THE EFFECT OF MASS TRANSFER ON MULTI-SPAN LATERAL DYNAMICS OF A UNIFORM WEB

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ABSTRACT

This paper will show that the acceleration equation used in multi-span lateral dynamic models is a consequence of mass transfer between spans[1]. Mass transfer effects fully account for the equation currently used in Euler-Bernoulli models and provides an analytical pathway to an acceleration equation that incorporates shear deformation. It also ties together contributions from three other researchers – John Shelton, who pioneered the use of beam theory in models of lateral web dynamics, Lisa Sievers, who proposed the principle of continuity of bending angle and Richard Benson, who was the first to publish an acceleration equation that correctly incorporates shear deformation.

The acceleration equation is applied in conjunction with the normal entry rule to convert information about web shape to a time-based differential equation. Several versions of the acceleration equation have been proposed that include one or more terms to account for shear. All but Benson's lead to results that either contradict observed web behavior or else fail to provide meaningful solutions.

Consideration of mass transfer arises from the moment of force that develops in a web when it is displaced by an upstream disturbance or by movement of a roller that is transporting it. Any moment at the entry to a roller causes the longitudinal tension to vary in a linear fashion across the width of the web. Analysis of the effect of this tension profile on mass flow leads to: 1) The acceleration equation that is currently used for models without shear deformation. 2) A new understanding of why this equation works and improved insight into how multi-span systems behave 3) An acceleration equation for models that include shear deformation 4) Identification of a new mechanism that can

[1] For purposes of this paper, the phrase "mass transfer between spans" does not mean movement of mass in a literal sense. It refers to the complementary changes in mass that occur in adjacent spans due to a change in longitudinal flow rate at a roller (mass on one side of the roller increasing as it decreases by the same amount on the other side). In this instance, the flow rate change is caused by cross-web variation in longitudinal tension.

cause micro-slip at the point of entry onto a roller. 5) Justification for the zero-moment steady state boundary condition at the downstream roller.

NOMENCLATURE

A	cross sectional area of web
E	elastic modulus
G	shear modulus
h	thickness of web
I	area moment of inertia
J	rotational inertia
L	span length
m	mass per unit length
n	Shear factor for Timoshenko beam
s	Laplace variable
t	time
T	tension in units of force
V_o	web velocity in machine direction
x	distance along length of web
y	lateral displacement of web
y_0	lateral web displacement at upstream roller, relative to ground
y_L	lateral web displacement at downstream roller, relative to ground
z	lateral displacement of roller relative to ground
θ_L	angle between web plane and plane of roller motion at entry to roller
θ_0	angle between web plane and plane of roller motion at exit of roller
β	boundary defect angle
γ	angle of roller axis
ρ	density
ϕ	rotation of cross section (bending angle)
ψ	shear angle
$_0$	subscript indicating value of variable at $x = 0$
$_L$	subscript indicating value of variable at $x = L$

INTRODUCTION

The paper is organized as follows.

1. The equations for the static elastic curve of the web between rollers are recapped, following Lisa Sievers treatment.
2. The velocity and acceleration equations, currently used to convert the web shape equation to a function of time, are presented and some of the problems with the acceleration equation are discussed.
3. The effects of mass flow are explained and analyzed.
4. A new acceleration equation for the Timoshenko model is introduced.
5. The new acceleration equation is used to create a multi-span differential equation.
6. A method for isolating and testing a single span is described.
7. An example of the effect of the new acceleration equation is illustrated by plotting the frequency response for fixed parallel idlers.
8. Benson's acceleration equation is discussed.

9. Other work on the acceleration equation is reviewed
10. Validity tests for the acceleration equation are summarized.
11. Equations for calculating projections of the roller pivoting angles on entering and exiting spans are presented. [This seemingly trivial subject has a major impact on multi-span models.]
12. Transient behavior of the boundary defect creates a possibility of microslip at the entry to a roller. A criterion for its onset is developed.
13. Results are summarized.

MODELING WEB SHAPE

The Elastic Curve

Shelton was the first to use beam theory in models of lateral web dynamics. He derived equations for the elastic curve of single spans using both Euler-Bernoulli and Timoshenko beam theories [1, 2]. The method presented here is due to Lisa Sievers and is particularly suited to multi-span problems [3]. It begins by first observing that the bending and shear angles are additive[2]. The bending is defined as a cross-sectional plane that is normal to the beam centerline before the application of external forces, ϕ is the rotation angle of the face (bending angle), ψ is the shear angle.

$$\frac{dy}{dx}=\phi+\psi \qquad \{1\}$$

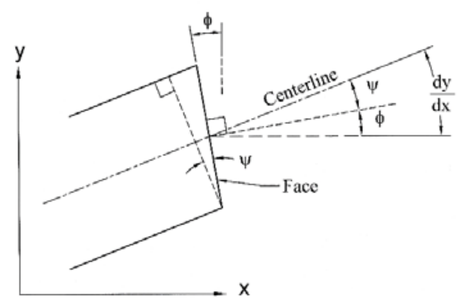

Figure 1 – Relationship of Slope, Shear Angle and Bending Angle

$\frac{dy}{dx}$= slope , ψ= shear angle , ϕ=bending angle

She then applied Hamilton's principle [4] to derive the equations of motion. This requires defining the kinetic energy K and potential energy V for the beam. The kinetic portion includes the effect of rotation of bending angle.

In this equation, m is the mass per unit of length, V_o is the transport velocity in the machine direction, y is the lateral deflection, J is the rotational inertia per unit of length and ϕ is the bending angle. The time derivatives are transformed to a Eulerian frame of reference.

[2] Note: Other names commonly used for the bending angle ϕ are face angle and angle of cross section rotation.

$$K=\frac{1}{2}\int_0^L m\left(V_o^2+\left(\frac{\partial y}{\partial t}+V_o\frac{\partial y}{\partial x}\right)^2\right)dx+\frac{1}{2}\int_0^L J\left(\frac{\partial \phi}{\partial t}+V_o\frac{\partial \phi}{\partial x}\right)^2 dx \qquad \{2\}$$

The potential energy is,

$$V=\frac{1}{2}\int_0^L EI\left(\frac{\partial \phi}{\partial x}\right)^2 dx+\frac{1}{2}\int_0^L \frac{AG}{n}(\psi)^2 dx+\frac{1}{2}\int_0^L T\left(\frac{\partial y}{\partial x}\right)^2 dx \qquad \{3\}$$

where A is the cross-sectional area, G is the shear modulus, n is the Timoshenko shear factor (approximately 1.2 for most web applications), E is the elastic modulus, I is the area moment of inertia, ϕ is the bending angle and ψ is the shear angle.

The first term is the energy due to the bending angle. The second one is the energy due to shear strain. The last one is the energy due to the interaction of longitudinal tension and slope.

This produces a solution that includes both time and spatial derivatives. The time derivatives are useful in determining the potential effect of natural vibrations. She found that the separation between the natural frequencies of the web and frequencies of interest in typical applications, while not as great as one might expect, are usually adequate to safely ignore the time-related terms. Details may be found in several references [3, 5, 6]. When the time-related terms are removed, the following two equations are left,

$$\left(1+\frac{nT}{AG}\right)\frac{d^2y}{dx^2}-\frac{d\phi}{dx}=0 \qquad \{4\}$$

$$EI\frac{d^2\phi}{dx^2}+\frac{AG}{n}\left(\frac{dy}{dx}-\phi\right)=0 \qquad \{5\}$$

These relationships can be manipulated to obtain the same fourth order differential equation found by Shelton.

$$\frac{d^4y}{dx^4}-K^2\frac{d^2y}{dx^2}=0 \qquad \{6\}$$

where,

$$K^2=\frac{T}{EI\left(1+\frac{nT}{AG}\right)} \qquad \{7\}$$

The solution to {6}, familiar to all web handling researchers, is

$$y(x)=C_1\sinh(Kx)+C_2\cosh(Kx)+C_3x+C_4 \qquad \{8\}$$

The solution just described applies to a Timoshenko beam model that includes the effects of shear deformation. It defaults to the Euler-Bernoulli (E-B) beam model if the shear factor n is set to zero.

