AN UPDATED MODEL FOR LATERAL DISPLACMENT OF NONUNIFORM WEBS

By

Jan Erik Olsen
SINTEF Materials & Chemistry
NORWAY

ABSTRACT

The theory for lateral mechanics of nonuniform webs in the open literature is believed to be flawed and an updated theory is presented. The theory is based on beam theory and a set of boundary conditions. The *so-called* fourth boundary condition is discussed and a condition is chosen based on the argumentation. A new term in the governing differential equations is derived. The updated theory predicts that the web moves to the slack side if there is sticking friction between web and roller.

INTRODUCTION

A web is a continuous, flexible strip of material such as paper, plastic film metal foil and more. Whether the material is paper or plastic film, the web will be transported at some stage through a web line. Some of these web lines are part of processes in which control of sideways motion is essential for the final quality of the product. Sideways motion is influenced by forces and bending moments transferred from rollers in contact with the web. As can be described by beam theory, the web moves sideways if a bending moment is applied to it. However, it has also been observed that a web may shift sideways if no apparent bending moment is applied. This is observed to happen to imperfect webs with widthwise variations in their material properties [1]. The variations may be variations in elastic modulus or in frozen-in strain. Webs with such widthwise variations are referred to as imperfect webs, non-uniform webs, cambered webs, baggy webs or curved webs.

In this paper we will discuss the theory of sideways motion of such webs which is known as lateral mechanics of imperfect webs. The theory is based on the work of Shelton [2][3] who studied lateral dynamics of perfect webs. The theory was generalized to include the effects of imperfect webs by the author [4]. The generalized theory is believed to be partly flawed. An alternative theory to the lateral mechanics of imperfect webs is presented below.

THEORY

The theory of lateral mechanics of webs is based on beam theory. A deflected beam has a stress profile described by

$$\sigma(x,y) = E\{\varepsilon_c + \kappa(x)y\} \qquad \{1\}$$

The stress σ is given by the elastic modulus E, the strain at the beam centroid ε_c and the curvature of the web κ. x is the coordinate along the beam axis and y is the coordinate in the normal direction. The curvature is defined as

$$\kappa \equiv -\frac{\partial^2 v}{\partial x^2} \qquad \{2\}$$

For an imperfect web, the theory needs to be generalized. Frozen-in strain ε_i and potential widthwise variations in the elastic modulus need to be accounted for. This is done by generalizing the equations above

$$\sigma(x,y) = E(y)\{\varepsilon_c + \kappa(x)y - \varepsilon_i\} \qquad \{3\}$$

The frozen-in strain ε_i is added as the integration constant obtained when integrating the definition of the elastic modulus. This constant is sometimes declared as a residual stress or a frozen-in stress. We declare it as a residual strain or a frozen-in strain since we assume that the stress free web has a known deflection or strain distribution. For simplicity we will focus on webs with no widthwise variations in elastic modulus and a linear variation in frozen-in strain. This is also known as uniform camber. Thus we assume

$$\begin{aligned} &E(y) \to E \\ &\varepsilon_i = \kappa_{web} y = \frac{1}{R_{web}} y \end{aligned} \qquad \{4\}$$

where κ_{web} is the inherent curvature and R_{web} is the inherent radius of curvature of the web. Since it is easier to relate to the radius of curvature we will use that quantity for the remaining parts of the text. It is assumed to be a known property of the web. Equation {3} is simplified to

$$\sigma = E\left\{\varepsilon_c + \left(\kappa - \frac{1}{R_{web}}\right)y\right\} \qquad \{5\}$$

Here we have also skipped the coordinate dependence in the notation for simplicity. Note that the curvature κ still depends on the x coordinate and the stress σ depends on both x and y. The overall curvature of the web is given by the inherent web curvature κ_{web} and the curvature added by bending forces κ_b

$$\kappa = \kappa_b + \kappa_{web} = \kappa_b + 1/R_{web} \qquad \{6\}$$

Bending due to shear forces is neglected for simplicity, but can be added according to previous work of the author [5]. We see that bending yields a contribution to the stress in addition to the web line tension which is captured by the centroidal strain. The essence of the generalized theory which accounts for the effect of imperfect webs is given by Eq.{5}. It may be generalized further to account for widthwise variations in elastic modulus and nonlinear frozen-in strains as in Eq.{3}. In Fig.1 we see how a web with inherent web camber is curved in its stress free state. If we stretch the web, it becomes straight with one side more stressed than the other. We refer to these sides as the slack and tight side.

