A LINEAR AND A HIGHER ORDER APPROXIMATION
OF HYPERSONIC LONGITUDINAL
DYNAMIC STABILITY

By

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PREFACE

In selecting a subject on which to do my graduate work, as a summer employee of Convair - Fort Worth, I asked my lead-man for advice concerning a thesis topic. Under the heading "Work Program for High Altitude, High Mach Number Study", he presented an outline for attempting to find the areas where conventional analysis is inadequate in determining dynamic characteristics of high speed, high altitude vehicles. This outline, when carried to completion, includes the very general case. It also suggested, in reaching the ultimate end, that analyses be made in successive degrees of complexity. This thesis is an attempt of the simplest of these complex analyses, with the equations of motion retained in the most general form as long as possible as an aid to the next step.

A note of explanation with reference to the list of symbols is in order. The symbols which are used in the body of this thesis but do not appear in the list of symbols have meaning only in the particular chapter or subtitle in which they occur, whereas, those appearing in the list of symbols apply throughout this study.

I would like to acknowledge my indebtedness to Mr. H. O. Ankenbruck and Mr. E. L. Kistler for guidance in developing the equations of motion, to Dr. L. Wayne Johnson and the

Oklahoma A. and M. College Computing Center for aid in solutions to the equations, to Mr. Charles Brown for programming the non-linear equations for the electronic computer, to Mr. Gary W. Reid and Rodene T. Capalongan for typing the study, and to Mr. L. J. Fila, my major advisor, for his guidance, encouragement, and understanding throughout this undertaking.

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LIST OF SYMBOLS

a	Acoustic velocity, feet per second
B	Wing span, feet
c	Mean aerodynamic chord, feet
cg	Center of gravity
C.P.	Distance from center of gravity to center of pressure of flat plate, feet
cp	Non-dimensional C.P., $\frac{C.P.}{c}$
D	Drag force, perpendicular to normal force along body axis, pounds; constant
d	Diameter of base of right circular cone, feet
g	Acceleration due to gravity, feet per second per second
H	Altitude, feet
h	Time interval in solution of non-linear equation of motion
J	Distance from center of gravity to center of pressure of cone, feet
j	Non-dimensional length J, $\frac{J}{c}$
J'	Product of inertia, slug feet ^{2}
L	Cone altitude, feet
l	Non-dimensional altitude L, $\frac{L}{c}$
M	Mach number; moment, foot pounds
m	Mass, slugs
N	Force normal to body axis, pounds
q	Dynamic pressure, pounds per square foot
R	Root chord, feet
r	Non-dimensional root chord, $\frac{R}{c}$

R'	Radius of earth, feet
S	Wing area, flat plate area, square feet
S_b	Cone base area, square feet
T	Thrust force, pounds
$C_{()}$	Non-dimensional coefficient of subscript
$()_0$	The quantity () when time is zero
α	Angle of attack, radians
γ	$\theta - \alpha$
δ	Cone semi-vertex angle, radians
$\dot{\theta}$	Angular velocity, radians per second
$\dot{\bar{\theta}}$	Non-dimensional velocity, $\frac{\dot{\theta} c}{2U}$
Λ	Leading edge sweepback angle, degrees
ρ	Density, slugs per cubic foot

Co-ordinate System

Axis	X	Y	Z
Linear velocity	U	V	W
Angular velocity	$\dot{\phi}$	$\dot{\theta}$	$\dot{\psi}$
Moments	L	M	N
Moment of inertia	I_x	I_y	I_z
Moment of momentum	H_x	H_y	H_z

CHAPTER I

INTRODUCTION

The official decision to attempt to materialize an earth satellite has brought the concept of space travel from the realm of fiction to present day possibility. A great deal of investigation and research will be required to predict and plan for the invasion of what presently is not well understood.

One of the problems of interest is the stability and control of a vehicle at high speeds and high altitudes. As higher and higher speeds are encountered, the neglected higher order effects of linear supersonic theory begin to grow until the linear theory no longer adequately describes real phenomena. It would be desirable to know to what extent the application of linear theory is satisfactory as far as dynamic stability is concerned.

As an approach to this end, a simple hypersonic theory with corrections for supersonic flow is applied to a simple configuration, and an investigation of the dynamic properties is pursued. These properties are then compared to those found by ordinary linear theory applied to the same configuration in the same flight condition.

For the non-linear investigation, the corpuscular theory as outlined by Isaac Newton, or Newtonian Impact

Theory, is employed. Although there exist several theories which are more accurate, perhaps, in the flight regimes of interest here, the Newtonian theory is chosen because of its simplicity. Since Newtonian theory is exact at infinite Mach number, an appropriate correction to it is used to compensate for finite speeds.

The equations of motion are developed in the general case; the ones pertaining to longitudinal motion are retained. These equations are applied to the relatively simple flight conditions, constant altitude and constant Mach number. After dynamic characteristics are examined for these conditions, more realistic conditions should be attempted, namely varying altitude, varying Mach number, varying altitude and Mach number sumultaneously, and finally, varying altitude, Mach number, and mass simultaneously. It is hoped that the investigation of the most simple flight conditions yields some experience and insight to suggest means of handling the more complicated flight conditions. Only the constant altitude and constant Mach number condition will be investigated here.

CHAPTER II

DERIVATION OF EQUATIONS OF LONGITUDINAL MOTION

From Newtonian mechanics:

1. The time rate of change of linear momentum equals the applied external force.
2. The time rate of change of moment of momentum equals the applied external torque.

General motion consists of translation of the center of gravity of the body and rotation about the center of gravity, the axes system being inertial or fixed in space.

These following assumptions are utilized for the analysis:

1. The vehicle is a rigid body.
2. The earth is fixed in space.
3. The atmosphere is stationary relative to the earth.

Under these assumptions, Newton's equations may be applied without modification.

For convenience a right handed Cartesian co-ordinate system is established with the origin at the center of gravity of the vehicle, the positive x axis fixed relative to the vehicle along the body axis pointing upstream. The z axis is perpendicular to the x axis, and it is positive downward. It is necessary to relate these Euler axes moving with the vehicle to the inertial axes fixed in space.

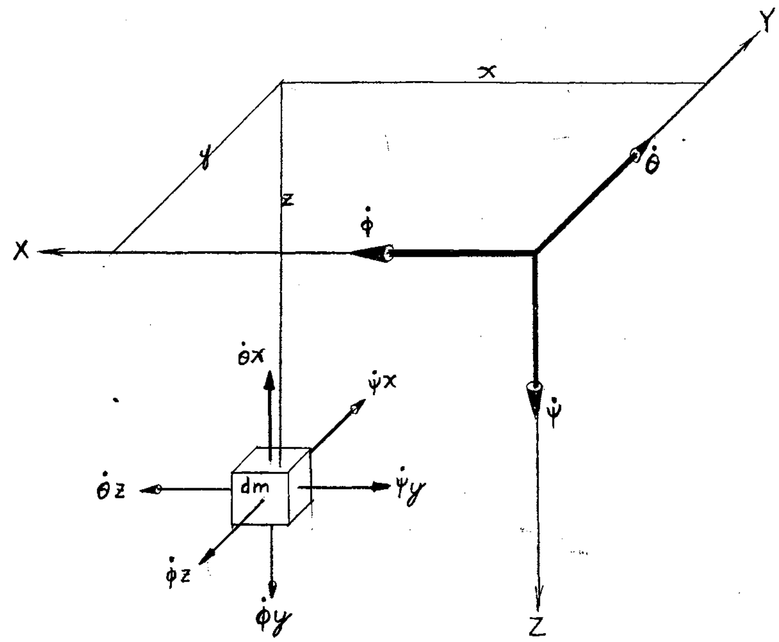

Fig. 1

VELOCITIES WITH RESPECT TO EULER AXES

The equations for moments of momentum are obtained from figure 1.

$$dh_x = y\dot\phi y\,dm + z\dot\phi z\,dm - z\dot\psi x\,dm - y\dot\theta x\,dm$$
$$= \dot\phi(y^2+z^2)\,dm - \dot\psi xz\,dm - \dot\theta xy\,dm$$

If the angular velocities $\dot\phi, \dot\theta, \dot\psi$ are independent of changes in mass, the moments of momentum are

$$h_x = \dot\phi\int(y^2+z^2)\,dm - \dot\psi\int xz\,dm - \dot\theta\int xy\,dm$$

(1a) $$= \dot\phi I_x - \dot\psi J'_{xz} - \dot\theta J'_{xy}$$

(1b) $$h_y = \dot\theta I_y - \dot\psi J'_{yz} - \dot\phi J'_{xy}$$

(1c) $$h_z = \dot\psi I_z - \dot\phi J'_{xz} - \dot\theta J'_{yz}.$$

It must be remembered that the preceding equations are taken with respect to axes moving with the vehicle.

Figure 2 shows the accelerations due to the velocities $u, v, w, \dot\phi, \dot\theta, \dot\psi$ in the general case. Figure 3 shows the moments of momentum due to the momentums h_x, h_y, h_z and the velocities

$\dot{\phi}, \dot{\psi}, \dot{\theta}$ in the general case. In both figures the motion is referred to the moving axes.

