72-14,115

NUNLEY, Clyde Dale, 1938-
ON CERTAIN CLASSES OF REGULAR NEAR-RINGS.

The University of Oklahoma, Ph.D., 1971
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

ON CERTAIN CLASSES OF REGULAR NEAR-RINGS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

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Norman, Oklahoma

1971

TABLE OF CONTENTS

Chapter | Page

I. INTRODUCTION. 1

1. Historical Background 1
2. Basic Definitions and Concepts. 2
3. Preliminary Results 3

II. SPECIAL AND SPECIAL REGULAR NEAR-RINGS. . . . 11

1. Motivation and Definitions. 11
2. Preliminary Results and Examples. 17
3. Some Structure Theorems 24
4. Some Decomposition Theorems 42
5. The Set of Distributors 46

III. SPECIAL β NEAR-RINGS. 50

1. Definition and Examples 50
2. Basic Results 55
3. Two Decomposition Theorems. 62

IV. WEAKLY COMMUTATIVE REGULAR NEAR-RINGS 65

1. Introduction. 65
2. Main Results. 66

LIST OF REFERENCES. 76

ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to Dr. Gene Levy, Chairman of the doctoral committee, for his invaluable guidance during the preparation of this work. Appreciation is also extended to the other members of the committee for their assistance.

ON CERTAIN CLASSES OF REGULAR NEAR-RINGS

CHAPTER I

INTRODUCTION

1. Historical Background

A left near-ring, denoted by $(N,+,\cdot)$ is an algebraic system consisting of a set N together with two operations called addition and multiplication and denoted respectively by $+$ and $\cdot$ such that

(a) $(N,+)$ is a group, not necessarily abelian,

(b) $(N,\cdot)$ is a semigroup,

(c) $x\cdot(y+z) = x\cdot y + x\cdot z$ for all $x,y,z \in N$.

A similar definition may be given for a right near-ring, the only difference being the obvious change in property (c). In this work, we shall deal exclusively with left near-rings. Thus, in the sequel the term "near-ring" shall mean "left near-ring." When there can be no ambiquity concerning the operations in the near-ring $(N,+,\cdot)$ we shall use the abbreviated symbol N to denote the system. We adopt the usual convention of denoting $x\cdot y$ by xy.

Many of the studies in the theory of near-rings have paralleled studies in ring theory. The goal has often been to extend results from ring theory to the more general setting of near-rings. Clay and Lawver [6] defined the concept of a Boolean near-ring and studied some of the properties of this class of objects. Noting that p-rings are generalizations of Boolean rings, Ratliff [18] defined the concept of a p-near-ring and used the technique employed by Clay and Lawver [6] to study certain classes of p-near-rings. Regular rings are, in some sense, generalizations of p-rings. Using the definition of a regular near-ring from a paper by Beidleman [1] and a technique similar to that employed in the previously mentioned works, we have studied certain classes of regular near-rings.

2. Basic Definitions and Concepts

The concept of a regular ring was first discussed by von Neumann [17]. For further work in this area see also Forsythe and McCoy [8] and McCoy [16]. An element r in a ring $(R,+,\cdot)$ is a <u>regular element</u> if there is an $r' \in R$ such that $rr'r = r$. The element r' will be called a <u>regularity companion</u> of r. The prime symbol on an element will be used exclusively to denote any regularity companion of that element. Regularity companions are not unique. The element 0 in a ring is always regular and any element is a regularity companion for 0. If a ring has an identity element, then that element is regular and is its own unique regularity companion. Any element which has a multiplicative inverse is regular. While an element may not have a unique regularity companion, multiplication of an element by any of its regularity companions always produces the same result as we shall show later.

A ring is said to be a <u>regular ring</u> if each of its elements is regular. An element n in a near-ring $(N,+,\cdot)$ is regular if there is an n' in N such that $nn'n = n$. If every element in a near-ring is regular then the near-ring is said to be regular. A <u>p-ring</u> is a ring $(R,+,\cdot)$ for which there is a prime number p such that $x^p = x$ and $px = 0$ for all x in R. A <u>p-near-ring</u> is a near-ring $(N,+,\cdot)$ for which there is a prime number p such that $x^p = x$ and $px = 0$ for all x in N. Clearly, any p-ring is a regular ring and any p-near-ring is a regular near-ring.

Before proceeding further, we mention some examples of the systems just defined. Any division ring, hence any field, is an example of a regular near-ring. Of course, any regular ring is a regular near-ring and any regular near-ring which is a ring is a regular ring. Beidleman

[1] has mentioned the following example of a regular near-ring. Let M denote the set of all functions from an additive group, not necessarily abelian, into itself which commute with the zero function. Then $(M,+,\cdot)$ is a regular near-ring where the operations are pointwise addition and composition of functions. The integers with addition and multiplication defined modulo a prime number p constitute an example of a p-ring. Many examples of p-near-rings may be found in the work of Ratliff [18] . There are examples of regular near-rings which are not p-near-rings. Such examples are provided by classes (27), (35), (48) and (52) in the list of Clay [5] where the additive group is $(Z_6,+)$.

We now give the definitions of some additional terms which are used throughout this work. We shall be concerned with the ideal structure of certain classes of near-rings. A <u>left ideal</u> of a near-ring $(N,+,\cdot)$ is a normal subgroup L of $(N,+)$ such that for all n in N and x in L, nx is in L. A <u>right ideal</u> of a near-ring $(N,+,\cdot)$ is a normal subgroup R of $(N,+)$ such that for all m and n in N and x in R, $(m + x)n-mn$ is in R. If I is both a left ideal and a right ideal then I is an <u>ideal</u> of $(N,+,\cdot)$. Ideals are frequently defined to be kernels of near-ring homomorphisms. Blackett [3] has shown that the latter definition is equivalent to the former.

The terms maximal sub-C-ring and maximal sub-Z-ring are defined in Berman and Silverman [2]. We shall have occasion to refer to an abelian near-ring. By this we mean a near-ring such that the additive group is abelian. In Chapters II and III we deal almost exclusively with abelian near-rings. It should be realized that in these instances all subgroups of the additive group are normal. This fact is used frequently without being mentioned. A sub-near-ring of a near-ring $(N,+,\cdot)$ is a subgroup of $(N,+)$ which is a sub-semi-group of $(N,\cdot)$. A subring of a near-ring is a sub-near-ring which is a ring. The factor near-ring of a near-ring $(N,+,\cdot)$ by an ideal I will be denoted by N/I. The coset corresponding to an element x will be denoted by $\bar{x}$. Factor near-rings are discussed in Berman and Silverman [2]. An element s in a near-ring $(N,+,\cdot)$ is right distributive if $(x + y)s = xs + ys$ for all x and y in N. For any right distributive element s we have $0s = 0$. We say that a near-ring $(N,+,\cdot)$ is distributively generated if N contains a multiplicative semigroup S whose elements generate $(N,+)$ and are right distributive. A near-ring $(N,+,\cdot)$ is said to be weakly commutative if $xyz = yxz$ for all x, y, z in N. A discussion of subdirect irreducibility for rings may be found in McCoy [14]. A similar discussion for the near-ring case may be found in Fain

[7] . Since a nonzero near-ring $(N,+,\cdot)$ is subdirectly irreducible if an only if the intersection of all nonzero ideals of N is nonzero we take this condition as the definition of <u>subdirectly irreducible</u>. On several occasions we use the symbol 0 to denote the set whose only element is 0. This should cause no confusion.

We conclude this section with the remark that in any left near-ring $(N,+,\cdot)$ we have $-(xy) = x(-y)$ for all x and y in N and $x0 = 0$ for all x in N.

3. Preliminary Results

Some of the following theorems are near-ring parallels of established ring theory results.

THEOREM 1.1. Let $(N,+,\cdot)$ be a near-ring and x a regular element in N. Let x' be a regularity companion of x. Then x'x is idempotent.

PROOF. $(x'x)(x'x) = x'(xx'x) = x'x.$
The next theorem is due to Ratliff [18] .

THEOREM 1.2. Let $(N,+,\cdot)$ be a weakly commutative near-ring. Then for every x and y in N and for every positive integer k, $(xy)^k = x^k y^k$.

THEOREM 1.3. Let $(N,+,\cdot)$ be a weakly commutative near-ring. Let x be a nonzero regular element of N. Then x is not nilpotent.

PROOF. Suppose x is a regular nilpotent element. Then there is a positive integer k such that $x^k = 0$. Let x' be any regularity companion of x. Then, since $x'x$ is idempotent $x'x = (x'x)^k = (x')^k x^k = (x')^k 0 = 0$. So $x = xx'x = x(x'x) = x0 = 0$.

COROLLARY 1.4. A weakly commutative regular near-ring has no nonzero idempotents.

PROOF. Every element is regular.

THEOREM 1.5. Let $(N,+,\cdot)$ be a weakly commutative near-ring and let x be a regular element in N. If x' and x'' are two regularity companions for x then $x'x = x''x$.

PROOF. $x'x = x'(xx''x) = x'(x''xx) = (x'x''x)x = (x''x'x)x = x''(x'xx) = x''(xx'x) = x''x$.

The following result is stated because of its frequent use in the sequel.

THEOREM 1.6. Let $(N,+,\cdot)$ be a weakly commutative near-ring and x an idempotent element in N. If x' is any regularity companion of x then $x'x = x$.

PROOF. $x'x = x'xx = xx'x = x$.

We shall have occasion later to mention a commutative regular ring with identity. The field of real numbers provides us with an example of such a ring which is not a p-ring. Other examples are provided by class (27) in the list of Clay [5] where the additive group is $(Z_6, +)$. With the help of the following lemma we are able to characterize those finite rings with identity which are commutative regular rings.

LEMMA 1.7. Let $(R,+,\cdot,1)$ be a commutative regular ring with identity. Let x be in N and x' a regularity companion of x. Then for any positive integer k, $x = (x')^{k-1}x^k$.

<u>PROOF</u>. If $x = 0$ the result follows. Suppose $x \neq 0$. If $x' = 0$ then $x = xx'x = x0x = 0$. But $x \neq 0$ so $x' \neq 0$. If $k = 1$ then $(x')^{k-1}x^k = (x')^0\ x = x$ where by definition $a^0 = 1$ if a is any nonzero element in R. Now suppose the result holds for $k = m$ where m is some positive integer. Then $(x')^m x^{m+1} = ((x')^{m-1}x^m)x'x = xx'x = x$. Thus, by induction the result holds for all positive integers k.

REMARK. From the proof of this lemma we note that in a ring if $x \neq 0$ then 0 is not a regularity companion of x. We also note that in a commutative ring the condition $xx'x = x$ may be written as $x'x^2 = x$. We shall use this consequence of commutativity freely.

THEOREM 1.8. Let $(R,+,\cdot,1)$ be a finite ring with identity. Then R is a commutative regular ring if and only if for each x in R there is an integer $n(x) > 1$ such that $x^{n(x)} = x$.

PROOF. Suppose that for each x in R there is an integer $n(x) > 1$ such that $x^{n(x)} = x$. Then by a well-known theorem of Jacobson, R is commutative. Let x be an element in R. If $x = 0$ then x is regular. If $x \neq 0$ then $x^{n(x)-2}x^2 = x$ and x is regular. Hence R is a commutative regular ring. Conversely, suppose R is a commutative regular ring. Let q be the number of elements in R and x in R. If $x = 0$ or 1 then $x^2 = x$ and the condition is satisfied. Thus we need be concerned only with x in R such that $x \neq 0$ and $x \neq 1$. If there is no such element the proof is complete. Otherwise, consider the set $\{x, x^2, \cdots, x^{q-1}\}$ where x is neither 0 nor 1. None of these elements is zero since by Corollary 1.4 there are no nonzero nilpotent elements. If the elements are distinct then they must constitute the set of nonzero elements of R. In this case, the identity must be among these elements. Since $x \neq 1$ there must be some integer k such that $1 < k \leq q-1$ and $x^k = 1$. Then $x^{k+1} = x$ where $2 < k+1 \leq q$. If the members of the set are not distinct then there must be positive integers m and n such that $1 \leq m < n \leq q-1$ and $x^m = x^n$. Let $k = n - m > 0$. Then $x^m = x^n = x^{m+k} = x^m x^k$. If $m = 1$, then $x = x^{k+1}$ and the proof is complete. Suppose $m > 1$. Let x' be a regularity companion of x. Since $x \neq 0$ then $x' \neq 0$ by the remark following Lemma 1.7. Thus, by Corollary 1.4, $(x')^{m-1} \neq 0$. Hence $x = (x')^{m-1}x^m = (x')^{m-1}x^m x^k = xx^k = x^{k+1}$ and the proof is complete.

CHAPTER II

SPECIAL NEAR-RINGS AND SPECIAL REGULAR NEAR-RINGS

1. Motivation and Definitions

The class of Boolean near-rings studied by Clay and Lawver [6] was, in some sense, related to a Boolean ring with identity. Ratliff [18] studied the more general case of a class of p-near-rings related in a similar way to a p-ring with identity. In each instance the procedure was to begin with a ring $(R,+,\cdot,1)$ having the desired property and define a new multiplication denoted by $*$ on $(R,+)$ so as to obtain a near-ring $(R,+,*)$. The new multiplication was expressed in terms of the original multiplication and addition and the identity by defining $x * y$ to be a polynomial in x and y with fixed coefficients from R for arbitrary x and y in R.

