METHOD FOR CALCULATING CATENARY WEB SPANS

By

James N. Dobbs and Daniel H. Carlson
3M Company
USA

ABSTRACT

The methods for calculating the catenary droop of a wire or cable hanging from fixed supports are well understood. For many common web materials, catenary droop or sag is not a consideration but for webs having a significant lineal mass density and low web tension, catenary droop may be significant. Applying the well known catenary equations to a web handling system with roller supports poses some interesting computational challenges and also some unexpected results. A computational method will be presented that can reliably solve the catenary equations for an arbitrary web span geometry. Catenary sag (from a straight line web path), web tension along the web span and the length of web hanging in the span can all be obtained from this solution method.

NOMENCLATURE

A, (x_A,y_A)	Starting point of catenary span (i.e. at point A or Roller A)
B, (x_B,y_B)	Ending point of catenary span (i.e. at point B or Roller B)
C	Lowest point of catenary span
c, c_A, c_B	Catenary parameter value
h	Catenary sag at C
O, (x_O,y_O), (x_{O_A},y_{O_A}), (x_{O_B},y_{O_B})	Catenary origin
s	Length of catenary span section
T, T_A, T_B, T_C	Web tension (at top of catenary, Roller A, Roller B or point C)
T_{X_A}, T_{Y_A}, T_{X_B}, T_{Y_B}	Vector components of tension T_A and T_B
w, W	Weight per unit length of web, weight of catenary span section
(x_{T_A},y_{T_A}), (x_{T_B},y_{T_B})	Web tangent points corresponding to θ_{C_A} and θ_{C_B}
Δ	Catenary fit parameter minimized by iterating θ_{C_A} and θ_{C_B}
θ_{C_A}, θ_{C_B}	Catenary angles on Rollers A and B – varied to minimize Δ
θ_{L_A}, θ_{L_B}	Minimum angle bounds on θ_{C_A} and θ_{C_B}
θ_{U_A}, θ_{U_B}	Maximum angle bounds on θ_{C_A} and θ_{C_B}

BACKGROUND

Catenary spans are those characterized by a load distributed uniformly along a *length* of cable or web having minimal[1] stiffness or resistance to flexure. For many common web materials, catenary droop is not a consideration but for webs having a significant lineal mass density and low web tension, catenary droop or sag may be significant.

The solution to the shape of a catenary span hanging from two fixed points is well understood as will be described using Figure 1.

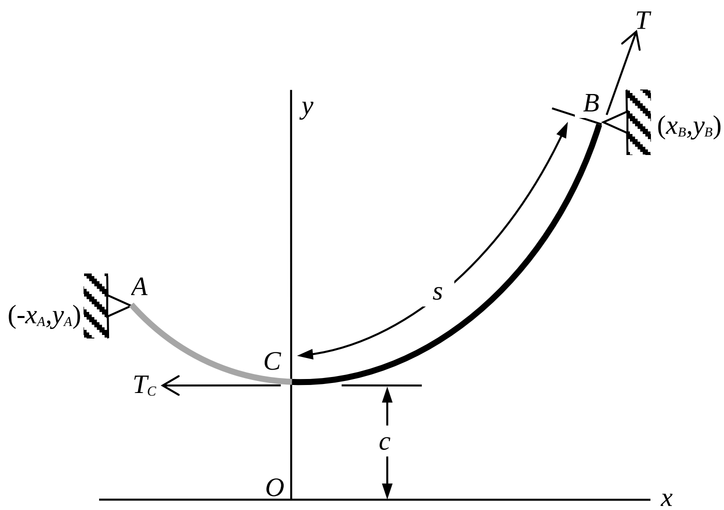

Figure 1 – Catenary span hanging between fixed points *A* and *B*.

A web having a load per unit length, *w*, is shown hanging between two fixed points, *A* and *B*. Focusing only on the right hand portion of the catenary span in Figure 1, we can measure the length of web, *s*, between points *C* and *B*. The total distributed load, W, on this portion of the catenary span is given by,

$$W = w \cdot s \tag{1}$$

while tensile loads on this span segment at *C* and *B* are T_C and *T* respectively. It is convenient to choose a coordinate origin *O* located at a distance *c* below the bottom of the catenary span *C*. This distance, *c*, often referred to as the parameter of the catenary, simplifies the following equations[2] which describe the form of the catenary solution.

For the height of the web hanging in the catenary we have

$$y = c \cdot \cosh(x/c) \tag{2}$$

[1] Stiffness is minimal at least in relation to the length of cable or web in the span.