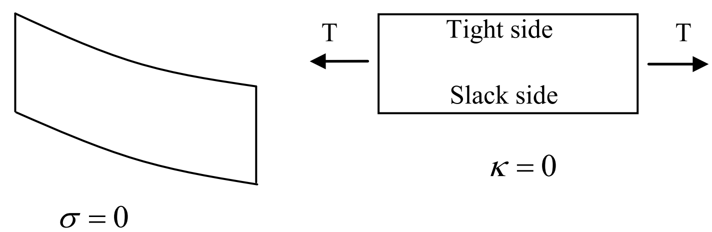

Figure 1 – Cambered web (left) and straightened cambered web (right).

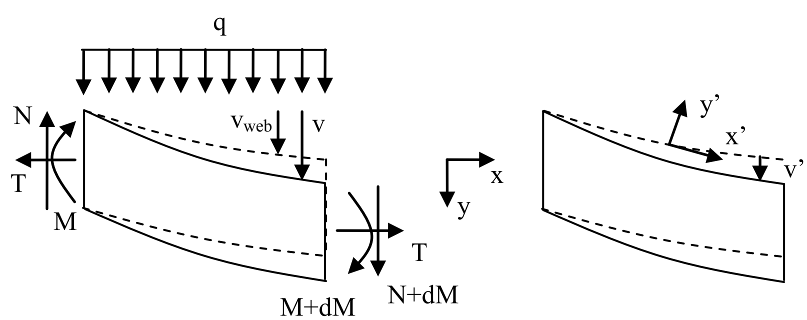

Figure 2 – Force balances and deflection definitions on a curved web.

A failure made in the previously published work on lateral mechanics of imperfect webs [4] was to neglect the influence of the frozen-in web curvature on the equilibrium conditions which the governing equations is derived from. A balance of forces and moments on an imperfect web as illustrated in Fig.2 must be made with reference to the relaxed state of the web. Equilibrium of forces and moments then yields:

$$q = \frac{\partial N}{\partial x} + T\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{R_{web}}\right) \qquad \{7\}$$

$$N = \frac{\partial M}{\partial x} \qquad \{8\}$$

where q is lateral load on the web, N is the shear force, T is the tension and M is the moment. This may be explained as a balance of forces and moments in a coordinate system (x' and y') defined along the relaxed beam axis of the web as seen on the right hand side of Fig.2 and then transformed back to the coordinate system of the web line. Substituting Eq.{8} into Eq.{7} gives

$$\frac{\partial^2 M}{\partial x^2} + T\frac{\partial^2 v}{\partial x^2} = q - \frac{T}{R_{web}} \qquad \{9\}$$

An expression for the moment M, can be found by integrating Eq.{5}. Then we get

$$M = \int_{-W/2}^{W/2} \sigma \cdot y dA = \frac{1}{12} EhW^3 \left(\kappa - \frac{1}{R_{web}} \right) \qquad \{10\}$$

which is a generalized beam equation. Substituting Eq.{2} for κ in Eq.{10} and performing a derivation of the moment, M, twice with respect to the x-coordinate gives the following expression for the moment:

$$\frac{\partial^2 M}{\partial x^2} = -\frac{EhW^3}{12}\frac{\partial^4 v}{\partial x^4} \qquad \{11\}$$

