PULL ROLLERS: PLAIN, VACUUM AND UNPORTED

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ABSTRACT

The role of a pull roller in a web machine is identified. The ability of pull rollers, plain or vacuum type, to permit tension differences is explained. Three types of pull rollers are described. Among them is the un-ported roller that utilizes holes and grooves to increase effectiveness. This paper, while providing theoretical and experimental details on pull roller design, will also serve as a useful tutorial for those interested in learning about the practical design and operational aspects of pull rollers.

NOMENCLATURE

b	web width, m
d	hole diameter, m
D	pull roller diameter, m
f	coefficient of friction
F	tension, N
K	vacuum pull roller vacuum efficiency factor
N	number of holes per m^2 of vacuum pull roller surface
Q	air flow rate, SCMM per m^2 of vacuum pull roller surface
V	vacuum pull roller vacuum, Pa
θ	wrap angle, rad

Subscripts

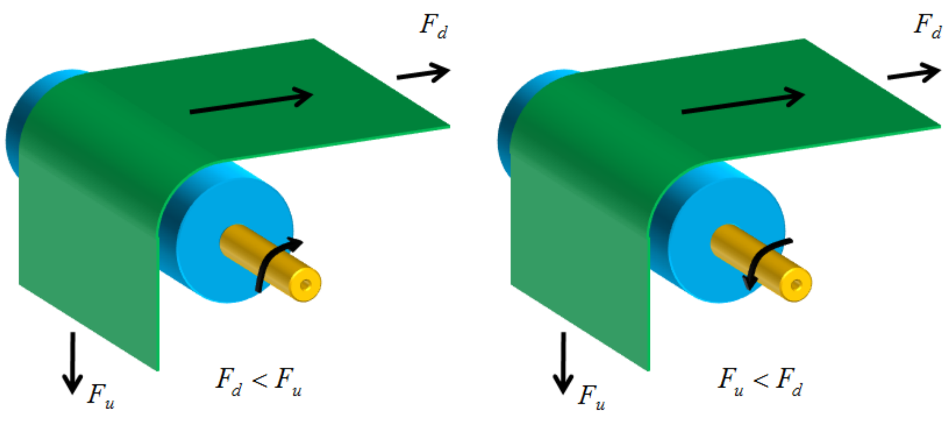
d, u	downstream, upstream
H, L	high tension side, low tension side
i	interface between the web and vacuum pull roller
0	zero speed
s	accounting for operational safety factor

PULL ROLLER OVERVIEW

A web is defined as a structure which is long, thin and flexible. Common web materials include paper, film, foil, nonwovens and textiles. Web handling is defined as any process that seeks to transport and handle a web while maintaining and preserving all of the web's properties with minimum web damage. Web converting is any process that takes one or more webs and permanently alters them in some fashion. Examples of web handling include unwinding, splicing, roller conveyance, air conveyance, lateral control, tension statics and dynamics, roll starts and winding. Examples of web converting include extrusion, drafting, tentering, surface treatment, coating, drying, slitting and laminating.

Successful web converting depends on successful web handling. In turn, successful web handling depends on successful implementation of several key web handling processes. One of the most important of these is tension control since tension is one of the key process factors impacting web handling. Tension control is the art and science of placement and control of motors and drives within a conveyance line so that optimum maintenance of tension throughout the line is assured.

Tension control consists of many subcomponents. A key one of these is the pull roller. Operationally, a pull roller is a conveyance component that can either be driven (e.g., motored) or braked so as to either remove or impart tension into the conveyance line (Fig. 1). This capability is often required to satisfy web handling scenarios. A case where motoring would be required is after a long sequence of idling conveyance rollers. In this situation, tension is rising as the web progresses downstream due to the cumulative effect of air and bearing drag, inertial and possibly elevation changes. The presence of a motor driven pull roller will serve to lower tension downstream of the pull roller thereby eliminating the possibility of tension becoming excessively high. A case where braking would be required would typically be upstream of a winder. Here, tension typically needs to be higher than in the central portion of the machine; hence, the pull roller must act in a braking mode.



Part a: Motoring Part b: Braking

Figure 1 – The Pull Roller Function

Regardless of which case is considered, the question arises as to how large this tension difference can be. For a plain pull roller, the *Capstan Formula* gives the answer.

If the web slips at known tension values of F_H and F_L (where "H" stands for "high" and "L" stands for "low" tension), at a given wrap θ (radians), the coefficient of friction, f, may be calculated for this situation. Roisum [1] provides more detail on this equation.

$$\frac{F_H}{F_L} = e^{f\theta} \ or \ f = \frac{1}{\theta} ln \frac{F_H}{F_L} \qquad \{1\}$$

Corresponding values of the high and low tension limits depend on whether the pull roller is motoring ($F_H = F_u$, $F_L = F_d$) or braking ($F_H = F_d$, $F_L = F_u$).

The coefficient of friction at zero conveyance speed is equivalent to the static coefficient of friction and for a plain pull roller can easily be computed by performing a simple experiment. First, clamp the pull roller with its rotation axis horizontal to prevent it from rotating. Wrap a web around it and then load both ends of the web with known weights equal to F_L. Increase the load on one end gradually, until the web slips. The weight at that instant is F_H. Plug the data into {1} and obtain f. This is the "coefficient of friction at zero web speed, f_0" or static coefficient of friction. Fig. 2 shows a schematic of this experiment.

