A BELATED APPRECIATION OF LISA SIEVERS' THESIS

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ABSTRACT

In her 1987 thesis [1] and a follow-up paper in the IEEE Transactions of Automatic Control [2], Lisa Sievers described three dynamic multi-span models for lateral web behavior.

- Convecting string with zero bending stiffness
- Euler-Bernoulli beam with bending stiffness and no shear
- Timoshenko beam with both bending and shear

The last two transferred the bending portion of lateral deformation across rollers. In the Timoshenko model, which included shear and is the main result of her work, only bending deformation is transferred.

Although she built on the work of everyone who preceded her, Lisa creatively reanalyzed everything she used and put it on a more rigorous mathematical footing. This deserves a wider appreciation.

Three notable features of the thesis are,

- Use of Hamilton's method for deriving the governing equations. This is certainly not the first instance of its use in web handling. It is nevertheless an excellent example.
- Derivation of the downstream boundary conditions for lateral velocity by application of the material derivative. So far as I know, this has never been done before.
- Use of spectral separation to justify use of static web shape in a dynamic model. This is a good idea. However, results from it suggest that there is not as much separation as we might like.

In an effort to make the results of her thesis more accessible, this paper will review it and recast the Timoshenko model into a form which facilitates analytical comparison with Euler-Bernoulli beam models in current use.

NOMENCLATURE

A	cross sectional area of web
E	elastic modulus
G	shear modulus
h	thickness of web
I	area moment of inertia
J	rotational inertia
L	span length
m	mass per unit length
n	Shear coefficient for Timoshenko beam
t	time
T	tension
v_o	web velocity in machine direction
v_y	lateral web velocity
x	distance along length of web
y	lateral displacement of web
y_0	lateral web displacement at upstream roller, relative to ground
y_L	lateral web displacement at downstream roller, relative to ground
z	lateral displacement of roller relative to ground
γ	angle of roller axis
θ	slope of web
ρ	density
ϕ	rotation of cross section
ψ	shear angle
$_0$	subscript indicating value of variable at x = 0
$_L$	subscript indicating value of variable at x = L

The notation used in this paper is different than that used in the thesis. Sievers' referenced variables to rollers using a subscript to identify the roller and a superscript to indicate whether the variable applied to the upstream side or downstream side. For example, y_i^u refers to the value of y on the upstream side of the *ith* roller. A superscript *d* is used to indicate the downstream side. In this paper and much of the current literature, variables are referenced to spans. A variable labeled y_{02} would indicate the value of y at $x = 0$ in span 2.

FRAMES OF REFERENCE (MATERIAL DERIVATIVE)

The material derivative has been around for a long time. Leonard Euler first introduced the concept as a method for analyzing fluid flow. However, Siever's thesis is the first place I've ever seen it used to derive the normal entry rule. Before getting to that, here is a brief review of the material derivative.

The web is modeled as a static rectangle of material. But, in reality it is moving uniformly in the machine direction. So, the thing which we call a web is really a fixed frame of reference for observation of material that is moving past the observer.

A physical law applied to a moving medium is usually defined in a Lagrangian frame of reference. This means that it applies to a property $y(x,t)$ in a particular volume of material *as it moves*. This implies that the position of the point of observation is moving with it as a function of time. Call this *x(t)*. We can create a fixed frame of reference by applying the chain rule to time derivatives.

$$\frac{dy(x,t)}{dt} = \frac{\partial y(x,t)}{\partial t} + \frac{\partial y}{\partial x}\frac{dx(t)}{dt} \qquad \{1\}$$

In the case of a web, *dx(t)/dt* is the transport velocity, v_o . So, {1} becomes,

$$\frac{dy(x,t)}{dt} = \frac{\partial y(x,t)}{\partial t} + \frac{\partial y(x,t)}{\partial x} v_o \qquad \{2\}$$

The total derivative on the left is often called the "material" derivative and is said to be in a Lagrangian frame of reference. The first term on the right applies to a fixed position because that is how the partial derivative is defined. It is said to be in a Eulerian frame of reference. The last term completes the material derivative by adding the effect of position variation caused by transport motion.

The material derivative for acceleration gets a little more complicated. It is,

$$\begin{aligned}\frac{d^2(x,t)}{dt^2} &= \frac{\partial}{\partial t}\left(\frac{\partial y(x,t)}{\partial t}+\frac{\partial y(x,t)}{\partial x}v_o\right)+\frac{\partial}{\partial x}\left(\frac{\partial y(x,t)}{\partial t}+\frac{\partial y(x,t)}{\partial x}v_o\right)\frac{dx(t)}{dt} \\ &= \frac{\partial^2 y(x,t)}{\partial t^2} + 2\frac{\partial y(x,t)}{\partial x \partial t}v_o + \frac{\partial^2 y(x,t)}{\partial x^2}v_o^{\ 2}\end{aligned} \qquad \{3\}$$

DERIVATION OF THE NORMAL ENTRY RULE

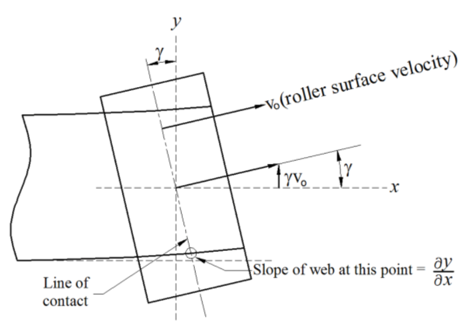

Figure 1 – Effect of misaligned roller

One of the most important equations in lateral web behavior is the normal entry rule. It defines lateral behavior at the entry to rollers and is usually explained using a vector diagram like the one in Figure 1. Sievers, however, showed that it is a direct consequence of changing from a Lagrangian to a Eulerian frame of reference. Particles of the web approaching the line of contact at a roller are pulled in the direction of the roller surface. If the roller is inclined to the coordinate *y* by an angle γ, there will be a Lagrangian component of velocity in the *y* direction equal to γv_o . Thus,

$$\frac{dy}{dt} = v_o \gamma = \frac{\partial y}{\partial t} + v_o \frac{\partial y}{\partial x} \quad \{4\}$$

Solving for the Eulerian velocity at the line of contact,

$$\frac{\partial y}{\partial t} = v_o \left(\gamma - \frac{\partial y}{\partial x} \right) \quad \{5\}$$

The quantity *y* in equation {5}is interpreted as being in relationship to the roller. If the roller is simultaneously being displaced laterally by a distance *z*, then {5}becomes,

$$\frac{\partial y}{\partial t} = v_o \left(\gamma - \frac{\partial y}{\partial x} \right) + \frac{\partial z}{\partial t} \quad \{6\}$$

and the *y* on the left is interpreted as being in relationship to the ground.

As the development of the model proceeds it will be necessary to establish a relationship between web curvature and lateral acceleration.

THE TROUBLESOME ACCELERATION EQUATION

Since the normal entry rule establishes a relationship between slope and velocity, it seems reasonable to define a companion relationship for acceleration. Sievers and Shelton both did this. Sievers applied the material derivative again.