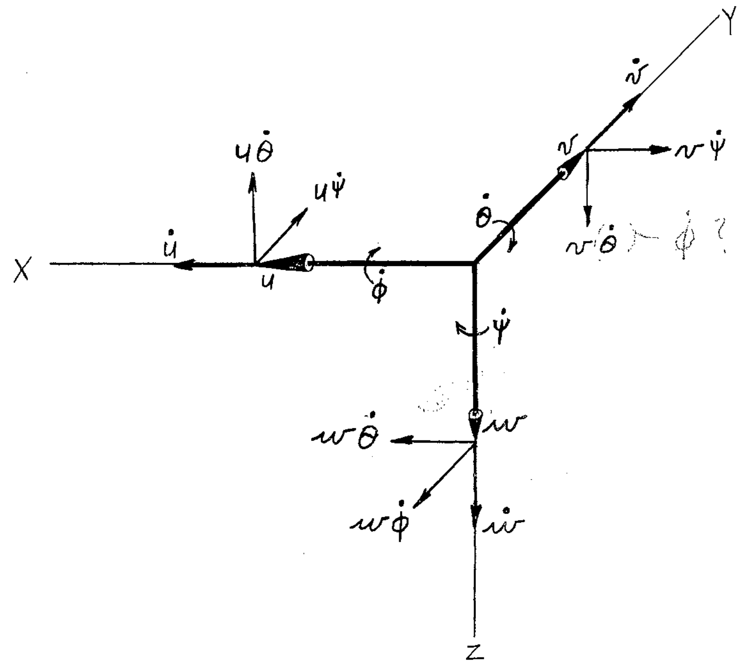

Fig. 2

ACCELERATION COMPONENTS REFERRED TO MOVING AXES

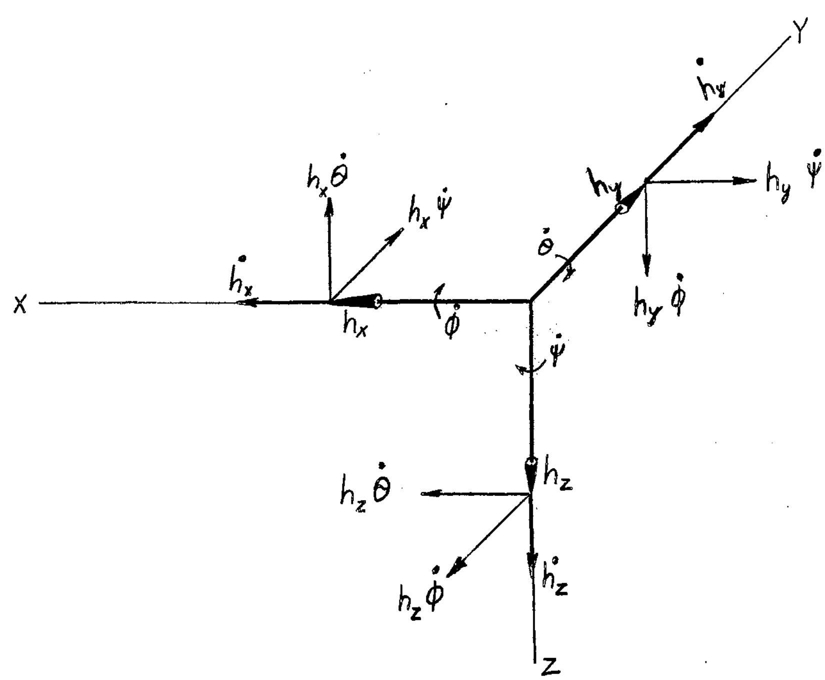

Fig. 3

MOMENT OF MOMENTUM COMPONENTS REFERRED TO MOVING AXES

With respect to fixed axes, the accelerations from figure 2 become

(2a) $$\frac{dU}{dt} = \dot{u} - v\dot{\psi} + w\dot{\theta}$$

(2b) $$\frac{dV}{dt} = \dot{v} - w\dot{\phi} + u\dot{\psi}$$

(2c) $$\frac{dW}{dt} = \dot{w} - u\dot{\theta} + v\dot{\phi}.$$

From figure 3 the rates of change of moment of momentum relative to fixed axes become

(3a) $$\frac{dH_x}{dt} = \dot{h}_x - h_y\dot{\psi} + h_z\dot{\theta}$$

(3b) $$\frac{dH_y}{dt} = \dot{h}_y - h_z\dot{\phi} + h_x\dot{\psi}$$

(3c) $$\frac{dH_z}{dt} = \dot{h}_z - h_x\dot{\theta} + h_y\dot{\phi}.$$

Using equation (2a), the longitudinal equation of motion changes from

$$\sum F_x = \frac{d}{dt}(mU) = m\dot{U} + U\dot{m}$$

to

(4a) $$\sum F_x = m(\dot{u} - v\dot{\psi} + w\dot{\theta}) + U\dot{m}.$$

Using equation (2c)

$$\sum F_z = \frac{d}{dt}(mW) = m\dot{W} + W\dot{m}$$

becomes

(4b) $$\sum F_z = m(\dot{w} - u\dot{\theta} + v\dot{\phi}) + W\dot{m}.$$

The summation of moments is

$$\sum M = \frac{d}{dt}(H_y) = \dot{h}_y - h_z\dot{\phi} + h_x\dot{\psi}$$

which, with equations (1) become

$$\sum M = \dot{\theta}\dot{I}_y + \ddot{\theta}I_y - \dot{\psi}\dot{J}'_{yz} - \ddot{\psi}J'_{yz} - \dot{\phi}\dot{J}'_{xy}$$
$$- \ddot{\phi}J'_{xy} - \dot{\phi}\dot{\psi}I_z + \dot{\phi}^2 J'_{xz} + \dot{\phi}\dot{\theta}J'_{yz} + \dot{\psi}\dot{\phi}I_y - \dot{\psi}J'_{xz} - \dot{\theta}J'_{xy}.$$

If the XZ plane is assumed to be the plane of symmetry, and since the origin is at the center of gravity, the terms $\dot{J}'_{yz}$, J'_{yz}, $\dot{J}'_{xy}$, J'_{xy} vanish.

(4c) $$\sum M = I_y \ddot{\theta} + \dot{I}_y \dot{\theta} + \dot{\phi}\dot{\Psi}(I_x - I_z) + J'_{xz}(\dot{\phi}^2 - \dot{\Psi}^2)$$

It is assumed that the symmetric degrees of freedom do not couple with the asymmetric degrees of freedom. Longitudinal motions have a negligible tendency to excite or effect lateral or rolling motions.[1] The converse is probably not true. Since this study is in the interest of longitudinal disturbances, the above assumption seems to be tenable. The equations of motion then reduce to

(5a) $$\sum F_x = m(\dot{u} + w\dot{\theta}) + U\dot{m}$$

(5b) $$\sum F_z = m(\dot{w} - u\dot{\theta}) + W\dot{m}$$

(5c) $$\sum M = I_y \ddot{\theta} + \dot{I}_y \dot{\theta}$$

where the left sides of the equations are the external aerodynamic forces or moments.

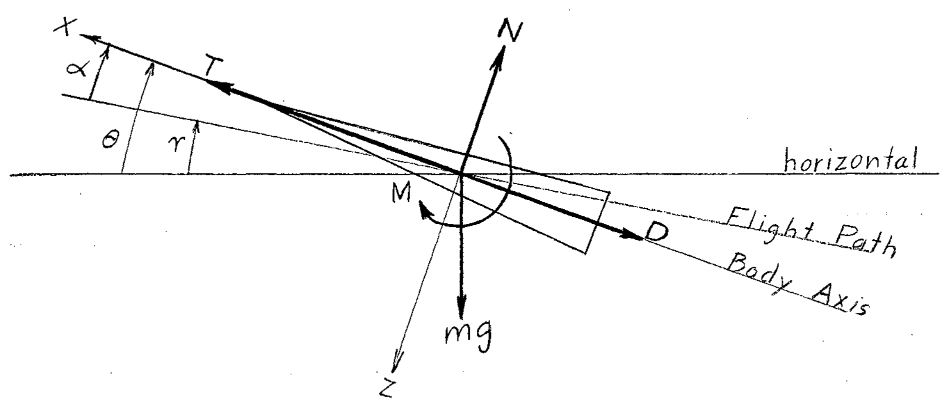

Fig. 4

AERODYNAMIC FORCES ON VEHICLE

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[1]Courtland D. Perkins, Robert E. Hage, Airplane Performance, Stability and Control, (New York, 1949), p. 383.

The aerodynamic forces and moments on the vehicle in flight are shown in figure 4. The equilibrium equations written from this figure are

(6a) $$\sum F_x = T - D - mg \sin\theta$$

(6b) $$\sum F_z = mg \cos\theta - N$$

(6c) $$\sum M = M .$$

It is emphasized that the drag does not act along the flight path, but it is defined as acting along the negative x axis.

CHAPTER III

NON-LINEAR AERODYNAMIC FORCES - WING

It is assumed that the coefficient of lift force can be expressed as

$$C_N = C_{N_{(\alpha)}} + C_{N_{(\dot{\alpha})}} + C_{N_{(u)}} + C_{N_{(\delta e)}} + C_{N_{(\dot{\delta e})}} + C_{N_{(\theta)}} + C_{N_{(\dot{\theta})}}$$

where $C_{N_{(\delta e)}}$ is the coefficient of lift due to elevator deflection, $C_{N_{(\dot{\delta e})}}$ is the coefficient of lift due to a velocity of the elevator deflection, etc. A similar expression for C_D and C_m is assumed to exist; the thrust coefficient C_T in the real case probably will be a function of time. Other parameters which may contribute to the aerodynamic forces are assumed to be negligibly small for this study.

The contributions of some of these parameters may be obscure. They would require a complex investigation which is beyond the scope of this study.

There are several aerodynamic theories which could be utilized for finding the effects of the various parameters. Some of these theories are basic while others are complex combinations of two or more fundamental ones. In this study, the corpuscular theory as outlined by Isaac Newton is chosen because of its simplicity[1] and its applicability[2]. This

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[1]A. F. Zahm, "Superaerodynamics", Journal of the Franklin Institute, CCXVII, Feb. 1934, p.154.

[2]G. Grimminger, E. P. Williams, G. B. W. Young, "Lift on Inclined Bodies of Revolution in Hypersonic Flow", Institute of Aeronautical Sciences, XVII, 1950, p. 677.

corpuscular theory or Newtonian impact theory is exact at infinite Mach number. To adapt this theory to Mach numbers less than infinite, a correction term is added.

The Effects of Angle of Attack - Flat Plate

Newtonian impact theory yields

$$C_N = 2 \sin^2 \alpha .$$

At hypersonic speeds the angle of attack is necessarily small because of power considerations. For small angles, $\sin \alpha$ is approximately equal to α.

$$C_N = 2 \alpha^2 \qquad \alpha > 0$$

Linear supersonic theory yields

$$C_L = \frac{dC_L}{d\alpha} \alpha = \frac{4}{M} \alpha .$$

This term is the one to be used here as a correction to the Newtonian impact theory.

(7) $$C_N = 2\alpha^2 + \frac{4}{M}\alpha .$$

This expression is valid only for angles of attack greater than zero. It is desirable to have a continuous function for positive and negative values of angles of attack. The squared term in equation (7) is approximated by a cubic of the form $A_1\alpha^3$ for the range of angles of attack of interest in this study. Such a substitution would permit one function to represent C_N for all values of angle of attack.[3]

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[3] Another alternative would be $C_N = 2\alpha|\alpha| + \frac{4}{M}\alpha$, but, because of the absolute value sign, later mathmetical manipulations become difficult. Various methods utilized by high speed computing machinery can handle this type of expression, and it is for this reason the alternate expression is shown.

To determine the coefficient A_1, it is assumed a priori that the maximum angle of attack to be experienced will not be greater than 0.35 radians or approximately 20°. The constant is to be chosen in such a manner that the approximate cubic function will be exact with respect to the squared function at $\alpha=0$ and $\alpha=0.35$ radians.

$$2\alpha_{max}^2 = A_1\alpha_{max}^3$$

$$2(0.35)^2 = A_1(0.35)^3$$

$$A_1 = 5.7$$

$$C_N = 5.7\alpha^3 + \frac{4}{M}\alpha \tag{8}$$

The maximum difference between equation (8) and equation (7) occurs when $\alpha=0.234$, the exact error depending on Mach number. The largest this error could possibly become is approximately 30 per cent, which occurs at infinite Mach number. At a Mach number of ten, this maximum error is approximately 15 per cent. An error of this magnitude is not intolerable for a study of this kind. Equation (8), then, is the angle of attack contribution to the normal force.

The Effects of Angular Velocity - Flat Plate

The effect of angular velocity is divided into two parts: the apparent increase in angle of attack due to an angular velocity and the effects of pure rotation, $\dot{\theta}_1$ and $\dot{\theta}_2$ respectively.

