SPATIALLY DEPENDENT TRANSFER FUNCTIONS FOR WEB LATERAL DYNAMICS

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ABSTRACT

In this paper we derive spatially dependent transfer functions for web span lateral dynamics which provide web lateral position and slope as outputs at any location in the span; the inputs are guide roller displacement, web lateral position disturbances from upstream spans, and disturbances due to misaligned rollers. This is in sharp contrast to the existing approach where only web lateral position response is available on the rollers. We describe the inherent drawbacks of the existing approach and how the new approach overcomes them. The new approach relies on taking the 1D Laplace transform with respect to the temporal variable of both the web governing equation and the boundary conditions. One can also obtain the web slope at any location within the web span with the proposed approach. A general span lateral transfer function, which is an explicit function of the spatial position along the span, is obtained first followed by its application to different intermediate guide configurations.

1 INTRODUCTION

There have been many studies on modeling the lateral behavior of moving webs, dating back to about 60 years. The first seminal work on the topic was reported by Shelton in his Ph.D. thesis in 1968 [1]. For the purpose of deriving the governing equations of the web lateral position on rollers, Shelton treated the moving web between two rollers as a tensioned Euler-Bernoulli beam. He correctly argued that, for most webs, the web mass is negligible, i.e., the force due to acceleration of web mass is negligible when compared to web tension, and, thus, treating the lateral motion of the web between two rollers as the motion of a static beam. He considered four boundary conditions (web lateral position and slope on each roller) to solve the differential equation describing the lateral motion of the web. Shelton also used a key observation/principle that was prevalent in the transport of belts literature – a belt approaching a roller aligns itself normal to the axis of rotation of the roller. He used this principle to setup two normal entry conditions, for web lateral velocity and acceleration in terms of roller lateral velocity and acceleration, web entry angle at the roller, and angle of the roller.

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Based on this approach, he derived transfer functions from the guide roller lateral position (input variable) to the web lateral position on the roller (controlled or output variable) for various guide roller mechanisms, such as the end-pivoted guide, center-pivoted guide, offset pivot guide, remotely pivoted guide, etc. Subsequent work in modeling and control of web lateral dynamics based on this treatment was reported in [2–7].

In this paper, we further investigate and discuss some of the inherent assumptions of the aforementioned approach that have implications for modeling and control of web lateral behaviour. First, the solution for the beam equation was obtained by assuming constants (time-invariant) boundary conditions, and subsequently the time derivative of the solution is used in the normal entry conditions to determine the transfer function. We argue that the purpose of the guide roller is to modify the boundary conditions, and therefore the time-invariant assumption on the boundary conditions is counter to the notion that the axis of rotation of the guide roller is utilized to change the web boundary conditions on the roller. Second, the existing approach provides a governing equation only for the lateral position behavior on the roller and not for any position within the span. Third, existing methods focus on controlling just the lateral position (and not slope), so that lateral oscillations are not propagated downstream of the guide roller. We will discuss these three aspects and present a method to obtain lateral transfer functions that address some of the aforementioned issues.

Our approach relies on taking the 1D Laplace transform of the span governing equation (static beam equation) and the boundary conditions with respect to the temporal variable. We consider the web lateral position and slope on the upstream roller to the span as two boundary conditions and the two normal entry conditions (for lateral velocity and lateral acceleration) on the downstream roller as the other two boundary conditions. The idea of incorporating the normal entry conditions as boundary conditions is not new, and has been considered in [3] where the normal entry conditions are applied to a system modeled using a dynamic beam equation; a 2D Laplace transform (for both spatial and temporal variables) was applied to the dynamic beam equation; this provides a solution to the beam equation in the frequency domain. Due to the complexity of determining the inverse Laplace transform of the resulting solution, the frequency domain solution was ignored and the spatial derivatives in the beam model were discretized using a finite difference method to obtain a set of ordinary differential equations for web lateral response on the roller; a two span system example was used to illustrate the procedure. In our approach, we not only employ the 1D Laplace transform in the temporal variable for the beam governing equation, but also for the boundary conditions. This allows us to solve the resulting equations, and obtain spatially dependent lateral transfer functions. The application of 1D Laplace transforms in the temporal variable for distributed parameter systems is provided in [8].

In many guide control systems, the feedback for the guide controller is the measurement from a lateral position sensor that is located immediately downstream of the guide roller; the ability of such a control system with traditional guide mechanisms to precisely control the lateral position and prevent propagation of lateral position and angle errors to spans downstream of the guide roller has been questioned; for example, see [2, 7]. A guiding apparatus is presented in [7] that uses four displacement sensors to solve four simultaneous equations to compute the four coefficients of the lateral governing equations and calculate the web lateral position and slope; two or more sensors are used as inputs for the PID controllers for lateral and slope control loops. Since the proposed approach will allow for obtaining not only position but also slope at any location in the span; thus, eliminating the need for the use of multiple sensors to determine the position and slope of the web; further, one can also use the transfer function to control the web lateral position at any location along the span using a traditional, intermediate web guide.

2 DISCUSSION OF EXISTING APPROACH AND ASSUMPTIONS

The governing equation for the web lateral dynamics is given by [1]

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} - T\frac{\partial^2 y(x,t)}{\partial x^2} = 0. \quad \{1\}$$

The above equation is similar to treating the web as a tensioned Euler-Bernoulli static beam, i.e., the web between two rollers is treated as a static beam by assuming that the web mass is negligible. The general solution is given by

$$y(x,t) = C_1\sinh(Kx) + C_2\cosh(Kx) + C_3 x + C_4 \quad \{2\}$$

where $K^2 = T/EI$. Four boundary conditions are required to obtain the coefficients in the solution {2}. These are typically assumed to be the lateral positions (y) and slopes ($\partial y/\partial x$) at the two ends of the web span, which are assumed to be known and given by

$$\begin{aligned} y(0,t) &= y_0(t); \quad \left.\frac{\partial y}{\partial x}\right|_{x=0,t} = \theta_{w0}(t) \\ y(L,t) &= y_L(t); \quad \left.\frac{\partial y}{\partial x}\right|_{x=L,t} = \theta_{wL}(t); \end{aligned} \quad \{3\}$$

Note that these boundary conditions imply that both the ends of the tensioned beam are free.

