SHEAR EFFECTS AND LATERAL DYNAMICS OF IMPERFECT WEBS

by

J. E. Olsen
SINTEF Materials Technology
NORWAY

A mathematical model describing the lateral dynamics of an imperfect web has been derived. The model includes the effect of shear forces. This was enabled by the development of a theory for beams of inhomogeneous materials. Calculations show that an inhomogeneous beam or a web with widthwise variations in material properties will bend towards the low tension side. The shear effect is significant.

NOMENCLATURE

A	cross-sectional area
E	MD modulus
H	secondary coefficient
h	web thickness
I	moment of inertia
K	web parameter
L	span length
M	applied bending moment
N	shear force
n	shear coefficient
m_i	secondary coefficient
q	lateral load
S	web stiffness
S_0	secondary coefficient
T	axial force
t	time
t_i	secondary coefficient
u	lateral deflection
V_x	MD velocity of web
x	MD position
y	CD position
z	lateral position of downstream roller
γ	shear angle
γ_i	frozen-in shear angle
ε	strain
ε_c	strain at centroid
ε_i	frozen-in strain
θ	angle
κ	curvature
κ_L	selfinduced curvature
σ	stress
Ω	secondary coefficient

INTRODUCTION

A web is a thin, continuous, flexible strip of material like paper, metal foil, plastic film or textiles. In production and converting processes the web is transported through a web line such as a coater, annealing line, paper machine or printing press. In many of these processes it is important to have strict control of lateral position. In a four color printing unit, the different colors will not match on top of each other if lateral movement is out of control.

A web moving through a web line is influenced by forces and bending moments transferred from rollers in contact with the web. As can be described by beam theory, the web moves sideways if a bending moment is applied to it. However, it has also been observed that a web may shift sideways even if no bending moment is applied. It is believed that this sideways shift is caused by widthwise variations in material properties. These variations may be variations in stiffness or in frozen-in strain. The latter is the cause of baggy webs. A web with widthwise variations in material properties is referred to as a nonuniform web or an imperfect web.

Deflection without applied bending moment can not be explained by elementary beam theory. A more general beam theory was thus derived by the author[1]. The theory verified that widthwise variations in material properties cause a sideways shift. The theory did not account for shear effects and dynamical behaviour. These effects are incorporated below.

BEAM THEORY

A beam parallel to the x-axis, with the beam axis defined as the centroidal axis, may deflect in the y-direction. The deflection is caused by the forces acting on the beam. The relation between the axial force T, the shear force N, the bending moment M and a lateral load q can be found from force balances as indicated in Fig.1. A force balance in the y-direction yields

$$q = \frac{\partial N}{\partial x} + T\frac{\partial^2 u}{\partial x^2} \tag{1}$$

and a balance of moments yields

$$N = \frac{\partial M}{\partial x} \tag{2}$$

as shown by Timoshenko [2]. Eqs.(1) and (2) can be combined into the following equation

$$\frac{\partial^2 M}{\partial x^2} + T\frac{\partial^2 u}{\partial x^2} = q \tag{3}$$

In the following we will express the moment M in terms of the deflection u in order to obtain a differential equation for the deflection.

Bernoulli's hypothesis of deformation implies that the strain parallel to the beam axis varies linearly with coordinate y. Mathematically we can express this with the following equation:

$$\varepsilon(y) = \varepsilon_c + \kappa y \tag{4}$$

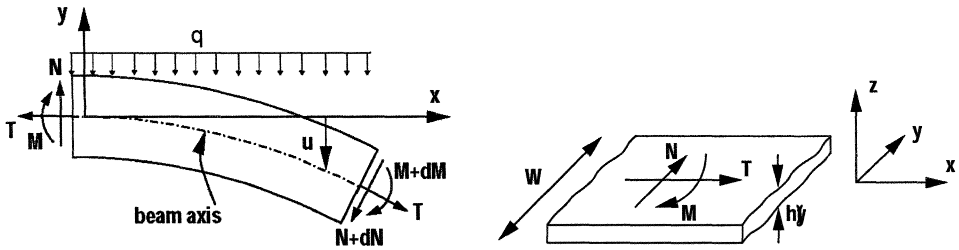
Here ε_c is the strain at the centroid of the cross sectional area of the beam and κ is the curvature of the beam. The curvature is defined by:

$$\kappa \equiv -\frac{\partial^2 u}{\partial x^2} \tag{5}$$

For a linear elastic material the stress field is thus

$$\sigma(y) = E(y)\varepsilon_c + E(y)\kappa y \tag{6}$$

Here E is the elastic modulus in the direction of the beam axis. Generally it may vary with widthwise position y.



