, 81

VITA

Jack Duane Testerman

Candidate for the Degree of

Master of Science

Thesis: ON THE UNBIASED ESTIMATE OF INTRA-CLASS CORRELATION

Major : Mathematics

Minor : Statistics

Biographical and Other Items:

Personal Data: Born near Marietta, Oklahoma, December 13, 1933, the son of Jesse D. and Lillie Ann Testerman.

Undergraduate Study: Murray State School of Agriculture, 1951-53; O. A. M. C., 1953-55.

Graduate Study: O. A. M. C., 1955-57.

Experiences: IBM 650 Electronic Computer Programmer at Tinker Field Air Force Base, June, 1956 - September, 1956; Graduate teaching assistant in the Mathematics Department, 1955-56, 1956-57.

Member of Pi Mu Epsilon, Institute of Mathematical Statistics, Phi Theta Kappa, and Mathematical Association of America.

Date of Final Examination: May, 1957

THESIS TITLE: ON THE UNBIASED ESTIMATE OF INTRA-CLASS CORRELATION

AUTHOR: Jack Duane Testerman

THESIS ADVISER: Dr. Franklin A. Graybill

TYPIST: W. Arlene Starwalt