Combining Eqs.{9} and {11} provides us with the governing differential equation for lateral mechanics of imperfect webs:

$$\frac{\partial^4 v}{\partial x^4} - K^2 \frac{\partial^2 v}{\partial x^2} = \frac{K^2}{R_{web}} \qquad \{12\}$$

where

$$K^2 = \frac{12T}{EhW^3} \qquad \{13\}$$

We have neglected the lateral load q, which normally is insignificant since webs are so thin that lateral loads do not affect them. The differential equation is a fourth order linear equation which thus requires four boundary conditions. It includes the effect of the imperfection of the web in the term on the right hand side. This was not included in the previous work of the author since the importance of performing the force balances on the relaxed coordinate system was not acknowledged.

Boundary Conditions

We need four boundary conditions for the mathematical description of lateral mechanics to be well posed. If we study the deflection of the web between two rollers, we can pose two boundary conditions at each roller. First we define the coordinate system according to the web position and roller alignment at the downstream roller [2][3]:

$$v_0 = 0 \qquad \{14\}$$

$$\left.\frac{\partial v}{\partial x}\right|_0 = 0 \qquad \{15\}$$

At the upstream roller we apply the principle of normal entry

$$\left.\frac{\partial v}{\partial x}\right|_L = \theta \qquad \{16\}$$

where θ is the angle of misalignment of the downstream roller. The principle of normal entry is based on the assumption of sticking contact between the web and the roller such that surface particles of the web will follow the path of the surface particles on the roller[2][3].

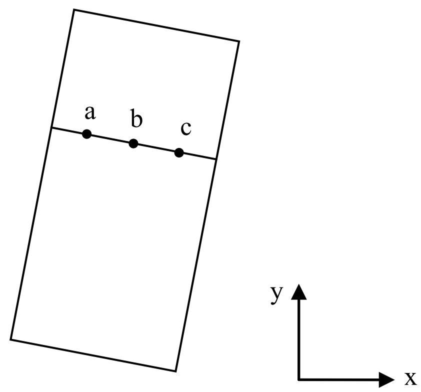

Figure 3 – Path line of web particles on a cylindrical roller.

Much of the debate concerning the lateral mechanics of imperfect webs has evolved around the fourth boundary condition. Shelton [2][3] showed that the curvature at of the web at the downstream roller was zero, $\kappa_L = 0$. For a perfect web this is equivalent to a zero moment at the downstream roller. For an imperfect web, zero downstream curvature and zero downstream moment are not equivalent as proved by Eq.{10}. It has been argued that the fourth boundary condition for an imperfect web is zero curvature at the downstream roller, zero moment at the downstream roller or something in between.

We can argue that a web particle will follow the surface motion of the roller if there is sticking contact between the web and the roller. If we consider a cylindrical roller which generally may be misaligned as seen in Fig.3, the web particle touching the roller at point *a*, will continue to move to position *b* and then position *c*. This path line is a straight line, and a straight line has zero curvature. Thus the author believes that the fourth boundary condition is zero curvature at the downstream roller:

$$\left.\frac{\partial^2 v}{\partial x^2}\right|_L = 0 \qquad \{17\}$$

This is in principle the same boundary conditions as originally applied by Shelton for a perfect web. The boundary condition is based on the fact that surface particles of the web follow the trajectory of the surface particles of the roller. This is true for both a perfect web and an imperfect web.

Solution

The fourth order differential equations, Eq.{12}, and the four boundary equations, Eqs.{14}-{17} constitute a mathematically well posed problem. The general solutions of the differential equation is

$$v(x) = C_1 \sinh(Kx) + C_2 \cosh(Kx) + C_3 x + C_4 + C_p x^2 \qquad \{18\}$$

where the last term indicates the particular solution due to the nonzero right hand side of Eq{12}. Applying the boundary conditions and inserting the particular solution yields the following expression for the web deflection:

$$\begin{aligned} v(x) = \theta L \Bigg\{ & \frac{\cosh(KL)}{\cosh(KL)-1}\left[\frac{x}{L} - \frac{\sinh(Kx)}{KL}\right] \\ & + \frac{\sinh(KL)}{\cosh(KL)-1}\frac{1}{KL}\left[\cosh(Kx)-1\right]\Bigg\} \\ + \frac{L}{KR_{web}} \Bigg\{ & \frac{KL\cosh(KL)-\sinh(KL)}{\cosh(KL)-1}\left[\frac{x}{L} - \frac{\sinh(Kx)}{KL}\right] \\ & + \left(\frac{KL\sinh(KL)}{\cosh(KL)-1} - 1\right)\frac{1}{KL}\left[\cosh(Kx)-1\right]\Bigg\} - \frac{1}{2}\frac{x^2}{R_{web}} \end{aligned} \qquad \{19\}$$

We assume perfect roller alignment ($\theta = 0$) and focus on the web deflection at the downstream roller ($x = L$) due to the inherent web curvature. The deflection at the downstream roller is

$$v_L = \frac{L}{KR_{web}}\left(\frac{2}{KL} + \frac{\frac{1}{2}KL\left(\cosh(KL)+1\right) - 2\sinh(KL)}{\cosh(KL)-1}\right) \qquad \{20\}$$

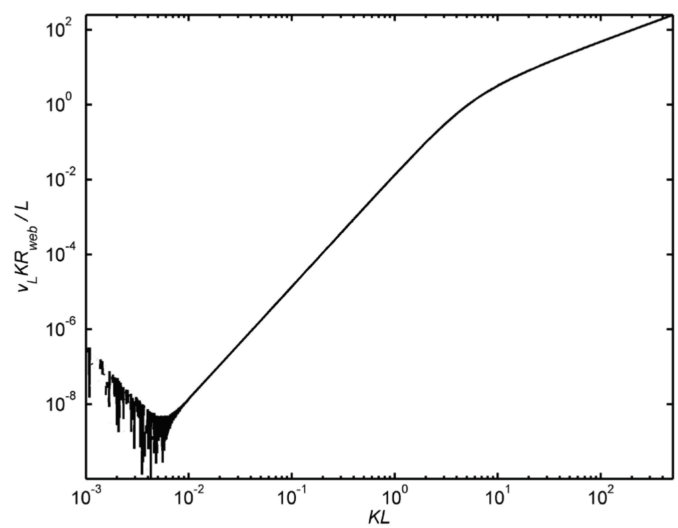

Figure 4 – Dimensionless deflection at downstream roller

	PET	SC Paper
Modulus - GPa	4.14	3.89
Thickness - µm	23.4	71
Width - m	0.3	0.5
Span Length - m	0.67	1.0
Radius of Curvature - m	528	880

Table 1 – Web properties for PET film and supercalendered paper

PREDICTIONS

By moving the product in front of the parentheses in Eq.{20} to the left hand side, we can plot the downstream deflection in a dimensionless manner. This is seen in Fig.4. where a dimensionless deflection at the downstream roller is plotted as a function of *KL*. The plot indicates that at very low numbers of *KL*, the theory seems to produce some irregularities. Since these values of *KL* are unrealistically low, we do not need to worry about that. The plot also shows that for all values of *KL*, the deflection is positive, which indicates that the web moves sideways to the slack side of the web. This is in agreement with observations [1].

In order to study some real case scenarios, we will apply the theory to PET film and supercalendered paper with properties as given in Table 1. The downstream deflection for these webs as a function of web tension is seen in Fig.5. The deflection seems to increase linearly with web tension for values covering the typical range applied to these webs. Note however that the deflection is very small. The webs move to the slack side, but the deflection is in principle insignificant.