If $P(x,y)$ is any such polynomial we note that in order for the left distributive law to hold we must have $P(x,y+z) = P(x,y) + P(x,z)$. Thus we restrict our attention to those polynomials of the following type:

$$P(x,y) = a_n x^n y + a_{n-1} x^{n-1} y + \dots + a_1 xy + a_0 y$$

where the a_i are fixed elements from the ring. In order that the associative law hold we must have $P(P(x,y),z) = P(x,P(y,z))$. If we attempt to write out these expressions for the polynomial given above we see that the problem is unmanageable unless additional restrictions are placed on the ring or the coefficients or both. The rings considered by Clay and Lawver [6] and also those considered by Ratliff [18] were commutative. In his work Ratliff [18] used conditions on the coefficients similar to $a_i a_j = 0$ if $i \neq j$ and $a_0{}^2 = a_0$. If we arbitrarily adopt these restrictions then for the case $n = 2$ associativity requires that $a_2{}^3x^4y^2z = a_2{}^2x^2y^2z$ for all x, y, z in the ring. If we begin with a Boolean ring this condition holds. This suggests that we need place restrictions on our original ring in addition to those already adopted if we wish to insure that the new multiplication is associative and, hence, that $(R,+,*)$ is a near-ring. The situation is more agreeable if we consider the simpler polynomial corresponding to $n = 1$.

THEOREM 2.1. Let $(N,+,\cdot)$ be a weakly commutative ring. Define a multiplication $*:N \times N \rightarrow N::(x,y) \longrightarrow x * y = axy + by$ where $a, b \in N$. Then $(N,+,*)$ is a weakly commutative near-ring if and only if $b(x * z) = bz$ for all $x, z \in N$.

PROOF. We note that the polynomial $P(x,y)$ used to define the multiplication is linear in y. Therefore, the left distributive law holds. Suppose $(N,+,*)$ is a weakly commutative near-ring. Then $*$ is associative. Hence $x * (y * z) = (x * y) * z$ for all x, y, z in N. Now $x * (y * z) = x * (ayz + bz) = ax(ayz + bz) + b(ayz + bz)$ $a^2xyz + abxz + abyz + b^2z$ and $(x * y) * z = a(x * y) z + bz =$ $a^2xyz + abyz + bz$. Thus $a^2xyz + abxz + abyz + b^2z =$ $a^2xyz + abyz + bz$ and so $abxz + b^2z = bz$. Hence $b(x * z) =$ $abxz + b^2z = bz$ for all $x, z \in N$. These steps are clearly reversible. So if $b(x * z) = bz$ for all $x, z \in N$ then $*$ is associative and $(N,+,*)$ is a near-ring. Also, $b(y * z) =$ $abyz + b^2z = bz$. So for all $x, y, z \in N$, $abxz = bz - b^2z$ and $abyz = bz - b^2z$. So $abxz = abyz$. Then $x * y * z =$ $a^2xyz + abyz + bz = a^2yxz + abxz + bz = y * x * z$ for all $x, y, z \in N$.

COROLLARY 2.2. Let $(N,+,\cdot)$ be a weakly commutative ring. Define $*: N \times N \rightarrow N :: (x,y) \longrightarrow x * y =$ $axy + by$. If $ab = 0$ and $b^2 = b$, then $(N, +, *)$ is a weakly commutative near-ring.

PROOF. Let $x, y \in N$. Then $b(x * y) =$ $b(axy + by) = abxy + b^2y = by$. The conclusion follows by Theorem 2.1.

If $(N,+,\cdot)$ is a weakly commutative ring and a and b are fixed elements of N such that $ab = 0$ and $b^2 = b$

and a new multiplication $*$ is defined on $(N,+)$ by $x * y = axy + by$ for all $x, y \in N$ then the near-ring $(N,+,*)$ will be called the special near-ring determined by a and b. The ring $(N,+,\cdot)$ will be referred to as the base ring. Note that a special near-ring as discussed in this work is not necessarily a special near-ring in the sense of Fain [7]. As an example we may take a ring from class (27) in the list of Clay [5] where the additive group is $(Z_6, +)$. Choose $a = 3$ and $b = 4$. The resulting special near-ring is in class (52). It is not a special p-near-ring as described in [18].

The rings in the studies by Clay and Lawver [6] and Ratliff [18] had identities. For these rings we are able to show that the arbitrary conditions which we placed on the coefficients in our polynomial are necessary. Suppose $(N,+,\cdot,1)$ is a weakly commutative ring with identity. Then N is commutative for $xy = xy1 = yx1 = yx$ for all x, y in N. Suppose a and b belong to N and let $(N,+,*)$ be the special near-ring which they determine. Then $*$ is associative so $0 * (0 * 1) = (0 * 0) * 1$. But $0 * (0 * 1) = 0 * (a01 + b1) = 0 * b = a0b + bb = b^2$ and $(0 * 0) * 1 = 0 * 1 = a01 + b1 = b$. Thus $b^2 = b$. Also $1 * (0 * 1) = (1 * 0) * 1$. Now $1 * (0 * 1) = 1 * (a01 + b1) = 1 * b = a1b + b^2 = ab + b$ and $(1 * 0) * 1 = 0 * 1 = a01 + b1 = b$. Thus

$ab + b = b$ and $ab = 0$.

THEOREM. 2.3. Let $(N,+,\cdot,1)$ be a commutative ring with identity. Let a be a regular element in $(N,+,\cdot,1)$ and let a' be a regularity companion for a. Define a new multiplication $*$ on N by $x * y = axy + (1 - a'a)y$. Then $(N,+,*)$ is a weakly commutative near-ring and a is regular in $(N,+,*)$.

<u>PROOF</u>. Let $b = 1 - a'a$. Then $b^2 = (1 - a'a)^2 = 1 - a'a - a'a + (a'a)^2$. By Theorem 1.1 $a'a$ is idempotent. Hence $b^2 = 1 - a'a - a'a + (a'a)^2 = 1 - a'a = b$. Also $ab = a(1 - a'a) = a - aa'a = a - a = 0$. By Corollary 2.2, $(N,+,*)$ is a weakly commutative near-ring. To show a is regular note

$$
\begin{aligned}
a * (a')^3 * a &= (a * (a')^3) * a \\
&= a\,(a * (a')^3)\,a + (1-a'a)a \\
&= a\,(aa(a')^3 + (1-a'a)(a')^3)\,a + (a-a'aa) \\
&= a^4\,(a')^3 + a^2\,(1-a'a)(a')^3 + 0 \\
&= a^4\,(a')^3 + 0 + 0 = a.
\end{aligned}
$$

<u>A special near-ring determined by a</u> is a special near-ring such that the base ring has an identity, a is regular and $b = 1 - a'a$.

COROLLARY 2.4. Let $(N,+,\cdot,1)$ be a commutative regular ring with identity. Let $a \in N$. Then the special near-ring determined by a is regular.

PROOF. Since a is regular in $(N,+,\cdot,1)$ we know by Theorem 2.3 that a determines a weakly commutative near-ring $(N,+,*)$. To show that $(N,+,*)$ is regular we show that each of its elements is regular. Let $z \in N$. Then z is regular in $(N,+,\cdot,1)$. Let z' denote a regularity companion of z. Then we have

$$\begin{aligned} z*(a')^2z' * z &= (z * (a')^2z') * z \\ &= a(z * (a')^2z')z + bz \\ &= a(az(a')^2z' + b(a')^2z')z + bz \\ &= (aa')^2zz'z + ab(a')^2z'z + bz \\ &= aa'z + bz = (a'a+b)z = 1z = z. \end{aligned}$$

If the base ring is regular then the special near-ring determined by an element a will be called a <u>special regular near-ring</u>. Special regular near-rings are weakly commutative regular near-rings. The latter class of near-rings is studied in Chapter IV. It should be noted that the special p-near-rings studied by Ratliff [18] are special regular near-rings. In this sense, many of the results in the current chapter are generalizations of results he obtained.

2. Preliminary Results and Examples

In this section we deal with the questions of when a special near-ring determined by an element a has an identity, when it is commutative, when it is a C-ring and when it is distributively generated. We also consider the question of when an element is right distributive.

LEMMA 2.5. Let $(N,+,*)$ be a special near-ring determined by a. Then a' is a left identity in $(N,+,*)$. Furthermore, a' is a right identity if and only if $a'a = 1$.

PROOF. Let x be any element of N. Then $a' * x = aa'x + (1 - a'a)x = a'ax + x - a'ax = x$. Hence a' is a left identity. Suppose now that $a'a = 1$. In this case, $x * a' = axa' + (1 - a'a)a' = a'ax + 0a' = 1x + 0 = x$ for all $x \in N$. Conversely, suppose a' is a right identity. Then $x * a' = x$ for all $x \in N$. Take $x = 1 + a'$. Then $(1 + a') * a' = a(1 + a')a' + (1 - a'a)a' = 1 + a'$. So $aa' + aa'a' + a' - a'aa' = aa' + a' = 1 + a'$. Hence $a'a = 1$. It should be noted that since a' is always a left identity it is the identity if one exists.

THEOREM 2.6. Let $(N,+,*)$ be a special near-ring determined by a, and $N \neq 0$.

Then the following statements are equivalent:

(a) $(N,+,*)$ has an identity,

(b) a has a unique regularity companion in $(N,+,\cdot,1)$,

(c) a is neither zero nor a zero divisor in $(N,+,\cdot,1)$,

(d) a has a multiplicative inverse in $(N,+,\cdot,1)$,

(e) $*$ is commutative,

(f) $(N,+,*)$ is a C-ring,

(g) $(N,+,*,a')$ is isomorphic to $(N,+,\cdot,1)$.

<u>PROOF</u>. (a) implies (b). Suppose $(N,+,*)$ has an identity, e. By the remark preceding this theorem $e = a'$. Suppose a'' is any regularity companion for a. Then by Lemma 2.5, a'' is a left identity. Hence $a' = a'' * a' = a''$. Thus a' is the unique regularity companion of a.

(b) implies (c). Suppose a' is the unique regularity companion of a in $(N,+,\cdot,1)$. Then $a \neq 0$ since every element is a regularity companion of 0 and $N \neq 0$. Now suppose $ax = 0$ for some $x \in N$. Then $a^2x = 0$ and $a^2x + a = a$. Hence, $a^2x + a^2a' = a^2(x + a') = a$. Thus $x + a'$ is a regularity companion of a. By uniqueness $x + a' = a'$. Therefore, $x = 0$. So a is not a zero divisor in $(N,+,\cdot,1)$.

(c) implies (d). Suppose a is neither zero nor a zero divisor in $(N,+,\cdot,1)$. Then $a(1 - a'a) = a - aa'a = a - a = 0$. Thus $1 - a'a = 0$ and $a'a = 1$. So a has a multiplicative inverse in $(N,+,\cdot,1)$.

(d) implies (e). Suppose a has a multiplicative inverse in $(N,+,\cdot,1)$. Denote this multiplicative inverse by u. Then $ua = 1$. Hence $ua^2 = a$ so u is a regularity companion for a. Then by Theorem 1.5, $a'a = ua = 1$ and a' is a multiplicative inverse for a in $(N,+,\cdot,1)$. Thus $a' = a'ua = ua'a = u$. We conclude that a' is the unique multiplicative inverse of a in $(N,+,\cdot,1)$. Now let $y \in N$. Since $a'a = 1$, $a'ay = y$ and $(1 - a'a)y = 0$ for all $y \in N$. Thus $x * y = axy + (1 - a'a)y = axy = ayx + (1 - a'a)x = y * x$ for all $x, y \in N$. So $*$ is commutative.

(e) implies (f). If $*$ is commutative then $0 * x = x * 0 = 0$ for all $x \in N$. Hence, $(N,+,*)$ is a C-ring.

(f) implies (g). Suppose $(N,+,*)$ is a C-ring. Then $0 = 0 * 1 = a01 + (1 - a'a)1 = 1 - a'a$. So $a'a = 1$ and we see that $a \neq 0$ and $a' \neq 0$. Also $(a'a)a' = 1a' = a'$. Now $x * y = axy + (1 - a'a)y = axy$ for all $x, y \in N$. Hence, $a' * x = x * a' = x$ for all $x \in N$. Define $g:(N, +, \cdot, 1) \longrightarrow (N,+,*,a')::x \longrightarrow a'x$. Then g is a mapping and if $g(x) = g(y)$ we have $a'x = a'y$. So $x = aa'x = aa'y = y$. Thus g is 1 - 1. Let $n \in N$. Then

$g(an) = a'an = n$. So g is onto. Now $g(x + y) = a'(x + y) = a'x + a'y = g(x) + g(y)$ and $g(xy) = a'xy = (a'aa')xy = a(a'x)(a'y) = ag(x)g(y) = g(x) * g(y)$. Thus g is an isomorphism between $(N,+,\cdot,1)$ and $(N,+,*,a')$.

(g) implies (a). If $(N,+,*,a')$ is isomorphic to $(N,+,\cdot,1)$ then clearly $(N,+,*)$ has an identity.

The next four results relate to the question of distributive generation. A theorem of Fröhlich [9] states in part that a distributively generated near-ring $(R,+,\cdot)$ with identity is a ring if and only if $(R,+)$ is abelian. With the aid of a pair of simple lemmas it is elementary to show that an arbitrary distributively generated near-ring $(R,+,\cdot)$ is a ring if and only if $(R,+)$ is abelian.

LEMMA 2.7. Let $(R,+,\cdot)$ be an abelian near-ring. Then $-(x + y) = (-x) + (-y)$ for all $x, y \in R$.

PROOF. In any group $-(x+y) = (-y) + (-x)$. Since $(R,+)$ is abelian the result follows.

LEMMA 2.8. Let $(R,+,\cdot)$ be an abelian near-ring. If $z \in R$ is right distributive then $-z$ is right distributive.

PROOF. Suppose $z \in R$ is right distributive.

Then for all $x, y \in R, (x + y)(-z) = -[(x + y)z] = -[xz + yz] = [-(xz)] + [-(yz)] = x(-z) + y(-z)$.
Thus $-z$ is right distributive.

Ligh [13] has used the name α near-ring to describe a near-ring with the property that the negatives of right distributive elements are right distributive. The above result shows that any abelian near-ring is an α near-ring.

