MODELING OF THE TRANSPORT BEHAVIOR OF LOW MODULUS WEBS

By

Pramod R. Raul and Prabhakar R. Pagilla
Oklahoma State University
USA

ABSTRACT

We consider three key issues in this study for low modulus webs. First, transport in the high strain region is considered when the materials exhibit nonlinear and viscoelastic behavior. Second, in-plane biaxial strain is considered in the development of the governing equation for web strain in the transport direction. Governing equations for web strain and tension are developed and evaluated us-

ing parameter sensitivity analysis and both time and frequency domain computer simulations under typical scenarios of transport and machine and environment induced disturbing forces. Third, the effect of moisture and heat diffusion in fibrous and porous non-woven webs on longitudinal web strain is considered. Moisture and heat diffusion is studied using known non-Fickian models which better represent the diffusion behavior in porous, fibrous materials.

NOMENCLATURE

A	:	Area of cross-section of web
B	:	Biot number of moisture diffusion
C	:	Moisture concentration in web
c	:	Heat capacity of web material
D	:	Moisture diffusion constant
E	:	Modulus of elasticity of web material
G	:	Transfer function
h	:	Thickness of web
L_i	:	Length of i^{th} web span
R_i	:	Radius of i^{th} roller
S	:	Sensitivity function
s	:	Laplace variable
T	:	Temperature of web material
T_∞	:	Temperature of surrounding
t	:	Time parameter
t_i	:	Actual web tension in i^{th} web span
v_i	:	Velocity of i^{th} roller
w	:	Width of web
x_i	:	Position of i^{th} roller
y	:	Web thickness direction
α	:	Sorption rate of fiber
β	:	Desorption rate of fiber
ε_i	:	Actual web strain in i^{th} web span
ν	:	Poisson ratio
ω_i	:	Angular velocity of i^{th} roller
ϕ	:	Porosity of web material
θ	:	Process parameter
ρ	:	Density of web material
σ_i	:	Web stress in i^{th} web span

Subscripts:

i	:	Span index, $i = 0, 1, 2, \ldots$
r	:	Reference value
u	:	Related to unstretch state

INTRODUCTION

Web materials may be classified based on their modulus. All metals, such as aluminum, copper, steel, and fiber glass are categorized as high modulus web materials with modulus of about 10^7 psi. Medium modulus webs, such as polyester, polyethylene, Tyvek, have modulus around 10^5 psi. Tissues, rubber, and some non-woven materials are categorized as low modulus materials and have significantly lower modulus in the range $10^3 \sim 10^4$ psi. The study of transport behavior of low modulus webs is considered in this paper.

The commonly used mathematical model for longitudinal web tension behavior of a free web span is developed by taking into account various assumptions, such as the web strain is small (much smaller than unity), prevalence of unidirectional stress in the transport direction, web is treated as continuum material, etc. The validity of some of these assumptions for low modulus webs is not well justified because they are typically transported in the high strain region of the stress-strain curve. An objective of this work is to obtain a governing equation for web strain without the small strain assumption, and conduct subsequent model analysis to compare it with the existing model.

There may be a significant change in dimensions when low modulus materials are transported under high strain. In particular, web width may be significantly reduced during transport. The commonly used longitudinal web strain equation neglects cross machine directional stress effects and considers only uniaxial stress. To develop an accurate governing equation for web strain for low modulus materials, Biaxial strain is considered in this paper.

A governing equation for longitudinal web strain is developed by relaxing small strain assumption and considering in-plane biaxial stress in order to evaluate the behavior of low modulus webs. Time and frequency domain simulations are conducted to evaluate the new equations. A parameter sensitivity analysis is performed on the linearized governing equations to observe the effect of system parameters on the process output (web tension).

Another related issue is the behavior of porous and nonhomogeneous low modulus non-woven webs in certain processes such as printing, coating, etc., which require humid and heated environment. Non-woven materials are typically manufactured by putting fibers together in the form of web and binding them with processes such as air-laid, meltblown, and spunbound. Therefore, non-woven materials are bonded fiber networks with considerable porous space. Longitudinal web strain is a complex function of applied stress, moisture and temperature. Simultaneous diffusion of moisture and temperature in low modulus, porous and nonhomogeneous webs is studied using a non-Fickian law. A second order non-Fickian and Fourier models are considered to obtain moisture concentration and temperature distribution in non-woven webs.

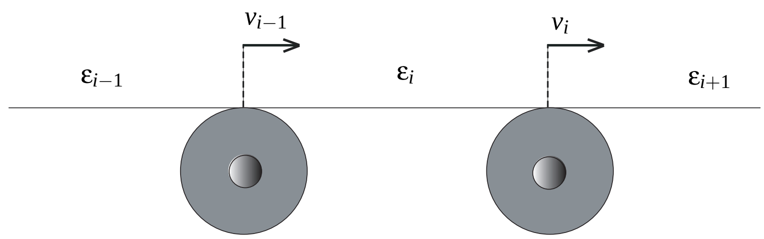

Figure 1: Web Span Between Two Rollers

WEB STRAIN GOVERNING EQUATIONS FOR LOW MODULUS WEBS

The commonly used governing equation for web strain in a span of fixed length has been derived from first principles under a number of simplifying assumptions. The primary assumption being the small strain assumption. In this section, we first present and discuss a governing equation for web strain without the small strain assumption. We also present governing equations for web tension by considering three constitutive relations between web strain and web tension: (*i*) linearly elastic, (*ii*) nonlinearly elastic, and (*iii*) viscoelastic. Second, we present a governing equation for longitudinal web strain that consider in-plane biaxial stress.