Figure 5a shows the orientation of $\dot{\theta}$ with respect to the vehicle. Figure 5b is a simplification of 5a showing the vertical velocity of the flat plate due to the angular velocity $\dot{\theta}_1$ and the arm C. P. Figure 5c is the velocity diagram taken with respect to the flat plate. From trigonometric relations the increase in angle of attack is

$$(9) \qquad \Delta\alpha = \tan^{-1}\frac{C.P.\,\dot{\theta}}{U} = \frac{C.P.\,\dot{\theta}}{U}.$$

Non-dimensionalizing the length C. P. by dividing by the mean aerodynamic chord c, and non-dimensionalizing the angular velocity $\dot{\theta}$ by multiplying by $\frac{c}{2U}$, equation (9) becomes

$$(10) \qquad \Delta\alpha = 2\,cp\,\dot{\theta}.$$

The $C_{N_{(\dot{\theta}_1)}}$ is

$$(11) \qquad C_{N_{(\dot{\theta}_1)}} = C_{N_{(\alpha+\Delta\alpha)}} - C_{N_{(\alpha)}}.$$

$$(12) \qquad C_{N_{(\alpha+\Delta\alpha)}} = 5.7\,(\alpha + 2cp\dot{\theta})^3 + \frac{4}{M}(\alpha + 2cp\dot{\theta})$$

$$(13) \qquad C_{N_{(\alpha)}} = 5.7\alpha^3 + \frac{4}{M}\alpha$$

Utilizing equations (12) and (13), equation (11) becomes

$$(14) \qquad C_{N_{(\dot{\theta}_1)}} = 68.4\,cp^2\dot{\theta}^2\alpha + 34.2\,cp\,\dot{\theta}\alpha^2 + 45.6\,cp^3\dot{\theta}^3 + \frac{8cp\dot{\theta}}{M}.$$

$$C_{m_{(\dot{\theta}_1)}} = C_{N_{(\dot{\theta})}}\,cp$$

$$(15) \qquad = 68.4\,cp^3\dot{\theta}^2\alpha + 34.2\,cp^2\dot{\theta}\alpha^2 + 45.6\,cp^4\dot{\theta}^3 + \frac{8cp^2\dot{\theta}}{M}$$

Drag due to $\dot{\theta}_1$ is assumed zero.

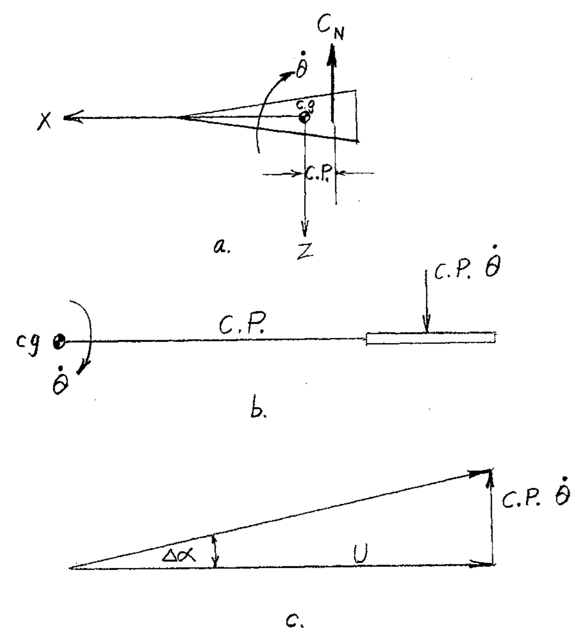

Fig. 5

EFFECTS OF $\dot{\theta}_1$ - FLAT PLATE

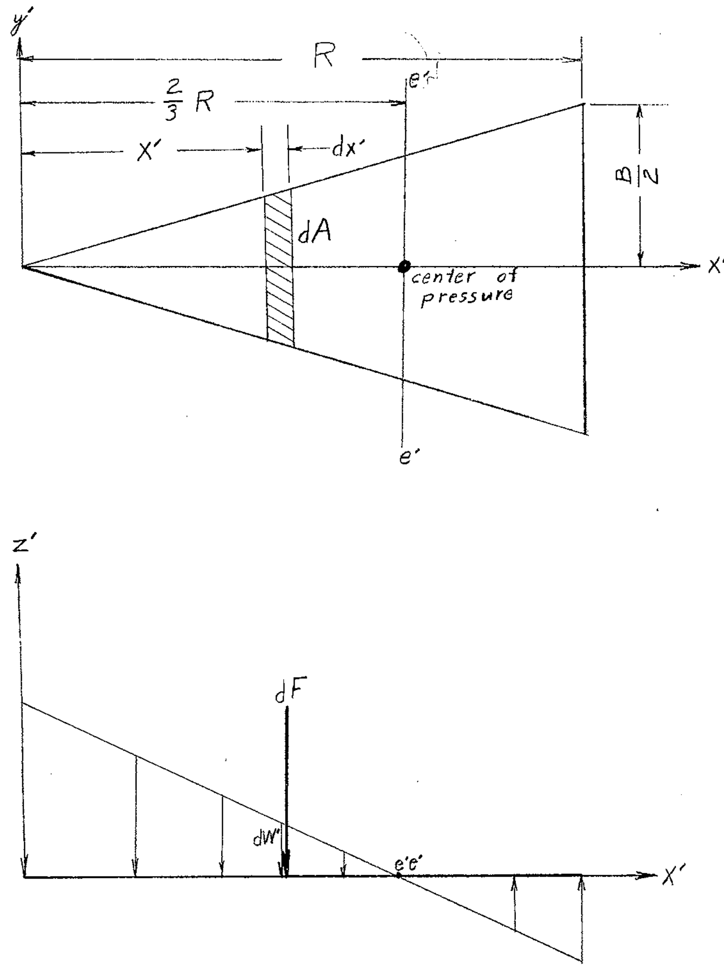

Fig. 6

EFFECTS OF $\dot{\theta}_2$ - FLAT PLATE

Next to be considered are the effects of $\dot{\theta}_2$ or pure rotation about the center of pressure of the flat plate. In figure 6, the axes x', y', and z' are used briefly to describe conveniently the flat plate and are not to be confused with the Euler axes. A triangular planform flat plate is chosen. The following refer to figure 6.

$$y' = \frac{B}{2R} x'$$

$$dA = 2y' \, dx' = \frac{B}{R} x' \, dx'$$

w' = increase in normal component of velocity (in excess of normal component due to α) due to $\dot{\theta}$

The force on the area dA due to an angular velocity $\dot{\theta}$ about the horizontal axis e'e' is

$$dF = (\rho\, dA\, W')\, dW$$

(16) $$= \left[\rho\, dA \left(x' - \tfrac{2}{3}R\right)\dot{\theta}\right] \left|\left(x' - \tfrac{2}{3}R\right)\dot{\theta}\right| .$$

The moment resulting from the forces on the area dA is

(17) $$dM = \left|x' - \tfrac{2}{3}R\right| dF.$$

Equation (1b) is substituted into (17) and the result is integrated.

$$M = \frac{B}{R}\rho\,\dot{\theta}\left|\dot{\theta}\right| \int_0^R x'\left(x' - \tfrac{2}{3}R\right)^3 dx'$$

(18) $$= -\frac{1}{270}\rho B\,\dot{\theta}\left|\dot{\theta}\right| R^4$$

Non-dimensionalizing by dividing by qSc and utilizing $r = \frac{3}{2}$ from the geometry, equation (18) becomes

$$C_{m_{(\dot{\theta}_2)}} = \frac{M}{qSc} = -\frac{2}{135}\,\frac{\dot{\theta}\left|\dot{\theta}\right| R^3}{U^2 c} = -\frac{8}{135}\, r^3 \dot{\theta}\left|\dot{\theta}\right|$$

(19) $$= -\frac{1}{5}\,\dot{\theta}\left|\dot{\theta}\right| .$$

The Effects of a Change in Forward Velocity - Flat Plate

Equation (8) is differentiated with respect to Mach number.

(8) $$C_N = 5.7\alpha^3 + \frac{4}{M}\alpha$$

$$\frac{dC_N}{dM} = -\frac{4\alpha}{M^2}$$

$$\frac{dC_N}{dU}a = -\frac{4\alpha a^2}{U^2}$$

$$dC_N = -\frac{4\alpha a}{U^2}$$

The preceding equation when integrated becomes

$$C_N = \frac{4\alpha a}{U} + c'$$

when $\alpha = 0$ $C_N = 0$ $\therefore c' = 0$

$$C_N = \frac{4\alpha a}{U}.$$

This shows the effect on C_N due to a change in forward velocity U in the general case, but it is not added to the C_N function of angle of attack (since it was here that the $C_{N_{(u)}}$ was derived).

In a point dynamic study with a given angle of attack using small pertubations about a constant Mach number, the $C_{N_{(u)}}$ function would be added to the $C_{N_{(\alpha)}}$ function. However, the value given above for $C_{N_{(u)}}$ would not be the correct one. This function of $C_{N_{(u)}}$ is not well understood at the present and would require involved study. It is assumed that the $C_{N_{(u)}}$ function is already included in the expression $C_{N_{(\alpha)}}$ previously derived.

The Effects of Other Parameters - Flat Plate

The term $C_{m_{(\delta e)}}$ is expressed as $C_{m_{trim}}$. This term is the coefficient of moment necessary to trim the vehicle at a particular set of given flight conditions, and it embraces any means whatever to obtain the necessary moment including elevator or canard deflection, jet reaction, or gyroscopic action. The aerodynamic effects of all other parameters are assumed either to be negligible or to have been considered in parameters previously discussed.

As a summary of the aerodynamic effects on a flat plate, Table I is shown below.

TABLE I

AERODYNAMIC FORCES AND MOMENTS

FLAT PLATE

Parameter	C_N	C_D	C_m
α	$5.7\alpha^3 + \frac{4}{M}\alpha$	0	$\left(5.7\alpha^3 + \frac{4}{M}\alpha\right)\varphi$
$\dot{\alpha}$	0	0	0
u	0	0	0
δ_e	0	0	0
$\dot{\delta}_e$	0	0	0
θ	0	0	0
$\dot{\theta}$	$68.4\varphi^2\dot{\theta}^2\alpha + 34.2\varphi\dot{\theta}\alpha^2 + 45.6\varphi^3\dot{\theta}^3 + \frac{8}{M}\varphi\dot{\theta}$	0	$68.4\varphi^3\dot{\theta}^2\alpha + 34.2\varphi^2\dot{\theta}\alpha^2 + 45.6\varphi^4\dot{\theta}^3 + \frac{8}{M}\varphi^2\dot{\theta} - \frac{1}{5}\dot{\theta}\lvert\dot{\theta}\rvert$

CHAPTER IV

VEHICLE CONFIGURATION

Before proceeding to find the aerodynamic effects of the body of the vehicle, the configuration must be known. In accordance with some assumptions already made the wing is chosen to be a triangular flat plate. It is realized that a flat plate wing is not structurally possible, but for purposes of preliminary investigation such as the present case, the simplicity of the flat plate lends itself more readily to analysis. The structural aspects are therefore deliberately ignored. To get a somewhat realistic configuration the following criteria are assumed:

a. The body is a cone with a slenderness ratio of 8.

b. The wing is a triangular flat plate with a 70° leading edge sweepback.

c. The vehicle weight is 20,000 pounds.

d. The entire volume is occupied by fuel with a specific gravity of unity.

e. The wing loading is 100 pounds per square foot.