Boundary Conditions

In this model, as in all other multi-span models to-date, the interaction of the web with rollers is greatly simplified. The width of the contact zone in the process direction is assumed to be zero.

Four boundary conditions are required. Lateral position at the upstream and downstream rollers provide two of them.

Sievers observed that the presence of shear deformation in the Timoshenko model causes both the slope and the shear angle to be discontinuous at rollers. Bending angle, however is continuous, even where a point force is applied. So, she used bending angle for the other pair of boundary conditions. Thus, both boundary conditions at the downstream end of a span become the boundary conditions for the upstream end of the next span. This is a crucial feature of the Timoshenko multi-span model.

Expressions for shear angle ψ and bending angle ϕ are derived from equations {1}, {4} and {5}.

$$\psi = -EIa\frac{n}{AG}\frac{d^3y}{dx^3} \tag{9}$$

where

$$a = 1 + \frac{nT}{AG} \tag{10}$$

and,

$$\phi = \frac{dy}{dx} + EIa\frac{n}{AG}\frac{d^3y}{dx^3} \tag{11}$$

So, the boundary conditions of the Timoshenko beam model will be,

$$\begin{array}{ll} y\big|_{x=0} = y_0 & y\big|_{x=L} = y_L \\ \left.\frac{dy}{dx}\right|_{x=0} + EIa\frac{n}{AG}\left.\frac{d^3y}{dx^3}\right|_{x=0} = \phi_0 & \left.\frac{dy}{dx}\right|_{x=L} + EIa\frac{n}{AG}\left.\frac{d^3y}{dx^3}\right|_{x=L} = \phi_L \end{array} \tag{12}$$

Equation {8} and its derivatives are substituted into the four equations of {12} which are then solved simultaneously for C_1, C_2, C_3 and C_4.

The Static Equation of Web Shape

Inserting values {12} into {8} and collecting terms,

$$y(x) = y_0 + \left(y_0 - y_L\right)g_4(x) + \phi_L g_5(x) + \phi_0 g_6(x) \tag{13}$$

where,

$$g_4(x) = \frac{\cosh(Kx)+\cosh(KL)-\cosh(KL-Kx)-Kax\sinh(KL)-1}{KLa\sinh(KL)-2(\cosh(KL-1)}$$

$$g_5(x) = \frac{KLa(\cosh(Kx)-1)-Kax(\cosh(KL)-1)-\sinh(Kx)-\sinh(KL-Kx)+\sinh(KL)}{Ka[KLa\sinh(KL)-2(\cosh(KL-1)]} \qquad \{14\}$$

$$g_6(x) = \frac{\sinh(Kx)-\sinh(KL)+\sinh(KL-Kx)-KLa(\cosh(KL-Kx)-1)+Ka(L-x)(\cosh(KL)-1)}{Ka[KLa\sinh(KL)-2(\cosh(KL-1)]}$$

Equations {14} are called shape functions.

Following the example of Young, Shelton and Kardimilas (YSK) [7], y_o appears twice in expression {13}. This reduces the number of shape functions from four to three.

Two other equations that will be needed later are the first and second derivatives of {13} at $x = L$.

$$\left.\frac{dy(x)}{dx}\right|_L = \left(y_0 - y_L\right)\frac{h_1}{L} + \phi_L h_2 + \phi_0 h_3 \qquad \{15\}$$

$$\left.\frac{d^2y(x)}{dx^2}\right|_L = \left(y_0 - y_L\right)\frac{g_1}{L^2} + \phi_L\frac{g_2}{L} + \phi_0\frac{g_3}{L} \qquad \{16\}$$

where,

$$h_1 = \frac{KLa\sinh(KL)(1-a)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$

$$h_2 = \frac{(a+1)(1-\cosh(KL))+KLa\sinh(KL)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]} \qquad \{17\}$$

$$h_3 = \frac{(a-1)(1-\cosh(KL))}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$

$$g_1 = \frac{K^2L^2a(\cosh(KL)-1)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$

$$g_2 = \frac{KL(KLa\cosh(KL)-\sinh(KL))}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]} \qquad \{18\}$$

$$g_3 = \frac{KL(\sinh(KL)-KLa)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$

The slope equation {15} reveals an important difference between the E-B and Timoshenko models. In the E-B model, nT/AG is zero. This means that $a = 1$ and that in turn causes $h_1 = h_3 = 0$ and $h_2 = 1$. This reduces {15} to,

$$\left.\frac{dy(x)}{dx}\right|_L = \phi_L \qquad \{19\}$$

So, in the E-B model, the slope of the web at $x = L$ is equal to ϕ_L and, unlike the Timoshenko model, is not a function of y_L, y_0 or ϕ_0.

THE TIME EQUATIONS THAT CONVERT THE STATIC WEB SHAPE TO A DYNAMIC EQUATION[3]

In “Lateral Dynamics of a Moving Web” [1], John Shelton introduced two relationships that are used to convert the equation for the elastic curve of a static web to a dynamic equation.

The Velocity Equation

The first equation, which is usually called the normal entry equation, is based on a principle that has been known since the earliest days of roll-to-roll processing. It defines the lateral velocity of the web at the downstream roller. In its simplest form, it states that a web entering onto a roller at an angle will “track” laterally on the roller surface until it is in perpendicular alignment with the roller axis. It is,

$$\frac{dy_L}{dt} = V_o\left(\gamma_L - \frac{dy_L}{dx}\right) + \frac{dz_L}{dt} \qquad \{20\}$$

The first group of terms on the right side is the lateral velocity due to the interaction of surface velocity of the roller and the slope of the web. The last term is the lateral velocity of the roller itself. The velocity term on the left is the lateral velocity of the web relative to ground. It is important to remember that {20} is true only at $x = L$ and should not, therefore, be differentiated with respect to x.

The Acceleration Equation

The second equation relates acceleration to web curvature. For a web without shear deformation (Euler- Bernoulli beam), it is,

$$\frac{d^2y_L}{dt^2} = V_o^2\frac{d^2y_L}{dx^2} + \frac{d^2z_L}{dt^2} \qquad \{21\}$$

These equations are used to create a time-based differential equation by first solving equation {15} for ϕ_L and substituting the result into equation {16}. Then, equations {20} and {21} are used to replace the first and second order spatial derivatives with time derivatives.

Issues with the Acceleration Equation

Shelton notes that the acceleration equation isn’t merely the derivative of {20} because this would produce an extra term containing the velocity of the roller swiveling, $d\gamma/dt$ and there can be no $d\gamma/dt$ term. He says,

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[3] In a model like this, it is easy to get partial and ordinary derivatives confused. There are many places where derivatives of y apply only at $x = 0$ or $x = L$. In those cases, partial derivatives evaluated at those locations will be written as ordinary derivatives with subscripts 0 and L. For example,

$$\left.\frac{\partial y}{\partial x}\right|_{x=L} \quad \textit{will be written as} \quad \frac{dy_L}{dx}$$

"…no acceleration can occur as an instantaneous result of roller swiveling, but only indirectly as the web curvature changes. A suddenly swiveling roller instantaneously swivels the downstream end of the web an equal amount, so that no instantaneous change in steering rate occurs, in contrast to the first-order theory of Chapter III."[4]

He derives it by assuming the web slope at the point of entry onto the roller changes because the curvature upstream of the point of entry is transported onto the roller by the web's longitudinal motion. As will be seen, this leads to the correct result for the E-B beam, however, his rationale provides little insight into how to incorporate the effect of shear. This may be why the equation he proposed for the Timoshenko beam was later shown to produce results that contradict observed behavior. Attempts by others will be discussed near the end of the paper.

There is another thing about the acceleration equation that seems odd. The velocity equation {20} will produce acceleration under the influence of input parameters, z_L and γ. So, why is an additional source of acceleration necessary?

The clue to the answer is the word "source" in the last sentence. Perhaps attention should be directed away from acceleration itself and towards some physical phenomenon that alters the normal entry condition. A good place to look is the effect of moment.