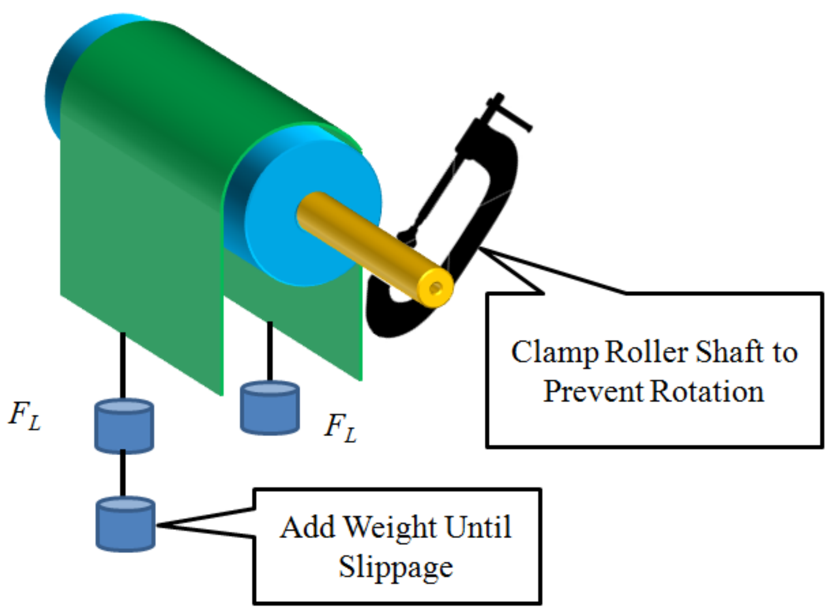


Figure 2 – Measurement of Static Coefficient of Friction

Once the coefficient of friction is known, expression {1} can be used to compute the maximum tension ratio that a plain pull roller can sustain prior to web slippage. For example, for a typical value of a static coefficient of friction of 0.3 and a wrap angle of 180° or π radians, the maximum tension ratio is approximately equal to 2.5. This is a respectable value but not high enough for many practical applications. Of additional importance is the fact that the coefficient of friction is not a constant in practice. At zero conveyance speed, the coefficient of friction is equivalent to the static coefficient of friction between the web and the roller. However, as conveyance speed increases, this will no longer be the case.

To demonstrate this behavior, one can now repeat the previous experiment where now the plain pull roller is placed in a machine where the roller can be motored (or braked) with a variable speed motor, and where there is independent control of the two tensions. Keeping the two tensions approximately equal, accelerate or decelerate to a desired web speed, then increase or decrease one of the two tensions until slip is detected. Again, calculate the coefficient of friction from {1}. Repeat this for several speeds. A typical plot of coefficient of friction against web speed will look like the curve shown in Fig. 3. This nontrivial experiment will show coefficient of friction dropping as speeds go up. Why is this?

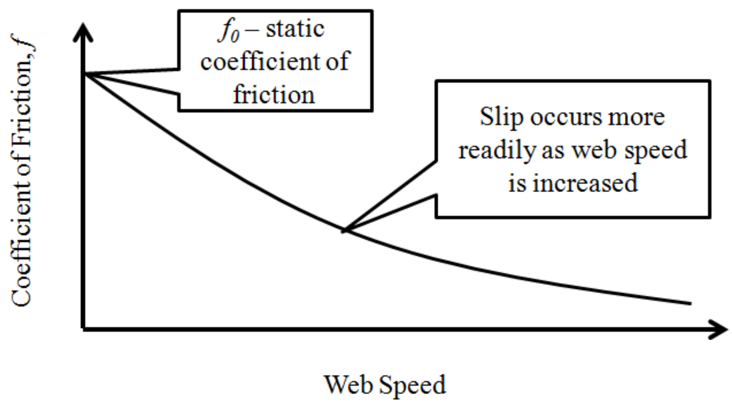


Figure 3 – Smooth Plain Pull Roller Coefficient of Friction vs. Speed

It's because of air lubrication. Air has viscosity meaning that it resists being sheared. In addition, it sticks to surfaces. Both the web and roller move through air and their surfaces drag air along. The layer of air carried along is called the "boundary air". The higher the speeds of the surfaces relative to the air, the thicker the boundary layer will be.

Now consider the entering nip where the web comes into contact with the roller. Two surfaces meet there: the roller surface and the "inside" web surface. Both of the surfaces drag air with them into the nip. At the nip, the air has three places to go: it can flow back out of the nip, it can escape laterally or it can stay with the surfaces and be carried through the nip point into the interface between the web and roller.

Air flow takes time. At low speeds not much air is carried along, and what small amount is dragged into the nip has time to escape. Very little air is trapped into the web-roller interface. At high speed, the boundary layer is thick, a lot of air is carried along and a large volume of air bunches up in the nip. There is little chance for the air to escape back upstream of the nip. Lateral escape is limited and consequently, a lot of air gets pumped into the web-roller interface.

The air in the interface acts like a lubricant, lowering the coefficient of friction. The more air there is in the interface, the lower the friction. Consequently, the sustainable tension ratio is reduced according to {1}.

What can be done to destroy air lubrication? It is very easy to get rid of this undesirable effect. All that has to be done is to give the interface air a place to go. One way to do this is to roughen up the roller surface. For example, the results of Fig. 4 compare a polished surface with a rough surface.