Using the relationship developed in{3},

$$\frac{d^2 y}{dt^2} = \frac{\partial^2 y(x,t)}{\partial t^2} + 2 \frac{\partial y(x,t)}{\partial x \partial t} v_o + \frac{\partial^2 y(x,t)}{\partial x^2} v_o^{\ 2} = v_o \frac{\partial \gamma}{\partial t} \quad \{7\}$$

The term on the right is the Lagrangian acceleration due to the rate of change of roller angle. If every roller is fixed, that term isn't needed (that's what Sievers did), but the rollers in guiding mechanisms must pivot, so, in the general case, the material derivative for acceleration should include it.

The cross derivative in {7} can be eliminated by taking the derivative of {5} with respect to *x*.

$$\frac{\partial^2 y}{\partial x \partial t} = -v_o \frac{\partial^2 y}{\partial x^2} \quad \{8\}$$

Substituting into{7}, simplifying and adding acceleration relative to ground yields,

$$\frac{\partial^2 y}{\partial t^2} = v_o^{\ 2} \frac{\partial^2 y}{\partial x^2} + v_o \frac{\partial \gamma}{\partial t} + \frac{\partial^2 z}{\partial t^2} \quad \{9\}$$

This seems very logical, but I can assure you it is wrong for the problem at hand.

A displacement guide turns out to be a good test for lateral dynamic models. When a displacement guide is pivoted, the two rollers and the web rotate about the pivot point as a rigid body (assuming no disturbance in *y* at the first roller). So, if the transfer function for the guide doesn't collapse to unity (no dynamics) when it's pivoted, something is wrong and that is just what happens. The transfer function for a displacement guide does not collapse to unity when the $d\gamma/dt$ term is included. Other things will also turn out wrong. For example, the steady state location of the effective center of rotation for a web approaching an inclined roller will not agree with the value found from static analysis.

The rate of change of angle term mustn't be there, but it's not easy to understand why. Sievers and Shelton each employed different strategies for getting rid of it.

Here is how Shelton did it in his 1968 dissertation [3]. In setting up the dynamics equations for his Bernoulli beam model, he said on page 104,

"Note that equation 4.1.5 [identical to {9} without the angle term] is not merely the derivative of equation 4.1.2 [equation {6} above]; differentiation of the latter equation results in an extra term containing the velocity of roller swiveling, $d\theta r/dt$ [θ_r is represented by γ in this paper]. Because of the assumption that shear deflection is negligible, no acceleration can occur as an instantaneous result of roller swiveling, but only indirectly as the web curvature changes. A suddenly swiveling roller instantaneously swivels the downstream end of the web an equal amount, so that no instantaneous change in steering rate occurs, in contrast to the first order theory of Chapter (III)."

This is true, but why not let the model tell us this rather than arbitrarily enforcing it? Furthermore, it may have led him to make a mistake. The rationale used for eliminating the angle term in the Euler Bernoulli model doesn't work for a model with shear. So when he got to the Timoshenko model he was compelled to use this equation (equation 4.3.1 on page 117).

$$\frac{\partial^2 y}{\partial t^2} = v_o \left.\frac{\partial^2 y}{\partial x^2}\right|_L + \frac{d^2 z_L}{dt^2} - v_o \frac{d\theta_{Ls}}{dt} \qquad \{10\}$$

where θ_{Ls} is the angle of shear. He doesn't give a rationale for the shear term and it may be wrong.

Sievers deftly skirted the issue altogether. The only pivoting rollers on her machine were those of the displacement guide shown in Figure 2. She recognized that these rollers and the span between them move together as a rigid body. So, she rotated the coordinate system for the span between the guide rollers and treated it as though it was passing between fixed parallel rollers. For the purposes of her thesis, this worked. However, it is not a good general solution because it doesn't work for a misaligned roller or for an ordinary steering guide.

So, what is the problem with{9} and{10}? The trouble is that the relationship between acceleration and curvature shouldn't be treated as an independent boundary condition. The acceleration equations should be eliminated. Acceleration will be treated when the occasion arises later in the development of the model and at that point, the material derivative will be seen to be irrelevant.

THE TEST MACHINE

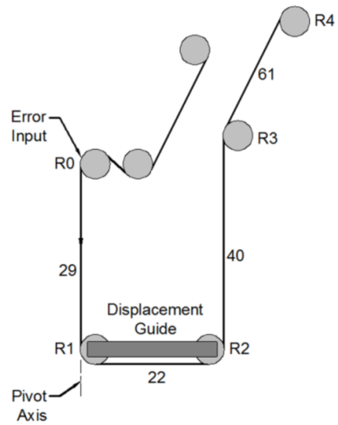

Figure 2 – Web path of test machine. Span lengths in inches.

There is ambiguity in the values reported for EI and GA/n. Material parameters in Chapter 2 are listed as,

Tension –	0.75 to 2 Lb/in
Velocity –	100 to 1000 ft/min
Weave frequency –	0.01 to 0.3 Hz
Web width –	12 to 60 inches
Web thickness –	0.0015 to 0.009 inches
Web density –	0.04 to 0.05 lbm/in^3
Young's Modulus –	5 x 10^5 to 6x10^5 lb/in^2
Poisson's ratio –	0.3

In chapter 8, the parameters for experiments are described as,

Material –	Acetate
EI	1.76 x 10^8 lbf-in^2
GA/n	3.42 x 10^5 Lbf
ρI	1.18 lbm-in
ρA	7.12 x 10^{-3} lbm/in
W	44.5 inches

Acetate has a S.G of about 1.3.That would make ρ = 0.047 lbm/in^3. With the given value of ρA and W, the thickness, h would be 0.0034 inch. If that value of h is used to calculate I as 25 in^4, its product with ρ is exactly equal to the value shown, 1.18 lbm-in. Thickness h = 0.0034 inch is also consistent with the thickness range shown in Chapter 2. Using the values above, however, produces a value for EI between 1.257x10^7 and 1.5x10^7 lbf-in^2. That is less than 1/10 the value shown.

Also since $G = E/(2(1+0.3))$, G should be between 1.9x10^5 and 2.3x10^5 psi. The value of the shear coefficient, n isn't mentioned in the thesis, but it's generally accepted as 1.2 for a rectangular cross section. Using these quantities, the value of GA/n is between 2.4x10^4 and 2.9x10^4 lbf. That is less than 1/10 the value shown.

For EI and GA/n to have the values shown in the thesis, the web thickness would have to be on the order of 0.04 inch. That is obviously inconsistent with the values for ρA

and ρI and is also outside the design ranges shown in Chapter 2. Therefore, it will be assumed that the thickness h is 0.0034 inch and the values for EI and GA/n are adjusted accordingly.

RELATIONSHIP OF SHEAR TO BENDING

Development of the Timoshenko model begins in Section 3.3 by first observing that the shear and bending deflections are additive as illustrated in Figure 3.

