MODELING OF LAMINATED WEBS

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ABSTRACT

A dynamic model of the longitudinal behavior of a laminated web is developed. A single web model that takes into account both thermal and hygral strains is developed first from first principles; the model assumes heat transfer in the region of wrap and free web span and moisture diffusion in the free web span. A classical one-dimensional heat equation is considered in the transverse direction to determine the heat transfer in the region of wrap. In the free web section, a lumped capacitance model is used to investigate heat transfer from the web surface. Moisture diffusion from the web surface is assumed to follow Fickian diffusion, which is used to determine hygral strain in the web. Mechanical and physical properties of a laminated web consisting of two isotropic webs of different material are stated using the rule-of-mixtures. The developed single web model and the laminate properties are used to derive a dynamic model for a laminated web span immediately downstream of the laminator rolls.

NOMENCLATURE

A	area of cross section of the web [m^2]
b	viscous damping coefficient [N sec/m]
c	specific heat of the web material [J/(kg K)]
E	modulus of elasticity (Young's modulus) [N/m^2]
F	force [N]
h	web thickness [m]
K_h	moisture diffusion constant
K_ϑ	thermal conductivity constant ($=\frac{k_t}{\rho c}$)
k, k_1	spring constants [N/m]
k_t	thermal conductivity [W/(m K)]
L	free span web length [m]

M	specific moisture concentration $\left(=\dfrac{H}{\rho}\right)$ [ppmv/kg/m^3]
$\overline{M}$	average moisture concentration
m	mass of the web [kg]
p_0, p_1,q_0,q_1	parameters in the viscoelastic model
R	radius of the roller [m]
$T_{w,i}$	span time constant $(L_i/\bar{v})$ [sec]
T_ϑ	thermal time constant $(=\rho hc/\gamma^*)$ [sec]
t	time [sec]
t_i	web tension in the *i*-th span [N]
$\bar{v}$	average web velocity [m/s]
v	web velocity (or peripheral velocity of the roller) [m/s]
w	web width [m]
x	web transport direction (length)
y	transverse direction (thickness)
z	lateral direction (width)
α	coefficient of thermal expansion [K−1]
β	coefficient of hygral expansion [W/(m K)]
γ^*	coefficient of heat transmission [W/(m2 K)]
δ	exponential function of the ratio of time constants (= exp($-T_{w,i},/T_\vartheta$))
ε	strain
θ	temperature, function of t only [K]
ϑ	temperature, function of x and t [K]
ν	volume fraction
ρ	density [kg/m^3]
σ	stress [N/m^2]
ϕ	angle of wrap [rad]
ω	roller angular velocity [rad/sec]

Subscripts

0	reference state
A,B	layers of webs
c	pertaining to the composite web
i	span or roller number
j,n	indices used in infinite sums
N	state after web wrap
R	pertaining to the roller
s	stretched state
U	input
w	pertaining to the web

INTRODUCTION

Composite materials have been in use as structural materials for a long time. Composite materials are formed from two or more distinct materials with different macroscopic properties. The mechanical and physical properties of the constituent materials dictate the properties of the composite material. Lamination of two or more webs to form a composite web is an important process in many web handling industries. The lamination process typically involves transporting multiple layers of web together into a loaded nip roller system to form a single cohesive laminated web. The properties of the laminated web depend on the upstream conditions of the constituent webs. For example, if the final product of the laminate that is produced in the rolled form is to be cut into flat rectangular sheets, then one would require that the resulting laminate in the unstretched state be flat without curling. One way to ensure this is to maintain equal strains in the individual layers. Therefore, the dynamic behavior of the individual webs upstream of the lamination roller, the physical and mechanical properties of the materials forming the laminate, and the lamination process play a critical role in forming a composite web with desirable properties.

Although web lamination is a common process in many web processing industries, web tension behavior during and after the process of lamination is not well understood. Further, to the best of authors knowledge there is no existing work in the literature on dynamic modeling of laminated webs; the work in this paper takes the first step towards this effort. Therefore, the goal is to develop a dynamic model for web tension in the laminated web span and investigate its properties. Relevant assumptions used in the derivation of the model as well as the conditions used to derive laminate properties based on the properties of the materials making up the laminate are discussed.

Much work can be found in the literature on longitudinal modeling with elastic webs for a number of situations [1–6]. A survey of the literature revealed that efforts to include thermal and hygral strains were cursory at best without a detailed analysis, except for the work of Brandenburg [1]. Brandenburg systematically applied well-known principles of continuum mechanics to investigate web tension models that include thermal effects from heated rollers and ambient air; he did not include hygral effects; heat transfer in the nip and free web span was modeled in the web transport direction by a single temperature equation similar to the approach taken in the lumped capacitance method. If one side of a web is exposed to a heated/cooled roller by means of a wrap on the roller and the other side to the surrounding, the temperature difference between the two ends of the web along the thickness is often large. Thus, heat transfer in the transverse direction (thickness) is of importance in the region of wrap. In the free web span, heat transfer takes place largely due to surface convection, and hence a lumped capacitance model in the transport direction is appropriate.

The paper is organized as follows. A mathematical model for web tension in a span by considering the thermal and hygral effects is developed first. A dynamic equation for mechanical strain is derived followed by the derivation of the web tension dynamic model. Both elastic and viscoelastic constitutive relations that relate mechanical strain and web tension are considered. Using the rule of mixtures on the mechanical and physical properties of the individual webs, properties of the laminated web are derived. Based on the mathematical model for single webs and composite properties of the laminated web, a dynamic model for web tension in the laminated web span is derived.

