INTRODUCTION TO COLLEGE MATHEMATICS

FOR SECONDARY SCHOOLS

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in fulfillment of the requirements
for Education 510 Seminar
May, 1958

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PREFACE

In the complex world of today the study of mathematics is of increasing importance since mathematical knowledge is the key to our scientific advances. Many new ideas have arisen in recent decades that have given new insights into old problems as well as forming the basis for radically new techniques. Before a person can hope to succeed in the numerous new courses being offered by college mathematics departments, he must have a firm grasp of certain tool subjects and some understanding of many new concepts. It shall be the purpose of this course to provide some of the needed tools.

This shall be but an introductory course for use in the second semester of twelfth grade. It is intended only for those students having a thorough background in algebra, plane geometry, and trigonometry. The author realizes that the material here presented is somewhat sketchy with very few exercises given, and only bare coverage of some ideas; however he feels that the teacher can easily supplement this material with examples from his own experience or by reference to the various sources mentioned in footnotes or bibliography.

The need for studies of this type is recognized by many agencies and is being advanced by three outstanding study groups, namely: The University of Illinois Committee on High School Mathematics, The Commission on Mathematics of the College Entrance Examination Board, and The National Science Foundation.

Indebtedness is acknowledged to Drs. Robert L. Swain and Richard E. Johnson, for many of the ideas contained in the chapters on "Sets" and "Calculus", respectively, were taken nearly verbatim from class lectures. Further acknowledgement is due to Dr. James H. Zant for criticisms and suggestions. The author is also indebted to his wife for her continued cheerfulness during the compilation of the study and for her typing of the final manuscript.

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CHAPTER I

THEORY OF SETS

I. 1. Introduction and Examples.

Most of modern mathematics is founded upon the concept of sets.[1] It shall be the purpose of this chapter to introduce some of the basic vocabulary and algebra of the theory of sets. This is in no way a complete treatment of the subject; but should contribute a nucleus of ideas for the better understanding of the later chapters of this course.

The idea of set is probably as old as man himself. Surely early man thought in terms of collections of things -- items both similar and dissimilar in nature. He spoke of a "flock" of sheep and a "herd" of cattle. Our language has a large number of such collective words: pair, class, group, family, school, state, set, and many others. The set is such a mental concept formed by thinking of several things as constituting a coherent entity. The items making up the set are said to belong to it, and are called elements or members of the set.

Example 1. The people in your immediate family (father, mother, you, and your sisters and brothers) form a set. Each person in the family is a member or element of the set.

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[1]The information for the chapter on sets is taken mainly from four sources: Robert L. Swain, Understanding Arithmetic (New York, 1957), pp. 27-66; E. J. McShane "Operating With Sets", Insights Into Modern Mathematics (Washington, 1957), pp. 36-64; Commission on Mathematics, College Entrance Examination Board, Introductory Probability and Statistical Inference for Secondary Schools (New York, 1957), pp. 131-151; High School Mathematics First Course (Urbana, 1957), Unit Four.

Other examples of sets are: The group of students in your school who are in the band, the faculty of your school, the city police department, the football team, the counties in your state, and the pieces of furniture in a room.

Example 2. The set of numbers $\{4, 7, 10, 13, 16, 19\}$. Each number (element) of this set is three more than a number in the set $\{1, 4, 7, 10, 13, 16\}$.

Example 3. The set of roots of the algebraic equation

$$(x - 7)(x + 5)(x + 3)(x^2 + 3) = 0$$

is

$$\{7, -5, -3, \sqrt{-3}, -\sqrt{-3}\}$$

This set contains subsets which are of interest, for example:

set of real roots: $\{7, -5, -3\}$

set of negative real roots: $\{-5, -3\}$

set of imaginary roots: $\{\sqrt{-3}, -\sqrt{-3}\}$

Example 4. The set of points whose x and y coordinates satisfy the equation

$$y = 4 - x$$

is a straight line L. Its graph is shown in Fig. 1.

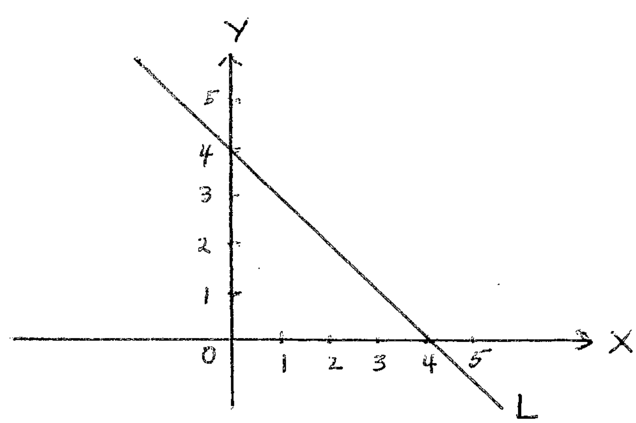

Fig. 1

Exercises for Section I. 1

1. Describe and sketch the sets of points whose x and y coordinates satisfy each of the following conditions:

   a. $x^2 + y^2 = 9$

   b. $x^2 + y^2 > 9$

   c. $x + y \leq 6$

   d. $x^2 + y^2 < 9$

2. For each of the following categories, give three examples of sets whose elements are:

   a. animals

   b. four legged animals

   c. geometrical figures

   d. letters of the Greek alphabet

3. Find the set, in each case, whose elements are the roots of the algebraic equations:

   a. $x^2 + 4x + 3 = 0$

   b. $x^2 - 36 = 0$

   c. $x^3 - 64x = 0$

   d. $x^2 + 17 = 0$

   e. $x^2 + \frac{1}{8} = 0$

I. 2. Ways of Specifying Sets.

Sets are usually specified in two different ways:

a. By listing all the names of each element of the set;

b. By stating the requirements which an item must meet in order to be an element of the set.

Frequently a single letter is used to represent a whole set; and we enclose the names of the elements of the set or the requirements for membership between braces. For example, suppose that S represents the set consisting of all integers less than 9. One method of designating this set is

$$S = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

The other method might be $S = \{x \mid x \text{ is an integer and } 1 \leq x < 9\}$, which would be read "S is the set of all elements x such that x is an integer and $1 \leq x < 9$".

<u>Example 1</u>. Let S be the set of odd integers between 1 and 9 inclusive, and let T be the set given by

$$T = \{x^3 \mid x \text{ is an element of } S\}.$$

Then we also have

$$T = \{1, 27, 125, 343, 729\}.$$

<u>Example 2</u>. The set of all fractions which are equal to $\frac{1}{4}$ may be written:

$$S = \left\{\frac{x}{y} \mid x \text{ and } y \text{ are integers, } y \neq 0, \text{ and } y = 4x\right\}.$$

<u>Example 3</u>. The set of all men who are members of the Masons could be designated:

$$S = \{x \mid x \text{ is a member of the Masons}\}.$$

Exercises for Section I. 2.

1. Use both: (a) "the method of listing all elements," and (b) "the stating of requirements method" to designate the following sets:

a. The odd integers less than 21

b. The vowels

c. The members of your mathematics class

d. The subjects you are enrolled in this semester.

2. Represent the following sets by one of the methods used in

exercise 1 and give reasons why the other method is difficult or impossible.

a. The set of all odd integers

b. The set of citizens of Nevada

c. The set consisting of all words in Webster's Dictionary

I. 3. Universal Set and Subsets.

In some instances we may wish to think of items belonging to a large set. In geometry, we might want to talk about properties of quadrilaterals, such as similarity or congruence. We could think of the set of all quadrilaterals as the universal set. From this set we could then select particular subsets, as: the set of all squares, the set of all rectangles, the set of all parallelograms, the set of all trapezoids having equal midlines, and so on.

Another example might be:

U = set of all babies born in the United States in 1956.

A = set of all boy babies born in United States in 1956.

B = set of all boy babies born in 1956 (in the United States) to parents having no previous children.

C = set of all sets of twins born in the United States in 1956.

In above example, U, is the universal set, and A, B, and C are subsets of U. Are there other possible subsets of U? of B? of C?

Exercise for Section I. 3.

1. Make up at least six further examples of universal sets and some of their subsets; taking cases such as sets of places, plants, objects, people, or ideas.

Inclusion is an important relationship between sets. If each element of a set A is also an element of a set B, then we say "A is contained in B" or "B contains A". These expressions are normally written:

$$A \subset B \text{ and } B \supset A.$$

In other words A is a subset of B.

Venn Diagram. Sometimes it is helpful to have a schematic representation of the universal set and its subsets. One such drawing is known as a Venn Diagram as in Fig. 2. The rectangle represents the Universal set U. The elements of U can be represented by the rectangle. Sets of elements of U can be shown by circles within the rectangle, as for example A and B in the figure. If every element of a set C is also an element of set A, then C is a subset of A. This is also represented in the diagram.

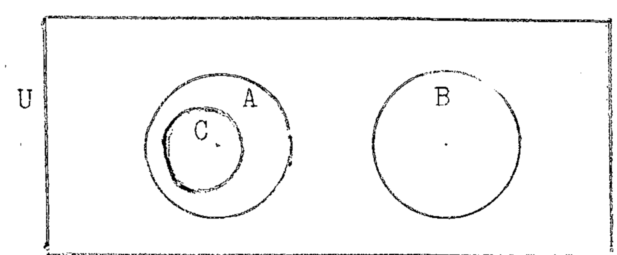

Fig. 2 Venn Diagram

Example.1. Let U be the set of all airplanes in the United States. Let A be the set of all military planes in the United States. Let B be the set of all B-52 bombers in the United States. Then B is a subset of A, and A is a subset of U. Is B a subset of U?

Finite and Infinite Sets. A set is said to be finite if it has some natural number of elements in it. All other sets are said to be infinite.

Example 2. (Finite sets) The set of all babies born in New York City on a given date; the set of all odd integers between 1 and 49 inclusive; the set of all girls in the Boy Scouts of America.

Example 3. (Infinite sets) The set of all fractions; the set of all rhombuses; the set $\{(x, y) \mid x \text{ and } y \text{ are real members and } y = x^2\}$.

Subsets of a Given Set. Suppose we start with a finite universal set U. If we know how many elements the set U contains, can we say how many different subsets U has? Our reply is "yes", and we will demonstrate how to find the number of such subsets. First we will show the case where $U = \{a, b, c\}$, just three elements. Then we will extend our reasoning to the case where U has any finite number n of elements.

Example 4. Let the universal set by $U = \{a, b, c\}$. We can select subsets of U in the following manner. We decide for each element of U whether or not we shall include it in our subset. If we decide to include it, we will show that fact by writing an x below it; and if we do not wish to include it we will place an O below it. Thus, the subset $A = \{a, c\}$ corresponds to

a	b	c
x	o	x

and the subset

$$B = \{a\}$$

corresponds to

a	b	c
x	o	o

To obtain all the possible subsets of U, we just write all possible arrangements of three zeros and/or x's as in the following table:

Case	U = {a,	b,	c}	Subset
1	x	x	x	$A_1 = \{a, b, c\}$
2	x	x	o	$A_2 = \{a, b\}$
3	x	o	o	$A_3 = \{a\}$
4	x	o	x	$A_4 = \{a, c\}$
5	o	x	x	$A_5 = \{b, c\}$
6	o	x	o	$A_6 = \{b\}$
7	o	o	x	$A_7 = \{c\}$
8	o	o	o	$A_8 =$?

Since there are exactly $2 \times 2 \times 2 = 2^3 = 8$ ways of filling the three spaces corresponding to the three elements of U, we have exhausted all possible cases. Let us discuss further, cases 1 and 8.

The subset "A_1" is identical with the universal set U. Should it be called a subset? It does satisfy the definition of subset -- namely: every element of A_1 is an element of U, so we shall say it is a subset of itself. Therefore, "sub" does not necessarily mean "smaller" in this context.

The subset "A_8" ended up with nothing. We describe this as the <u>empty</u> set, or the <u>null</u> set. Frequently it is denoted by the symbol $\emptyset$, which is read as "null". We shall agree that if S is any set whatever, then the null set is a subset of S. This makes it possible for us to write a very simple formula for the total number of subsets which can be made from the elements of any given finite set U.

> Theorem. Let U be a finite set containing n elements. Then there are 2^n different subsets of U (including U itself, as well as the null set).

The <u>proof</u> is left to the student. <u>Hint</u>: The number of subsets of U is equal to the number of ordered arrangements $(x_1, x_2, \ldots, x_n)$, where each x_i is either o or 1; $i = 1, 2, \ldots, n$.

Further Exercises for Section I. 3

2. Let U be the set of all positive integers smaller than 7. How many different subsets can you make from U; (a) in case at least one element must be included? (b) in case the null set is allowed?

3. How many non-empty subsets can be formed from a set of 5 elements? From a set of s elements?

4. A proper subset of U is defined to a non-empty subset which does not contain all the elements of U. How many proper subsets can be formed from a set of 5 elements? From a set of y elements?

5. Let set $U = \{R, S, T\}$. Let set $A = \{R, T\}$. Let set $B = \emptyset$. Tell which of the following statements are true and which false.

(a) U is a proper subset of U. (b) $B \not\subset A$ (c) $U \subset U$

(d) $B \subset U$ (e) $U \supset A$ (f) $B \subset A \subset U$

(g) $B \supset U \supset A$

I. 4. Operations with Sets.

Let U designate our universal set, and let A, B, C, ..., S be subsets of U. To deal with these sets, we must learn to deal with combinations of them, just as with numbers we deal with combinations which we call addition, difference, etc. Three important set operations are: union, intersection, and complementation. Let us define these terms, and illustrate each by means of drawings.

> Definition 1. The union of two sets, A and B, is the set of all elements of U which belong either to A or to B or to both.

We denote the union by $A \cup B$, which is read either "A union B" or "A cup B".

The union is designated by the shaded portion in the following diagram.

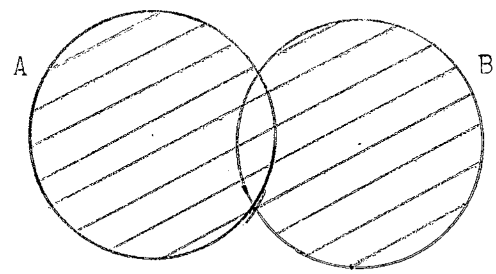

Fig. 3. Union of A and B

> Definition 2. The intersection of two sets is the set of all elements belonging to both of the sets.

We denote the intersection by $A \cap B$, which is read "A intersect B" or "A cap B".

The intersection is designated by the shaded portion in the diagram below.

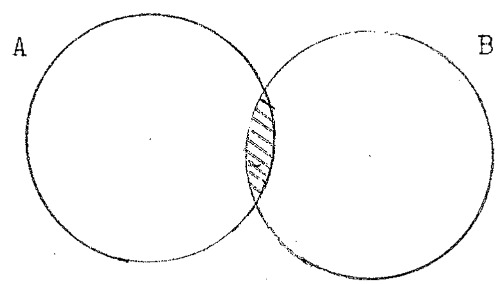

Fig. 4. Intersection of A and B

> Definition 3. Two sets S and T are said to be disjoint or mutually exclusive if they contain no common elements.

In symbols, S and T are disjoint if and only if $S \cap T = \emptyset$, that is, if their intersection is the null set.

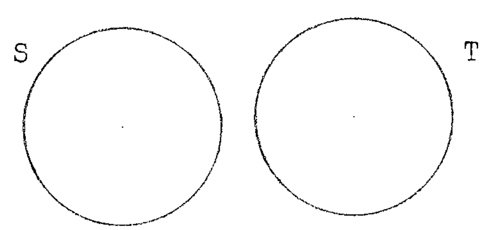

Fig. 5. Disjoint sets

Definition 4. The complement of A is the set of all elements in U which are not in A. We denote the complement of A by $\overline{A}$ which is read "A bar".