[2] The solution to the catenary problem is presented without derivation and is a staple feature of many introductory courses in engineering statics. The presentation and nomenclature presented in this paper follows that referenced in [1].

where cosh() is the hyperbolic cosine function. Likewise for the length of web hanging in the catenary span we have

$$s = c \cdot \sinh(x/c) \tag{3}$$

where sinh() is the hyperbolic sine function. Using the trigonometric relationships for hyperbolic functions, the following equation can be derived.

$$y^2 - s^2 = c^2 \tag{4}$$

Web tension in a catenary span is proportional to the height of the web above the coordinate origin, leading to the following expressions for tension at *C* or at any other arbitrary height *y*.

$$T_C = w \cdot c \quad and \quad T = w \cdot y \tag{5}$$

Catenary droop or sag is defined as the distance between a straight line or non-catenary span between the supports, *A* and *B*, and the actual height of the web given in {2}. For the catenary span shown in Figure 1, we can derive an expression for the sag, *h*, at the center of the catenary span, *C*.

$$h = \frac{y_A \cdot x_B - y_B \cdot x_A}{x_B - x_A} - c \tag{6}$$

CATENARY SPANS WITH ROLLERS

Hanging a web span on rollers presents additional challenges because we no longer have fixed locations in space that can be used to define our problem. For example in Figure 2, we can see that even with a simple symmetrical web span connecting Roller A with Roller B there may be multiple potential solutions to the catenary equation, each having unique points of origin, (x_T, y_T), on each roller's surface. These points of origin represent the tangent points where the web contacts the roller surface.

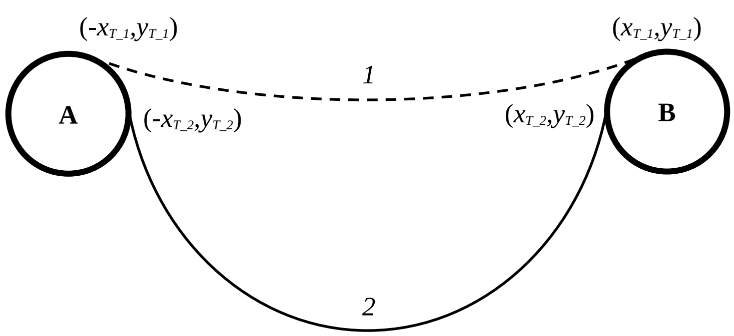

Figure 2 – Symmetrical web span between rollers A and B showing potential solutions 1 (dash line) and 2 (solid line) for the catenary equation.

In the figure above, I have presented solutions 1 and 2 as having a common coordinate origin. From the background discussion on catenaries, each catenary solution will have a unique coordinate origin, (x_O, y_O), and parameter value, *c*. Reconciling machine coordinates with solution coordinates is one of the challenging aspects of working with catenaries. As it turns out, the key to solving the general problem of a catenary hang between two rollers is to project a proposed solution independently from each roller and iterate until both of these solutions share a common origin (x_O, y_O) and catenary parameter, c. A multi-parameter optimizing routine such as Microsoft's Solver™ [2] can be used to find a common solution for both rollers.

Efficient multi-parameter optimization requires the identification of parameters that map directly into the solution space and have carefully defined upper and lower bounds that avoid false optimization results. As web line threading is inherently geometric, web wrap angle on the rollers turns out to be an ideal optimization parameter for catenary solutions as shown in Figure 3.

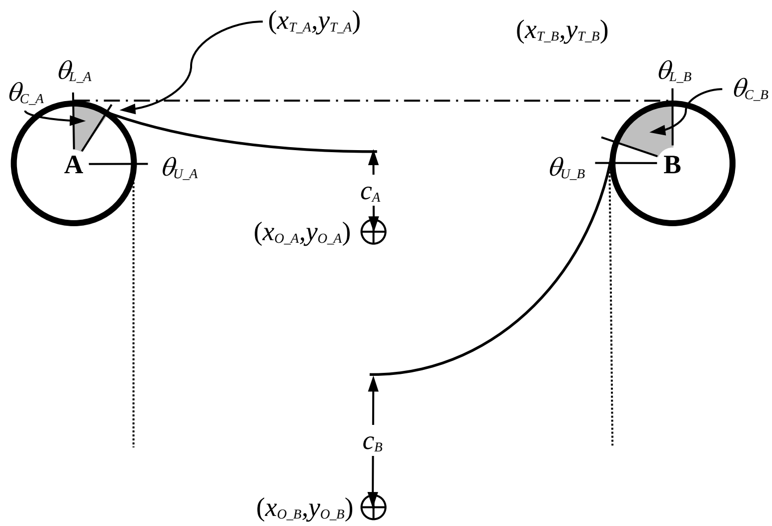

Figure 3 – Independent catenary solutions (solid lines) are projected from Rollers A and B based on web wrap angles, θ_{C_A} and θ_{C_B} respectively. Viable wrap angles are bounded by straight web path (dot-dash line) and vertical tangents (dot lines) to the rollers.

