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Title of Study: GEOMETRIC EQUIVALENTS TO CHANGING RATES

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Scope of Study: The teaching of the concepts involved in the study of acceleration and its effect on velocity and distance is difficult when working with students of general science and physics in high school, because their training in mathematics is necessarily limited by their ages. This study develops a method of solution of problems involving changing rates by the use of figures of reference having dimensions equivalent to the rates and constants encountered in the problems. The principles employed are similar to those used by Galileo and Descartes. The limitation of dimensions is overcome by reducing multiple dimensional figures to an equivalent line for use in representing figures requiring more than three dimensions. The development is by the use of simple drawings accompanied by explanations which require little more than the knowledge of the formulas for the areas and volumes of simple straight line figures, since the material is for high school use, and no attempt is made to proceed further than the derivation of a concept of the basis for the indefinite integral and the derivative.

James H. Zant

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GEOMETRIC EQUIVALENTS TO CHANGING RATES

By

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GEOMETRIC EQUIVALENTS TO CHANGING RATES

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PREFACE

Indebtedness is acknowledged to the National Science Foundation, without whose support this paper and the year's study accompanying it would have been impossible. Special appreciation is expressed for the helpful advice of Dr. James Zant, the faculty advisor, and the understanding inspiration of Dr. Richard Johnson.

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CHAPTER I

INTRODUCTION

This paper presents a simple method of approach to the solution of problems involving changing rates by solving equivalent areas and volumes. The method discussed is not new, but it is, apparently, little used, although the writer has used it for a number of years in teaching certain relationships in general science and physics in high school. The use of geometric equivalents in solving problems was introduced by the Greeks, but the application of the method to problems in rates was not made until Galileo used it in the representation of the relationship between distance and velocity through a period of time in dealing with problems in the study of falling bodies.[1] The discovery and development of the calculus apparently rendered this method obsolete so far as mathematicians are concerned, since the calculus may be used more universally for such problems and since the assignment of space dimensions is necessarily limited.

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[1]Paul F. Schmidt. "Models of Scientific Thought." American Scientist, XLV (March 1957), 141-143.

The concept of geometric equivalence is, however, useful in that it (1) simplifies the instruction of students in problems involving changing rates, even though they may lack knowledge of mathematics other than simple arithmetic; (2) serves as a quick method by which students may verify answers obtained algebraically; and (3) encourages students to study additional years of mathematics in high school, since the concept of integrals and derivatives may be approached without the formal use of rigorous proofs.

The presentation of the material is purposely simple in this report, since it is designed primarily for use with students who have not had opportunity to develop much background. In the interest of brevity, the material used for both discussion and illustration deals with changing rates in velocity, but the same geometric concepts may be used for many other types of rate problems.

Chapter II and Chapter III of this paper present the mathematical basis of the concept. This material has not in practice been submitted to classes of students unless there was exhibited a definite desire for proof. Chapter IV demonstrates the practical application of the method and constitutes the usual method of classroom presentation. Chapter V shows an approach to very simple integration as well as a simple concept of the derivative and, again, has not in the past been submitted to classes.

CHAPTER II

TWO DIMENSIONAL EQUIVALENTS

Areas may be used to represent many products other than areas, since one may frequently show them to be numerically equivalent, as is illustrated in Fig. 1, which demonstrates the relationship between the area shown and the equation $s = vt$. The distance, s, represented by the area of the rectangle, is 6 units of distance; the velocity, v, represented by its length, is 3 units of length per unit of time; and the time, t, represented by the width, is 2 units of time. It need not be confusing that the area shown represents a distance, since one is not concerned with an area as such but rather with an area which is equivalent to a product. The same area might easily represent the product of two pens and three cows per pen, in which event there should be no confusion. The concept is the same; each case is a product of two factors, one of which is expressed in terms of one unit and the other of which is expressed in terms of a rate with the unit of the former in its denominator, ie., hours x miles/hour = miles; pens x cows/pen = cows; sacks x apples/sack = apples, etc.[2]

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[2]M. Wiles Keller and James H. Zant. Basic Mathematics. Boston: Houghton Mifflin Co., 1948, 160-162.