Governing Equation with Small Strain Assumption (Model 1)

Considering the assumption such as small strain assumption ($\varepsilon_i \ll 1, 1/(1+\varepsilon_i) \approx 1-\varepsilon_i$), adjacent rollers to be stationary ($\dot{x}_i = 0, \dot{x}_{i-1} = 0$), and uniform strain along the length of the span, that is $\varepsilon_x(x,t) = \varepsilon_{x_i}(t) = \varepsilon_i(t)$, we obtain the strain equation

$$\dot{\varepsilon}_i = \frac{v_i(1-\varepsilon_i) - v_{i-1}(1-\varepsilon_{i-1})}{L_i}. \qquad \{1\}$$

Assuming the web to be linearly elastic ($t_i = EA\varepsilon_i$), the governing equation for web tension is given by

$$\dot{t}_i = \frac{v_i(EA-t_i) - v_{i-1}(EA-t_{i-1})}{L_i}. \qquad \{2\}$$

Governing Equation without Small Strain Assumption (Model 2)

Relaxing the small strain assumption and considering the rollers to be stationary and uniform strain along the length of span, the strain equation can be expressed as

$$\dot{\varepsilon}_i = \frac{v_i(1+\varepsilon_i) - \frac{v_{i-1}}{1+\varepsilon_{i-1}}(1+\varepsilon_i)^2}{L_i}. \qquad \{3\}$$

Assuming the web to be linearly elastic ($t_i = EA\varepsilon_i$), the web tension equation is given by

$$\dot{t}_i = \frac{v_i(EA + t_i) - \frac{v_{i-1}}{EA+t_{i-1}}(EA + t_i)^2}{L_i}. \qquad \{4\}$$

Typical stress-strain behavior of low modulus material under the application of load is shown in Figure 2.

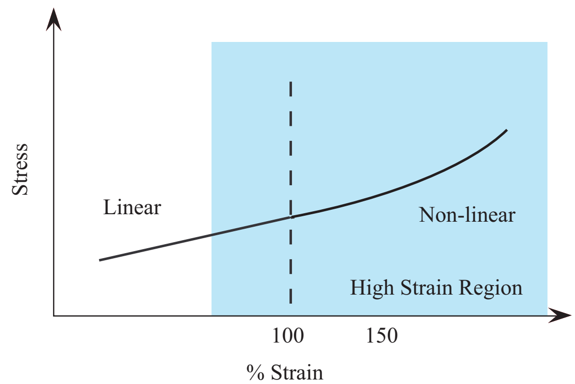

Figure 2: Low Modulus Material Behavior

Assuming, the web behavior to be nonlinear elastic (assume parabolic behavior; $t_i = EA\varepsilon_i^2$), web tension dynamics is given by

$$\dot{t}_i = \frac{2v_i\sqrt{t_i}(\sqrt{EA} + \sqrt{t_i}) - \frac{2v_{i-1}\sqrt{t_i}}{\sqrt{EA}+\sqrt{t_{i-1}}}(\sqrt{EA} + \sqrt{t_i})^2}{L_i}. \qquad \{5\}$$

Note that the equation {5} is used as Model 2 in further analysis.

Effect of Viscoelastic Behavior of Web Material (Model 3)

Low modulus materials also show viscoelastic behavior during transport on rollers. A constitutive relation between strain and tension may be developed by assuming the web to exhibit linear viscoelastic behavior. Assuming a Maxwell model and a linear spring in parallel as shown in Figure 3, the stress-strain relation for the viscoelastic effect is given by

$$p_0\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon} \qquad \{6\}$$

where $\sigma = \frac{t_i}{A}$, $p_0 = \frac{1}{b}$, $p_1 = \frac{1}{k}$, $q_0 = \frac{k_1}{b}$, and $q_1 = 1 + \frac{k_1}{k}$

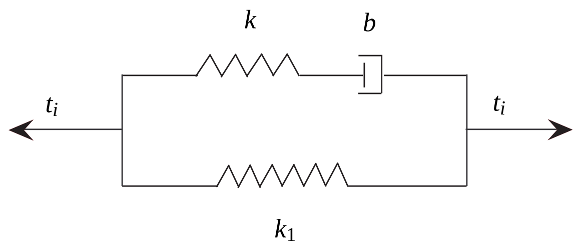

Figure 3: Viscoelastic Model

By substituting the viscoelastic relation {6} into equation {3}, one can write the governing equation for tension as

$$\dot{t}_i = -\frac{p_0}{p_1}t_i + \frac{Aq_0}{p_1}\varepsilon_i + \frac{Aq_1}{p_1 L_i}\left(v_i(1+\varepsilon_i) - \frac{v_{i-1}}{1+\varepsilon_{i-1}}(1+\varepsilon_i)^2\right). \qquad \{7\}$$

Effect of In-Plane Biaxial Stress (Model 4)

The cross machine direction dimensional stability is significantly affected for low modulus materials when subjected to stress in longitudinal direction. The change in web width is more significant compared to that of thickness. Therefore, the two dimensional in-plane stress is considered for developing a new governing equation for web strain.

We assume the following relation between strains in machine and cross-machine directions:

$$\varepsilon_z = -\nu\varepsilon_x \qquad \{8\}$$

where ε_x is web strain in the longitudinal direction (x), ε_z is web strain in the cross-machine direction (z), ν is Poisson's ratio.

The cross section area of a small infinitesimal web element in stretched condition, neglecting the change in thickness, is given by

$$A = d \times h = [(1+\varepsilon_z)d_u] \times h \qquad \{9\}$$

By substituting equation {9} in the mass balance equation {33} (given in the Appendix), we get

$$\frac{d}{\ } \qquad \frac{}{(1+\varepsilon_x)(1+\varepsilon_z)}dx = \frac{\rho_{u_{i-1}} h_{u_{i-1}} d_{i-1} v_{i-1}(t)}{(1+\varepsilon_{x_{i-1}})(1+\varepsilon_{z_{i-1}})} - \frac{\rho_{u_i} h_{u_i} d_i v_i(t)}{(1+\varepsilon_{x_i})(1+\varepsilon_{z_i})}. \qquad \{10\}$$

Assuming uniform strain along the length of the span, and the density of the web material and cross section area are constants for the web material, the longitudinal strain dynamics in the i^{th} span is given by

$$\dot{\varepsilon}_i = \frac{-v_{i-1}[(1+\varepsilon_i)(1-\nu\varepsilon_i)]^2}{L_i(1+\varepsilon_{i-1})(1-\nu\varepsilon_{i-1})(-\nu+1-2\nu\varepsilon_i)} + \frac{v_i(1+\varepsilon_i)(1-\nu\varepsilon_i)}{L_i(-\nu+1-2\nu\varepsilon_i)}. \qquad \{11\}$$