By defining $\Delta = K(KL\sinh(KL) + 2 - 2\cosh(KL))$ and employing the boundary conditions, the coefficients are given by

$$\begin{aligned} C_1 &= \frac{1}{\Delta}\left[K\sinh(KL)(y_0 - y_L) + (KL\sinh(KL) + 1 - \cosh(KL))\,\theta_{w0} + (\cosh(KL) - 1)\theta_{wL}\right] \\ C_2 &= \frac{1}{\Delta}\left[K(\cosh(KL) - 1)(y_L - y_0) + (\sinh(KL) - KL\cosh(KL))\theta_{w0} + (KL - \sinh(KL))\theta_{wL}\right] \\ C_3 &= \theta_{w0} - C_1 K \\ C_4 &= y_0 - C_2 \end{aligned} \quad \{4\}$$

The effect of the rollers on the lateral position of the web is modeled as follows. Adequate friction between the roller and web surfaces is assumed such that the web surface immediately aligns perpendicular to the roller axis of rotation when it makes contact with the roller. This observation is well known among researchers in both the web and the belt communities; this is typically referred to as the ”normal entry rule” in the web handling community. Figure 1 provides a line

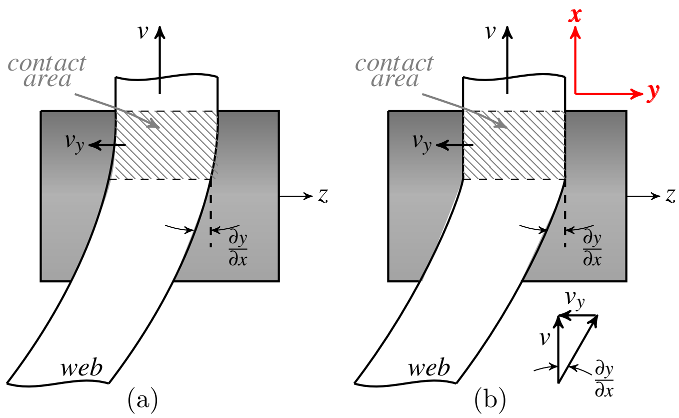

Figure 1 – Web behavior at roller entry: (a) "real web"; (b) "entry rule interpretation"

sketch showing the relationship between the web slope and the lateral displacements of the roller. The relation between the lateral velocities, including the lateral velocity of the roller, is given by

$$\frac{\partial y_i(t)}{\partial t} = v\left(\theta_i(t) - \frac{\partial y(t)}{\partial x}\Big|_{x=i}\right) + \frac{\partial z_i(t)}{\partial t} \qquad \{5\}$$

where t denotes time, v is the web transport speed, θ_i is the angle between the roller axis and the perpendicular to the transport direction, and z_i is the lateral position of the roller. The subscript i is used to denote the entry ($i=0$) and exit rollers ($i=L$) of the web span. Due to the forced perpendicular entry to the roller, the web suffers a sudden change in slope between the free span and its contact with the roller, which can be expressed as

$$\frac{\partial^2 y_i(t)}{\partial t^2} = v^2 \frac{\partial^2 y(t)}{\partial x^2}\Big|_{x=i} + \frac{\partial^2 z_i(t)}{\partial t^2} \qquad \{6\}$$

Note that the normal entry rule is used as a mechanism by which a guide roller can control the lateral position of the web via the rotation and translation of the guide roller about a pivot point by an actuating mechanism. The lateral web position response on a roller is analyzed for two separate conditions: for fixed rollers ($z=0, \theta=0$) and steering (or guide) rollers. The responses are combined by assuming that the principle of superposition applies to this situation. The governing equation for the evolution of the lateral position for the two conditions is obtained as follows. First, the second partial derivative of {2} with respect to x is substituted into {6}. The resulting equation contains the web angle, θ_{wi}, which is replaced by the slope term from the entry rule given by {5}. Taking the Laplace transform of the resulting equation with respect to time results in the following input-output transfer functions in the frequency domain [4]

$$y_L(s) = \left(\frac{-\frac{f_3(KL)s}{\tau} + \frac{f_1(KL)}{\tau^2}}{s^2 + \frac{f_2(KL)s}{\tau} + \frac{f_1(KL)}{\tau^2}} \right) y_0(s) + \left(\frac{\frac{v f_3(KL)}{\tau}}{s^2 + \frac{f_2(KL)s}{\tau} + \frac{f_1(KL)}{\tau^2}} \right) \theta_0(s)$$
$$+ \left(\frac{\frac{v f_2(KL)}{\tau}}{s^2 + \frac{f_2(KL)s}{\tau} + \frac{f_1(KL)}{\tau^2}} \right) \theta_L(s) + \left(\frac{\frac{f_3(KL)s}{\tau}}{s^2 + \frac{f_2(KL)s}{\tau} + \frac{f_1(KL)}{\tau^2}} \right) z_0(s)$$
$$+ \left(\frac{s^2 + \frac{f_2(KL)s}{\tau}}{s^2 + \frac{f_2(KL)s}{\tau} + \frac{f_1(KL)}{\tau^2}} \right) z_L(s) \qquad \{7\}$$

where $\tau = L/v$, $\Delta_f = KL\sinh(KL) + 2(1-\cosh(KL))]$,
$f_1(KL) = (KL)^2(\cosh(KL)-1)/\Delta_f$, $f_2(KL) = KL[KL(\cosh(KL)-\sinh(KL))]/\Delta_f$, and $f_3(KL) = KL[\sinh(KL)-KL]/\Delta_f$. For a Remotely Pivoted Guide (RPG), and for small angle changes, roller angle is approximated as $\theta_L(t) = z_L(t)/X_1$, where X_1 is the distance between the guide roller and the pivot point. Then, one can assume that the control action is provided through changes in z_L. A similar approach is considered for the Offset-Pivot Guide (OPG) where the typical pivot distance is the span length, $X_1 = L$.