(a) (b)

Fig.1: Deflected beam with applied forces (a) and web with beam forces (b).

Due to different reasons, a web may have frozen-in strains, sometimes referred to as initial strain or bagginess. Frozen-in strains are not accounted for by elementary beam theory. From the definition of the elastic modulus, we get the following generalization of the Hookes law:

$$\sigma = E\,(\varepsilon - \varepsilon_i) \tag{7}$$

where ε_i is frozen-in strain. For a beam with frozen-in strains, the stress field is thus

$$\sigma(y) = E(y)\varepsilon_c - E(y)\varepsilon_i(y) + E(y)\kappa y \tag{8}$$

The axial force T and the bending moment about the z-axis M are given by

$$T = \int_A \sigma(y)\, dA = \int \sigma(y)\, h(y)\, dy \tag{9}$$

and

$$M = \int_A \sigma(y) y\, dA = \int \sigma(y)\, y\, h(y)\, dy \tag{10}$$

The integrals are carried out over the width of the web. Inserting Eq.(8) in Eqs.(9) and (10) yields

$$T + t_i = \varepsilon_c S_0 + \kappa H \tag{11}$$

$$M + m_i = \varepsilon_c H + \kappa \Omega \tag{12}$$

where

$$t_i = \int \varepsilon_i\, E\, h\, dy \qquad m_i = \int \varepsilon_i\, E\, y\, h\, dy \tag{13}$$

$$S_0 = \int E\, h\, dy \qquad H = \int E\, y\, h\, dy \qquad \Omega = \int E\, y^2\, h\, dy$$

Thickness h, frozen-in strain ε_i and modulus E are generally functions of the widthwise position y. The coefficients defined by Eqs.(13) will herafter be referred to as the secondary coefficients.

In terms of ε_c and κ, Eqs.(11) and (12) reduce to the following expressions:

$$\varepsilon_c = \frac{(T + t_i)\Omega - (M + m_i)H}{\Omega S_0 - H^2} \tag{14}$$

$$\kappa = \frac{(M + m_i)S_0 - (T + t_i)H}{\Omega S_0 - H^2} \tag{15}$$

For a perfect homogeneous beam ($E(y) = E_0$) with constant thickness and no frozen-in strain (ε_i=0), Eq.(15) is reduced to

$$\kappa = \frac{M}{E_0 I} \tag{16}$$

which is the expression found from *elementary* beam theory.

The above equations do not account for the effect of shear deflections. Only deflection due to bending is concidered. The deflection due to shear u_s is given by the definition of the shear angle

$$\frac{\partial u_s}{\partial x} = \frac{nN}{AG_c} + \gamma_i \tag{17}$$

where N is shear force, A is the area of the cross section, G is the shear modulus, n is the shear coefficient from Timoshenko's beam theory [2] and γ_i is the frozen-in shear strain. We see that the shear effect on an imperfect web has a contribution from shear force and frozen-in shear strain. A perfect web has no frozen-in shear, and thus the shear effect on an imperfect web might be different from the shear effect on a perfect web. Appendix A discusses the shear coefficient, and the frozen-in shear is given by conditions of compatibility:

$$\gamma_i = x\frac{\partial \varepsilon_i}{\partial y} \tag{18}$$

The derivative of Eq.(17) with Eq.(1) inserted gives an expression for the curvature due to shear. The total curvature is then

$$\kappa = \frac{(M{+}m_i)S_0 - (T{+}t_i)H}{\Omega S_0 - H^2} + \frac{nT}{AG_c}\frac{\partial^2 u}{\partial x^2} - \frac{\partial \varepsilon_i}{\partial y} - \frac{nq}{AG} \tag{19}$$

or

$$\kappa = \frac{(M{+}m_i)S_0 - (T{+}t_i)H - (\Omega S_0{-}H^2)\left(\frac{\partial \varepsilon_i}{\partial y} + \frac{nq}{AG}\right)}{(\Omega S_0{-}H^2)\left(1{+}\frac{nT}{AG_c}\right)} \tag{20}$$