Assuming parabolic behavior for the stress-strain curve in the high strain region ($t_i = EA\varepsilon_i^2$), the governing equation for tension is given by

$$\dot{t}_i = \frac{-2v_{i-1}\sqrt{t_i}[(\sqrt{EA}+\sqrt{t_i})(\sqrt{EA}-\nu\sqrt{t_i})]^2}{L_i(\sqrt{EA}+\sqrt{t_{i-1}})(\sqrt{EA}-\nu\sqrt{t_{i-1}})(-\nu\sqrt{EA}+\sqrt{EA}-2\nu\sqrt{t_i})} + \frac{2v_i\sqrt{t_i}(\sqrt{EA}+\sqrt{t_i})(\sqrt{EA}-\nu\sqrt{t_i})}{L_i(-\nu\sqrt{EA}+\sqrt{EA}-2\nu\sqrt{t_i})}. \qquad \{12\}$$

COMPARISON OF DIFFERENT MODELS VIA NUMERICAL SIMULATIONS

Numerical simulations in time and frequency domains are conducted to evaluate and compare the new and existing governing equations for tension. The following parameter values are considered for model simulations: $t_r = 25$ lbf (high strain region); modulus constant, $E = 6930$ psi; web width, $w = 4.75$ in; web thickness, $h = 0.0045$ in; reference speed, $v_r = 850$ fpm; Poisson's ratio, $\nu = 0.25$ and 0.45; sinusoidal disturbance: amplitude 5 fpm, frequency 2 Hz.

Time Domain Simulation Results

For time domain simulations, a non-ideal situation is considered by injecting a sinusoidal disturbance in web transport velocity. Three different transport conditions are considered: (*i*) constant line speed, (*ii*) line acceleration, and (*iii*) line deceleration.

The tension response of the developed models (with and without small strain assumption) is shown in Figure 4. Simulation results show that the model without the small strain assumption (Model 2) has higher amplitude and slightly increased frequency of oscillations compared to the model with the small strain assumption (Model 1) to sinusoidal disturbance. Model 2 is more sensitive to disturbances in the high strain region. Figures 5 and 6 show comparison of tension response of both models to acceleration and deceleration of the web line. The amplitude of oscillations increases during deceleration and decreases during acceleration.

Model 2 is more sensitive to speed disturbances compared to Model 1. Behavior similar to Model 2 is exhibited when considering viscoelastic webs without the small strain assumption (Model 3) and, therefore, results are not presented.

Numerical model simulations are performed for the model considering in-plane biaxial stress (Model 4). The tension response of Model 1 and Model 4 are shown in Figure 7. Model simulation results indicate that cross-machine direction stresses significantly affect longitudinal behavior when low modulus webs are transported in the high strain region.

Frequency Domain Simulation Results

The frequency response study is performed for low modulus web material to verify the linearized model developed without the small strain assumption and

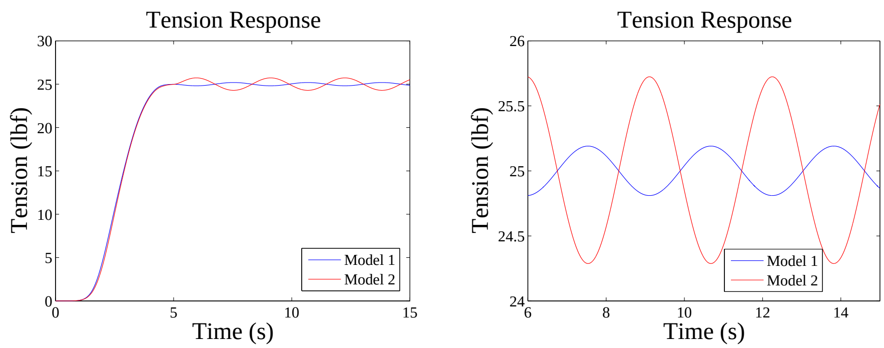

Figure 4: Tension Response; Constant Web Line Speed

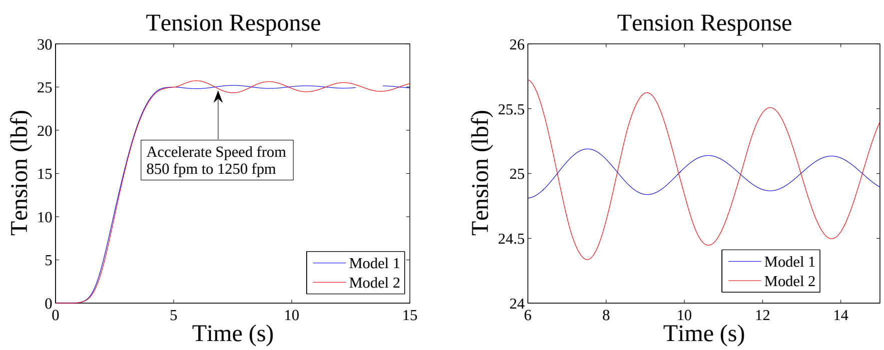

Figure 5: Tension Response; Accelerated Web Line Speed

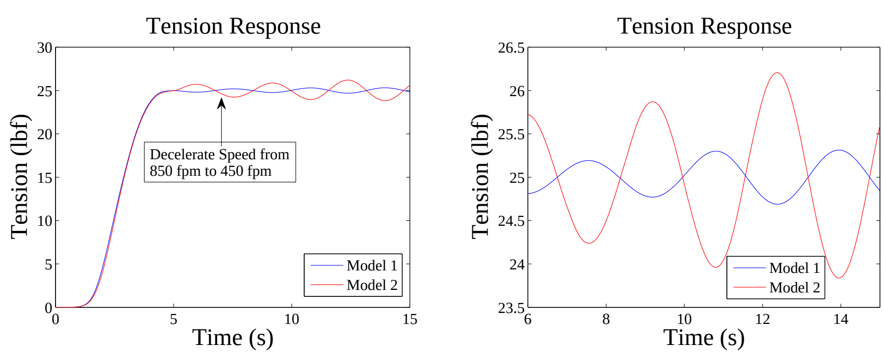

Figure 6: Tension Response; Decelerated Web Line Speed

considering cross machine direction strain. The new governing equations affect the resonant frequencies of a system of idle rollers and spans that is analyzed in this study.