PRIMARY ASSUMPTIONS

To simplify the mathematics and make the modeling problem tractable for analysis, the following assumptions are made:

A1) The length of the contact region between the web material and rollers is negligible compared to the length of the free web span between the rollers.

A2) The web thickness is very small compared to the radius of the rollers.

A3) The weight of the material is neglected (sufficient tension in web spans to prevent sag between rollers).

A4) Each individual web material is isotropic, homogeneous and of uniform thickness.

A5) There is no change in the density and Young's modulus within a web span.

A6) The web material is in a state of uni-axial stress (machine direction stress prevails).

A7) The strains are small.

A8) The laminated web is perfectly bonded.

A9) The mechanical and physical properties of the laminated web (moduli, density, thermal and hygral expansion coefficients) are approximated using the rule of mixtures.

DYNAMICS OFWEBS WITH THERMAL AND HYGRAL EFFECTS

The dynamic behavior of a web between two rollers (span) is derived using mass balance in a control volume enclosing the length of the web span. The modeling is typically carried out in two steps: (i) finding a dynamic model of web strain and (ii) relating strain and tension via a constitutive relation and then using it to find tension dynamics; for example, Hooke's law is used to relate strain and tension under the assumption that the web is perfectly elastic. Since the process of lamination of several webs often includes one or more webs being heated/cooled using either hot/chilled rollers or via changes in temperature/moisture of the surrounding air, it is important to investigate modeling of single-web longitudinal dynamics by including thermal and hygral effects.

The goal of this section is to derive a dynamic model of web longitudinal behavior which includes strain induced by thermal and hygral effects in addition to the mechanical strain. Heat transfer via conduction along the thickness is assumed in the region of wrap on hot/chilled rollers whereas heat transfer by surface convection is assumed in the free web span. Moisture diffusion from the web surface within the span is modeled using Fick's law. The modeling procedure and relevant discussions are given in the following. The web mass in the control volume shown in Figure 1 is given by

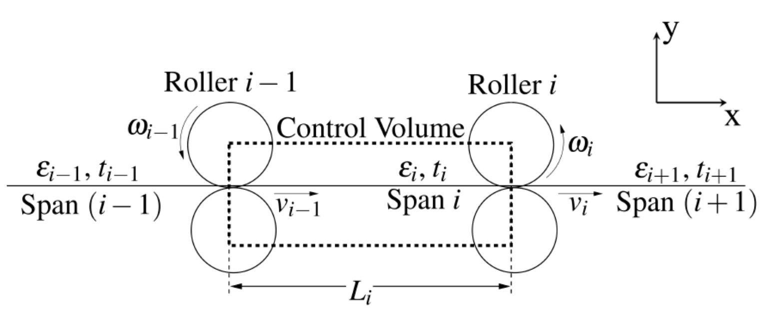

Figure 1: Control volume and roller-span notation

$$m_{cv} = \int_0^{L_i} \rho_{i.s}(x,t) A_{i.s}(x,t) dx \,. \qquad \{1\}$$

where the subscript s denotes the stretched state. The mass accumulated in the control volume during a time interval of Δt is given by

$$\Delta m_{cv} = \Delta\left[\int_0^{L_i} \rho_{i.s}(x,t) A_{i.s}(x,t) dx\right]. \qquad \{2\}$$

In the same time interval Δt the mass entering and leaving the control volume i given by $m_{i-1} = \rho_{i-1,s} A_{i-1,s} v_{i-1} \Delta t$ and $m_i = \rho_{i,s} A_{i,s} v_i \Delta t$, respectively. Application of mass balance in the control volume gives $\Delta m_{cv} = m_{i-1} - m_i$. Therefore,

$$\Delta\left[\int_0^{L_i} \rho_{i.s}(x,t) A_{i.s}(x,t) dx\right] = \rho_{i-1,s} A_{i-1,s} v_{i-1} \Delta t - \rho_{i,s} A_{i,s} v_i \Delta t \,. \qquad \{3\}$$

The mass of a representative volume element of the web in the stretched state is $m_{i,s} = \rho_{i,s} A_{i,s} \Delta x_{i,s}$ and in the unstretched state is $m_i = \rho_i A_i \Delta x_i$. Since $m_{i,s} = m_i$,

$$\frac{\rho_{i,s} A_{i,s}}{\rho_i A_i} = \frac{\Delta x_i}{\Delta x_{i,s}} = \frac{1}{1+\varepsilon_i(x,t)} \qquad \{4\}$$

where $\Delta x_{i,s} = (1+\varepsilon_i(x,t))\Delta x_i$ is used. Substitution of {4} into {3} results in

$$\Delta\left[\int_0^{L,i} \frac{\rho_i(x,t) A_i(x,t)}{1+\varepsilon_i(x,t)} dx\right] = \left[\frac{\rho_{i-1} A_{i-1} v_{i-1}}{1+\varepsilon_{i-1}} - \frac{\rho_i A_i v_i}{1+\varepsilon_i}\right] \Delta t \,. \qquad \{5\}$$