THEOREM 2.9. Let $(R,+,\cdot)$ be a distributively generated near-ring. Then $(R,+,\cdot)$ is a ring if and only if $(R,+)$ is abelian.

PROOF. Clearly, if $(R,+,\cdot)$ is a ring then $(R,+)$ is abelian. Conversely, suppose $(R,+)$ is abelian. Let $x, y, z \in R$. Then $z = z_1 + \ldots + z_n$ where either z_i is right distributive or z_i is the negative of a right distributive element for each $i \in \{1,2,\ldots,n\}$. By Lemma 2.8, z_i is right distributive for $i \in \{1,2,\ldots,n\}$. Thus $(x + y) z = (x + y)(z_1 + \ldots + z_n) = (x + y)z_1 + \ldots + (x + y)z_n = (xz_1 + yz_1) + \ldots + (xz_n + yz_n) =$
$(xz_1 + \ldots + xz_n) + (yz_1 + \ldots + yz_n) = xz + yz$.
Thus, the right distributive law holds and $(R,+,\cdot)$ is a ring.

THEOREM 2.10. Let $(N,+,*)$ be a special near-ring determined by a. Then $(N,+,*)$ is distributively generated if an only if $(N,+,*)$ is a ring.

PROOF. If $(N,+,*)$ is a ring then it is distributively generated. Conversely, suppose $(N,+,*)$ is distributively generated. Since $(N,+)$ is abelian we have by Theorem 2.9 that $(N,+,*)$ is a ring.

If $(N,+,*)$ is a special near-ring determined by an element a, it is natural to ask under what conditions an element in N will be right distributive. The next two theorems relate to this question.

THEOREM 2.11. Let $(N,+,\cdot,1)$ be a commutative ring with identity. Let a be a regular element in N and $b = 1 - a'a$. Then $N = aN \oplus bN$.

PROOF. It is well-known that aN and bN are ideals of $(N,+,\cdot,1)$. Now let $x \in aN \cap bN$. Then $x = an$ and $x = bm$ for some $n, m \in N$. So we have $x = an = (aa'a)n = (aa')an = (aa')bm = a'(ab)m = a'0m = 0$. Thus $aN \cap bN = 0$. Clearly $aN \oplus bN \subset N$. Let $x \in N$. Then $x = x1 = x(a'a + b) = aa'x + bx \in aN \oplus bN$. So $N \subset aN \oplus bN$. We conclude that $N = aN \oplus bN$.

THEOREM 2.12. Let $(N,+,*)$ be a special near-ring determined by a. Let $z \in N$. Then z is right distributive if and only if $z \in aN$.

PROOF. Suppose $z \in aN$. Then $z = an$ for some $n \in N$. So $(1 - a'a)z = (1 - a'a)an = (a - aa'a)n = 0n = 0$. Thus $(x + y) * z = a(x + y)z + (1 - a'a)z = axz + ayz = axz + (1 - a'a)z + ayz + (1 - a'a)z = x * z + y * z$. So we have established the fact that z is right distributive. Conversely, suppose z is right distributive. Then $0 = 0 * z = a0z + (1 - a'a)z = z - a'az$. So $z = a(a'z) \in aN$.

Before turning to a discussion of the ideal structure of special near-rings in general and special regular near-rings in particular we consider some examples. Let $(N,+)$ be the Klein 4-group and let $(N,+,\cdot,c)$ be the representative given for class (4) in the list of Clay [5]. Then $(N,+,\cdot,c)$ is a commutative ring with identity element c. It is not regular. The regular elements are, in Clay's notation, 0, b and c. The element 0 determines a near-ring in class (23) which is a 2-near-ring. The element b determines a near-ring isomorphic to the base ring. The element c determines the base ring. As another example consider $(Z_6,+,\cdot,1)$ from Clay's class (27). This is a commutative regular ring with identity. For each $x \in Z_6$, $x^3 = x$ and $x'x = x^2$. We obtain the following special regular near-rings.

If $(a,b) = (1,0)$ then $x * y = xy$. This produces the base ring.

If $(a,b) = (0,1)$ then $x * y = y$. This produces class (48).

If $(a,b) = (2,3)$ then $x * y = 2xy + 3y$. This produces class (35).

If $(a,b) = (3,4)$ then $x * y = 3xy + 4y$. This produces class (52).

If $(a,b) = (4,3)$ then $x * y = 4xy + 3y$. This produces class (35).

If $(a,b) = (5,0)$ then $x * y = 5xy$. This is isomorphic to the base ring.

There are regular near-rings which are not special regular near-rings. An example is found in Clay's class (8) where the additive group is $(Z_5, +)$.

3. Some Structure Theorems

In this section we consider the ideal structure of special near-rings. For any regular element t in a commutative ring with identity $(N,+,\cdot,1)$ we define the set $P(t) = \{x \in N : xt't = x\}$. This set apparently depends on t' as well as t but the dependence is superficial as shown by Theorem 1.5. If t is idempotent then we see from Theorem 1.6 that $P(t) = \{x \in N : xt = x\}$.

THEOREM 2.13. Let $(N,+,\cdot,1)$ be a commutative ring with identity and let t be regular in N. Then P(t) is an ideal of $(N,+,\cdot,1)$ with identity element $t't$. Furthermore, if t is idempotent the identity in $(P(t),+,\cdot)$ is t.

PROOF. Let $x, y \in P(t)$. Then $x = t'tx$ and $y = t'ty$. Thus $x - y = t'tx - t'ty = t't(x - y)$. Hence, $x - y \in P(t)$. So $(P(t),+)$ is a normal subgroup of $(N,+)$. Let $n \in N$. Then $nx = n(xt't) = (nx)t't$. Thus P(t) is an ideal. The remaining assertions are obvious.

Let $(N,+,\cdot,1)$ be a commutative ring with identity. Let t be a regular element in N and L a subset of N. Now define $L(t) = \{x \in N : x = st't \text{ for some } s \in L\}$. As before the dependence on t' is superficial and if t is idempotent $L(t) = \{x \in N : x = st \text{ for some } s \in L\}$.

THEOREM 2.14. Let $(N,+,\cdot,1)$ be a commutative ring with identity and t regular in N.

(a) If $(L,+)$ is a subgroup of $(N,+)$ then $(L(t),+)$ is a subgroup of $(N,+)$.

(b) If $(L,+,\cdot)$ is a subring of $(N,+,\cdot)$ then $(L(t),+,\cdot)$ is a subring of $(N,+,\cdot)$.

(c) If $(L,+,\cdot)$ is an ideal of $(N,+,\cdot)$ then $(L(t),+,\cdot)$ is an ideal of $(N,+,\cdot)$.

PROOF. (a) Suppose $(L,+)$ is a subgroup of $(N,+)$ and $x, y \in L(t)$. Then $x = s_1t't$ and $y = s_2t't$ for some $s_1, s_2 \in L$. So $x - y = (s_1 - s_2)t't$. Since $(L,+)$ is a subgroup $s_1 - s_2 \in L$. Thus $x - y \in L(t)$ and $(L(t),+)$ is a subgroup of $(N,+)$.

(b) Suppose $(L,+,\cdot)$ is a subring of $(N,+,\cdot)$. Then by part (a) $(L(t),+)$ is a subgroup of $(N,+)$. If $x, y \in L(t)$ then $x = s_1t't$ and $y = s_2t't$ for some $s_1, s_2 \in L$. Thus $xy = s_1s_2(t't)^2 = s_1s_2t't \in L(t)$ since $s_1s_2 \in L$. Hence, $(L(t),+,\cdot)$ is a subring of $(N,+,\cdot)$.

(c) Suppose $(L,+,\cdot)$ is an ideal of $(N,+,\cdot)$. Let $x \in L(t)$ and $n \in N$. Then $x = st't$ for some $s \in L$. Thus $nx = nst't \in L(t)$ since $ns \in L$. Hence $(L(t),+,\cdot)$ is an ideal of $(N,+,\cdot)$.

THEOREM 2.15. Let $(N,+,\cdot,1)$ be a commutative ring with identity. Let t be regular in N and L a subset of N. Then $L(t) \subset P(t)$.

PROOF. Let $x \in L(t)$. Then for some $s \in L$, $x = st't$. Hence $xt't = (st't)t't = s(t't)^2 = st't = x$. Thus $x \in P(t)$.

THEOREM 2.16. Let $(N,+,*)$ be a special near-ring determined by a. Let N_z denote the maximal sub-Z-ring of $(N,+,*)$ and let N_c denote the maximal sub-C-ring of $(N,+,*)$. Then $N_z = P(1 - a'a)$ and $N_c = P(a)$.

PROOF. Recall that $P(t)$ is defined only in the event that t is regular. Since $(1 - a'a)^2 = 1 - a'a$, then $1 - a'a$ is regular in $(N,+,\cdot,1)$. Thus $P(1 - a'a)$ is defined. From Berman and Silverman [2] we know that $N_z = \{x \in N: n * x = x \text{ for all } n \in N\}$ and $N_c = \{x \in N: 0 * x = 0\}$. It is immediate that 0 is in both N_z and N_c and is the only such element. Since $1 - a'a$ is idempotent $P(1 - a'a) = \{x \in N: x(1 - a'a) = x\}$. Let $x \in N_z$. Then $n * x = x$ for all $n \in N$. So $0 * x = x$. Thus $x = 0 * x = a0x + (1 - a'a)x = (1 - a'a)x = x(1 - a'a)$. Hence, $x \in P(1 - a'a)$ and $N_z \subset P(1 - a'a)$. Now let x be any element in $P(1 - a'a)$. Then $x = x(1 - a'a)$. So for any $n \in N$, $n * x = anx + (1-a'a)x = anx(1 - a'a) + (1 - a'a)x = nx(a - aa'a) + x = x$. Thus $x \in N_z$ and $P(1 - a'a) \subset N_z$. The two inclusions show $P(1 - a'a) = N_z$. Now we show that $P(a) = N_c$. Let $x \in N_c$. Then $0 = 0 * x = a0x + (1 - a'a)x = x - a'ax$. So $x = a'ax =$

$xa'a$ and $x \in P(a)$. Thus $N_c \subset P(a)$. Conversely, suppose $x \in P(a)$. Then $x = xa'a$ and $0 = x - a'ax = (1 - a'a)x$. So $0 * x = a0x + (1 - a'a)x = 0$. Therefore, $x \in N_c$ and $P(a) \subset N_c$. The two inclusions show that $P(a) = N_c$.

THEOREM 2.17. Let $(N,+,*)$ be a special near-ring determined by a. If t is regular in N then $(P(t),+,*)$ is an ideal of $(N,+,*)$.

<u>PROOF</u>. By Theorem 2.13 we know that $(P(t),+)$ is a subgroup of $(N,+)$. Let $n \in N$ and $x \in P(t)$. Then $x = xt't$. So $(n * x)t't = [anx + (1 - a'a)x]t't = an(xt't) + (1 - a'a)(xt't) = anx + (1 - a'a)x = n * x$. Thus $n * x \in P(t)$. Now let $n, m \in N$ and $x \in P(t)$. Then $[(n + x) * m - n*m]t't = [anm + axm - anm]t't = (axm)t't = am(xt't) = amx = axm = (n + x) * m - n * m$. Thus $(n + x) * m - n * m \in P(t)$ and $(P(t),+,*)$ is an ideal.

It should be noted, in particular, that $(P(a),+,*)$ and $(P(1-a'a),+,*)$ are ideals of $(N,+,*)$.

THEOREM 2.18. Let $(N,+,*)$ be a special near-ring determined by a and let t be regular in N.

(a) If $(L,+,*)$ is a left ideal of $N,+,*)$ then $(L(t),+,*)$ is a left ideal of $(N,+,*)$.

(b) If $(L,+,*)$ is a right ideal of $(N,+,*)$ then $(L(t),+,*)$ is a right ideal of $(N,+,*)$.

(c) If $(L,+,*)$ is an ideal of $(N,+,*)$ then $(L(t),+,*)$ is an ideal of $(N,+,*)$.

PROOF. (a) Let $(L,+,*)$ be a left ideal of $(N,+,*)$. Let $n \in N$ and $x \in L(t)$. Then $x = st't$ for some $s \in L$. So $n * x = anx + (1 - a'a)x =$ $anst't + (1 - a'a)s\, t't = \left[ans + (1 - a'a)s\right] t't =$ $(n * s)t't$. Now $n * s \in L$ since L is a left ideal. Thus $n * x \in L(t)$. So $(L(t),+,*)$ is a left ideal.

(b) Let $(L,+,*)$ be a right ideal of $(N,+,*)$. Let $m, n \in N$ and $x \in L(t)$. Then $x = st't$ for some $s \in L$. Since $(L,+,*)$ is a right ideal $(m + s) * n - m * n = a(m + s)n + (1 - a'a)n - amn - (1 - a'a)n =$ $asn \in L$. Thus $(m + x) * n - m * n = a(m + x)n + (1 - a'a)n -$ $amn - (1 - a'a)n = axn = a(st't)n = (asn)t't \in L(t)$. Hence, $(L(t),+,*)$ is a right ideal of $(N,+,*)$.

(c) This result follows immediately from parts (a) and (b).

In particular, the above theorem gives the result that $(L(a),+,*)$ and $(L(1 - a'a),+,*)$ are ideals (left, right) of $(N,+,*)$ whenever $(L,+,*)$ is an ideal (left, right) of $(N,+,*)$. Actually, weaker hypotheses guarantee that $(L(1 - a'a),+,*)$ is an ideal of $(N,+,*)$ as the next theorem shows.

THEOREM 2.19. Let $(N,+,*)$ be a special near-ring determined by a and let $(L,+)$ be a subgroup of $(N,+)$. Then $(L(1 - a'a),+,*)$ is an ideal of $(N,+,*)$.