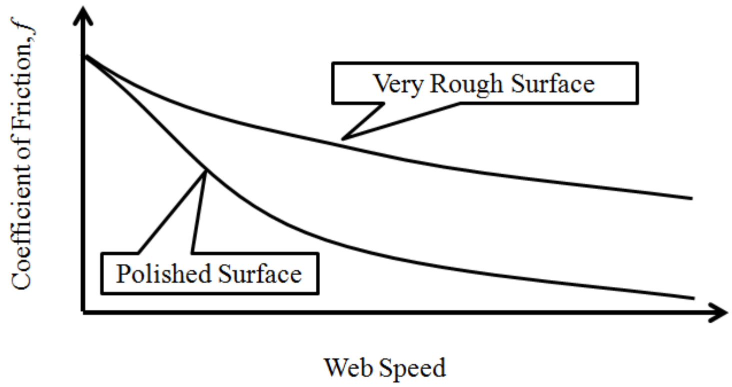


Figure 4 – Effect of Surface Roughness on Plain Pull Roller Coefficient of Friction

Another way is to provide grooves in the surface. These can be arranged circumferentially, helically or axially. The only thing that matters is that they be closely spaced. Improvements similar to those illustrated in Fig. 4 can be expected from grooving.

Are there secondary effects? Yes, there are. Expression {1} does not take into account web speed. This is taken care of by considering "*f*" to be a "parameter". It depends on speed, as has already been discussed. It can also be made to take into account the secondary effects of web width and pull roller diameter. It has already been mentioned that boundary layer air has a better chance to escape from the nip by flowing laterally if the web is narrow. Hence, for narrow webs, the onset of air lubrication is delayed, which results in a flatter f-versus-speed curve than for wider webs (see Fig. 5a).

The air in the web-roller interface is under pressure because the web is tensioned. The pressure is not very high, typically less than 7 kPa. It is lower for larger roller diameters than for smaller diameters. The interface air pressure tends to push the air toward the web edges and the nips, causing air flow in those directions. Obviously, a higher interface pressure pushes air out faster; hence a small diameter roller has better speed performance (see Fig. 5b).

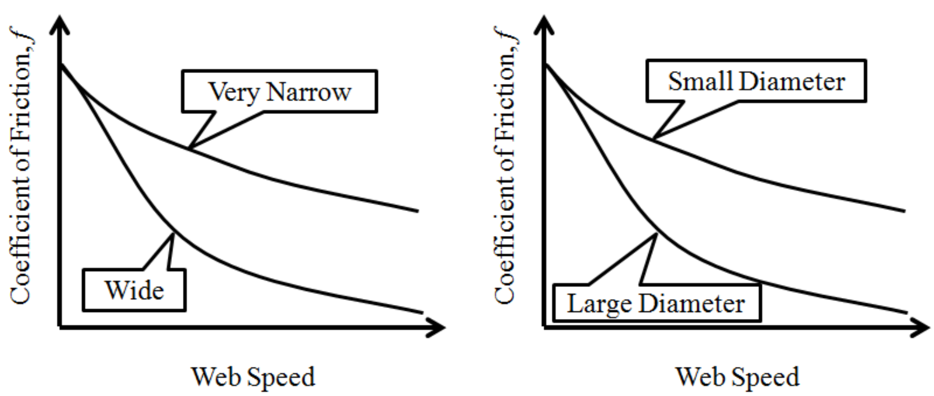


Figure 5a, 5b – Effect of Web Width and Diameter on Plain Pull Roller Coefficient of Friction

VACUUM PULL ROLLER

Even though the effects of air lubrication can be mitigated by proper design of the surface of a plain pull roller, the maximum tension ratio that can be achieved is still limited by the static coefficient of friction and wrap angle according to {1}. Often times, this is not adequate to insure robust performance. In cases such as this, significantly enhanced performance can be achieved by utilizing a *vacuum pull roller*. What is a vacuum pull roller? It is a hollow roller, the surface of which is perforated to permit air to flow through it. The ends are closed and equipped with bearings to allow the roller to turn. Air is sucked from the hollow interior through some kind of rotating joint or valve, mounted concentric with the roller. Fig. 6 shows a schematic of a typical vacuum pull roller.