For the purpose of simulation, the unwind section of the Euclid Web Line (EWL), shown in Figure 8, is considered.

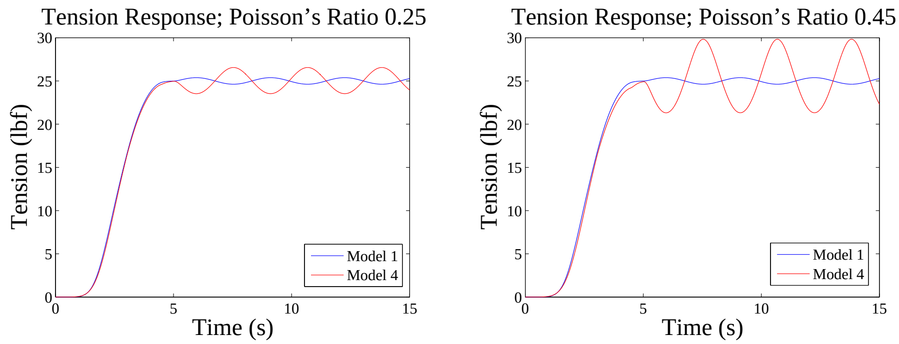

Figure 7: Cross Machine Direction Strain Effect; Tension Response at Constant Web Line Speed; Left: Poisson's Ratio = 0.25, Right: Poisson's Ratio = 0.45

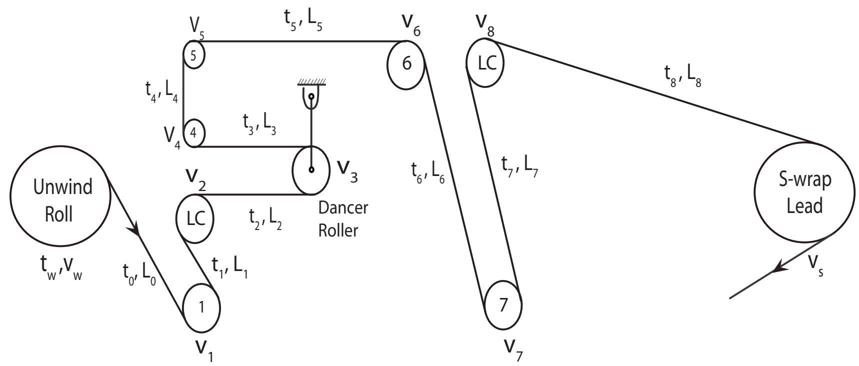

Figure 8: Euclid Web Line Unwind Section

Frequency response simulations are performed for the models 1, 2, and 4. Sinusoidal disturbance is injected into the velocity reference of the S-wrap roller and the web tension output is recorded. The bode magnitude plot of the transfer function from the velocity input to the tension output is shown in Figure 9. The peaks in the plot indicate resonant frequencies. It is evident that the minimum resonant frequency is higher for Model 2 compared to Model 1. When we consider biaxial stress (Model 4), the minimum frequency is increased further. Frequency response numerical simulations with medium modulus webs for the various models did not indicate any appreciable change in the resonant frequencies.

PARAMETER SENSITIVITY ANALYSIS

Parameter sensitivity analysis is performed to analyze the effect of web system parameter variations on web tension (system output). For this analysis we consider linearized governing equations with and without small strain assumption and the web material to be elastic. Sensitivity functions are obtained and studied in frequency domain to understand the effect of web transport system parameter

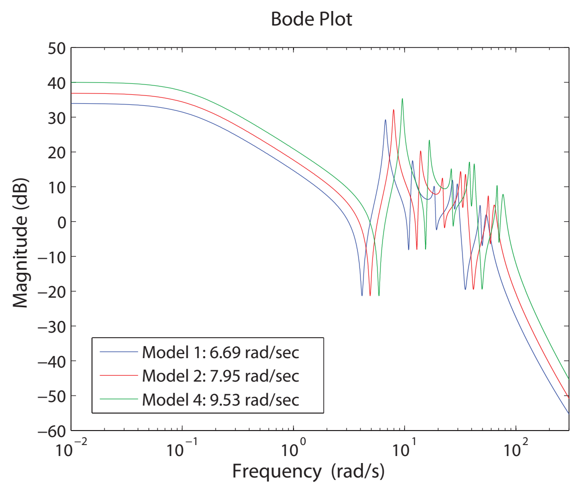

Figure 9: Bode Plot of Tension Dynamic Models; Low Modulus Material

variation on the plant output response.

We consider the classical definition of parameter sensitivity which is defined as the ratio of the change in the system transfer function to an incremental change in a process parameter of interest. Let the open loop transfer function of the process be

$$G(s,\theta) = \frac{N(s,\theta)}{D(s,\theta)} \qquad \{13\}$$

where θ is the process parameter of interest. The sensitivity function with respect to θ is given by

$$S_{\theta}^{G} = \frac{\partial G/G}{\partial\theta/\theta} \qquad \{14\}$$

The parameters of interest are elastic modulus (EA), span length (L_i), reference velocity (v_{ir}), and roll radius (R_i). Frequency response of the sensitivity functions is obtained and used to compare the derived tension models.

The tension equation derived with and without small strain assumption are linearized around the reference values of tension and velocity. The transfer function from velocity to tension with the small strain assumption is given by

$$G_s(s) = \frac{T_i}{V_i} = \frac{\frac{EA-t_{ir}}{v_{ir}}}{\frac{L_i}{v_{ir}}s+1} \qquad \{15\}$$

The transfer function without the small strain assumption is given by

$$G_w(s) = \frac{T_i}{V_i} = \frac{\frac{EA+t_{ir}}{v_{ir}}}{\frac{L_i}{v_{ir}}s + \frac{2v_{i-1r}(EA+t_{ir})}{v_{ir}(EA+t_{i-1r})} - 1} \qquad \{16\}$$

The denominator term of the transfer function G_w reflects the effect of the downstream roller velocity (v_{i-1r}). Therefore, it can accurately model the process, such as a velocity draw in consecutive driven tension zones.