We can organize Eq.(20) with respect to the moment M. Then we have the moment expressed by the deflection. This expression is derivated twice and the result is inserted into Eq.(3). If we define

$$K^2 = \frac{TS_0}{(\Omega S_0 - H^2)\,(1 + nT/AG_c)} \tag{21}$$

we get the following differential equation for the deflection

$$\frac{\partial^4 u}{\partial x^4} - K^2\frac{\partial^2 u}{\partial x^2} = \frac{K^2}{T}q(x) \tag{22}$$

This differential equation needs four boundary conditions in order to constitute a well-posed problem. In the following section these are derived for a web moving between two rollers in a web line.

LATERAL DYNAMICS OF A WEB

In web handling terminology, the deflection in the y-direction is referred to as lateral deflection or lateral movement. The phenomenon is described by the differential equation above and four bondary conditions. A web moving between two cylinders or rollers as illustrated in Fig.2, will satisfy two boundary conditions at each roller. If the coordinate system is defined so that the upstream roller is positioned normal to MD we have

$$u = 0 \qquad \text{at } x = 0 \tag{23}$$

and

$$\frac{\partial u}{\partial x} = \gamma_o \qquad \text{at } x = 0 \tag{24}$$

where γ_o is the shear angle at the upstream roller. The shear angle, Eq.(17), is proportional to the shear force

$$N = -\frac{T}{K^2}\frac{\partial^2 u}{\partial x^2} \tag{25}$$

as derived from Eqs.(2), (20) and (21).

At the downstream roller we assume that there is contact between the web and the roller. That implies that the surface particles of the web must move in the same direction as the surface particles of the roller. Based on this Shelton [3] derived the following dynamic boundary condition at the downstream roller

$$\frac{\partial u}{\partial x} = \theta + \frac{1}{V_x}\frac{\partial z}{\partial t} - \frac{1}{V_x}\frac{\partial u}{\partial t} \qquad \text{at } x = L \tag{26}$$

Here V_x is the MD velocity of the web and z is the lateral position of the downstream roller.

Historically there has been some controversy regarding the fourth boundary condition. A solution to the debate was provided by the author [1] for static conditions. For imperfect webs under static conditions we have

$$\frac{\partial^2 u}{\partial x^2} = -\kappa_L \tag{27}$$

where κ_L is the self-induced curvature due to widthwise variations in web properties. It is found from Eq.(20) with $M = 0$. For a perfect web under dynamic conditions Shelton [3] derived the following

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{V_x^2}\frac{d^2 u}{dt^2} - \frac{1}{V_x^2}\frac{d^2 z}{dt^2} - \frac{1}{V_x}\frac{d\gamma}{dt} \qquad \text{at } x = L \tag{28}$$

We assume that the dynamic condition for an imperfect web is the added contributions from the static conditions of an imperfect web and dynamic conditions for a perfect web. Thus we get

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = \frac{1}{V_x^2}\frac{d^2 u_L}{dt^2} - \kappa_L - \frac{1}{V_x^2}\frac{d^2 z}{dt^2} - \frac{1}{V_x}\frac{d\gamma_L}{dt} \tag{29}$$

as the fourth boundary condition.

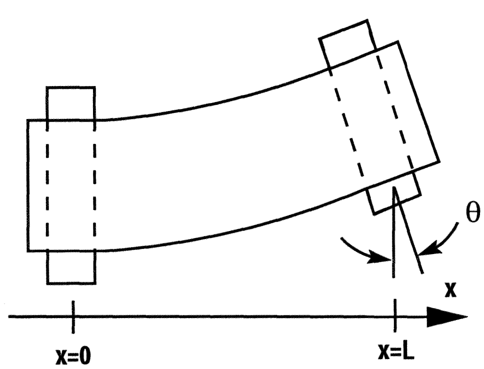


Fig.2: Web moving between rollers.