As can be seen in Figure 3, the natural boundaries for the minimum angle at which a catenary span contact the rollers are those given by the straight line web path (no catenary sag) connecting the rollers. Furthermore, the catenary will not hang past the vertical tangent to the rollers. These two angles, for each roller respectively, provide the natural search space for finding a single catenary solution that satisfies the boundary condition imposed at each roller. Catenary projection angles, θ_{C_A} and θ_{C_B}, correspond to a tangent points, (x_{T_A},y_{T_A}) and (x_{T_B},y_{T_B}) where the web makes contact with Rollers A and B respectively. These tangent points are analogous to fixed points A and B in Figure 1. A free body diagram of one of the catenary projections from Figure 3 is shown in Figure 4.

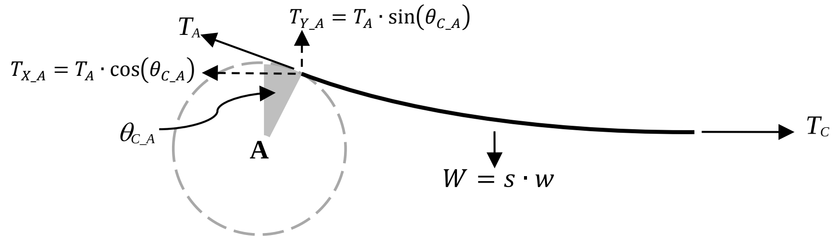

Figure 4 – Free body diagram of catenary solution projected from Roller A in Figure 3. Forces acting on the span are: web tension at Roller A, $T_{\_A}$; web span weight, W; and the web tension at the bottom of the catenary span, T_C. Web tension vector components, T_{X_A} and T_{Y_A}, are shown as dashed arrows.

Following Figure 4, we see that the tension at the bottom of the loop, T_C, must equal $T_A \cdot \cos(\theta_{C\_A})$. Using {5} and a load (weight) per unit length of web, w, we can derive the catenary parameter c_A as a function of the catenary projection angle θ_{C_A} and web tension, T_A.

$$c_A = \frac{T_C}{w} = \frac{T_A \cdot \cos(\theta_{C\_A})}{w} \tag{7}$$

Using {2} and {5}, we can solve for the catenary origin (x_{O_A}, y_{O_A}) as referenced from tangent point (x_{T_A}, y_{T_A}) shown in Figure 3,

$$x_{O\_A} = x_{T\_A} + \cosh^{-1}\left(\frac{1}{\cos\theta_{C\_A}}\right) \cdot c_A \tag{8}$$

$$y_{O\_A} = y_{T\_A} - {}^{T_A}\!/_{w} \tag{9}$$

where c_A is obtained from {7}. In deriving {8} we make use of the relationship $\frac{y_{T\_A}}{c} = \frac{T_A}{T_C} = \frac{1}{\cos\theta_{C\_A}}$ from {5}. Because the tangent point is a function θ_{C_A}, the catenary parameter c_A and origin (x_{O_A}, y_{O_A}) are now defined by the geometry of Roller A, web tension T_A, web load w and our iteration variable θ_{C_A}.

Solutions for the catenary solution projected from Roller B in Figure 3 may be obtained in an analogous manner to {7}, {8} and {9} with one additional parameter, namely the tension, T_B, of the catenary at Roller B. Employing {5} once again, we can write

$$T_B = T_A + (y_{T\_B} - y_{T\_A}) \cdot w \tag{10}$$

and

$$c_B = \frac{T_C}{w} = \frac{T_B \cdot \cos(\theta_{C\_B})}{w} \tag{11}$$

$$x_{O\_B} = x_{T\_B} + \cosh^{-1}\left(\frac{1}{\cos\theta_{C\_B}}\right) \cdot c_B \tag{12}$$

$$y_{O\_B} = y_{T\_B} - {T_B}/{w} \qquad \{13\}$$

resulting in a catenary parameter c_B and origin (x_{O_B}, y_{O_B}) defined in terms of the geometry of Roller B, web tension T_B, web load w and our second iteration variable θ_{C_B}.