$\overline{A}$ is the shaded area in the diagram below.

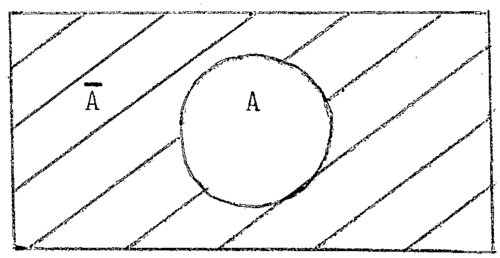

Fig. 6. Complement of a set

Example 1. Let U consist of the numbers 1, 2, 3, ..., 7 and the letters of the alphabet a, b, c, ..., m.

Let

$A = \{1, 2, 3, 5, a, c, f\}$

$B = \{1, 2, 3, 4, 5, a, b, c, d, g\}$

Then

$A \cap B = \{1, 2, 3, 5, a, c\}$

$A \cup B = \{1, 2, 3, 4, 5, a, b, c, d, f, g\}$

$\overline{A} = \{4, 6, 7, b, d, e, g, h, i, j, k, l, m\}$

$\overline{B} = \{6, 7, e, f, h, i, j, k, l, m\}$

$\overline{A} \cap B = \{4, b, d, g\}$

Exercises for Section I. 4

1. Let U be a universal set and S a subset of U. Show by Venn Diagrams that:

(a) $S \cup \overline{S} = U$ (c) $S \cup S = S$

(b) $S \cap \overline{S} = \emptyset$ (d) $S \cap S = S$

2. Let A, B, and C be subsets of a finite universal set U.

Suppose:

$U = \{x \mid x \text{ all integers less than } 100\}$

$B = \{x \mid x \text{ all odd integers less than } 67\}$

$C = \{x \mid x \text{ all even integers below } 85\}$

$A = \{x \mid x \text{ all integers less than } 51\}$

How many elements in each of the following sets?

(a) $\overline{A}$ (b) $A \cap B$ (c) $A \cup C$ (d) $\overline{C}$

(e) $B \cup C$ (f) $\overline{B}$ (g) $A \cap C$ (h) $B \cap C$

(j) $\overline{A} \cap \overline{B}$ (k) $\overline{B} \cup \overline{C}$ (l) $(A \cup \overline{B}) \cup C$

(m) $A \cup (B \cup C)$ (n) $A \cap (B \cap C)$

3. Let A and B be subsets of a universal set U. A and B are not disjoint. Show by a Venn Diagram that $(\overline{\overline{A} \cup \overline{B}}) = A \cap B$

4. $\overline{\overline{A}}$ is read "the complement of the complement of A". If A is a subset of a universal set U, does $\overline{\overline{A}} = A$?

5. If A denotes the set of all domestic cats over 100 years of age and B denotes the set of all women presidents of the United States, does $A = B$? Explain.

*I. 5. <u>Some Further Remarks on Sets</u>.

This section on Remarks is starred to denote information that is not developed to any great degree of completeness in this paper, but is herein given to illustrate some further concepts and to allow more coverage of set notions if so desired by the instructor.

1. In algebra we learn that for ordinary number a, b, and c, the commutative, associative, and distributive laws hold:

Commutative Law: $a + b = b + a$ $ab = ba$

Associative Law: $(a + b) + c = a + (b + c)$ $ab(c) = a(bc)$

Distributive Law: $a(b + c) = ab + ac$

Similar laws hold for sets A, B, and C, and the operations $\cup$ and $\cap$. They can be illustrated by Venn Diagrams.[2]

Commutative Law: $A \cup B = B \cup A$, $A \cap B = B \cap A$

Associative Law: $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. If S is a set with m elements, and T is a set with n elements, and S and T are disjoint, the $S \cup T$ has $m \neq n$ elements.

If a set S has m elements, and n is different from m, then S does not have n elements.

3. If A and B are sets, and $A \subset B$ and $B \subset A$, then $A = B$.

Some Further Exercises for Chapter I

For all the following problems let us say we have given the following sets:

A = {School, Church, Theater, Teacher, Pastor, Actor}

B = {Chair, Desk, Curtain, Pulpit}

C = {Teacher, Church, Actor}

D = {Theater, School, Actor, Pastor}

E = {Chair, Desk}

F = {Pulpit, Actor, Church}

G = {Church}

H = {Pastor, Actor}

J = {Teacher, Actor}

---

[2]The illustrating of the laws is left for the student.

1. Designate how many elements there should be after fulfilling the following set operations:

(a) $A \cap B = ?$ (b) $A \cup B = ?$ (c) $A \cap A = ?$

(d) $B \cup D = ?$ (e) $(G \cup A) \cap C = ?$ (f) $E \cup E = ?$

(g) $H \cup F = ?$ (h) $(D \cap A) \cup H = ?$ (j) $(C \cup C) \cap C = ?$

(k) $A \cup (B \cap C) = ?$ (l) $B \cap E = ?$

2. Answer the following questions:

(a) Does $D \cap H = A \cap C$?

(b) Does $J \cup G = C$?

(d) Is the statement true: $E \subset B$?

(e) Is it true that: $J \subset C$? $D \supset H$? $F \subset A$?

(f) Does $B \cup E = B$?

(g) Does $C \cap J = J$?

(h) Does $H \cap J = D \cap F$?

(j) Is $C \cap E = \emptyset$?

(k) Does $(B \cup C) \cup D = A \cup B$?

CHAPTER II

STATISTICS AND PROBABILITY

II. 1. What are Statistics and Probability?

Probability as defined by mathematicians is: "In the doctrine of chance, the likelihood of the occurrence of any particular form of an event, estimated as the ratio of the number of ways in which that form might occur to the whole number of ways in which the event might occur in any form (all such, elementary forms being assumed as equally probable.)"[1]

Statistics is the science of the collection and classification of facts on the basis of relative number of occurrences as a ground for induction. The facts thus collected are used to show the truth of laws inferred from the observed particular cases. It is demonstrated that if the law holds in a certain case it must hold in the next similar case, and therefore in the next, and so endlessly. This is referred to as mathematical induction.

Both the theory of probability, and the field of statistical inference are of extreme importance in today's scientific processes. Some of the basic ideas dealing with both will be presented in this chapter.[2]

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[1]William Allan Neilson, ed., Webster's New International Dictionary of the English Language (Springfield, Massachusetts, 1946), p. 1971.

[2]The data for this chapter is taken mainly from: Reginald Stevens Kimball, ed., Practical Mathematics (New York, 1945), II, pp. 506-562; Herbert Robbins, "The Theory of Probability," Insights Into Modern Mathematics (Washington, 1957), pp. 336-371; Introductory Probability and Statistical Inference for Secondary Schools (New York, 1957), pp. 1-130, 152-180; Merle W. Tate, Statistics in Education (New York, 1955), 1-525.

In the theory of probability, we deduce the probable composition of the sample from the composition of the original entire group from which it was taken. Statistics makes it possible for us to reverse this process so that we can infer the composition of the original entire group from the composition of a properly chosen sample.

Statistics is further concerned with the analysis and presentation of data, decisions forming, and experimental design.

II. 2. Putting Figures to Work.

Frequently statistical data must be organized in tables or graphs before the meaning becomes clear. These visual aids reduce the bulk of the data to a size that can be understood; and at the same time, make it easier to recognize similarities and differences within the presented data.

In this section of the chapter we shall discuss some of the methods of organizing and presenting statistical data.

One of the most common and simplest tables is the frequency distribution. The following illustrative problem will show what is meant by frequency distribution.

A random sample of 30 aluminum castings, when tested, yielded the following tensile strength in pounds per square inch to the nearest 100 pounds:

29,300	34,900	36,800	30,100	34,000
30,800	35,400	31,300	32,200	33,400
37,700	34,900	26,700	34,800	38,000
25,700	25,800	26,500	28,000	24,600
25,800	23,700	28,700	32,400	28,200
34,000	34,500	29,200	28,700	29,800

One description of the data could be a dot frequency diagram, which represents each of the measurements by a dot, the entire 30 dots giving a graphical description of how the values are distributed. Such a diagram arranges the measurements in an order from least to greatest. It further provides information about the grouping of the values as well as the extent of distribution. In this sample the measurements fall between 23,000 and 38,000, and they cluster arnoud 29,000 and 34,500.

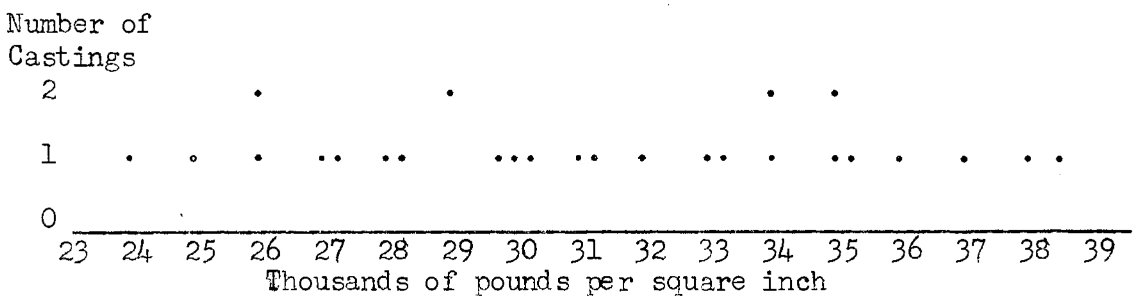


Fig. 7 Dot Frequency Diagram of Tensile Strength of 30 Aluminum Castings

In figure 8, the ordinate of the step-like graph for any given abscissa x gives on the left-hand scale the frequency F, and on the right-hand scale the percent p, of the castings having tensile strength less than or equal to that x. A comparison of the cumulative frequency and cumulative percent scales shows that the 30 measurements represent the whole sample, or 100%. Likewise a frequency $F = 6$ corresponds to a percent $p = 20$.

It is convenient for practical purposes to assign a value of x for every value of p. A rule for obtaining such a value of x is:

(1) pick any percent p on the cumulative percent scale; (2) draw a horizontal line until it intersects a jump in the step-line graph, or until it strikes one of the plotted points and;

(3) then draw a line vertically downward from the intersection or point to the x-axis.

The intersection with the x - axis is called the p - th percentile. Some special statistics' terms can now be explained. The 50th percentile is called the median of the measurements. The median tensile

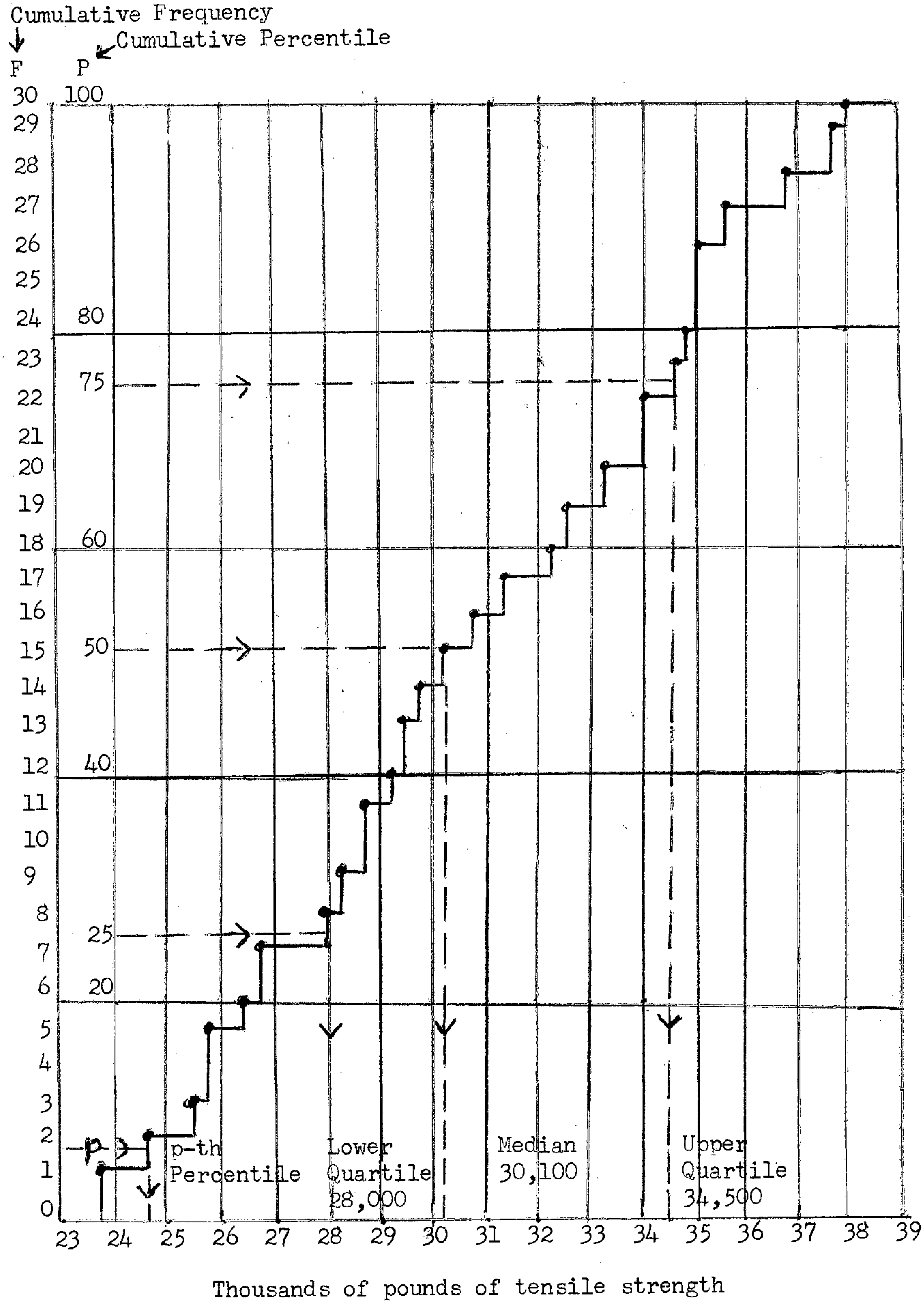


Fig. 8 Cumulative Graph of Tensile Strength of 30 Aluminum Castings

strength in the example in Fig. 8 is 30,100. The 25th percentile is usually called the lower quartile; and the 75th percentile is called the upper quartile. The lower quartile in Fig. 8 is 28,000 and the upper quartile is 34,500.

The difference between the largest and smallest measurements in a sample is called the range of the sample. In the example, the largest measurement is 38,000 pounds per square inch and the smallest is 23,700 pounds per square inch. The range in pounds per square inch is 38,000-23,700 = 14,300.

Exercises for Section II. 2

1. The following figures of production were reported by the foreman in a munitions factory:

404	401	399	401	402	400	403	399	399	403
403	402	401	404	407	403	400	403	401	403
401	405	405	403	407	398	401	400	401	406

Make a dot frequency diagram and a cumulative graph. What is the median production? The lower quartile? The upper quartile? The range?

2. The members of a rifle team made the following scores from the prone position:

80	82	89	86	83	79	91	90	86	75
87	93	88	83	90	89	93	78	81	84

Make a dot frequency diagram and a cumulative graph. What is the median score? The upper quartile? The lower quartile? The range?