These criteria are used merely to obtain the general size and shape of the vehicle and need not be rigidly followed.

Criteria c and d require that the vehicle contain 320 cubic feet. Letting L be the cone altitude and d be the base diameter, criterion a requires $L = 8d$. The expression for the volume is

$$\frac{\pi d^2 L}{12} = 320,^2$$

from which d = 5.37 feet, L = 42.4 feet. For convenience make L = 40 feet and d = 5 feet. The moment of inertia of a right circular cone about an axis parallel to the base through the center of gravity is

$$I = \frac{3m}{80}(d^2 + L^2) = 380{,}000 \text{ ft. lb. sec.}^2.$$

Criterion e requires the wing area S be

$$S = \frac{20{,}000 \text{ lb}}{100 \text{ lb./ft}^2} = 200 \text{ square feet.}$$

In accordance with criterion b and the wing area S the geometry from figure 7 yields

(20) $S = 200 = \frac{1}{2}BR$

(21) $\tan 20^\circ = \frac{B}{2R} = 0.364.$

Simultaneous solution of (20) and (21) yields

$R = 23.5$ feet

$B = 17.1$ feet

$c = \frac{2}{3}R \approx 15.5$ feet.

Figure 8 is the top view of the vehicle.

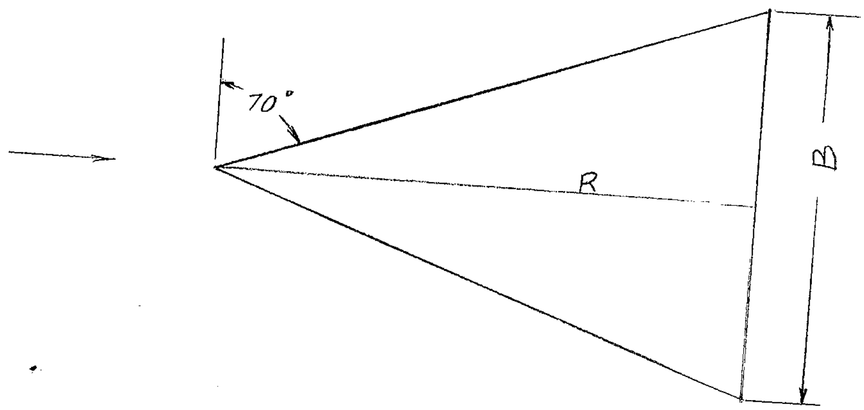

Fig. 7

WING PLANFORM

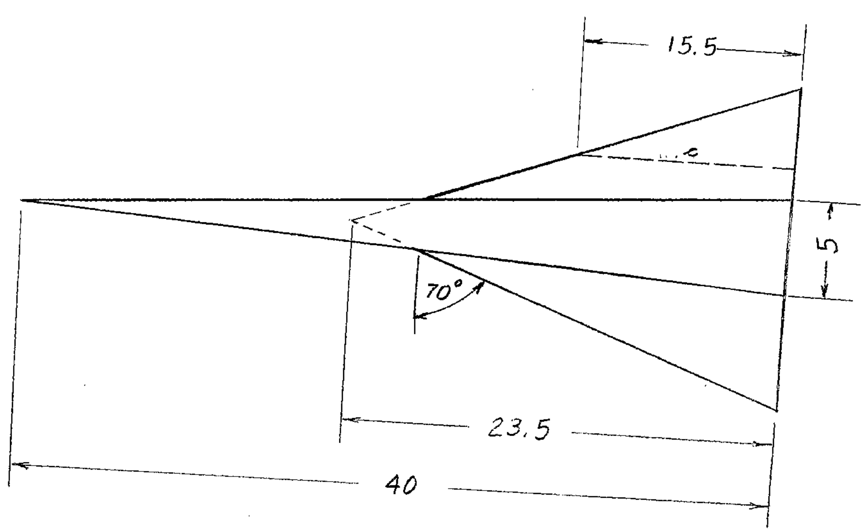

Fig. 8

VEHICLE CONFIGURATION

CHAPTER V

NON-LINEAR AERODYNAMIC FORCES - BODY

Effects of Angle of Attack - Body

In much the same manner used in finding the normal force on a flat plate,

$$C_N = k \sin^2 \alpha + \frac{dC_N}{d\alpha} \alpha$$

is assumed for the cone. The slope of the lift curve is

$$\frac{dC_N}{d\alpha} = 2 \cos^2 \delta$$ [1]

where δ = cone semivertex angle and

$$C_N = \frac{N}{q S_b}.$$

The effects of the second order terms in relation to the linear term are negligible for the cone shaped body.[2] To satisfy this the constant k must vanish. The $C_{N_{(\alpha)}}$ term is

$$C_{N_{(\alpha)}} = 2\alpha \cos^2 \delta$$

and is based on the cone base area. Since it is desirable to have the aerodynamic effects for both the wing and body non-dimensionalized with respect to a common area,

$$(22) \quad C_{N_{(\alpha)}} = 2\alpha \frac{S_b}{S} \cos^2 \delta.$$

For a cone of this slenderness ratio, the semivertex angle δ is small and $\cos \delta \approx 1$.

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[1] Ibid., p. 681.

[2] Z. Kopal, "Supersonic Flow Around Cones of Large Yaw", Massachusetts Institute of Technology Technical Report 5, 1949, p. xvi.

Equation (22) then becomes

(23) $C_{N_{(\alpha)}} = 2\alpha \frac{S_b}{S}$.

The cone's contribution to drag is[3]

(24) $C_D = 2\sin^2\delta + \sin^2\alpha\,(1 - 3\sin^2\delta)$.

The term $(1 - 3\sin^2\delta)$ is replaced by the term $(1 - \sin^2\delta)$. With a cone of slenderness ratio of 8, this substitution can be performed with a resulting error less than one per cent. In accordance with the assumptions previously made, equation (23) becomes

(25) $C_D = 2\delta^2 + \alpha^2 + \text{friction drag}$.

The friction drag is very small in comparison to wave drag at hypersonic speeds and is considered zero. Equation (24) based on wing area becomes

(26) $C_{D_{(\alpha)}} = (2\delta^2 + \alpha^2)\frac{S_b}{S}$.

The center of pressure on a right circular cone is located at a point one-third the height of the cone[4], and the center of gravity is located at a point one-fourth the height of the cone. The moment coefficient due to angle of attack is

$$C_{m_{(\alpha)}} = C_N\left(\frac{L}{3} - \frac{L}{4}\right)\frac{1}{c}$$

(27) $= \frac{1}{6}\alpha \ell \frac{S_b}{S}$.

It is noticed that the right circular cone has an unstable moment, that is, $\frac{dC_m}{d\alpha}$ is positive.

[3]Grimminger, Williams, Young, p. 681.

[4]Ibid.

The Effects of Angular Velocity - Body

As in the case of the flat plate wing, the cone effects of angular velocity are separated into two parts, $\dot{\theta}_1$ and $\dot{\theta}_2$.

In obtaining the effects of $\dot{\theta}_1$ reference is made to figure 9 where the geometry of the body is shown in a, the vertical velocity due to $\dot{\theta}$ is shown in b, and the velocity diagram as seen by an observer on the cone in c. Trigonometric relations from figure 9c yield

(28) $$\Delta\alpha = \tan^{-1}\frac{\dot{\theta}J}{U} = \frac{\dot{\theta}J}{U}.$$

The distance J from figure 9a is

$$J = \frac{L}{12}.$$

Non-dimensionalizing J and $\dot{\theta}$ and substituting into (28) yield

(29) $$\Delta\alpha = \frac{1}{6}\ell\dot{\theta}.$$

Substituting equation (29) into (23), $C_{N_{(\dot{\theta}_1)}}$ becomes

(30) $$C_{N_{(\dot{\theta}_1)}} = 2(\Delta\alpha)\frac{S_b}{S} = \frac{1}{3}\ell\dot{\theta}\frac{S_b}{S}.$$

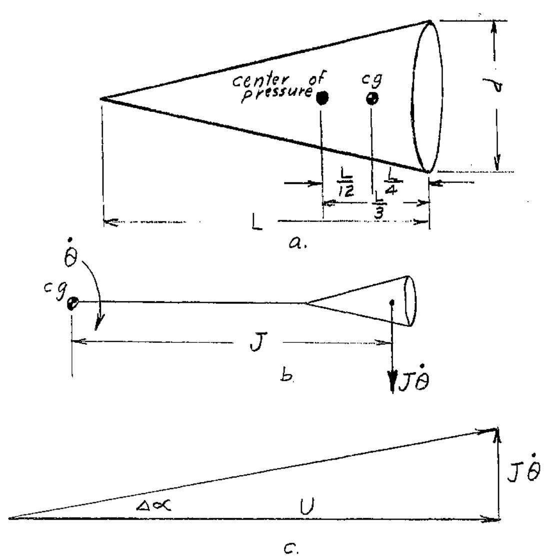

Fig. 9

EFFECTS OF $\dot{\theta}_1$ - BODY

The drag contribution is

(31) $C_{D_{(\dot{\theta}_1)}} = C_{D_{(\alpha+\Delta\alpha)}} - C_{D_{(\alpha)}}$

(32) $C_{D_{(\alpha+\Delta\alpha)}} = \left[2\delta^2 + (\alpha+\Delta\alpha)^2\right]\frac{S_b}{S}$

(33) $C_{D_{(\alpha)}} = (2\delta^2 + \alpha^2)\frac{S_b}{S}.$

(34) $C_{D_{(\dot{\theta}_1)}} = \left(2\alpha + \frac{1}{6}\ell\dot{\theta}\right)\frac{1}{6}\ell\dot{\theta}\frac{S_b}{S}$

The moment contribution is

(35) $C_{m_{(\dot{\theta}_1)}} = -C_{N_{(\dot{\theta}_1)}}\,j = -\frac{1}{36}\ell^2\dot{\theta}\frac{S_b}{S}.$

Now the effects of pure rotation are considered. The normal and drag forces are assumed zero. The effect of $\mathring{\theta}_2$ may be visualized as the moment resulting from the cone being rotated about a fixed axis parallel to the base and passing through the center of gravity while the atmosphere surrounding the cone is stationary but corresponding to the flight conditions of the free stream.