To simplify {5}, it is assumed that the change in the cross-sectional area and density within a span as functions of time and displacement in the transport direction are negligible; this is generally true in most cases except where the cross-sectional area changes are induced intentionally, such as in a drawing or hot rolling process. Under this assumption, dividing {5} by Δt and taking the limit as $\Delta t \to 0$, the following equation is obtained:

$$\frac{d}{dt}\left[\int_0^{L,i} \frac{dx}{1+\varepsilon_i(x,t)}\right] = \frac{v_{i-1}}{1+\varepsilon_{i-1}(t)} - \frac{v_i}{1+\varepsilon_i(t)} \,. \qquad \{6\}$$

Note that $\varepsilon_i(t)$ in the right-hand-side of the above equation is the strain at the exit end of the control volume, whereas, $\varepsilon_i(x,t)$ is the strain within the control volume which is a function of both x and t. Therefore, in the steady-state, the velocity-strain relationship between the two spans is given by

$$\frac{\bar{v}_i}{\bar{v}_{i-1}} - \frac{1+\bar{\varepsilon}_i}{1+\bar{\varepsilon}_{i-1}} \,. \qquad \{7\}$$

where $\bar{v}$ and $\bar{\varepsilon}$ denote the steady-state values of velocity and strain, respectively.

The total strain ($\varepsilon_i(x, t)$) is composed of three terms: tension-dependent, ε_{ti} , temperature-dependent, $\varepsilon_{\vartheta i}$, and moisture-dependent, ε_{hi}. Note that there is no stress associated with either thermal or hygral strains as they are purely dilational. To determine thermal strains, the web temperature at the end of the free web span will be computed based on the heat transfer in the region of wrap on the previous roller and the free web section. Hygral strain is determined based on the average moisture content in the web using Fickian diffusion.

Thermal Effects

To determine the temperature distribution J(x, t) within a span, we first consider heat transfer in the region of wrap on the roller and then evaluate the temperature distribution in the free span. Heat conduction in the region of wrap of the web over a hot/chilled roller is modeled by a one-dimensional heat equation in the transverse direction. In the free web span, heat transfer takes place primarily due to convective heat transfer from the surface with the surrounding and is modeled via a lumped capacitance model.

In the region of wrap, the heat equation is given by

$$K_\vartheta \frac{\partial^2 \vartheta}{\partial y^2} = \frac{\partial \vartheta}{\partial t}. \qquad \{8\}$$

The web is treated within the region of wrap as a slender rectangular bar with variable end temperatures; the heat transfer in the transport direction of the material is assumed to be negligible and hence ignored. Note that in many situations, due to web transport speed being much larger than the angle of wrap and roller radius, the resident time of a representative volume element (RVE) of the web on the roller is small. The initial and boundary conditions for the heat equation are the following:

$$\text{for } 0 < y < h, \qquad \vartheta(y,0) = \theta_{w,i-1},$$
$$\text{for } t > 0, \qquad \vartheta(0,t) = \theta_{U,i-1}, \vartheta(h,t) = \theta_{R,i-1},$$

where $\theta_{w,i-1}$ is the temperature of the web prior to engaging the ($i-1$)-th roller, $\theta_{R,i-1}$ is the temperature of the roller, and $\theta_{U,i-1}$ is the ambient temperature in the region of wrap. The solution of {8} with the above initial and boundary conditions is given by

$$\vartheta_{N,i-1}(y,t) = \theta_{U,i-1} + \frac{\theta_{R,i-1} - \theta_{U,i-1}}{h} y + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\theta_{w,i-1} - \theta_{U,i-1} + (-1)^n (\theta_{R,i-1} - \theta_{w,i-1})}{n} \sin \frac{n\pi y}{h} \exp\left(- \frac{-n^2 \pi^2 K_\vartheta t}{h^2} \right). \qquad \{9\}$$

The temperature distribution in a RVE at the beginning of the i-th web span ($x = 0$) is given by setting $t = R_{i-1}\phi_{i-1}/\bar{v}$, which is the resident time of a RVE on the roller. For short times ($t << 1$), the infinite sum in the above expression can be reasonably approximated by the first few terms ($n = 1,2$). One rough approximation is to simply include the first two terms of the solution, which means that the temperature distribution is assumed to be a linear function of the thickness. To make the problem more tractable, one can consider the temperature of the web at the beginning of the i-th web span to be an average evaluated over the thickness of the web, that is,

$$\theta_{N,i}(t) = \frac{1}{h}\int_0^h \vartheta_{N,i-1}(y,t)dy \ . \qquad \{10\}$$

Within the free web span, the heat transfer to the surrounding is assumed to be convective from the web surface and is modeled using a lumped capacitance model given by

$$\frac{d\vartheta_i(x,t)}{dt} := \frac{\partial\vartheta_i}{\partial t} + v(x,t)\frac{\partial\vartheta_i}{\partial x} = -\frac{1}{T_\vartheta}\left(\vartheta_i(x,t) - \vartheta_{U,i}(t)\right) \qquad \{11\}$$

where $T_\vartheta = \rho_{hc}/\gamma^*$ is the thermal time constant. To simplify the solution of the heat equation {11} it is assumed that the velocity $v(x, t) = \bar{v}$ is a constant and the ambient temperature $\vartheta_{U,i}(t)$ is a function of time only. With this assumption, the solution can be written as

$$\vartheta_i(x,t) = \vartheta_i(0,t)\exp\left(\frac{-x}{\bar{v}T_\vartheta}\right) + \left(1 - \exp\left(\frac{-x}{\bar{v}T_\vartheta}\right)\right)\vartheta_{U,i}(t) \qquad \{12\}$$

where the boundary condition $\vartheta_i(0,t) = \theta_{N,i}(t)$ is applied. Note that the temperature of an RVE at any location within the span depends on the time constant of that location $(x/\bar{v})$ and the thermal time constant. To simplify the analysis, the thermal strain is assumed to depend linearly on the change in temperature,

$$\varepsilon_{\vartheta i}(x,t) = \alpha\left(\vartheta_i(x,t) - \vartheta_0\right). \qquad \{13\}$$

Figure 2 gives the temperature notation in the web, roller, and ambient air at different locations.