3. The annual salaries of the office employees of a certain company were given as:

$2500	$3000	$1900	$2100	$2750
2350	2975	2475	2250	2500
2950	2825	2225	2875	2950
2350	2475	2250	2750	2500
2850	2925	2475	2100	1950

Make a dot frequency diagram and a cumulative graph. What is the median salary? The upper quartile? The lower quartile? The range?

II. 3. Other Graphical Presentations of the Frequency Distribution.

When the sample becomes fairly large, the procedures for constructing a dot frequency diagram or a cumulative graph become very tedious; therefore other means of presenting the data should be used. We can still show the main features of the sample by first grouping the measurements, and then constructing a frequency histogram instead of the dot frequency diagram, and a cumulative polygon instead of a cumulative graph.

Our first work is to construct a frequency table. To proceed we cut the x - axis into intervals of equal length. Try to choose the intervals so they will have a rather simple midpoint, and so the total number of intervals is somewhere between 10 and 25.

Returning to the table used in the previous section, it is found that the most convenient interval is one with length 1000 pounds and midpoints 24,000, 25,000, ..., 38,000 on the x - axis. This choice gives 15 intervals which include all of the measurements, the boundaries being 23,500, 24,500, ..., 38,500. Where a measurement falls on a point of division, we assign it to the interval lying at its left.

We will now set up columns (a) and (b) in Table 1. Column (c) is set up next as a convenience, if we do not have a dot frequency diagram. Column (d) is obtained by counting the tallies in column (c). The entries in column (e) are those of column (d) expressed as decimal fractions of the total of the frequencies. Column (f) shows the cumulative frequences obtained by the successive summation of column (d). The final entry in this column should equal the total sample 30. Column (g) is the successive summation of column (e). It should total 1.00 when completed.

The frequencies in Table 1 can be represented by what is called a frequency histogram as in Figure 9. We lose some information here because we record only the intervals in which a measurement falls rather than its exact value.

Table I

Frequency Distribution for Grouped Measurements of Tensile Strengths of 30 Aluminum Castings

(a) Internal boundaries	(b) Internal midpoints	(c) Tallied frequency	(d) Frequency	(f) Relative frequency	(f) Cumulative frequency	(g) Cumulative relative frequency
23500-24500	24,000	/	1	.0333	1	.0333
24500-25500	25,000	/	1	.0333	2	.0666
25500-26500	26,000	////	4	.1332	6	.1998
26500-27500	27,000	/	1	.0333	7	.2331
27500-28500	28,000	//	2	.0666	9	.2997
28500-29500	29,000	////	4	.1332	13	.4329
29500-30500	30,000	//	2	.0666	15	.4995
30500-31500	31,000	//	2	.0666	17	.5661
31500-32500	32,000	//	2	.0666	19	.6327
32500-33500	33,000	/	1	.0333	20	.6660
33500-34500	34,000	///	3	.0999	23	.7659
34500-35500	35,000	////	4	.1332	27	.8991
35500-36500	36,000		0	.0000	27	.8991
36500-37500	37,000	/	1	.0333	28	.9324
37500-38500	38,000	//	2	.0666	30	1.0000

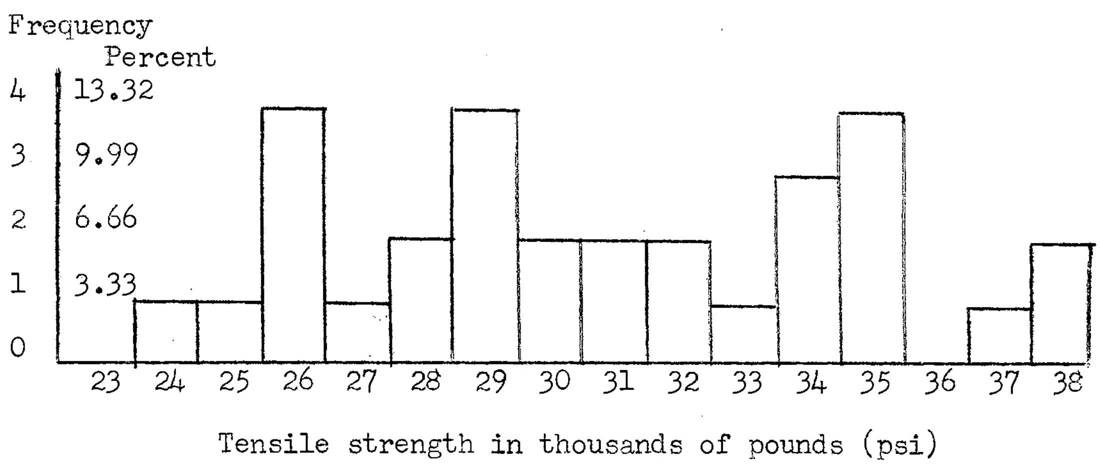
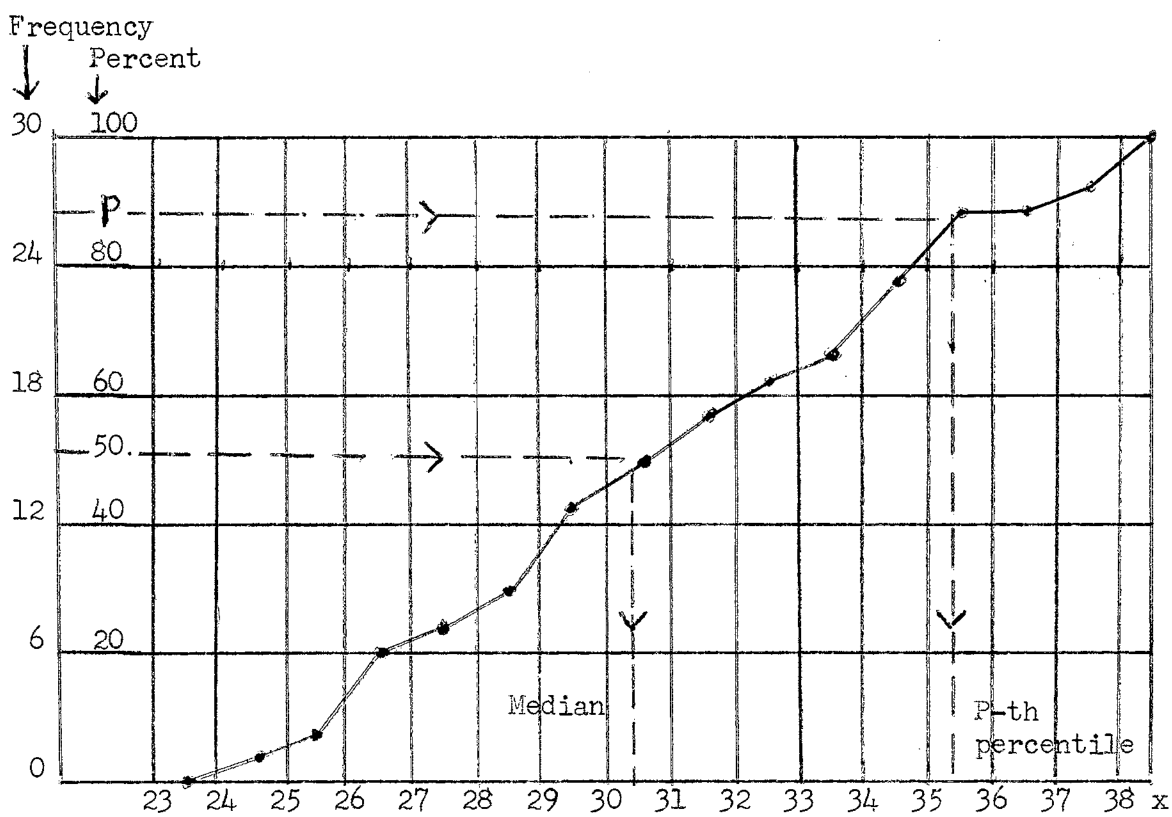


Fig. 9 Frequency Histogram of Frequencies in Table I

Another graphical way of presenting the data in Table I which is often useful is the cumulative polygon as shown in Fig. 10. This graph is obtained by first plotting points whose abscissas as the right-hand end points of the intervals in column (a), and whose ordinates are the cumulative frequencies in column (f). Then these points are connected by straight line segments. If we think of the right-hand end points of each interval as x and the cumulative frequency as y, we have 16 points: (23,500, 0),(24,500, 1),(25,500 2), (26,500 6), ..., (38,500, 30). The cumulative polygon is such that for any x, the corresponding y gives, approximately, the number, or percent, or measurements less than or equal to that x. For instance, suppose x = 35,5000, then 27 is the number of measurements less than or equal to 35,500, or 90 percent.



Tensile strength in thousands of pounds (psi)

Fig. 10 Cumulative Polygon for the Cumulative Frequencies in Table I

Exercises for Section II. 3

1. Suppose the test scores of 200 seniors of a certain school range from 245 to 780. What interval boundaries and interval mid-points would you set up for a frequency table? Make up columns (a) and (b) for such a table. (See Table I).

2. For one of the problems in Section II. 2 construct a frequency table, a histogram, and a cumulative polygon from the measurements given in the problem. Indicate the two quartiles and the median on the cumulative polygon.

II. 4. Statistical Inference Based on the Frequency Distribution.

Our analysis of the castings measurements has served to illustrate some general ways of summarizing data about a collection of measurements; however, we can further use the measurements from this random sample to make statistical inferences concerning the population of measurements from which the sample was drawn.

These 30 aluminum castings had been taken at random from the specified population, namely, all the aluminum castings produced at this plant on a certain day. We can use the properties of the sample to estimate properties of the population. The percentile derived from the frequency distribution of the sample are useful estimates of the corresponding percentiles of the complete frequency distribution of the output for the entire day. For example, we note that 75% of the sample had a tensile strength of at least 28,000 pounds per square inch. We could estimate that the entire population would roughly be likewise.

II. 5. The Mean and Standard Deviation.

We have noticed that the first step in the reduction and description of a long series of measurements is classifying the data in a frequency distribution and constructing the various diagrams. After thies is done, we can, by inspection, note points of similarity and difference. These comparisons tend to be inexact, and it is difficult to obtain agreement on them. As a rule, we use quantitative methods, rather than graphical methods, to compare frequency distributions. We shall now discuss two important descriptive statistics which are more suited to computation and interpretation: the arithmetic mean and the standard deviation. The first measures the magnitude and the latter measures the variability of a set of measurements.

The definition of the arithmetic mean $\bar{x}$ (read "x-bar") of a set of measurements is simply, the sum of the values in a set divided by the number of items. Using $x_1, x_2, x_3, \ldots, x_n$ to represent the values of the respective n items in a set, the definition may be written

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

It may be stated more simply as:

$$\bar{x} = \frac{\sum x}{n} \quad \text{or more precisely} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

which states explicitly that all of the items in the set are summed. The symbol $\sum$ always refers to sum in statistics. No new ideas are involved. The mean of n numbers is simply their average. Given four numbers 1, 5, 9, 13, the mean is:

$$\bar{x} = \frac{1 + 5 + 9 + 13}{4} = \frac{28}{4} = 7.$$

The mean is used widely as a statistic, so it will be worth emphasizing some of its properties.

From the definition of the mean, we have

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Therefore: $n\bar{x} = \sum_{i=1}^{n} x_i$.

Hence the sum of the measurements can be recovered by multiplying the mean by the number of measurements. Thus an employer would doubtlessly prefer the mean to the median as a measure of magnitude of salaries paid, since he can easily find out the total size of his payroll by multiplying the mean by the number of employees, whereas the median would seldom, if ever, present a true picture of total salary expense.

Example. Suppose salaries average $10 per week for a factory with 200 workers, and $110 per week for another factory with 40 workers. Then the total payroll in dollars for the two factories is 200 x 70 + 4- x 100 = 18,000. The mean salary in dollars for the workers in the two factories is 18,000/240 = 75.

More generally, suppose we have k sets of measurements, the sets having $n_1, n_2, \ldots, n_k$ measurements, and corresponding means $\bar{x}_1$, $\bar{x}_2$, $\ldots$, $\bar{x}_k$. Then the grand mean of all the $n_1 + n_2 + \ldots + n_k = n$ measurements is

$$\bar{x} = \frac{1}{n}(n_1 \bar{x}_1 + n_2 \bar{x}_2 + \ldots + n_k \bar{x}_k) = \frac{\sum n_i \bar{x}_i}{n},$$

where $\sum n_i \bar{x}_i$ is the sum over the k sets of the products $n_i \bar{x}_i$. This formula gives the weighted mean of k means.

Another important property of the mean can be demonstrated as follows: Consider a set of measurements 3, 5, 10. The mean is 6. If we subtract the mean from one of the measurements, we get the deviation of that item from the mean. Thus for our set, the deviations are:

$$\begin{aligned} 3 - 6 &= -3 \\ 5 - 6 &= -1 \\ 10 - 6 &= \underline{\;4\;} \\ \text{sum} &= \;\;0 \end{aligned}$$

Note that the sum of the deviations is 0.

In general, if x_i is a measurement from some set with mean $\bar{x}$, the deviation of x_i from $\bar{x}$ is defined as $x_i - \bar{x}$. Moreover, the sum of the deviations from $\bar{x}$ is always zero. For,

$$\sum_{i=1}^{n}(x_i - \bar{x}) = (x_1 - \bar{x}) + (x_2 - \bar{x}) + \dots + (x_n - \bar{x})$$
$$= x_1 + x_2 + \dots + x_n - n\bar{x}$$
$$= \sum x_i - \sum x_i = 0$$

We conclude that the mean typifies all the measurements since the sum of all the positive deviations equals the sum of all the negative deviations.

Again the mean of a random sample can be used to estimate the mean of the population from which it is drawn.

The <u>standard deviation</u> is the most used and important measure of variability (the scatter of the items about the average value). The best way to describe this statistic is to state the operations by which it is calculated. In order of operations it is found as follows:

a. Find the deviation of each value from the mean.
b. Square these deviations
c. Sum the squares
d. Divide the sum by n
e. Extract the square root of the quotient

To illustrate, let us find the standard deviation of a set of test scores.

Score x	Deviation from Mean $x - \bar{x}$ or d	Deviation from Mean squared or d^2
24	-5	25
22	-7	49
27	-2	4
32	3	9
40	11	121
$\sum x = 145$	$\sum d = 0$	$\sum d^2 = 208$
$\bar{x} = 29$		

The standard deviation of this set, in which n = 5, is the square root of 208/5. Thus, S. D. = $\sqrt{41.6}$ or = 6.45.

When we represent the deviation of an item from the mean of its set by d, we may define the standard deviation:

$$S.D. = \sqrt{\frac{\sum d^2}{n}}$$

If we square both sides of this equation we have $(S.D.)^2 = \sum d^2/n$. This is designated the variance, and the quantity $\sum d^2$ is designated the <u>sum of squares</u>. These are technical terms in statistics that are always defined as such. In other words, the sum of squares of a set is the sum of the deviations from the mean squared; the variance is the sum of squares divided by n; and the standard deviation is the square root of the variance.

Exercises for Section II. 5

1. The number of pupils in seven different general math classes are 22, 26, 37, 27, 28, 34, 36. Find the mean number of pupils per class, and compare it with the median number of students.

2. In a certain math class the following grades were made on a test: 3 students got 53, 4 -- 64, 4-- 69, 6 -- 75, 5 -- 78, 3 -- 84, 3 -- 93, and 2 -- 95. Find the mean grade of all the students. How does this compare with the median grade for the class?