Shelton, in his thesis, touched on the role of moment in an intuitive proof of the zero-moment steady state boundary condition. On page 29 he says,

"*Imagine that an initially straight and uniform web in its steady state position has a residual negative moment as it contacts a roller. The web has a finite radius of curvature, as shown in Figure 2.1.3, [a diagram showing a curved web as it enters onto a roller] which means that the web is longer on one side than the other. If the roller were composed of many short, independent rollers instead of a single body, the roller at the left side of the web would turn fastest, because more length of web is passing over it per unit time. Similarly, the roller on the right side would turn slowest. But both ends of the single roller must turn at the same speed. Thus, because the left side of the web in Figure 2.1.3 would be trying to turn the roller faster and the right side slower, the roller would exert a positive moment on the web until the negative moment was cancelled because of the web movement, if the friction forces were sufficient. The initial assumption of a steady-state moment was incorrect, so the steady-state moment at the downstream end of the free span must be zero."*

This is a good explanation of how the web arrives at its steady state and contains the germ of the idea that is at the core of this paper.

MASS FLOW EFFECTS IN A BEAM MODEL WITHOUT SHEAR DEFORMATION

Qualitative Description

Analysis of the elastic curve of the web is usually carried out as though the material in the span, in its relaxed state, has a constant shape and is not moving. For steady state analysis, this is quite natural because at each instant of time, the web looks the same,

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[4] He was referring to an E-B model and the reason he was concerned about the $d\gamma/dt$ term, is that including it in {21} leads to results that contradict observed behavior. This will be explained later.

even though each part of it is moving downstream and being replaced by the portion behind it. Until now, this assumption has also been applied to dynamic analysis.

The explanation that follows will show how a non-uniform tension profile, resulting from a moment at a downstream roller, changes the reference shape of a span, making it a function of time. It will also show how this change in shape can be interpreted as a change in the bending angle boundary condition.

Figure 2 (a) shows a web moving continuously in a steady state between parallel fixed rollers with tension σ_o.

Figure 2 (b) the relaxed (reference) stationary web is shown. This is what would be seen if the web in (a) were suddenly frozen in place and cut transversely along its line of contact with the downstream roller and then unfrozen. At the upstream end, it is assumed to be anchored to the line of contact in the *x*-direction. The downstream end will now fall short of its original line of contact with the roller. The reason for showing the reference shape at this point is that we are interested in how mass transfer will affect its shape.

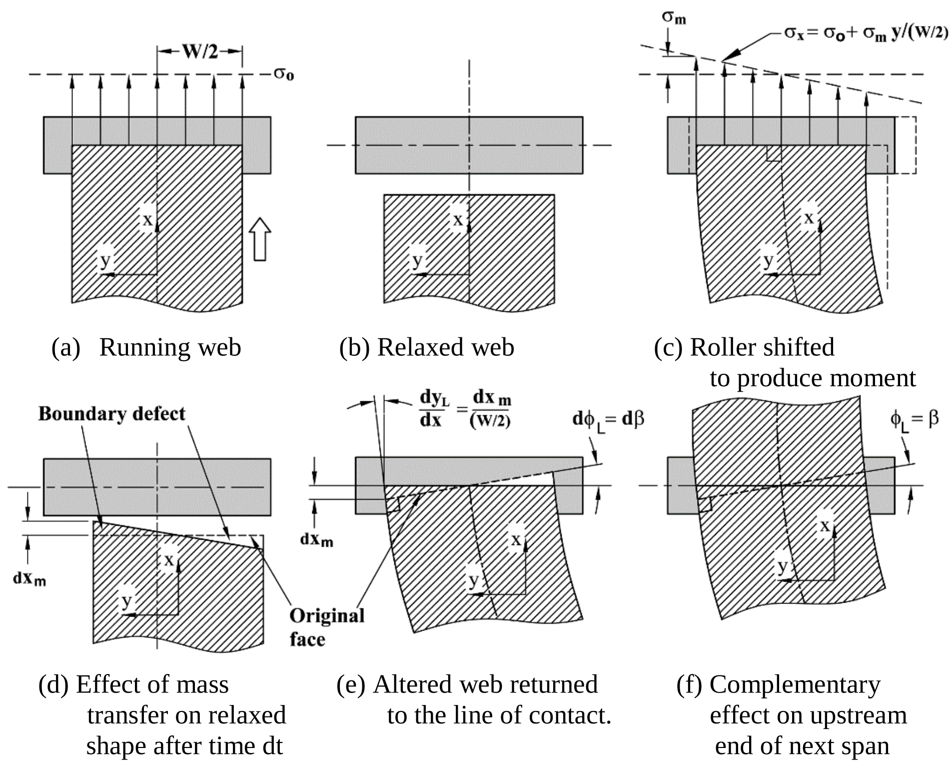

Figure 2 – Effect of Mass Flow on Downstream Boundary Condition for E-B Beam Model

Figure 2 (c). The web of (a) is shown after the roller has been shifted to the left to create a negative moment. The stress profile will be linear and symmetrical, provided that the radius of curvature produced by the moment is much larger than the dimensions of the span. In typical web handling applications, this will always be the case. At the left edge, the tension will be higher than σ_o by some value σ_m. At the right edge, it will be lower by the same amount. The value at the center will be unchanged from (a). Since the model for the web is assumed to be an E-B beam, the web centerline will initially be perpendicular to the roller axis, as will every particle path that was originally parallel to the *x*-axis in (a).

If it were not for what is going to be described next, the web would continue to run forever in the state shown in (c) with all particles following curved paths that are perpendicular to the roller axis at the line of entry onto the roller.

Figure 2 (d). In (d) the condition described in (c) has been allowed to operate for a time increment, *dt*. Then, once again, we assume that the web is suddenly frozen in place and cut transversely along its line of contact with the downstream roller before being unfrozen. Also, as before, the upstream end, is assumed to be anchored to the line of contact in the *x*-direction. The downstream boundary of the relaxed web will now no longer be perpendicular to the centerline because the stress profile changed the cross-sectional area and density of the web at each point across its width. This modified the rate of mass flow onto the roller, causing higher flow on the side with lower tension and lower flow on the side with higher tension. Therefore, the left edge of the relaxed web grew longer by an increment dx_m while the right edge was shortened by the same amount. At intermediate points, mass flow varied in proportion to the distance *y* from the centerline, so that the new relaxed edge will be straight but rotated relative to the web centerline. The net effect is to alter the shape of the web being analyzed, making it a function of time. This potentially jeopardizes the entire scheme of shape analysis by invalidating the variational method behind it. Fortunately, there is a way to make the shape change look like a change in one of the boundary conditions.

Figure 2 (e). In (e), the modified edge of the web is returned to the line of contact of the roller (where it had been before it was frozen and cut). It is apparent that original face (before the boundary defect developed) is offset from the roller axis by the boundary defect angle β.

Thus, (*and this is critical to all that follows*) the effect of the shape change can be approximated by defining the bending angle boundary condition as the boundary defect angle β plus the roller angle *applied to the original reference shape*. So,

$$\phi_L = \beta + \gamma_L \tag{22}$$

Since the pivoting motion of the roller is not necessarily in the plane of the web span, γ_L is the projection of the roller angle onto the plane of the web.

The boundary defect angle β can be calculated as follows.