PROOF. By Theorem 2.14, $(L(1 - a'a),+)$ is a subgroup of $(N,+)$. Let $n \in N$ and $x \in L(1 - a'a)$. Since $L(1 - a'a) \subset P(1 - a'a) = N_Z, n * x = x \in L(1 - a'a)$. Hence, $(L(1 - a'a),+,*)$ is a left ideal. Now let $m, n \in N$ and $x \in L(1 - a'a)$. Then $x = s(1 - a'a)$ for some $s \in L$. Thus $(m + x) * n - m * n = a(m + x)n + (1 - a'a)n - amn - (1 - a'a)n = axn = as(1 - a'a)n = s(a - aa'a)n = s0n = 0 \in L(1 - a'a)$. Hence, $(L(1 - a'a),+,*)$ is a right ideal and, therefore, an ideal of $(N,+,*)$.

THEOREM 2.20. Let $(N,+,*)$ be a special near-ring determined by a.

(a) $(P(a),+,*)$ is a subring of $(N,+,*)$.

(b) If $(L,+,*)$ is a left ideal of $(N,+,*)$ then $(L(a),+,*)$ is a subring of $(N,+,*)$.

PROOF. By Theorem 2.17 $(P(a),+,*)$ is an ideal of $(N,+,*)$. Let $x, y, z \in P(a)$. Then $x * y \in P(a)$. Since $z \in P(a)$, $z = za'a$ and $(1 - a'a)z = 0$. Thus $(x + y) * z = a(x + y)z + (1 - a'a)z = axz + ayz = axz + (1 - a'a)z + ayz + (1 - a'a)z = x * z + y * z$.

(b) Suppose $(L,+,*)$ is a left ideal of $(N,+,*)$. Then by Theorem 2.18, $(L(a),+,*)$ is a left ideal of $(N,+,*)$. Now let $x, y, z \in L(a)$. Then $x * y \in L(a)$. Also $z = sa'a$ for some $s \in L$. So $(1 - a'a)z = (1 - a'a)sa'a = 0$. Therefore, $(x + y) * z = a(x + y)z + (1 - a'a)z = axz + ayz = axz + (1 - a'a)z + ayz + (1 - a'a)z = x * z + y * z$.

THEOREM 2.21. Let $(N,+,*)$ be a special near-ring determined by a. If $(L,+,*)$ is a left ideal of $(N,+,*)$ then $L = L(1 - a'a) \oplus L(a)$, a direct sum of left ideals of $(N,+,*)$. Conversely, if $R \subset P(1 - a'a)$ and $S \subset P(a)$ are left ideals of $(N,+,*)$ then $(R \oplus S,+,*)$ is a left ideal of $(N,+,*)$.

PROOF. Let $(L,+,*)$ be a left ideal of $(N,+,*)$. Then by Theorem 2.18 $(L(1 - a'a),+,*)$ and $(L(a),+,*)$ are left ideals of $(N,+,*)$. Also by Theorem 2.15 $L(1 - a'a) \subset P(1 - a'a)$ and $L(a) \subset P(a)$. So, $L(1-a'a) \cap L(a) \subset P(1-a'a) \cap P(a) = N_z \cap N_c = 0$. We now show that $L(1 - a'a) \oplus L(a) = L$. To this end let $x \in L(1 - a'a)$. Then $x = s(1 - a'a)$ for some $s \in L$. Then $(1 - a'a) * s = a(1 - a'a)s + (1 - a'a)s = (1 - a'a)s = x \in L$ since L is a left ideal. Hence $L(1 - a'a) \subset L$. Now suppose $x \in L(a)$. Then $x = sa'a$ for some $s \in L$. Thus $x =$

$sa'a = s - (1 - a'a)s = s - (a(1-a'a)s + (1-a'a)s) =$
$s - (1-a'a)*s \in L$ since L is a left ideal. Thus $L(a) \subset L$. Since $L(1 - a'a)$ and $L(a)$ are both subsets of L we have $L(1 - a'a) \oplus L(a) \subset L$. Now let $s \in L$. Then $s = s(1 - a'a) + sa'a$. Clearly, $s(1 - a'a) \in L(1 - a'a)$ and $sa'a \in L(a)$. Thus $s \in L(1 - a'a) \oplus L(a)$. Hence, $L \subset L(1 - a'a) \oplus L(a)$ and by double inclusion we have $L = L(1 - a'a) \oplus L(a)$.

Now consider the converse. Let $R \subset P(1 - a'a)$ and $S \subset P(a)$ be left ideals of $(N,+,*)$. Then as in the first part $R \cap S \subset P(1 - a'a) \cap P(a) = 0$. So $R \oplus S$ is a direct sum of left ideals. Let $n \in N$ and $x = r + s \in R \oplus S$. Then $n * x = n * (r + s) = n * r + n * s \in R \oplus S$ since R and S are left ideals of $(N,+,*)$. Thus $(R \oplus S,+,*)$ is a left ideal of $(N,+,*)$.

COROLLARY 2.22. Let $(N,+,*)$ be a special near-ring determined by a. If $(L,+,*)$ is an ideal of $(N,+,*)$ then $L = L(1 - a'a) \oplus L(a)$, a direct sum of ideals. Conversely, if $R \subset P(1 - a'a)$ and $S \subset P(a)$ are ideals of $(N,+,*)$ then $(R \oplus S,+,*)$ is an ideal of $(N,+,*)$.

<u>PROOF</u>. Let $(L,+,*)$ be an ideal of $(N,+,*)$. By Theorem 2.18, $(L(1 - a'a),+,*)$ and $(L(a),+,*)$ are ideals of $(N,+,*)$. By Theorem 2.21, $L = L(1 - a'a) \oplus L(a)$. Hence, L is a direct sum of ideals. Conversely, suppose $R \subset P(1 - a'a)$ and $S \subset P(a)$ are ideals of $(N,+,*)$.

Then by Theorem 2.21, $(R \oplus S,+,*)$ is a left ideal of $(N,+,*)$. Let $m,n \in N$ and $x = r + s \in R \oplus S$. Then $(m + x) * n - m*n = axn = a(r + s)n = arn + asn = \left[(m + r) * n - m * n\right] + \left[(m+s) * n - m * n\right]$. Since R and S are right ideals $(m + r) * n - m * n \in R$ and $(m + s) * n - m * n \in S$. Hence $(m + x) * n - m * n \in R \oplus S$. We conclude that the left ideal $R \oplus S$ is also a right ideal and, therefore, an ideal of $(N,+,*)$.

THEOREM 2.23. Let $(N,+,*)$ be a special near-ring determined by a and let $L \subset P(a)$ be a left ideal of $(N,+,*)$. If $t \in L$ is regular in $(N,+,\cdot,1)$ then $P(t) \subset L$.

<u>PROOF</u>. Let $(L,+,*)$ be a left ideal of $(N,+,*)$ such that $L \subset P(a)$. Let $t \in L$ be regular in $(N,+,\cdot,1)$ and $x \in P(t)$. Then $x = xt't$. Since $t \in P(a)$ we have $t = ta'a$. So $x = xt't = xt'(ta'a) = (xt't)a'a = xa'a$. Hence, $(a'xt') * t = aa'xt't + (1 - a'a)t = (xa'a)t't + (1 - a'a)ta'a = xt't + (1 - a'a)ata' = x + 0 = x \in L$ since L is a left ideal. Thus $P(t) \subset L$.

We note, in particular, that if $L \subset P(a)$ is a left ideal of any special regular near-ring determined by a then for every $t \in L$, $P(t) \subset L$.

THEOREM 2.24. Let $(N,+,*)$ be a special near-ring determined by a and $L \subset P(a)$. Then $(L,+,*)$ is an ideal of $(N,+,*)$ is and only if $(L,+,*)$ is a left ideal of $(N,+,*)$.

PROOF. If L is an ideal it is a left ideal. Consider the converse. Suppose $(L,+,*)$ is a left ideal of $(N,+,*)$. Then $y * s \in L$ for all $y \in N$ and $s \in L$. Let $x, y \in N$ and $s \in L$. Then $s \in P(a)$ so $s = sa'a$ and $(1 - a'a)s = 0$. Thus $(x + s) * y - x * y =$ $a(x + s)y + (1 - a'a)y - axy - (1 - a'a)y =$ $asy = ays + (1 - a'a)s = y * s \in L$ since L is a left ideal. Therefore, the left ideal L is also a right ideal and, hence, an ideal of $(N,+,*)$.

THEOREM 2.25. Let $(N,+,*)$ be a special near-ring determined by a and $L \subset P(1 - a'a)$. Then the following statements are equivalent:

(a) $(L,+,*)$ is an ideal of $(N,+,*)$,

(b) $(L,+,*)$ is a left ideal of $(N,+,*)$,

(c) $(L,+)$ is a subgroup of $(P(1 - a'a),+)$.

PROOF. Clearly (a) implies (b) and (b) implies (c). To show that (c) implies (a), let $(L,+)$ be a subgroup of $(P(1 - a'a),+)$. Let $x, y \in N$ and $s \in L$. Since $s \in P(1 - a'a)$, $s = s(1 - a'a)$. So $x * s = axs + (1 - a'a)s =$

$axs(1 - a'a) + (1 - a'a)s = 0 + (1 - a'a)s = s \in L$. Thus L is a left ideal. Also $(x + s) * y - x * y =$ $a(x + s)y + (1 - a'a)y - axy - (1 - a'a)y = asy =$ $as(1 - a'a)y = 0 \in L$. So the left ideal $(L,+,*)$ is also a right ideal and, hence, an ideal of $(N,+,*)$.

THEOREM 2.26. Let $(N,+,*)$ be a special near-ring determined by a. Then $(I,+,*)$ is an ideal of $(N,+,*)$ if and only if $(I,+,*)$ is a left ideal of $(N,+,*)$.

PROOF. If I is an ideal then I is a left ideal so consider the converse. Let $(I,+,*)$ be a left ideal of $(N,+,*)$. By Theorem 2.21, $I = I(1 - a'a) \oplus I(a)$ where $(I(1 - a'a),+,*)$ and $(I(a),+,*)$ are left ideals of $(N,+,*)$.

By Theorem 2.15, $I(1 - a'a) \subset P(1 - a'a)$ and $I(a) \subset P(a)$. By Theorem 2.25, $(I(1 - a'a),+,*)$ is an ideal of $(N,+,*)$. By Theorem 2.24, $(I(a),+,*)$ is an ideal of $(N,+,*)$. Then by Corollary 2.22, $I(1 - a'a) \oplus I(a) =$ I is an ideal of $(N,+,*)$.

LEMMA 2.27. Let $(N,+,*)$ be a special near-ring. If $x, y, z \in N$, then $(x + y) * z - y * z - x * z = -bz$.

<u>PROOF</u>. Let $x, y, z \in N$. Then

$(x + y) * z - y * z - x * z =$

$a(x + y)z + bz - axz - bz - ayz - bz = - bz$.

The next lemma contains some well-known results whioh will be useful in subsequent work.

LEMMA 2.28. Let $(N,+,\cdot)$ be an arbitrary near-ring and $(I,+,\cdot)$ any ideal of $(N,+,\cdot)$. Let N/I be the factor near-ring. Then for any $x, y, z \in N$ the following statements are equivalent:

(a) $(\bar{x} + \bar{y})\bar{z} = \bar{x}\,\bar{z} + \bar{y}\,\bar{z}$.

(b) $\overline{(x + y)}\bar{z} = \overline{xz} + \overline{yz}$.

(c) $\overline{(x + y)z} = \overline{xz + yz}$.

(d) $(x + y)z - yz - xz \in I$.

THEOREM 2.29. Let $(N,+,*)$ be a special near-ring and $(I,+,*)$ an ideal of $(N,+,*)$. If $z \in N$ then the statement

(e) $-bz \;\; I \in$ is equivalent to each of (a), (b), (c), (d), of Lemma 2.28.

PROOF. Since the addition in $(N,+,*)$ is commutative statement (d) of Lemma 2.28 is equivalent, in the present context, to the statement that $(x + y) * z - x * z - y * z \in I$ for all $x, y, z \in N$. But by Lemma 2.27 for any $x, y, z \in N$, $(x + y) * z - x * z - y * z = -bz$. Thus statement (e) is equivalent to statement (d) and, hence, to the other statements of Lemma 2.28.

THEOREM 2.30. Let $(N,+,*)$ be a special near-ring determined by a and let $(I,+,*)$ be an ideal of $(N,+,*)$. Then N/I is a ring if and only if $P(1 - a'a) \subset I$.

PROOF. Let N/I be a ring and let $z \in P(1 - a'a)$. The right distributive law holds in N/I so for all $\bar{x}, \bar{y}, \bar{z} \in N/I$, $(\bar{x} + \bar{y}) * \bar{z} = \bar{x} * \bar{z} + \bar{y} * \bar{z}$. Since $b = 1 - a'a$, by Lemma 2.28 and Theorem 2.29, $-bz = -(1 - a'a)z \in I$. Since $(I,+)$ is a subgroup of $(N,+)$, $(1 - a'a)z \in I$. But $z = (1 - a'a)z$ since $z \in P(1 - a'a)$. Thus $z \in I$ and $P(1 - a'a) \subset I$. Conversely, suppose $P(1 - a'a) \subset I$. For all $z \in N$, $-(1 - a'a)z \in P(1 - a'a) \subset I$. So by Theorem 2.29, $(\bar{x} + \bar{y}) * \bar{z} = \bar{x} * \bar{z} + \bar{y} * \bar{z}$ for all $x, y, z \in N$. Thus N/I is a ring.

COROLLARY 2.31. Let $(N,+,*)$ be a special regular near-ring determined by a and let $(I,+,*)$ be an ideal of $(N,+,*)$. Then N/I is a regular ring if and only if $P(1 - a'a) \subset I$.