The transfer function between tension variation T_i and velocity variation V_i can be expressed in terms of the radius parameter of the roll by replacing the reference velocity v_{ir} with the relation:

$$v_{ir} = R_i \omega_{ir}$$

The sensitivity functions for the transfer function G_s and G_w with respect to parameters EA, L_i, v_{ir}, and R_i are given in Table 1.

Web Parameter	S^{G_s}	S^{G_w}
EA	$\dfrac{EA}{EA - t_{ir}}$	$\dfrac{EA}{EA + t_{ir}} - \dfrac{EA\left(\dfrac{2v_{i-1r}}{EA + t_{i-1r}} - \dfrac{2v_{i-1r}(EA + t_{ir})}{(EA + t_{i-1r})^2}\right)}{L_i s - v_{ir} + \dfrac{2v_{i-1r}(EA + t_{ir})}{(EA + t_{i-1r})}}$
L_i	$\dfrac{-L_i s}{L_i s + v_{ir}}$	$\dfrac{-L_i s}{L_i s - v_{ir} + \dfrac{2v_{i-1r}(EA + t_{ir})}{(EA + t_{i-1r})}}$
v_{ir}	$\dfrac{-v_{ir}}{L_i s + v_{ir}}$	$\dfrac{v_{ir}}{L_i s - v_{ir} + \frac{2v_{i-1r}(EA + t_{ir})}{(EA + t_{i-1r})}}$
R_i	$\dfrac{-R_i \omega_{ir}}{L_i s + R_i \omega_{ir}}$	$\dfrac{R_i \omega_{ir}}{L_i s - R_i \omega_{ir} + \dfrac{2v_{i-1r}(EA + t_{ir})}{(EA + t_{i-1r})}}$

Table 1: Sensitivity Functions of Web Parameters

The sensitivity functions of the model G_w are more involved compared to model G_s. The additional terms reflect the draw as well the reference web tension for neighboring web spans which are not reflected in the sensitivity functions of G_s.

The transfer function between output span tension T_i and input velocity V_i is formulated for the linearized web tension dynamics derived considering cross machine directional stress. In this analysis, the parameter of interest is Poisson's ratio. For simplicity, consider the case with no draw and equal reference tension for all web spans. The transfer function with the small strain assumption and considering cross machine direction stress is

$$G_{sv} = \frac{T_i}{V_i} = \frac{\frac{(EA - t_{ir})(EA + \nu t_{ir})}{(EA - EA\nu + 2\nu t_{ir})}}{L_i s + v_{ir}}. \qquad \{17\}$$

The transfer function without the small strain assumption and considering cross machine direction stress is

$$G_{wv} = \frac{T_i}{V_i} = \frac{\frac{(EA + t_{ir})(EA - \nu t_{ir})}{(EA - EA\nu - 2\nu t_{ir})}}{L_i s + v_{ir}}. \qquad \{18\}$$

The sensitivity functions for the transfer function G_{sv} and G_{wv} with respect to Poisson's ratio ν are given in Table 2.

Sensitivity Function	ν
$S^{G_{sv}}$	$\nu\left(\frac{t_{ir}(EA-EA\nu+2\nu t_{ir})}{EA+\nu t_{ir}}+(EA-2t_{ir})\right)$
$S^{G_{wv}}$	$\nu\left(\frac{-t_{ir}(EA-EA\nu-2\nu t_{ir})}{EA-\nu t_{ir}}+(EA+2t_{ir})\right)$

Table 2: Sensitivity Functions for Poisson's Ratio

The sensitivity functions are analyzed through the frequency response in the following subsection.

Simulation Results

The two roller system shown in Figure 1 is considered and frequency response is obtained for models with and without small strain assumption for low modulus and medium modulus webs. The following parameter values are considered for model simulations: $t_r = 25$ lbf (high strain region); modulus constant, $EA = 150$ lbf (low modulus web) and $EA = 2800$ lbf (medium modulus web); reference speed, $v_r = 850$ fpm; draw: 5%. The Bode magnitude plots for the two derived models (G_s and G_w) are shown in Figure 10. The two models differ in terms of the magnitude for the low modulus webs, while the response is almost similar for the medium modulus webs.

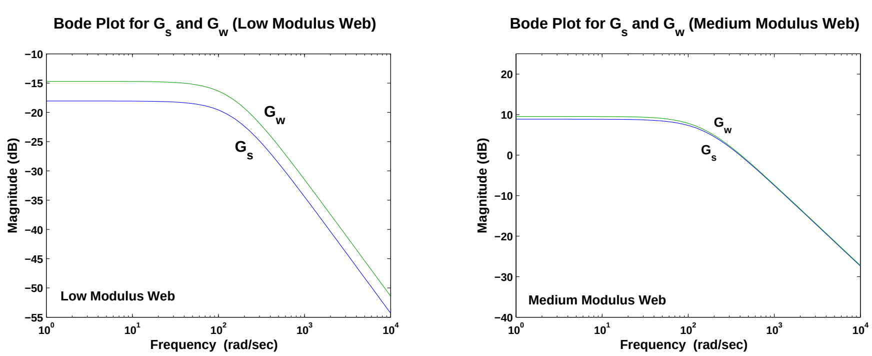

Figure 10: Transfer Function Bode Plot; Left: Low Modulus Web, Right: Medium Modulus Web

The frequency response of the sensitivity functions for a low modulus web is shown in Figure 11. The sensitivity function response slightly differ for both the models. The effect of modulus change (which may be due to humidity or temperature) is the same across all frequencies.