Remark 2.1 *This approach has several issues: (1) The developed governing equation provides the evolution of lateral position only on the roller; it does not provide lateral web behavior within the span. (2) The effect of the rollers is not included in solving the beam equation {1}; to determine the coefficients in the solution, free end boundary conditions are assumed for the tensioned beam. (3) Although lateral position at the downstream roller is of interest, this is assumed as a known boundary condition in the development. (4) Shelton and subsequent researchers used the normal entry conditions in an indirect manner in the sense that they assumed the knowledge of the lateral position and slope on the downstream roller (resulting in the response at a specific location) as a boundary condition and then applied the resulting solution to fit the normal entry conditions.*

3 SPATIALLY DEPENDENT TRANSFER FUNCTIONS

In the existing approach summarized in Section 2, the effect of the boundary conditions on web lateral behavior within the free span is not clear. The "normal entry rules" are applied on rollers adjacent to the span to introduce a dynamical behavior on the roller and to obtain a relationship between the lateral web position on the guide roller and the control action (guide motion). In this work, instead of assuming the downstream boundary conditions for lateral position and slope (at $x = L$) (whose evolution is one of our interests), the two normal entry conditions will be employed as downstream boundary conditions. This will allow for directly incorporating the effect of the roller into the solution of the beam equation. Further, this will also allow for directly coupling the dynamic effects of the rollers with the span lateral dynamics.

In this work, a free span is defined as the length of material between two rollers that is not wrapped on the rollers. For the upstream roller, we establish the boundary conditions for the span at the exit of the region of wrap of the upstream roller. Due to the application of the entry rule to the upstream roller, which

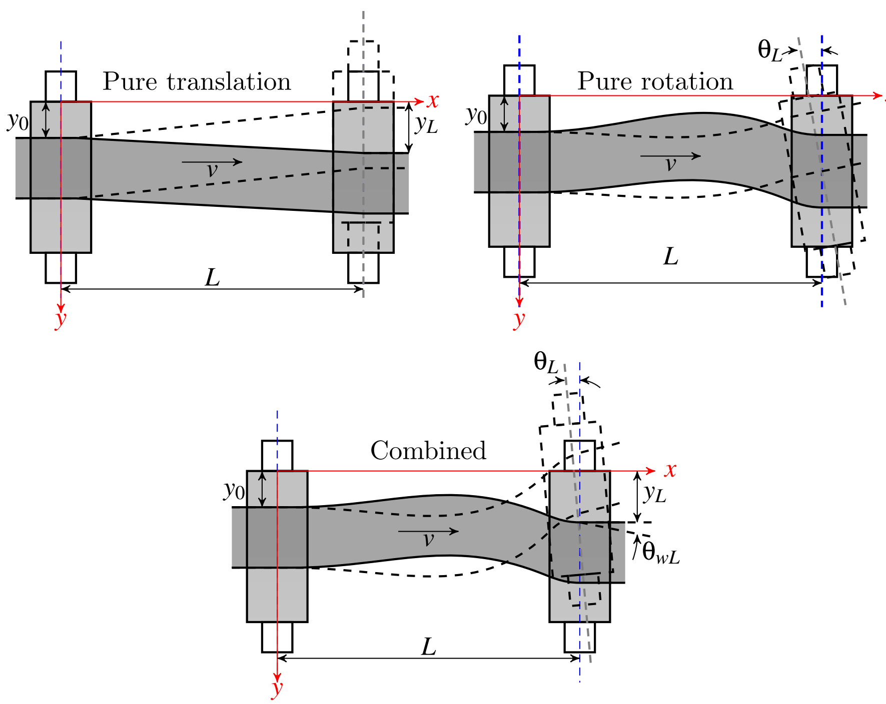

Figure 2 – Web lateral movement due to roller translation and rotation

stipulates that the web aligns perpendicular to the roller at contact, the web leaves this roller perpendicularly. Then, the upstream roller angle becomes a boundary condition for web slope. In the region of wrap of the upstream roller, the lateral displacement remains the same throughout which is taken as the second boundary condition at the beginning of the span. These conditions are shown in Figure 2.

To solve the beam equation, we apply a 1D Laplace transform in the temporal variable for both the beam equation and the boundary conditions. The beam equation is given by {1} and the boundary conditions are rewritten compactly as:

$$\begin{aligned} y(0,t) &= y_0(t); \\ \frac{\partial y(0,t)}{\partial x} &= \theta_0(t); \\ \frac{\partial y(L,t)}{\partial x} &= \theta_L(t) + \frac{1}{v}\frac{\partial z(t)}{\partial t} - \frac{1}{v}\frac{\partial y(L,t)}{\partial t}; \\ \frac{\partial^2 y(L,t)}{\partial x^2} &= \frac{1}{v^2}\left(\frac{\partial^2 y(L,t)}{\partial x^2} - \frac{\partial^2 z(t)}{\partial x^2}\right). \end{aligned} \qquad \{8\}$$

The first condition represents any web lateral position present at the entry of the span (exit of the region of wrap of the upstream roller). The second condition represents the web slope present at the entrance of the span; this is the same as the upstream roller angle $\theta_0(t)$, due to the interpretation of the entry rule for the web on that roller. The third and fourth conditions are the normal entry rules at the entry of the region of wrap for the downstream roller. We will apply the following 1D Laplace transform for the time variable:

$$\mathcal{L}\{f(x,t)\} = \hat{f}(x,s) = \int_0^\infty e^{-st} f(x,t)dt \tag{9}$$

to the web governing equation {1} and the boundary conditions {8} to obtain

$$\frac{\partial^4 \hat{y}(x,s)}{\partial x^4} - K^2 \frac{\partial^2 \hat{y}(x,s)}{\partial x^2} = 0 \tag{10}$$

and

$$\begin{aligned}
\hat{y}(0,s) &= \hat{y}_0(s) \\
\frac{\partial \hat{y}(0,s)}{\partial x} &= \hat{\theta}_0(s); \\
\frac{\partial \hat{y}(L,s)}{\partial x} &= \hat{\theta}_L(s) + \frac{s}{v}\hat{z}_L(s) - \frac{s}{v}\hat{y}_L(s); \\
\frac{\partial^2 \hat{y}(L,s)}{\partial x^2} &= \frac{s^2}{v^2}\hat{y}_L(s) - \frac{s^2}{v^2}\hat{z}_L(s).
\end{aligned} \tag{11}$$

The general solution of {1} is given by

$$\hat{y}(x,s) = C_1(s)\sinh(Kx) + C_2(s)\cosh(Kx) + C_3(s)x + C_4(s). \tag{12}$$