3. A dairy farmer found that his herd of cows averaged 210 gallons per day in a certain week. His records for six days of the week show the following totals: 190, 205, 220, 200, 230, 204, but he lost his record for the other day. What must it have been?

4. a. We know that the mean of the measurements 3, 5, 10, is 6. If 5 is added to each of these measurements, what is the mean of the

new set of measurements?

b. Given that the numbers x_1, x_2, x_3, have mean $\bar{x}$; if a constant k is added to each number to form a new set of numbers, what will be the mean of the new set?

c. From parts (a) and (b) generalize on the mean of a new set formed by adding a constant k to each member of a set of n elements with mean $\bar{x}$.

5. a. Given the set of numbers 3, 5, 10, with mean 6, form a new set by multiplying each of the original elements by 7. Find the mean of the new set and relate it to the mean of the original set.

b. Given the set of elements x_1, x_2, ... x_n, with mean $\bar{x}$, form the new set $z_i = cx_i$, where $i = 1, 2, \ldots, n$, and show that $\bar{z} = c\bar{x}$.

6. a. Given the set of measurements 2, 7, 9, find the mean. Also find the mean of the new set obtained by multiplying each measurement by 5 and then adding 7 to the result. Relate the new and old means.

b. Given the set x_1, x_2, ..., x_n, with mean $\bar{x}$, form the new set $z_i = cx_i + k$, where $i = 1, 2, \ldots, n$, and then find the mean $\bar{z}$ of the new set.

7. a. What is the mean of the set of numbers: 1, 3, 5, 7, 9?

b. What is the mean of the set of numbers: 1, 9, 25, 49, 81?

c. Square your answer to problem 7a and compare it with your answer to problem 7b. Is the square of the mean of a set always equal to the mean of the squares?

8. If the foul-shooting averages of the five first string players of the school basketball team are 0.70, 0.25, 0.33, 0.65, and 0.66, in 30, 16, 27, 40, and 21 tries, respectively, what is the foul-shooting average for the first team?

9. Compute the variance and standard deviation of the measurements in problem 1.

10. Compute the standard deviation for the seven measurements in problem 3.

11. Given the measurements 6, 10, 12, 22, 25, compute the variance and the standard deviation. Then subtract 3 from each measurement and compute the mean and standard deviation of the new set.

12. Compute the variance of the three measurements 3, 5, 10. If 7 is added to each of these measurements, what is the variance of the new set?

II. 6. Experimenting with a Coin.

Let us consider the toss of a coin.[1] It is often said "the chances are 50-50 that it comes up heads". By this people mean that, if the coin were tossed say 100 times, we should expect 50 heads and 50 tails. But we do not expect exactly 50 heads in every 100 tosses: the number will vary slightly from one set of 100 tosses to another.

In the language of probability, we translate "50-50 chance for a head" as follows: "the probability of the outcome, the coin comes up heads, is $\frac{1}{2}$". Similarly, the probability for the outcome "tails" is $\frac{1}{2}$. We then say that the probability that a toss gives either "heads" or "tails" is $\frac{1}{2} + \frac{1}{2}$ or 1. While the probability that neither "heads" or "tails" comes is 0. In general, the numbers used for probabilities are positive or zero, and the sum of the probabilities of all mutually exclusive outcomes is unity.

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[1]Aaron Bakst, Mathematics Its Magic and Mastery (New York, 1952), pp. 329-353; gives a good discussion on probability in this chapter titled "How to Have Fun with Lady Luck".

To get the probability of an outcome we count the number of possible favorable outcomes and divide by the number of possible outcomes. We might write, in the case of a coin:

$$\frac{\text{number of favorable outcomes (heads)}}{\text{number of possible outcomes (heads or tails)}} = \frac{f}{n} = \frac{1}{2}$$

where f is the number of favorable outcomes and n the total number of possible outcomes.

II. 7. Experimenting with a Deck of Cards.

Suppose that we draw a card from a shuffled deck of bridge playing cards. What is the probability of drawing a spade? Since there are 13 spades out of the 52 cards in the deck, there are 13 "favorable outcomes." It follows that

$$\frac{f}{n} = \frac{13}{52} = \frac{1}{4}$$

Exercises for Section II. 7

1. The ordinary die has faces numbered 1 to 6. Such a die is thrown. What is the probability that an odd number appears on the top face? What is the probability that a number less than 5 turns up?

2. Suppose we draw a card from a shuffled deck of pinochle cards. What is the probability of drawing a spade? Of drawing an ace? Of drawing the 9 of spades?

3. What would be the probability of drawing a card of rank less than 10 in the previous exercise?

4. What would be the probability of drawing a black card from a bridge deck? Of drawing a diamond?

II. 8. Sample Spaces.

If we toss a coin, there are two possible outcomes which we can represent thus:

H, T,

Where H stands for "heads" and T for "tails". If we toss a single die, there are six possible outcomes which may be represented by a list of the numbers on the faces:

1, 2, 3, 4, 5, 6.

Such listings of all possible outcomes of an experiment are called sample spaces.

Suppose a bag contains a number of balls, alike in every way except that some are red (R), some yellow (Y), and some (G). We draw two balls from the bag without replacement. What is the sample space for this experiment? It would be a listing of all possible outcomes for the first and second ball drawn. Here is the list:

RR	RY	RG
YR	YY	YG
GR	GY	GG

Suppose that we toss two dice, one yellow and one blue, and that we note the numbers on the top faces. The yellow die has six possible outcomes. So does the blue die. If we list the number appearing on the top face of the yellow die by y, and that on the blue die by b, then the outcome of a single throw of the two dice can be represented by the ordered number-pair (y,b). How many such ordered number-pairs are there? Since there are 6 possible values for y and, for each value of y, 6 for b, there are 6 x 6 or 36 ordered number-pairs. The student should arrange these ordered number-pairs in a table.

The array shown in the table will be the sample space for the experiment of tossing two dice. It is also referred to as the universal set for the experiment. The ordered number-pairs (y, b) are elements of the sample space or of the universal set. They are called sample points, or briefly, points.

II. 9. Events and Their Probabilities.

A point or a set of points of the sample space is called an event. If we assume that each point in the sample space for the tossing of two dice has an equal chance of happening, then the probability of any one of these events is 1/36. If the event we are considering includes more than one point of the sample space, we find its probability as follows:

a) count the number of points in it to get f, the number of favorable outcomes; and
b) take f/n as the probability of the event where n is the number of points in the whole sample space.

For example, consider the event described by the phrase "the sum of the dots on the two dice is 7". This event contains the points: (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), and (1, 6) and thus $f = 6$, $n = 36$, and its probability is 6/36 or 1/6.

Notation: In the sample space for the two dice problems, the student has listed the points in ordered pair notation as (1, 2), (2, 3) and so on. Instead of the sentence "the probability of the event (2, 3) is 1/36", we write, in symbols

$$P((2, 3)) = \frac{1}{36},$$

If no chance for confusion is involved we use only one set of parentheses. The inner set was needed for the notation of an ordered pair, so if we denoted the ordered pair by a symbol e, we would use P (e) for the probability. Here the parenthesis correspond to the outer set in the original notation. We would now write:

$$P\ (2,\ 3) = \frac{1}{36}.$$

Using the sample space table for the two dice problems, and the concept of an event, we can answer some interesting probability questions by simple counting.

Example 1. What is the probability that one die gives a 5 and the other a 4?

Since the event contains the two points (5, 4) and (4, 5) we have

$$P\ ((5,\ 4),\ (4,\ 5)) = 2/36 = 1/18$$

Example 2. What is the probability that the yellow die gives a 3? Counting we see that there are 6 points with $y = 3$. Thus the probability is $6/36 = 1/6$.

Exercises for Section II. 9

1. In the two-dice problems, what is the probability that the yellow die gives a number less than 2, and the blue die a number greater than 3?

2. A man has a penny, a nickle, a dime, a quarter, and a half-dollar in his pocket. He takes two coins out of his pocket, one after the other. List the sample space. Assuming all ordered pairs are equally likely, what is the probability that both coins are silver? What is the probability that the coins total value is less than 40¢? Less than 20¢? More than 20¢? A prime number? A number divisible by 10?

3. Suppose you plan to make a survey of families having three children (single births). List an appropriate sample space. How many "points" does it have? How many of these correspond to families having

two boys and one girl? How many to families in which the first born is a boy? Suppose that each point in the sample space is given the same probability. What is the probability that the first two are girls and the third a boy? What is the probability that at least two are girls?

II. 10. Probability and Set Notions.

In the foregoing discussions, we can use the language of sets. An event A is a set of points in the sample space S. We define probability of the event P (A) as the sum of the probabilities of its points.

First we must identify the set of points involved. Since we are considering sample spaces of n points, where each probability is considered equal to $\frac{1}{n}$, we can obtain the probability of an event by counting the number of points in it and multiplying that number by $\frac{1}{n}$. From the yellow and blue dice sample space, we shall take examples.

Example: What is the probability that $y \leq 2$ or $b \leq 3$? in the two dice problem?

For the event $y \leq 2$ the yellow die has to show either 1 or 2. The corresponding set A consists of 12 points. For the event $b \leq 3$ the blue die has to show either 1 or 2 or 3. The corresponding set B consists of 18 points. But we cannot just add these numbers, because 6 points are in both sets, and should not be counted twice. Thus for the event $y \leq 2$ or $b \leq 3$ the count of different points is: $12 + 18 - 6 = 24$. Therefore the desired probability is $24/36 = 2/3$.

Note that in our calculation, 12 is the number of points in A, 18 those in B, and 6 both in A and B. If we had divided by 36, we would have had: $\frac{12}{36} + \frac{18}{36} - \frac{6}{36} = \frac{24}{36}$.

In the example we could have said:

$$P(A) + P(B) - P(A \text{ and } B) = P(A \text{ or } B).$$

II. 11. Mutually Exclusive Events.

Example: What is the probability that the throw of the dice gives a total of 6 or 11?

There are five sample points where the total is 6, and there are two where the total is 11. These sets have no common points, therefore the probability is 7/36.

When two events have no common points, they are called mutually exclusive, or disjoint. In set language, let A be the set of points for which the sum is 6 and B the set for which the sum is 11. Then we have (when the sets are disjoint)

$$P(A \text{ or } B) = P(A) + P(B) = \frac{5}{36} + \frac{2}{36} = \frac{7}{36}.$$

II. 12. Complementary Events.

An event A and the event which consists of all other points of the same sample space are called complementary events.

Any event A and its complement $\bar{A}$ together comprise the whole sample space. Therefore:

$$P\,(A \text{ or } \bar{A}) = 1.$$

Since A and $\bar{A}$ are disjoint, we can see that:

$$P\,(A \text{ or } \bar{A}) = P(A) + P(\bar{A}),$$

so we would know that:

$$P(A) = 1 - P(\bar{A}).$$

For the above reasons, complementary events are often used in arriving at the probability of any given event.

Example: What is the probability in the two-dice experiment that $y + b \neq 3$?

Let A be the event "the dots on the faces do not add to 3". Then $\bar{A}$ is the event "the dots add to 3". The set $\bar{A}$ consists of the two points (1, 2), (2, 1). Therefore:

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{2}{36} = \frac{17}{18}.$$

Exercises for Sections II. 10 to II. 12

1. In the two-dice experiment of Section II. 8, find the probability that:

a. The sum of the spots is not 9.

b. The two dice show only the numbers 2 or 5.

c. Neither 3 nor 5 appears.

d. Each die shows 3 or more spots.

e. At least one die shows fewer than 4 spots.

f. $y \leq 2 + b$.

g. $y \geq 3 + b$.

2. Five boys in a club want to select a committee of three. The boys' names are: Tom, Dick, Larry, Pete, and Lou. The committee is chosen "by lot" in such a way that all 10 possible committees are equally probable.

a. Set up a sample space of 10 points to represent the 10 possible committees.

b. What is the probability that Lou is on the committee? That Tom is not on the committee?

c. What is the probability that Lou is on the committee and Tom is not?

d. What is the probability that neither Lou nor Tom is on the committee?

e. What is the probability that Tom, Pete, and Lou are not all on the committee?

f. Suppose Tom and Pete are cousins, and that Dick and Lou are cousins. What is the probability that the committee has two cousins on it?

II. 13. Concluding Remarks on the Chapter.

This has been an intuitive discussion of probability and it is hoped it will make the student desire further work in this area. The author believes the more formal approach should be left for college study and is passing over it in favor of work in elementary calculus.

CHAPTER III

INTRODUCTORY CALCULUS

III. 1. Variables, and Functions.

A variable is a quantity to which an unlimited number of values can be assigned in an investigation; while a quantity whose value is fixed (does not change) in any investigation is called a constant.[1] In the geometric formula for the circumference of a circle $C = 2\pi r$, we call C and r variables, and 2 and π constants. As r varies so will the value of C change.

When two variables are so related that the value of the first variable is determined when the value of the second variable is known, then the first variable is said to be a function of the second. It might be said that a function is a correspondence that associates with each number of a given collection of numbers a unique number. For example, the formula $C = 2\pi r$ defines a function. To each positive real number r, there corresponds a unique positive real number C given by the formula.

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[1]This section on "Introductory Calculus", unless otherwise noted, is taken mainly from these sources: Reginald Stevens Kimball, ed., Practical Mathematics (New York, 1945), II, pp. 427-478; A. Albert Klaf, Calculus Refresher for Technical Men (New York, 1956), pp. 1-88, 163-216; William A. Granville, Percey F. Smith, and William R. Longley, Elements of the Differential and Integral Calculus (New York, 1941), 1-337; and Richard E. Johnson, syllabus for course in "Calculus for the High School Science and Mathematics Teacher", unpublished material used in class during fall semester of 1957-58 academic year, Institute of National Science Foundation at Oklahoma State University.

If $r = 3$, then $C = 6\pi$; if $r = 5$, then $C = 10\pi$; if $r = 7$, then $C = 14\pi$; and so on.

The linear equation $y = 5 - 3x$ defines a function that associates with each real number x, a real number y given by the equation. If $x = 0$, then $y = 5$; if $x = -4$, then $y = 17$; and so forth. The graph of this equation would give a straight line through the points (0, 5) and $(\frac{5}{3}, 0)$, with slope -3.

The collection (or set) of numbers over which a function is defined is called the domain of the function. For example, the domain of the function defined by the equation $C = 2\pi r$ is the set of all positive real numbers; while that of the function defined by the equation $y = 5 - 3x$ is the set of all real numbers.

Just as letters x, y, z, and so on are used for convenience to designate numbers, so it is convenient to designate functions by letters f, F, g, G, and so on. If f is used to designate a function, then f(x) read "f of x", is used to designate the number associated with the number x by the function f. That is, a function f associates with each number x in its domain a unique number f(x).

For example, if f is the function defined by the formula $A = \pi r^2$, then associated with each positive number r is the number f(r), where $f(r) = \pi r^2$. Thus, $f(2) = \pi \cdot 2^2 = 4\pi$, $f(4) = \pi \cdot 4^2 = 16\pi$, $f(\sqrt{7}) = \pi \cdot (\sqrt{7})^2 = 7\pi$, and so on.