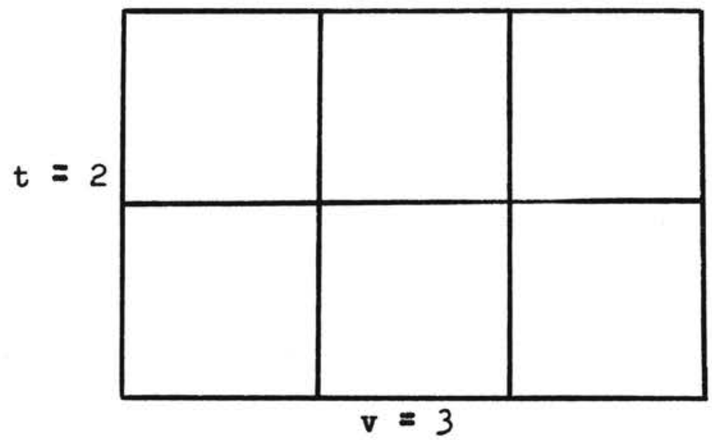

Fig. 1

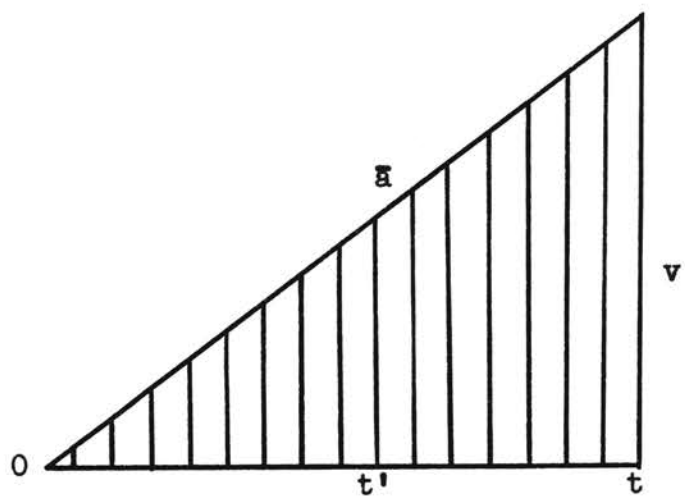

Fig. 2

It is often necessary to use areas other than rectangles to conveniently represent some types of equivalence. Such an occasion occurs when it is necessary to consider a product such that one factor is of the nature of a changing rate, as in Fig. 2, in which the distance, s, is the product of a time, t', and a constantly changing velocity, v. The vertical lines represent the values of the velocity, v, in terms of units of linear measurement per unit of time at various times, t. The total time, t', is expressed in units of time after $t = 0$ for the case illustrated. Intuitively, the distance, s, should be equivalent to the shaded area formed. This may be proven true, if the assumption is made that the distance is a product of the average velocity and the time, by showing that the area of a triangle is the product of the altitude and the average width. This concept may be approached more easily by reference to Fig. 3.

Given: triangle ABC with altitude, a, and base, AC.

By Construction: DE, $\perp$ bisector of a; IJ $\perp$ DE at D; HK $\perp$ DE at E; HI $\parallel$ AC through B.

To Prove: (1) DE is the average width of triangle ABC, and (2) Rectangle HIJK, with length, a, and width, DE, is equal in area to triangle ABC.

Proof: (1) for each line, $d + b + c$, constructed any distance, x, from and $\parallel$ DE, there exists some line, $d' + b' + c'$, which may be constructed a distance, x', from and $\parallel$ DE such that $x = x'$ but in the opposite direction. Then $d = d'$, $c = c'$, $DE = b' = d + b + c = b + d' + c'$; thence, $2(DE) = b + d' + b' + c'$, and $DE = \frac{1}{2}(b + d' + b' + c')$ for each pair, including the pair of lines represented by the line AC and the point B; therefore, DE is the average width of triangle ABC;