PROOF. The result follows from Theorem 2.30 and the fact that for any regular near-ring R and any ideal A of R, R/A is regular. To see the latter assertion let $\bar{z} \in R/A$. Since z is regular in $(R,+,\cdot)$ then $\bar{z}\bar{z}'\bar{z} = \overline{zz'z} = \bar{z}$.

The next theorem states conditions under which two special regular near-rings are isomorphic.

THEOREM 2.32. Let $(N,+,\cdot,1)$ be a commutative regular ring with identity and let a and c be regular in N. Let $(N,+,*a)$ and $(N,+,*c)$ be the special regular near-rings determined by a and c respectively. If there is an automorphism of $(N,+,\cdot,1)$ which maps a to c then $(N,+,*_a)$ and $(N,+,*_c)$ are isomorphic.

PROOF. Let f denote an automorphism of $(N,+,\cdot,1)$ which has the property that $f(a) = c$ and let c' denote any regularity companion of c. Let $x, y \in N$. From the hypotheses we have $f(x + y) = f(x) + f(y)$. Since $f(a) = c$, we obtain $c = f(a) = f(a'a^2) = f(a')f(a^2) = f(a')(f(a))^2 = f(a')c^2$. By Theorem 1.5, $f(a')c = c'c$. So $f(x *_a y) = f(axy + (1 - a'a)y) = f(axy) + f((1 - a'a)y) = f(a)f(x)f(y) + f(1-a'a)f(y) = cf(x)f(y) + (f(1) - f(a'a))f(y) = cf(x)f(y) + (1 - f(a')f(a))f(y) = cf(x)f(y) + (1-f(a')c)f(y) = f(x) *_c f(y)$. Hence, $(N,+,*_c)$ is isomorphic to $(N,+,*_a)$.

THEOREM 2.33. Let $(N,+,\cdot,1)$ be a commutative ring with identity and let $(A,+,\cdot)$ be an ideal of $(N,+,\cdot,1)$. Then A is a direct summand of N if and only if $A = P(u)$ for some regular u in N.

<u>PROOF</u>. Let $A = P(u)$ where u is regular in N. Since $N = P(1 - u'u) \oplus P(u)$, $A = P(u)$ is a direct summand of N. Conversely, let A be a direct summand of N. Then $N = A \oplus B$, where $(B,+,\cdot)$ is also an ideal. Now $1 \in N = A \oplus B$ so $1 = u + v$ where $u \in A$ and $v \in B$. Then $u = u1 = u(u + v) = u^2 + uv$. Since $uv \in A \cap B = 0$, $uv = 0$. Thus $u = u^2$ and u is regular in N. If $x \in A$, then $x = x\,1 = x(u + v) = xu + xv = xu + 0 = xu$. Thus $x \in P(u)$ and $A \subset P(u)$. If $y \in P(u)$ then $y = yu \in A$ since A is an ideal. Therefore, $P(u) \subset A$ and we conclude that $A = P(u)$.

THEOREM 2.34. Let $(N,+,*)$ be a special near-ring determined by a and let $(B,+,*)$ be an ideal of $(N,+,*)$. Then B is a direct summand of N if and only if $B(a)$ and $B(1 - a'a)$ are direct summands of $P(a)$ and $P(1 - a'a)$ respectively.

<u>PROOF</u>. Suppose $B(a)$ is a direct summand of $P(a)$ and $B(1 - a'a)$ is a direct summand of $P(1 - a'a)$. Then $P(1 - a'a) = A \oplus B(1 - a'a)$ and $P(a) = B(a) \oplus C$ for some ideals A and C. Since $(B,+,*)$ is an ideal

$B = B(1 - a'a) \oplus B(a)$. Hence $N = P(1 - a'a) \oplus P(a) = A \oplus B(1 - a'a) \oplus B(a) \oplus C = A \oplus B \oplus C$. Thus B is a direct summand of N. Conversely, suppose B is a direct summand of N. Then $N = B \oplus C$ for some ideal C. Since $B = B(1 - a'a) \oplus B(a)$ and $C = C(1 - a'a) \oplus C(a)$ it follows that $N = B \oplus C =$
$B(1 - a'a) \oplus B(a) \oplus C(1 - a'a) \oplus C(a) =$
$B(1 - a'a) \oplus C(1 - a'a) \oplus B(a) \oplus C(a)$. By Theorem 2.15, $B(1 - a'a)$ and $C(1 - a'a)$ are subsets of $P(1 - a'a)$ and $B(a)$ and $C(a)$ are subsets of $P(a)$. Hence, $B(1 - a'a) \oplus C(1 - a'a) \subset P(1 - a'a)$ and $B(a) \oplus C(a) \subset P(a)$. Let $x \in P(a)$. Then $x = xa'a$. Since $x \in N = B \oplus C$, $x = b + c$ where $b \in B$ and $c \in C$. So $x = xa'a = (b + c)a'a = ba'a + ca'a$. But $ba'a \in B(a)$ and $ca'a \in C(a)$. Hence $x \in B(a) \oplus C(a)$ and we conclude that $P(a) \subset B(a) \oplus C(a)$. By double inclusion we have that $P(a) = B(a) \oplus C(a)$. In a similar manner we may show that $P(1 - a'a) = B(1 - a'a) \oplus C(1 - a'a)$.

REMARK. The proof of the second half of the theorem shows that if $(N,+,*)$ is a special near-ring determined by a and N is a direct sum of ideals, say $N = B \oplus C$, then $P(a) = B(a) \oplus C(a)$ and $P(1 - a'a) = B(1 - a'a) \oplus C(1 - a'a)$.

THEOREM 2.35. Let $(N,+,*)$ be a special near-ring determined by a. Let $(I,+,*)$ be an ideal of $(N,+,*)$. If I is a direct summand of N then there is a regular $t \in P(a)$ such that $I = P(t) \oplus M$ where $P(t) \subset P(a)$ is an ideal and $M \subset P(1 - a'a)$ is an ideal.

PROOF. If I is a direct summand of N then $N = I \oplus L$ where L is an ideal. From previous work we know that $I = I(a) \oplus I(1 - a'a)$ where $I(a)$ and $I(1 - a'a)$ are ideals. By Theorem 2.34, $I(a)$ is a direct summand of $P(a)$. Since $(P(a),+,\cdot,a'a)$ is a commutative ring with identity and $I(a)$ is an ideal of $P(a)$ which is a direct summand of $P(a)$ we know by Theorem 2.33 that $I(a) = P(t)$ for some t regular in $P(a)$. Thus $I = I(a) \oplus I(1 - a'a) = P(t) \oplus I(1 - a'a)$. Clearly $P(t) = I(a) \subset P(a)$ is an ideal and $I(1 - a'a) \subset P(1 - a'a)$ is an ideal.

We are able to establish the converse of Theorem 2.35 for a certain class of special near-rings determined by elements, namely that class in which the additive group of any member has the property that each of its subgroups is pure and bounded.

THEOREM 2.36. Let $(N,+,*)$ be a special near-ring determined by a. Let I be an ideal of $(N,+,*)$. If there is a regular element $t \in P(a)$ such that $I = P(t) \oplus M$ where $P(t) \subset P(a)$ is an ideal and $(M,+)$ is a pure and bounded subgroup of $(P(1 - a'a),+)$ then I is a direct summand of N.

PROOF. By previous work we know that $P(a) = P(t) \oplus A$ for some ideal A. Kaplansky [10] proves that a pure and bounded subgroup is a direct summand. Thus $P(1 - a'a) = M \oplus B$ for some ideal B. Then $N = P(a) \oplus P(1 - a'a) = P(t) \oplus A \oplus M \oplus B = P(t) \oplus M \oplus A \oplus B = I \oplus A \oplus B$.

4. Some Decomposition Theorems

As mentioned earlier, any field is a commutative regular ring with identity. Therefore, any field may be used as the base ring in which we define a new multiplication to obtain a special regular near-ring. It is natural to ask what types of near-rings are obtained in this case.

THEOREM 2.37. Let $(F,+,\cdot,1)$ be a field and $a \in F$. Let $(F,+,*)$ be the special regular near-ring determined by $a \in F$. Then either $x * y = y$ for all $x, y \in F$ or $(F,+,*,a^{-1})$ is isomorphic to $(F,+,\cdot,1)$.

PROOF: If $a = 0$ then $x * y = axy + (1 - a'a)y = y$ for all $x, y \in F$. If $a \neq 0$ then a has a multiplicative inverse which we denote by a^{-1}. In this case a^{-1} is the unique regularity companion of a. Then $x * y = axy + (1 - a^{-1}a)y = axy = ayx + (1 - a^{-1}a)x = y * x$ for all $x, y \in F$. Note that $a^{-1} * y = aa^{-1}y = y$ for all $y \in F$. Since $*$ is commutative, a^{-1} is the identity in $(F,+,*)$. Also, $F \neq 0$ so by Theorem 2.6,

$(F,+,*, a^{-1})$ is isomorphic to $(F,+,\cdot,1)$.

Theorem 2.37 leads to a decomposition theorem for special regular near-rings. In order to obtain such a theorem we make use of the following which may be found in McCoy [14].

THEOREM 2.38. Every commutative regular ring of more than one element is isomorphic to a subdirect sum of fields.

Theorem 2.38 restricts the following discussion to special regular near-rings of more than one element but this restriction is not significant.

THEOREM 2.39. Let $(N,+,*)$ be a special regular near-ring determined by a. Then $(N,+,*)$ is isomorphic to a subdirect sum of near-rings N_i where each N_i is one of the following two types:

(a) N_i is a field,

(b) the additive group of $(N_i,+,*_i)$ is that of a field and $x_i *_i y_i = y_i$ for all $x_i, y_i \in N_i$.

<u>PROOF</u>. By Theorem 2.38, the commutative regular ring with identity, $(N,+,\cdot,1)$ is isomorphic to a subdirect sum of fields F_i where i is in some index set I. Hence, there is some subring, say S, of the direct sum of the F_i, which is isomorphic to N.

Denote the isomorphism by α. Let $a \in N$. Then $\alpha(a) = [a_i]_{i\in I}$ where $a_i \in N_i$ and $N_i = F_i$. Then a determines a near-ring $(N,+,*)$ and $\alpha(a)$ determines a near-ring $(S,+,*)$. It is straightforward to show that $(N,+,*)$ is isomorphic to $(S,+,*)$ under α. Now each N_i is the near-ring $(F_i,+,*_i)$ where the multiplication $*_i$ is determined by $x_i *_i y_i = (x*y)_i$ where the i^{th} components of x and y are x_i and y_i respectively. Hence, $(N,+,*)$ is isomorphic to a subdirect sum of near-rings N_i where each N_i is $(F_i,+,*_i)$.

Let $a = [a_i]_{i\in I}$ and $b = [b_i]_{i\in I}$ where $b = 1 - a'a$. Since $b^2 = b$ and b_i is in the field F_i, b_i must be either 0 or 1 for each $i \in I$. Since $ab = 0$, $a_i = 0$ when $b_i = 1$. Also since $b + a'a = 1$, $(a'a)_i = 0$ if $b_i = 1$ and $(a'a)_i = 1$ if $b_i = 0$. Let $i \in I$ and consider N_i. If $b_i = 0$ then $(a'a)_i = 1$. Now $a_i \neq 0$ for, otherwise, $(a'a)_i = a_i'a_i = a_i' 0 = 0$ which contradicts the fact that $(a'a)_i = 1$. For $x = [x_i]_{i\in I}$ and $y = [y_i]_{i\in I}$, elements of N, $x * y = [(x * y)_i]_{i\in I}$. Then $(x * y)_i = a_i x_i y_i + [1 - (a'a)_i] y_i = a_i x_i y_i$. By the same kind of argument used in the proof of Theorem 2.37, N_i is isomorphic to $(F_i,+,\cdot,1)$. Thus N_i is a field. If $b_i = 1$, then $a_i = 0$ as stated above. Then $x_i *_i y_i = (x * y)_i = a_i x_i y_i + [1 - (a'a)_i] y_i = y_i$.

Thus the additive group of $(N_i,+,*_i)$ is the additive group of the field F_i and $x_i *_i y_i = y_i$ for all $x_i, y_i \in N_i$.

The near-rings N_i in the above decomposition theorem are not necessarily subdirectly irreducible. We can obtain a decomposition into subdirectly irreducible near-rings by appealing to a result of Ligh [11]. First we need to recall the following definitions. A near-ring $(N,+,\cdot)$ is <u>small</u> if and only if for each $x \in N$ either $xy = y$ for all $y \in N$ or $xy = 0y$ for all $y \in N$. A near-ring $(N,+,\cdot)$ is called a <u>β near-ring</u> if it is weakly commutative and $x^2 = x$ for all $x \in N$. Any β near-ring is a regular near-ring. The next theorem is due to Ligh [11].

THEOREM 2.40. Every β near-ring $(N,+,\cdot)$ is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i where each N_i is either a two element field or a small near-ring.

THEOREM 2.41. Let $(N,+,*)$ be a special regular near-ring determined by a. Then $(N,+,*)$ is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i where each N_i is either a field or a small near-ring.

PROOF. Consider a decomposition guaranteed by Theorem 2.39. If N_i is a field it is clearly a subdirectly irreducible near-ring. If N_i is the second type of near-ring mentioned in Theorem 2.39 then it is easily shown to be a β near-ring, and, hence, by Theorem 2.40 it is isomorphic to a subdirect sum of subdirectly irreducible near-rings which are either fields or small. The conclusion then follows.

5. The Set of Distributors

Let $(N,+,\cdot)$ be an arbitrary near-ring. Then any element of the form $(x + y)z - yz - xz$, for $x, y, z \in N$ is called a distributor [4]. The set $D = \{(x + y)z - yz - xz : x, y, z \in N\}$ is called the set of distributors. In case D is an ideal it will be called the distributor ideal.