Example 1. Let f be the function defined by the equation $f(x) = x^2 + x - 3$. The domain of f is the set of all real numbers; f associates with each real number x the real number $x^2 + x - 3$. Thus:

$$f(0) = 0^2 + 0 - 3 = -3; \quad f(1) = 1^2 + 1 - 3 = -1.$$

$$f(2) = 2^2 + 2 - 3 = 3; \quad f(-2) = -2^2 - 2 - 3 = -1.$$

Example 2. Let F be the function defined by the equation $F(y) = \sqrt{y - 5}$. Since $\sqrt{y - 5}$ is a real number if and only if $y \geq 5$, the domain of F is the set of all numbers $x \geq 5$. We have:

$$F(5) = \sqrt{5 - 5} = 0; \quad F(9) = \sqrt{9 - 5} = 2;$$

$$F(12) = \sqrt{12 - 5} = \sqrt{7}; \quad F(21) = 4.$$

III. 2. Polynomial Functions.

If for the function f and the number c, $f(x) = c$, for every real number x, then f is called a constant function. Thus, if $f(x) = 7$ for every x then f is a constant function.

If there exists some numbers a and b such that $f(x) = ax + b$ for every number x, the f is called a linear function. The function f defined by $f(x) = 3x + 5$ is a linear function.

A function F is called a quadratic function if $F(x) = ax^2 + bx + c$, $a \neq 0$, for some numbers a, b, and c, and every number x.

Likewise, if $g(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, g is called a cubic function and so on.

The constant, linear, quadratic, and cubic functions are special cases of polynomial functions. If there exists numbers a_0, a_1, a_2, ..., a_n such that:

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

for every real number x, then f is called a polynomial function. Therefore if the functions F, g, and G are defined by

$$F(x) = x^7 - x^5 + x^2 - 7,$$

$$g(x) = x^5 - x^4 + x^3 + x^2 - 1,$$

$$G(x) = x^3 - 9,$$

then F, g, and G are polynomial functions.

III. 3. Combining Functions.

We can combine functions by the operations of addition, subtraction, multiplication, and division, just as we combine numbers in arithmetic.

Given two functions f and g, the function F edfined by $F(x) = f(x) + g(x)$ is called the sum of f and g.

If $F(x) = f(x) - g(x)$, then F is called the difference of f and g. In each of these cases, the domain of F is the set of numbers in both the domain of f and that of g.

When $F(x) = f(x) \cdot g(x)$, the function F is called the product of f and g.

Similarly $F(x) = \frac{f(x)}{g(x)}$; F is called the quotient of f and g. Again the domain of the product or quotient function is the set of all numbers common to the domain of f and g, except that the numbers x such that $g(x) = 0$ must be excluded from the domain of the quotient.

Example: The function F defined by $F(x) = 5x^2 + 7x - 9$ is the sum of the function of f and g, where

$$f(x) = 5x^2, \; g(x) = 7x - 9.$$

Likewise, $f(x) = G(x) \cdot H(x)$, where $G(x) = 5$, $H(x) = x^2$;

and $g(x) = D(x) - E(x)$, where $D(x) = 7x$, $E(x) = 9$.

It is obvious that H is the product of two functions

$$H(x) = I(x) \cdot I(x), \text{ where } I(x) = x,$$

and $D(x) = 7I(x)$.

The function I such that $I(x) = x$ for every number x is called the identity function.

When a function is defined as a quotient

$$F(x) = \frac{f(x)}{g(x)}$$

where f and g are polynomial functions, then F is called a rational

<u>function</u>. Thus, if F is defined by

$$F(x) = \frac{5x^2 + 3}{x + 4},$$

then F is a rational function.

The function F defined by: $F(x) = \sqrt{3x + 5}$ illustrates another way of combining functions. If $g(x) = 3x + 5$ then $F(x) = \sqrt{g(x)}$. Or, if we let $f(x) = \sqrt{x}$, then $f(g(x)) = \sqrt{g(x)}$, that is, $F(x) = f(g(x))$. In this example, the domain of f is the set of all non-negative numbers; that of g the set of all numbers; and that of F the set of all numbers $x \geq -5/3$.

If f and g are functions and if $F(x) = f(g(x))$ for every x such that $g(x)$ is in the domain of f, then the function F is called the <u>composite</u> of f and g.

<u>Example</u>: If $f(x) = x^6$, $g(x) = x^2 + 3$, then the composite F of f by g is defined by:

$$F(x) = f(g(x)) = [g(x)]^6 = (x^2 + 3)^6.$$

Also, the composite G of g by f would be:

$$G(x) = g(f(x)) = (x^6)^2 + 3 = x^{12} + 3.$$

Observe that F and G are different functions. The composite K of g and g would be:

$$K(x) = g(g(x)) = (x^2 + 3)^2 + 1 = x^4 + 6x^2 + 10.$$

Exercises for Section III. 1 -- III. 3.

If $f(x) = x^2 - 5x + 4$, find:

1. $f(0)$; $f(3)$; $f(-2)$; $f(-\sqrt{5})$; $f(a)$
2. $f(1 + h)$; $f(x + h)$; $f(-3 + h)$
3. $\frac{f(1 + h) - f(1)}{h}$, $h \neq 0$; $\frac{f(x + h) - f(x)}{h}$, $h \neq 0$

If $F(x) = \dfrac{x - 2}{x + 2}$, $x \neq -2$, find:

4. $F(0)$; $F(1)$; $F(2)$; $F(a)$; $F(y)$

5. $F(1 + b)$; $F(x + h)$

6. $\dfrac{F(1 + h) - F(1)}{h}$, $h \neq 0$; $\dfrac{F(x + h) - F(x)}{h}$, $h \neq 0$.

7. $\dfrac{F(5) - F(3)}{3}$; $\dfrac{F(x) - F(3)}{x - 3}$, $x \neq 3$.

If $f(x) = 3x^2 - 7x$, find the equation of the line through the following two points:

8. $(0, f(0))$ and $(2, f(2))$

9. $(3, f(3))$ and $(3 + h, f(3 + h))$, $h \neq 0$

10. $(a, f(a))$ and $(a + h, f(a + h))$, $h \neq 0$

If $f(x) = x^2 - 4$ and $g(x) = 5x + 1$, define:

11. (a) The sum of f and g; (b) The difference of f and g.

12. (a) The product of f and g; (b) The quotient of f by g.

13. (a) The composite of f by g; (b) The composite of g by f.

14. If $f(x) = x^3$, find a function g such that $f(g(x)) = x$. Is $g(f(x)) = x$ also?

15. If $f(x) = \sqrt{x}$, find a function g such that $f(g(x)) = x$. Is $g(f(x)) = x$ also?

III. 4. Average Rate of Change of a Function.

You found out in algebra how to write the equation of a line which passes through two given points. You know that the slope m of the line is the difference between the y values divided by the difference between the x values of the coordinates (points).

In finding the equation of a line through the points (h, k) and (h', k'), you find the slope m of the line is $\dfrac{k - k'}{h - h'}$. Then the equation of the line is $y - k = \dfrac{k - k'}{h - h'}(x - h)$.

It is convenient to use the Greek letter delta (Δ) to indicate "Difference between values of". Thus Δy means "difference between values of y", that is, $k - k'$, and Δx means "difference between values of x", that is, $h - h'$. Using this notation we have $m = \frac{\Delta y}{\Delta x}$. That is, the slope of the straight line through the points (h, k) and (h', k') is the rate of change in the value of y.

Example: In the function $y = 3x^2 - 2x + 6$, find the average rate of change iny as x increases from -5 to -3.

$$\text{When } x = -5,\ y = 3(-5)^2 - 2(-5) + 6 = 91$$

$$\text{When } x = -3,\ y = 3(-3)^2 - 2(-3) + 6 = 39$$

$$\Delta x = -3 - (-5) = 2 \quad \Delta y = 39 - 91 = -52$$

$$\text{Then } \frac{\Delta y}{\Delta x} = m = -26$$

The average rate of change of the function is therefore negative. This means that, as the value of x increases from -5 to -3, the value of y decreases.

In general, the average rate of change of a function f between x and x_1 can be defined as:

$$\frac{f(x_1) - f(x)}{x_1 - x}$$

Example: If $f(x) = x^2 - 5$ find the average rate of change of f:
(a) between $x = 2$ and $x = 5$; (b) between x and $x + h$.

Solving we would have:

$$\frac{f(5) - f(2)}{5 - 2} = \frac{(25 - 5) - (4 - 5)}{3} = 7$$

$$\frac{f(x + h) - f(x)}{(x - h) - x} = \frac{[(x + h)^2 - 5] - [x^2 - 5]}{h} = 2x + h$$

We can check (a) by using (b). Thus if $x = 2$ and $h = 3$, we get 7.

Exercises for Section III. 4

1. Find the average rate of increase in the circumference of a circle as the radius change from 5 in. to 10 in.

In the following functions find $\frac{\Delta y}{\Delta x}$:

2. $y = x^2 - 5x + 6$

   a. from $x = 2$ to $x = 4$ b. from $x = 0$ to $x = 3$

3. $f(x) = x^3 - 4$

   a. from $x = 0$ to $x = 2$ b. from $x = 4$ to $x = 6$

4. $g(x) = \frac{2}{3} x - 4$

   a. from $x = 3$ to $x = 6$ b. from $x = 9$ to $x = 15$

5. $f(x) = 2 + x - x^2$

   a. from $x = 2$ to $x = 4$ b. from $x = a$ to $x = a + h$

6. Find the average rate of change of the force of attraction of two particles between $d = 10$ cm and $d = 12$ cm.

III. 5. Limits.

The idea of a variable approaching a limit occurs in geometry in extablishing a formula for the area of a circle. The area of the regular inscribed polygon with any number of sides n is considered, and n is then assumed to increase indefinitely. The variable area then approaches a limit, and this limit is defined as the area of the circle. In this case the variable v(the area) increases constantly, and difference $a - v$, where a is the area of the circle, diminishes and finally becomes less than any preassigned number, however small.

<u>Definition</u>. If a variable x approaches more and more closely a constant value c, so that $c - x$ eventually becomes and remains less, in absolute value, than any preassigned positive number, however small,

the constant c is said to be the limit of x.

Example 1. Let the values of x be $2 + 1$, $2 + \frac{1}{2}$, $2 + \frac{1}{4}$, ..., $2 + \frac{1}{2n}$, ..., without end. Then, obviously, limit $x = 2$, or x 2.

If we mark on a straight line the point P corresponding to the limit 1, and from P lay off on each side a length ε, however small, then the points determined by x will finally all lie within the segment corresponding to the interval $[1 - \varepsilon, 1 + \varepsilon]$.

Example 2. Where $f(x) = 2x + 5$, it is obvious that if x is close to -2, then f(x) is close to 1. Thus, $f(-1.9) = 1.2$, $f(-2.01) = -.98$, and so on. Since f(x) is close to 1 when x is close to -2, we shall say that the limit of f(x) as x approaches -2 equals 1, and write

$$\lim_{x \to -2} f(x) = 1$$

In both cases thus far the limits have been rather obvious; but this is not always the case as further examples will show.

Example 3. Find: $\displaystyle \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

We can not guess the value of this problem by letting $x = 9$, since the fraction $(\sqrt{9} - 3)/(9 - 9)$ is not defined. But, we can change the form of the quotient as follows:

$$\frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \frac{1}{\sqrt{x} + 3}, \quad x \neq 9.$$

Now it is evident that $\sqrt{x} + 3$ is close to 6 when x is close to 9, and

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6}.$$

In evaluating $\lim_{x \to a} f(x)$, f(a) need not be defined in order for the limit to exist. The value of the limit is a number b that f(x) is close to when x is close to (but not equal to) a. Even if f(a) is defined, the limit of f(x) as x approaches a might be different from f(a).

Example 4. Find $\lim_{x \to -3} \dfrac{\frac{1}{x} + \frac{1}{3}}{x + 3}$

Again, the limit can not be found by just letting x = -3. We, therefore, simplify as follows:

$$\frac{\frac{1}{x} + \frac{1}{3}}{x + 3} = \frac{\frac{3 + x}{3x}}{x + 3} = \frac{1}{3x}, \quad x \neq -3$$

then

$$\lim_{x \to -3} \frac{\frac{1}{x} + \frac{1}{3}}{x + 3} = \lim_{x \to -3} \frac{1}{3x} = -\frac{1}{9}$$

since $\frac{1}{3x}$ is close to $-\frac{1}{9}$ when x is close to -3.

Exercises for Section III. 5.

1. $\lim_{x \to 2} \dfrac{x^2 - 9}{x + 2}$

2. $\lim_{x \to 2} x^2 + 4x$

3. $\lim_{x \to 2} \dfrac{x^2 - 4}{x + 2}$

4. $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

5. $\lim_{x \to -2} \dfrac{x^2 - 4}{x + 2}$

6. $\lim_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

7. $\lim_{y \to 1} \dfrac{y - 1}{y^2 - 1}$

8. $\lim_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

9. $\lim_{h \to 0} \dfrac{(x + h)^3 - x^3}{h}$

10. If $f(x) = x^2 - 2x + 3$, find:
(a) $\lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$
(b) $\lim_{h \to 0} \dfrac{f(1 + h) - f(1)}{h}$

11. If $F(x) = 25 - x^2$, find:
$\lim_{x \to 4} \dfrac{F(x) - F(4)}{x - 4}$

12. Find the average rate of change, f(h), of the surface area of a sphere from r = 3 to r = 3 + h. Then find $\lim_{h \to 0} f(h)$.

III. 6. Continuous and Discontinuous Functions.

In many cases it is noted that the limit of f(x) as x approaches a is just f(a). Thus if $f(x) = x^2 + 4x$,

then: $\lim_{x \to 2} (x^2 + 4x) = 12,$

we see that the answer is the value of the function for $x = 2$. Here the limiting value of the function when x approaches 2 as a limit is equal to the value of the function for $x = 2$. The function is said to be <u>continuous</u> for $x = 2$. The definition is:

> A function $f(x)$ is said to be continuous for $x = a$ if the limiting value of the function when x approaches a as a limit is the value assigned to the function for $x = a$. In symbols, if
> $$\lim_{x \to a} f(x) = f(a)$$
> then $f(x)$ is continuous for $x = a$.

The function is said to be <u>discontinuous</u> for $x = a$ if this condition is not satisfied.

Consider the function $f(x) = \frac{x^2 - 4}{x - 2}$. For $x = 1$, $f(x) = f(1) = 3$. Moreover, as x approaches 1 as a limit, the function $f(x)$ approaches 3 as a limit. Hence the function is continuous for $x = 1$.

However the function is not defined for $x = 2$ (since then there would be division by zero). But for every other value of x,

$$\frac{x^2 - 4}{x - 2} = x \neq 2; \quad \text{and} \lim_{x \to 2} (x \neq 2) = 4;$$

therefore $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.

Although the function is not defined for $x = 2$, if we arbitrarily assign to it the value 4 for $x = 2$, it becomes continuous for this value. That is, if $f(x)$ is not defined for $x = a$ and if $\lim_{x \to a} f(x) = B$, then $f(x)$ will be continuous for $x = a$, if B is assumed as the value of $f(x)$ for $x = a$. A function $f(x)$ is said to be continuous in an interval when it is continuous for all values of x in this interval.

III. 7. <u>The Limit Theorems</u>.

There are fundamental limit theorems that state the limits of various combinations of functions. The proofs of these theorems are left for the student. The theorems are as follows:

If the limits of f(x) and g(x) as x approaches a exist, then:

(a) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(b) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(c) $\lim\limits_{x \to a} [f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

(d) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ if $\lim\limits_{x \to a} g(x) \neq 0$

(e) $\lim\limits_{x \to a} [f(x)]^n = [\lim\limits_{x \to a} f(x)]^n$, n a positive integer.

(f) $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$, n a positive integer.