The relationship shown in Figure 3 is fundamental to everything that follows. The total slope is equal to the sum of bending and shear[1].

$$\frac{\partial y}{\partial x} = \phi + \psi \qquad \{11\}$$

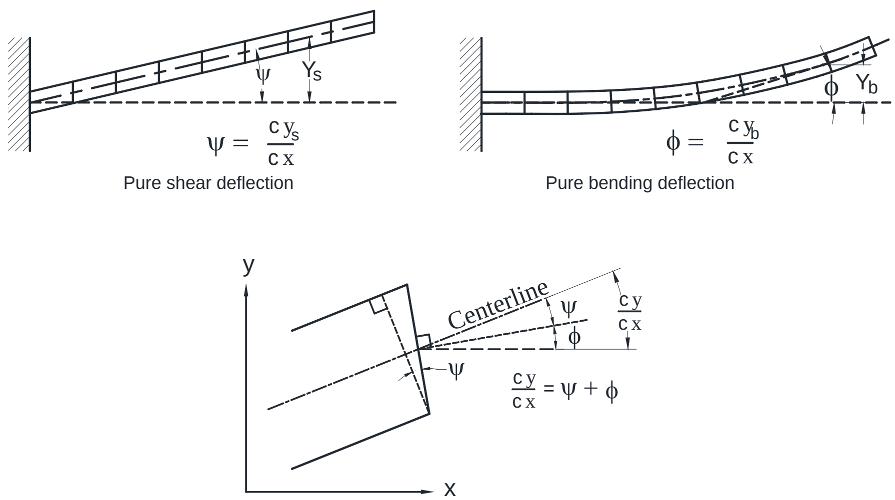

Figure 3 – Relationship of Slope, shear and rotation of cross section

$\frac{\partial y}{\partial x} = \text{slope}$, $\psi = \text{shear angle}$, $\phi = \text{angle of cross section}$

HAMILTON'S PRINCIPLE

Hamilton's principle [4] works particularly well for this problem. It begins in section 3.3 by defining kinetic energy K and potential energy V for the beam. The kinetic portion includes the effect of rotation.

[1] Note: Sievers used the symbol θ to represent rotation of the cross section rotation (which she called face angle). In this paper ϕ is used, because θ is used in most of the literature as another way to represent $\frac{\partial y}{\partial x}$.

$$K=\frac{1}{2}\int_0^L m\left(v_o{}^2+v_y{}^2\right)dx+\frac{1}{2}\int_0^L J\left(\frac{\partial\phi}{\partial t}\right)^2 dx \qquad \{12\}$$

In this equation, m is the mass per unit of length, v_o is the transport velocity in the machine direction, v_y is the velocity of lateral deflection, J is the rotational inertia per unit of length and ϕ is the rotation of cross section. Before using it, the variables in equation {12} are transformed to a Eulerian frame of reference. So, {12} becomes,

$$K=\frac{1}{2}\int_0^L m\left(v_o{}^2+\left(\frac{\partial y}{\partial t}+v_o\frac{\partial y}{\partial x}\right)^2\right)dx+\frac{1}{2}\int_0^L J\left(\frac{\partial\phi}{\partial t}+v_o\frac{\partial\phi}{\partial x}\right)^2 dx \qquad \{13\}$$

The potential energy is,

$$V=\frac{1}{2}\int_0^L EI\left(\frac{\partial\phi}{\partial x}\right)^2 dx+\frac{1}{2}\int_0^L \frac{AG}{n}(\psi)^2 dx+\frac{1}{2}\int_0^L T\left(\frac{\partial y}{\partial x}\right)^2 dx \qquad \{14\}$$

The first term is the energy due to rotation of the cross section (which Sievers calls face angle). The second one is the energy due to shear strain. The last one is the energy due to the interaction of longitudinal tension and slope.

Hamilton's principle applies the calculus of variations [5] to find the relationship between the variables ψ, ϕ, and y which will minimize $(K - V)$. Before doing this, the number of variables is reduced by replacing ψ with $\partial y/\partial x - \phi$.

After many integrations by parts and applications of the chain rule, two equations are produced. [I can't imagine doing this without the help of symbolic manipulation software.]

$$-m\left(\ddot{y}+2v_o\dot{y}'+v_o{}^2y''\right)+\left(\frac{AG}{n}+T\right)y''-\frac{AG}{n}\phi'=0 \qquad \{15\}$$

$$J\left(\ddot{\phi}+2v_o\dot{\phi}'+v_o{}^2\phi''\right)+EI\phi''+\frac{AG}{n}(y'-\phi)=0 \qquad \{16\}$$

The dot represents differentiation with respect t and a quote mark indicates differentiation with respect to x.

In the thesis there is an important mistake in these equations. The shear angle ψ appears where face angle ϕ (θ in the thesis) should have been. This is probably a typographical mistake because all of the subsequent equations are correct.

To anyone familiar with the transverse vibrations of a traveling string, equation {15} looks very familiar [6]. The first quantity in parenthesis on the left side is the Eulerian acceleration and if the terms involving *AG/n* are eliminated, what is left is *exactly* the traveling string wave equation. Equation {16} has the same form. So, elimination of the *AG/n* term in that case would leave a traveling string equation for rotational vibrations (rotation in the plane of the web). Therefore, it's appropriate to consider {15} and {16} as coupled equations for describing a traveling, tensioned, Timoshenko beam.

A lot of work could have been saved if the transformation to a Eulerian frame of reference had been postponed until after the application of Hamilton's principle.

Equations {15} and {16} can be combined into one involving only y. First, equation {15} is solved for ϕ' and then differentiated by x or t to produce values for $\ddot{\phi}'$, $\dot{\phi}''$ *and* ϕ''' in terms of y and its derivatives. Then, the second equation is differentiated once by x and the values from equation {15} are substituted to eliminate ϕ. The result is,

$$\begin{aligned} 0 = &\left(Jv_o^2 - EI\right)\left(\frac{mnv_o^2}{AG} - \frac{nT}{AG} - 1\right)\frac{\partial^4 y}{\partial x^4} + \left(mv_o^2 - T\right)\frac{\partial^2 y}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} \\ &+ \left(\frac{6mnJv_o^2}{AG} - \frac{E\,\mathrm{Im}\,n}{AG} - \frac{JnT}{AG} - J\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} \\ &+ \left(\frac{4Jmnv_o^3}{AG} - \frac{2E\,\mathrm{Im}\,nv_o}{AG} - \frac{2Jv_o nT}{AG} - 2Jv_o\right)\frac{\partial^4 y}{\partial x^3 \partial t} \\ &+ \frac{Jmn}{AG}\frac{\partial^4 y}{\partial t^4} + \frac{4Jmnv_o}{AG}\frac{\partial^4 y}{\partial t^3 x} + 2mv_o\frac{\partial^2 y}{\partial x \partial t} \end{aligned} \qquad \{17\}$$

In the thesis there is a minor mistake in this equation. On the second line, the first term should have a *J* in it and the fourth term on the same line should be positive. Those two terms then combine to equal the first term on the second line as shown here.

SPECTRAL SEPARATION

This concept was new to me, but seems immediately obvious. The idea is that if the natural vibration frequencies of an elastic object are much higher than the operating frequencies of an elastic system, it is acceptable to treat it quasi-statically. Another way of looking at it is that separation means that the imposed elastic deformations occur at frequencies which are too low for their inertial reactions to be significant.

Since the thesis compares results for three models, beginning with a flexible string, it used the first mode of the flexible string as the lower bound for all three models. This makes sense in the context of the thesis because it is the least stiff and should, therefore, have the lowest natural frequency. It is,

$$f_o = \frac{1}{2L}\sqrt{\frac{T}{m}} \qquad \{18\}$$

Using worst case parameters for the test machine, (L = 61 inches, $m = \rho A$ = 7.12x10-3 lbm/in, T = 44.5 lbf), f_o is 12.7 Hz. The thesis says 70 Hz. I can't explain the difference. It could be a problem with units; 12.7 Hz would be 80 radians/sec.