THEOREM 2.42. Let $(N,+,\cdot)$ be a near-ring of prime order. Then either $D = 0$ or $D = N$.

PROOF. Suppose $D \neq 0$. Let d be a nonzero element of D. Then $d = (x + y)z - yz - xz$ for some $x, y, z \in N$. Let k be a positive integer. Then,

$$kd = [(x + y)z - yz - xz] + \ldots + [(x + y)z - yz - xz]$$

where there are k terms in the sum. Since $(N,+)$ is abelian $kd = [(x + y)z + \ldots + (x + y)z] - [yz + \ldots + yz] - [xz + \ldots + xz] = (x + y)[z + \ldots + z] - [y(z + \ldots + z)] - [x(z + \ldots + z)] = (x + y)(kz) - y(kz) - x(kz) \in D$. Since N is of prime order N has no non-trivial subgroups. Thus $D = N$.

For each of the examples in the list of Clay [5], D is a subgroup. Class (33) where the additive group is $(S_3,+)$ provides us with the only examples in Clay's list where D is not a normal subgroup. This same example shows that D is not always an ideal.

THEOREM 2.43. Let $(N,+,\cdot)$ be a near-ring. If the set D of distributors is a normal subgroup, then D is a left ideal.

PROOF. Suppose $(D,+)$ is a normal subgroup of $(N,+)$ and let $d \in D$. Then $d = (x + y)z - yz - xz$ for some $x, y, z \in N$. Let $n \in N$. Then $nd = n[(x + y)z - yz - xz] = n(x + y)z + n[-(yz)] + n[-(xz)] = (nx + ny)z - (ny)z - (nx)z \in D$. Thus, $(D,+,\cdot)$ is a left ideal of $(N,+,\cdot)$.

THEOREM 2.44. Let $(N,+,\cdot)$ be an arbitrary near-ring and D the set of distributors. If D is a normal subgroup then D is an ideal if and only if $dn \in D$ for all $d \in D$ and $n \in N$.

PROOF. Let D be a normal subgroup of N. Then by Theorem 2.43, D is a left ideal. Thus we need to show that D is a right ideal if and only if $dn \in D$ for all $d \in D$ and $n \in N$. Suppose D is a right ideal and let $d \in D$ and $n \in N$. For all $x \in N$, $(x + d)n - dn - xn \in D$. But $(x + d)n - dn - xn = (x + d)n - xn + xn - dn - xn \in D$. Since D is a right ideal $(x + d)n - xn \in D$. Thus we have $xn - dn - xn \in D$. Since D is normal $-dn \in D$ and since D is a group $dn \in D$. Now consider the converse. Suppose for all $d \in D$ and $n \in N$, $dn \in D$. As above, for any $x \in N$, $(x + d)n - dn - xn = (x + d)n - xn + xn - dn - xn \in D$. Since $dn \in D$, $-dn \in D$ and since D is normal $xn - dn - xn \in D$. Thus we conclude that for all $x, n \in N$, $d \in D$, $(x + d)n - xn \in D$. Therefore, D is a right ideal.

THEOREM 2.45. Let $(N,+,*)$ be a special near-ring. Then the set D of distributors is an ideal.

PROOF. By Lemma 2.27, $D = \{-bz : z \in N\}$. Let $c, d \in D$. Then $c = -bn$ and $d = -bm$ for some $n,m \in N$. So $c-d = -b(n-m) \in D$. Thus D is a normal subgroup. Let $x \in N$. Then $dx = -b(mx) \in D$. Thus by Theorem 2.44, D is an ideal.

THEOREM 2.46. Let $(N,+,\cdot)$ be an abelian near-ring. If the set D of distributors is an ideal then N/D is a ring.

<u>PROOF</u>. Since N is an abelian near-ring, N/D is an abelian near-ring. Thus we need only show that the right distributive law holds in N/D. Let $\bar{x}, \bar{y}, \bar{z} \in N/D$. Then since $(x + y)z - yz - xz \in D$, $(\bar{x} + \bar{y})\bar{z} - \bar{y}\bar{z} - \bar{x}\bar{z} = \overline{(x + y)z - yz - xz} = D$. Hence, $(\bar{x} + \bar{y})\bar{z} = \bar{x}\bar{z} + \bar{y}\bar{z}$ and N/D is a ring.

COROLLARY 2.47. Let $(N,+,*)$ be a special near-ring and let D be the distributor ideal. Then N/D is a ring.

<u>PROOF</u>. Since $(N,+)$ is abelian, the result is immediate from Theorem 2.46.

We remark that this corollary could be obtained for special near-rings determined by single elements through an application of Theorem 2.30 since $P(1 - a'a) \subset D$.
As a matter of fact, in the setting of a special near-ring determined by a we have $D = P(1 - a'a)$.

CHAPTER III

SPECIAL β NEAR-RINGS

1. Definition and Examples

In Chapter II we began with a commutative ring, $(N,+,\cdot,1)$, with identity and used an arbitrary regular element, a, of N to define a new multiplication * on N such that $(N,+,*)$ was a near-ring. To define this new multiplication we used a polynomial in two variables with the fixed coefficients of the terms of the polynomial determined by the fixed element a. In the present chapter we use a different technique to obtain a near-ring dependent in some sense upon a certain type of ring, namely a commutative regular ring with identity. The technique used is similar to the one employed in Chapter II, but differs from it in that the polynomial used to define the new multiplication has coefficients which are dependent upon the factors being multiplied as well as upon a fixed constant.

Because the coefficients are not constants, the multiplication formula of the current chapter is much more difficult to work with than the formula of Chapter II. Therefore, the results we are able to obtain here are not as extensive as those in Chapter II. The multiplication formula which we shall use is presented in the first theorem.

THEOREM 3.1. Let $(N,+,\cdot,1)$ be a commutative regular ring with identity. Let a, b be fixed elements of N such that $ab = 0$ and $b^2 = b$. Define $*: N \times N \longrightarrow N::$ $(x,y) \longrightarrow x * y = a'ax'xy + by$. Then $(N,+,*)$ is a weakly commutative near-ring.

PROOF. We need to show that $*$ is associative and distributes over + from the left. To this end, let $x, y, z \in N$. Notice first that $(x * y)^2y' =$ $(a'ax'xy + by)^2y' = \left[(a'a)^2(x'x)^2y^2 + b^2y^2\right]y' =$ $(a'ax'xy^2 + by^2)y' = a'ax'xy^2y' + by^2y' =$ $a'ax'xy + by = x * y$. Thus by Theorem 1.5, $(x * y)'(x * y) =$ $y'(x * y)$. Now we demonstrate that $*$ is associative. First, $(x * y) * z = a'a(x * y)'(x * y)z + bz =$ $a'ay'(x * y)z + bz = a'ay'(a'ax'xy + by)z + bz =$ $a'ax'xy'yz + bz$. Now $x * (y * z) =$ $a'ax'x(y * z) + b(y * z) = a'ax'x(a'ay'yz + bz) +$ $b(a'ay'yz + bz) = a'ax'xy'yz + b^2z = a'ax'xy'yz + bz$.

Thus $(x * y) * z = x * (y * z)$ and $*$ is associative. From $x * y * z = a'ax'xy'yz + bz$ we observe that interchanging x and y will not affect the product. Hence, $x * y * z = y * x * z$ and $(N,+,*)$ is weakly commutative. Since the expression for $x * y$ is linear in the second variable, the left distributive law holds. Thus the conclusion follows.

THEOREM 3.2. Let $(N,+,\cdot,1)$ be a commutative regular ring with identity and let a be a fixed element of N. Let $b = 1 - a'a$. Define $*: N \times N \longrightarrow N :: (x,y) \longrightarrow x * y = a'ax'xy + by$. Then $(N,+,*)$ is a β near-ring.

<u>PROOF</u>. Since $b = 1 - a'a$, $ab = a(1 - a'a) = a - aa'a = 0$ and $b^2 = (1 - a'a)^2 = 1 - a'a - a'a + (a'a)^2 = 1 - a'a - a'a + a'a = 1 - a'a = b$. Thus by Theorem 3.1, $(N,+,*)$ is a weakly commutative near-ring. Let $x \in N$. Then $x * x = a'ax'xx + (1 - a'a)x = a'ax + x - a'ax = x$.

A near-ring $(N,+,*)$ is a <u>special β near-ring</u> if and only if there exists a commutative regular ring with identity $(N,+,\cdot,1)$ such that for some $a \in N$, $x * y = a'ax'xy + (1 - a'a)y$ for all $x, y \in N$. We say that the special β near-ring is determined by a. Before proceeding to study some of the properties of the class of special β near-rings we consider some examples. First, recall Ratliff's [18] definition that a p-near-ring $(N,+,*)$ is an (α,β) p-near-ring if and only if there exists a p-ring with identity $(N,+,\cdot,1)$ and, $\alpha,\beta \in N$ such that $\alpha\beta = 0, \beta^2 = \beta$ and $x * y = (1 - \alpha^{p-1} - \beta)x^{p-1}y + \alpha xy + \beta y$ for all $x, y \in N$. It is clear that the class of special β near-rings includes those (α,β) p-near-rings for which $\alpha = 0$ and $\beta = 1 - a'a$ for some $a \in N$. Near-rings in Clay's [5] class (8) where the additive group is $(Z_5,+)$ are examples of this situation.

Consider the near-rings defined on the cyclic group of order 6. Those in class (27) are commutative regular rings each of which has an identity element. Thus, we may use a ring in class (27) to determine some special β near-rings. Since the ring we are considering is of order 6 it is not a p-ring. Hence, the near-rings we obtain will be special β near-rings which are not (α,β) p-near-rings. The ring under consideration is $(Z_6,+,\cdot,1)$ where the addition and multiplication are, as usual, modulo 6. In this ring $x^3 = x$ for all $x \in Z_6$. So for any $x \in Z_6$ one regularity companion for x is x itself. We now list the possible choices for a, the resulting multiplications and the near-rings thus obtained.

If $(a,b) = (0,1)$ then $x * y = y$. This produces class (48).

If $(a,b) = (1,0)$ then $x * y = x^2 y$. This produces class (53).

If $(a,b) = (2,3)$ then $x * y = 4x^2y + 3y$. This produces class (49).

If $(a,b) = (3,4)$ then $x * y = 3x^2y + 4y$. This produces class (52).

If $(a,b) = (4,3)$ then $x * y = 4x^2y + 3y$. This produces class (49).

If $(a,b) = (5,0)$ then $x * y = x^2y$. This produces class (53).

Classes (49) and (53) provide examples of special β near-rings which are not special regular near-rings. Classes (48) and (52) provide examples of near-rings which are both special β near-rings and special regular near-rings. Classes (27) and (35) provide examples of special regular near-rings which are not special β near-rings. Those near-rings in class (7) where the additive group is $(Z_5,+)$ provide examples of regular near-rings which are neither special regular near-rings nor special β near-rings.

2. Basic Results

We now begin a study of some of the properties of the class of special β near-rings.

THEOREM 3.3. Let $(N,+,*)$ be a special β near-ring determined by a. Then the following statements are equivalent:

(a) $(N,+,*)$ is commutative,

(b) $a = 1$ and $x^2 = x$ for all $x \in N$,

(c) 1 is a right identity, hence the identity in $(N,+,*)$,

(d) $(N,+,*,1)$ is isomorphic to $(N,+,\cdot,1)$.

PROOF. (a) implies (b). If $(N,+,*)$ is commutative then $x * y = y * x$ for all $x, y \in N$. So $0 = 1 * 0 = 0 * 1 = a'a0'01 + (1 - a'a)1 = 1 - a'a$. Hence $a'a = 1$. Then $x * y = a'ax'xy + (1 - a'a)y = x'xy$ for all $x, y \in N$. Also

$y * x = y'yx$ for all $x, y \in N$. Thus $x'xy = y'yx$ for all $x, y \in N$. Choose $y = 1$. Then $x'x = x$ for all $x \in N$. Since $x'x = x$ we have $x = x(x'x) = xx = x^2$ for all $x \in N$. In particular $a^2 = a$ and $a = a'a^2 = a'a = 1$.

(b) implies (c). Suppose $a = 1$ and $x^2 = x$ for all $x \in N$. Then $1 = a = a'a^2 = a'aa = a'a1 = a'a$. Let $x \in N$. Then $x * 1 = a'ax'x1 + (1 - a'a)1 = x'x = x'x^2 = x$ for all $x \in N$. Thus 1 is a right identity for $(N,+,*)$. Also $1 * x = a'a1'1x + (1 - a'a)x = a'ax + x - a'ax = x$. Thus 1 is a left identity in any special β near-ring. Hence, in this case, 1 is the identity for $(N,+,*)$.

(c) implies (d). Suppose 1 is the identity in $(N,+,*)$. Then $0 = 0 * 1 = a'a0'01 + (1 - a'a)1 = 1 - a'a$. So $a'a = 1$. Then $x * y = a'ax'xy + (1 - a'a)y = x'xy$ for all $x, y \in N$. It then follows that $x = x * 1 = x'x1 = x'x$ for all x in N and $x * y = x'xy = (x'x)y = xy$ for all $x, y \in N$. Hence, $(N,+,*,1)$ is isomorphic to $(N,+,\cdot,1)$.

(d) implies (a). Suppose $(N,+,*,1)$ is isomorphic to $(N,+,\cdot,1)$. Then $*$ is commutative since $\cdot$ is commutative.

THEOREM 3.4. Let $(N,+,*)$ be a special β near-ring determined by a. Then $(N,+,*)$ is a C-ring if and only if $a'a = 1$.