NUMERICAL SOLUTION

The differential equation, Eq.(22), with its boundary conditions, Eqs.(23), (24), (26) and (29), has an analytical solution eqivalent to that of a perfect web [3] if K is a constant. In most cases K is a constant, but sometimes it may vary with MD position, x. Thus a numerical solution has been developed.

The length between the two rollers or cylinders is devided into n_x increments of length Δx and with discretisized positions denoted by x_j. A time increment Δt separates the times t_i at which calculations are carried out. The first and second boundary condtitions, Eqs.(23) and (24), yield

$$u_{1,i} = 0 \tag{30}$$

and

$$0 = \tfrac{nT}{AG_c} u_{4,i} - \tfrac{3nT}{AG_c} u_{3,i} + \left(K^2 \Delta x^2 + \tfrac{3nT}{AG_c}\right) u_{2,i} - \left(K^2 \Delta x^2 + \tfrac{nT}{AG_c}\right) u_{1,i} \tag{31}$$

The differential equation, Eq.(22), is numerically expressed by

$$0 = u_{j-2,i} - (4 + K^2 \Delta x^2) u_{j-1,i} + (6 + 2K^2 \Delta x^2) u_{j,i} - (4 + K^2 \Delta x^2) u_{j+1,i} + u_{j+2,i} \tag{32}$$

and applies to $2 < j < n_x - 2$. The boundary conditions at the downstream roller, Eqs.(26) and (29), yield

$$(V_x \Delta t + \Delta x) u_{n_x,i} - V_x \Delta t \, u_{n_x-1,i} = V_x \Delta t \Delta x \, \theta + \Delta t \Delta x \, \dot{z}_i + \Delta x \, u_{n_x,i-1} \tag{33}$$

and

$$\begin{aligned} &\left(\tfrac{1}{\Delta x^2} - \tfrac{1}{V_x^2 \Delta t^2} + F_s\right) u_{n_x,i} - \left(\tfrac{2}{\Delta x^2} + 3F_s\right) u_{n_x-1,i} + \left(\tfrac{1}{\Delta x^2} + 3F_s\right) u_{n_x-2,i} - 3F_s u_{n_x-3,i} = \\ &\quad \left(F_s - \tfrac{2}{V_x^2 \Delta t^2}\right) u_{n_x,i-1} - F_s (3u_{n_x-1,i-1} - 3u_{n_x-2,i-1} + u_{n_x-3,i-1}) \\ &\quad + \tfrac{1}{V_x^2 \Delta t^2} u_{n_x,i-2} + \frac{L}{V_x} \frac{\partial}{\partial t} \frac{\partial \varepsilon_i}{\partial y} - \kappa_L - \frac{\ddot{z}}{V_x^2} \end{aligned} \tag{34}$$

where F_s is

$$F_s = \frac{n_x T}{K^2 V_x A G_c \Delta x^3 \Delta t} \tag{35}$$

For each time step we get a set of n_x equations with n_x unknowns which has a straightforward solution. The solution of each time step depends upon the solution of the previous two time steps.

NUMERICAL EXAMPLES

As an example we will make calculations for a paper web moving between two rollers. In the paper industry widthwise variations in properties are known as CD profiles (cross directional profiles). Idealized representations of typical CD profiles for stiffness and frozen-in strain in edge rolls are seen in Fig.3. Note that stiffness is a term used in the paper industry for the product of elastic modulus and paper thickness. In all expressions above, the modulus occurs together with thickness as a product. Thus it is straightforward to implement the terminology of the paper industry. Widthwise variations in stiffness $S = Eh$ have been modeled with the expression

$$S = E_0 h_0 \left(1 - \{1 - F_E\} \frac{e^{B_E y} + e^{-B_E W/2}}{2 \sinh(B_E W/2)}\right) \tag{36}$$

Here E_0 and h_0 is the modulus and thickness at the left side ($y = -W/2$) of the web, F_E is the ratio between the stiffness at the right hand side and the left hand side of the paper web, B_E is a factor describing the shape of the variations and W is the width of the web. Similarly the