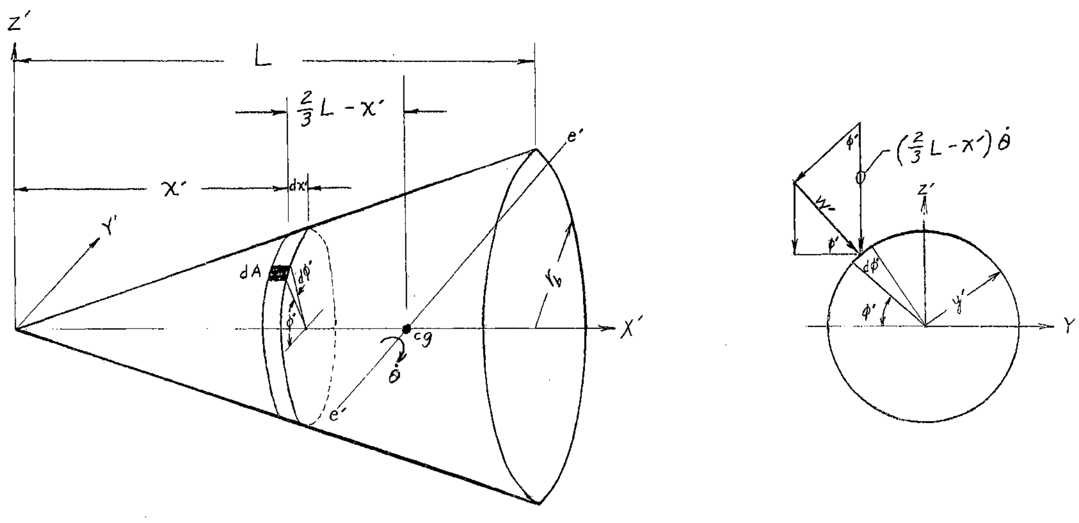

Fig. 10

EFFECTS OF $\mathring{\theta}_2$ - BODY

In figure 10, the axes system X'Y'Z' and the values ϕ', x', y', and W' are used briefly to describe conveniently the effects of $\dot{\theta}_2$. They have meaning only in this subtitle section. The term W' is the component of velocity normal to the area dA. The moment about the axis e'e' due to the angular rotation $\dot{\theta}$ is expressed as

(36) M = (vertical component of) [component of force normal to surface dA] multiplied by moment arm.

The following are referred to figure 10.

$$y' = \frac{r_b}{L} x'$$

$$W' = \left(\tfrac{2}{3}L - x'\right)\dot{\theta} \sin \phi'$$

$$dA = y' \, d\phi' \, dx' = \frac{r_b}{L} x' \, d\phi' \, dx'$$

The bracketed term in equation (36) is

(37) [component of force normal to surface dA] $= \rho (dA) W' \, dW'$

$$= \rho \left(\frac{r_b}{L} x' \, d\phi' \, dx'\right)\left(-\left[\tfrac{2}{3}L - x'\right]\dot{\theta} \sin \phi'\right)\left|\left(\tfrac{2}{3}L - x'\right)\dot{\theta} \sin \phi'\right|.$$

To obtain the vertical component of (37) it is necessary to multiply (37) by

(38) $\left| \sin \phi' \right|$.

The moment arm is

(39) $\tfrac{2}{3}L - x'$.

Then substituting (38), (39) and the right hand side of (37) into the proper place and taking the necessary summations, equation (36) becomes

$$M = -\rho \frac{r_b}{L} \dot{\theta} \left|\dot{\theta}\right| \int_0^L \int_0^\pi \left(\tfrac{2}{3}L - x'\right)^3 x' \sin^3 \phi' \, d\phi' \, dx',$$

which, after the indicated integrations, becomes

$$M = -\frac{2}{405}\rho\,\dot{\theta}\,|\dot{\theta}|\,r_b\,L^4.$$

A slenderness ratio of 8 requires

$$r_b = \frac{L}{16}.$$

$$M = -\frac{1}{3240}\rho\,\dot{\theta}\,|\dot{\theta}|\,L^5$$

This moment when non-dimensionalized and changed to coefficient form becomes

$$C_{m_{(\dot{\theta})_2}} = -\frac{1}{405}\,\frac{\ell^5 c^2}{S}\,\dot{\theta}\,|\dot{\theta}|. \tag{40}$$

Effects of a Change in Forward Velocity - Body

The quantity $\frac{dC_N}{d\alpha}$ at $\alpha = 0$ shows very little variation for increasing Mach numbers.[5] Since the variation of C_N with respect to angle of attack is a linear variation, the normal force coefficient does not vary with Mach number nor u.

Effects of Other Parameters - Body

The aerodynamic effects of all other parameters are assumed either to have been considered in parameters already discussed or to be negligible.

Table II shows a summary of the aerodynamics on the cone shaped body.

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[5] Grimminger, Williams, Young, p. 676.

TABLE II

AERODYNAMIC FORCES AND MOMENTS - CONE BODY

Parameter	C_N	C_D	C_m
α	$2\alpha \frac{S_b}{S}$	$(2\delta^2+\alpha^2)\frac{S_b}{S}$	$\frac{1}{6}\alpha l \frac{S_b}{S}$
$\dot{\alpha}$	0	0	0
u	0	0	0
δ_e	0	0	0
$\dot{\delta}_e$	0	0	0
θ	0	0	0
$\dot{\theta}$	$\frac{1}{3} l \dot{\theta} \frac{S_b}{S}$	$(2\alpha + \frac{1}{6} l \dot{\theta})\frac{1}{6} l \dot{\theta} \frac{S_b}{S}$	$-\frac{1}{36} l^2 \dot{\theta} \frac{S_b}{S}$ $-\frac{1}{405}\frac{l^5}{S} c^2 \dot{\theta}\lvert\dot{\theta}\rvert$

CHAPTER VI

NON-LINEAR AERODYNAMIC FORCES - VEHICLE

The aerodynamic effects of the various parameters, α $\theta, \dot{\theta}, u, \delta_e, \dot{\delta}_e$, etc., are to be summed, and this summation is to represent the total aerodynamic forces and moments acting on the vehicle. The effects of interference are neglected even though they may be of the same order of magnitude as the aerodynamic forces themselves. The present method of approach to this problem does not readily suggest a theory for considering these effects.

A set of typical, realistic conditions are assumed and the total aerodynamic forces and moments are calculated to investigate the individual effects of the various parameters. The following conditions are assumed:

$M = 10$	$H = 200{,}000$ feet
$\alpha = 0.10$ radians	$\dot{\theta} = 0.03$ radians per second
$\varphi = -0.30$	$\delta = 0.087$ radians.

The configuration in figure 8 is utilized.

$$
\begin{aligned}
C_N = {} & 5.7\alpha^3 \left[5700 \times 10^{-6}\right] + \frac{4}{M}\alpha \left[40{,}000 \times 10^{-6}\right] \\
& + 68.4\,\varphi\dot{\theta}^2\alpha \left[0.000222 \times 10^{-6}\right] + 34.2\,\varphi\dot{\theta}\alpha^2 \left[-1.95 \times 10^{-6}\right] \\
& + 45.6\,\varphi^3\dot{\theta}^3 \left[0.00000000844 \times 10^{-6}\right] + \frac{8}{M}\varphi\dot{\theta} \left[4.55 \times 10^{-6}\right] \\
& + 2\alpha\frac{S_b}{S} \left[13{,}600 \times 10^{-6}\right] + \frac{1}{3}\ell\frac{S_b}{S}\dot{\theta} \left[1.6 \times 10^{-6}\right]
\end{aligned}
$$

The numerical value in the bracket is the value of the term adjacent to it for the above chosen conditions; the brackets are used here merely to simplify notation.

$$
\begin{aligned}
C_D = {} & 2\delta^2 \frac{S_b}{S} \left[1480 \times 10^{-6}\right] + \alpha \frac{S_b}{S} \left[980 \times 10^{-6}\right] \\
& + \frac{1}{3} \alpha \ell \dot{\theta} \frac{S_b}{S} \left[0.16 \times 10^{-6}\right] + \frac{1}{36} \ell^2 \dot{\theta}^2 \frac{S_b}{S} \left[0.00000654 \times 10^{-6}\right] \\
C_m = {} & 5.7\alpha^3 cp \left[-1710 \times 10^{-6}\right] + \frac{4}{M} \alpha\, cp \left[-12000 \times 10^{-6}\right] \\
& + \frac{1}{6} \alpha \ell \frac{S_b}{S} \left[4220 \times 10^{-6}\right] + 68.4\, cp^3 \dot{\theta}^2 \alpha \left[-0.0000666 \times 10^{-6}\right] \\
& - 34.2\, cp^2 \dot{\theta}^2 \alpha^2 \left[-0.00001112 \times 10^{-6}\right] - 45.6\, cp^4 \dot{\theta}^3 \\
& \left[0.0000000000000000253 \times 10^{-6}\right] - \frac{8}{M} cp^2 \dot{\theta} \left[-1.35 \times 10^{-6}\right] \\
& - \frac{1}{5} \dot{\theta} |\dot{\theta}| \left[-0.0000723 \times 10^{-6}\right] - \frac{1}{36} \ell^2 \frac{S_b}{S} \dot{\theta} \left[-0.344 \times 10^{-6}\right] \\
& - \frac{1}{405} \frac{\ell^5}{5} c^2 \dot{\theta} |\dot{\theta}| \left[-0.000122 \times 10^{-6}\right]
\end{aligned}
$$

The relative sizes of the various terms for this particular configuration at the above assumed conditions indicate that the terms containing $\dot{\theta}$ and $\dot{\theta}^2$ are smaller than the terms containing α by several orders of magnitude. These small terms are omitted. The summations of the aerodynamic forces and moments are shown in equation (41); the terms to be considered negligibly small are crossed out.

$$
\begin{aligned}
(41a) \quad C_N = {} & 5.7\alpha^3 + \frac{4}{M}\alpha + 2\alpha \frac{S_b}{S} \cancel{+ 68.4\, cp^2 \dot{\theta}^2 \alpha - 34.2\, cp\, \dot{\theta} \alpha^2} \\
& \cancel{- 45.6\, cp^3 \dot{\theta}^3 - \frac{8}{M} cp\, \dot{\theta}} \\
& \cancel{+ \frac{1}{3} \ell \frac{S_b}{S} \dot{\theta}}
\end{aligned}
$$

$$
(41b) \quad C_D = (2\delta^2 + \alpha^2) \frac{S_b}{S} \cancel{+ \left(2\alpha + \frac{1}{6} \ell \dot{\theta}\right) \frac{1}{6} \ell \frac{S_b}{S} \dot{\theta}}
$$

$$
\begin{aligned}
(41c) \quad C_m = {} & \left(5.7\alpha^3 + \frac{4}{M}\alpha\right) cp + \frac{1}{6} \alpha \ell \frac{S_b}{S} + C_{m_t} \\
& \cancel{+ 68.4\, cp^3 \dot{\theta}^2 \alpha - 34.2\, cp^2 \dot{\theta}^2 \alpha^2} \\
& \cancel{- 45.6\, cp^4 \dot{\theta}^3 - \frac{8}{M} cp^2 \dot{\theta} - \frac{1}{5} \dot{\theta} |\dot{\theta}|} \\
& \cancel{- \frac{1}{36} \ell^2 \frac{S_b}{S} \dot{\theta} - \frac{1}{405} \frac{\ell^5}{5} c^5 \dot{\theta} |\dot{\theta}|}
\end{aligned}
$$

It is realized that only with this very particular set of conditions are the crossed over terms negligible, but because of the very large difference between the crossed over terms and the retained terms in this general case, it is believed that the crossed over terms will never become of appreciable magnitude in flight regimes of interest.

CHAPTER VII

LINEAR AERODYNAMIC FORCES - VEHICLE

In determining the linear aerodynamic forces and moments on the vehicle, it is assumed that the effects deemed negligible in the non-linear case are also negligible in the linear case. It is necessary then to consider only the linear effects of angle of attack on the vehicle. This is accomplished by determining the angle of attack effects for the wing and body separately, then taking their summation and neglecting interference effects as in the non-linear case.