(2) triangle AJD $\cong$ triangle DIB and triangle BHE $\cong$

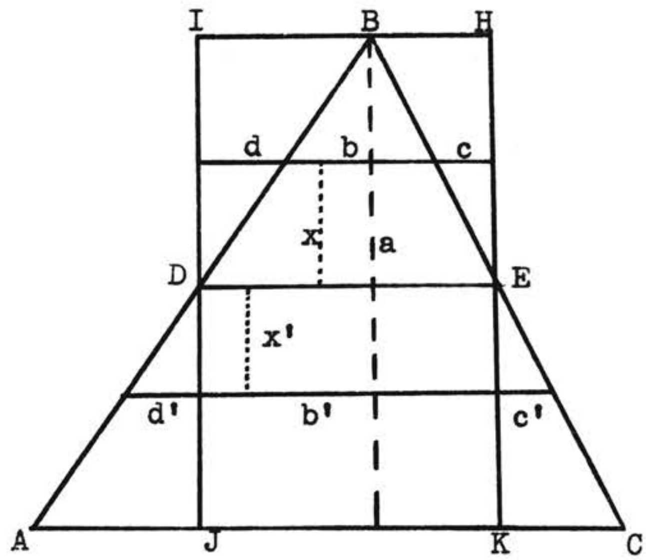

Fig. 3

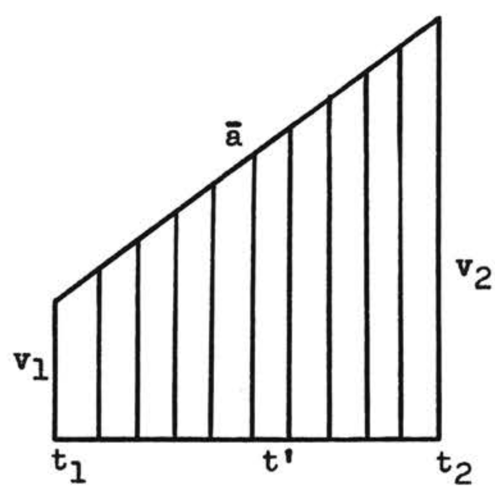

Fig. 4

> triangle KEC; and triangle ABC ≡ triangle ADJ + triangle KEC + triangle BED + rectangle JDEK, and rectangle HIJK ≡ triangle BED + triangle HEB + rectangle JDEK; therefore, triangle ABC = rectangle HIJK, ie., the area of a triangle is the product of the altitude and the average width.

This proof, together with the correspondence between the width of the triangle with the velocities represented, indicate that one may now consider the shaded area of Fig. 2 as representing the equation, $s = vt'/2$; s indicating the distance, v the final velocity, and t' the total time.

A consideration of trapezoid equivalence is convenient in determining the effect of changing rates when the initial velocity is other than zero. Such a case is illustrated in Fig. 4. The shaded area in this figure represents the distance covered by a body initially moving at a velocity, v_1, at a time, t_1, as it accelerates constantly to a velocity, v_2, at a time, t_2. This area may be solved by four general methods, and, since each may be used in various equivalent circumstances, each will be examined briefly. (1) Consider the area as a product of the average velocity, $(v_1 + v_2)/2$, and the time, t' $(t' = t_2 - t_1)$. (2) Consider the area of a trapezoid as being the sum of two triangles having the same altitude, as in Fig. 6, the area of one being $s = v_1t'/2$ and the other being $s = v_2t'/2$. In both triangles, the time, t' is the difference between t_1 and t_2 as in (1). Since the area of the trapezoid may be expressed, $s = v_1t'/2 + v_2t'/2$, then $s = t'[(v_1 + v_2)/2]$, the same as in (1). (3) Consider the special nature of the trapezoids used in representing distances in this paper, ie., each one has one