Since the natural frequency of the string model is so low, I decided to investigate further. The FEA tools available to me aren't capable of handling equation {17} in its full glory, so I modeled it with with v_o, J and n equal *to 0*. This is a stationary, tensioned Euler Bernoulli beam. Using parameters for the last span on the test machine, and applying boundary conditions for clamped ends, the frequency of the fundamental mode is *46 Hz*.

L = 61 inches	T = 44.5 lbf
h = 0.0034 inches	E = 500,000 psi
W =44.5 inches	ρ = 0.036 lbm/in^3 (S.G. = 1.3)

I was able go a step further and model a tensioned, stationary beam with shear by setting v_o and *J to 0* but with *n = 1.2*. With clamped ends, the fundamental mode for this is *28 Hz* (the Timoshenko beam is less stiff than the Euler Bernoulli).

The low natural frequencies of the beam models surprises me. It's true that the sinusoidal disturbances studied in the thesis are more than an order of magnitude lower than 12.7 Hz, so it's acceptable for this particular case. However, there are many web processes for which such values could be an issue.

THE STATIC EQUATION FOR WEB SHAPE

It's easier to skip equation {17} and work with {15} and {16}. Eliminating the leftmost acceleration terms yields,

$$\left(\frac{AG}{n}+T\right)y'' - \frac{AG}{n}\phi' = 0 \qquad \{19\}$$

$$EI\phi'' + \frac{AG}{n}\left(y'-\phi\right) = 0 \qquad \{20\}$$

To obtain a single equation involving only the variable *y*, equation {19}is solved for ϕ' and then differentiated by *x* to produce values for ϕ'' and ϕ''' which are used to eliminate ϕ from {20}after it is differentiated once with respect to *x*. The result is the familiar equation,

$$\frac{d^4y}{dx^4}-K^2\frac{d^2y}{dx^2}=0 \qquad \{21\}$$

where,

$$K^2=\frac{T}{EI\left(1+\frac{nT}{AG}\right)} \qquad \{22\}$$

The solution to {21}is the familiar equation,

$$y(x)=C_1\sinh\left(Kx\right)+C_2\cosh\left(Kx\right)+C_3x+C_4 \qquad \{23\}$$

<u>Boundary conditions that determine static web shape</u>

The coefficients of {23}are determined from static boundary conditions. Four are needed. Lateral position at each end is an obvious choice for all three models and for a multi-span model it must obviously be continuous across rollers. That takes care of two. For the Euler Bernoulli beam, slope is the other logical choice because it will also be continuous across a roller.

$$y\big|_{x=0} = y_0 \qquad y\big|_{x=L} = y_L$$
$$\left.\frac{dy}{dx}\right|_{x=0} = \theta_{w0} \qquad \left.\frac{dy}{dx}\right|_{x=L} = \theta_{wL} \tag{24}$$

For the Timoshenko beam, the presence of shear makes it possible for slope to be discontinuous across a roller. So, it isn't preserved as the web passes from entry to exit. However, the cross section rotation is preserved, making it the logical choice for the other pair of boundary conditions.

Expressions for rotation of cross section ϕ and shear angle ψ are needed. First, differentiate {19} with respect to x and solve for ϕ''.

$$\phi'' = y'''a \tag{25}$$

where

$$a = 1 + \frac{nT}{AG} \tag{26}$$

Replacing $y' - \phi$ in {20}with ψ and using {25},

$$\psi = -EIa\frac{n}{AG}y''' \tag{27}$$

Now, using {11},

$$\phi = y' + EIa\frac{n}{AG}y''' \tag{28}$$

So, the boundary conditions of the Timoshenko beam model will be,

$$y\big|_{x=0} = y_0 \qquad y\big|_{x=L} = y_L$$
$$\left.\frac{dy}{dx}\right|_{x=0} + EIa\frac{n}{AG}\left.\frac{d^3y}{dx^3}\right|_{x=0} = \phi_0 \qquad \left.\frac{dy}{dx}\right|_{x=L} + EIa\frac{n}{AG}\left.\frac{d^3y}{dx^3}\right|_{x=L} = \phi_L \tag{29}$$

To determine the coefficients of {23}, it is differentiated to provide expressions for the derivatives in ϕ_o and ϕ_L,

$$\frac{dy}{dx} = C_3 + C_1K\cosh(Kx) + C_2K\sinh(Kx) \tag{30}$$

$$\frac{d^3y}{dx^3} = C_1K^3\cosh(Kx) + C_2K^3\sinh(Kx) \tag{31}$$

Using {23} again for y_0 and y_L provides the four boundary conditions.

$$
\begin{aligned}
&\phi_0 = C_3 + C_1 Ka \\
&\phi_L = C_3 + Ka\left(C_1 \cosh(KL) + C_2 \sinh(KL)\right) \\
&y_0 = C_2 + C_4 \\
&y_L = C_1 \sinh(KL) + C_2 \cosh(KL) + C_3 L + C_4
\end{aligned}
\qquad \{32\}
$$

These are solved simultaneously for C_1, C_2, C_3 *and* C_4.

Inserting these values into {23}and collecting terms,

$$
y(x) = y_0 + \left(y_0 - y_L\right) g_4(x,L) + \phi_L g_5(x,L) + \phi_0 g_6(x,L) \qquad \{33\}
$$

where,

$$
\begin{aligned}
g_4(x) &= [\cosh(Kx) + \cosh(KL) - \cosh(KL - Kx) - Kax \sinh(KL) - 1] / R \\
g_{5(x)} &= [KLa\left(\cosh(Kx) - 1\right) - Kax\left(\cosh(KL) - 1\right) - \sinh(Kx) \\
&\quad - \sinh(KL - Kx) + \sinh(KL)] / KaR \\
g_6(x) &= [\sinh(Kx) - \sinh(KL) + \sinh(KL - Kx) - KLa\left(\cosh(KL - Kx) - 1\right) \\
&\quad + Ka(L - x)(\cosh(KL) - 1)] / KaR
\end{aligned}
\qquad \{34\}
$$

and,

$$
R = KLa \sinh(KL) - 2\left(\cosh(KL - 1\right) \qquad \{35\}
$$

Following the example of Young, Shelton and Kardimilas (YSK) [7], y_o appears twice in expression {33}. This reduces the number of shape factors from four to three.

In the thesis, Sievers shows the value of *a* as,

$$
a = 1 + K^2 \frac{EIn}{AG}\left(1 + \frac{nT}{AG}\right) \qquad \{36\}
$$

This simplifies to {26}.

A FORK IN THE ROAD

At this point there are two different approaches to creating a numerical model. In the method used by Sievers, two quantities are calculated at the downstream end of a span. One is the lateral displacement and the other is the cross section rotation. Both of these quantities are then passed to the calculations for the next span. In the method used by Young Shelton and Kardamilas , only the lateral displacement is passed to the next span. Since all of the information about its derivatives is implicit in the lateral displacement, it is possible to reconstruct the cross section rotation from the previous span by mathematical manipulation. This may seem redundant. However, it has the advantage of making it possible to describe the system with transfer functions. The disadvantage is that the numerical solution of the underlying differential equation is problematic because it involves third order derivatives. Each method has its advantages. The 2nd order equations of the Sievers approach are much easier to solve with FEA software.