Analysis of the Effect of Mass Transfer in the E-B model

At any location across the web, tension σ_x causes an increment of area of thickness *h* and width *dy* to be changed to $dy(1+\varepsilon_y)h(1+\varepsilon_z)$ where ε_y and ε_z are strains in the *y* and *z* directions respectively (these strains are not part of the E-B model, but will be included to remove any doubt about unrecognized effects). The surface velocity of the roller, V_o will be the same as it was in (b) because the average tension hasn't changed. So, the increment of mass *dq* passing through the area in time increment *dt* at any location along the line of entry onto the roller is,

$$dq = \rho\, dt\, V_O\, dy\left(1+\varepsilon_y\right)h\left(1+\varepsilon_z\right) \tag{23}$$

where ρ is the density of the web when it is under tension, V_o is the surface speed of the roller. The density ρ at location y will be affected by tension (for any material whose Poisson ratio is not equal to 0.5). It is equal to,

$$\rho = \frac{\rho_o}{(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_y)} \qquad \{24\}$$

where ρ_o is the density of the relaxed web. The increment of mass dq is therefore,

$$dq = \rho_o V_o \, dt \, dy \, h \frac{1}{1+\varepsilon_x} \qquad \{25\}$$

At the center of the web, ε_x is equal to ε_o, the strain due to σ_o. At the left edge, $\varepsilon_x = \varepsilon_o + \varepsilon_m$, where ε_m is the strain due to the increment of stress σ_m (assuming small strains). Using these values for ε_x in {25}, the difference in the mass flow rates at the center and left edge can now be calculated as

$$\frac{dq_m}{dt} = \frac{V_o \, \rho_o \, dy \, h \, \varepsilon_m}{(1+\varepsilon_o)(1+\varepsilon_o+\varepsilon_m)} \qquad \{26\}$$

A piece of web at $y = W/2$ with length dx_m and the same cross section as in {26} will have a mass of,

$$dq_m = \frac{dx_m \, dy \, h \, \rho_o}{1+\varepsilon_o+\varepsilon_m} \qquad \{27\}$$

Equating the mass increments in {26} to {27} and solving for dx.

$$dx_m = -\frac{V_o \, dt \, \varepsilon_m}{1+\varepsilon_o} \qquad \{28\}$$

The denominator of {28} is present because the value of dx has been calculated for the web when it is under tension. Since it causes only a 2^{nd} order effect on the calculation of the boundary defect, it will be dropped. Furthermore, dx has been calculated for mass that has moved out of the current span and into the next. The negative sign indicates a reduction in length relative to the center in the next span. Therefore, the sign of dx_m must be reversed. So,

$$dx_m = V_o \, dt \, \varepsilon_m \qquad \{29\}$$

In Figure 2 (e) the web is shown after tension has been restored and the face has been returned to the line of contact. One way to look at the situation is that the original face has become misaligned with the roller axis by the defect angle β.

The rate of change of the boundary defect angle β can now be calculated, using {29}.

$$\frac{d\beta}{dt} = \frac{dx_m}{dt} \frac{1}{W/2} = 2\frac{\varepsilon_m V_o}{W} \qquad \{30\}$$

Next, we look for a way to calculate β in terms of the moment. This can be done by finding an expression for curvature in terms of ε_m. The negative moment M can be expressed in terms of the stress profile as,

$$M = -\int_{-W/2}^{W/2} y\left(\sigma_o + \sigma_m \frac{2y}{W}\right) h\, dy = \frac{-W^2 h \sigma_m}{6} = \frac{-W^2 h\, \varepsilon_m E}{6} \qquad \{31\}$$

The moment is also, by definition,

$$M = EI\frac{d^2 y_L}{dx^2} = E\frac{W^3 h}{12}\frac{d^2 y_L}{dx^2} \qquad \{32\}$$

Equating {31} to {32} and solving for the curvature,

$$\frac{d^2 y_L}{dx^2} = -2\frac{\sigma_m}{EW} = -2\frac{\varepsilon_m}{W} \qquad \{33\}$$

Using equations {30} and {33} ,

$$\frac{d\beta}{dt} = -V_o\frac{d^2 y_L}{dx^2} \qquad \{34\}$$

In the case of an E-B beam, where the face and slope are always at right angles, the slope will be equal to the angle of the face. So, equation {22} becomes,

$$\phi_L = \frac{dy_L}{dx} = \beta + \gamma_L \qquad \{35\}$$

Without β in equation {35}, the slope of the web would be equal to the roller angle and the web would be locked in a condition of perpetual normal entry, unable to move laterally on the roller.

In the case of the shifted parallel roller described at the beginning of the present section, the growth of β will cause the web to begin moving back toward its original position and as it moves, the moment that produced β will begin to decrease until both the moment and β will be zero when the web arrives at its original position.

Acceleration Equation for the E-B Model

Finally, it will be shown that all of this is subsumed in Shelton's original acceleration equation.

Taking the time derivative of {35} and using {34}.

$$\frac{d}{dt}\left(\frac{dy_L}{dx}\right) = -V_o\frac{d^2 y_L}{dx^2} + \frac{d\gamma_L}{dt} \qquad \{36\}$$

The cross derivative on the left side of {36} can be eliminated by substituting the time derivative of the velocity equation {20}. The result is Shelton's acceleration equation {21}.

$$\frac{d^2y_L}{dt^2}=V_o^2\frac{d^2y_L}{dx^2}+\frac{d^2z_L}{dt^2} \qquad \{37\}$$

Thus, it's seen that the acceleration equation is a mathematical consequence of the effect of bending moment on mass flow.

Equation {37} has been derived for shifting the downstream roller in a parallel pair. However, it applies equally well for any combination of lateral shift and pivot. The pivot angle has already been included in equation {35} and is always small (on the order of a degree or less), so the quantities used in equations {23} to {35} will be good approximations to corresponding values at a pivoted roller. Furthermore, the procedure above doesn't depend on how the moment is created. It could also be caused by an upstream disturbance in lateral position, but this doesn't change the calculations and will lead to equation {37}.

Finally, in Figure 2(f) both spans are shown (without regard to the angle of wrap) and it is apparent that the net effect of the boundary defect is that the bending angles in the two spans match one another, so that the web remains continuous.

The Meaning of β in the E-B Model

In the E-B model, the angle β is not an elastic deformation. It is entirely due to a transient change in mass distribution that arises when a moment causes the rate of mass flow from one span to the next to vary linearly across the width (while the average remains constant). Because of it, the face of the web at the roller (the cross-sectional plane that is perpendicular to the web centerline when the web is relaxed) becomes misaligned with the axis of the roller. A complementary effect occurs at the exit of the roller so that there is a matching change in the angle of the face of the web (forward on the side with lower tension and backward on the other). For the web as a whole, there is no net gain or loss of mass and the result is that the boundary between the two spans becomes skewed by β. For a roller that is not misaligned, β becomes the effective angle of the face at both the entry and exit of the roller. In other words, $\phi_{L1} = \phi_{02} = \beta$ where the subscript *L1* refers to the downstream end of a span and subscript *02* refers to upstream end of the next span. When the roller is misaligned $\phi_{L1} = \beta + \gamma_{L1}$ and $\phi_{02} = \beta + \gamma_{02}$, where γ_{L1} and γ_{02} are projections of the roller angle onto their respective spans[5].

For the E-B model there is a relationship that should not be overlooked. Equation {35} can be inserted in the normal entry equation {20} and it is seen that $-\beta$ is identical to the entry angle.

$$\frac{dy_L}{dt}= V_o(-\beta)+\frac{dz_L}{dt} \qquad \{38\}$$

And this means that,

$$\beta=\frac{1}{V_o}\left(\frac{dz_L}{dt}-\frac{dy_L}{dt}\right) \qquad \{39\}$$

Thus, another feature of β in the E-B model is that it will always decay to zero in the steady state.

[5] Discussed in detail later in the paper.

ANALYSIS OF MASS FLOW IN A BEAM MODEL WITH SHEAR

Changes in the Mass Flow Analysis Due to Shear Deformation

The effect of adding shear deformation to the model is illustrated in Figure 3. It shows the effect of shear on Figure 2 (e) when the altered web is returned to the line of contact. The most important difference from the E-B model is that the angle of the face, β, is no longer equal to the slope α.

The label for the angle of the face that was called β in the E-B model has been changed to ϕ_L to agree with the shape analysis.