Example 1. Use the limit theorems in evaluating:

$$\lim_{x \to 3} (x^2 + 7x - 5) = \lim_{x \to 3} x^2 + \lim_{x \to 3} 7x + \lim_{x \to 3} -5$$

$$= \lim_{x \to 3} x \cdot \lim_{x \to 3} x + 7 \lim_{x \to 3} x - 5$$

$$= 3 \cdot 3 + 7 \cdot 3 - 5 = 25$$

If we let $f(x) = x^2 + 7x - 5$, then $f(3) = 25$. Since $\lim\limits_{x \to 3} f(x) = f(3)$, f is a continuous function at 3.

Example 2. Use the limit theorems in finding:

$$\lim_{x \to -3} \frac{x + 1}{x - 1} = \frac{\lim\limits_{x \to -3} x + 1}{\lim\limits_{x \to -3} x - 1} = \frac{\lim\limits_{x \to -3} x + \lim\limits_{x \to -3} 1}{\lim\limits_{x \to -3} x - \lim\limits_{x \to -3} 1} =$$

$$\frac{-3 + 1}{-3 - 1} = \frac{-2}{-4} = \frac{1}{2}$$

III. 8. Infinity (∞).

If the numerical value of a variable x ultimately becomes and remains greater than any preassigned positive number, however large, we say x becomes <u>infinite</u>. If x takes on only positive values, it becomes positively infinite; if negative values only, it becomes negatively infinite. In these cases x does not approach a limit as previously defined. The notation limit $x = \infty$ or $x \to \infty$, must be read "x becomes infinite", not "x approaches infinity". Infinity is not a limit, since infinity is not a number at all.

We can now write, for example,

$\lim\limits_{x \to 0} \frac{1}{x} = \infty$, meaning that $\frac{1}{x}$ becomes infinite when x approaches zero.

If $\lim\limits_{x \to a} f(x) = \infty$, that is, if f(x) becomes infinite as x approaches a as a limit, then f(x) is discontinuous for $x = a$.

A function may have a limiting value when the independent variable becomes infinite. For example, $\lim\limits_{x \to \infty} \frac{1}{x} = 0$

In general if f(x) approaches a constant value A as a limit when $x \to \infty$ we write: $\lim\limits_{x \to \infty} f(x) = A$

Certain special limits occur frequently. The constant c is not zero in these given below.

(a) $\lim\limits_{x \to 0} \frac{c}{x} = \infty$ (b) $\lim\limits_{x \to \infty} cx = \infty$

(c) $\lim\limits_{x \to \infty} \frac{x}{c} = \infty$ (d) $\lim\limits_{x \to \infty} \frac{c}{x} = 0$

These special limits are useful in finding the limits of the quotient of two polynomials when the variable becomes infinite.

<u>Example</u>: Find: $\lim\limits_{x \to \infty} \frac{2x^3 - 3x^2 + 4}{5x - x^2 - 7x^3}$

Divide numerator and denominator by x^3, the highest power of x present in either. Then we have: $\lim\limits_{x \to \infty} \frac{2 - \frac{3}{x} + \frac{4}{x^3}}{\frac{5}{x^2} - \frac{1}{x} - 7} = -\frac{2}{7}$

since the limit of each term in numerator or denominator containing x is 0.

Exercises for Section III. 6 -- III. 8.

1. Find: $\lim\limits_{x \to \infty} \frac{5 - 2x^2}{3x + 5x^2}$

2. Find: $\lim\limits_{x \to 0} \frac{4x^2 + 3x + 2}{x^3 + 2x - 6}$

3. Find: $\lim\limits_{x \to a} \frac{x^4 - a^4}{x^2 - a^2}$

4. Find: $\lim\limits_{y \to 2} \frac{y^2 + y - 6}{y^2 - 4}$

5. Given: $f(x) = x^2$, find

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

6. Given: $f(x) = ax^2 + bx + c$, find

$$\lim_{b \to 0} \frac{f(x + b) - f(x)}{b}$$

III. 9. Derivatives.

The instantaneous rate of change of a function is of sufficient importance in the calculus to warrant a special name. It is called the derivative of the function. The derivative of a function f at a number a, designated $f'(a)$, is defined as:

$$\text{A)} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if the limit exists. If the limit does not exist, f does not have a derivative at a.

The domain of f' is the set of all numbers a such that $f'(a)$ exists.

An alternate definition of the derivative, which is usually simpler to apply is:

$$\text{B)} \quad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Example 1. Find f', if the function f is defined by $f(x) = 3x^2 - 2x + 6$

$$f'(x) = \lim_{h \to 0} \frac{[3(x + h)^2 - 2(x + h) + 6] - [3x^2 - 2x + 6]}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h + 6 - 3x^2 + 2x - 6}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}$$

$$= \lim_{h \to 0} (6x + 3h - 2) = 6x - 2$$

Example 2. Find g', if the function g is defined by $g(x) = x^2 - 7$

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} \frac{[(x + h)^2 - 7] - [x^2 - 7]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 7 - x^2 + 7}{h} = \lim_{h \to 0} 2x + h = 2x$$

Exercises for Section III. 9.

Use definition A to find the derivative of each of the following functions.

1. $f(x) = 2 - 3x$
2. $f(x) = 3x^2 - x^3$
3. $g(x) = x^2 + \frac{3}{x}$

Use definition B to find the derivative of each of the following functions.

4. $f(x) = 4x^2 + 2x^3$
5. $f(x) = 3x^2 - 5$
6. $f(x) = \frac{c}{x^2}$

Find the derivative of each of the following by applying either definition A or B.

7. $f(x) = x^3$
8. $g(x) = x$
9. $f(x) = \frac{1}{x}$
10. $g(x) = \frac{1}{x^3}$

11. $G(x) = \frac{1}{\sqrt{x}}$ 12. $F(x) = \frac{1}{x^4}$

13. $f(t) = \frac{1}{t\sqrt{t}}$

III. 10. Formulas for Differentiation.

The process of finding the derivative of a function is called differentiation. Some basic formulas will be given in this section.

Sometimes it is convenient to use another notation for the derivative of a function. Thus, $f'(x)$ will also be designated $D_x f$ read "the derivative of the function f at the number x."

If each of the functions f and g has a derivative at x, then so does the function $f + g$, and

I. $D_x(f + g) = D_x f + D_x g$

That is, the derivative of a sum is equal to the sum of the derivatives.

Proof: Recall that the sum of the functions f and g is the function $f + g$ such that $(f + g)(x) = f(x) + g(x)$. Then $D_x(f + g)$

$$= \lim_{h \to 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{[f(x + h) - f(x)] + [g(x + h) - g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h}$$

$$= D_x f + D_x g$$

Similarly: II. $D_x(f - g) = D_x f - D_x g$

What is the derivative of a constant? The derivative of a constant is zero. That is: III. $D_x c = 0$, since c does not change while the variable grows.

Example. If $f(x) = 6$, then $f' = 0$

The identity function I (such that $I(x) = x$ for every x) may be differentiated thus:

$$D_x I = \lim_{h \to 0} \frac{I(x + h) - I(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x + h) - x}{h}$$

$$= \lim_{h \to 0} 1 = 1$$

That is: IV. $D_x I = 1$

By this an immediate further rule is that: $D_x I^n = nx^{n-1}$ for every integer n. This can be written: V. $D_x x^n = nx^{n-1}$

<u>Example</u>. Find $D_x(x^5) = 5x^4$

Three other formulas are: The derivative of the product of two functions is the first function times the derivative of the second one plus the second function times the derivative of the first one.
VI. $D_x f \cdot g = f(x)D_x g + g(x)D_x f$.

The derivative of the quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the square of the denominator. As a formula:

$$\text{VII.} \quad D_x \frac{f}{g} = \frac{g(x)D_x f - f(x)D_x g}{[g(x)]^2}$$

A special case of formula VI occurs if f and g are equal. Then we get

$$D_x f \cdot f = f(x)D_x f + f(x)D_x f = 2f(x)D_x f.$$

We shall designate the function $f \cdot f$ as f^2. More generally, we designate by f^n the n the power of the function f for any integer n. Thus, the function f^n is defined by $f^n(x) = [f(x)]^n$.

We saw that $D_x f^2 = 2f(x)D_x f$. Then,

$$D_x f^3 = D_x f^2 \cdot f$$
$$= f^2(x)D_x f + f(x)D_x f^2$$
$$= f^2(x)D_x f + f(x) \cdot 2f(x)D_x f$$
$$= 3f^2(x)D_x f$$

Whenever it is true that:

$$D_x f^{n-1} = (n - 1)f^{n-2}(x)D_x f$$

for the positive integer n, then:

$$D_x f^n = D_x f^{n-1} \cdot f$$
$$= f^{n-1}(x)D_x f + f(x)D_x f^{n-1}$$
$$= f^{n-1}(x)D_x f + f(x) \cdot (n - 1)f^{n-2}(x)D_x f$$
$$= f^{n-1}(x)D_x f + (n - 1)f^{n-1}(x)D_x f$$
$$= nf^{n-1}(x)D_x f$$

Thence: VIII. $D_x f^n = nf^{n-1}(x)D_x f$.

<u>Example 1</u>. Find: $D_x(x^2 - 5)$

$$D_x(x^2 - 5) = D_x x^2 - D_x 5$$
$$= 2x$$

<u>Example 2</u>. Find: $D_x(7x^4 + 5x^3 - 3x^2 + 7x - 1)$

$$D_x(7x^4 + 5x^3 - 3x^2 + 7x - 1) = D_x 7x^4 + D_x 5x^3 - D_x 3x^2 + D_x 7x - D_x 1$$
$$= 7D_x x^4 + 5D_x x^3 - 3D_x x2 + 7D_x x - D_x 1$$
$$= 7 \cdot 4x^3 + 5 \cdot 3x^2 - 3 \cdot 2x + 7 \cdot 1 - 0$$
$$= 28x^3 + 15x^2 - 6x + 7$$

<u>Example 3</u>. Find: $D_x(5 - x^2)^6$

$$D_x(5 - x^2)^6 = 6(5 - x^2)^5 D_x(5 - x^2)$$
$$= 6(5 - x^2)^5 [D_x 5 - D_x x^2]$$
$$= 6(5 - x^2)^5 [0 - 2x]$$
$$= -12x(5 - x^2)^5$$

Exercises for Section III. 10

Find each of the following derivatives.

1. $D_x(7x^6 - 5x^5 + 3x^3 - x + 9)$

2. $D_y(y^2 - \frac{1}{y^2})$

3. $D_x(3x + 1)^3$

4. $D_x \frac{(x^2 - 2x + 2)}{x - 1}$

5. $D_y(y^{27} - 7y^6)$

6. $D_x\left(\frac{x - 1}{x^2 - 2x + 2}\right)$

III. 11. Higher Derivatives.

The derivative f' of a function f is called the <u>first derivative</u>. In turn, the derivative of f' is designated f'' and is called the <u>second derivative</u> of f. Likewise f''' is the <u>third derivative</u> of f and so on. These could be denoted by $D_x{}^2f$, $D_x{}^3f$, and so on.

<u>Example</u>: If $f(x) = x^4 - 5x^2 + \frac{1}{x}$, find the first four derivatives of f.

We have

$$D_x(x^4 - 5x^2 + x^{-1}) = 4x^3 - 10x - x^{-2}$$

$$\begin{aligned} D_x{}^2(x^4 - 5x^2 + x^{-1}) &= D_x(4x^3 - 10x - x^{-2}) \\ &= 12x^2 - 10 + 2x^{-3} \\ D_x{}^3(x^4 - 5x^2 + x^{-1}) &= D_x(12x^2 - 10 + 2x^{-3}) \\ &= 24x - 6x^{-4} \\ D_x{}^4(x^4 - 5x^2 + x^{-1}) &= D_x(24x - 6x^{-4}) \\ &= 24 + 24x^{-5} \\ &= 24 + \frac{24}{x^5} \end{aligned}$$

Exercises for Section III. 11

Find the first three derivatives of each of the following functions.

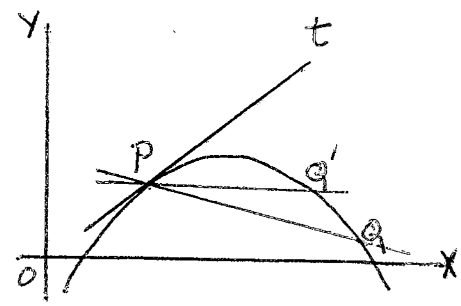
1. $g(y) = y^3 - \frac{1}{y}$

2. $f(x) = x^2 + \frac{2}{x^2}$

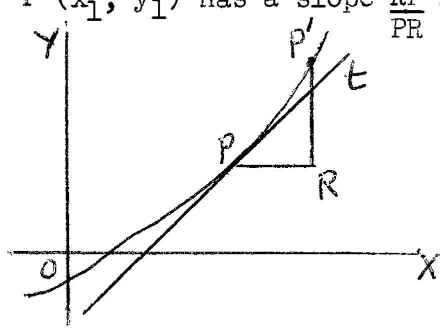
3. $g(x) = x^7 - 6x^5 + 3x^2 - 5$ 4. $G(y) = y^7 - y^4 + y^3 - y$

5. $f(x) = \frac{2x - 1}{2x + 1}$

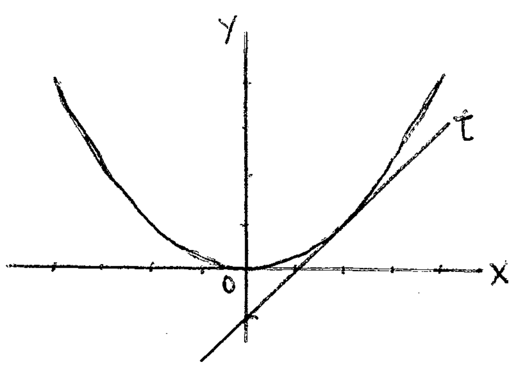
III. 12. Geometric Applications of the Derivative.



Consider a secant which intersects a curve in two points P and Q, and which revolves about the point P so that point Q appraoches point P along the curve. The limiting position of PQ is called the tangent to the curve at P. The derivative at P is the slope of the tangent and of the curve. Thus the secant through points P(x, y) and $P'(x_1, y_1)$ has a slope $\frac{RP'}{PR} = \frac{\Delta y}{\Delta x}$. The limiting value of this, as Δx approaches zero, is $\frac{dy}{dx}$, the slope of the tangent at P. For, as Δx approaches zero, P' approaches P and P'P approaches the tangent t.



Example 1. Find the slope of $f(x) = \frac{1}{4}x^2$ at the point (2, 1).



$f(x) = \frac{1}{4}x^2$ $f'(x) = \frac{1}{4} \cdot 2x = \frac{1}{2}x$

thus, when x = 2, f'(x) = 1.

The slope of the curve at the point (2, 1) is 1.

What is the equation of the tangent line t? Since the slope of t = 1, an equation is given by $y - 1 = 1(x - 2)$ or $y - 1 = x - 2$, which is $x - y - 1 = 0$.

Example 2. Find every point on the graph of $y = x^2 - 4$ at which the tangent line has a slope 2.

The graph of the given equation is the graph of the function f, where $f(x) = x^2 - 4$. At the point $(x, f(x))$ on this graph, the tangent line has a slope $f'(x) = 2x$.

Then $f'(x) = 2$ if and only if $2x = 2$, or $x = 1$. The point $(1, -3)$ is where $f'(x) = 2$. The tangent line has the equation:

$$y - (-3) = 2(x - 1)$$
$$y + 3 = 2x - 2$$
$$2x - y - 5 = 0$$

Exercises on Section III. 12

In each of the following, find an equation of the tangent line to the graph of the given function at the given point. Sketch the graph of the function and the tangent line.