Two versions of Sievers' method will be described in this paper. The Young, Shelton, Kardamilas method will be described in a companion paper [8].

In the model presented in her thesis Sievers treated the displacement guide as though the pivoting rollers were fixed and parallel to the other rollers. She then used a coordinate rotation to account for the pivoting motion. The transformation is described in Appendix A.2 of the thesis.

The modified model, described first, handles pivoting rollers without a coordinate transformation. The two models produce exactly the same numerical results, but it is the opinion of the author that the modified version is preferable because a) the coordinate rotation technique cannot accommodate a single misaligned roller or a single-roller steering guide and b) it is consistent with more recent work [7] [9].

MODIFIED SIEVERS MODEL

The lateral dynamic behavior of a web is primarily driven by its geometry as it enters onto rollers. At the downstream end of a span, the normal entry rule defined in {6} controls lateral velocity. At the exit of the upstream roller, conditions are defined by what happened in the previous span. Either the slope (for the Euler Bernoulli model) or the cross section rotation (for the Timoshenko model) is transported from the entry of the upstream roller. So, attention is focused on behavior at $x = L$. The controlling equations are,

$$\frac{\partial y_L}{\partial t} = v_o\left(\gamma_L - \frac{\partial y_L}{\partial x}\right) + \frac{dz_L}{dt} \quad \text{or} \quad \frac{\partial y_L}{\partial x} = \frac{1}{v_o}\left(\frac{dz_L}{dt} - \frac{\partial y_L}{\partial t}\right) + \gamma_L \qquad \{37\}$$

$$\frac{dy_L}{dx} = \left(y_0 - y_L\right)\frac{h_1}{L} + \phi_L h_2 + \phi_0 h_3 \qquad \{38\}$$

$$\frac{d^2 y_L}{dx^2} = \left(y_0 - y_L\right)\frac{g_1}{L^2} + \phi_L \frac{g_2}{L} + \phi_0 \frac{g_3}{L} \qquad \{39\}$$

Equations {37}are two versions of the normal entry equation{5}. The one on the right expresses the web slope in terms of time.

Equation {38}is the web slope based on the first derivative of {33}with respect to *x*. Equation {39}is the web curvature based on the same equation with respect to *x*. Note that these are not partial derivatives. There is only one independent variable. This is what makes the model quasi-static. The variable *z* is the lateral displacement of the roller relative to ground.

$$g_1 = L^2\left(g_4''(L)\right) = \frac{K^2L^2a\left(\cosh(KL)-1\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$g_2 = L\left(g_5''(L)\right) = \frac{KL\left(KLa\cosh(KL)-\sinh(KL)\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]} \qquad \{40\}$$

$$g_3 = L\left(g_6''(L)\right) = \frac{KL\left(\sinh(KL)-KLa\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$h_1 = L\left(g_4'(L)\right) = \frac{KLa\sinh(KL)(1-a)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$h_2 = g_5'(L) = \frac{(a+1)\left(1-\cosh(KL)\right)+KLa\sinh(KL)}{a\left[KLA\sinh(KL)-2\left(\cosh(KL)-1\right)\right]} \qquad \{41\}$$

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The *h* and *g* coefficients (shape factors) are based on the values of the *x* derivatives of $g_4(x)$, $g_5(x)$ and $g_6(x)$ at $x = L$. The factors L^2 and *L* are used to make the shape functions dimensionless. The subscripts *L* and *0* represent the values of *y*, z and ϕ or their derivatives at $x = L$ and $x = 0$ respectively. The model defaults to the Euler Bernoulli beam if a = 1.

These equations can be used to form a time dependent O.D.E. for y_L as a function of y_o, ϕ_o, z_L, γ_L and γ_o.

Equation {38}is solved for ϕ_L .

$$\phi_L = \frac{1}{h_2}\left(\frac{\partial y_L}{\partial x} - h_3\phi_0 - \frac{h_1}{L}\left(y_0 - y_L\right)\right) \qquad \{42\}$$

The values for ϕ_o and y_o in {42} will come from the previous span.The value of ϕ_L from {42} is substituted in{39}.

$$\frac{d^2y_L}{dx^2} = \left(y_0 - y_L\right)\frac{1}{L^2}\left(g_1 - \frac{g_2h_1}{h_2}\right) + \frac{g_2}{h_2}\frac{1}{L}\left[\frac{dy_L}{dx}\right] + \frac{\phi_0}{L}\left(g_3 - \frac{g_2h_3}{h_2}\right) \qquad \{43\}$$

The next, and final step, is to convert {43}into a dynamic equation by replacing the two spatial derivatives with their time-based equivalents. The first derivative of y_L with respect to *x* is replaced with{37}. We now need a relationship for the second derivative of y_L.This is where the dilemma of the material derivative, discussed at the beginning of the paper, is resolved. It’s clear at this point that what is needed has nothing to do with the time derivative of the roller angle. The desired relationship can be found by simply considering the finite differences of velocity and slope of two closely space points and then allowing them to become infinitesimally close at $x = L$. Shelton did this on page 103 of his thesis, but without an appropriate justification. So,

$$\frac{d^2 y_L}{dt^2} = v_o{}^2 \frac{d^2 y_L}{dx^2} \qquad \{44\}$$

Equation {44}agrees with the steady state 4th boundary condition discovered by Shelton. When the time derivative at $x = L$ is zero (steady state), the curvature is zero.

There is no need to worry about partial derivatives in {44} because x is fixed. Adding the z acceleration to reference the lateral displacement to ground,

$$\frac{d^2 y_L}{dt^2} = v_o{}^2 \frac{d^2 y_L}{dx^2} + \frac{d^2 z_L}{dt^2} \qquad \{45\}$$

Finally, {37}and {45} are used to replace x with t as the independent variable. The result is a time dependent O. D. E.

$$\frac{d^2 y_L}{dt^2} = \left(y_0 - y_L\right)\frac{v_o{}^2}{L^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right) + \frac{g_2}{h_2}\left[\frac{v_o}{L}\left(\frac{dz_L}{dt} - \frac{dy_L}{dt}\right) + \frac{v_o{}^2}{L}\gamma_L\right] + \frac{v_o{}^2 \phi_0}{L}\left(g_3 - \frac{g_2 h_3}{h_2}\right) + \frac{d^2 z_L}{dt^2} \qquad \{46\}$$

When the values of y_o and ϕ_o depend on y_L and ϕ_L from the preceding span, equation {46} cannot be solved for a single span. And since those values will usually depend, in turn, on values farther upstream, it can, in that case, only be solved as part of a set of simultaneous equations that include the starting point where both y_L and ϕ_L are known. Furthermore, when there is an upstream disturbance influencing y_o and ϕ_o, there is no practical way to express {46} as a transfer function.

When y_o and ϕ_o are fixed and the web is being influenced only by inputs z_L, γ_o or γ_L , single span solutions are possible and transfer functions can be written. Guide rollers are in this category. In a steering guide, roller angle γ_L is controlled by the same mechanism that controls lateral movement. So, $\gamma_L = z_L/x_1$, where x_1 is the radius of the pivoting motion. In a displacement guide, γ_o is also controlled by z_L. So, $\gamma_o = z_L/x_1$, In a displacement guide, x_1 is the distance between the rollers, L.

There is also an effect of roller axis angle on ϕ_o that must be considered.