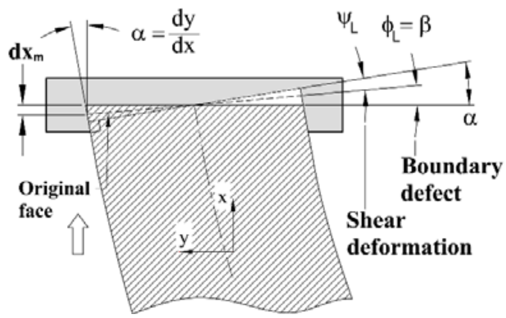

Figure 3 – Effect of Mass Flow on Downstream Boundary Condition for Timoshenko Beam Model

The analysis proceeds in the same way as for the E-B beam until equation {32} where the equation for moment becomes,

$$M = aEI\frac{d^2y_L}{dx^2} = aE\frac{W^3h}{12}\frac{d^2y_L}{dx^2} \qquad \{40\}$$

So,

$$\frac{d\beta}{dt} = -aV_o\frac{d^2y_L}{dx^2} \qquad \{41\}$$

Taking the time derivative of {22} and using {41},

$$\frac{d\phi_L}{dt} = -aV_o\frac{d^2y_L}{dx^2} + \frac{d\gamma_L}{dt} \qquad \{42\}$$

Since equation {42} conveys all the consequences of mass transfer, it could be called the mass transfer equation. See the discussion related to Figure 7 for more discussion of its interpretation.

The Acceleration Equation for the Timoshenko Model

It is at this point where the biggest change from the E-B model occurs, because the web face no longer has a simple perpendicular relationship to the web centerline. This can be handled by using the slope equation {15}. Taking its time derivative and substituting {42}for the time derivative of ϕ_L.

$$\frac{d}{dt}\left(\frac{dy_L}{dx}\right)=\left(\frac{dy_o}{dt}-\frac{dy_L}{dt}\right)\frac{h_1}{L}+\left(-aV_o\frac{d^2y_L}{dx^2}+\frac{d\gamma_L}{dt}\right)h_2+\frac{d\phi_o}{dt}h_3 \qquad \{43\}$$

The cross derivative is replaced as before by using the time derivative of the normal entry equation {20}. Solving for acceleration produces the acceleration equation for the Timoshenko beam model.

$$\frac{d^2y_L}{dt^2}=aV_o^2\frac{d^2y_L}{dx^2}h_2+V_o\left[\frac{dy_L}{dt}\frac{h_1}{L}-\frac{d\phi_o}{dt}h_3+\frac{d\gamma_L}{dt}\left(1-h_2\right)-\frac{dy_o}{dt}\frac{h_1}{L}\right]+\frac{d^2z_L}{dt^2} \qquad \{44\}$$

This looks complicated, but it's reassuring to note:

1. When $a = 1$, then $h_2 = 1$, $h_1 = h_3 = 0$ and it defaults to the E-B acceleration equation, as it should.
2. It passes all three of the validity tests described at near the end of this paper.
3. When all of the time related terms are zero, equation {44} becomes the 4th boundary condition for the Timoshenko steady state model – zero curvature at $x = L$.

It should be noted that the boundary defect is responsible only for the curvature term on the right side of {44}. The other terms, involving y_0, y_L, γ_L and ϕ_0 exist because of shear deformation in the Timoshenko beam.

The Meaning of β in the Timoshenko Model

It can be shown with the use of {22} and {1} that the velocity equation can be written as,

$$\frac{dy_L}{dt}=V_o\left(-\beta-\psi_L\right)+\frac{dz_L}{dt} \qquad \{45\}$$

This reveals that, although β in the Timoshenko model was derived from the same considerations of mass flow as the E-B model, it can have non-zero values in the steady state whenever there is a side force necessary to bend the web – for example, at an inclined roller (non-zero ψ).

It can also be shown, using {15}, {42} and{1}, that the time derivative of {45} produces the acceleration equation {44}.

As in the E-B model, $\phi_{L1} = \beta + \gamma_{L1}$ and $\phi_{02} = \beta + \gamma_{02}$, where γ_{L1} and γ_{02} are projections of the roller angle onto their respective spans.

It is notable that equation {41} says that the time rate of change of the boundary defect β is equal to V_oM/EI. This means that when a moment is present at the downstream roller, the slope of the web at the entry to the roller will be changing and that, in turn means the web will be in an unsteady state, moving laterally on the roller. That's kind of the whole story in a nutshell.

Effect on the Next Span

In Figure 4, the web is drawn without regard to the wrap on the roller.

In the E-B model of Figure 4 (a), the slope is continuous as the web passes from the entry to the exit of the roller, but in the Timoshenko model of (b) there is a discontinuity in the slope due to the presence of shear deformation. Correspondingly, in the E-B model, the face of the web will be out of alignment with the roller axis only when the web is not in a steady state; but in the Timoshenko model, the face can be out of alignment in both steady-state and non-steady-state conditions.

In both cases, $\phi_{L1} = \phi_{02} = \beta$ (continuity of bending angle) where the subscript *L1* refers to the downstream end of a span and subscript *02* refers to upstream end of the next span. When the roller is misaligned and the effect of wrap is included, $\phi_{L1} = \beta + \gamma_{L1}$ and $\phi_{02} = \beta + \gamma_{02}$, where γ_{L1} and γ_{02} are projections of the roller angle onto their respective spans.

Sievers, in her multi-span model proposed that bending angle should be assumed to be continuous across a roller – tacitly ignoring the effect of wrap angle. It is now apparent that it would be better to say that the boundary defect, β, is continuous across rollers and that the bending angle is defined as the boundary defect plus the projection of roller wrap angle.

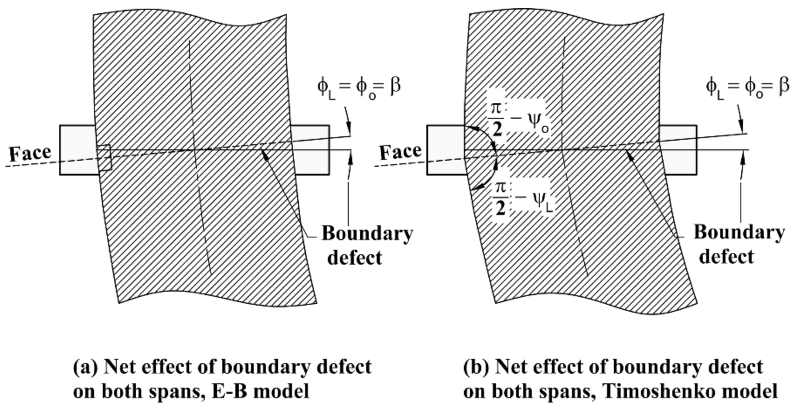

Figure 4 – Net Effect of Boundary Defect on Both Spans

THE RELATIONSHIP BETWEEN THE ACCELERATION EQUATION AND THE TIME DERIVATIVE OF THE VELOCITY EQUATION

One of reasons, given at the beginning of this paper, for questioning the acceleration equation was its redundancy. The velocity equation {20} will generate acceleration under the influence of parameters, such as z_L and γ. So, why is an additional source of acceleration even necessary? This question can now be answered.

If the time derivative of the velocity equation is equated to the acceleration equation of the Timoshenko model, the result is,

$$\frac{d}{dt}\phi_L = -a\frac{d^2 y_L}{dx^2} + \frac{d\gamma_L}{dt} \qquad \{46\}$$

And this is exactly the result that is obtained in equation {42} of the analysis of mass flow.

THE MULTI-SPAN DIFFERENTIAL EQUATION

Solution for Lateral Displacement at the Downstream End of a Span

A differential equation for lateral displacement at the downstream end of a span may be created by first solving equation {15} for ϕ_L and substituting the result into equation {16}. Then, equations {20} and {44} are used to replace the first and second order spatial derivatives with time derivatives.

The resulting Timoshenko model is,

$$\begin{aligned}\frac{d^2 y_L}{dt^2} &= -\frac{1}{\tau}\left(ag_2 - h_1\right)\frac{dy_L}{dt} - \frac{h_1}{\tau}\frac{dy_0}{dt} + \frac{a}{\tau^2}\left(g_1 h_2 - g_2 h_1\right)\left(y_0 - y_L\right) - V_o h_3 \frac{d\phi_0}{dt} \\ &+ \frac{V_o}{\tau} a\left(g_3 h_2 - g_2 h_3\right)\phi_0 + V_o\left(1 - h_2\right)\frac{d\gamma_L}{dt} + \frac{V_o}{\tau} a\, g_2 \gamma_L + \frac{d^2 z_L}{dt^2} + \frac{ag_2}{\tau}\frac{dz_L}{dt}\end{aligned} \qquad \{47\}$$

where $\tau = L/V_o$. This becomes the E-B equation when $a = 1$, $h_3 = h_1 = 0$.