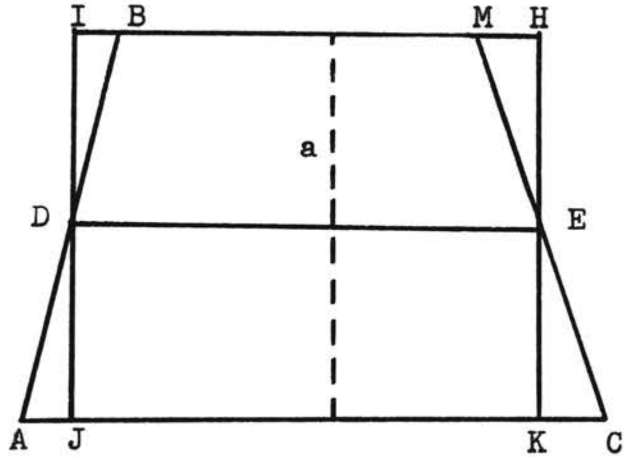

Fig. 5

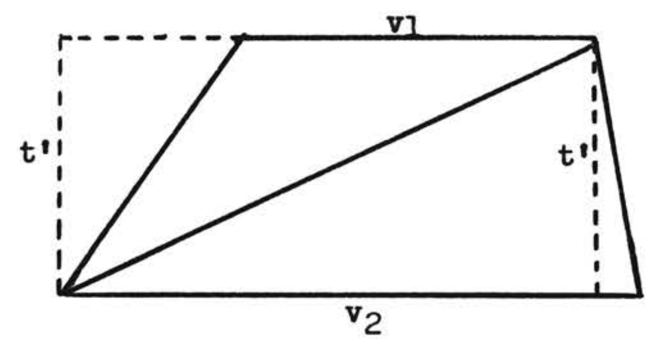

Fig. 6

end which is perpendicular to the parallel sides, as in Fig. 7. This trapezoid is composed of a triangle having its area represented as $s = t' (v_2 - v_1)/2$ and a rectangle having its area represented as $s = t'v_1$. The total area may then be represented as $s = t'v_1 + t' (v_2 - v_1)$; $s = t' [v_1 + (v_2 - v_1)/2]$; $s = t' (2v_1 + v_2 - v_1)/2$; or $s = t' (v_1 + v_2)/2$. Attention is called to the additive nature of the areas in the last two figures under consideration, since this nature gives the most convenient approach to the next method. (4) Consider the trapezoid, as in Fig. 8, as part of a large triangle, OBt_2, and its area the difference between that triangle and the small triangle, OAt_1. It is clear that any one of these areas and all its dimensions may be solved if the other two are known or are solvable. In further consideration of the additive nature of areas of reference, Fig. 9 illustrates a situation in which the velocity, v, becomes zero. Either v_1 or v_2 is negative and the other is positive, the difference in sign corresponding to a difference in direction. The triangle below the line, $v = 0$, corresponds to the distance in one direction and the one above it the distance in the other.

In each of the figures involving time and velocity, one line has been labeled $\bar{a}$. The length of this line has no significance in this paper, but its slope is convenient, since from it one may determine the relationship between velocity and time. The differences in figures 1, 2, and 8 serve to illustrate this idea. In Fig. 1, there is no change in velocity; therefore, the slope, a, of the line, $\bar{a}$,