Effect of roller axis angle on ϕ_0

The variable γ_L is the projection of roller alignment onto the plane of the web. In Figure 4 it is assumed that the angular motion of the roller is in a plane that is parallel to the cross machine direction. In that case,

$$\gamma_{L1} = \gamma \cos(\alpha) \quad \text{and} \quad \gamma_0 = \gamma \cos(\beta) \qquad \{47\}$$

It is evident from {47}that the angle of misalignment, as seen at the entry to the roller will, in general, be different than the angle as seen at the exit. In the case of roller R1 in the test machine, γ_L is zero and γ_o is the pivot angle of the guide assembly. So, ϕ must change as the web passes over it.

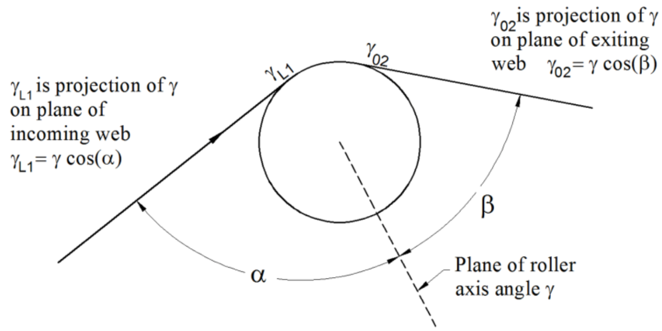


Figure 4 – Relationship of plane of roller axis motion and web planes

The value of ϕ_0

The value of ϕ_o in {46} is based on the value of ϕ_L from the previous span, but it has to be adjusted for the roller axis angle. Referring to Figure 4

$$\phi_{02} = \phi_{L1} - \gamma_{L1} + \gamma_{02} = \phi_{L1} - \gamma\cos(\alpha) + \gamma\cos(\beta) \qquad \{48\}$$

The value of ϕ_{L1} is found from equation{42}.

The same treatment must be given to the slope in the Euler Bernoulli model. It should be noted that this change is a reorientation of the web that does not involve any changes in deformation while on the roller.

The relationship in Figure 4 is something that Sievers got wrong. She assumed,

$$\phi_{02} = \phi_{L1} + \frac{\alpha}{\pi/2}\left(\gamma_{02} - \gamma_{L1}\right) \qquad \{49\}$$

where α is the wrap angle measured from the entry point. This didn't affect her thesis results for two reasons. First, it is correct when the wrap angle is 90 degrees and that was the case for the only two rollers that pivoted on her test machine and, second, her thesis model treated the pivoting rollers as though they were fixed and parallel to the other rollers, making γ equal to zero in all cases.

MODEL USED IN THE THESIS

Equation {46}already includes a coordinate transformation which adds the lateral translation, z_L of the roller relative to ground. The pivoting transformation is effected by,

$$y_L = \hat{y}_L + z_L \qquad \{50\}$$

where $\hat{y}$ is lateral displacement in the coordinate system that pivots with the web. The last two are already present in {46}. So, the only change from {46}is to add z_L to the first term and remove y_L. The resulting equation labeled A.30, appears on page 105 in the thesis. It is,

$$\frac{\partial^2 y_L}{\partial t^2} = \left(y_0 - y_L + z_L\right)\frac{v_o{}^2}{L^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right) + \frac{g_2}{h_2}\left[\frac{v_o}{L}\left(\frac{\partial z_L}{\partial t} - \frac{\partial y_L}{\partial t}\right)\right] + \frac{v_o{}^2\phi_0}{L}\left(g_3 - \frac{g_2 h_3}{h_2}\right) + \frac{\partial^2 z_L}{\partial t^2} \qquad \{51\}$$

This equation applies only to the span between rollers R1 and R2. For the other spans equation {46}is used. When it is used, there is no need to make adjustments for roller angle described in equation {48}.

TRANSFER FUNCTIONS FOR STEERING AND DISPLACEMENT GUIDES

Displacement guide

For a displacement guide,

$$\phi_o = \gamma_0 = \gamma_L \qquad \gamma_L = \frac{z_L}{L} \qquad y_0 = 0 \qquad \{52\}$$

Taking the Laplace transform of {46} and assuming all initial conditions equal to zero,

$$y_L(s) = z_L \frac{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 + \frac{g_2}{h_2} - \frac{g_2 h_3}{h_2}\right)}{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right)} \qquad \{53\}$$

It can be shown that $g_2 + g_3 = g_1$ and $h_3 + h_2 - 1 = h_1$ and that this leads to,

$$g_1 + \frac{g_2}{h_2} - \frac{g_2 h_3}{h_2} = g_1 - \frac{g_2 h_1}{h_2} \qquad \{54\}$$

So, the transfer function for a displacement guide is,

$$y_L = z_L \qquad \{55\}$$

Steering guide

For a steering guide,

$$\phi_o = \gamma_0 = 0 \qquad \gamma_L = \frac{z_L}{x_1} \qquad y_0 = 0 \qquad \{56\}$$

where x_1 is the radius of the pivoting motion.

The Laplace transform of {51}, is then,

$$y_L(s) = z_L \frac{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\frac{g_2}{h_2}\frac{1}{x_1}}{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right)} \qquad \{57\}$$

It can be shown that,

$$y_L = z_L \quad \text{when} \quad x_1 = LK_c \qquad \{58\}$$

where,

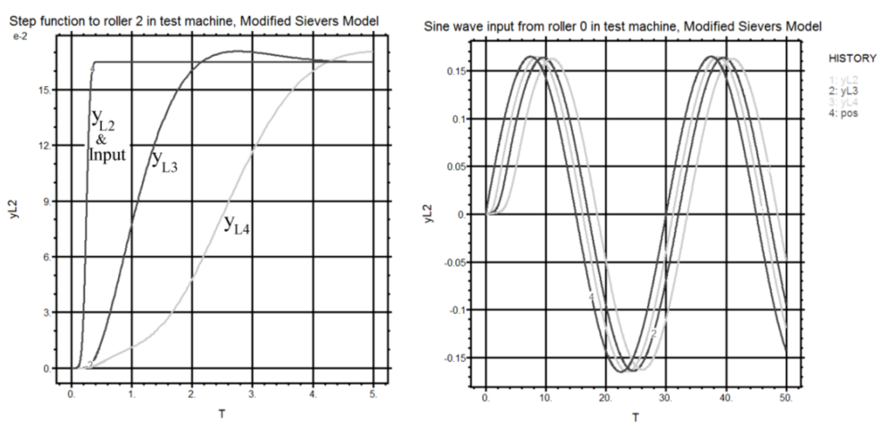
$$K_c = \frac{g_2}{g_1 h_2 - g_2 h_1} = \frac{1}{KL}\frac{\sinh(KL) - KLa\cosh(KL)}{1 - a\cosh(KL)} \qquad \{59\}$$

Equation {59}is identical to the relationship derived by Shelton for a static web[2].

NUMERICAL VALIDITY TEST

The following simulations were produced with four simultaneous ODEs running on FlexPDE. These simulations run very quickly. The model run time for Figure 6 was four minutes.

As mentioned earlier, a displacement guide is a good test of a lateral dynamic model because the transfer function should collapse to unity.