Note that the coefficients C_i are functions of the frequency domain variable 's'. Substituting the boundary conditions into {12} results in the following solution:

$$\begin{aligned}
\hat{y}(x,s) = &- \left(\frac{s^2}{v^2}g_2(x) + \frac{s}{v}g_1(x)\right)\hat{y}_L(s) + \left(\frac{s^2}{v^2}g_2(x) + \frac{s}{v}g_1(x)\right)\hat{z}_L(s) \\
&+ \hat{y}_0(s) + g_1(x)\hat{\theta}_L(s) + (x - g_1(x))\hat{\theta}_0(s)
\end{aligned} \tag{13}$$

where $\hat{y}_L(s) = \hat{y}(L,s)$ and

$$\begin{aligned}
g_1(x) &= \frac{\sinh(KL)[\cosh(Kx) - 1] - \cosh(KL)[\sinh(Kx) - Kx]}{K[\cosh(KL) - 1]}, \\
g_2(x) &= \frac{[\cosh(KL) - 1][\cosh(Kx) - 1] - \sinh(KL)[\sinh(Kx) - Kx]}{K^2[\cosh(KL) - 1]}
\end{aligned} \tag{14}$$

Note that the slope and moment at any location inside the span are given by

$$\begin{aligned}
\frac{\partial \hat{y}}{\partial x}(x,s) = &- \left(\frac{s^2}{v^2}g_{2x}(x) + \frac{s}{v}g_{1x}(x)\right)\hat{y}_L(s) + \left(\frac{s^2}{v^2}g_{2x}(x) + \frac{s}{v}g_{1x}(x)\right)\hat{z}_L(s) \\
&+ g_{1x}(x)\hat{\theta}_L(s) + (1 - g_{1x}(x))\hat{\theta}_0(s)
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\partial^2 \hat{y}}{\partial x^2}(x,s) = &- \left(\frac{s^2}{v^2}g_{2xx}(x) + \frac{s}{v}g_{1xx}(x)\right)\hat{y}_L(s) + \left(\frac{s^2}{v^2}g_{2xx}(x) + \frac{s}{v}g_{1xx}(x)\right)\hat{z}_L(s) \\
&+ g_{1xx}(x)\hat{\theta}_L(s) - g_{1xx}(x)\hat{\theta}_0(s)
\end{aligned} \tag{16}$$

where $g_{lx}(x)$ is the first partial derivative of $g_l(x)$ with respect to x $(l = 1,2)$ and $g_{lxx}(x)$ is the second partial derivative.

When $x = L$, Equation {13} can be expressed as

$$\hat{y}_L(s) = -\left(\frac{s^2}{v^2}g_2(L) + \frac{s}{v}g_1(L)\right)\hat{y}_L(s) + g_1(L)\hat{\theta}_L(s) + \hat{y}_0(s)$$
$$+\left(\frac{s^2}{v^2}g_2(L) + \frac{s}{v}g_1(L)\right)\hat{z}_L(s) + (L - g_1(L))\hat{\theta}_0(s) \qquad \{17\}$$

And it can be simplified to:

$$\hat{y}_L(s) = \left(\frac{s^2 + v\frac{g_1(L)}{g_2(L)}s}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\right)\hat{z}_L(s) + \left(\frac{\frac{g_1(L)}{g_2(L)}v^2}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\right)\hat{\theta}_L(s)$$
$$+ \left(\frac{\frac{L - g_1(L)}{g_2(L)}v^2}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\right)\hat{\theta}_0(s) + \left(\frac{\frac{1}{g_2(L)}v^2}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\right)\hat{y}_0(s) \qquad \{18\}$$

Substituting {18} into {13} and simplifying we obtain

$$\boxed{\begin{aligned}\hat{y}(x,s) =& \frac{P_1(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_0(s) + \frac{P_2(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s)\\ &+ \frac{P_3(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_L(s) + \frac{\frac{g_2(x)}{g_2(L)}\left(s^2 + v\frac{g_1(x)}{g_2(x)}s\right)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{z}_L(s)\end{aligned}} \qquad \{19\}$$

where

$$\begin{aligned}P_1(s) =& \frac{1}{g_2(L)}\left[((x - g_1(x))g_2(L) - (L - g_1(L))g_2(x))\,s^2\right.\\ &\left.+v\,(g_1(L)(x - g_1(x)) - g_1(x)(L - g_1(L)))\,s + (x - g_1(x))v^2\right]\\ P_2(s) =& \frac{1}{g_2(L)}\left[(g_2(L) - g_2(x))s^2 + v(g_1(L) - g_1(x))s + v^2\right]\\ P_3(s) =& \frac{1}{g_2(L)}\left[(g_1(x)g_2(L) - g_1(L)g_2(x))s^2 + v^2 g_1(x)\right]\end{aligned} \qquad \{20\}$$

Note that Equation {19} is a convenient transfer function representation of the solution given by Equation {12}. Further, one can obtain both the slope and moment at any location along the web span. One can substitute {18} in {15} and {16} to obtain the slope and moment (or take the partial derivatives of {19} with respect to x). The slope is given by

$$\boxed{\begin{aligned}\frac{\partial\hat{y}}{\partial x}(x,s) =& \frac{P_{1x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_0(s) + \frac{P_{2x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s)\\ &+ \frac{P_{3x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_L(s) + \frac{\frac{g_{2x}(x)}{g_2(L)}\left(s^2 + v\frac{g_{1x}(x)}{g_{2x}(x)}s\right)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{z}_L(s)\end{aligned}} \qquad \{21\}$$

where

$$\begin{aligned}
P_{1x}(s) =& \frac{1}{g_2(L)}\left[\left((1-g_{1x}(x))g_2(L)-(L-g_1(L))g_{2x}(x)\right)s^2\right.\\
&\left.+v\left(g_1(L)(1-g_{1x}(x))-g_{1x}(x)(L-g_1(L))\right)s+(1-g_{1x}(x))v^2\right]\\
P_{2x}(s) =& -\frac{1}{g_2(L)}\left(g_{2x}(x)s^2+vg_{1x}(x)s\right)\\
P_{3x}(s) =& \frac{1}{g_2(L)}\left[\left(g_{1x}(x)g_2(L)-g_1(L)g_{2x}(x)\right)s^2+v^2g_{1x}(x)\right]
\end{aligned} \tag{22}$$

Equations {19} and {21}, respectively, are the general expressions for the web lateral position and slope; further simplification of these equations can be achieved by considering the specific roller configuration corresponding to a given situation.