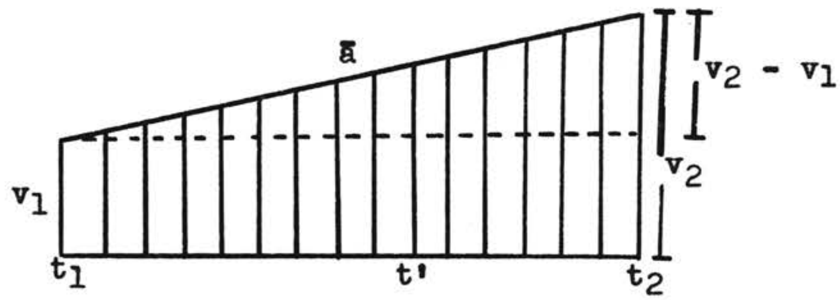

Fig. 7

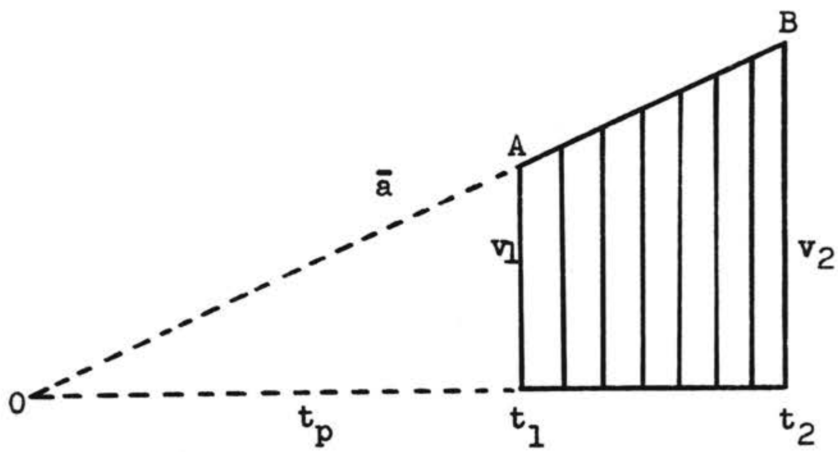

Fig. 8

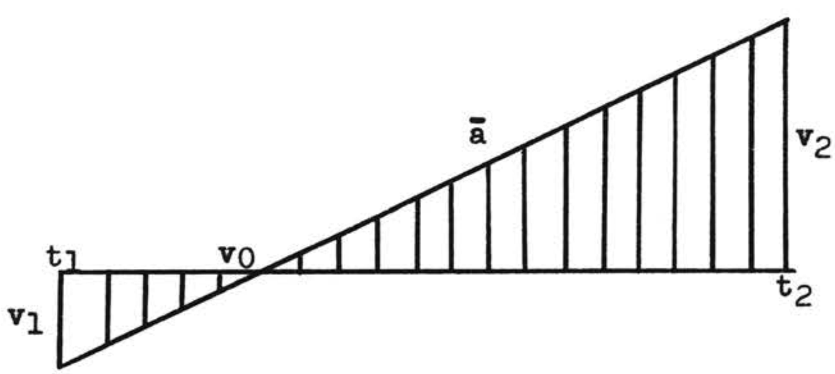

Fig. 9

is zero, ie., all of the distance is the result of the initial velocity through the time indicated. In Fig. 2, there is no initial velocity involved; therefore all of the velocity is related to the time involved, t', and the slope, a, such that $v = at$. The time, t', is in this case the same numerically as t, since the initial time is zero. All of the distance is a result of a changing velocity due to the effects of the slope. We have shown previously that $s = vt^2/2$, but $v = at$; therefore, $s = at^2/2$. Remembering the additive nature of areas in equivalence, it is convenient to apply this same approach to the solution in Fig. 8, the difference between two triangles. The small triangle may be represented as $s = at_1^2/2$ and the large triangle as $s = at_2^2/2$. The trapezoid, then, has area $s = at_2^2/2 - at_1^2/2$; hence $s = a(t_2^2/2 - t_1^2/2)$; or one may say that the distance, s, is the product of the acceleration, a, and the value of $t^2/2$ between $t = t_1$ and $t = t_2$.

CHAPTER III

THREE DIMENSIONAL EQUIVALENTS

In the previous chapter, in connection with Fig. 2, the relationship between the distance and its contributing factors was discussed as being equivalent to a two dimen sional figure, the triangle. It is often convenient to think of this relationship as being three dimensional in nature. The distance, s, was first defined as a product of the time and velocity, as in Fig. 1, but in Fig. 2, the velocity was defined as a product of the time and the acceleration. One may, then think of this velocity as being equivalent to an area, v, with dimensions equivalent to the acceleration, a, and the time, t, as indicated in Fig. 10. Since the acceleration is constant in this case, the thickness of the figure is constant. The time is changing; therefore the height of the figure increases as the length increases, since both dimensions represent the same time. The figure is clearly one-half of a rectangular prism; therefore its volume may be expressed as $s = at^2/2$, ie., the volume of the figure corresponds to the distance.

Not all accelerations are constant, and it is now possible for one to consider accelerations which, themselves, change at a constant rate. Such a case may be brought to