(a) (b)

Figure 5 – (a) Step function applied to 2nd roller of displacement guide
(b) sine wave disturbance at R0, 2 cycles/min
modified Sievers model.

[2] The equation published by Shelton on page 64 of his dissertation is incorrect. He informed me of the corrections in a private communication in the late '90s.

In Figure 5 (a), the curve labeled yL2 shows the lateral displacement of the web at the second roller, R2 of the displacement guide in Sievers' test machine. The input, z_L is labled "Input". It overlays y_{L2} so closely that the two curves look like one. The curves labeled y_{L3} & y_{L4} are displacements at rollers R3 and R4. Figure 5 (b) shows the displacement at R2, R3 and R4 with a very slow sine wave disturbance at R0 and web guide control loop open. This is what one would expect. The model used in the thesis produces identical results.

WEAVE REGENERATION

The primary motivation for the thesis was to design a multi-span control system to eliminate slow oscillatory errors in lateral position called weave regeneration. Lateral control systems can reduce weave to any desired level at a single point in the process, but the weave may reappear downstream. Figure 6 is a simulation that shows how it looked on the test machine.

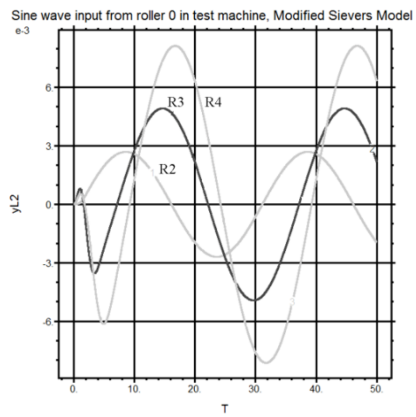


Figure 6 – Simulation of weave amplitudes at rollers R2, R3 and R4
For Timoshenko beam (n = 1.2) with closed loop control at R2

A weave disturbance with an amplitude of 0.165 inch was introduced at roller R0. A guiding system consisting of rollers R1 and R2 with a sensor at R2 reduced the weave to the value labeled R2 in Figure 6 – one sixtieth of its original magnitude (the gain of the proportional controller was 60). It can be seen that the weave reappears at rollers R3 and R4.

Figure 6 was produced using the modified model described above. The parameters are those listed for Experiment 1. Web speed was 200 ft/min, tension 1 pli and frequency 2 cycles/minute. It shows the same qualitative behavior reported in the thesis – the weave gets progressively larger. At R3 it is larger than at R2 and at R4 it is larger than at R3. The peak value at R4 is 0.008, which is less than the value of 0.013 reported in the thesis. This could be due to a difference in some parameter that isn't documented, such as the value of n. As noted in the section on the test machine, there is ambiguity in the parameters.

The control system was,

$$z_L = -G y_L \quad \{60\}$$

where z_L & y_L are at R2. The thesis reported the control equation was $\ddot{z}_L = -G(y_L)$. This couldn't have been right. In the IEEE Transactions paper [2] it was reported as shown in {60}.

Results for the Euler Bernoulli model (n = 0) with the same parameters also show regeneration, however, it doesn't become progressively larger. The simulation graphic in Figure 7 looks exactly the same as shown in the thesis.

There are numerous experiments documented in the thesis showing close qualitative and quantitative agreement with the Timoshenko model.

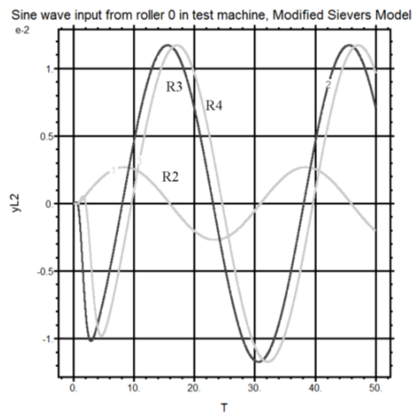


Figure 7 – Simulation of weave amplitudes at rollers R2, R3 and R4
For Euler Bernoulli beam (n = 0) with closed loop control at R2

CONCLUSIONS

The normal entry rule can be derived by application of the material derivative concept.

Lateral position errors can regenerate downstream of a guiding system because variations in slope or cross section angle at the point of control are not eliminated by simple position control systems.

Both beam models exhibit weave regeneration. Data from four different experiments, each with different operating parameters, show better qualitative agreement with the Timoshenko model simulations than with the Euler Bernoulli model.

Sievers' model can be extended and improved to,

- Eliminate special treatment for pivoting rollers (rotating coordinate system)
- Handle misaligned fixed rollers

The curvature factor for the modified Sievers model is the same as the value Shelton derived for his static, single span Timoshenko model.

The lateral acceleration equation should not be viewed in the same light as the normal entry rule.

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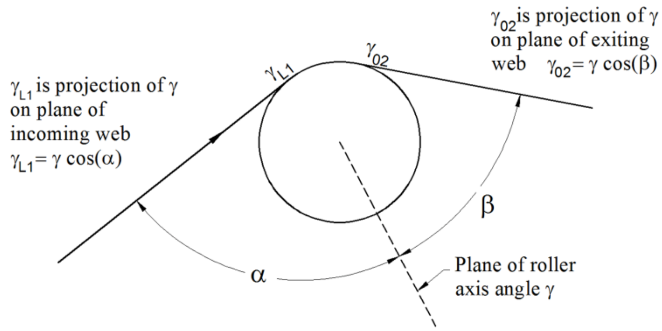

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$$y_L = \hat{y}_L + z_L \qquad \{50\}$$

where $\hat{y}$ is lateral displacement in the coordinate system that pivots with the web. The last two are already present in {46}. So, the only change from {46}is to add z_L to the first term and remove y_L. The resulting equation labeled A.30, appears on page 105 in the thesis. It is,

$$\frac{\partial^2 y_L}{\partial t^2} = \left(y_0 - y_L + z_L\right)\frac{{v_o}^2}{L^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right) + \frac{g_2}{h_2}\left[\frac{v_o}{L}\left(\frac{\partial z_L}{\partial t} - \frac{\partial y_L}{\partial t}\right)\right] + \frac{{v_o}^2 \phi_0}{L}\left(g_3 - \frac{g_2 h_3}{h_2}\right) + \frac{\partial^2 z_L}{\partial t^2} \qquad \{51\}$$

This equation applies only to the span between rollers R1 and R2. For the other spans equation {46}is used. When it is used, there is no need to make adjustments for roller angle described in equation {48}.

TRANSFER FUNCTIONS FOR STEERING AND DISPLACEMENT GUIDES

Displacement guide

For a displacement guide,

$$\phi_o = \gamma_0 = \gamma_L \qquad \gamma_L = \frac{z_L}{L} \qquad y_0 = 0 \qquad \{52\}$$

Taking the Laplace transform of {46} and assuming all initial conditions equal to zero,

$$y_L(s) = z_L \frac{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 + \frac{g_2}{h_2} - \frac{g_2 h_3}{h_2}\right)}{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right)} \qquad \{53\}$$

It can be shown that $g_2 + g_3 = g_1$ and $h_3 + h_2 - 1 = h_1$ and that this leads to,

$$g_1 + \frac{g_2}{h_2} - \frac{g_2 h_3}{h_2} = g_1 - \frac{g_2 h_1}{h_2} \qquad \{54\}$$

So, the transfer function for a displacement guide is,

$$y_L = z_L \qquad \{55\}$$

Steering guide

For a steering guide,

$$\phi_o = \gamma_0 = 0 \qquad \gamma_L = \frac{z_L}{x_1} \qquad y_0 = 0 \qquad \{56\}$$

where x_1 is the radius of the pivoting motion.