3.1 Comparison with Existing Transfer Functions

Since the approach in this work provides transfer functions from control and disturbance inputs to lateral position output at any location in the span, we can compare it with the general transfer functions given in [4] when $x = L$. Note that at $x = L$, we can write the following relations: $g_{1x}(L) = 1$, $g_{2x}(L) = 0$, $g_{1xx}(L) = 0$, $g_{2xx}(L) = -1$, and

$$\begin{aligned}
g_1(L) =& \frac{KL\cosh(KL)-\sinh(KL)}{K[\cosh(KL)-1]}\\
g_2(L) =& \frac{KL\sinh(KL)+2[1-\cosh(KL)]}{K^2[\cosh(KL)-1]} && \left(=\frac{L^2}{f_1(KL)}\right)\\
\frac{L-g_1(L)}{g_2(L)} =& \frac{K\sinh(KL)-KL}{KL\sinh(KL)+2[1-\cosh(KL)]} && \left(=\frac{f_3(KL)}{L}\right)\\
\frac{g_1(L)}{g_2(L)} =& \frac{K[KL\cosh(KL)-\sinh(KL)]}{KL\sinh(KL)+2[1-\cosh(KL)]} && \left(=\frac{f_2(KL)}{L}\right)
\end{aligned} \tag{23}$$

Substitution of these relations in {18} results in the web lateral position response on the roller at $x = L$:

$$\begin{aligned}
y_L(s) =& \frac{s^2+\frac{f_2(KL)s}{\tau}}{s^2+\frac{f_2(KL)s}{\tau}+\frac{f_1(KL)}{\tau^2}}\hat{z}_L(s)+\frac{\frac{vf_2(KL)}{\tau}}{s^2+\frac{f_2(KL)s}{\tau}+\frac{f_1(KL)}{\tau^2}}\hat{\theta}_L(s)\\
&+\frac{\frac{f_1(KL)}{\tau^2}}{s^2+\frac{f_2(KL)s}{\tau}+\frac{f_1(KL)}{\tau^2}}\hat{y}_0(s)+\frac{\frac{vf_3(KL)}{\tau}}{s^2+\frac{f_2(KL)s}{\tau}+\frac{f_1(KL)}{\tau^2}}\hat{\theta}_0(s)
\end{aligned} \tag{24}$$

This differs slightly from Equation {7} because in the existing approach the free span definition included the region of wrap also, and the entry rule was applied to the upstream roller in the following manner:

$$\begin{aligned}
&\hat{y}(0,s) = \hat{y}_0(s)\\
&\frac{\partial\hat{y}}{\partial x}(0,s) = \left.\frac{\partial\hat{y}}{\partial x}\right|_{x=0,s} = \hat{\theta}_{w0}(s) = -\frac{s}{v}\hat{y}_0(s)
\end{aligned} \tag{25}$$

However, in our approach, the free span does not include the region of wrap in either the upstream or downstream rollers; the interpretation of the normal entry

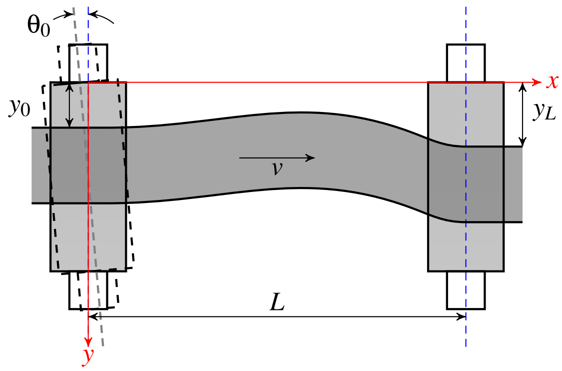

Figure 3 – Web span with downstream fixed roller

rule is that the web will acquire the roller angle and keep it for the entire region of wrap; due to this the first term in the numerator of $y_0(s)$ and $z_0(s)$ cancels each other.

The following remarks provide some observations based on the results of this section.

Remark 3.1 *By employing normal entry conditions on the downstream roller of the span as the boundary conditions, we incorporate the effect of web/roller contact into the solution of the governing equation. The proposed method allows us to directly obtain higher spatial partial derivatives of the lateral response, and thus can be used for obtaining web slope, moment, shear force, etc. The method also further opens up the opportunity to develop controllers for processes which require control of lateral position within the span.*

Remark 3.2 *The proposed method can be extended to include shear by establishing appropriate boundary conditions; the inclusion of shear modifies the boundary condition for the lateral acceleration. Inclusion of shear introduces an additional pole and zero to the lateral transfer functions.*

4 SPATIAL TRANSFER FUNCTIONS FOR SPECIFIC GUIDE CONFIGURATIONS

4.1 Response Due to a Downstream Fixed Roller

A web span with a fixed roller at the exit of the span is provided in Figure 3; the angle and lateral position of the fixed roller are both zero, i.e., $\hat{\theta}_L(s) = 0$ and $\hat{z}_L(s) = 0$. Note that lateral position, slope or moment in a span with a fixed exit roller depend only on the perturbations $(\hat{y}_0(s), \hat{\theta}_0(s))$ at the entrance of the span. Additionally, the roller angle $\hat{\theta}_0(s)$ can be a function of the time or a constant. Therefore, for controlling the lateral position for any point downstream of a guide roller, one has to control both position and slope; much of the existing work has focused on obtaining zero error for the lateral position at the exit of the guide without accounting for the roller angle. Note that in most guide installations, the plane of the guiding span and the span downstream of it are perpendicular to each other; if this is not the case the roller angle will affect downstream lateral position.