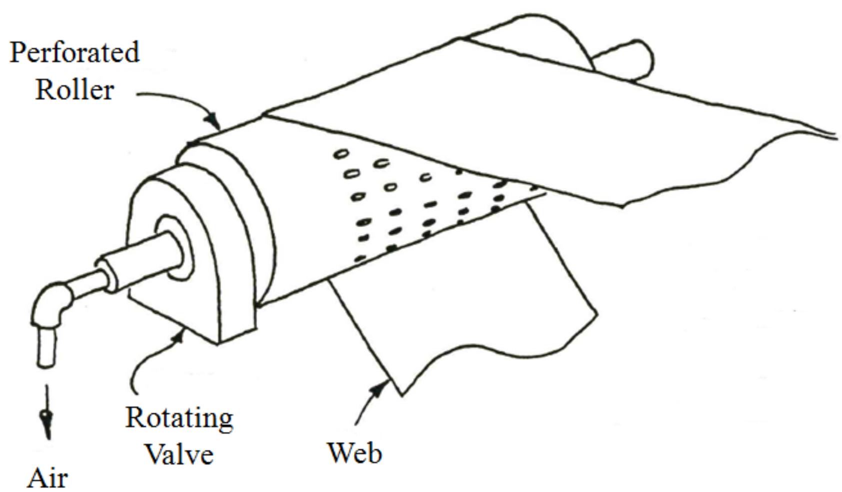


Figure 6 – A Typical Vacuum Pull Roller

Why the vacuum? Consider Fig. 7 which illustrated the operating principle of the vacuum pull roller action. Web is shown wrapped on the roller and air is sucked from the interior of the roller. The web closes off the perforations in the surface and the sucking action creates a vacuum in the interior of the roller. This vacuum generates an air pressure difference across the thickness of the web and thus, the web is pressed in contact with the drum surface by air pressure. This tends to lock the web to the drum and helps to keep it from slipping on the drum.

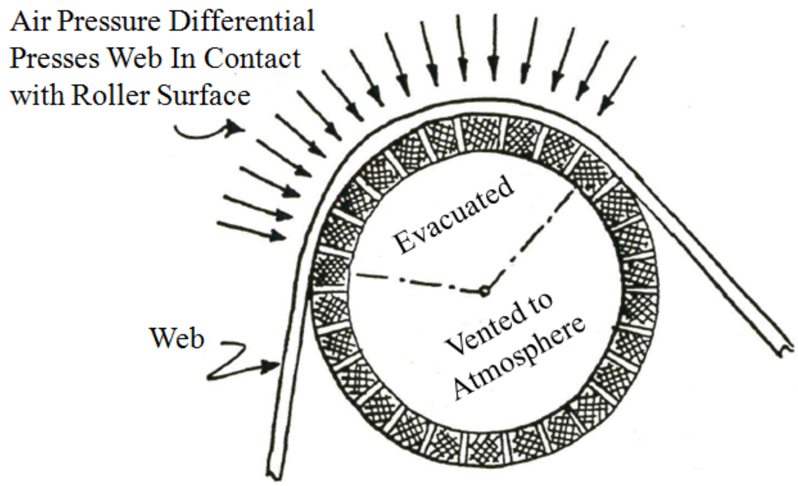


Figure 7 – Principle of Vacuum Pull Roller Action

The increased tension isolation capability relative to plain pull rollers is one of the main advantages of vacuum pull rollers compared to plain pull rollers. In appendix A, we show how to derive a formula, analogous to expression {1}, which takes the effect of vacuum into account. This formula is:

$$F_H = F_L e^{f_0\theta} + \frac{K}{2}(VbD)\left(e^{f_0\theta} - 1\right) \qquad \{2\}$$

where V = vacuum, measured inside the roller (Pa), K = "vacuum efficiency" (more about this later, dimensionless), b = web width (m) and D = roller diameter (m).

Note that if the vacuum is equal to zero that this expression reduces to {1]. There are two additive terms in expression {2}. The first one is the "*wrap effect*" and the second is the "*vacuum effect*". The vacuum effect, at higher values of V (which cannot exceed about 8.5E04 Pa as a practical matter), is much bigger than the wrap effect: this is how vacuum helps.

For example, with F_L/b = 87.6 N/m, f_0 = 0.3, θ = 180°, K = 0.5, D = 0.152 m, V = 3.39E04 Pa, the wrap effect is F_H/b = 228 N/m and the vacuum effect is F_H/b = 2050 N/m.

More insight as to the relative importance of each of the variables and of the relative strength of the vacuum effect with respect to the wrap effect can be seen by rewriting {2} in non-dimensional form by equating the two terms on the right hand side to within a scale factor. Defining η as the vacuum effect divided by the wrap effect and the scale factor γ as the pull roller vacuum divided by the low tension side belt wrap pressure, we obtain the following:

$$\gamma = \frac{V}{\frac{2}{D}\frac{F_L}{b}} = \frac{1}{K}\left(\frac{1}{1 - exp^{-f_0\theta}}\right)\eta \qquad \{3\}$$

Expression {3} is plotted in Fig. 8 for three levels of η. From this graph, it is seen that larger values of friction times wrap angle is preferred since increasing this product reduces γ. In addition, as expected, smaller values of η result in smaller values of γ. However, the strength of the vacuum effect is illustrated in that the value calculated previously corresponds to a value of γ equal to 29.4, η equal to 9.0 and a vacuum equal to 33% of atmospheric pressure (1.013E05 Pa) and so even fairly high values of γ are easily attained with relatively low vacuum levels.