The Laplace transform of {51}, is then,

$$y_L(s) = z_L \frac{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\frac{g_2}{h_2}\frac{1}{x_1}}{s^2 + s\frac{1}{\tau}\frac{g_2}{h_2} + \frac{1}{\tau^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right)} \qquad \{57\}$$

It can be shown that,

$$y_L = z_L \quad \text{when} \quad x_1 = LK_c \qquad \{58\}$$

where,

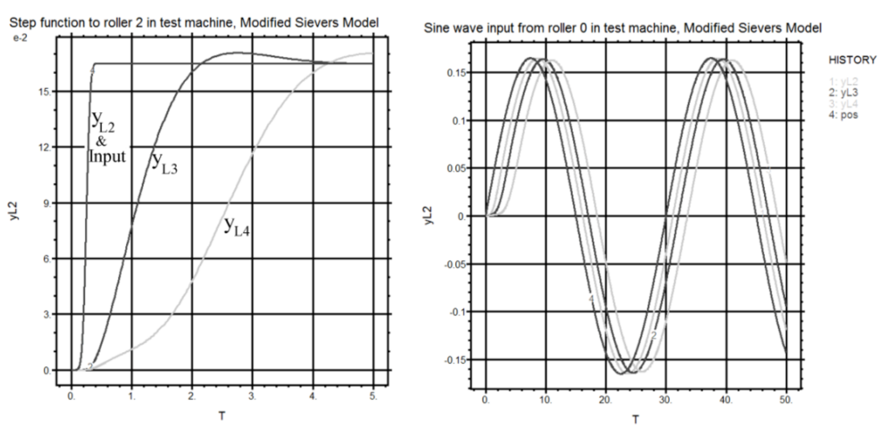
$$K_c = \frac{g_2}{g_1 h_2 - g_2 h_1} = \frac{1}{KL}\frac{\sinh(KL) - KLa\cosh(KL)}{1 - a\cosh(KL)} \qquad \{59\}$$

Equation {59}is identical to the relationship derived by Shelton for a static web[2].

NUMERICAL VALIDITY TEST

The following simulations were produced with four simultaneous ODEs running on FlexPDE. These simulations run very quickly. The model run time for Figure 6 was four minutes.

As mentioned earlier, a displacement guide is a good test of a lateral dynamic model because the transfer function should collapse to unity.



(a) (b)

Figure 5 – (a) Step function applied to 2nd roller of displacement guide
(b) sine wave disturbance at R0, 2 cycles/min
modified Sievers model.

[2] The equation published by Shelton on page 64 of his dissertation is incorrect. He informed me of the corrections in a private communication in the late '90s.

In Figure 5 (a), the curve labeled yL2 shows the lateral displacement of the web at the second roller, R2 of the displacement guide in Sievers' test machine. The input, z_L is labled "Input". It overlays y_{L2} so closely that the two curves look like one. The curves labeled y_{L3} & y_{L4} are displacements at rollers R3 and R4. Figure 5 (b) shows the displacement at R2, R3 and R4 with a very slow sine wave disturbance at R0 and web guide control loop open. This is what one would expect. The model used in the thesis produces identical results.

WEAVE REGENERATION

The primary motivation for the thesis was to design a multi-span control system to eliminate slow oscillatory errors in lateral position called weave regeneration. Lateral control systems can reduce weave to any desired level at a single point in the process, but the weave may reappear downstream. Figure 6 is a simulation that shows how it looked on the test machine.

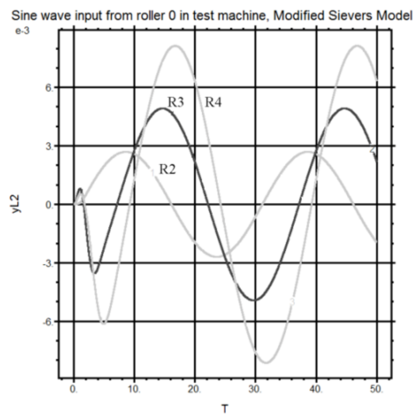


Figure 6 – Simulation of weave amplitudes at rollers R2, R3 and R4
For Timoshenko beam (n = 1.2) with closed loop control at R2

A weave disturbance with an amplitude of 0.165 inch was introduced at roller R0. A guiding system consisting of rollers R1 and R2 with a sensor at R2 reduced the weave to the value labeled R2 in Figure 6 – one sixtieth of its original magnitude (the gain of the proportional controller was 60). It can be seen that the weave reappears at rollers R3 and R4.

Figure 6 was produced using the modified model described above. The parameters are those listed for Experiment 1. Web speed was 200 ft/min, tension 1 pli and frequency 2 cycles/minute. It shows the same qualitative behavior reported in the thesis – the weave gets progressively larger. At R3 it is larger than at R2 and at R4 it is larger than at R3. The peak value at R4 is 0.008, which is less than the value of 0.013 reported in the thesis. This could be due to a difference in some parameter that isn't documented, such as the value of n. As noted in the section on the test machine, there is ambiguity in the parameters.

The control system was,

$$z_L = -Gy_L \qquad \{60\}$$

where z_L & y_L are at R2. The thesis reported the control equation was $\ddot{z}_L$ = -G(y_L). This couldn't have been right. In the IEEE Transactions paper [**Error! Bookmark not defined.**] it was reported as shown in {60}.

Results for the Euler Bernoulli model (n = 0) with the same parameters also show regeneration, however, it doesn't become progressively larger. The simulation graphic in Figure 7 looks exactly the same as shown in the thesis.

There are numerous experiments documented in the thesis showing close qualitative and quantitative agreement with the Timoshenko model.

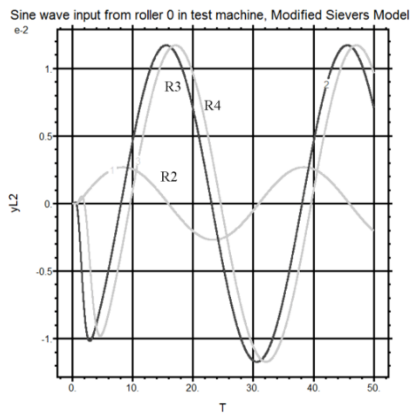


Figure 7 – Simulation of weave amplitudes at rollers R2, R3 and R4
For Euler Bernoulli beam (n = 0) with closed loop control at R2

CONCLUSIONS

The normal entry rule can be derived by application of the material derivative concept.

Lateral position errors can regenerate downstream of a guiding system because variations in slope or cross section angle at the point of control are not eliminated by simple position control systems.

Both beam models exhibit weave regeneration. Data from four different experiments, each with different operating parameters, show better qualitative agreement with the Timoshenko model simulations than with the Euler Bernoulli model.

Sievers' model can be extended and improved to,

- Eliminate special treatment for pivoting rollers (rotating coordinate system)
- Handle misaligned fixed rollers

The curvature factor for the modified Sievers model is the same as the value Shelton derived for his static, single span Timoshenko model.

The lateral acceleration equation should not be viewed in the same light as the normal entry rule.

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