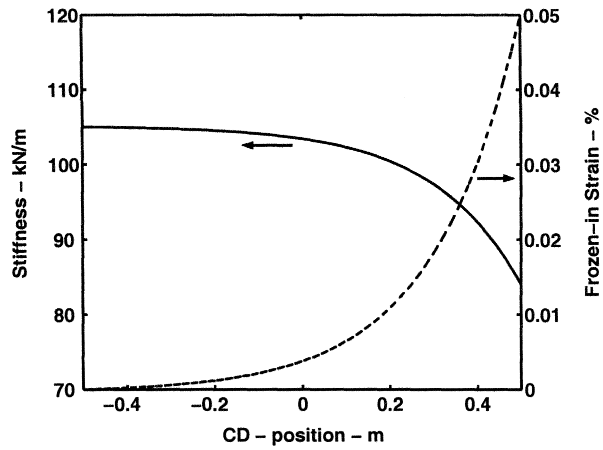


Fig.3: Stiffness and frozen-in strain as functions of CD-position.

frozen-in strain is modeled by

$$\varepsilon_i = \frac{\Delta\varepsilon_i}{2\sinh(B_\varepsilon W/2)}\left\{e^{B_\varepsilon y} - e^{-B_\varepsilon W/2}\right\} \tag{37}$$

where $\Delta\varepsilon_i$ is the change in frozen-in strain over the entire width of the web and B_ε is a factor describing the shape of the variations. Note that low values (≤ 0.1) of B_E and B_ε yield linear profiles.

For the numerical examples we have chosen parameters as given in Table 1. Note that axial force T is web line tension T_w multiplied by web width W. With these parameters we can calculate the secondary coefficients of Eqs.(13) for different values of $\Delta\varepsilon_i$ and F_E. We have chosen $B_\varepsilon = 1$ and $B_E = 5$. Results for ten different cases are seen in Table 2 and *Case 5* is plotted in Fig.3. *Case 10* is quite unrealistic in terms of paper webs, but it demonstrates possible values of secondary coefficients at extreme conditions. We see that S_0, H and Ω only depend upon variations in elastic modulus, wheras t_i and m_i is also influenced by frozen-in strain. This is consistent with their definitions, Eqs.(13). The shear coefficient n is not affected by variations in frozen-in strain. Thus an imperfect web with variations in frozen-in strain (camber) and no variations in elastic modulus has a shear coefficient similiar to that of a perfect web. Remember that the shear effect of an imperfect web also has a contribution from frozen-in shear strain. Thus the shear effect is not neccessarily equal to that of a perfect web, even if the shear coefficient is equal.

TABLE 1: Primary Parameters

Property	Value	Unit
L	1.0	m
W	1.6	m
E_0	1.5	GPa
h_0	70	μm
T_w	300	N/m
V_x	10	m/s
q	0	N/m

TABLE 2: Secondary Coefficients and Shear Coefficient

Case	$\Delta\varepsilon_i$ %	F_E	t_i N	m_i Nm	S_0 kN	H kNm	Ω kNm^2	n
1	0.00	1.0	0.0	0.0	168.0	0.0	35.83	1.20
2	0.00	0.8	0.0	0.0	163.8	-2.5	35.83	1.10
3	0.00	0.6	0.0	0.0	159.6	-5.0	35.83	1.01
4	0.05	1.0	31.3	10.7	168.0	0.0	35.83	1.20
5	0.05	0.8	28.5	9.7	163.8	-2.5	35.83	1.10
6	0.05	0.6	25.8	8.7	159.6	-5.0	35.83	1.01
7	0.10	1.0	62.5	21.5	168.0	0.0	35.83	1.20
8	0.10	0.8	57.1	19.5	163.8	-2.5	35.83	1.10
9	0.10	0.6	51.6	17.4	159.6	-5.0	35.83	1.01
10	0.50	0.2	203.7	66.8	151.2	-10.0	35.82	0.83

As a numerical example we have chosen a web with the properties in *Case 5* of Table 2. This web is subjected to two different movements of the downstream roller. The downstream roller moves laterally before it rotates back and forth as seen in Fig.4. Numerical calculations yield lateral deflections of the web at the downstream roller as seen in Fig.5. Calculations including the entire shear effect, calculations including the shear effect from the shear force only and calculations excluding the entire shear effect are displayed. As previously demonstrated [1] the web moves towards the low tension side. The shear force increases the sideways shift and the frozen-in shear decreases it. For higher span lengths, Fig.6, the effect of the shear force is less significant. The effect of the frozen-in shear seems to be significant also for higher span lengths. This is confirmed by analyses on the effect of the length to width ratio.