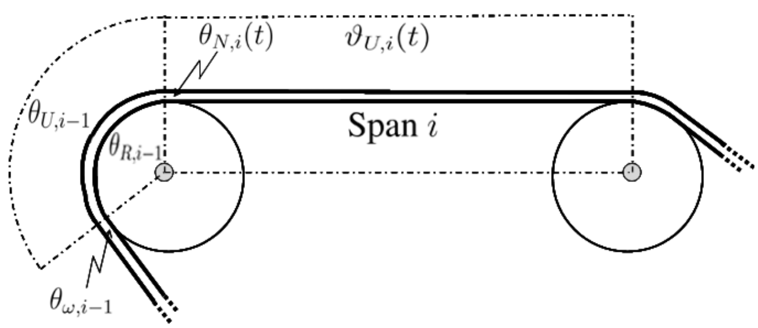

Figure 2: Temperature notation

<u>Hygral Effects</u>

Moisture diffusion in the laminate is assumed to follow Fick's law,

$$K_h\frac{\partial^2 M}{\partial y^2} = \frac{\partial M}{\partial t} \qquad \{14\}$$

where $M = H/\rho$ is the specific moisture concentration. Consider the following boundary conditions:

$$M = M_0 \text{ at } t = 0 \text{ for } 0 < y < h,$$
$$M = M_\infty \text{ at } t > 0 \text{ for } y = 0, y = h.$$

In the above boundary conditions, M_0 denotes initial moisture concentration in the web material and M_∞ is the concentration in the surrounding air. For these boundary conditions, the solution to {14} is

$$\frac{M - M_0}{M_\infty - M_0} = 1 - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\sin\frac{(2j+1)\pi y}{h}\exp\left(-\frac{(2j+1)^2\pi^2 K_h t}{h^2}\right). \qquad \{15\}$$

It is common to work with the average moisture content $\overline{M}$ defined by

$$\overline{M} = \frac{1}{h}\int_0^h M dx\,, \qquad \{16\}$$

instead of M, with the boundary conditions $\overline{M} = M_0$ at time $t = 0$ and $M = M_\infty$ at time $t = \infty$. The corresponding equation for $\overline{M}$ is

$$\frac{\overline{M} - M_0}{M_\infty - M_0} = 1 - \frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{\exp\left(-\frac{(2j+1)^2\pi^2 K_h t}{h^2}\right)}{(2j+1)^2}. \qquad \{17\}$$

Depending on the time horizon, one can approximate the infinite series in the above expression and obtain the following closed-form approximations:

$$\frac{\overline{M} - M_0}{M_\infty - M_0} = \begin{cases} 1 - \frac{8}{\pi^2}\exp\left(-\frac{\pi^2 k_h t}{h^2}\right) & \text{for large time } t, \\ 4\sqrt{\frac{\mathrm{K_h}}{\pi h^2}} & \text{for short time } t. \end{cases} \qquad \{18\}$$

The hygral strain is assumed to depend linearly on change in moisture concentration,

$$\varepsilon_{hi}(t) = \beta\left(\overline{M}_i(t) - M_0\right). \qquad \{19\}$$

Strain Dynamics

Equation {6} is further simplified by assuming that the strain is small ($\varepsilon << 1$) which can be used to approximate $1/(1 + \varepsilon)/(1 - \varepsilon)$. This approximation can be used in {6} either before or after evaluating the derivative leading to two slightly different approximations of strain dynamics [6]. Use of the small strain approximation prior to the evaluation of the integral and derivative in the left-hand-side of {6}, and assuming that the length L_i is constant, results in the following simplified strain dynamics:

$$\frac{d}{dt}\left(\int_0^{L_i}\varepsilon_i(x,t)dx\right)=v_i(1-\varepsilon_i)-v_{i-1}(1-\varepsilon_{i-1}). \qquad \{20\}$$

Substitution of $\varepsilon_i(x, t) = \varepsilon_{t,i}(t) + \varepsilon_{\vartheta,i}(x, t) + \varepsilon_{h,i}(t)$ into the dynamic equation, and upon simplification using the previously derived equations on thermal and hygral strains, results in

$$L_i\frac{d\varepsilon_{t,i}}{dt}=(v_i-v_{i-1})+(v_{i-1}\varepsilon_{t,i-1}-v_i\varepsilon_{t,i})+(v_{i-1}\varepsilon_{\vartheta,i-1}-v_i\varepsilon_{\vartheta,i})+(v_{i-1}\varepsilon_{h,i-1}-v_i\varepsilon_{h,i}) \\ -L_i\beta\frac{d\overline{M}_i}{dt}-\frac{d}{dt}\left(\int_0^{L_i}\varepsilon_{\vartheta,i}(x,t)dx\right). \qquad \{21\}$$