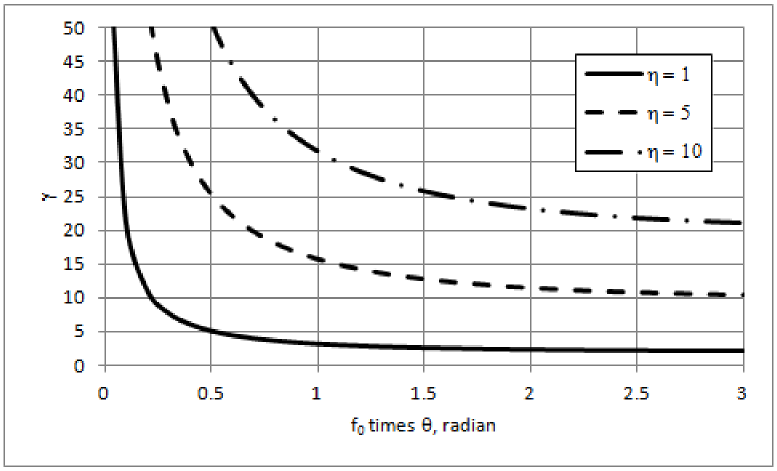


Figure 8 – γ versus $f_0\theta$, K = 0.5 from expression {3}

Because of the increased tension isolation capability of vacuum pull rollers, they consequently provide another useful function in that they can operate as a "*web metering device*". In this capacity, a vacuum pull roller can accurately measure web length or govern the speed of a web passing over it. We merely have to assure that no slip exists and count the roller's revolutions. In fact, it is often the case that a vacuum pull roller will operate performing both of these functions. For example, on a slitter/rewinder, a vacuum roller positioned between the unwind and slitter knives, driven by a variable-speed motor, governs the speed of the web (is a metering device). If the unwind is braked and the tension is low compared to the total windup tension, the vacuum pull roller acts as a tension isolator as well.

There are several different basic types of vacuum pull rollers but only three will be discussed in this paper. The "*holed, ported*" type consists of a roller with relatively large diameter holes through its surface. A system of rotating ports admits vacuum only to those holes which are covered by the web. Fig. 7 is an example of this type. The "*porous, un-ported*" type consists of a roller with a very large number of extremely small holes. Rollers of this type are often made from porous sleeves, such as air filter materials. A third option, which combines the benefits while eliminating most of the negatives of each of the first two is what is referred to as the "*holed and grooved, un-ported*" type. In the following sections, we discuss each of these in more detail.

PORTED VACUUM PULL ROLLER – HOLED SURFACE

When using relatively large holes through the roller, a porting system must be used to avoid excessive vacuum and air flow requirements since the portion of the pull roller not covered by the web will vent directly to atmosphere. While simple in concept, a porting system tends to make rollers of this type expensive to build and can cause maintenance problems. There are two possible arrangements: external and internal.

The internal design uses a stationary core, running the full length of the roller, and the roller is a cylindrical sleeve turning about this core. The core has flutes which communicate with the rows of holes in the sleeve as it rotates about the core. Air is sucked from those flutes where the web covers the holes. There are typically no flutes in the unwrapped portion. This design is applicable to variable width web use if the flutes are equipped with axially moveable plugs or dams. The construction is very expensive. Fig. 9 shows a schematic of this type of vacuum roller.

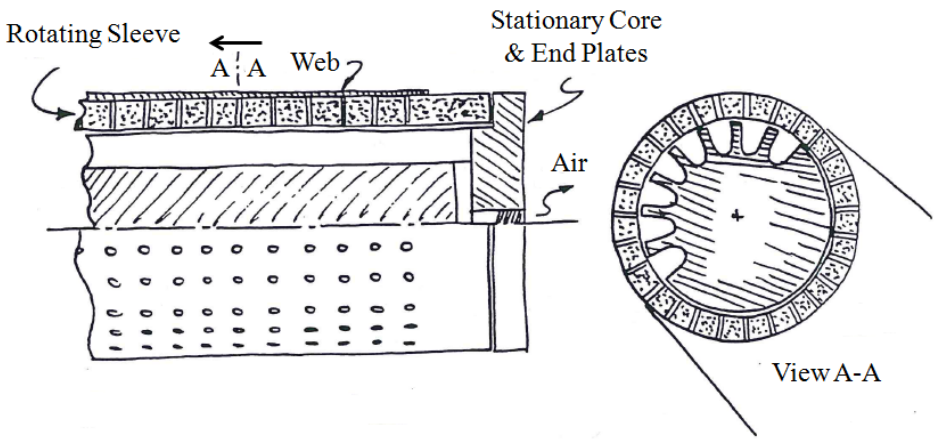


Figure 9 – Internal Design of a Porting System

The external design uses a stationary plate at one or both ends of the drum. Fig. 10 shows a schematic of this type of vacuum roller. The roller is solid and each axial row of holes is drilled radially into an axial bore which feeds each row. Looking at the roller from one end (view B-B), one sees a circle of holes, which are the individual bores connecting to the axial row of holes. The stationary plate has two recesses in its face (view A-A). One communicates with those bores which serve the web covered holes and it is connected to the vacuum source. The other plate recess is vented to atmosphere.

In either design, two surfaces are involved, one stationary and the other moving. This presents an additional air sealing problem, which can be solved in one of two ways:

- The surfaces can be allowed to be in contact if suitable low friction materials are chosen. In use, this can present problems of wear, dirt generation, and heat build-up at higher speeds. Care must be taken in the design, to correctly take into account the non-uniform end loading of the plate against the roller end. The vacuum pulls the plate in contact with the roller end only in a limited sector of the plate.
- A non-contacting, clearance type seal may be used. In order to keep air leakage at a reasonable level, clearances have to be small. This can lead to problems with binding. Close tolerances are involved, which make this seal expensive to manufacture.