An iteration optimizing parameter may now be defined as the least squares difference between our two catenary solutions.

$$\Delta = \sqrt{(c_A - c_B)^2 + \left(x_{O\_A} - x_{O\_B}\right)^2} \qquad \{14\}$$

As y_{O_A} and y_{O_B} do not depend on θ_{C_A}. or θ_{C_B}, it is not necessary to include them in {14}. Finding a common solution to the catenary equation for Roller A and Roller B consists of varying iteration parameters θ_{C_A}. or θ_{C_B}, using {7} through {13} to compute a new optimizing parameter Δ using {14} and continuing this process until Δ is minimized to a value near zero.

EXCEL IMPLEMENTATION

The procedure described in the previous section was coded into a Microsoft Excel ™ workbook. As it is difficult to visualize purely numerical results from the computation, routines were written to scaled drawings of two rollers and catenary solution connecting them. Required input to the workbook consisted of: roller geometry (origin, diameter), web tension T_A at Roller A and the weight per unit length of web w. With the exception of web tension, a common set of parameters was used and are listed in Table 1. The iteration procedure was found to be stable and provided good convergence except in cases of very low web tension or very high web weight.

Catenary Parameters Used for Computation		
Roller Diameter	76 mm	3 in.
Roller Spacing	609.6 mm	24 in.
Web Width	228 mm	9 in.
Weight per Unit Length	0.0099 kg/cm	0.056 lb/in.

Table 1 – Catenary parameters used for Excel ™ implementation of catenary solution.

In some cases, as shown in Figure 5, the catenary iteration procedure was found to converge on two different solutions depending on the initial values chosen for θ_{C_A}. and θ_{C_B}. Note that these two solutions are for the exact same input conditions of roller geometry, web weight per unit length and web tension. Solution convergence was stable provided the same initial angle values were provided to the solver and by careful selection of those input angles, the solver could be made to converge reliably to either the upper or lower solution.

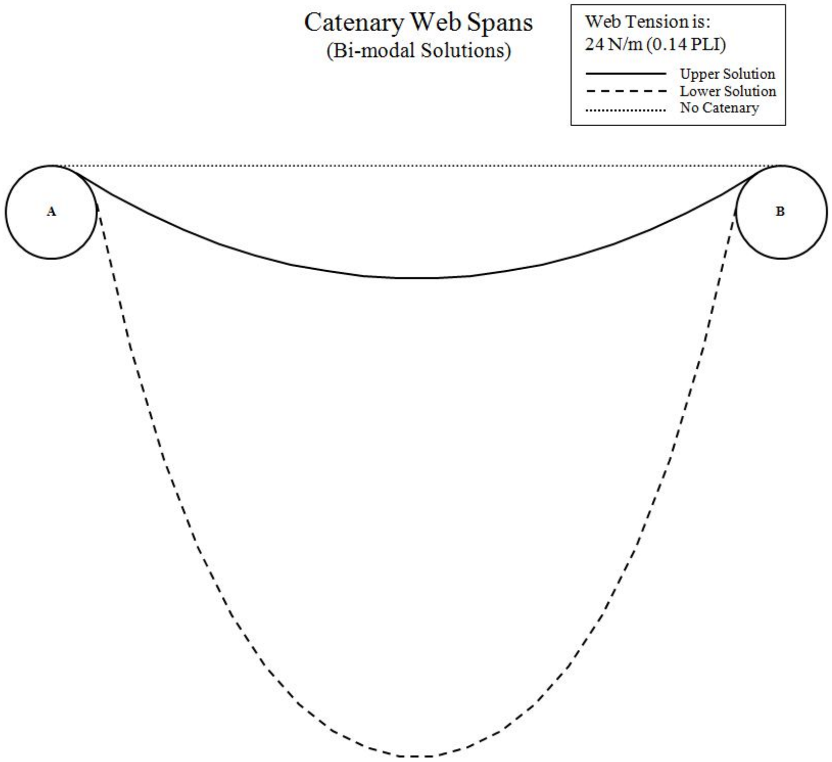

Figure 5 – Graph of catenary web spans for web tension (at Rollers A and B) of 24 N/m (0.14 PLI). Solution is bi-modal having an upper solution (solid line) and lower solution (dashed line). Straight line non-catenary web path is shown as a dotted line.