In deriving the above equation it was implicitly assumed that only thermal strain is spatially dependent on x. This is reasonable because the spatial gradient of the mechanical strain and hygral strain in the transport direction within a web span is negligible; this need not be true for long spans in metals where the web sags due to its own weight. The last term in the above dynamics can be written as follows:

$$\frac{d}{dt}\left(\int_0^{L_i}\varepsilon_{\vartheta,i}(x,t)dx\right)=\alpha\overline{v}T_\vartheta(1-\delta_i)\frac{d\vartheta_i(0,t)}{dt}+\alpha(L_i-\overline{v}T_\vartheta(1-\delta_i))\frac{d\vartheta_{U,i}(t)}{dt} \qquad \{22\}$$

where $\delta_i = \exp(-T_{w,i}/T_\vartheta)$ with $T_{w,i} = L_i/\overline{v}$. Substituting this expression into {21}, ignoring higher-order terms, and after some simplification, results in the following mechanical strain dynamics:

$$L_i\frac{d\varepsilon_{t,i}}{dt}=(v_i-v_{i-1})+\overline{v}(\varepsilon_{t,i-1}-\varepsilon_{t,i})+\overline{v}\beta(\overline{M}_{i-1}-\overline{M}_i)-L_i\beta\frac{d\overline{M}_i}{dt} \\ +\overline{v}\alpha(\vartheta_{i-1}(t)-\vartheta_i(t))-\alpha\overline{v}T_\vartheta(1-\delta_i)\frac{d\vartheta_i(0,t)}{dt} \\ -\alpha(L_i-\overline{v}T_\vartheta(1-\delta_i))\frac{d\vartheta_{U,i}(t)}{dt}. \qquad \{23\}$$

In the above strain dynamics the three time-derivatives in the right-hand-side can be readily obtained. The time-derivative of average moisture concentration can be evaluated using either {17} or one of the approximations given in {18}. Since average temperature is used at the beginning of the span ($x = 0$), $\vartheta_i(0, t)$ can be replaced by $\theta_{w,i}(t)$. Hence, $d\vartheta_i(0, t)/dt \approx d\theta_{N,i}(t)/dt$, which can be evaluated using {10}. The last derivative in the above equation is the time-derivative of the ambient air temperature which is an input to the system, and hence known.

Tension Dynamics

We can relate web tension and mechanical strain by assuming the web to be either elastic or viscoelastic.

- For elastic webs, one can model using a linear constitutive relation between strain and tension,

$$\varepsilon_{t,i} = \frac{t_i}{EA}.$$

- For viscoelastic webs, a simple model that consists of a Maxwell model (spring and damper in series) and a linear spring in can be used (Fig. 3 shows the mechanical analog); this model has been known to agree reasonably well with observed viscoelastic response. The stress-strain relationship for this model is

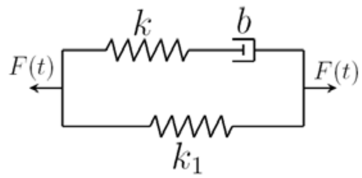

Figure 3: Viscoelastic model (Maxwell model and a linear spring in parallel)

given by

$$p_0\sigma_i + p_1\dot{\sigma}_i + q_0\varepsilon_i + q_1\dot{\varepsilon}_i \qquad \{24\}$$

where $p_0 = 1/b$, $p_1 = 1/k$, $q_0 = k_1/b$, $q_1 = (1+k_1/k)$ and $\sigma_i = t_i/A$.

The web tension dynamics under the assumption that the web is elastic is given by

$$\frac{L_i}{EA}\frac{dt_i}{dt} = (v_i - v_{i-1}) + \frac{\bar{v}}{EA}(t_{i-1} - t_i) + f_{h,i}(t) + f_{\vartheta,i}(t) \qquad \{25\}$$

where

$$f_{h,i}(t) = \bar{v}\beta(\overline{M}_{i-1} - \overline{M}_i) - L_i\beta\frac{d\overline{M}_i}{dt} \qquad \{26\}$$

$$f_{\vartheta,i}(t) = \bar{v}\alpha(\vartheta_{i-1}(t) - \vartheta_i(t)) - \alpha\bar{v}T_\vartheta(1-\delta_i)\frac{d\vartheta_i(0,t)}{dt} - \alpha(L_i - \bar{v}T_\vartheta(1-\delta_i))\frac{d\vartheta_{U,i}(t)}{dt}. \qquad \{27\}$$

If the web is viscoelastic, one can obtain the web tension dynamics by using {24}.

MECHANICAL AND PHYSICAL PROPERTIES OFWEB LAMINATES

Consider the web lamina consisting of two materials A and B as shown in Fig. 4. A simple rule-of-mixtures approach is used to develop the mechanical and physical properties of the laminate. A constant stress or constant strain condition can be used to determine the modulus of elasticity of the web lamina in the *x* and *y* directions. Since a uniform displacement is observed, and required, in the machine direction (*x*-direction) for perfect bonding between the two layers, a constant strain approximation is considered for determining the modulus of elasticity in the machine direction. Since the width of the two layers is the same, the total stress in the lamina in the transport direction is given by

$$\sigma_{cx} = \frac{\sigma_{Ax} h_A + \sigma_{Bx} h_B}{h_A + h_B}. \qquad \{28\}$$

