LATERAL MECHANICS OF AN IMPERFECT WEB

by

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ABSTRACT

A model describing the lateral mechanics of an imperfect web has been derived which brings forward information on the much debated fourth boundary condition. The model is based upon a generalized beam theory. Calculations show that a web with CD profile in stiffness and/or frozen-in strain (camber) will shift towards the low tension side. The deflection increases with increasing tension and decreasing stiffness. The dependence upon tension is related to the stiffness profile. A web with a constant profile in stiffness, but with a varying profile in frozen-in strain, has a very weak influence of tension upon deflection.

NOMENCLATURE

A	cross-sectional area
E	MD modulus
H	material parameter
h	web thickness
K	web parameter
L	span length
M	applied bending moment
m_i	material parameter
S_0	global web stiffness
T	axial force
t_i	material parameter
u	lateral deflection
x	MD position
y	CD position
ε	strain
ε_0	strain at centroid
ε_i	frozen-in strain
θ_L	angle
κ	curvature
σ	stress
Ω	bending stiffness

INTRODUCTION

A web moving through a web line is influenced by forces and bending moments transfered from rollers in contact with the web. As can be described by beam theory, the web moves laterally if a bending moment is applied to it. However, it has also been observed that a web may shift sideways even if no bending moment is applied. This can not be explained by elementary beam theory. It is believed that this sideway shift is caused by widthwise variations in material properties. These variations may be variations in elastic modulus or in frozen-in strain. The latter is sometimes referred to as camber, especially in the steel industry. Elementary beam theory assumes no variations in these properties. Therefore we need to derive a more general beam theory.

THEORY FOR WEB

A beam parallel to the x-axis, with the beam axis defined as the centroidal axis, may deflect in the y- and z-direction. See Fig.1. In general the deflections in the y- and z-direction are coupled. However, for a very thin beam the deflection in the z-direction may be neglected when studying lateral deflection, i.e. deflection in the y-direction. The lateral deflection u is given by the definition of curvature κ

$$\kappa \equiv -\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Finding an expression for the curvature is thus essential when calculating the deflection.

Bernoulli's hypothesis of deformation implies that the strain parallel to the beam axis varies linearly with coordinate y. Mathematically we can express this with the following equation:

$$\varepsilon(y) = \varepsilon_0 + \kappa y \tag{2}$$

Here ε_0 is the strain at the centroid of the cross sectional area of the beam.

Due to different reasons, a web may have frozen-in strains, sometimes referred to as camber. In order to account for this phenomenon, it needs to be accounted for by the beam theory. Frozen-in strains are not accounted for by elementary beam theory. Thus a more general theory is needed. For practical reasons the frozen-in strains are defined to be zero at the centroid. For a beam with frozen-in strain the above equation generalizes to

$$\varepsilon(y) = \varepsilon_0 - \varepsilon_i(y) + \kappa y \tag{3}$$

where ε_i is frozen-in strain. For a linear elastic material the stress field is thus

$$\sigma(y) = E(y)\varepsilon_0 - E(y)\varepsilon_i(y) + E(y)\kappa y \tag{4}$$

Here $E(y)$ is the elastic modulus in the direction of the beam axis, which is the machine direction (MD) in web handling terminology. Thus it is known as the MD modulus. Generally it may vary with widthwise position y.

The axial force T and the bending moment about the z-axis M are given by

$$T = \int_A \sigma(y)\, dA = \int \sigma(y)\, h(y)\, dy \tag{5}$$

and

$$M = \int_A \sigma(y) y\, dA = \int \sigma(y)\, y\, h(y)\, dy \tag{6}$$

Inserting Eq.(4) in Eqs.(5) and (6) yields

$$T + t_i = \varepsilon_0 S_0 + \kappa H \tag{7}$$

$$M + m_i = \varepsilon_0 H + \kappa \Omega \tag{8}$$

where

$$t_i = \int \varepsilon_i \, E \, h \, dy \qquad m_i = \int \varepsilon_i \, E \, y \, h \, dy \qquad S_0 = \int E \, h \, dy$$

$$H = \int E \, y \, h \, dy \qquad \Omega = \int E \, y^2 \, h \, dy \tag{9}$$

Note that thickness h, frozen-in strain ε_i and modulus E are functions of the widthwise position y. t_i is the MD force required to stretch out the average frozen-in strain without bending the web ($\kappa = 0$). m_i is the moment required to straighten out the web without stretching the web (ε_0=0). S_0 is the global stiffness of the web. H is a constant related to the bending and stretching of the web. Ω is the bending stiffness of the web.