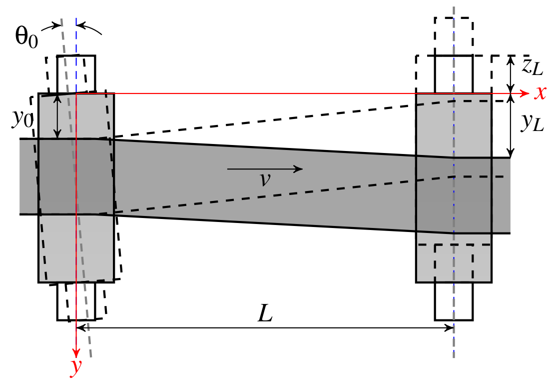

Figure 4 – Web span with pure lateral displacement of downstream roller

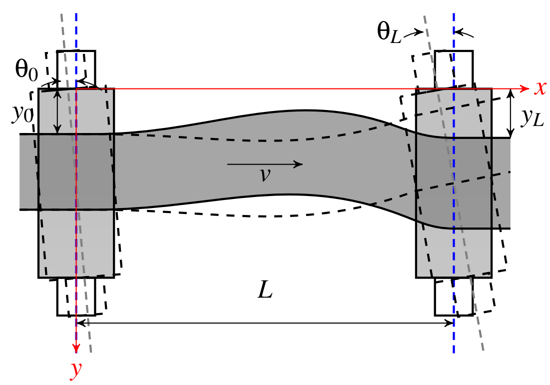

Figure 5 – Span with pure rotation of downstream roller

4.2 Response Due to a Pure Roller Displacement

Figure 4 provides a sketch of the span with a pure displacement action on the downstream roller. The roller displacement is given by $z_L(t)$. For this case we set $\hat{\theta}_L = 0$ in the general transfer function.

4.3 Response due to a Roller under Pure Rotation

This configuration is presented in Figure 5, where the lateral displacement $z(t) = 0$. Notice that we treat a fixed misalignment as a rotating guide with constant angle value in time. In the case of variable angle change, it also can be considered as a pivoting guide, but with the pivoting axis located in the center of the roller face.

4.4 Remotely Pivoted Guide Roller

The remotely pivoted guide roller corrects the web position by rotating the roller around a point in the entering span at a distance X_1 from the guide roller; see Figure 6. Considering small angular rotation of the guide roller, we can write $z(t) = X_1\theta_L(t)$. We will use the roller angle instead of the lateral position, since this

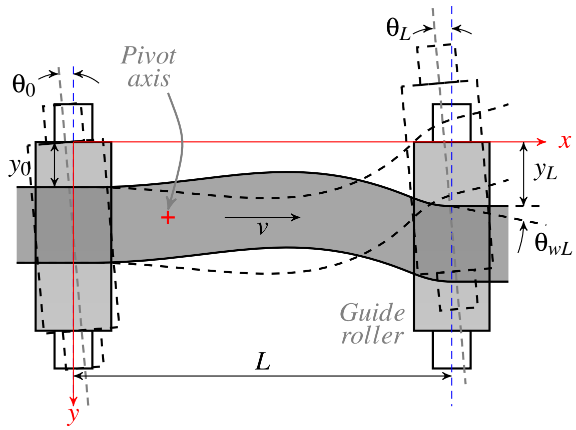

Figure 6 – Web span with remotely pivoted guide

angle is what is directly controlled by the guide motor. With these considerations, Equation {19} can be written as

$$\hat{y}(x,s) = \frac{P_1(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_0(s) + \frac{P_2(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s) + \frac{P_3(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_L(s) + \frac{\frac{g_2(x)}{g_2(L)}\left(s^2 + v\frac{g_1(x)}{g_2(x)}s\right)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}X_1\hat{\theta}_L(s) \qquad \{26\}$$

$$\boxed{\hat{y}(x,s) = \frac{P_1(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_0(s) + \frac{P_2(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s) + \frac{P_4(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_L(s)} \qquad \{27\}$$

where,

$$\begin{aligned} P_4(x,s) &= \frac{1}{g_2(L)}\left[P_3(s) + X_1 g_2(x)s^2 + X_1 g_2(x) v g_1(x) s\right] \\ &= \frac{1}{g_2(L)}\left[\left(g_1(x)g_2(L) - g_1(L)g_2(x) + X_1 g_2(x)\right)s^2 + X_1 g_1(x) v s + v^2 g_1(x)\right] \end{aligned} \qquad \{28\}$$

and similarly, Equation {21}:

$$\boxed{\frac{\partial\hat{y}}{\partial x}(x,s) = \frac{P_{1x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_0(s) + \frac{P_{2x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s) + \frac{P_{4x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}_L(s)} \qquad \{29\}$$

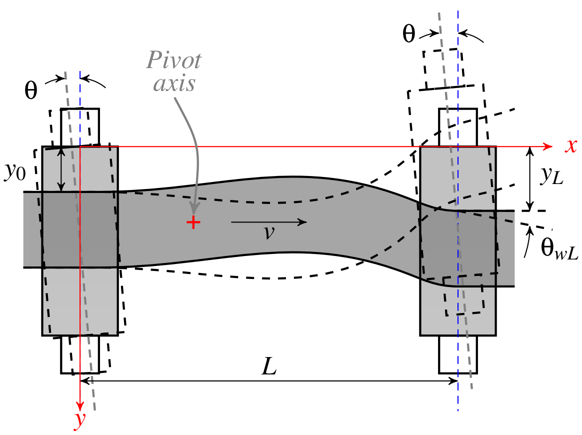

Figure 7 – Web span with offset pivot guide

where,

$$P_{4x}(x,s) = \frac{1}{g_2(L)}\left[P_{3x}(s) + X_1 g_{2x}(x)s^2 + X_1 v g_{1x}(x)s\right]$$
$$= \frac{1}{g_2(L)}\left[\left(g_{1x}(x)g_2(L) - g_1(L)g_{2x}(x) + X_1 g_{2x}(x)\right)s^2 + X_1 g_{1x}(x)vs + v^2 g_{1x}(x)\right] \quad \{30\}$$