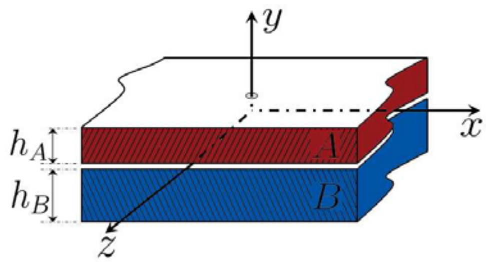

Figure 4: Lamination of two webs

Uniform strain model in the *x* direction gives $\varepsilon_{cx} = \varepsilon_{Ax} = \varepsilon_{Bx}$. The stress in individual layers can be expressed as $\sigma_{Ax} = E_A \varepsilon_{Ax}$ and $\sigma_{Bx} = E_B \varepsilon_{Bx}$. Therefore the modulus of elasticity of the lamina in the transport direction is given by

$$E_{cx} = \frac{\sigma_{cx}}{\varepsilon_{cx}} = \frac{E_A h_A + E_B h_B}{h_A + h_B}. \qquad \{29\}$$

Assuming constant stress condition in the *y*-direction gives the modulus of elasticity as

$$E_{cy} = \frac{E_A E_B (h_A + h_B)}{E_A h_A + E_B h_B}. \qquad \{30\}$$

The physical properties such as the density, thermal and hygral coefficients are derived in a similar manner using the rule of mixtures. The equivalent density of the lamina is given by

$$\rho_c = \frac{\text{Total mass per unit length}}{\text{Total volume per unit length}} = \frac{\rho_A h_A w + \rho_B h_B w}{(h_1 + h_2) w} = \frac{\rho_A h_A + \rho_B h_B}{h_A + h_B}. \qquad \{31\}$$

For a lamina consisting of isotropic layers, the thermal expansion coefficients are given by [7]

$$\alpha_{cx} = \frac{E_A \alpha_A h_A + E_B \alpha_B h_B}{E_A h_A + E_B h_B} \qquad \{32\}$$

$$\alpha_{cy} = \alpha_A v_A (1 + v_A) + \alpha_B v_B (1 + v_B) - \alpha_V (v_A^2 + v_B^2) \qquad \{33\}$$

For hygral expansion coefficients β_{cx} and β_{cy}, the quantity a in the above two equations is replaced by β.

DYNAMICS OF A LAMINATED WEB

Application of the law of conservation of mass to a control volume of the laminated web span between two rollers (see Fig. 5) results in

$$\frac{d}{dt}\left[\int_0^{L_c} \rho_{c,s}(x,t) A_{c,s}(x,t)\,dx\right] = \left(\rho_{A,s} A_{A,s} v_1 + \rho_{B,s} A_{B,s} v_1\right) - \rho_{c,s} A_{c,s} v_2 \qquad \{34\}$$

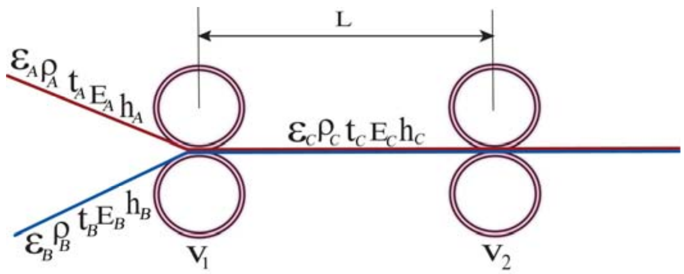

Figure 5: Lamination of two webs

The equations corresponding to {4} for the individual web layers and the composite web are

$$\frac{\rho_{A,s} A_{A,s}}{\rho_A A_A} = \frac{1}{1+\varepsilon_A(t)}, \quad \frac{\rho_{B,s} A_{B,s}}{\rho_B A_B} = \frac{1}{1+\varepsilon_B(t)}, \quad \text{and} \quad \frac{\rho_{c,s} A_{c,s}}{\rho_c A_c} = \frac{1}{1+\varepsilon_c(x,t)} \qquad \{35\}$$

Therefore, equation {34} can be written as

$$\frac{d}{dt}\left[\int_0^{L_c} \frac{\rho_c(x,t) A_c(x,t)}{1+\varepsilon_c(x,t)}\,dx\right] = \frac{\rho_A A_A v_1}{1+\varepsilon_A(t)} + \frac{\rho_B A_B v_1}{1+\varepsilon_B(t)} + \frac{\rho_c A_c v_2}{1+\varepsilon_c(t)} \qquad \{36\}$$

Assuming within any span $\rho_c(x, t) = \rho_c$, $Ac(x, t) = A_c$, and similarly for the individual web layers,

$$\rho_c A_c \frac{d}{dt}\left[\int_0^{L_c} \frac{1}{1+\varepsilon_c(x,t)}\,dx\right] = \frac{\rho_A A_A v_1}{1+\varepsilon_A(t)} + \frac{\rho_B A_B v_1}{1+\varepsilon_B(t)} - \frac{\rho_c A_c v_2}{1+\varepsilon_c(t)} \qquad \{37\}$$

Assuming small strain, $1 - \varepsilon^2 \approx 1$,

$$\begin{aligned}\rho_c A_c \frac{d}{dt}\left[\int_0^{L_c} \left(1-\varepsilon_c(x,t)\right)dx\right] &= \left(\rho_A A_A v_1\right)\left(1-\varepsilon_A(t)\right) + \left(\rho_B A_B v_1\right)\left(1-\varepsilon_B(t)\right) \\ &\quad - \left(\rho_c A_c v_2\right)\left(1-\varepsilon_c(t)\right)\end{aligned} \qquad \{38\}$$