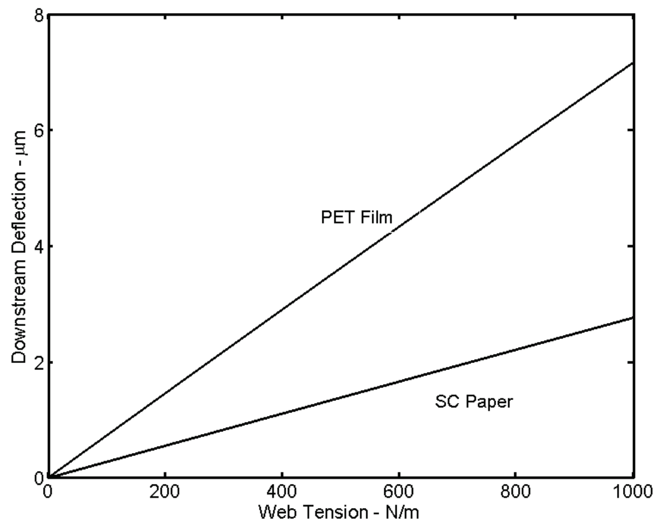

Figure 5 – Downstream deflection as a function of web tension for PET film and SC paper.

It has previously been showed that at low tensions the deflection increases as the web tension is too small to stretch out the bagginess of the slack side [6]. In order to capture this aspect, one has to acknowledge that most webs do not carry negative tension. They tend to buckle out of plane instead. This is captured by generalizing Eq.{3} or Eq.{5} so that the stress has a minimum value of 0:

$$\sigma = MIN\left(0, E\left\{\varepsilon_c + \left(\kappa - \frac{1}{R_{web}}\right)y\right\}\right) \qquad \{21\}$$

The web has no compressive stresses. This result in an effective web width given by:

$$W_e = MIN\left(W, \sqrt{\frac{2TR_{web}}{Eh}}\right) \qquad \{22\}$$

The effective web width is the width of the web which is carrying tension. In Eq.{22} it is assumed that the web is stretched out and do not have any significant curvature. A more general approach results in a more complicated differential equation [6]. For simplicity we apply Eq.{22} in the expression for K in Eq.{13}. This changes the deflection at lower tensions as seen in Fig.6. We see that the web still moves to the slack side for all tensions, but now we see an increase for the lower tensions. This is due to the reduction in effective web width at lower tensions. Still the deflection is in principle insignificant.

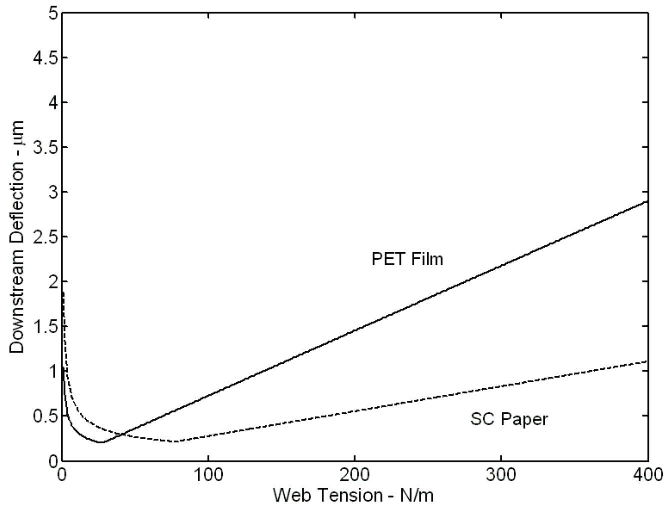

Figure 6 – Downstream deflection as a function of web tension for PET film and SC paper with theory for no compressive stresses.

DISCUSSION

The theory presented above shows that a baggy or cambered web moves to the slack side under the assumption of sticking friction at the downstream roller. The sideways deflection for the cases investigated is however so small that it can be considered insignificant. One might ask if this is in contradiction to observations.