4.5 Offset Pivot Guide Roller

In this guide mechanism, two consecutive rollers rotate around a pivot point that is located at a distance X_p from the entering roller; see Figure 7. Since both rollers rotate around the same pivot point, we have $(\theta_0(t) = \theta_L(t) = \theta(t))$. For small angle rotations, the lateral displacement of the downstream roller is $z_L(t) = (L - X_p)\theta_L(t)$. For this situation, Equation {19} can be simplified to the following:

$$\hat{y}(x,s) = \frac{P_1(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}(s) + \frac{P_2(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s)$$
$$+ \frac{P_3(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}(s) + \frac{\frac{g_2(x)}{g_2(L)}\left(s^2 + v\frac{g_1(x)}{g_2(x)}s\right)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}(L - X_p)\hat{\theta}(s)$$

$$\boxed{\hat{y}(x,s) = \frac{P_5(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}(s) + \frac{P_2(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s)} \quad \{31\}$$

where

$$P_5(x,s) = \frac{1}{g_2(L)}\left[P_1(s) + P_3(s) + (L - X_p)g_2(x)s^2 + (L - X_p)vg_1(x)s\right]$$
$$= \frac{1}{g_2(L)}\left[\left(xg_2(L) - X_p g_2(x)\right)s^2 + \left(xg_1(L) - g_1(x)X_p\right)vs + xv^2\right] \quad \{32\}$$

The spatially dependent web slope is given by

$$\boxed{\frac{\partial \hat{y}}{\partial x}(x,s) = \frac{P_{5x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{\theta}(s) + \frac{P_{2x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s)} \qquad \{33\}$$

where

$$\begin{aligned} P_{5x}(x,s) =& \frac{1}{g_2(L)}\left[P_{1x}(s) + P_{3x}(s) + (L - X_p)g_{2x}(x)s^2 + (L - X_p)vg_1(x)s\right] \\ =& \frac{1}{g_2(L)}\left[(g_2(L) - X_p g_{2x}(x))\,s^2 + (g_1(L) - g_{1x}(x)X_p)vs + v^2\right] \end{aligned} \qquad \{34\}$$

The commonly used guides have $X_p = 0$, which results in the following:

$$\hat{y}_L(s) = L\hat{\theta}(s) + \frac{\frac{v^2}{g_2(L)}}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s) \qquad \{35\}$$

$$\hat{y}(x) = x\hat{\theta}(s) + \frac{P_2(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s) \qquad \{36\}$$

$$\frac{\partial \hat{y}}{\partial x}(x,s) = \hat{\theta}(s) + \frac{P_{2x}(s)}{s^2 + v\frac{g_1(L)}{g_2(L)}s + \frac{v^2}{g_2(L)}}\hat{y}_0(s) \qquad \{37\}$$

Note that Equation {35} matches with Equation {27} if we consider $\hat{\theta}_0(s) = \hat{\theta}_L(s) = \hat{\theta}(s)$ and $X_1 = L$.

5 Simulations

Definition	**Symbol**	**Value**	**Units**
Entry Span Length	L_o	3.833 (1.1684)	$ft(m)$
Exit Span Length	L	3.2808 (1)	$ft(m)$
Integral Gain	ki	10	
Pivoting Distance	X_1	2.5833 (0.7874)	$ft(m)$
Proportional gain	kp	90	
Tension	T	10 (44.48)	$lbf(N)$
Transport Speed	v	500 (2.54)	$ft/min(m/s)$
Web width	w	5.4 (137.16)	in (mm)
Web thickness	h	0.005 (0.127)	in (mm)
Young's Module	E	0.40466 (2.76×10^9)	Mpsi (Pa)

Table 1 – Parameter values used in the simulations

Model simulation were conducted with the proposed transfer functions. The values of the web and guide parameters used in the simulations are provided in Table 1. The system considered for simulations consists of two spans, the first span is the entry span with guide roller (RPG) as the downstream roller and the second span is the exit span of the guide, see Figure 8. Lateral web position at the guide roller is controlled using a Proportional Integral controller. A sinusoidal position disturbance (5 mm amplitude) at the upstream roller is employed to evaluate the response. Figure 9 provides the response for different locations along the entry span and Figures 10 and 12 and **??** shows the guide roller angle input ($\theta_L(t)$). For the case of a wrap angle of 90° and 0° misalignment of the guide roller with respect to the plane of the exit span, response at several locations in the exit span are provided in Figure 11; for the 80° and 2° misalignment, the response is provided in Figure 13. Note that as the distance increases from the guide roller along the exit span, the amplitude of the oscillations amplify.

6 CONCLUDING REMARKS

In this paper, we have derived spatially dependent transfer functions for web lateral dynamics. The obvious benefits are that one can obtain the evolution of lateral position response at any location in the span as well as all higher-order spatial partial derivatives, such as, slope, moment, shear force, etc. Further, these transfer functions may be used to control the lateral position and slope at a prescribed location in the span other than on the roller. In addition, Roll-to-Roll (R2R) manufacturing of flexible and printed electronics requires positioning the web precisely in both lateral and longitudinal directions. Traditional printing systems have relied solely on longitudinal registration for printing presses with multiple print units. With the goal of achieving print registration accuracy within a few microns in R2R printing of electronics, this work is expected to aid in a more precise analysis of lateral behavior and facilitate the design of model-based lateral control systems for achieving tight regulation of lateral print registration.

ACKNOWLEDGEMENTS

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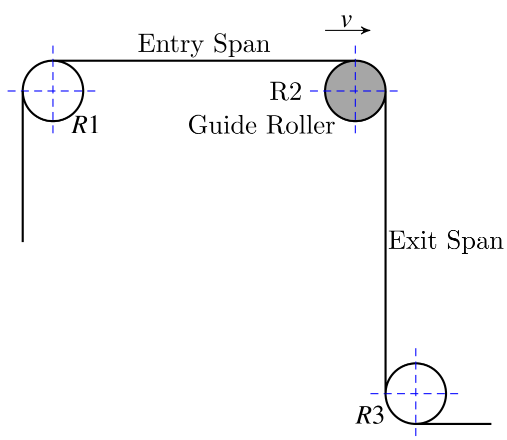

Figure 8 – Representation of two spans in a remotely pivoted guide for the simulation