A multi-span model may be created by representing each span with a different version of equation {47}, using appropriate physical parameters. The values of ϕ_0 and y_0 come from the previous span. The bending angle $\phi_0 = \beta + \gamma_0$, where β is from the previous span (a value common to the entry and exit sides of the roller) and γ_0 is the roller angle projected onto the plane of the span of interest. The lateral displacement y_0 is equal to y_L of the previous span. This is true for both the E-B and Timoshenko versions of {47}. Although it is possible to get a closed-form solution for {47}, it is convenient to solve the system of differential equations numerically, using a variety of commercially available software tools.

For control applications, it is helpful to look at {47} as a sum of transfer functions. Applying the Laplace transform,

$$\begin{aligned} y_L(s) &= \frac{-\frac{h_1}{\tau}s + \frac{a}{\tau^2}\left(g_1 h_2 - g_2 h_1\right)}{s^2 + \frac{1}{\tau}\left(a g_2 - h_1\right)s + \frac{a}{\tau^2}\left(g_1 h_2 - g_2 h_1\right)} y_0(s) + \frac{-V_o h_3 s + \frac{V_o a}{\tau}\left(g_3 h_2 - g_2 h_3\right)}{s^2 + \frac{1}{\tau}\left(a g_2 - h_1\right)s + \frac{a}{\tau^2}\left(g_1 h_2 - g_2 h_1\right)} \phi_0(s) \\ &+ \frac{V_o s\left(1 - h_2\right) + \frac{V_o}{\tau} a g_2}{s^2 + \frac{1}{\tau}\left(ag_2 - h_1\right)s + \frac{a}{\tau^2}\left(g_1 h_2 - g_2 h_1\right)} \gamma_L(s) + \frac{s^2 + \frac{a}{\tau} g_2 s}{s^2 + \frac{1}{\tau}\left(ag_2 - h_1\right)s + \frac{a}{\tau^2}\left(g_1 h_2 - g_2 h_1\right)} z_L(s)\end{aligned} \qquad \{48\}$$

There is no transfer function for z_0 in {48} because z_0 is equal to z_L of the previous span, where it is included in the normal entry equation {20}. Thus, it's implicit in y_0 and isn't needed.

Generally, y_0 and ϕ_0 must be applied in tandem because they both depend on corresponding values at the downstream end of the previous span. So, when y_0 changes, so does ϕ_0. There are exceptions (some are noted below), but in any numerical implementation of {48} it is best to make this assumption.

In the case of the E-B model, there is a very useful simplification that can remove the connection between y_0 and ϕ_0. Equation {39} is used to replace β in the relationship $\phi_0 = \beta + \gamma_0$. So,

$$\phi_0(s)=\frac{1}{V_o}\left(z_0-y_0\right)s+\gamma_o(s) \qquad \{49\}$$

Substituting and collecting terms, the E-B model becomes,

$$\begin{aligned}y_L(s) = {} & \frac{-\frac{g_3}{\tau}s+\frac{g_1}{\tau^2}}{s^2+\frac{g_2}{\tau}s+\frac{g_1}{\tau^2}}y_0(s)+\frac{\frac{V_o}{\tau}g_3}{s^2+\frac{g_2}{\tau}s+\frac{g_1}{\tau^2}}\gamma_0(s)+\frac{s^2+\frac{1}{\tau}\left(g_2+g_3\right)s}{s^2+\frac{g_2}{\tau}s+\frac{g_1}{\tau^2}}z_L(s)\\ & +\frac{\frac{V_o}{\tau}g_2}{s^2+\frac{g_2}{\tau}s+\frac{g_1}{\tau^2}}\gamma_L(s)+\frac{\frac{g_3}{\tau}s}{s^2+\frac{g_2}{\tau}s+\frac{g_1}{\tau^2}}z_0\end{aligned} \qquad \{50\}$$

The technique used to eliminate ϕ_0, along with an early version of {50} were first reported by Young, Shelton and Kardimilas [7]. An equation identical to {50} appears in later paper by Seshradi and Pagilla [8]. Its virtue is that it depends only on parameters for the span under consideration. It isn't always possible to do this for the Timoshenko model of {48} because β must be calculated from equation {45} which depends on upstream values of ϕ and y from the previous span and those values will, in turn, depend on values from the span before that. Thus, ϕ_0 becomes a function of not only the parameters of the current span, but also those of every span preceding it.

Note that unlike {48}, a term for z_0 appears in {50}. It is there because of {49} and must be used, even though it is duplicated by z_L in the previous span.

Solving for Variables at Other Locations in a Span

As values of y_L, y_0, ϕ_L and ϕ_0 are generated by solution of equations {47} or {48}, equation {13} and its derivatives can be used to calculate values for y, ϕ, ψ, slope or curvature at intermediate locations in the spans being modeled.

ISOLATING A SPAN FOR TESTING

In web handling tests, attention is naturally focused on lateral displacements, usually at the downstream end of a particular span. It should be evident from this analysis, however, that the effect of upstream variations in internal stresses caused by bending can have a significant effect. Variations in lateral position will almost always be accompanied by variations in bending angle ϕ and this can be propagated over rollers as weave regeneration. Figure 5 illustrates a method for varying y_o with minimal effect on ϕ_o. Shelton, in his dissertation [1], used an arrangement similar to this to test the response at a fixed roller. The input, y_0 was created by varying z_0 at roller B and the output y_L was measured at roller C.

Care must still be taken to ensure that a uniform stress profile is maintained upstream of the guide.

When equation {50} is used to model an isolated span with an arrangement like Figure 5, y_0 is used as the input and ϕ_0 is set to zero. Both z_0 and the negative term in the numerator of the y_0 transfer function are set to zero because they are actually a part of ϕ_0 (which is zero) through equation {49}.

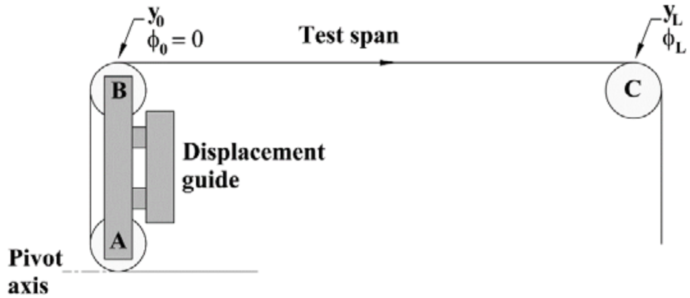

Figure 5 – Test Setup for Varying y_0 while Keeping $\phi_0 = 0$

Ideally, a span with an unwind at the upstream end can be treated as an isolated span (provided the unwind axis is aligned with the roller axis and the web is very uniform). In practice, wound in tension anomalies probably have significant effects on behavior in the first few spans.

AN EXAMPLE OF THE EFFECT OF SHEAR DEFORMATION.

In Figure 6, the effect of the new acceleration equation on response at a fixed idler for $L/W = 1$, is compared with other models.

- The solid curves show amplitude and phase for the y_0 transfer function of equation {48} with $n = 1.2$
- The dashed curves are for the E-B model (y_0 transfer function of equation {50} with $n = 0$)
- The dotted curves are for Sievers' shear deformation model [6] (which used Shelton's acceleration equation {21}).

The span is assumed to be isolated using the technique described in the previous section. $KL = 0.2$, $L = 3$ inches, $T = 46$ *Lbf*, $W = 3$ inches, $h = 0.009$ inch, $E = 510{,}000$ psi.

The biggest change is in the phase response of equation {48} with n = 1.2 for values of ωT_1 above 2.0. This is due to the $-h_2 s/\tau$ term in the numerator of the y_0 transfer function. There is a similar term in the y_0 transfer function of the E-B model, but it must be removed when the isolation scheme of the previous section is used because it is due to ϕ_0 which is zero in an isolated span.