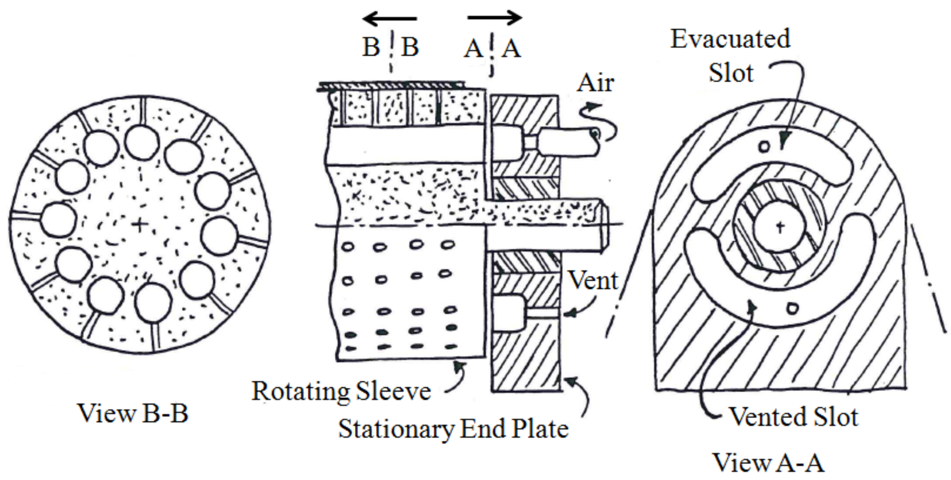


Figure 10 – External Porting System Design

Another disadvantage of this design is that time is required for the vacuum drum to operate effectively. This is so because it takes time to not only evacuate the flutes but also the interface between the web and the drum. It is this interface vacuum which is important. A vacuum platen, Fig. 11, can be used to show this time effect. The platen has a central hole and a peripheral groove, both connected to a vacuum supply via a valve. If a web is laid on the platen, and then the valve is opened, seconds elapse for the interface vacuum to approach the supply vacuum value. We will return to this subject later when we discuss the un-ported vacuum pull roller with holes and grooves. It is sufficient to say at this point that this consideration, along with those previously discussed, renders this design option seriously lacking from an operational perspective, particularly at higher speeds.

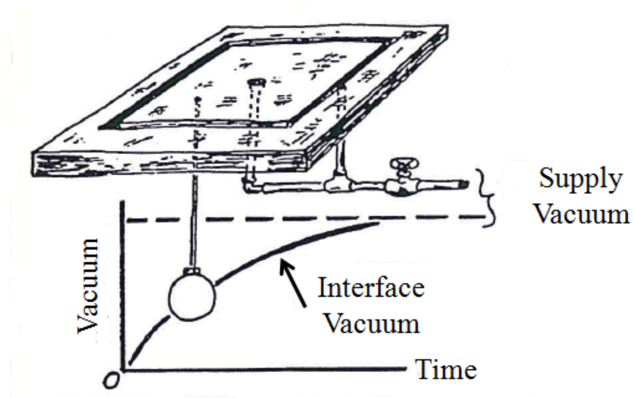


Figure 11 – Vacuum Evacuation Principles

UNPORTED VACUUM PULL ROLLER – POROUS SURFACE

Porous rollers are very simple in principle. No complicated and expensive porting system is needed. They are ideal for variable width web applications. However, there are also some serious disadvantages:

- Because the roller material is essentially an air filter, porous rollers are dirt catchers, tend to plug up with time and then are almost impossible to clean.
- Porous materials are very difficult to machine, because machining tends to close the pores.
- Porous materials are usually soft and subject to damage through carelessness.
- Because the porous material offers high resistance to air flow, a high vacuum is required to make the roller work.

Because of these and other considerations, the porous non-ported option for a vacuum pull roller is not usually the preferred option.

UNPORTED VACUUM PULL ROLLER – HOLED AND GROOVED SURFACE

The third alternative, which combines the benefits of the ported holed roller while avoiding the disadvantages of the porous sleeve is the un-ported holed and grooved vacuum pull roller. The reason for the grooves is two-fold. First, by proper selection of groove and hole size and patterns, the air flow is small even if there is no web on the roller. This dispenses with the need for porting or valving, which substantially reduces construction and maintenance costs. Second, vacuum efficiency, K, is high thanks to the grooves since the presence of the grooves facilitates interface evacuation by shortening the air evacuation paths. Hence, low vacuum levels are usual, typically less than 2.0E04 Pa, even at speeds in several hundreds of m/min.

When we design a vacuum roller so its interface can "*breathe well*" (e.g., very quickly evacuate the web/roller interface), by specifying many small-diameter holes and a system of surface grooves, we therefore accomplish two purposes:

- We keep the coefficient of friction, f, from dropping, as previously explained.
- We create a vacuum in the interface which, averaged over the entire wrap, is higher than it would be if the air in the interface could not be sucked away as readily.

The vacuum in the interface which, averaged over the entire area of contact between the web and drum, is the vacuum which causes additional normal force on which the holding power of the vacuum depends. This average vacuum is always lower than the vacuum measured inside the drum. "K" is "*vacuum efficiency*", and is the ratio:

$$K = \frac{average\ interface\ vacuum}{vacuum\ inside\ the\ roller} \qquad \{4\}$$

It is around 0.1 for poorly-breathing rollers with large diameter holes. For well-breathing rollers with proper hole and groove size and spacing, a value of 0.5 is typical.

It remains to provide some practical guidelines on the design of an un-ported pull roller with holes and grooves. Much more detailed information can be found in [2]. The following steps should be followed in the design of such a pull roller.