A practical method was required to seed the solver with input values that would lead to a desired solution, upper or lower. Using a Visual Basic for Applications™ (VBA) macro running in Excel™, a search space covering the entire range of possible catenary solution angles for was gridded[3]. For the problem illustrated in Figure 3, catenary solution angles θ_{C_A}. and θ_{C_B} can range from 0 to 1.57 radians and correspond to θ_{L_A}. and θ_{U_A} (Roller A) and θ_{L_B}. and θ_{U_B} (Roller B). The catenary fit parameter {14} was then computed as a function of solution angles θ_{C_A}. and θ_{C_B}, as shown in Figure 6 using surface (a) and contour graphs (b). Web tension input was 24 N/m (0.14 PLI).

Two minima can be observed in the catenary fit parameter Δ, obtained from {14}, corresponding to the upper and lower catenary solutions. It is a relatively easy task to search this solution space for local minima. Starting values for an upper solution will be a local minimum with the smallest values for θ_{C_A}. and θ_{C_B}, (U on Figure 6b) while a lower solution will be a minimum with the largest values for θ_{C_A}. and θ_{C_B} (L on Figure 6b).

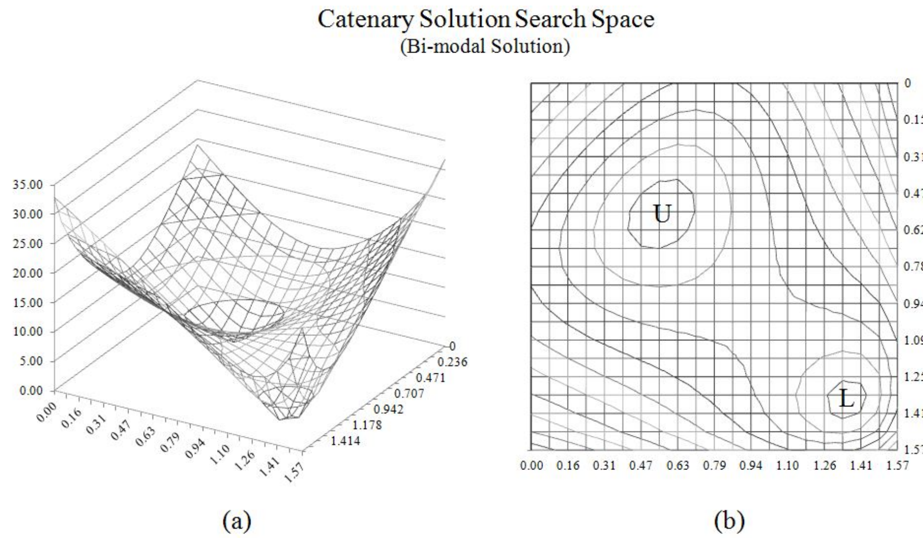

Figure 6 – Graph of solution search space showing surface (a) and contour (b) plots for catenary fit parameter Δ as a function of catenary solution angles θ_{C_A}. and θ_{C_B}. Upper and lower catenary solutions are shown on (b) as U and L respectively.

Surface (a) and contour (b) graphs of the catenary fit parameter Δ using a somewhat higher input web tension of 78 N/m (0.44 PLI) are shown in Figure 7. Even with a single minimum in the search space, it was found advantageous to seed the solver with θ_{C_A}. and θ_{C_B} values corresponding to this minimum.