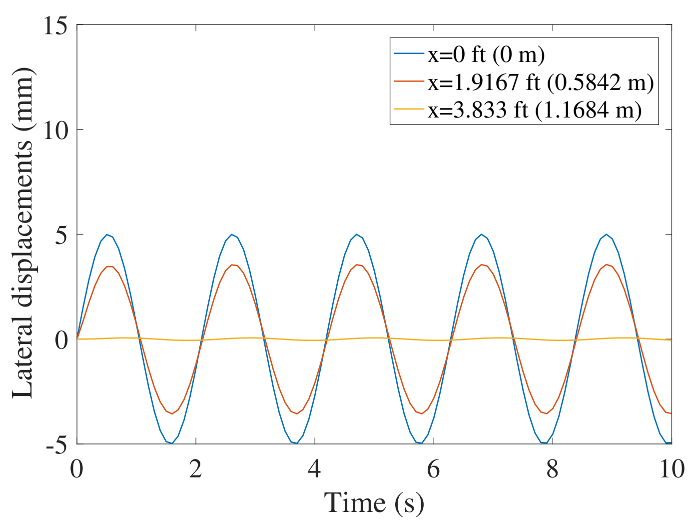

Figure 9 – Response at different locations (x) in the entry span; span length = 3.833 ft (1.1684 meter)

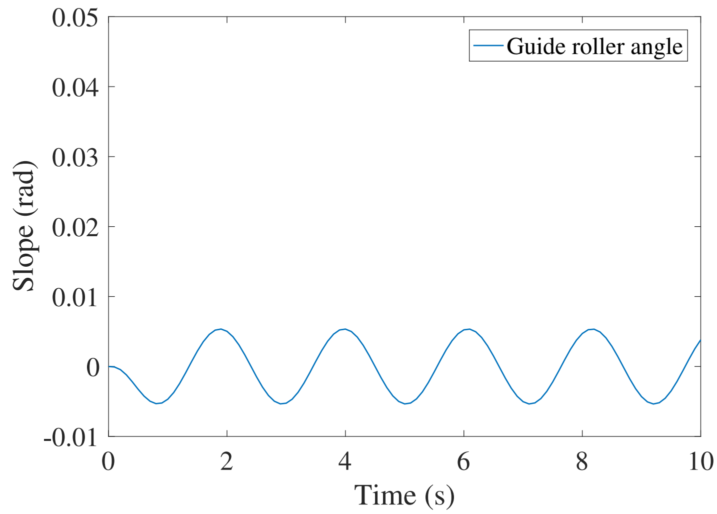

Figure 10 – Guide roller angle $\theta_L(t)$ (input)

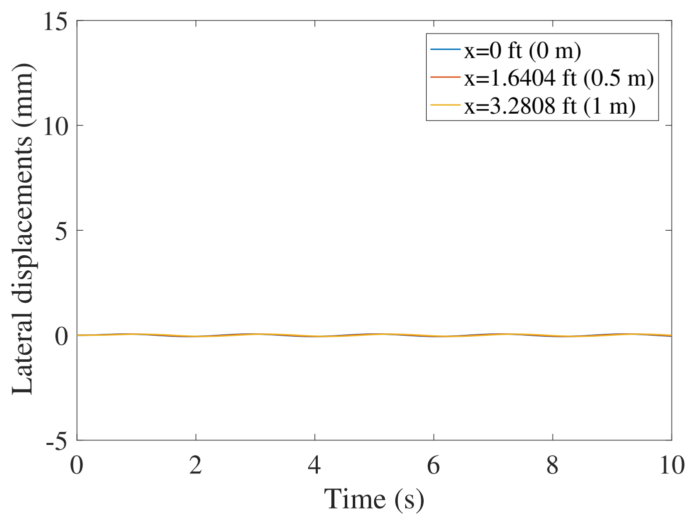

Figure 11 – Response at different locations (x) in exit span for 90° wrap on guide roller and 0° misalignment in the plane of the exist span; span length = 3.2808 ft (1 meter)

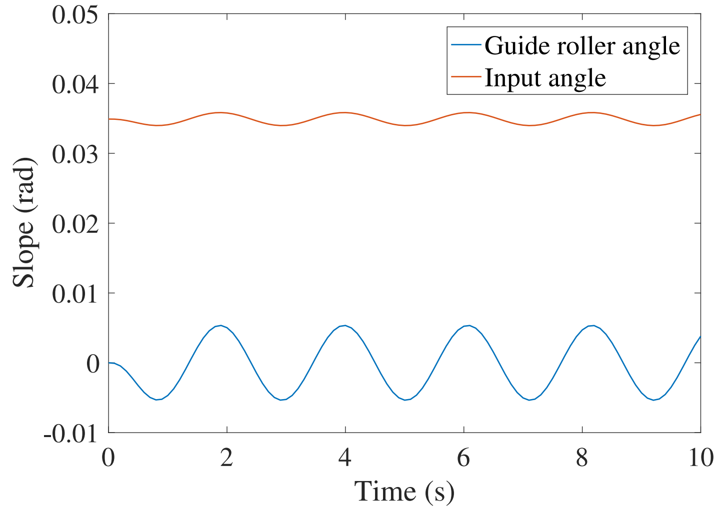

Figure 12 – Guide roller angle ($\theta_L(t)$) for wrap angle on guide roller of 80°; guide roller misalignment as seen in the exit span = 2°

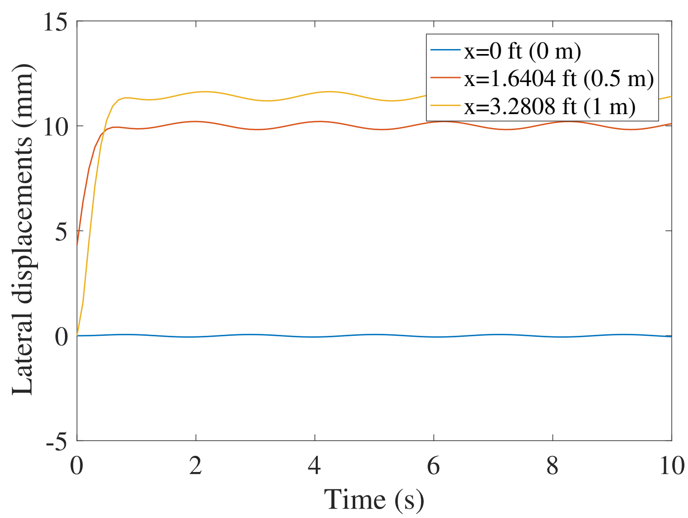

Figure 13 – Response at different locations (x) in exit span for 80° wrap on guide roller and 2° misalignment in the plane of the exist span; span length = 3.2808 ft (1 meter)