1. $f(x) = 2x^2$; $(2, f(2))$.

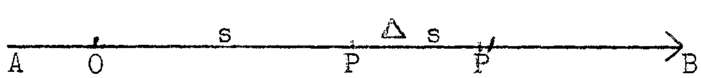
2. $g(x) = x^3$; $(2, g(2))$.

3. $G(x) = x^4$; $(1, G(1))$.

4. Find the points on the graph of $y = 3x^2 - 4x^3$ at which the tangent lines have slope -6.

III. 13. Other Applications of the Derivative.

Important applications arise when the independent variable in a rate is the time. The rate is then called a time-rate. Velocity in rectilinear motion affords a simple example.



Consider the motion of a point P on the straight line AB. Let s be the distance measured from some fixed point, as O, to any position of P, and let t be the corresponding elapsed time. To each value of t corresponds a position of P and therefore a distance s. Hence s will be a function of t, and we may write $s = f(t)$.

Now let t take on an increment Δt; then s takes on an increment Δs, and $\frac{\Delta s}{\Delta t}$ = the average velocity of P when the point moves from P to P', during the time interval Δt. If P moves with uniform motion (constant velocity), the above ratio will have the same value for every interval of time and is the velocity at any instant.

For the general case of any kind of motion, uniform or not, we define the velocity (time - rate of change of s) at any instant as the limit of the average velocity as Δt approaches zero as a limit; that is $v = \frac{ds}{dt}$.

The <u>velocity</u> at any instant is the derivative of the distance with respect to the time, or the time - rate of change of the distance.

When v is positive, the distance s is an increasing function of t, and the point P is moving in the direction AB. When v is negative, s is a decreasing function of t, and P is moving in the direction BA.

If we think of the point P as having the coordinate $s(t)$; then the velocity $v(t)$ of P at the time t is $v(t) = s'(t)$, the derivative of s at t.

Since the <u>acceleration</u> $a(t)$ of the moving point P is the instantaneous rate of change of the velocity function at t, $a(t) = v'(t)$, the derivative of v at t; or $a(t) = s''(t)$.

<u>Example 1</u>. The position function s of an object moving along a line is given by $s(t) = 6t + 1$. Describe the motion of the object.

The velocity of the object is $v(t) = s'(t) = 6$.

Clearly $a(t) = 0$; since $a(t) = v'(t)$. Starting at $t = 0$, $s(0) = 1$ and the object is at the point with coordinate 1. The object then moves with a constant velocity of 6 towards the right, and is at the point with coordinate 7 at $t = 1$, 13 at $t = 2$, and so on. The motion is shown in Fig. 11.

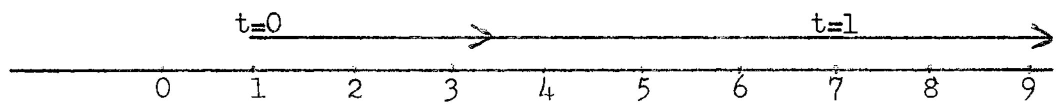


Fig. 11

Example 2. A ball thrown vertically upward from the ground is $s(t)$ ft. above the ground after t-seconds, where $s(t) = 400t - 16t^2$. Describe the motion of the ball.

The velocity of the ball is $v(t) = 400 - 32t$, and the acceleration is $a(t) = -32$, which is the acceleration of gravity. The initial velocity is $v(0) = 400$ ft/sec, the velocity at time $t = 0$. How high does the ball go? The ball will come to rest when it reaches its highest point; that is, its velocity will be zero at the highest point of its path. Since $v(t) = 0$ if and only if $t = 25/2$ sec, evidently the highest point will be reached 25/2 sec after it is thrown. Thus $s(t) = 400\ (25/2) - 16\ (25/2)^2 = 2500$ ft. as the maximum height. The ball will reach the ground 25 sec after it is thrown, since $s(t) = 0$ if and only if $16t(25 - t) = 0$, that is, if and only if $t = 0$ or $t = 25$.

Besides the use of the derivative in solving problems dealing with tangent lines, velocity and acceleration, many other uses are found in economics, physics, chemistry and other sciences. Some illustrations are now given.

Example 3. A gas is expanding according to Boyle's law, $PV = C$, where P is the pressure and V is the volume of the gas, and C is a constant. At a certain instant, $P = 2000$ lb/ft^2 and $V = 4$ft^3. If at this instant the volume is increasing at the rate of 2 ft^3/min (that is $D_tV = 2$), find the rate of change of the pressure.

PV = C, therefore C = 8000 in the given instance. The derivative of the volume with respect to the pressure is given by $D_PV = -\frac{C}{P^2}$ since $V = \frac{C}{P}$. The derivative with respect to the time t is given by:

$$D_tV = -\frac{C}{P^2} D_tP$$

But $D_tV = 2$ so we can solve for D_tP:

$$2 = -\frac{C}{P^2} D_tP \quad D_tP = -\frac{2P^2}{C} = -\frac{2(2000)^2}{8000} = -1000 \text{ lbs/ft}^2$$

That is, as the volume increases the pressure decreases.

In economics, if C(x) is the cost of producting x units of some commodity, then C is called a <u>cost function</u>. We can assume that the domain of C is the set of all positive real numbers. The instantaneous rate of change of C at x is called the <u>marginal cost</u> at x; it is the approximate cost of producing one more unit at the instant x units are being produced. Clearly $C'(x)$, the derivative of C at x, is then the marginal cost at x. We will call C the <u>marginal cost function</u>.

The <u>demand function</u> D is defined as: D(p) equals the number of units of a commodity that can be sold if the price of each unit is P. On the other hand, the <u>price function</u> p is such that P(x) is the price that can be asked for each unit of a commodity in order that x units may be sold. In an ideal economy, the domain of D and P would be the set of positive real numbers.

The <u>revenue function</u> R is defined by R(x) = xP(x), where x is the number of units sold and P(x) is the price per unit. The function R' is called the <u>marginal revenue function</u>. It is obvious that $R'(x) = xP'(x) \neq P(x)$.

If S(p) designates the supply of a certain item, if the selling price is p, then S is called the <u>supply function</u>. Equilibrium conditions will exist for an item selling at price p if $S(p) = D(p)$, that is, if supply equals demand. The price p of an item is called the equilibrium price when $S(p) = D(p)$.

Example 4. Let us assume that the price $P(x)$, in cents, of a bushel of potatoes is given by $P(x) = 220 - \frac{5x}{10^7}$, where x is the number of bushels of potatoes grown in the United States. Find the revenue and marginal revenue functions for potatoes.

We have $R(x) = x(220 - \frac{5x}{10^7}) = 220x - \frac{5x^2}{10^7}$,

or $R(x) = 220x - \frac{x^2}{2 \cdot 10^6}$,

$R'(x) = 220 - \frac{x}{10^6}$.

If 10^8 bushels of potatoes are produced, $R(10^8) = 120$, that is, an additional bushel of potatoes produced increases the revenue by approximately \$1.20.

Exercises for Section III. 13

1. Describe the motion of an object whose position function s, on a line, is given by $s(t) = 4t + 1$.

2. Describe the motion of an object moving on a line whose position function is given by $s(t) = 4t - t^2$.

3. Describe the motion of a ball thrown vertically downward from a point 200 ft above the ground, if the distance $s(t)$ of the ball above the ground t-seconds after it is thrown is given by $s(t) = 200-40t-16t^2$?

4. Sand is being poured on the ground at the rate of $12ft^3/min$. If the pile always has the form of a right circular cone having the same height as diameter of the base, find: (a) $h(t)$, the height of the

pile after t minutes; (b) The time it takes to get a pile 4 ft high; (c) The rate of change of h when the pile is 4 ft high.

5. The equilibrium constant of the reaction $PCl_5 \rightleftarrows PCl_3 + Cl_2$ is given by $K = \frac{x^2}{(a - x)V}$, where x is the amount of Cl_2 and PCl_3, V is the volume of the gas, and a is the original amount of PCl_5. Find the rate of change of V with respect to x.

6. In special relativity of a free particle P of mass m,

$$E^2 = c^2 (m^2c^2 + p^2),$$

where E is the energy of P, p is the momentum of P, and c is the velocity of light. If $v = D_pE$, show that $p = mv/\sqrt{1 - v^2/c^2}$

7. The cost C(x), in dollars, of producing x dozens of pairs of socks in a certain hosiery mill is estimated to be given by $C(x) = 3000 + 2x$. Find the marginal cost function. If the price P(x), in dollars, asked for each dozen pairs of socks is given by $P(x) = 2.30 + \frac{10^2}{x^2}$, where x is the number of dozens of socks produced, find the revenue and marginal revenue functions. Also find $R'(10^4)$.

8. The demand D(p), in millions of bushels, for wheat in a certain nation is given by $D(p) = \frac{6}{\sqrt{p}}$, where p is the price of a bushel of wheat in crowns. Find the revenue and marginal revenue functions for wheat. If $S(p) = 2\sqrt{p}$, what is the equilibrium price of wheat?

III. 14. <u>Maxima and Minima</u>.

Fig. 12 shows the graph of the function $y = x^3 + 4x^2 - 4$. Its derivative is $3x^2 + 8x$.

This means that, at any point x_1, the rate of change of the function is $3x_1^2 + 8x_1$.

Putting the derivative equal to zero and solving, we have $3x^2 + 8x = 0$; $x = 0, -\frac{8}{3}$.

At any value of x less than $-\frac{8}{3}$, $3x^2 + 8x$ is positive. Thus for values of x less than $-\frac{8}{3}$ the function is increasing, as from A to B. The derivative is also positive for values of x greater than 0. Thus it is seen that the function is also increasing from C to D.

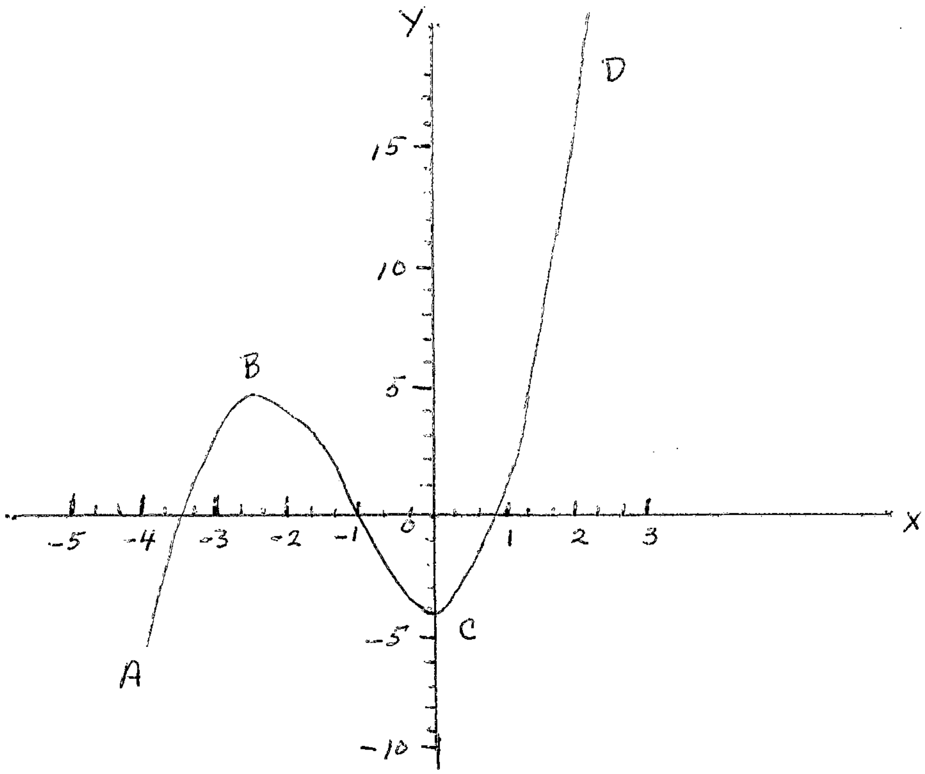


Fig. 12

For values of x greater than $-\frac{8}{3}$ and less than 0 the derivative is negative and the function is decreasing, as from B to C.

At point B$(x = -\frac{8}{3})$ and at point C$(x = 0)$ the value of the derivative is zero. The values of the function at these points are called <u>critical</u> values. Thus, at point B, $x = -\frac{8}{3}$, $y = (-\frac{8}{3})^3 + 4(-\frac{8}{3})^2 - 4$, or $5\frac{13}{27}$. Since the value of the function is less than $5\frac{13}{27}$ on each side of this point, the value $y = 5\frac{13}{27}$ is called a maximum value of the function.

At point C, $x = 0$, and $y = -4$. Since the function is greater than -4 on each side of this point, the value $y = -4$ is called a <u>minimum</u>

value of the function.

At a critical point the slope is zero or infinite.

Using the first derivative we can test for maxima or minima as follows:

First find $f'(x)$ and the critical values by setting $f'(x)$ equal to zero. Then for a critical value $x = x_1$,

$f(x)$ has a maxima $= f(x_1)$ if $f'(x)$ changes from $+$ to $-$,

$f(x)$ has a minima $= f(x_1)$ if $f'(x)$ changes from $-$ to $+$,

$f(x)$ has neither if $f'(x)$ does not change sign.

A second derivative test is sometimes used also.

First find $f'(x)$ and the critical values. Then find $f''(x)$. For a critical value $x = x_1$,

$f(x)$ has a maxima $= f(x_1)$ if $f''(x_1) < 0$,

$f(x)$ has a minima $= f(x_1)$ if $f''(x_1) > 0$,

the test fails however if $f''(x) = 0$ or becomes infinite.

Example 1. Find the maximum and minimum values of the function f defined by $f(x) = x^3 - 12x^2 + 36x$.

$$f'(x) = 3x^2 - 24x + 36 = 3(x - 6)(x - 2)$$

Then $f'(x) = 0$ if and only if $x = 6$ or $x = 2$. Thus these are the only critical points. We can check to see which points they are -- maxima or minima. By the following table:

x	1	2	3	6	7
f'(x)	15	0	-9	0	15

f is increasing at 1 and decreasing at 3; therefore $f(2) = 32$ is a maximum value of f. And f is decreasing at 3 and increasing at 7, so that $f(6) = 0$ is a minimum value of f.

Exercises for Section III. 14.

Find the maximum and minimum values of the following functions and tell where the function is increasing and where decreasing. Draw the graphs. Be sure to show the work where you test for maxima or minima.

1. $f(x) = x^2 - 10$
2. $f(x) = x^2 - 6$
3. $f(x) = x^4 - 2x^2$
4. $f(x) = x^4 - 32x^2$
5. $f(x) = x^3 + 14x + 49$
6. $f(x) = x^3 - 3x^2$

III. 15. Applications of Maxima and Minima:

Many problems in mathematics ask for the largest or least value of some function. Several types shall be used here to illustrate such problems.

Example 1. A farmer has 200 rd. of fence which he wishes to use in enclosing a rectangular pasture to have the greatest possible area. What must be the dimensions?

Let x be the number of rods in one dimension of the field. Since the perimeter is 200 rd., the other dimension will be 100 - x. Then the area A will be:

$$A = x(100 - x) = 100x - x^2.$$

Since we wish to have a maximum value of A, we must find the derivative and set it equal to zero.

$$D_xA = 100 - 2x \qquad 100 - 2x = 0$$
$$x = 50$$
$$\text{and } 100 - 5 = 50$$

The dimensions therefore are 50 rods by 50 rods.