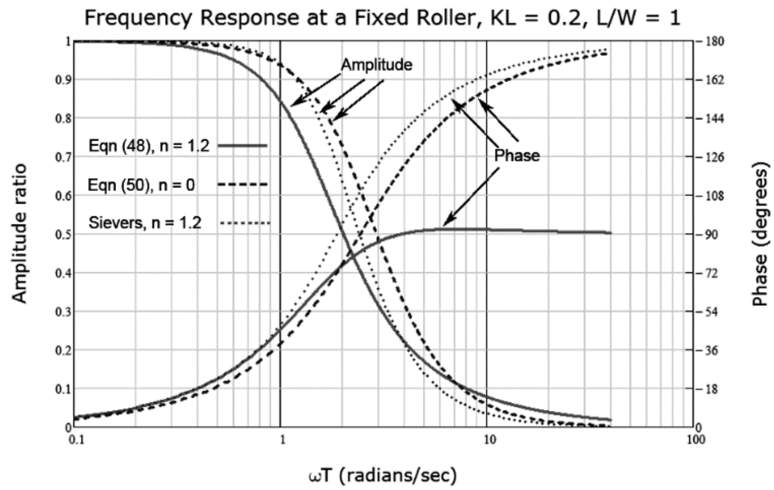

Figure 6 – Effect of the New Acceleration Equation on Response at a Fixed Roller

VALIDITY TESTS

There are three mathematical tests that any lateral dynamic beam model should pass. Passing them is necessary, but not sufficient for validity, nevertheless, they are useful for weeding out many mistakes.

1. A Timoshenko beam model should default to an E-B model when the shear factor *n* is reduced to zero.
2. The transfer function for a displacement guide should be unity (no dynamics) for a z_L input to equation {48} with $y_0 = 0$, $\phi_0 = \gamma_L$ and $\gamma_L = z_L/L$. This should happen regardless of whether the effect of shear deformation is included. A change in z_L rotates both rollers as a rigid assembly so that the entry span is not deformed as it is rotated. Therefore, if only the response of the entry span is considered, there is nothing that can cause the web to move laterally relative to the roller, so the lateral displacement y_L should equal z_L. This may sound trivial, but many candidates for new acceleration equations will produce transfer functions that fail this test.
3. The steady state gain for the transfer function of a pivoting roller should equal the curvature factor K_c derived in a steady state shear deformation model that doesn't rely on an acceleration equation [9]. This is,

$$K_C = \frac{KLa\cosh(KL) - \sinh(KL)}{KL\left(a\cosh(KL) - 1\right)}$$

BENSON'S ACCELERATION EQUATION

In his paper, "Lateral Dynamics of a Moving Web With Geometrical Imperfection" [10], Richard Benson observed that the velocity equation, "is the result of imposing a velocity match between the downstream roller and the centerline of the web." He then suggested that "It is further expected that the web will stick to the roller for all points of first contact. – not just at the web's centerline. To achieve that, we must also match the rotational velocities of the roller [axis] and web ‘face.’” This is the basis for his equation {22},

$$\frac{d\gamma_L}{dt} = \frac{D\phi_L}{Dt} = \frac{d\phi_L}{dt} + V_o \frac{d\phi_L}{dx} \tag*{\{51\}}$$

The symbol D is used to indicate the material derivative[6].

Since,

$$\frac{d\phi}{dx} = a\frac{d^2 y_L}{dx^2} \tag*{\{52\}}$$

equation {51} is equivalent to equation {42}.

Using equation {1} to replace ϕ_L and using the time derivative of the normal entry equation {20} to eliminate the cross derivative, equation {51} becomes, (his equation (23)).

$$\frac{d^2 y_L}{dt^2} = V_o^2 \frac{d^2 y_L}{dx^2} + \frac{d^2 z_L}{dt^2} - V_o\left(\frac{d\psi_L}{dt} + V_o\frac{d\psi_L}{dx}\right) \tag*{\{53\}}$$

It can be shown (using $d\psi/dx = (a\text{-}1)d^2y_L/dx^2$ and equations {15},{42} and {22}) that this is equivalent to the acceleration equation for the Timoshenko model of this paper and it is obvious that it reduces to Shelton’s acceleration equation for the E-B model, (shear strain terms equal to zero).

Benson used many of the same defining relationships as this paper (for example, equations {4} and {5}). Even so, the fact that he could get the correct acceleration equation without any consideration of mass flow came as quite a surprise to me. The connection can be understood with the help of Figure 7.

The diagram in Figure 7 (a) shows a web immediately after it is bent by a lateral shift like that in Figure 2 (c). The dashed lines are cross-sectional planes that define the bending angle ϕ. Figure 7 (b) shows the web after a short time interval, *dt*.

Several things are evident. First, the spacing between the cross-sectional planes becomes progressively greater in going from the bottom edge to the top. This causes a change in mass per unit length (longitudinal compression in the bottom half and stretching in the top half). Second, the curvature causes the angle ϕ to change with x. In fact, moment is equal to $EI(d\phi/dx)$. Third, the surface velocity of the roller advances the web at a uniform rate all along its face and this causes the angle of the face at the line of entry to change with time. Since the face in (b) corresponds to the original face when the web was in its relaxed state, the value of β shown here is the same as that calculated in the mass flow analysis.

[6] Although it is correct, I’m not convinced that equation {51} should be attributed to the material derivative. It happens because of β and constant V_o at the line of entry.

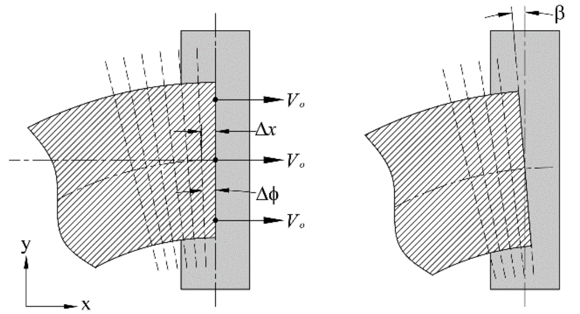

Figure 7 – Benson's Velocity Matching Criterion

So, although Benson never mentioned the effects of mass flow, it was implicitly included in his analysis.

OTHER VERSIONS OF THE TIMOSHENKO ACCELERATION EQUATION

Sievers

Sievers [3] derived the acceleration equation using the material derivative. This gave her equation {21}. However, this technique only worked for fixed parallel rollers. She could do this because the only pivoting rollers on her machine were those of a displacement guide. She recognized that these rollers and the span between them move together as a rigid body. So, she mathematically rotated the coordinate system for the span between the guide rollers and treated the web as though it was passing between fixed parallel rollers. For her E-B model, this worked. However, the method provided no insight into incorporating shear. Furthermore, it was not a good general solution because it cannot be applied to a misaligned roller or an ordinary steering guide.

Sievers used Shelton's acceleration equation for both the E-B and Timoshenko models. None of the validity tests detect the fact that it is incorrect for the Timoshenko model.

Brown

When Brown tried to apply Sievers' material derivative method to pivoting rollers, it produced an extra term involving the rate of change of roller angle [6].

$$\frac{d^2y}{dt^2} = V_o^2 \frac{d^2y_L}{dx^2} + V_o \frac{d\gamma_L}{dt} + \frac{d^2z_L}{dt^2} \tag{54}$$

This fails test number 2.

Brown also attempted to develop a Timoshenko model [9] by using equation {11} for ϕ_0 and ϕ_L in equation {16}. This produced an equation with 1st, 2nd and 3rd order spatial derivatives that were then replaced with time derivatives. The normal entry rule {20} and Shelton's acceleration {21} were used for the 1st and 2nd derivatives. For the third derivative, the following relationship was used derived by repeated application of the chain rule,

$$\frac{d^3 y_L}{dt^3} = -v_o{}^3 \frac{\partial^3 y_L}{\partial x^3} \qquad \{55\}$$

There is no physical justification for this. Furthermore, it is mathematically incorrect because it involved a cross derivative that was removed by using the *x*-derivative of the normal entry equation - an invalid operation because that equation applies only at $x = L$. It is no surprise that this model also failed test number 2.