Expansion of the left-hand-side of the above equation with $\varepsilon_c(x, t) = \varepsilon_{t,c}(t) + \varepsilon_{h,c}(t) + \varepsilon_{\vartheta,c}(x, t)$, and simplifying results in the following equation:

$$L_c \frac{d\varepsilon_{t,c}(t)}{dt} = (v_2 - v_1) + \left(\frac{\rho_A h_A \varepsilon_A(t) + \rho_B h_B \varepsilon_B(t)}{\rho_A h_A + \rho_B h_B}\right) v_1 - \varepsilon_c(t) v_2$$
$$- \beta_c \frac{d\overline{M}_c}{dt} - \frac{d}{dt}\left[\int_0^{L_c} \varepsilon_{\vartheta,c}(x,t)dx\right]. \qquad \{39\}$$

The last term of the above equation can be written as

$$\frac{d}{dt}\left[\int_0^{L_c} \varepsilon_{\vartheta,c}(x,t)dx\right] = \alpha_c T_{\vartheta,c}\bar{v}(1-\delta_c)\frac{d\vartheta_c(0,t)}{dt} + \alpha_c\left(L_c - T_{\vartheta c}\bar{v}(1-\delta_c)\right)\frac{d\vartheta_{U,c}(t)}{dt}. \qquad \{40\}$$

Substitution of {40} into {39}, and replacing the velocity by average velocity $\bar{v}$ in the terms involving the product of strain and velocity, results in

$$L_c \frac{d\varepsilon_{t,c}(t)}{dt} = (v_2 - v_1) + \left(\frac{\rho_A h_A \varepsilon_{t,A}(t) + \rho_B h_B \varepsilon_{t,B}(t)}{\rho_A h_A + \rho_B h_B}\right)\bar{v} - \left(\varepsilon_{h,c}(t) + \varepsilon_{\vartheta,c}(t) + \varepsilon_{t,c}(t)\right)\bar{v}$$
$$+ \left(\frac{\rho_A h_A\left(\varepsilon_{h,A}(t) + \varepsilon_{\vartheta,A}(t)\right) + \rho_B h_B\left(\varepsilon_{h,B}(t) + \varepsilon_{\vartheta,B}(t)\right)}{\rho_A h_A + \rho_B h_B}\right)\bar{v} - \beta_c \frac{d\overline{M}_c}{dt} \qquad \{41\}$$
$$- \alpha_c T_{\vartheta c}\bar{v}(1-\delta_c)\frac{d\vartheta_c(0,t)}{dt} - \alpha_c\left(L_c - T_{\vartheta c}\bar{v}(1-\delta_c)\right)\frac{d\vartheta_{U,c}(t)}{dt}.$$

Laminated Web Tension Dynamics

Assuming that the individual layers as well as the laminate to be linearly elastic, that is,

$$\varepsilon_{t,c}(t) = \frac{t_c(t)}{E_c A_c}, \quad \varepsilon_{t,A}(t) = \frac{t_A(t)}{E_A A_A} \quad \text{and} \quad \varepsilon_{t,B}(t) = \frac{t_B(t)}{E_B A_B},$$

the web tension dynamics in the laminated span is given by

$$\frac{dt_c}{dt} = \frac{E_c A_c}{L_c}(v_2 - v_1) + \frac{E_c A_c}{L_c}\left(\frac{\frac{\rho_A h_A}{E_A A_A} t_A + \frac{\rho_B h_B}{E_B A_B} t_B}{\rho_A h_A + \rho_B h_B}\right)\bar{v} - \frac{t_c \bar{v}}{L_c}$$
$$+ \left(\frac{\rho_A h_A\left(\varepsilon_{h,A}(t) + \varepsilon_{\vartheta,A}(t)\right) + \rho_B h_B\left(\varepsilon_{h,A}(t) + \varepsilon_{\vartheta,A}(t)\right)}{\rho_A h_A + \rho_B h_B}\right)\bar{v} - \left(\varepsilon_{h,c}(t) + \varepsilon_{\vartheta,c}(t)\right)\bar{v} \qquad \{42\}$$
$$- \beta_c \frac{d\overline{M}_c}{dt} - \alpha_c T_{\vartheta c}\bar{v}(1-\delta_c)\frac{d\vartheta_c(0,t)}{dt} - \alpha_c\left(L_c - T_{\vartheta c}\bar{v}(1-\delta_c)\right)\frac{d\vartheta_{U,c}(t)}{dt}.$$