PROOF. Suppose $(N,+,*)$ is a C-ring. Then $0 = 1 * 0 = 0 * 1 = a'a0'01 + (1 - a'a)1 = 1 - a'a$. Thus $a'a = 1$. Conversely, suppose $a'a = 1$. Then $0*x = a'a0'0x + (1 - a'a)x = 0$ for all $x \in N$. Hence, $(N,+,*)$ is a C-ring.

THEOREM 3.5. Let $(N,+,*)$ be a special β near-ring determined by a. Then $(N,+,*)$ is distributively generated if and only if $(N,+,*)$ is a ring.

PROOF. If $(N,+,*)$ is a ring then $(N,+,*)$ is distributively generated. So consider the converse. Suppose $(N,+,*)$ is distributively generated. Recall that $(N,+)$ is abelian. Then by Theorem 2.9, $(N,+,*)$ is a ring.

THEOREM 3.6. Let $(N,+,*)$ be a special β near-ring determined by a. Then $P(1 - a'a)$ is the maximal sub-Z-ring, N_Z, and $P(a)$ is the maximal sub-C-ring, N_C.

PROOF. Recall $N_z = \{x \in N : n * x = x \text{ for all } n \in N\}$ and $N_c = \{x \in N : 0 * x = 0\}$. We first show $P(1 - a'a) = N_z$. Let $x \in P(1 - a'a)$. Then $x = (1 - a'a)x$. For any $n \in N$, $n * x = a'an'nx + (1 - a'a)x = a'an'n(1 - a'a)x + x = x$. Thus $x \in N_z$ and $P(1 - a'a) \subset N_z$. Now let x be any element in N_z. Then for all $n \in N$, $n * x = x$. So take $n = 1 - a'a$. This produces $x = (1 - a'a) * x = a'a(1 - a'a)'(1 - a'a)x + (1 - a'a)x$. Since $(1 - a'a)^2 = (1 - a'a)$, $(1 - a'a)'(1 - a'a) = (1 - a'a)$. Then $x = (1 - a'a) * x = a'a(1 - a'a)x + (1 - a'a)x = (1 - a'a)x$. Thus, $x \in P(1 - a'a)$. Hence, $N_z \subset P(1 - a'a)$. Therefore, by double inclusion $P(1 - a'a) = N_z$.

Now we show $P(a) = N_c$. Let $x \in P(a)$. Then $x = a'ax$. So $0 * x = a'a0'0x + (1 - a'a)x = x - a'ax = x-x = 0$. So $x \in N_c$ and $P(a) \subset N_c$. Now let x be any element in N_c. Then $0 * x = 0$. So $0 = 0 * x = a'a0'0x + (1 - a'a)x$. Thus $x = a'ax$. Therefore, $x \in P(a)$ and $N_c \subset P(a)$. Hence, by double inclusion $P(a) = N_c$.

LEMMA 3.7. Let $(N,+,*)$ be a special β near-ring determined by a. If $t \in N$ then $(P(t),+,*)$ is a left ideal of $(N,+,*)$.

PROOF. From previous results we know that $P(t),+)$ is a normal subgroup of $(N,+)$. Let $x \in P(t)$ and $n \in N$. Then since $x = t'tx$ we have $(n * x)t't = [a'an'nx + (1 - a'a)x]t't = a'an'nxt't + (1 - a'a)xt't = a'an'nx + (1 - a'a)x = n * x$. Thus, $n * x \in P(t)$

and $(P(t),+,*)$ is a left ideal of $(N,+,*)$.

LEMMA 3.8. Let $(N,+,*)$ be a special β near-ring determined by a. If $(S,+,*)$ is a left ideal of $(N,+,*)$ then for any $t \in N$, $(S(t),+,*)$ is a left ideal of $(N,+,*)$.

PROOF. From previous results we know that $(S(t),+)$ is a normal subgroup of $(N,+)$. Let $x \in S(t)$ and $n \in N$. Then $x = st't$ for some $s \in S$. Since S is a left ideal of $(N,+,*)$, $n * s = a'an'ns + (1 - a'a)s$ belongs to S. Hence, $n * x = a'an'nx + (1 - a'a)x = a'an'nst't + (1 - a'a)st't = [a'an'ns + (1 - a'a)s]t't = (n * s)t't$ belongs to $S(t)$. Thus $(S(t),+,*)$ is a left ideal of $(N,+,*)$.

THEOREM 3.9. Let $(N,+,*)$ be a special β near-ring determined by a. If $(S,+,*)$ is a left ideal of $(N,+,*)$ then $S = S(1 - a'a) \oplus S(a)$, a direct sum of left ideals of $(N,+,*)$. Conversely, if $U \subset P(1 - a'a)$ and $V \subset P(a)$ are left ideals of $(N,+,*)$ then $U \oplus V$ is a left ideal of $(N,+,*)$.

PROOF. Let $(S,+,*)$ be a left ideal of $(N,+,*)$. Then by Lemma 3.8, $(S(1 - a'a),+,*)$ and $(S(a).+,*)$ are left ideals of $(N,+,*)$. Now we have the result that $S(1 - a'a) \cap S(a) \subset P(1 - a'a) \cap P(a) = 0$.

Let $x \in S(1 - a'a)$. Then $x = s(1 - a'a)$ for some $s \in S$. Since $(S,+,*)$ is a left ideal of $(N,+,*)$, $0 * s = (1 - a'a)s = x \in S$. Thus $S(1 - a'a) \subset S$.

Now let x be any element of $S(a)$. Then $x = sa'a$ for some $s \in S$. Now $s - (1 - a'a) * s \in S$ since $(S,+,*)$ is a left ideal of $(N,+,*)$. But $(1 - a'a) * s =$ $a'a(1 - a'a)'(1 - a'a)s + (1 - a'a)s = a'a(1 - a'a)s + (1 - a'a)s$ since $(1 - a'a)'(1 - a'a) = 1 - a'a$. Thus $(1 - a'a) * s =$ $- a'as + s = -x + s$. Hence, $x = s - (1 - a'a) * s \in S$. Therefore, $S(a) \subset S$. Clearly, then, $S(1 - a'a) \oplus S(a) \subset S$. We now show these two sets are actually equal by showing the complementary inclusion. To this end let $s \in S$. Then $s = (1 - a'a)s + a'as$. Now $(1 - a'a)s \in S(1 - a'a)$ and $a'as \in S(a)$. Hence, $s \in S(1 - a'a) \oplus S(a)$ and $S \subset S(1 - a'a) \oplus S(a)$. The two inclusions show that $S = S(1 - a'a) \oplus S(a)$, a direct sum of left ideals of $(N,+,*)$.

Conversely, suppose $U \subset P(1 - a'a)$ and $V \subset P(a)$ are left ideals of $(N,+,*)$. Then we have the result $U \cap V \subset P(1 - a'a) \cap P(a) = 0$. So $U \oplus V$ is at least a direct sum of left ideals. Now let $n \in N$ and $x = u+v \in U \oplus V$ where $u \in U$ and $v \in V$. Then $n * x =$ $n * (u + v) = n * u + n * v \in U \oplus V$ since $(U,+,*)$ and $(V,+,*)$ are left ideals of $(N,+,*)$. Therefore, $(U \oplus V,+,*)$ is a left ideal of $(N,+,*)$.

THEOREM 3.10. Let $(N,+,*)$ be a special β near-ring determined by a. Let $(R,+,*)$ and $(S,+,*)$ be ideals of $(N,+,*)$. Then $N = R \oplus S$ if and only if $P(1 - a'a) = R(1 - a'a) \oplus S(1 - a'a)$ and $P(a) = R(a) \oplus S(a)$.

PROOF. If $P(1 - a'a) = R(1 - a'a) \oplus S(1 - a'a)$ and $P(a) = R(a) \oplus S(a)$ then $N = P(1 - a'a) \oplus P(a) =$
$R(1 - a'a) \oplus S(1 - a'a) \oplus R(a) \oplus S(a) =$
$R(1 - a'a) \oplus R(a) \oplus S(1 - a'a) \oplus S(a) = R \oplus S$.

Conversely, suppose $N = R \oplus S$ where R and S are ideals of $(N,+,*)$. Then $N = R(1 - a'a) \oplus R(a) \oplus S(1 - a'a) \oplus S(a) = R(1 - a'a) \oplus S(1 - a'a) \oplus R(a) \oplus S(a)$. From Chapter II, we know that $R(1 - a'a) \subset P(1 - a'a)$ and $S(1 - a'a) \subset P(1 - a'a)$. Thus $R(1 - a'a) \oplus S(1 - a'a) \subset P(1 - a'a)$. Also $R(a) \subset P(a)$ and $S(a) \subset P(a)$ so $R(a) \oplus S(a) \subset P(a)$. Now let $x \in P(a)$. Then $x \in N = R \oplus S$ so $x = r + s$ where $r \in R$ and $s \in S$. Thus $x = xa'a = (r + s)a'a = ra'a + sa'a \in R(a) \oplus S(a)$. Hence, $P(a) \subset R(a) \oplus S(a)$. The two inclusions show $P(a) = R(a) \oplus S(a)$. Similarly, suppose $x \in P(1 - a'a)$. Then $x = x(1 - a'a)$. Since $x \in N = R \oplus S$, $x = r + s$ where $r \in R$ and $s \in S$. Then $x = x(1 - a'a) = (r + s)(1 - a'a) = r(1 - a'a) + s(1 - a'a) \in R(1 - a'a) \oplus S(1 - a'a)$.

Hence, $P(1 - a'a) \subset R(1 - a'a) \oplus S(1 - a'a)$. The two inclusions show $P(1 - a'a) = R(1 - a'a) \oplus S(1 - a'a)$.

3. Two Decomposition Theorems

In this section we present two decomposition theorems for special β near-rings. In the first theorem we do not claim that the component near-rings in the decomposition are subdirectly irreducible. We do, however, show explicitly the multiplication in the component near-rings. In the second theorem the component near-rings are subdirectly irreducible.

THEOREM 3.11. Let $(N,+,*)$ be a special β near-ring determined by a where the base ring has at least two elements. Then N is isomorphic to a subdirect sum of near-rings N_i each of which is one of the following two types:

(a) $(N_i,+,*_i)$ is a small near-ring where $(N_i,+)$ is the additive group of a field and $0 *_i y_i = 0$ for all $y_i \in N_i$ and $x_i *_i y_i = y_i$ for all $y_i \in N_i$ if $x_i \neq 0$,

(b) $(N_i,+,*_i)$ is a small near-ring where $(N_i,+)$ is the additive group of a field and $x_i *_i y_i = y_i$ for all $y_i \in N_i$.

PROOF. Let $(N,+,*)$ be a special β near-ring determined by a where the base ring $(N,+,\cdot,1)$ is a commutative regular ring with at least two elements. Then by Theorem 2.38 N is isomorphic to a subdirect sum of fields F_i for i in some index set I. If $x \in N$ then $x = [x_i]_{i \in I}$ where $x_i \in N_i$ and $N_i = F_i$. Hence, $(N,+,*)$ is isomorphic to a subdirect sum of near-rings N_i where $i \in I$ and each N_i is the near-ring $(F_i, +, *_i)$ where $*_i$ is some multiplication determined by $*$ as in the proof of Theorem 2.39. Let $a = [a_i]_{i \in I}$ and $b = [b_i]_{i \in I}$ where $b = 1 - a'a$. Since $b^2 = b$ and b_i is in the field F_i for each $i \in I$, b_i must be 0 or 1 for each $i \in I$. Since $ab = 0$, $(ab)_i = a_ib_i = 0$ for all $i \in I$. Thus $a_i = 0$ if $b_i = 1$. Also since $b + a'a = 1$, $(a'a)_i = a_i'a_i = 1$ if $b_i = 0$ and $a_i'a_i = 0$ if $b_i = 1$.

Let $i \in I$ and consider N_i. There are two cases corresponding to $b_i = 0$ and $b_i = 1$. First suppose $b_i = 0$. Then $a_i'a_i = 1$. For $x = [x_i]_{i \in I}$ and $y = [y_i]_{i \in I}$ elements of N, $x * y = [(x * y)_i]_{i \in I}$ Then $x_i *_i y_i = (x * y)_i = a_i'a_ix_i'x_iy_i + (1 - a_i'a_i)y_i = x_i'x_iy_i$. If $x_i = 0$ then $x_i *_i y_i = 0$. If we denote the near-ring N_i by $(N_i,+,*_i)$ then $0 *_i y_i = 0$ for all $y_i \in N_i$. If $x_i \neq 0$ then $x_i' = x_i^{-1}$. Thus $x_i'x_i = 1$. So $x_i *_i y_i = x_i'x_iy_i = y_i$ for all $y_i \in N_i$. Note that $(N_i,+,*_i)$ is a small near-ring.

Now consider the case $b_i = 1$. Then, as stated above, $a_i = 0$. So $x_i *_i y_i = (x * y)_i = a_i'a_i x_i'x_i y_i + (1 - a_i'a_i)y_i = y_i$ for all $y_i \in N_i$. Again we note that $(N_i, +, *_i)$ is a small near-ring.

The second decomposition theorem to which we have reference has been stated earlier as Theorem 2.40. It is due to Ligh [11] .

CHAPTER IV

WEAKLY COMMUTATIVE REGULAR NEAR-RINGS

1. Introduction

In this chapter we study the class of weakly commutative regular near-rings. The special regular near-rings of Chapter II and the special β near-rings of Chapter III were weakly commutative regular near-rings. Thus the results of this chapter apply to those classes of near-rings studied in Chapter II and Chapter III. Ratliff [18] studied a class of near-rings which he called ν near-rings. A near-ring N is a $\underline{\nu}$ near-ring if and only if for every $x \in N$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$ and for all $x, y, z \in N$, $xyz = yxz$. Every β near-ring is a ν near-ring and, thus, Ratliff's work generalized some of the results of Ligh [11]. Any ν near-ring is a weakly commutative regular near-ring and so the results which we obtain here generalize corresponding results for ν near-rings established by Ratliff [18].