Walton

Walton [11] found fault with equation {21} because the numerical models based on it appeared to violate the assumption of no web slippage on rollers. He stated that in numerical models based on Sievers' equations, "a step in lateral position on one roller will instantaneously result in a face angle change in the web at all the downstream rollers in a multi-roller set of web spans." He proposed the equation {56} as an alternative. It is equation {42} with the roller axis angle, $\gamma_L = 0$ which is appropriate for Sievers formulation.

$$\frac{d\phi}{dt} = -V_o a \frac{d^2 y_L}{dx^2} \qquad \{56\}$$

It may be that the problem he observed (instantaneous response of face angle to a step) was because the rollers were modeled with zero transit time.

Shelton

On pages 104 and 105 of his thesis [1], Shelton develops an expression for lateral acceleration that includes the effect of shear. He says, "The lateral velocity due to the angle of shear deflection may be deduced from the discussion of Section 3.1 and Figure 3.1.1 [First-Order Dynamics of a Massless Web] to be equal to -$V\theta_{LS}$, where θ_{LS} is the slope due to changing the angle of shear deflection of the web at the downstream roller. The lateral acceleration due to a changing angle of shear deflection is found by differentiation with respect to time." From this, he concludes that the acceleration equation at the downstream roller is,

$$\frac{d^2 y_L}{dt^2} = V_o{}^2 \frac{d^2 y_L}{dx^2} + \frac{d^2 z}{dt^2} - V_o \frac{d\psi_L}{dt} \qquad \{57\}$$

This is similar to Benson's equation, but lacks the term involving the spatial derivative of ψ_L. Based on the earlier discussion of Benson's work, it's apparent that this is not equivalent to the acceleration equation {44} and it fails test number 3.

PROJECTION OF ROLLER ANGLES

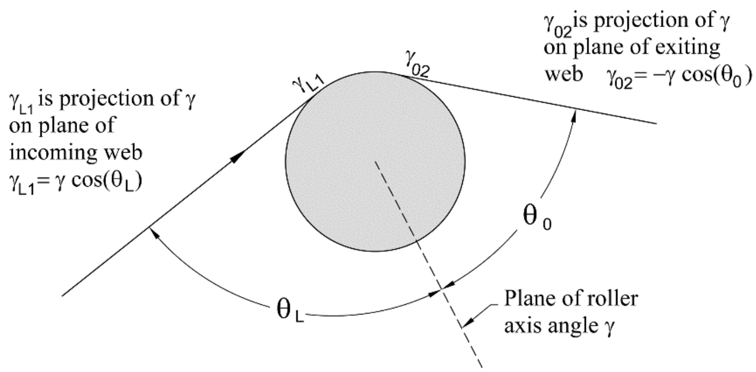

Figure 8 – Relationship between Plane of Roller Motion and Angle Experienced by Web

The roller angle γ must consider both the angle of wrap and whether it applies to the downstream or upstream end of a span.

The angles γ_{L1} and γ_{02} in Figure 8 above are projections of roller alignment onto the plane of the web. If the angular motion of the roller is in a plane that is parallel to the cross-machine direction, the following relationships hold.

$$\gamma_{L1} = \gamma\cos(\theta_L) \quad \text{and} \quad \gamma_{02} = -\gamma\cos(\theta_0) \tag{58}$$

The effect of β on traction

The moment that creates the boundary defect can also affect traction at the downstream roller by causing time-varying changes in the tension profile. If tension changes rapidly at the entry, the resulting circumferential gradient on the roller can exceed the friction necessary to prevent micro-slipping (slipping due to the strain gradient) on the entry side. Whitworth and Harrison described how variations in the average line tension could cause such slipping in their 1983 paper, “Tension Variations in Pliable Material in Production Machinery” [12]. The maximum gradient that can be supported at the entry to a roller without slipping is,

$$\left|\frac{d\sigma_x}{dx}\right| \le \mu\frac{\sigma_x}{R} \tag{59}$$

The relationship between the time rate of change at the entry to a roller and the spatial gradient on the surface is,

$$\left.\frac{\partial\sigma_x}{\partial x}\right|_{x=L} = \frac{1}{V_o}\left.\frac{\partial\sigma_x}{\partial t}\right|_{x=L} \tag{60}$$

The maximum tension at the web edge, due to a moment, is found from equation{33} (with a factor of a to account for shear)

$$\sigma_m = -\frac{EW}{2}a\frac{d^2y_L}{dx^2} \tag{61}$$

So, the cross-web profile of the time rate of change of tension is,

$$\frac{\partial \sigma_m}{\partial t}\frac{2y}{W}=\left(-\frac{EW}{2}a\frac{d}{dt}\frac{d^2y_L}{dx^2}\right)\frac{2y}{W}=-Eya\frac{d}{dt}\frac{d^2y_L}{dx^2} \qquad \{62\}$$

And the condition for no slipping is,

$$\left|\frac{1}{V_o}\frac{d}{dt}a\frac{d^2y_L}{dx^2}Ey\right|\leq\frac{\mu}{R}\left(-a\frac{d^2y_L}{dx^2}Ey+\sigma_o\right) \qquad \{63\}$$

Suggested terminology for the right side of {63} is Friction Rate and for the left side, Stress Gradient. The implications of this relationship won't be explored here.

Even if the tension profile doesn't produce slipping at the entry, it will pass over the roller to the exit where it will contribute to micro-slipping there.

ROLLERS ARE BEING NEGLECTED

All multi-span analyses to-date, including this one, assume the distance traveled on the roller is so small compared to span lengths that anything happening on the roller, including transit time, will have little effect. However, as the ratio of span length to width of the web decreases, this assumption becomes less realistic and micro-slip due to moments at both the entry and exit of the roller may have a bigger impact on boundary conditions.

SUMMARY

It has been shown that considerations of mass flow lead to the acceleration equation that was experimentally verified for the E-B model in Shelton's 1968 dissertation [1]. The same considerations, when applied to the Timoshenko beam, lead to a model that, although not experimentally proven, is consistent with three validity tests.

A new parameter has been introduced, identified here as the boundary defect angle β. It is seen to be due to the mass changes that cause the face of the web (the face that existed at the roller entry before mass flow changed it) to become misaligned with the axis of the roller. A complementary effect occurs at the exit of the roller so that any mass gained by one span is lost by the other and the result is that the boundary between the two spans becomes skewed by β. When the roller is misaligned $\phi_{L1} = \beta + \gamma_{L1}$ and $\phi_{02} = \beta + \gamma_{02}$, where γ_{L1} and γ_{02} are projections of the roller angle onto their respective spans. In E-B models, β is purely transient, decaying to zero in the steady state.

Use of the mass flow concept removes the need for making an a priori assumption about the behavior of bending angle as the web passes over rollers.

This dynamic analysis provides theoretical justification for steady state boundary conditions of zero moment at the downstream roller for both the E-B and Timoshenko models.

It has been shown that the same phenomenon that leads to the boundary defect can also produce a spatial strain gradient on the roller surface that causes microslipping at the roller entry when it exceeds a threshold set by a relationship between friction coefficient, longitudinal tension and roller radius. Even when no microslipping occurs at the entry, the tension profile that produces the boundary defect can be transported to the exit where it will affect microslip conditions there.

It has been discovered that Richard Benson, in his paper on webs with geometrical imperfections [10], proposed a method that leads to the correct acceleration equation for a Timoshenko beam model without explicitly relying on mass flow analysis.

Multi-span lateral models for low values of *L/W* may fall short of expectations without better models for behavior of the web when it is on the roller.

ACKNOWLEDGEMENTS

Thanks to Sinan Muftu who, after hearing my presentations at the last IWEB conference, drew my attention to Richard Benson's work.

Also, thanks to Dilwyn Jones who was kind enough to review and critique this paper as it was being written. He made many helpful suggestions and is the one who discovered that Benson's acceleration equation is equivalent to acceleration equation {44} (even though it looks completely different) because it is based on the same relationship between bending angle, roller angle and curvature as derived from the mass flow analysis (equation {42}).

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