Published observations are very few and the author is only aware of one such publication. Swanson [1] showed experimentally that a baggy web of PET film moves to the slack side. The reported sideways shift was typically 0.1-0.3mm which is several orders of magnitude higher than what was found by the theory presented here. This might indicate that something is not correct in the theory. However, if we look more carefully at the results of Swanson, we see that the deflection depends upon the coefficient of friction between the web and the roller. This indicates that we do not have sticking friction between web and roller. Thus the results of Swanson should not be compared to any theory which assumes sticking friction. The assumption of sticking friction is often applied since this is used for studies of perfect webs. For a baggy web the slack side may often have small tensile stresses even if the web is stretched out. Small tensile stresses results in low contact pressure towards rollers and thus increased possibilities of slippage. Thus the assumption of sticking friction may need to be abolished in order to study real scenarios.

The theory presented results in a differential equation and a set of boundary conditions. The influence of the web bagginess is introduced through the differential equations. Formerly published theory introduces the web bagginess through the *so-called* fourth boundary condition with no terms for bagginess in the differential equation. The former theory results in much higher sideways deflections. The author is however convinced that the fourth boundary conditions should state a zero curvature at the downstream roller as argued above. Thus the theory presented here is favoured by the author. Note that this has not been critically examined by other experts and might be debated in the future.

CONCLUSIONS

A new and updated theory for lateral mechanics of nonuniform webs have been presented and applied to baggy webs. The theory predicts that the web moves to the slack side if there is sticking friction between web and roller. The sideways deflection is however insignificantly small for the cases studied. It is hard to validate the theory since the only available experimental results seems to have been conducted with slipping friction.

To bring this research into a conclusive theory, the author believes that experiments with sticking friction need to be performed and/or that boundary conditions for slipping friction and moment transfer must be developed.

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An Updated Model for Lateral Displacement of Nonuniform Webs	**J. E. Olsen,** SINTEF Materials & Chemistry, NORWAY

Name & Affiliation	**Question**
Dave Roisum, Finishing Technologies, Inc.	If I might add two data points, coming from the metal industry, both of them giving identical results where we actually controlled camber on the fly. Metal was heating up a lot and it turns out the system is bi-stable. If it is a little off center, it continues to drive in that direction, which indicates that it is moving essentially to this tight side and about the order of magnitude that would be predicted by a simple thermal expansion model with a bi-metallics element. The second data point was even more convincing because one of the facilities had the ability to adjust the end heaters. They could steer to the left wherever they wanted by heating up one side. It only took a couple of degrees to move it quite a great deal. I'm not talking microns, I'm talking inches. The span was fairly long.
Name & Affiliation	**Answer**
Jan Erik Olsen, SINTEF Materials & Chemistry	You see the theory is very dependent on K times L parameter. I've only looked at what seems to be more normal span lengths.
Name & Affiliation	**Question**
John Shelton, Oklahoma State University	The metal industry commonly uses crown rollers, which is a whole new ball game. I would also like to mention that we know that we are talking about a cambered web that is undisturbed by external forces in the span that we are dealing with. For example, in an air flotation oven, Ron Swanson and Doug Kedl were trying to predict the amount of steering toward the tight side in an air flotation oven. The force of the air pressure is pushing it toward the tight side. An air flotation oven is for drying, not just running a web through it. There are other considerations, drying considerations, of course. For a large L/W, air flotation ovens, it usually goes toward the tight side.
Name & Affiliation	**Answer**
Jan Erik Olsen, SINTEF Materials & Chemistry	That comment would be that the theories we are working with assume that you don't change the material properties in the span you are looking at. So heating or drying are not taken into account in these models.
Name & Affiliation	**Question**
Ming Yang, Xerox Corporation	In the John Shelton's original paper, he stated the boundary condition is greatly influenced by velocity. How do you feel about this effect?

Name & Affiliation	**Answer**
Jan Erik Olsen, SINTEF Materials & Chemistry	My presentation was based on static conditions. If you consider dynamic conditions, you have to take into account the speed of your web or your rotating roller. That is possible. You just take the updated theory and go through the same procedure as John Shelton did in his work and you will get a dynamic version of this model.