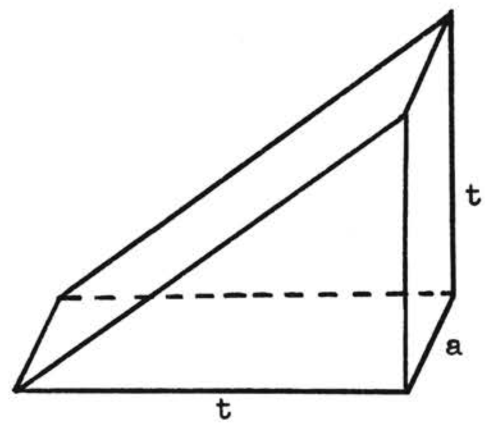

Fig. 10

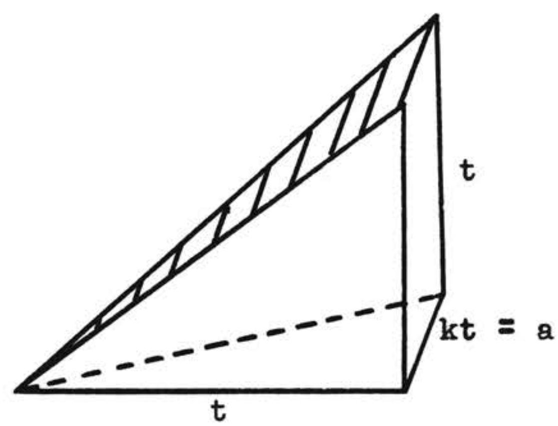

Fig. 11

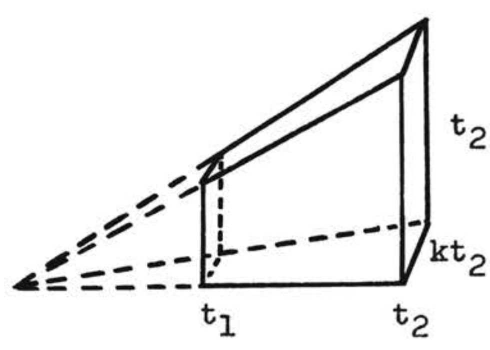

Fig. 12

mind if one considers that force is a product of the mass and the acceleration, ie., $F = ma$. If the mass remains the same while the force is constantly increased, the acceleration will increase. The acceleration will also increase if the force which causes it remains constant while the mass is decreased, as in a propulsive device which burns its own load as fuel. If the mass decreases as time increases according to a constant rate, k, then one may think of the acceleration, a, as being the product of the time, t, and the rate, k, as in Fig. 11. The shaded area is formed by lines which represent the corresponding values of the acceleration, a, at various times, t. The figure formed is a rectangular pyramid, the volume of which is expressed as $s = Bh/3$, where B is the base; however, B represents the final velocity, $v = at = kt^2$, and $h = t$; therefore, $s = kt^3/3$.

It is again necessary to consider situations in which there exists an initial velocity, as in Fig. 12, which illustrates the truncated pyramid as being a part of the entire pyramid. The distance covered between the two times indicated may be expressed, $s = kt_2{}^3/3 - kt_1{}^3/3$; thence $s = k/3(t_2{}^3 - t_1{}^3)$; or one may express the distance, s, as being the product of k and $t^3/3$ between $t=t_1$ and $t= t_2$.

Thus far, the approach of this paper is made by associating an area or a volume with a distance. In order to proceed further in assigning dimensions, it becomes necessary to consider a converse type of thing to the extent of reducing a volume to an area. In establishing this concept, it is convenient to consider the difference between Fig. 12 and Fig. 13. In Fig. 12, one of the dimensions, a, was expressed as the product of a constant rate, k, and the time, t. Clearly, this dimension itself could be expressed as an area, and yet there are no more space dimensions available, since three have already been used. Two of these dimensions, however, are identical and are combined as t^2 in Fig. 13, the constant, k, having been factored and established as a separate dimension. If k in each figure is first considered to be unity, it is clear that the volumes correspond numerically, since each possible vertical line of Fig. 13 corresponds to an area in Fig. 12. That this correspondence exists for any k is then just as clear.

The development of this concept, of course, causes one to consider the possibility that the rate of change, k, could, itself, be changing with time and leads one to picture a case as is illustrated in Fig. 14, in which $k=xt$. If, then, the volume of this figure may be proved, one considers the possibility of replacing t^2 in the figure with t^3 to create another figure, since x may be a product of t and some changing rate, t^3 with t^4, etc. It is convenient to consider a sequence or progression as proof of distances involved from reference to Figs. 1, 2, 12, and 14. From these may be derived the sequence: $vt^1/1$, $at^2/2$, $kt^3/3$, $xt^4/4, \ldots\ldots . yt^n/n$. From this sequence, it is held that