The frequency response of the sensitivity functions to span length (L_i) is shown in Figure 12. The transfer function is more sensitive to length changes in high

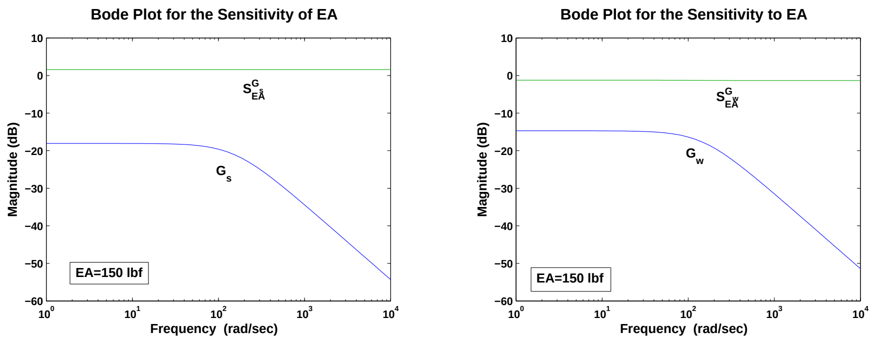

Figure 11: Low Modulus EA Sensitivity Function Bode Plot; Left: With Small Strain Assumption, Right: Without Small Strain Assumption

frequency region. The sensitivity to L_i is prominent in the vicinity of the corner frequency. This can be evident from the fact that tension is affected substantially by the rate of change of length due to an out-of-round or an eccentric roll.

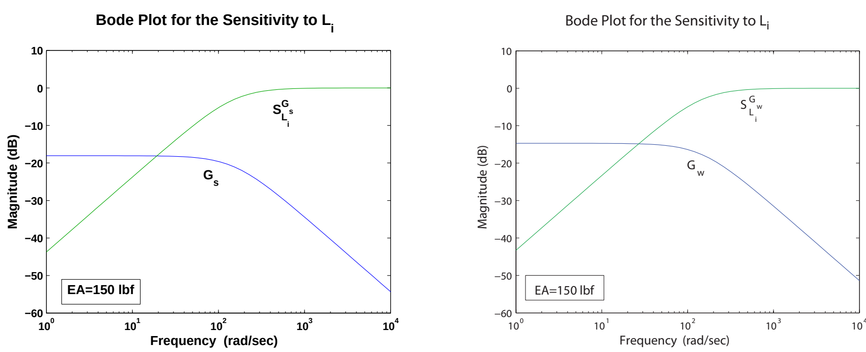

Figure 12: Low Modulus L_i Sensitivity Function Bode Plot; Left: With Small Strain Assumption, Right: Without Small Strain Assumption

The velocity reference may change continuously due to the correction signal provided by the outer tension controller in a speed-based tension control system. The frequency response of the sensitivity functions to velocity reference (v_{ir}) is shown in Figure 13. In the low frequency region, the tension transfer function is sensitive to variations in the reference speed. Both the sensitivity functions and transfer functions roll-off at around the vicinity of the corner frequency.

The change in roll radius affects tension output. The frequency response of the sensitivity functions to roll radius (R_i) is shown in Figure 14. In the low frequency region, the tension transfer function is sensitive to variations in the radius.

The frequency response of the sensitivity functions corresponding to Poisson's ratio (nu) for both models (G_{sv} and G_{wv}) is shown in in Figure 15. It is evident

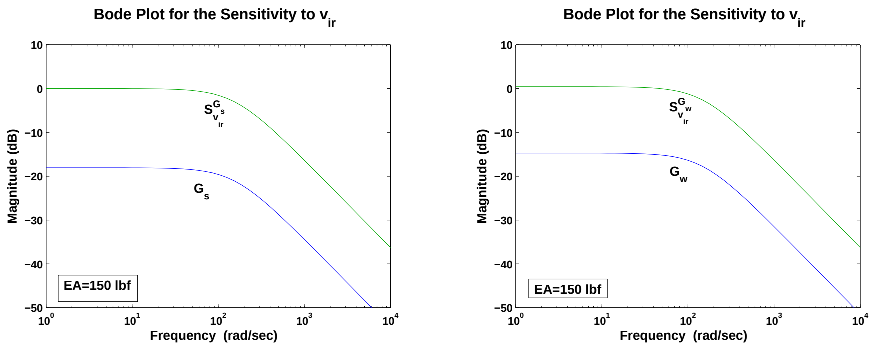

Figure 13: Low Modulus v_{ir} Sensitivity Function Bode Plot; Left: With Small Strain Assumption, Right: Without Small Strain Assumption

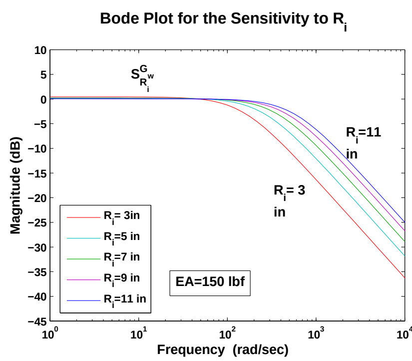

Figure 14: Low Modulus Radius Sensitivity Function Bode Plot for Model Without Small Strain Assumption

that the transfer function $G_{w\nu}$ is highly sensitive to changes in Poisson's ratio.

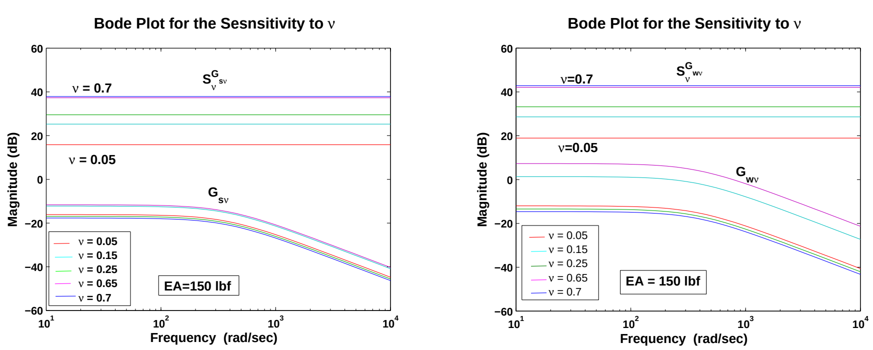

Figure 15: Low Modulus Poisson Ratio Sensitivity Function Bode Plot; Left: With Small Strain Assumption, Right: Without Small Strain Assumption

EFFECT OF MOISTURE AND TEMPERATURE ON TRANSPORT BEHAVIOR OF LOW MODULUS WEBS

Both moisture content and temperature significantly affect the transport behavior of low modulus, non-woven webs. Non-woven webs consists of fibers, bonded together to form a porous random network as shown in Figure 16.