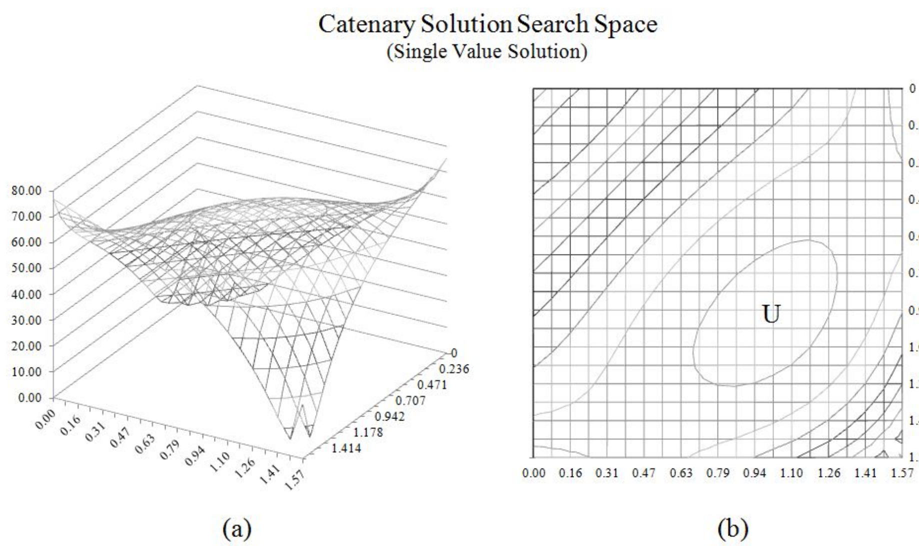

Figure 7– Graph of solution search space showing surface (a) and contour (b) plots for catenary fit parameter Δ as a function of catenary solution angles θ_{C_A}. and θ_{C_B}. A single catenary solution minima marked U is shown on (b).

To test the robustness of the catenary solution method, a wide range of web tension was applied to the catenary parameters of Table 1. Results are shown in Table 2.

Input Parameters		Upper Catenary Solution			Lower Catenary Solution		
Tension Input (N/m)	Wrap Angle (rad)	Catenary Sag (mm)	Tension Min. (N/m)	Span Length (mm)	Catenary Sag (mm)	Tension Min. (N/m)	Span Length (mm)
19	0.818	123	14	682	303	6.4	954
24	1.015	86	21	645	451	5.1	1220
39	1.239	50	38	621	824	3.7	1930
78	1.406	25	77	612	NA	NA	NA
156	1.488	12.5	155	610.3	NA	NA	NA
311	1.529	6.3	311	609.7	NA	NA	NA
623	1.550	3.1	622	609.7	NA	NA	NA

Table 2 – Table of catenary solutions for varying input web tension. Shown are the wrap angles found for each solution along with the catenary sag, minimum web tension (at bottom of catenary) and total length of web in the catenary. Where bi-modal solutions exist, both upper and lower results are shown.

Not surprisingly, the amount of catenary sag decreases and the length of web approaches the straight line web span length of 609.6 mm (24 in.) found in Table 1. Model convergence was not reliable below the lowest input tension of 19 N/m (0.11 PLI). At the highest tension modeled of 623 N/m (3.56 PLI) convergence was not an issue. It should be noted that even at that highest tension, web sag would still be a measurable 3.1 mm (0.124 in.). Selected catenary solutions are shown graphically in Figure 8 with high tension solutions omitted for graphical clarity.

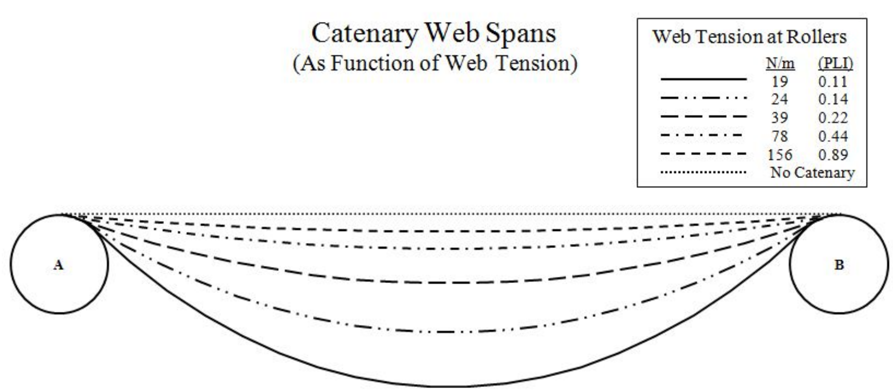

Figure 8 – Graph of selected catenary web span solutions as a function of web tension from Table 2. In cases where bi-model solutions exist, only the upper solution is shown.

A computational method for calculating catenary spans for webs on rollers has been presented and implemented in Microsoft Excel™. Catenary shape, droop or sag, length of web and web tension may all be obtained from this method. In some cases, bi-modal computational results were discovered and a strategy was presented to choose a particular solution. Numerical results for a catenary span have been presented, and the method has proven computationally robust for a wide range of input web tension.

REFERENCES

1. Beer, F. P., and Johnston, E. R. Jr., "Vector Mechanics for Engineers: Statics," McGraw-Hill Book Company, 2nd ed., 1972, pp. 282-284.
2. Solver is an Add-In application for Microsoft Excel. Although Solver is distributed with Excel, it is usually necessary to independently activate this Add-In after program installation. Solver and Excel are trademarks of Microsoft Corp., Redmond WA, www.microsoft.com