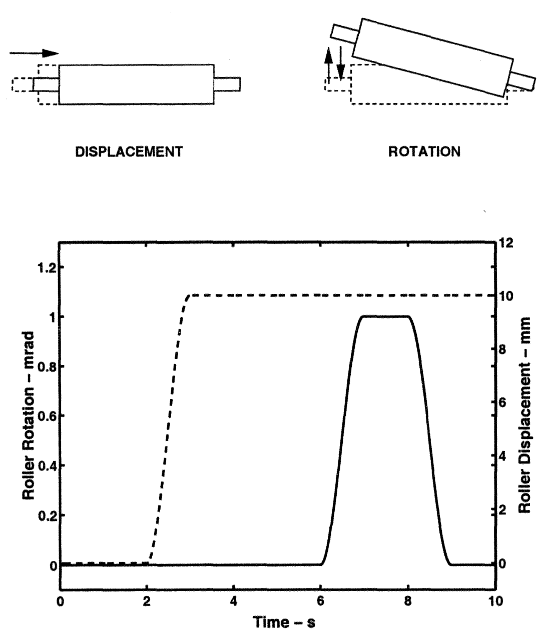


Fig.4: Illustration (top) and plot (bottom) of displacement (dashed line) and rotation of downstream roller.

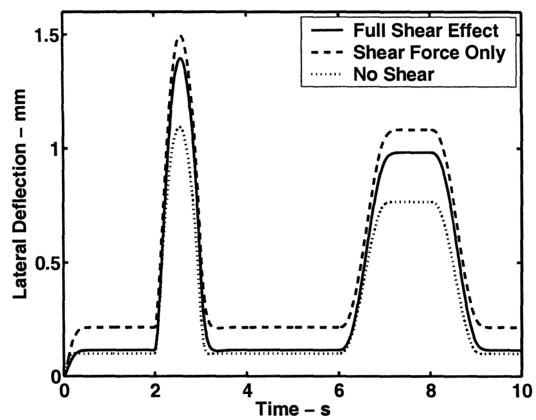


Fig.5: Lateral deflection at downstream roller for a web moving between two rollers 1.0m apart.

Web deflections at the downstream roller have been calculated for different aspect ratios (i.e. length to width ratios, L/W). Calculations have been carried out with and without contributions from shear forces and frozen-in shear. They have been applied to a perfect web and an imperfect web subjected to the rotation mentioned above and illustrated in Fig.4. They have also been applied to an imperfect web deflecting due to its selfinduced curvature (no rotation). By comparing calculations of deflection at the downstream roller for analyses including shear effects with analyses neglecting shear effects, we can calculate the contribution from the shear effect. This is shown in Fig.7. We see that for a perfect web the shear effect is only significant for small aspect ratios. For an imperfect web it is significant for all aspect ratios. This is due to the frozen-in shear which do not become insignificant at high aspect ratios.

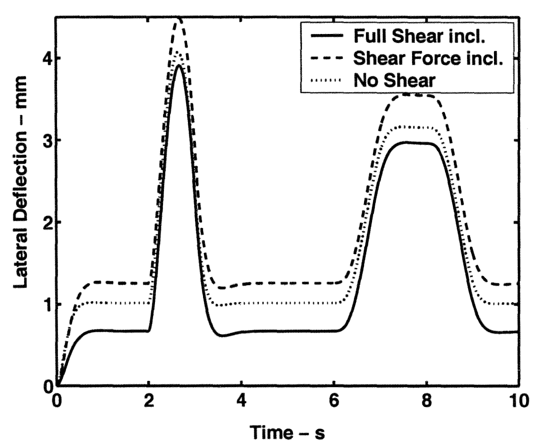


Fig.6: Lateral deflection at downstream roller for a web moving between two rollers 3.6m apart.