First, the requirements of the application should be determined. Included in this list is the web width, the roller diameter, the angle of wrap and the lower and higher of the two tensions. Drum diameter and web wrap should be reasonably large, typically 0.20 m or greater and 90° or larger respectively to insure time to adequately evacuate the interface air between the web and roller outer surface. Designs following these guidelines will operate effectively up to web conveyance speeds of 300 mpm. The values to be used for the lower and higher values of web tension should incorporate transient extremes.

Next, assume a static coefficient of friction. The object of the design is to produce a vacuum pull roller which does not let the web slip on its surface. The force which prevents this is a friction force whose calculation, according to {2} requires knowledge of the static coefficient of friction. It is advisable to measure this parameter according to the method illustrated in Fig. 2 or, if such a measurement cannot be readily made, assume a conservative estimate of 0.1.

The third step is to determine the vacuum required in the interface between the web and roller (averaged over the contact area) which prevents slip. Use the following formula for this computation:

$$V_i = \left(\frac{2}{bD}\right)\left(\frac{F_H - F_L e^{f_0\theta}}{e^{f_0\theta} - 1}\right) \qquad \{5\}$$

where V_i, in Pa, is the average effective interface vacuum (the numerator in {4}) to prevent slip. The fourth step is to determine the vacuum that must exist in the drum, which according to expression {4} is the average effective interface vacuum divided by the “vacuum efficiency, K”. As mentioned, for a properly designed vacuum roller, the vacuum efficiency is typically equal to 0.5 so that the vacuum necessary to be just sufficient to prevent web slipping, V, is equal to $2 \times V_i$.

We have now determined the vacuum, V, which is expected to be just sufficient to prevent web slipping. The vacuum actually supplied to the roller must be higher because we do not want marginal, slip-impending operation. Thus, the fifth step is to now determine the actual vacuum to be used inside of the roller so as to insure an adequate factor of safety. Towards this end, decide on a vacuum, V_s, higher than V, which can then be used to evaluate the maximum high tension, F_{HM}, which the roller will be able to handle in the presence of the specified low tension, F_L:

$$F_{HM} = F_L e^{f_0\theta} + 0.5KbDV_s\left(e^{f_0\theta} - 1\right) \qquad \{6\}$$

Similarly, evaluate the equation:

$$F_{LM} = \frac{1}{e^{f_0\theta}}\left[F_H - 0.5KbDV_s\left(e^{f_0\theta} - 1\right)\right] \qquad \{7\}$$

where F_{LM} is the smallest low tension which the roller will be able to handle in the presence of a specified high tension, F_H. In this calculation, F_{LM} may compute to be negative; if so, let it be equal to zero. From these results, compute the factors of safety and adjust the actual vacuum until both of these are at least equal to two.

$$FS(H) = \frac{F_{HM}}{F_H}, \quad FS(L) = \frac{F_L}{F_{LM}} \qquad \{8\}$$

Note that V_s is the vacuum which is to exist inside the roller. To assure that a vacuum source, remote from the roller and connected to it by piping, will indeed supply V_s, it is necessary to calculate pressure drop in the piping. Standard handbooks and texts on air flow show how to do this. The air flow to use in these calculations is now discussed.

Next, the number and size of holes has to be determined. Two requirements are to be met: (a) the more holes per unit area, the better and (b) the air flow, with the roller uncovered by the web, should be at least 1.5 SCMM per square meter of drum surface. With these requirements in mind, first decide on the smallest hole diameter which can be drilled through the roller wall. This will depend on the material used and the wall thickness. Holes larger than 1.5 mm should be avoided. Having decided on hole diameter, compute the number of holes per square meter using the following equation:

$$N = \frac{Q}{49d^2\sqrt{V_s}} \qquad \{9\}$$

where d is hole diameter (m), V_s is vacuum inside the roller (Pa) and Q is the air flow rate (SCMM per square meter of roller surface, at least 1.5).

The hole and groove pattern must now be specified. For this, the following simple rules should be followed:

- The distance from any point on the roller surface to the nearest groove should be no further than 1.3 mm.
- If grooves project beyond the web edge (and they may do so), holes must be at least 25 mm inside the web edge.
- Holes closer than 25 mm to either the entering or leaving nip must not be counted.
- Grooves are effective only if they are connected to holes and holes are effective only if they are connected to grooves.
- The most direct groove path between holes nearest each other must not exceed 0.15 m.
- The cross-sectional area of any groove should not be less than 0.032 square mm.
- The width of a groove should not be more than five times the thickness of the thinnest web to be run on the roller, and should never be wider than 0.75 mm.
- Grooves should be made as narrow as practically possible, provided they can be evacuated quickly enough to permit operation at the desired web speed.

Remaining considerations include roller surface finish and the details of the rotating joint. Regarding surface finish, steps must be taken to assure no sharp edges or burrs remain on the grooves. Also, the roller surface should be adequately polished so that it can be readily cleaned and so that it does not change with in-use wear. In addition, both of these specifications will minimize web damage if inadvertent slip should occur. Regarding the rotating joint, it must be large enough to not be a significant resistance to air flow. In addition, it is recommended that a section of flexibility be interposed between the rotary joint and the rigid supply piping to provide stress relief.

Some remaining caveats should be mentioned regarding the applicability of the design guidelines. Webs that are porous, are very rough, are textured, are thicker than 0.200 mm or thinner than 0.075 mm are special cases and require more detailed analysis to insure a robust design. Otherwise, the guidelines provide the basis for designing and building reliable un-ported pull rollers with holes and grooves.