Effects of Angle of Attack - Flat Plate

The C_N is obtained from ordinary linear supersonic theory.

(42) $C_N = \frac{4\alpha}{\sqrt{M^2-1}}$

Since high Mach numbers are of interest here, the following substitution is made.

$$\sqrt{M^2-1} = M$$

Hence, (42) becomes

(43) $C_{N_{(\alpha)}} = \frac{4}{M}\alpha.$

(44) $C_{D_{(\alpha)}} = 0$

(45) $C_{m_{(\alpha)}} = \frac{4}{M}\alpha\, cp.$

In addition to the C_m due to angle of attack term, the term

$C_{m_{trim}}$ is also introduced and has the same meaning as in the non-linear case.

Effects of Angle of Attack - Body

Based on cone base area, linear theory yields

(46) $C_{N_{(\alpha)}} = 2\alpha$.[1]

Based on wing area, equation (46) becomes

(47) $C_{N_{(\alpha)}} = 2\alpha \frac{S_b}{S}$.

(48) $C_{m_{(\alpha)}} = C_N \left(\frac{L}{3} - \frac{L}{4}\right) = \frac{1}{6} \ell \alpha \frac{S_b}{S}$

To be considered next is the drag component.

(49) $C_{D_{(\alpha)}} = \alpha^2$

Equation (49) is based on cone base area.[2] In terms of wing area (49) becomes

(50) $C_{D_{(\alpha)}} = \alpha^2 \frac{S_b}{S}$.

In addition to the drag effects due to angle of attack, Sears (1954) finds the drag effects due to body thickness,

(51) $C_D = 2\pi \ell^2 \delta^2 \left(\ln \frac{2}{M\delta} - \frac{1}{2}\right) \frac{S_b}{S}$.

Combination of the effects due to thickness with the effects of angle of attack yields

(52) $C_D = \left[\alpha^2 + 2\pi \ell^2 \delta^2 \left(\ln \frac{2}{M\delta} - \frac{1}{2}\right)\right] \frac{S_b}{S}$.

Equations (53) show the summation of the linear aerodynamic forces and moments with the effects of interference neglected.

(53a) $C_N = \left(\frac{4}{M} + 2\frac{S_b}{S}\right)\alpha$

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[1]Antonio Ferri, Elements of Aerodynamics of Supersonic Flows, (New York, 1949), p. 229.

[2]W. Sears, General Theory of High Speed Aerodynamics, VI, (Princeton University, 1954), p. 240.

(53b) $C_D = \left[\alpha^2 + 2\pi \ell^2 \delta^2 \left(\ln \frac{2}{M\delta} - \frac{1}{2}\right)\right] \frac{S_b}{S}$

(53c) $C_m = \frac{4}{M} \alpha cp + \frac{1}{6} \alpha \ell \frac{S_b}{S} + C_{m_t}$

CHAPTER VIII

DIFFERENTIAL EQUATIONS OF MOTION

After non-dimensionalizing equations (5) and (6) by dividing (5a), (5b), (6a), and (6b) by the quantity qS, (5c) and (6c) by qSc, then substituting equations (41) into (6) and equating this result to equations (5) the following differential equations of motion are obtained:

(54a) $\frac{m}{qS}(\dot{u} + w\dot{\theta}) + \frac{U\dot{m}}{qS} - C_T + C_D + \frac{mg}{qS}\sin\theta = 0$

(54b) $\frac{m}{qS}(\dot{w} - u\dot{\theta}) + \frac{W\dot{m}}{qS} + C_N - \frac{mg}{qS}\cos\theta = 0$

(54c) $\frac{I_y\ddot{\theta}}{qSc} + \frac{\dot{I}_y\dot{\theta}}{qSc} - C_m = 0.$

In a first attempt for a solution to these equations, the following assumptions are made:

a. Forward velocity is constant.

b. Vertical oscillations take place about a constant altitude.

c. Mass remains constant.

Assumption a immediately reduces the system to one with two degrees of freedom. The geometry of the body axes system yields

$$w = u\sin\alpha \approx u\alpha$$

$$\dot{w} = u\dot{\alpha} + \dot{u}\alpha.$$

Assumption a requires

$$\dot{u} = 0.$$

(55) $\therefore \quad \dot{w} = u\dot{\alpha}$

The equations of motion as they stand do not consider the effects of centrifugal force on the vehicle as it travels in level flight in its curved path about the earth. Since the speeds of interest approach orbital velocity, these centrifugal force effects can not be ignored. This correction is made by adding the term

$$\frac{m u^2}{q S (R'+H)} \cos \theta \quad [1]$$

where R' is the radius of the earth and H is the altitude, to the weight term in equation (54b). After substitution of this correction and (55) into (54) and applying the assumptions listed above, the differential equations become for the non-linear case,

$$\dot{\alpha} - \dot{\theta} + \frac{5.7 q S}{m u} \alpha^3 + \frac{q S}{m u}\left(\frac{4}{M} + 2\frac{S_b}{S}\right)\alpha - \left(\frac{g}{u} - \frac{u}{R'+H}\right) = 0 \tag{56a}$$

$$\ddot{\theta} - \frac{5.7 q S c}{I_y}\alpha^3 - \frac{q S c}{I_y}\left(\frac{4}{M} c_p + \frac{1}{6} \ell \frac{S_b}{S}\right)\alpha - \frac{C_{m_t} q S c}{I_y} = 0, \tag{56b}$$

and for the linear case,

$$\dot{\alpha} - \dot{\theta} + \frac{q S}{m u}\left(\frac{4}{M} + 2\frac{S_b}{S}\right)\alpha - \left(\frac{g}{u} - \frac{u}{R'+H}\right) = 0 \tag{57a}$$

$$\ddot{\theta} - \frac{q S c}{I_y}\left(\frac{4}{M} c_p + \frac{1}{6} \ell \frac{S_b}{S}\right)\alpha - \frac{C_{m_t} q S c}{I_y} = 0. \tag{57b}$$

where cosine θ is assumed to be unity.

[1]This term is the vertical component of the non-dimensionalized force due to the normal acceleration of a mass traveling in a circular path.

CHAPTER IX

SOLUTION OF EQUATIONS

This chapter is devoted to solving the equations of motion in both the linear case and the non-linear case. The linear equations are solved by ordinary analytical means. The non-linear equations are solved by a numerical method.

For convenience the equations of motion (56) and (57) are written with the following notations.

(58a) $\dot{\alpha} - \dot{\theta} + A\alpha^3 + B\alpha + C = 0$

(58b) $\ddot{\theta} + D\alpha^3 + E\alpha + F_{non\text{-}lin} = 0$

Equations (58) are the non-linear ones.

(59a) $\dot{\alpha} - \dot{\theta} + B\alpha + C = 0$

(59b) $\ddot{\theta} + E\alpha + F_{lin} = 0$

Equations (59) are the linear ones. The upper case letters in (58) and (59) are

(60) $A = \dfrac{5.7 q S}{m u}$ $\qquad D = -\dfrac{5.7 q S c}{I_y}$

$B = \dfrac{q S}{m u}\left(\dfrac{4}{M} + 2\,\dfrac{S_b}{S}\right)$ $\qquad E = -\dfrac{q S c}{I_y}\left(\dfrac{4}{M}\,cp + 2\,cp\,\dfrac{S_b}{S}\right)$

$C = -\left(\dfrac{g}{u} - \dfrac{u}{R' + H}\right)$ $\qquad F = -\dfrac{C_{m_t} q S c}{I_y}$.

The value of each of the items in equations (60) depend on the flight conditions. It is desirable to select a set of flight conditions which a vehicle such as the one chosen might encounter in high speed, high altitude, level flight. The following conditions are selected.

$M = 8$ $\qquad H = 100{,}000$ ft. $\qquad cp = -0.25$

m = 70% take-off mass = 439 slugs

From assumptions already made concerning the size and shape of the vehicle and these flight conditions, the following quantities are calculated.

(61) $S = 200$ ft.^{2} $\quad$ $u = 7768$ fps.

$S_b = 25\pi$ ft.^{2} $\quad$ $g = 32.2$ ft./sec.^{2}

$I_y = 26{,}800$ slug ft.^{2} $\quad$ $R' = 4000$ mi. = 21,120,000 ft.

$q = 1005$ lb./ft.^{2} (using standard NACA atmosphere)

$L = 40$ ft. $\quad$ $c = 15.5$ ft.

Using the flight conditions and equations (61), equation (60) becomes

(62a) $A = 0.33596$

$B = 0.075761$

$C = -0.003779$.

For steady state conditions the linear equation (59a) becomes

$$B\alpha + C = 0$$

from which steady state α is obtained in the linear case.

$$\alpha_{\text{steady state}} = \alpha_{\text{trim-lin.}} = -\frac{C}{B} = 0.04988 \text{ radians}$$

The non-linear equations (58a) in the steady state become

$$A\alpha^3 + B\alpha + C = 0$$

from which $\alpha_{\substack{trim\\ non\text{-}lin.}}$ is found to be 0.049295 radians.

It is found that the vehicle is statically unstable in these flight conditions for the linear case. This will be discussed in detail in Chapter X. In order to have a statically stable vehicle, the flight conditions should be changed. Static stability can be obtained readily by lowering the Mach number to below 5.92 or decreasing the c_p at least to -0.3378.

Since it is desirable to stay in the truly hypersonic region, the cp is changed rather than the Mach number. The cp is arbitrarily placed at -0.4000. Changing the cp (as cp is defined in this study) to a greater negative number, in effect, shifts the wing farther aft. Using the new value -0.4000 for cp, the remaining quantites in (60), in addition to $\alpha_{t \cdot lin}$ and $\alpha_{t_{non\text{-}lin}}$ are shown in (62b).

(62b) $E = -3.61382$ $\qquad\qquad$ $\alpha_{t \cdot lin.} = 0.04988$

$D = 265.0500$ $\qquad\qquad$ $\alpha_{t_{non\text{-}lin.}} = 0.049295$

$F_{non\text{-}lin} = -D\alpha^3_{t_{non\text{-}lin}} - E\alpha_{t_{non\text{-}lin}} = -.146396$

$F = -E\alpha_{t \cdot lin} = -0.180257$

Analytical solutions to the linear equations are pursued first. After differentiating equation (59a) with respect to time, the quantity $\ddot{\theta}$ is eliminated from (59a) and (59b).

(63) $$\ddot{\alpha} + B\dot{\alpha} + E\alpha + F_{lin} = 0$$

Equation (63) written in operator form becomes

$$(D_0^2 + BD_0 + E)\alpha = -F_{lin}.$$

The characteristic equation becomes

$$p^2 + Bp + E = 0$$

$$p = -\tfrac{1}{2}B \pm \tfrac{i}{2}\sqrt{4E - B^2} = a \pm bi$$

and

$$a = -\tfrac{1}{2}B = -0.038$$

$$b = \tfrac{1}{2}\sqrt{4E - B^2} = 1.900.$$

Therefore the general solution is

$$\alpha = c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} = e^{at}\left(c_1' \cos bt + c_2' \sin bt\right).$$

The particular solution is obtained from

$$\alpha = \frac{1}{(D_0^2 + BD + E)} F_{lin.}$$

$$= -\left(\frac{1}{E} - \frac{BD_0}{E^2} + \cdots\right) F_{lin} = -\frac{F_{lin}}{E}.$$

In order to determine the constants c_1, c_2, c_1', and c_2', it is necessary to establish initial conditions.