Example 2. What number subtracted from its square will give the least result?

Let x be the number. Then $S = x^2 - x$

$D_xS = 2x - 1$. When $2x - 1 = 0$, $x = \frac{1}{2}$

$S = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$.

Any other number such as 1/3 or 2/3 will give a greater result since $\frac{1}{9} - \frac{1}{3} = -\frac{2}{9}$ and $\frac{4}{9} - \frac{2}{3} = -\frac{2}{9}$, each of which is greater than $-\frac{1}{4}$.

Example 3. A farmer wishes to use 80 rd. of fence to enclose the greatest possible rectangular pasture along a river. If no fence is needed on the river side, what dimensions should he use?

Let x be the number of rods in one dimension of the field (the one parallel to the river). Since there is 80 rods of fencing, 80 - x rods is left for the other sides to be fenced, or $\frac{80 - x}{2}$ rods per side. The area will be:

$$A = x\left(\frac{80 - x}{2}\right) = 40x - \frac{x^2}{2}$$

The derivative $D_xA = 40 - x$

Set $40 - x = 0$, therefore $x = 40$, and $\frac{80 - x}{2} = 20$

The field should be 40 rods by 20 rods.

Exercises for Section III. 15

1. What positive number exceeds its cube by the greatest amount?

2. An open box is to be made by cutting out squares from the corners of a rectangular piece of cardboard and then turning up the sides. If the piece of cardboard is 12" by 24", what are the dimensions of the box of largest volume made in this way?

3. What number exceeds twice its square by the greatest amount?

III. 16. Integration and Applications.

This section on integration is given solely to introduce the student to the concept of this part of the calculus and some of its uses; it is not intended to show the derivation of the integral in a formal way.

We have learned how to find the exact rate of change of a function; that is, given a function, to find its derivative. Now we shall see how to integrate; that is, given the derivative, to find the original function.

The inverse process to differentiation is called integration. Integral calculus is the inverse of differential. Its fundamental object is to find the function, the realtion between the rates or differentials of the variables which enter it being given.

A function is called the integral of its differential, and the process by which we derive it is called integration. Differentiation, as its name implies, is closely related to the operation of taking differences. Integration, on the other hand, is closely allied to the operation of taking sums. In fact, the symbol, $\int$, which is read "integral of", is nothing more than a long S-sign, and stands for the Latin word, "summa", or sum.

The process of integration is just the reverse of the process of differentiation, thus:

x^3 has as its differential $3x^2\,dx$

x^3 therefore is the integral of $3x^2\,dx$.

In integral calculus, we write this as:

$$\int 3x^2\,dx = x^3,$$

read "the integral of $3x^2\,dx$ is x^3"; where dx specifies the variable.

If C is any constant, we have

$$\frac{dC}{dx} = 0, \text{ so that } \int 0\,dx = C.$$

The integral of zero is an arbitrary constant, and the integral of the sum of two functions is the sum of their integrals; therefore, the integrals which we give are not fully determined functions, but are determinate except for the possibility of an indeterminate constant. Hence, two expressions for the integral of a function, obtained in different ways, need not be exactly the same, but may differ by a constant. It is essential when integrating to add a constant, thus:

$$\int f'(x)dx = f(x) + C.$$

You know that the derivative of $3x^2$ is $6x$; an integral of $6x\,dx$ is $3x^2$. However, note:

$$\frac{d}{dx}\,3x^2 = 6x; \quad \frac{d}{dx}\,(3x^2 + 2) = 6x; \quad \frac{d}{dx}\,(3x^2 - 8) = 6x; \text{ etc.}$$

That is, $\frac{d}{dx}\,(3x^2 + k) = 6x$. Hence $\int 6x\,dx = 3x^2 + k$, where k represents any constant.

<u>Examples</u>:

(a) $\frac{d}{dx}\,(4x + 9) = 4; \quad \text{so,} \int 4dx = 4x + k$

(b) $\frac{d}{dx}\,(2x^2 + 3x) = 4x + 3; \quad \text{so,} \int (4x + 3)dx = 2x^2 + 3k + k$

(c) $\frac{d}{dx}\left(\frac{3}{x^2}\right) = -\frac{6}{x^3}; \quad \text{so,} \int\left(\frac{-6}{x^3}\right)dx = \frac{3}{x^2} + k$

The integral of the sum of any number of differentials is the sum of their integrals. This follows from the rule that the derivative of a sum equals the sum of the derivatives. Thus:

$$\int (du + dv) = \int du + \int dv.$$

The integral of a constant multiple of a variable is the constant multiplied by the integral of the variable. Thus:

$$\int C du = C \int du.$$

The integral of a variable with a constant exponent in the differential of the variable is the variable with an exponent increased by one, divided by the increased exponent. Thus:

$$\int u^n du = \frac{u^{n+1}}{n+1}, \quad (n \neq -1).$$

Examples:

(a) $\int x^6 dx = \frac{x^{6+1}}{6+1} + C = \frac{x^7}{7} + C$

(b) $\int 3x dx = 3 \int x dx = \frac{3}{2}x^2 + C$

(c) $\int (5x^2 + 4x) dx = \int 5x^2 dx + \int 4x dx$

$= 5\int x^2 dx + 4\int x dx$

$= \frac{5x^3}{3} + 2x^2 + C$

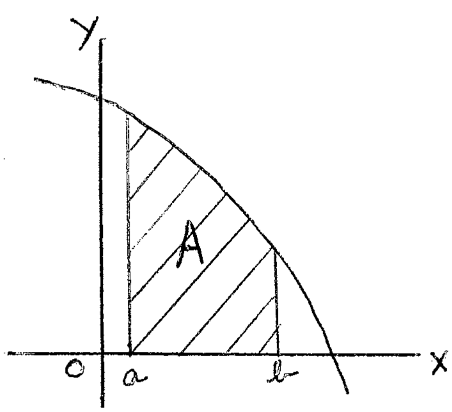
(d) $\int \frac{dx}{\sqrt{x}} = \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}} + C = 2\sqrt{x} + C$

(e) $\int \frac{2dt}{t^2} = 2\int t^{-2} dt = 2\,\frac{t^{-2+1}}{-1} + C = -\frac{2}{t} + C$

One of the practical uses of integration in mathematics is to find the area of a section under a curve.

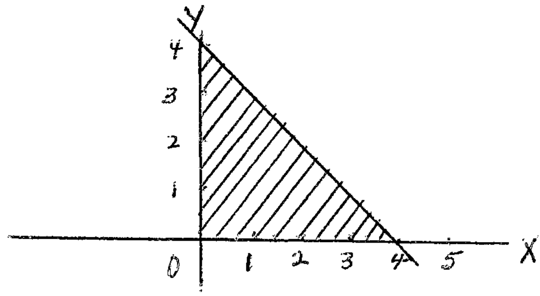
Without proof we shall take the formula:

$$A = \int_a^b y dx$$



where A is the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates at $x = a$, and $x = b$. The symbol $\int_a^b y dx$ means that the integral of $y dx$ is found and $x = b$ and $x = a$ are substituted. Then the difference between them is found. This is the area.

Example 1. Find the area bounded by the line $y = 4 - x$ and the axes.



The area extends from $x = 0$ to $x = 4$.

Hence $A = \int_0^4 y\,dx = \int_0^4 (4 - x)dx$

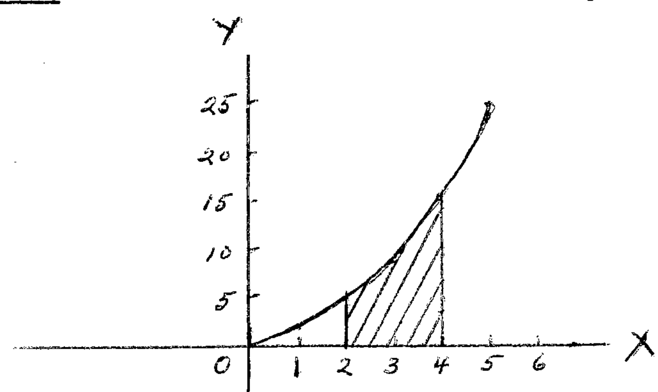
$\int(4 - x)dx = \int 4dx - \int x\,dx = 4x - \frac{x^2}{2} + k$

at $x = 4$: $4x - \frac{x}{2} + k = 8 + k$

at $x = 0$: $4x - \frac{x}{2} + k = k$

$(8 + k) - k = 8$. The area is 8 square units. (Note: It is not necessary to bring in the constant of integration, since it always disappears in subtracting. We will leave it off in the future.)

Example 2. Find the area under the curve $y = x^2$ from $x = 2$ to $x = 4$.



The area desired is $A = \int_2^4 x^2\,dx$

$\int x^2\,dx = \frac{x^3}{3} + k$

At $x = 4$: $\frac{x^3}{3} = \frac{64}{3}$

At $x = 2$: $\frac{x^3}{3} = \frac{8}{3}$

$A = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}$ square units.

The area enclosed by the graphs of two different curves can likewise be found by integration.

Example 3. Find the area of a region bounded by the graphs of functions f and g, where

$$f(x) = 2x - x^2, \quad g(x) = x - 2.$$

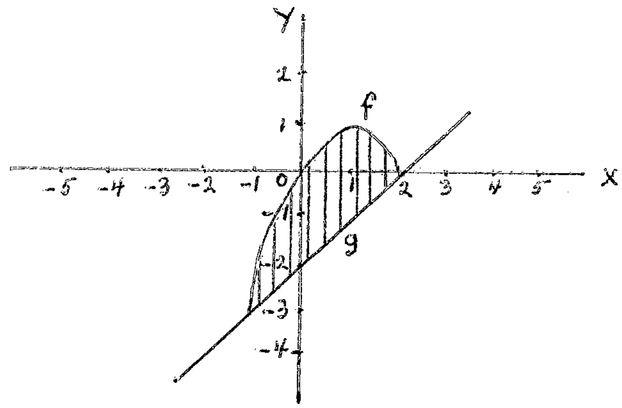


Fig. 13

Fig. 13 shows the graphs of the functions. The region between the graphs of f and g is between the lines $x = -1$ and $x = 2$. Hence,

$$A = \int_{-1}^{2} (f - g) = \int_{-1}^{2} [(2x - x^2) - (x - 2)]dx$$

$$= \int_{-1}^{2} (-x^2 + x + 2)dx$$

$$\int (-x^2 + x + 2)dx = \frac{-x^3}{3} + \frac{x^2}{2} + 2x$$

When $x = 2$: $-\frac{x^3}{3} + \frac{x^2}{2} + 2x = \frac{10}{3}$

When $x = -1$: $-\frac{x^3}{3} + \frac{x^2}{2} + 2x = -\frac{7}{6}$

Therefore the Area $= \frac{9}{2}$ square units.

Exercises for Section III. 16.

1. Find $\int_{1}^{4} x^2 \, dx$

2. Find $\int_{0}^{a} (a^2x - x^3)dx$

3. Find the area bounded by the parabola $y = x^2$, the x-axis, and the ordinates $x = 2$ and $x = 4$.

4. Find by integration the area of the triangle bounded by the line $y = 2x$, the x-axis, and the ordinate $x = 4$. Verify your answer by finding the area as half the product of the base and altitude.

5. Find the area of the trapezoid bounded by the line $x + y = 10$, the x-axis, and the ordinates $x = 1$ and $x = 8$. Verify your answer by finding the area as half the product of the sum of the parallel sides and the altitude.

6. Find the area bounded by the parabola $y^2 = 2x$ and the straight line $x - y = 4$.

This, then, has been a brief look at integration. It is true that many more uses of integration are possible; but it is hoped this very short section shall serve as a stimulus to further study by the interested pupil. If such is the case, the insertion shall have served its purpose.

The interested student can find much fuller development of the calculus, and the topics of the first two chapters, if he or she will but look at the various books listed in the bibliography at the end of the paper, or by referring to many other volumes dealing with advanced mathematics.

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APPENDIX A

Symbols Used in Set Theory

$A = \{x_1, \ldots, x_n\}$	A is a set containing the elements $x_1, \ldots, x_n$ and no others
$A \subset B$	A is a subset of B
$B \supset A$	B is a superset of A
$A \cap B$	Intersection of Sets A and B
$A \cup B$	Union of sets A and B
$\bar{A}$	Complement of set A
$\emptyset$	The empty of null set

Symbols Used in Statistics and Probability

$\bar{x}$	Arithmetic mean
$\sum$	Summation
d	Deviation of an item from the mean
$\sum d^2$	The Sum of squares

Symbols Used in Introductory Calculus

f(x), g(x), etc.	Functions of x
Δ	Difference between values of
m	Slope of a line
∞	Infinite
$f'(x)$ or $D_x f$	First derivative of the function f(x)
$\int$	Integral of
$\frac{d}{dx} f(x)$	Derivative of a function in regards to x
dx, dy, etc.	Differentials

APPENDIX B

Some Useful Formulas

$C = 2\pi r$; $C = \pi d$	Circumference of a circle with radius r, with diameter d
$A = \pi r^2$; $A = \frac{\pi}{4} d^2$	Area of a circle with radius r, with diameter d
$A = b^2$	Area of a square with side b.
$S = 6b^2$	Surface area of a cube with edge b.
$A = bh$	Area of a rectangle with sides b and h.
$A = \frac{1}{2} bh$	Area of a triangle with base b and altitude h.
$V = b^3$	Volume of a cube with edge b.
$S = 4\pi r^2$	Surface area of a sphere of radius r.
$V = \frac{4}{3}\pi r^3$	Volume of a sphere of radius r.
$A = nr^2 \tan\left(\frac{180}{n}\right)$	Area of a regular polygon of n sides circumscribed about a circle of radius r.
$V = \pi r^2 h$	Volume of a right circular cylinder of radius r and altitude h.
$V = 1/3\,\pi r^2 h$	Volume of a right circular cone of radius r and altitude h.
$S = 2\pi rh$	Lateral surface area of a right circular cylinder of radius r and altitude h
$F = \frac{mm_1}{d^2}$	Force of attraction of two particles of masses m and m_1, at a distance d units apart.
$F = \frac{qq_1}{d^2}$	Force of attraction (or repulsion) of two electrical charges q and q_1, at a distance d units apart. (A similar formula is used for the force of attraction of two magnetic poles.

APPENDIX C

Glossary of Frequently Used Terms

Arithmetic Mean	The simple average formed by adding quantities together in any order and dividing by their number.
Constant	Quantity whose value is fixed (does not change) in any one investigation.
Derivative	The instantaneous rate of change of a function.
Differential	The differential of the independent variable is its increment; while the differential of the dependent variable is the product of a derivative and the increment of the independent variable.
Differentiation	Process of finding the derivative.
Disjoint	Sets that contain no common elements.
Domain	Collection of numbers over which a function is defined.
Element	The items that belong to, are members of, or make up the set.
Function	A correspondence that associates with each number of a given collection of numbers, a unique number.
Integration	Process of finding the function from its derivative.
Median	The fiftieth percentile of a group of measurements.
Probability	When a given event can happen in h ways and fail to happen if f ways, and if each of the $h + f$ ways is equally likely to occur, then the probability of the event happening is $p = \frac{h}{h + f}$
Range	The difference between the largest and smallest measurement in a sample.
Set	Totality of all points or numbers that satisfy a given condition.
Statistics	The theories and techniques involved in collecting, summarizing, and interpreting numberical facts.
Variable	Quantity which can be assigned an unlimited number of values in an investigation.

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