Using $E_c = \frac{E_A h_A + E_B h_B}{h_A + h_B}$ and $A = (h_A + h_B)w$ in the above equation, and simplifying

$$
\begin{aligned}
\frac{dt_c}{dt} &= \frac{\left(E_A h_A + E_B h_B\right)w}{L_c}\left(v_2 - v_1\right) + \frac{1}{L_c}\left(\frac{E_A h_A + E_B h_B}{E_A w}\right)\left(\frac{\rho_A h_A}{\rho_A h_A + \rho_B h_B}\right)t_A \overline{v} \\
&+ \frac{1}{L_C}\left(\frac{E_A h_A + E_B h_B}{E_B w}\right)\left(\frac{\rho_B h_B}{\rho_A h_A + \rho_B h_B}\right)t_B \overline{v} - \frac{t_c \overline{v}}{L} \\
&+ \left(\frac{\rho_A h_A\left(\varepsilon_{h,A}(t) + \varepsilon_{\vartheta,A}(t)\right) + \rho_B h_B\left(\varepsilon_{h,A}(t) + \varepsilon_{\vartheta,A}(t)\right)}{\rho_A h_A + \rho_B h_B}\right)\overline{v} - \left(\varepsilon_{h,c}(t) + \varepsilon_{\vartheta,c}(t)\right)\overline{v} \\
&- \beta_c \frac{d\overline{M}_c}{dt} - \alpha_c T_{\vartheta c}\overline{v}\left(1 - \delta_c\right)\frac{d\vartheta_c(0,t)}{dt} - \alpha_c\left(L_c - T_{\vartheta c}\overline{v}\left(1 - \delta_c\right)\right)\frac{d\vartheta_{U,c}(t)}{dt}.
\end{aligned}
\qquad \{43\}
$$

The tension dynamic model is complete in the sense that it contains both the thermal and hygral effects in the individual web layers as well as the composite web; it also contains thermal effects of a heated laminator roll. Since the developed model is transparent in terms of different effects, based on a specific process, the machine designer/engineer can selectively choose thermal and hygral effects in certain spans of the web line. The upstream conditions (stresses and strains) of the individual webs in the process line that form the laminate are crucial for the lamination process. For example, if the final product of the laminate that is produced in the rolled form is to be cut into flat rectangular sheets, then one would require that the resulting laminate in the unstretched state be flat without curling. One way to ensure this is to maintain equal strains in the incoming web layers, that is, ensure that the tensions in the two individual webs prior to the lamination roller satisfy the relation $t_A/t_B = (E_A h_A)/(E_B h_B)$ (this ignores thermal and hygral effects). To include thermal and hygral effects, one has to equate strains in both layers and get a relationship between tensions t_A and t_B.

CONCLUSION

A dynamic model for web strain in a span that includes thermal and hygral effects was developed. Web strain and tension can be related by assuming the web to be elastic or viscoelastic; corresponding constitutive relations for elastic and viscoelastic cases are given and tension dynamics are derived based on these relations. Using the notion of rule-of-mixtures and the mechanical and physical properties of individual webs, properties of the laminated web were obtained. Based on the developed single web model and the properties of the composite web, a dynamic model for the laminated web span adjoining the laminator rolls is derived. Future research should focus on control of the laminator rolls in conjunction with control of individual webs upstream of the laminator rolls to obtain a laminated web required properties and desired quality.

ACKNOWLEDGEMENTS

This work was supported in part by the Web Handling Research Center at Oklahoma State University, Stillwater.

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P. R. Pagilla, K. N. Reid & J. Newton, Oklahoma State University, USA

Name & Affiliation	**Question**
Unknown	Thank you, very good presentation. My question concerning the heat transfer and how you take it into account. Why do you choose to take into account the thermal dependence of the strains in the webs but you ignore the effects of temperature in the viscoelactic model?
Name & Affiliation	**Answer**
P.R. Pagilla, Oklahoma State Univerity	We did consider that. That's why I separated all of this, step by step, so that other viscoelastic behaviors could be introduced. So if temperature dependence is needed in the viscoelastic behavior it is easily added. According to the literature the viscoelasticity is nonlinear as well.
Name & Affiliation	**Question**
Unknown	What is the nature of the contact between the rollers and the web itself? I there some thermal contact resistance defined, how can you quantify that resistance?
Name & Affiliation	**Answer**
P.R. Pagilla, Oklahoma State Univerity	I assume no contact resistance between the roller surface and the web when I do that analysis. But you can include that. That may not be that big of a problem. The problem comes in when you have a heated roll and the web is already laminated. Then there is a contact resistance between two layers of the web that must be considered. Most of the heat transfer books discuss this.
Name & Affiliation	**Question**
Sinan Muftu, Northeastern University	Very good model. I have a quick question about the heat transfer model. I think you considered conduction in the radial direction, but then you have two boundary conditions on the entering and exiting spans. When you were solving the equations, did you consider that too? I didn't understand how you handled that.
Name & Affiliation	**Answer**
P.R. Pagilla, Oklahoma State University	Yes there must be boundary conditions for that heat equation in the transfer direction. These boundary conditions affect the solution. I need to know the temperature of the web is before it got heated from the roller.
Name & Affiliation	**Question**
Paul Fussey, AET Films	One of the issues in laminating, if you are trying to apply this to real world situations, is the change in physical properties of the materials as they heat up and as they dry. We manufacture polypropylene film, and you go through a slight expansion, then you go into some rapid and almost disastrous shrink at some point. Young's modulus changes significantly with temperature. Many products undergo

	layer lamination, through an extrusion process, where it is heated to several hundred degrees, and all of a sudden the modulus is dropped, you're getting strain from the film trying to shrink in. Is there any thought of expanding your model to figure out how these material complexities affect the quality of the laminate?
Name & Affiliation	**Answer**
P.R. Pagilla, Oklahoma State University	Yes there is. We look to the sponsors of the Web Handling Research Center when we produce models such as these to give us input regarding important parameters and effects that should be included to make the models both realistic and useful.