The initial conditions are derived from the following statements. The vehicle is assumed to be in level flight in a pitch attitude 0.35 radians greater than trim for level flight with zero velocity of angle of attack. The vehicle is released from this attitude and allowed to move unhampered thereafter. These initial conditions are written as

$$\alpha = \theta = 0.35 + \alpha_t$$

$$\dot{\alpha} = 0 \; ; \quad t = 0$$

or for the non-linear case

$$\alpha_0 = \theta_0 = 0.399295$$

$$\dot{\alpha}_0 = 0$$

or for the linear case

$$\alpha_0 = \theta_0 = 0.39988$$

$$\dot{\alpha}_0 = 0.$$

Using the initial conditions, the constants in equations (64) become

$$c_1 = 0.175(1 + \tfrac{a}{b}i) \qquad c_1' = 0.35$$

$$c_2 = 0.175(1 - \tfrac{a}{b}i) \qquad c_2' = 0.00698 \; .$$

Equation (64a) becomes

(65) $\alpha = 0.175(1 + \frac{a}{b}i)\, e^{(a+bi)t} + 0.175(1 - \frac{a}{b}i)\, e^{(a-bi)t} - \frac{F_{non\text{-}lin}}{E}$.

The term $\dot{\alpha}$ is obtained by differentiating (65).

(66) $\dot{\alpha} = 0.175(1 + \frac{a}{b}i)(a+bi)\, e^{(a+bi)t} + 0.175(1 - \frac{a}{b}i)(a-bi)\, e^{(a-bi)t}$

Substitution equation (66) and (65) into (59a), then integrating with definite limits (θ from θ_0 to θ, t from 0 to t), after simplification θ is obtained as

$$\theta = e^{at}\left[0.35\left(1+\frac{2aB}{a^2+b^2}\right)\cos bt + 0.35\left\{-\frac{a}{b}+\frac{B}{b}\left(\frac{b^2-a^2}{b^2+a^2}\right)\right\}\sin bt\right] - 0.35\left[1+\frac{2aB}{a^2+b^2}\right] - \frac{BF_{lin}}{E} + C + \theta_0$$

which, when numerically evaluated becomes

(67a) $\theta = e^{-0.038t}\left[0.35 \cos 1.9t + 0.0209 \sin 1.9t\right] + 0.04988.$

Numerically evaluated, equation (64b) becomes

(67b) $\alpha = e^{-0.038t}\left[0.35 \cos 1.9t + 0.00698\ 1.9t\right] + 0.04988.$

Equations (67) are the solutions of the linear differential equations for this particular set of flight and initial conditions.

The non-linear equations, because of the difficulties encountered in analytical solution, are solved by a numerical method for systems of simultaneous equations.[1] Values of α and θ in the neighborhood of α and θ at time equal to zero are assumed to be represented by these Taylor's series.

(68a) $\alpha = \alpha_0 + \dot{\alpha}_0 t + \frac{\ddot{\alpha}_0 t^2}{\lfloor 2} + \frac{\dddot{\alpha}_0 t^3}{\lfloor 3} + \frac{\alpha_0^{iv} t^4}{\lfloor 4} + \frac{\alpha_0^{v} t^5}{\lfloor 5}$

(68b) $\theta = \theta_0 + \dot{\theta}_0 t + \frac{\ddot{\theta}_0 t^2}{\lfloor 2} + \frac{\dddot{\theta}_0 t^3}{\lfloor 3} + \frac{\theta^{iv} t^4}{\lfloor 4} + \frac{\theta^{v} t^5}{\lfloor 5}$

For convenience equations (58) are rearranged and written as

(69a) $\dot{\alpha} = \dot{\theta} - \left[A\alpha^3 + B\alpha + C\right]$

(69b) $\ddot{\theta} = -\left[D\alpha^3 + E\alpha + F_{non\text{-}lin}\right].$

The initial conditions provide enough information to calculate $\dot{\theta}_0$ from (69a) and $\ddot{\theta}_0$ from (69b).

$$\dot{\theta}_0 = \dot{\alpha} + A\alpha_0^3 + B\alpha_0 + C = 0.047860$$

$$\ddot{\theta}_0 = -\left[D\alpha_0^3 + E\alpha_0 + F_{non\text{-}lin}\right] = -18.17024$$

The derivative $\ddot{\alpha}_0$ is obtained by differentiating (69a), then substituting values now known.

$$\ddot{\alpha}_0 = \ddot{\theta}_0 - \left[3A\alpha_0^2 + B\right]\dot{\alpha}_0 = -18.17024$$

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[1]James B. Scarborough, Numerical Mathematical Analysis, (Baltimore, 1950), pp.271-273.

The derivative $\dddot{\theta}_0$ is obtained by differentiating (69b), then substituting values now known.

$$\dddot{\theta}_0 = -\left[3D\alpha_0^2 + E\right]\dot{\alpha}_0 = 0$$

The derivative $\dddot{\alpha}_0$ is obtained by differentiating $\ddot{\alpha}$, then substituting known values.

$$\dddot{\alpha}_0 = \dddot{\theta}_0 - \left[3A\alpha_0^2 + B\right]\ddot{\alpha}_0 - 6A\,\alpha_0\,\dot{\alpha}_0^2 = 4.29640$$

By differentiating $\dddot{\theta}_0$ the value of θ_0^{IV} is found.

$$\theta_0^{IV} = -\left[3D\alpha_0^2 + E\right]\ddot{\alpha}_0 - 6\,D\alpha_0\dot{\alpha}_0^2 = 2369.21$$

This alternating process is continued.

$$\alpha_0^{IV} = \theta_0^{IV} - \left[3A\alpha_0^2 + B\right]\dddot{\alpha}_0 - 18A\alpha_0\dot{\alpha}_0\ddot{\alpha}_0 - 6A\dot{\alpha}_0^3 = 2368.19$$

$$\theta_0^{V} = -\left[3D\alpha_0^2 + E\right]\dddot{\alpha}_0 - 18\,D\alpha_0\dot{\alpha}_0\ddot{\alpha}_0 - 6\,D\dot{\alpha}_0^3 = -560.2058$$

$$\alpha_0^{V} = \theta_0^{V} - \left[3A\alpha_0^2 + B\right]\alpha_0^{IV} - 36A\dot{\alpha}_0^2\ddot{\alpha}_0 + 18A\ddot{\alpha}_0^2\alpha_0 + 24A\alpha_0\dot{\alpha}_0\dddot{\alpha}_0 = -322.959$$

Equations (68) and (69) become

(70a) $$\theta = 0.399295 + 0.04786\,t - 9.08512\,t^2 + 98.7170\,t^4 - 4.66838\,t^5$$

(70b) $$\alpha = 0.399295 - 9.08512\,t^2 + 0.716066\,t^3 + 98.6747\,t^4 - 2.69132\,t^5$$

(70c) $$\dot{\alpha} = -18.17024\,t + 2.14819\,t^2 + 394.6988\,t^3 - 13.4566\,t^4$$

(70d) $$\dot{\theta} = \dot{\alpha} + 0.33596\,\alpha^3 + 0.075761\,\alpha - 0.003779$$

(70e) $$\ddot{\theta} = -265.0500\,\alpha^3 - 3.61382\,\alpha + 0.146396.$$

Selecting a time interval h of 0.01 seconds, equations (70a), (70b), and (70c) are calculated for values of t equal to -0.02, -0.01, 0, 0.01, 0.02. These values are shown in Table III. Equations (70d) and (70e) are calculated for the same values of t by using the proper values of α. The higher order differences $\Delta_1\dot\theta, \Delta_2\dot\theta, \Delta_3\dot\theta, \Delta_4\dot\theta$, etc., are found by taking the difference of the preceding two numbers in the preceding column. The new line of the table is found by

(71a) $\Delta\theta = h\left[\dot\theta - \frac{1}{2}\Delta_1\dot\theta - \frac{1}{12}\Delta_2\dot\theta - \frac{1}{24}\Delta_3\dot\theta - \frac{1}{38}\Delta_4\dot\theta\right]$

(71b) $\Delta\dot\theta = h\left[\ddot\theta + \frac{1}{2}\Delta_1\ddot\theta + \frac{5}{12}\Delta_2\ddot\theta + \frac{3}{8}\Delta_3\ddot\theta + \frac{1}{3}\Delta_4\ddot\theta\right]$

(71c) $\Delta\alpha = h\left[\dot\alpha - \frac{1}{2}\Delta_1\dot\alpha - \frac{1}{12}\Delta_2\dot\alpha - \frac{1}{24}\Delta_3\dot\alpha - \frac{1}{38}\Delta_4\dot\alpha\right]$

(71d) $\ddot\theta = -265.0500\alpha^3 - 3.61382\alpha + 0.146396$

(71e) $\dot\alpha = \dot\theta - 0.33596\alpha^3 - 0.075761\alpha + 0.003779.$

For the new line t = 0.03 equations (71) become

$$\Delta\theta = 0.002232 \qquad \ddot\theta = -17.359053$$

$$\Delta\dot\theta = -0.174375 \qquad \dot\alpha = -0.533135$$

$$\Delta\alpha = -0.002715.$$

After entering these values into Table III the new values for θ, α, $\dot\theta$, and the higher order differences are found, thereby completing the line for t = 0.03. The next line for t = 0.04 is found in the same manner using equations (71). In this way the values of θ, α, $\dot\theta$, $\dot\alpha$, and $\ddot\theta$ can be found at any time t by continuing to lengthen the table. This reiterative method of the solution of the non-linear equations readily lends itself to computation by high speed computing equipment. Since the accuracy of the answer depends on the smallness of the time interval h, such a means of computing will render answers as accurate as desirable over a lengthy time period without involving much more work. It is necessary, in any case, to have accurate starting values. Figure 11 shows α and θ for the first 900 lines.