CONCLUSIONS

The purpose and types of pull rollers have been described. The principles of operation have been presented and the benefits and shortcomings of each listed. Plain pull rollers rely on friction developed from radial pressure arising from web tension and consequently have limited capability since pressure developed in this fashion is often not sufficient to provide adequate friction at low speeds or to prevent air lubrication at higher speeds. Vacuum pull rollers overcome this limitation by providing an additional source of radial pressure thereby increasing friction capability. Two conventional types were described: un-ported porous and ported holed. However, manufacturing and operational issues render these designs costly to build and to reliably operate. These limitations are overcome with the third type of vacuum pull roller: the un-ported holed and grooved design. With this design, air flow is sufficiently small so that porting is not required and grooves are incorporated to insure maximum vacuum effectiveness. Guidelines were presented to enable the design and construction of un-ported holed and grooved vacuum pull rollers.

REFERENCES

1. Roisum, D.R., The Mechanics of Rollers, Tappi Press, 1996, p. 52.
2. U.S. Patent 3,630,424, “Drilled Non-Ported Vacuum Drum”, J.A. Rau, Dec. 28, 1971.

APPENDIX 1 – DERIVATION OF {2}

Figure 12 shows an element of the web in contact with the vacuum roller. The element subtends an angular increment $d\phi$, and has width *b*. Hence, the area of contact between the element and roller is:

$$dA = bRd\phi \qquad \{10\}$$

There is a force, caused by the vacuum *KV* in the interface, pushing the element into contact with the roller. This force is $KVdA$.

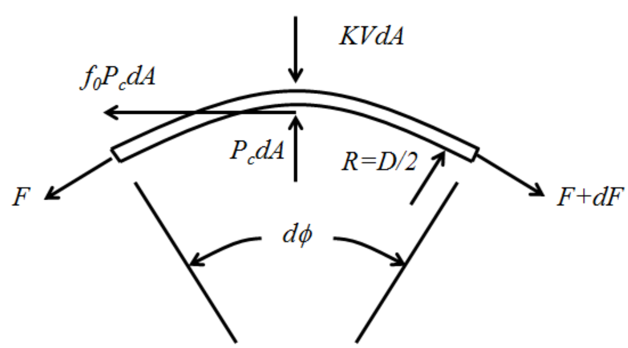


Figure 12 – Body Diagram of Elemental Web in Contact with the Roller

Assume that the tension at one end of the element is F and at the other end is greater than F by an incremental amount dF.

The three known forces, F, $F + dF$, and $KVdA$ are kept in equilibrium by the reaction force $P_c dA$, (where P_c is an unknown pressure) and by a friction force $f_0 P_c dA$ which resists impending motion, i.e., is directed opposite dF where dF is greater than zero by definition.

Sum forces horizontally and vertically, obtaining:

$$(F + dF) cos \frac{d\phi}{2} - F cos \frac{d\phi}{2} - f_0 P_c dA = 0 \tag{11}$$

$$P_c dA - KVdA - (F + dF) sin \frac{d\phi}{2} - F sin \frac{d\phi}{2} = 0 \tag{12}$$

Substitute {10} into {11} and {12}:

$$(F + dF) cos \frac{d\phi}{2} - F cos \frac{d\phi}{2} - f_0 P_c bRd\phi = 0 \tag{13}$$

$$P_c bRd\phi - KVbRd\phi - (F + dF) sin \frac{d\phi}{2} - F sin \frac{d\phi}{2} = 0 \tag{14}$$

Now $d\phi$ is a very small angle, for which $sin \frac{d\phi}{2} \approx \frac{d\phi}{2}$ *and* $cos \frac{d\phi}{2} \approx 1$. Substitution gives:

$$(F + dF) - F - f_0 P_c bRd\phi = 0$$

$$P_c bRd\phi - KVbRd\phi - (F + dF) \frac{d\phi}{2} - F \frac{d\phi}{2} = 0$$

Whence, simplifying, and recognizing also that the differential of higher order, $dFd\phi$, is approximately zero, obtain:

$$dF - f_0 P_c bRd\phi = 0 \tag{15}$$

$$P_c bR - KVbR - F = 0 \tag{16}$$

Eliminate P_c from {15} and {16}

$$dF = (KVbR + F) f_0 d\phi \tag{17}$$

whence

$$\frac{dF}{F + KVbR} = f_0 d\phi \tag{18}$$

Expression {18} can be integrated immediately. The limits of integration are:

$$\phi = 0 \rightarrow F = F_L \textit{ and } \phi = \theta \rightarrow F = F_H$$

whence

$$\int_{F_L}^{F_H} \frac{dF}{F + KVbR} = \int_0^{\theta} f_0 d\phi \qquad \{19\}$$

and

$$ln\left(\frac{F_H + KVbR}{F_L + KVbR}\right) = f_0\theta$$

and finally

$$\frac{F_H + KVbR}{F_L + KVbR} = e^{f_0\theta} \qquad \{20\}$$

which can be manipulated into

$$F_H = F_L e^{f_0\theta} + KVbR\left(e^{f_0\theta} - 1\right) \qquad \{21\}$$

Since $D = 2R$, we obtain from {21} the final result which is expression {2}.