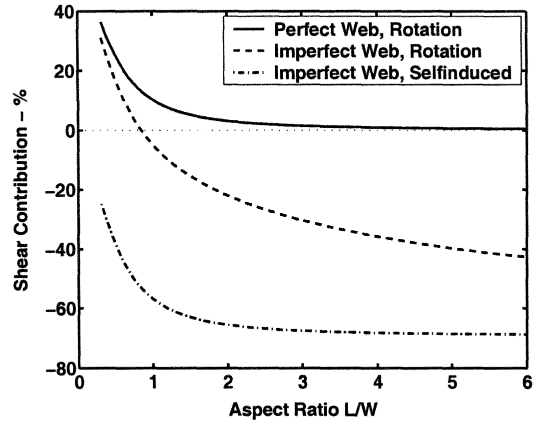


Fig.7: Effect of shear on lateral deflection at downstream roller as a function of the aspect ratio.

Calculations with different web velocities have been carried out. The results are displayed in Fig.8, and they show that lateral displacement is affected by the velocity. This is a dynamic effect. The response to different velocities depend upon whether the disturbance is a lateral shift or a rotation of the downstream roller. For a rotation, high velocities means quicker response and possible greater deflection of the web. The lateral movement of the downstream roller has a more complex response as seen by Eq.(26). Apparently low velocities yield greater web deflection.

CONCLUSIONS

A model describing the lateral dynamics of an imperfect web has been derived. The model accounts for shear effects. Calculations show that shear effects are significant for imperfect webs. The shear effect has a contribution from the shear force and frozen-in shear strain. The effect from the shear force become insignificant at high length to width ratios. Frozen-in shear strain is significant at all aspect ratios.

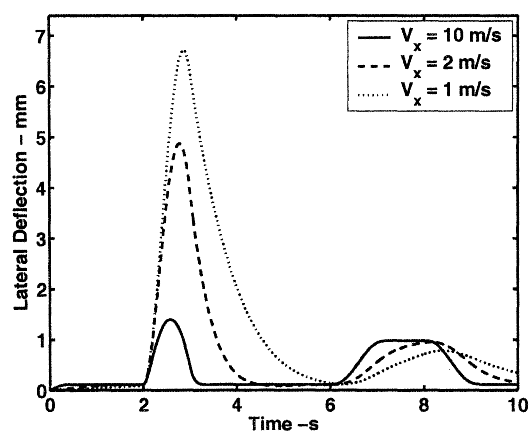


Fig.8: Lateral displacement at the downstream roller for a paper web moving at different velocities.

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REFERENCES

[1] Olsen,J.E. Lateral Mechanics of an Imperfect Web. *Journal of Pulp and Paper Science*, 28(9):310–314, 2002.

[2] Timoshenko,S. *"Strength of Materials - Part 1"*. D.Van Nostrand Company, 2 edition, 1940.

[3] Shelton,J.J. *"Lateral Dynamics of a Moving Web"*. PhD thesis, Oklahoma State University, 1968.

[4] Cowper,G.R. The Shear Coefficient in Timoshenko's Beam Theory. *Journal of Applied Mechanics*, 33(June):335–340, 1966.

[5] Dharmarajan,S. & McCutchen,H. Shear Coefficients for Orthotropic Beams. *Journal of Composite Materials*, 7:530–535, 1973.

Appendix A

SHEAR COEFFICIENT

The author have not been able to find an expression for the shear coefficient for an inhomogenous orthotropic material. However, it is possible to find an expression or a solution for an inhomogenous isotropic material and a homogenous orthotropic material.

Inhomogenous Isotropic Material

In order to calculate the shear coefficient, an expression for the shear stress is needed. The expression is derived as follows.