TABLE III

NUMERICAL SOLUTIONS OF NON-LINEAR EQUATIONS

t	θ	$\Delta\theta$	α	$\Delta\alpha$	$\dot\theta$	$\Delta_1\dot\theta$	$\Delta_2\dot\theta$	$\Delta_3\dot\theta$	$\Delta_4\dot\theta$
-0.02	.394720		.395671		.408112				
-0.01	.397909	.003189	.398387	.002716	.229167	-.178945			
0	.399295	.001386	.399295	.000908	.047860	-.181307	-.002362		
0.01	.398865	-.000430	.398389	-.000906	-.133446	-.181306	-.000001	.002361	
0.02	.396634	-.002231	.395683	-.002706	-.312380	-.178934	.002372	.002373	.000012
0.03	.394402	-.002232	.392968	-.002715	-.486755	-.174375	.004559	.002187	-.000186

t	$\dot\alpha$	$\Delta_1\dot\alpha$	$\Delta_2\dot\alpha$	$\Delta_3\dot\alpha$	$\Delta_4\dot\alpha$	$\ddot\theta$	$\Delta_1\ddot\theta$	$\Delta_2\ddot\theta$	$\Delta_3\ddot\theta$	$\Delta_4\ddot\theta$
-0.02	.361103					-17.701851				
-0.01	.181522	-.179581				-18.052096	-.350245			
0	0	-.181522	-.001941			-18.170250	-.118154	.232091		
0.01	-.181092	-.181092	.000430	.002371		-18.052368	.117882	.236036	.003945	
0.02	-.359391	-.178299	.002793	.002363	.000008	-17.703405	.348963	.231081	-.004955	-.008900
0.03	-.533135	-.143974	.034325	.031532	.029169	-17.359053	.344351	-.004612	-.235693	-.230738

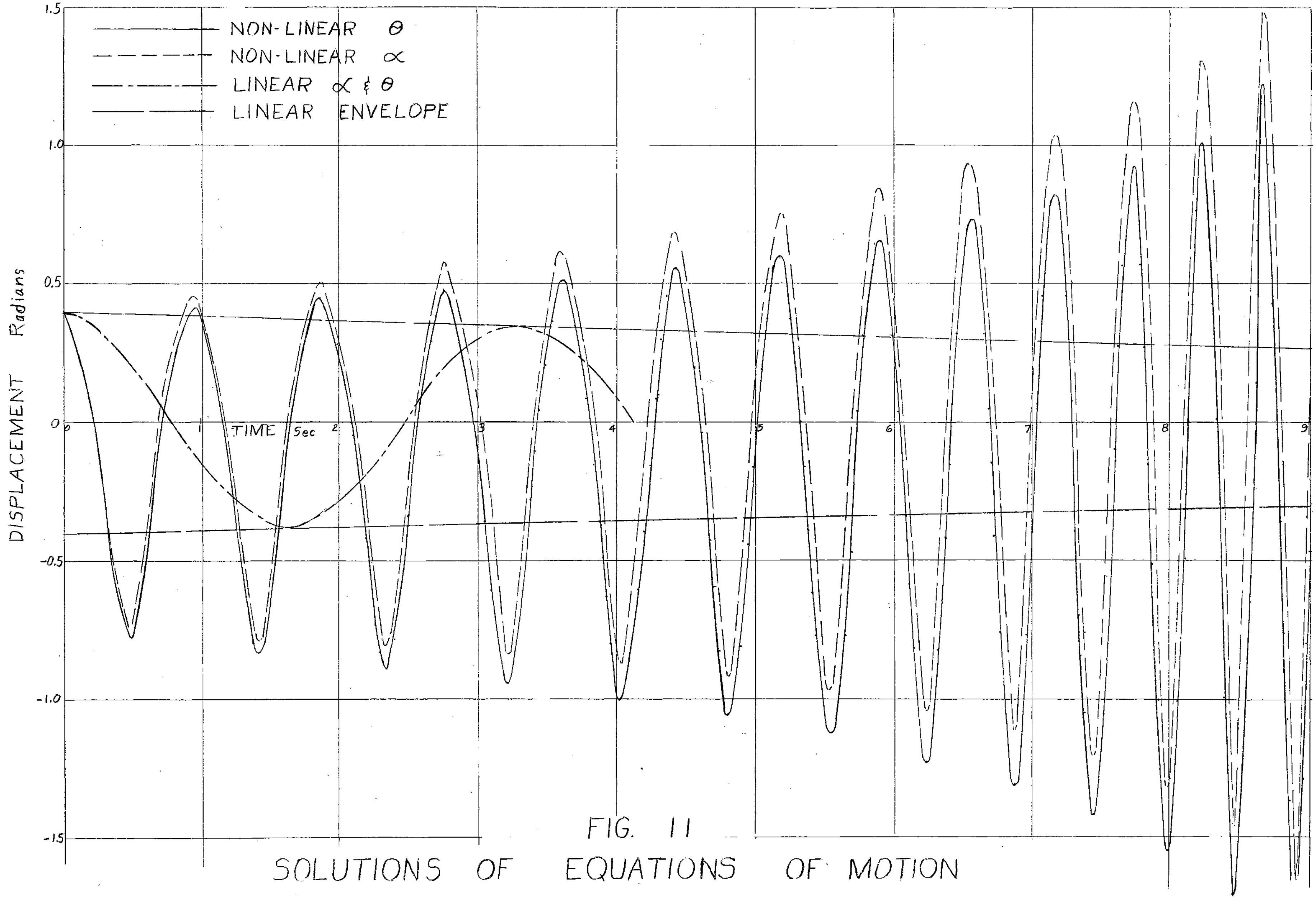

FIG. 11
SOLUTIONS OF EQUATIONS OF MOTION

CHAPTER X

CONCLUSIONS

Figure 11 is a displacement-time graph of θ and α for the linear and the non-linear cases for the first nine seconds after the vehicle has been released from disturbed conditions. This figure will be the basis on which the conclusions are drawn. The non-linear results, as they were found by the numerical method and shown in Table III, depend on the accuracy of the starting values and the time interval h. It is possible that the equations (71) for starting the new lines in Table III at some time a number of h intervals after the start no longer converge properly because of large higher order differences. However, since the machine computed values coincide with hand computed values, at least for the first new line ($t = 0.03$), and since the starting values are correct to six decimal places and the interval h is 0.01 seconds, it seems reasonable to assume the machine computed values are correct to two decimal places, the accuracy shown in figure 11.

In the introductory chapter it is suggested that it would be desirable to find the degree to which linear supersonic theory could be utilized in predicting dynamic stability at high speeds compared to Newtonian theory. In the development of the equations of motion, in finding the aerodynamic effects, and in solving the equations of motion, numerous assumptions are made. The results can be evaluated only within the limitations imposed by all these assumptions.

It might be desirable, at this point, after having applied these assumptions, to examine the actual situation under consideration. A simple configuration consisting of a flat plate and cone is disturbed approximately twenty degrees from trim, level, hypersonic flight and allowed to oscillate as a two degree of freedom system. The aerodynamic forces acting on the vehicle emanate from a straight line lift curve in the linear case and from a cubic lift curve fitted to a Newtonian impact theory plus a linear term curve in the non-linear case. This cubic approximation fits exactly at only three points. A particluar set of flight conditions is chosen and the results of the solutions of the equations of motion are shown in figure 11.

In selecting the flight conditions a rather interesting phenomenon is observed. For the conditions M = 8, H = 100,000 feet, c_p = -0.25, the aerodynamic moments for the linear case and the non-linear case, equations (53c) and (41c) respectively, are shown in figure 12.

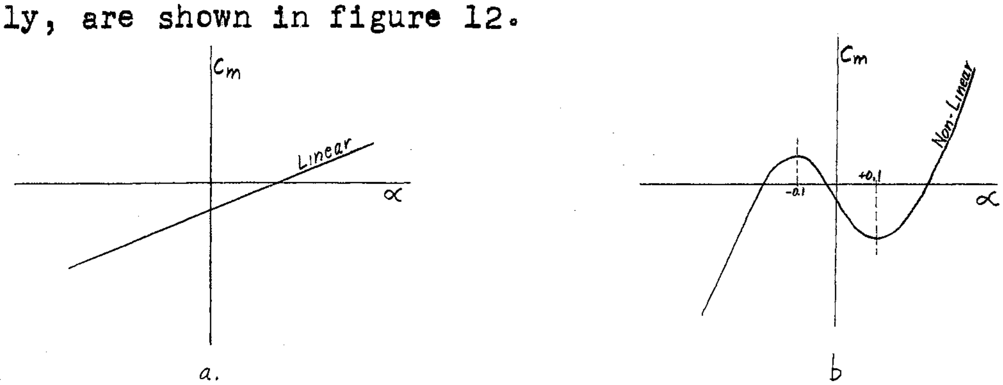

Fig. 12

MOMENT CURVES FOR c_p = -0.25

Since a positive slope in the C_m verses α curve indicates static instability, the linear case is everywhere unstable. The non-linear case, however, is statically unstable in a narrow band extending approximately 0.1 radians on each side of zero angle of attack. Outside this region the vehicle is statically stable. If the vehicle in the non-linear case trims at some angle of attack greater than 0.1 radians and never oscillates to angles of attack in this band, it would be statically stable and have possibilities of being dynamically stable. In these given flight conditions the vehicle trims at 0.049295 radians which is within this static instability band. The vehicle would move either to negative angles of attack or to positive angles of attack outside this static instability band but never trim at 0.049295 radians. This shows only one case in which a vehicle may be statically unstable in the linear case yet statically stable for the non-linear case and demonstrates one of the effects of higher order terms. Since dynamic stability presupposes static stability, the φ is changed to -0.4000 to obtain a statically stable vehicle in both the linear and non-linear cases.

Attention is now turned to figure 11 in which are shown the dynamic characteristics of the vehicle in flight conditions producing static stability. In the linear case the displacements θ and α are almost coincidental. This means that the vehicle maintains very nearly a level flight path. These displacements dampen to one half the initial amplitude in approximately eighteen seconds. The frequency of the linear oscillations is

$$f = \frac{1.9}{2\pi} = 0.303 \text{ cycles per second.}$$

The period is 3.30 seconds per cycle.

The non-linear displacements demonstrate a tendency of dynamic instability from the outset. During the first cycle, the period is approximately 0.96 seconds per cycle. The period becomes shorter for each successive cycle. The period at t = 9 seconds is approximately 0.37 seconds per cycle. The median of the θ oscillations tends to drift in the negative direction while the median of the α oscillations stays rather well centered along trim α. This drift seems to indicate a diving tendency of the flight path. There appears to be no positive correlation between the linear and the non-linear displacements. Since the displacements approach the order of magnitude of two radians, it seems of little value to consider any time after nine seconds since the Newtonian theory and the equations of motion are not valid for so large a displacement.

There are several possibilities for the large discrepancy in the linear and the non-linear results. The essential difference between the equations of motion in the non-linear case, equations (58), and the linear case, equation (59) is the non-existence of the constants A and D in (59). Examination of equation (62) with reference to the relative magnitudes of the constants reveal that D, the constant for the α^3 term in the moment equation, is almost a hundred times greater than the largest of the others. This makes

D an extremely powerful and most probably the controlling factor in the discrepancy between the displacements in figure 11. From equation (60) it is noticed that the value of D depends not only on q, S, c, and I_y, which are functions of the configuration and flight conditions, but, more importantly, from the discussion preceding equation (8), it depends on the manner in which the $C_{N(\alpha)}$ cubic is fitted to the Newtonian quadratic. Even though Newtonian theory has merits for predicting steady state aerodynamic forces at high speeds, it may not be readily applicable for predicting aerodynamic effects in dynamic situations. This question of applicability probably will remain for experimentation to answer. It is possible that quite different displacements for θ and α may have been attained if the system were treated with three degrees of freedom allowing fore and aft motion rather than limiting the vehicle to two degrees of freedom as in this study. This treatment might be considered for subsequent investigations. However, the extreme size of the constant D seems to offer the most plausible explanation for the contrast. It would be of interest to investigate several different means of approximating the $C_{N(\alpha)}$ quadratic to get a continuous function as well as investigating the results of using the quadratic itself. These investigations would be desirable before disregarding the Newtonian theory for dynamic situations, even though the manner in which Newtonian theory plus the correction and its cubic approximation has been used in this study suggest that such a discharge be made.

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