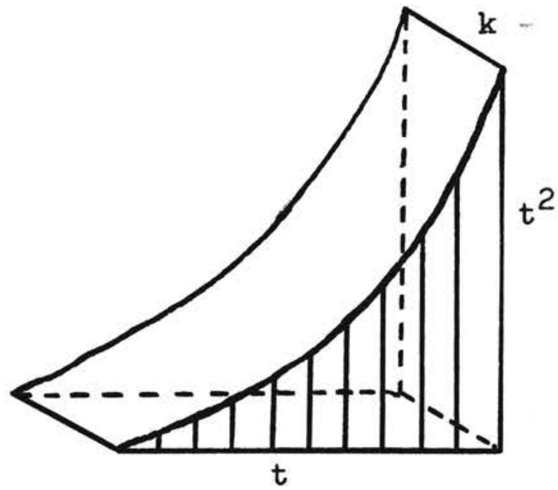

Fig. 13

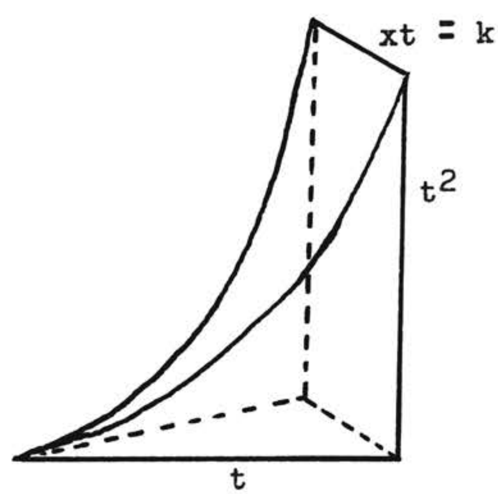

Fig. 14

a product may have many dimensions related to one changing factor, such as the factor, t, in the sequence given, and that the total effect of these related dimensions varies directly with their products and inversely with the number of times the changing factor occurs in the combined product. The volume of the figure in Fig. 14, then, should be expressed, $s = xt^4/4$, since time is the variable factor and occurs in four dimensions, one of which is modified by another factor, x.

CHAPTER IV

PRACTICAL APPLICATIONS

Problem: A train traveled for 9 hrs. at a constant velocity of 60 mph. How far did it go during the time? Solution: Since the velocity is constant, time is involved in only one dimension, and the distance is equivalent to a rectangle as in Fig. 1, ie., $s = vt$ or 540 mi.

Problem: An automobile accelerates constantly from a position of rest for ten seconds and at the end of that time has a velocity of 80 ft./sec. How far has it moved? Solution: since velocity contains time as one of its dimensions, time is involved in two dimensions as in Fig. 2., ie., $s = vt/2$.

Problem: A sled is started with an initial velocity of 10 ft./sec down a hill on which it accelerates 8 ft./sec. per second. How far will it go in 6 seconds? Solution: Part of the distance is as a result of time in two dimensions as in Fig. 7, ie., $s = v_1 t + at^2/2$, or 204 ft.

Problem: A conical tank with the point down is filled with water from the bottom. If the tank is 10 ft. deep and 20 ft. in diameter at the top, how many foot pounds of work will be needed to fill it?

Solution: as the tank fills, the height of the water increases. Concurrently, the area of the surface increases directly according to the square of the height modified by the constant, π, making the volume a result of a variable, h, in three dimensions and the constant, π; but the work required to lift a cubic foot of water varies with h modified by a constant density, making the total work a result of a variable, h, in four dimensions modified by the density and π expressed as constants. Therefore, $W = 62\frac{1}{2}\ h^4/4$, or 49,087.5 foot pounds.

CHAPTER V

CONCLUSION

In closing this report, it seems appropriate to point out that a non-rigorous approach to two fundamental concepts of the calculus, namely the integral and the derivative, may be made through consideration of the material discussed.

Throughout this paper, the term product has been used to designate the result of a multiplication type operation in which one or more factors are of the nature of a changing rate. Clearly, this is not true multiplication, and the answer is not a true product, unless the changing dimension exists in only one factor and the other factor remains constant, as in Fig. 1. The term integration more aptly applies to this type of operation in that it is more general and, in addition to true multiplication, contains the operation illustrated in all of the three dimensional figures wherein areas, the bases of the figures, become solids when combined with an additional variable dimension related to one or more dimensions of the area.

The inverse operation, ie., the determination of the base of the figure from the known volume and the dimension represented as the altitude, results in the derivative, hereto referred to as a factor. Only if the

the integral is a true product, as in Fig. 1, is the derivative a true factor, ie., a constant.

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Master of Science

Report: GEOMETRIC EQUIVALENTS TO CHANGING RATES

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