For an inhomogenous isotropic material it can be shown that the derivative of the longitudinal stress σ with respect to x-coordinate is

$$\frac{\partial \sigma}{\partial x} = \frac{\partial M}{\partial x} \frac{S_0 E y - H E}{\Omega S_0 - H^2} \tag{38}$$

Since the thickness is very small compared to the width of the beam, we will assume that there is no variations of the stress field in the thickness (z) direction. Equilibrium of forces is then mathematically expressed by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{39}$$

For a beam of length L exposed to a tip-load Q we have

$$Q = \partial M / \partial x \tag{40}$$

From this and Eqs.(38) and (39) we get

$$\tau_{xy} = \frac{Q f_I(y)}{\Omega S_0 - H^2} \tag{41}$$

where

$$f_I(y) = \left\{ H \left[\int_0^y E(\eta)\, d\eta - \int_0^{W/2} E(\eta)\, d\eta \right] - S_0 \left[\int_0^y E(\eta)\, \eta\, d\eta - \int_0^{W/2} E(\eta)\, \eta\, d\eta \right] \right\} \tag{42}$$

No shear stress on the surface ($\tau_{xy}(w/2) = 0$) has been applied as a boundary condition.

This expression for the shear stress is needed in the calculation of the shear coeffiecient. The shear deflection u_s of the beam is given by the strain energy U [2]

$$u_s = \frac{2U}{Q} \tag{43}$$

The strain energy is

$$U = \int_0^L \int_{-W/2}^{W/2} \frac{\tau_{xy}\gamma_{xy}}{2}\, h\, dy\, dx \tag{44}$$

Here γ_{xy} is the shear strain. For an isotropic elastic material the shear strain is given by

$$\gamma_{xy} = \frac{\tau_{xy}}{G} + \gamma_i \tag{45}$$

where G is the shear modulus and γ_i is the frozen-in shear strain. By inserting this expression in Eq.(44), we get

$$U = \frac{Q^2}{2(\Omega S_0 - H^2)^2} \int_0^L Q^2\, dx \int_{-W/2}^{W/2} \frac{f_I^2}{G}\, h\, dy \; + \; F_i \tag{46}$$

where F_i is the term accounting for the frozen-in shear strain. From Eq.(43) we get

$$\frac{\partial u_s}{\partial x} = \frac{Q}{(\Omega S_0 - H^2)^2} \int_{-W/2}^{W/2} \frac{f_I^2}{G} h\, dy + \frac{2}{Q}\frac{\partial F_i}{\partial x} \tag{47}$$

From the definition of the shear coefficient we have

$$\frac{\partial u_s}{\partial x} = \frac{nQ}{AG_c} + \gamma_i \tag{48}$$

where $A = Wh$ is the cross-sectional area. The first term on the right hand side in Eqs.(47) and (48) represents the effect from the shear stress. Together they yield

$$n = \frac{AG_c}{(\Omega S_0 - H^2)^2} \int_{-W/2}^{W/2} \frac{f_I^2}{G} h\, dy \tag{49}$$

For a beam with no widthwise variations in elastic modulus and thickness we have $H = 0$, $\Omega = EI$, $S_0 = EA$ and

$$f_I(y) = E^2 A \left\{ \frac{W^2}{8} - \frac{y^2}{2} \right\} \tag{50}$$

This yields $n = 1.2$

Values for material properties and their variations in the widthwise position can be inserted in the expressions above and the shear coefficient can be calculated. Values for different examples are presented in the main part of the paper. They show that widthwise variations in stiffness influence the shear coefficient, while widthwise variations in frozen-in strain do not.

Homogenous Orthotropic Material

For an orthotropic linear elastic material it is possible to generalize the theory of Cowper[4] for an isotropic linear elastic material. This has been carried out by Dharmarajan & McCutchen[5]. They found the following shear coefficient:

$$n = \frac{6}{5} - \frac{\nu_{xy}}{5}\frac{G_{xy}}{E_x} \tag{51}$$

For an isotropic material this reduces to

$$n = \frac{12 + 11\nu}{10(1+\nu)} \tag{52}$$

which is the same expression as that found by Cowper[4]. Note that these expressions are valid for homogenous beams only. Any widthwise variations in stiffness or in frozen-in strain is not accounted for. However, it is reasonable to assume that frozen-in strain (or bagginess) have no effect on the shear coefficient. At least that was the case for the inhomogenous isotropic beams.