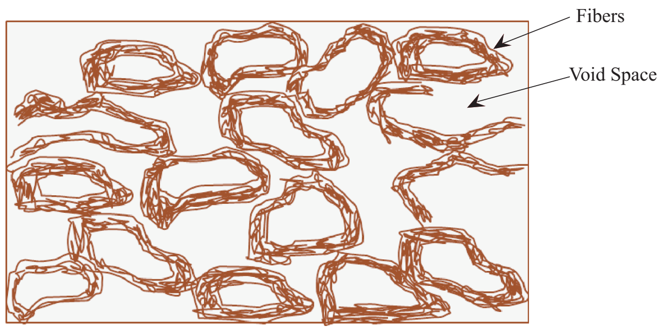

Figure 16: Fiber and Porous Structure in Non-woven Webs

The geometrical distortion observed in webs due to these effects is a complex function of web structure and hygro-thermal elastic properties of fibers. Absorption of moisture causes swelling of fibers. The fiber network transfers the resulting stress in machine and cross-machine directions which leads to web expansion or contraction. The dimensional changes due to the change in the relative humidity and temperature of the surrounding atmosphere and the process is called hygroexpansivity and thermoexpansivity and the resultant strain is referred to as hygroscopic and thermal strain.

Heat diffusion and moisture diffusion are studied by using the relations developed by Fourier for heat conduction and similar theory by Fick for moisture diffusion. Roisum (1993) proposed the use of Fick's law to study moisture process in wound rolls. Pagilla et al. (2007) used a second-order Fick's law for moisture diffusion as given in equation {19} and developed a governing equation for web span tension to evaluate moisture effect on web tension.

$$D\frac{\partial^2 C}{\partial y^2} = \frac{\partial C}{\partial t} \qquad \{19\}$$

where C is the moisture concentration and D is the diffusion constant.

Fick's law assumes the web material to be homogeneous and non-porous. Fick's law does not consider absorption and desorption phenomenon in the fibers. The interaction of moisture diffusion between porous space and fibers must be considered for non-woven materials. Several studies (Lescanne (1992), Hellen (2001)) in literature have shown that moisture diffusion in porous paper webs is non-Fickian. Several researchers have derived simultaneous moisture and temperature transport models for paper webs with porous structure. Nordon et al. (1967) and Foss et al. (2003) modeled heat and moisture transport in a paper sheet in response to variations of relative humidity and temperature surrounding the sheet. Green (1981) studied the effect of moisture, temperature, and stress on paper dimensions by observing the recoverable and nonrecoverable strains. Uesaka (1994) derived general formula for the hygroexpansion of paper.

In this paper, based on existing research, we present governing equations for moisture concentration and temperature, which will be subsequently used to obtain hygral and thermal web strains. The pore space of non-woven web is denoted by subscript p and solid fiber matrix by subscript f. The fiber and porous space may not necessarily be in equilibrium during web processing and hence the humidity inside fiber and porous space need to be evaluated separately. The production of moisture in porous space and fiber is assumed to be negligible. The mass balance of water vapor inside the pore space using a non-fickian moisture diffusion is given by

$$\frac{\partial C_p(y,t)}{\partial t} = D_p \frac{\partial^2 C_p(y,t)}{\partial y^2} - \frac{(1-\phi)k_m}{\phi}(C_{eq}(C_p,T) - C_f) \qquad \{20\}$$

where C_p is moisture concentration in porous space, C_f is moisture concentration in fiber space, C_{eq} is equilibrium moisture concentration in web, and y is thickness direction, D_p is moisture diffusion constant in pore space, ϕ is web porosity, k_m is interphase (pores to fibers) mass transfer coefficient. The second term in the right side of equation {20} represents the moisture interphase between fiber and porous space. The equilibrium moisture concentration is a function of moisture in pore space and temperature of web. Lescanne (1992) shows that moisture transport thorough the pore spaces is much more rapid compared to the fibers. Hence, neglecting diffusion in fiber and assuming there is just an exchange of water between porous space and fibers, the mass balance of water vapor in the fiber is given by

$$\frac{\partial C_f(y,t)}{\partial t} = k_m(C_{eq}(C_p,T) - C_f) \qquad \{21\}$$

The initial conditions are usually taken as, the porous space in equilibrium with the surrounding air and fiber is considered to be in equilibrium with porous space.

$$C_p(y,0) = C_\infty(0)$$
$$C_f(y,0) = C_{eq}(y,0)$$

where C_∞ is moisture concentration in surrounding air.

When the web is exposed to the same humidity condition on both sides, there is symmetry plane at the center of web

$$\frac{\partial C_p}{\partial t}\Big|_{y=0} = 0$$

The boundary conditions on the surface of the web for moisture transfer are

$$\frac{\partial C_p}{\partial y}\Big|_{y=\frac{h}{2}} = B_m(C_\infty(t) - C_p(\frac{h}{2},t)),$$
$$\frac{\partial C_p}{\partial y}\Big|_{y=\frac{-h}{2}} = B_m(C_\infty(t) - C_p(\frac{-h}{2},t))$$

where $C_p(\frac{h}{2},t) = C_p(\frac{-h}{2},t) = \phi C_\infty$ and B_m is Biot number for moisture mass transfer.

The heat absorption by the web is a function of the moisture content in the fiber. Assuming the temperature of porous space and fiber to be equal ($T_p = T_f = T$), the heat balance in the web material can be expressed as

$$[\phi\rho_p c_p + (1-\phi)\rho_f c_f]\frac{\partial T}{\partial t} = k_{eff}\frac{\partial^2 T}{\partial y^2} - (1-\phi)\rho_f H_{abs}\frac{\partial C_f}{\partial t} \qquad \{22\}$$

where T is the web temperature, c_p is the heat capacity of the porous medium, c_f is the heat capacity of fibers, ρ_p is the density of porous space, ρ_f is the density of fiber, k_{eff} is the effective thermal conductivity, and H_{abs} is the heat of absorption. The term $\phi\rho_p c_p + (1-\phi)\rho_f c_f$ represents volumetric heat capacity of web. The second term in the right side of equation indicates the effect of moisture content on web temperature.