Discussions in this chapter, unlike those in earlier chapters, will involve only one mathematical system and, hence, only one multiplication. For this reason we shall use the abbreviated notation N to denote a near-ring $(N,+,\cdot)$ without ambiguity. Also the letter "a" will not have the special significance here that it had in earlier chapters. We shall continue to use the prime notation to denote a regularity companion of a given element.

2. Main Results

Let N be a near-ring and $x \in N$. Define $A_x = \{a \in N : xa = 0\}$. Thus A_x is the set of right annihilators of x in N. The set A_x is always a right ideal [18]. The first theorem is due to Szeto [19].

THEOREM 4.1. If N is a weakly commutative near-ring and $x \in N$ then A_x is an ideal of N.

LEMMA 4.2. Let N be a near-ring and $e \in N$. If e is regular and $A_e = 0$ then $e'e$ is a left identity.

PROOF. Let $x \in N$. Then $e(e'e\,x - x) = ee'ex - e\,x = ex - ex = 0$. Thus $e'ex - x \in A_e$. Since $A_e = 0$, $e'ex - x = 0$ and $e'ex = x$ for all $x \in N$. Hence, $e'e$ is a left identity.

THEOREM 4.3. Let N be a regular near-ring. If $e \in N$ is a right identity then e is an identity.

PROOF. Let $x \in A_e$. Then $ex = 0$ and $x = xx'x = x(x'e)x = xx'(ex) = xx'0 = 0$. Thus $A_e = 0$. By Lemma 4.2, $e'e$ is a left identity. But since e is a right identity $e'e = e'$. Thus e' is a left identity. Then $e'e = e$. Hence, e is a left identity and, therefore, the identity in N.

THEOREM 4.4. Let N be a subdirectly irreducible regular near-ring. If A_x is an ideal for every $x \in N$, then N has a left identity.

PROOF. If $N = 0$, the result follows. Suppose $N \neq 0$. Define $R = \left\{x \in N : A_x \neq 0\right\}$. If R is empty then $A_x = 0$ for every $x \in N$. Thus by Lemma 4.2, for every $x \in N$, $x'x$ is a left identity and the proof is complete. Therefore, suppose R is not empty. Then define $A = \bigcap \left\{A_x : x \in R\right\}$. Now A is nonzero since N is subdirectly irreducible. Let $x \in A$ such that $x \neq 0$. Assume that R is all of N. In that case $x \in A_y$ for all $y \in N$. In particular, $x \in A_{xx'}$. Thus $x = xx'x = 0$ which contradicts the fact that x was chosen to be nonzero. Hence R must not be all of N. There must exist some $e \in N$ such that $A_e = 0$. Since

e is regular we have by Lemma 4.2 that $e'e$ is a left identity.

COROLLARY 4.5. Any subdirectly irreducible weakly commutative regular near-ring N has a left identity.

PROOF. By Theorem 4.1, A_x is an ideal for every $x \in N$. Then by Theorem 4.4, N has a left identity.

THEOREM 4.6. Let N be a subdirectly irreducible weakly commutative regular near-ring and let a be a nonzero element of N. If $A_a \neq 0$ then $ay = 0y$ for all $y \in N$ and $A_a = A_0$.

PROOF. Let $R = \{x \in N : A_x \neq 0\}$ and define $A = \bigcap \{A_x : x \in R\}$. Note that R is not empty since, by hypothesis, $a \in R$. Also $A \neq 0$ since N is subdirectly irreducible. Let w be a nonzero element of A. Since $w \in A_x$ for all $x \in R$, $xw = 0$ for all $x \in R$ and, in particular, $aw = 0$. If $A_w \neq 0$ then $w \in A$ and $A \subset A_w$. Now by Theorem 4.1, A_w is an ideal. So $w'w \in A_w$ if $w \in A_w$. Hence $w = w(w'w) = 0$. This contradicts the fact that w was chosen to be nonzero. Therefore, it must be that $A_w = 0$. Then by Lemma 4.2, $w'w$ is a left identity. Also $aw'w = w'aw = w'0 = 0$. Now let $y \in N$. Then $ay = a(w'wy) = (aw'w)y = 0y$. Since $ay = 0$ if and only if $0y = 0$, it is clear that $A_a = A_0$.

COROLLARY 4.7. Let N be a subdirectly irreducible weakly commutative regular near-ring such that $ON = 0$. Then (a) For every nonzero $x \in N$, $A_x = 0$ and, hence, $x'x$ is a left identity, (b) N has no nonzero zero divisors.

PROOF. If $N = 0$ the conclusions are obvious. Suppose $N \neq 0$. Let x be any nonzero element of N. If $A_x \neq 0$ then by Theorem 4.6, $xy = 0y$ for all $y \in N$. But $ON = 0$. Thus $0y = 0$ for all $y \in N$ and so $xy = 0$ for all y in N. Choose $y = x'x$. Then $x = x(x'x) = 0$ which contradicts the fact that x was chosen to be nonzero. Hence $A_x = 0$ and by Lemma 4.2, $x'x$ is a left identity.

Now let $a, b \in N$ such that $ab = 0$. If $a \neq 0$ then the preceding argument shows $a'a$ is a left identity. Thus $b = (a'a)b = a'(ab) = a'0 = 0$. So there are no left zero divisors. On the other hand, if $b \neq 0$ then $b'b$ is a left identity. Thus $a = b'ba = b'baa'a = b'aba'a = b'0a'a = 0$. So N has no nonzero zero divisors.

The following theorem is due to Fröhlich [9].

THEOREM 4.8. Let N be a distributively generated near-ring with identity. Then each of the following conditions is necessary and sufficient for N to be a ring:

(a) N is distributive,

(b) $(N,+)$ is commutative.

THEOREM 4.9. Let N be a subdirectly irreducible weakly commutative regular near-ring such that $0N = 0$. If there is a nonzero element, e, in N such that for every nonzero $x \in N$, $x'x = e$, then N is a field.

<u>PROOF</u>. Let x and y be nonzero elements of N. Then $xy = x(yy'y) = xy(y'y) = xye = yxe = yx(x'x) = y(xx'x) = yx$. Since $0x = 0$ for all $x \in N$, it is clear that if either x or y is zero then $xy = yx$. Thus $(N,\cdot)$ is commutative and the right distributive law holds in N. Hence, N is distributively generated. By Corollary 4.7, N has a left identity which, by commutativity, is a right identity. This identity is $x'x$ where x is any nonzero element. But since $x'x = e$ for every nonzero x, the identity is e. Thus N is a distributively generated near-ring with identity which is distributive. Then by Theorem 4.8, N is a ring.

By Corollary 4.7, N has no nonzero zero divisors. Thus N is an integral domain with identity e. Also, every nonzero x has a multiplicative inverse, namely x', since $x'x = e$. So N is a field.

The following theorem due to Fain [7] is stated here for future reference.

THEOREM 4.10. Every near-ring N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i.

The following definition is due to Ratliff [18]. A near-ring N is <u>almost small</u> if and only if the set $\{A_x : x \in N\}$ contains at most two distinct elements.

THEOREM 4.11. Every weakly commutative regular near-ring N is isomorphic to a subdirect sum of subdirectly irreducible weakly commutative regular near-rings N_i where each N_i is one of the following two types:

(a) N_i is a field,

(b) N_i is almost small.

<u>PROOF</u>. By Theorem 4.10, N is isomorphic to a subdirect sum of subdirectly irreducible near-rings N_i. Each N_i is a homomorphic image of N. It is immediate that weak commutativity and regularity are preserved under near-ring homomorphisms. Thus each N_i is a weakly commutative regular near-ring. We consider three cases.

(1) $0N_i = 0$ and there is a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x'x = e$.

Then by Theorem 4.9, N_i is a field.

(2) $ON_i = 0$ and there does not exist a nonzero $e \in N_i$ such that for every nonzero $x \in N_i$, $x'x = e$. Since $ON_i = 0$ we have $A_0 = N_i$. By Corollary 4.7, for every nonzero $x \in N_i$, $A_x = 0$. Thus $\{A_x : x \in N_i\} = \{0, N_i\}$. Thus, N_i is almost small.

(3) $ON_i \neq 0$. Let $x \in N_i$. If $x = 0$, then $A_x = A_0$. If $x \neq 0$ then either $A_x = 0$ or $A_x \neq 0$. If $A_x \neq 0$ we have by Theorem 4.6 that $A_x = A_0$. Thus if $x \neq 0$ then either $A_x = 0$ or $A_x = A_0$. In either case we conclude that N_i is almost small.

THEOREM 4.12. Let N be a subdirectly irreducible weakly commutative regular near-ring with a nonzero right distributive element r. Then N is a field.

PROOF. Let r be a nonzero right distributive element in N. Then $0r = (0 + 0)r = 0r + 0r$. Hence, $0r = 0$. If $A_r \neq 0$ then by Theorem 4.6, $ry = 0y$ for all $y \in N$. Then $r = r(r'r) = 0r'r = r'0r = r'0 = 0$ which contradicts the fact that r is nonzero. Thus $A_r = 0$ and by Lemma 4.2, $r'r$ is a left identity.

Define $L_r = \{a \in N : ar = 0\}$. Then L_r is the set of left annihilators of r. We know L_r is not empty since $0 \in L_r$. Let $x, y \in L_r$ and $m, n \in N$. Then $(x - y)r = xr + (-y)r = xr + (-y)r + yr - yr =$

$xr + (-y + y)r - yr = xr + 0r - yr = xr - yr = 0 - 0 = 0$. Thus $x - y \in L_r$ and $(L_r,+)$ is a subgroup of $(N,+)$. Also $(-n + x + n)r = (-n)r + xr + nr = (-n)r + 0 + nr = (-n)r + nr = (-n + n)r = 0r = 0 \in L_r$. Thus $-n + x + n \in L_r$ and $(L_r,+)$ is a normal subgroup of $(N,+)$. Now $(nx)r = n(xr) = n0 = 0 \in L_r$. So L_r is a left ideal. Finally, $\left[(m + x)n - mn\right]r = (m + x)nr + (-mn)r = n(m + x)r + (-mn)r = n(mr + xr) + (-mn)r = n(mr + 0) + (-mn)r = nmr + (-mn)r = (mn)r + (-mn)r = (mn - mn)r = 0r = 0 \in L_r$. So $(m + x)n - mn \in L_r$ and L_r is an ideal.

Now define $R = \{x \in N : A_x \neq 0\}$. Since $0r = 0$, $r \in A_0$. Thus A_0 contains a nonzero element so $A_0 \neq 0$. Hence, $0 \in R$ so R is not empty. Define $A = \bigcap\{A_x : x \in R\}$. Since N is subdirectly irreducible $A \neq 0$. Assume $A \cap L_r \neq 0$. Let w be a nonzero element of $A \cap L_r$. Now $A \cap L_r$ is an ideal so $w'w \in A \cap L_r$. Either $A_w=0$ or $A_w \neq 0$. We shall show that either case leads to a contradiction and, hence, that $A \cap L_r = 0$. Suppose $A_w \neq 0$. Then $A \subset A_w$. So $w'w \in A_w$. Thus $w = ww'w = 0$ which is a contradiction. Suppose $A_w = 0$. Then by Lemma 4.2, $w'w$ is a left identity. Thus $r = w'wr = 0$ since $w'w \in L_r$. Again we have reached a contradiction so we are forced to conclude that $A \cap L_r = 0$. If $L_r \neq 0$ then since $A \neq 0$, $A \cap L_r \neq 0$ because N is subdirectly irreducible. But $A \cap L_r = 0$. Hence, we must have $L_r = 0$.

Now let $y \in N$. If $y = 0$ then $yr = 0$. Conversely, if $yr = 0$ then $y \in L_r$. Hence $y = 0$. Therefore, $yr = 0$ if and only if $y = 0$. Now let x by any element in N. Then $(xr'r - x)r = xr'rr - xr = xrr'r - xr = xr - xr = 0$. So $xr'r - x = 0$ and $xr'r = x$ for all $x \in N$. Hence, $r'r$ is a right identity. Since $r'r$ is also a left identity, $r'r$ is the identity element in N. Now let $x, y \in N$. Then $xy = xyr'r = yxr'r = yx$. Thus $(N,\cdot)$ is commutative. This implies that N is distributive and, therefore, distributively generated. Then by Theorem 4.8, N is a ring and, hence, a commutative ring with identity. Since N is distributive $0N = 0$. Then by Corollary 4.7 N has no nonzero zero divisors and is, therefore, an integral domain with identity. Since $r \neq 0$, $r'r \neq 0$ for otherwise $r = r(r'r) = r0 = 0$. Now let x be any nonzero element of N. By Corollary 4.7, $x'x$ is a left identity. Hence $x'x = r'r$. Then by Theorem 4.9, N is a field.

COROLLARY 4.13. Let N be a subdirectly irreducible weakly commutative regular near-ring with a nonzero right identity e. Then N is a field.

PROOF. The result follows from Theorem 4.12 since e is a nonzero right distributive element.

THEOREM 4.14. Let N be a weakly commutative regular near-ring. N is a commutative ring if and only if every nonzero homomorphic image of N contains a nonzero right distributive element.

PROOF. If $N = 0$, the conclusion follows. Suppose $N \neq 0$. If N is a commutative ring then every nonzero homomorphic image of N is commutative and, thus, contains a nonzero right distributive element. Conversely, suppose every nonzero homomorphic image of N contains a nonzero right distributive element. Then by Theorem 4.11, N is isomorphic to a subdirect sum of subdirectly irreducible weakly commutative regular near-rings N_i. By hypothesis each N_i contains a nonzero right distributive element. Then by Theorem 4.12 each N_i is a field. Hence, the direct sum of the N_i is a commutative ring. Since N is isomorphic to a subring of this direct sum, N is a commutative ring.

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