At the surface of web, the temperature boundary conditions are given by

$$\frac{\partial T}{\partial y}\Big|_{y=\frac{h}{2}} = B_T(T_\infty(t) - T(\frac{h}{2},t)),$$
$$\frac{\partial T}{\partial y}\Big|_{y=\frac{-h}{2}} = B_T(T_\infty(t) - T(\frac{-h}{2},t))$$

where T_∞ is surrounding temperature, B_T is Biot number for heat transfer.

Changes in web moisture content and temperature result in dimensional change of fibers, each fiber transmits those changes to adjacent fibers in its network. The structure of the fiber network affects the degree of transfer of dimensional changes. The composite model for the length change can be expressed as

$$L = L_0 + f(\sigma - \sigma_i, T, C, t) \qquad \{23\}$$

where L is the deformed length of web, L_0 is the original length of web, σ is the applied stress, σ_i is the internal stress, C is the moisture concentration in web, and t is the time parameter. In the case of non-woven webs, which is transported at high tension, the internal stress is small compared to applied stress and can be neglected. Hence, the deformation is a function of applied stress, moisture content, temperature, and time. The total web strain in longitudinal direction may be expressed as

$$\varepsilon = \varepsilon_e + \varepsilon_h + \varepsilon_t \qquad \{24\}$$

where ε_e is the elastic strain induced by the applied stress, ε_t is the thermal strain, ε_h is the hygral strain.

For low humidity levels, there is a linear relationship between hygral strain and moisture concentration. But for higher humidity levels, the relationship is nonlinear and may be expressed by the following relation (Uesaka, 1994)

$$\varepsilon_h = \gamma e^{\xi(C_f - C_\infty)} \qquad \{25\}$$

where γ and ξ are constants.

Assuming a linear relationship between thermal strain and web temperature,

$$\varepsilon_t = \delta(T - T_\infty) \qquad \{26\}$$

where δ is constant.

Moisture content and web temperature also affect the elastic modulus. The modulus of web decreases with increase in moisture content and web temperature. The non-woven web elastic modulus under stress is a function of elastic strain, moisture content, web temperature ($E = f(\varepsilon_e, C, T)$). The total strain can substituted into the strain dynamics to obtain a governing equation for tension.

SUMMARY AND CONCLUSION

Governing equations for web strain and tension for low modulus webs are developed by relaxing the small strain assumption and considering in-plane biaxial strain. Numerical simulation results show that there is substantial difference in behavior between the model with small strain assumption and the one without, and by considering in-plane biaxial stress. A similar conclusion is reached after conducting a parameter sensitive analysis. Simultaneous diffusion of moisture and heat in the web material is studied for non-woven webs by taking into account moisture diffusion process to be non-Fickian. The governing equations for moisture concentration and temperature are presented and used to obtain hygral and thermal web strains. Future work will consider experimental validation of the new governing equations.

ACKNOWLEDGEMENTS

This work was supported by the Web Handling Research Center, Oklahoma State University, Stillwater, Oklahoma.

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APPENDIX

The derivation of the strain dynamic model is based on conservation of mass in a control volume: the change of web mass in a span is equal to the difference between the amount of web mass entering the span from the previous span and the amount of web mass leaving for the next span. The law of mass conservation can be expressed as

$$\frac{d}{dt}\int_{x_{i-1}(t)}^{x_i(t)} \rho(x,t)A(x,t)dx = \rho_{i-1}A_{i-1}v_{i-1} - \rho_i A_i v_i \qquad \{27\}$$

where $x_{i-1}(t)$ and $x_i(t)$ are the web positions on the $(i-1)^{th}$ and i^{th} rollers, v_{i-1} and v_i are the web velocities on the $(i-1)^{th}$ and i^{th} rollers, ρ is the density of web material, and A is the cross-section area of web.

Consider an infinite element of the web in the longitudinal direction. The length, width, and thickness of the element are given by

$$dx = (1+\varepsilon_x)dx_u, \tag{28}$$

$$w = (1+\varepsilon_w)w_u, \tag{29}$$

$$h = (1+\varepsilon_h)h_u \tag{30}$$

where ε_x is the strain along the length of the web, ε_w is the strain along the width of the web, ε_h is the strain along the thickness of the web, and the subscript u denotes the unstretched state of the web.

The mass of the infinitesimal element of the web can be expressed as

$$dm = \rho(x,t)A(x,t)dx = \rho_u(x,t)A_u(x,t)dx_u \tag{31}$$

Substituting the right side of {28} into {31} results in:

$$\frac{\rho(x,t)A(x,t)}{\rho_u(x,t)A_u(x,t)} = \frac{dx_u}{dx} = \frac{1}{1+\varepsilon_x(x,t)} \tag{32}$$

Therefore, the mass conservation equation can be expressed as

$$\frac{d}{dt}\int_{x_{i-1}}^{x_i} \frac{\rho_u(x,t)A_u(x,t)}{} \qquad \frac{}{1+\varepsilon_{x_{i-1}}(x,t)} - \frac{\rho_{u_i}(x,t)A_{u_i}(x,t)v_i(t)}{1+\varepsilon_{x_i}(x,t)} \tag{33}$$

Under the assumption that the density of the web material and cross section area are constants for the unstretched material, that is $\rho_u = \rho_{u_{i-1}} = \rho_{u_i}$ and $A_u = A_{u_{i-1}} = A_{u_i}$, equation {33} can be simplified to

$$\frac{d}{dt}\int_{x_{i-1}}^{x_i} \frac{1}{1+\varepsilon_x(x,t)}dx = \frac{v_{i-1}(t)}{1+\varepsilon_{x_{i-1}}(x,t)} - \frac{v_i(t)}{1+\varepsilon_{x_i}(x,t)} \tag{34}$$

The integral term in the equation {34} is evaluated by applying the Leibnitz rule with time dependent limits,

$$\frac{d}{dt}\left(\int_{\phi(t)}^{\psi(t)} f(x,t)dx\right) = \int_{\phi(t)}^{\psi(t)} \frac{\partial f(x,t)}{\partial t}dx - \frac{d\phi}{dt}f(\phi(t),t) + \frac{d\psi}{dt}f(\psi(t),t), \tag{35}$$