In terms of ε_0 and κ, Eqs.(7) and (8) reduce to the following expressions:

$$\varepsilon_0 = \frac{(T + t_i)\Omega - (M + m_i)H}{\Omega S_0 - H^2} \tag{10}$$

$$\kappa = \frac{(M + m_i)S_0 - (T + t_i)H}{\Omega S_0 - H^2} \tag{11}$$

For a perfect web, i.e. homogeneous web ($E(y)$=E_0) with no frozen-in strain (ε_i=0), we have H=t_i=m_i=0 and Ω=$E_0 I$, where I is the second moment of area about the z-axis. Thus Eq.(11) is reduced to

$$\kappa = \frac{M}{E_0 I} \tag{12}$$

which is the expression found from *elementary* beam theory. This is as expected for a perfect web.

A force balance as derived by Shelton[1] yields

$$\frac{\partial^2 M}{\partial x^2} + T \frac{\partial^2 u}{\partial x^2} = 0 \tag{13}$$

We can organize Eq.(11) with respect to M, carry out the derivation, and insert the result into Eq.(13). That leaves us with the following equation

$$\frac{\partial^2 \kappa}{\partial x^2} + K^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{14}$$

where the coefficient K is generalized

$$K^2 = \frac{T S_0}{\Omega S_0 - H^2} \tag{15}$$

Applying $\kappa = -\partial^2 u / \partial x^2$ results in the well-known differential equation with respect to the deflection u

$$\frac{\partial^4 u}{\partial x^4} - K^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{16}$$

The general solution of this differential equation is

$$u = C_1 \sinh(Kx) + C_2 \cosh(Kx) + C_3 x + C_4 \tag{17}$$

In order to decide the coeffiecients C_1, C_2, C_3 and C_4, four boundary conditions are required. We find two at the upstream roller. If the coordinate system is defined so that the upstream roller is positioned normal to MD we have

$$u = 0 \qquad u' = 0 \qquad \text{at } x = 0 \tag{18}$$

The downstream roller is positioned at a distance L from the upstream roller with an angle θ_L from the normal yielding

$$u' = \theta_L \qquad \text{at } x = L \tag{19}$$

Historically there has been some controversy regarding the fourth boundary condition. The issue is, however, solved by Eq.(11). Applying M=0 at x=L as verified by Shelton[1], in Eq.(11), we get

$$u'' = -\kappa_L = \frac{(T + t_i)H - m_i S_0}{\Omega S_0 - H^2} \qquad \text{at } x = L \tag{20}$$

as the fourth boundary condition. We see that for an imperfect web $\kappa_L \neq 0$. This is consistent with both Shelton[1] and Swanson[2]. By applying these boundary conditions to decide the coefficients of Eq.(17), we get

$$u = L\theta_L \left[\frac{\cosh(KL)}{\cosh(KL) - 1} \left(\frac{x}{L} - \frac{\sinh(Kx)}{KL} \right) + \frac{1}{KL} \frac{\sinh(KL)}{\cosh(KL) - 1} (\cosh(Kx) - 1) \right]$$

$$+ \frac{L\kappa_L}{K} \left[\frac{\sinh(KL)}{\cosh(KL) - 1} \left(\frac{x}{L} - \frac{\sinh(Kx)}{KL} \right) + \frac{1}{KL} (\cosh(Kx) - 1) \right] \tag{21}$$

Inserting above expressions, Eqs.(15) and (20), for K and κ_L, and assuming θ_L=0, we get the following deflection at the downstream roller (x=L):

$$u_L = -\frac{(T + t_i)\, H - m_i S_0}{T S_0} \left[\frac{2 - 2\cosh(KL) + KL \sinh(KL)}{\cosh(KL) - 1} \right] \tag{22}$$

From this we see that there is a deflection at the downstream roller even with θ_L=0 and a zero moment at the downstream roller.

NUMERICAL EXAMPLE

As an example we will study the case in which paper is unwound. To pull the paper out of the roll and over a roller an axial force T is applied. The angle of the downstream roller is zero (θ_L=0). See Fig.2 for illustration.

Inhomogeneous Paper

First we study the case of a an inhomogeneous paper without frozen-in strain. As a numerical example we have chosen material properties typical for paper. Nominal values of the modulus varies between 1.5GPa, 2.5GPa and 3.5GPa and the nominal value of thickness is chosen as 70μm. Typically properties deviate from these values by decreasing values towards the edge positioned at the outside of a roll set. In the paper industry one often measures the stiffness of the paper. The stiffness is the product of the modulus and the thickness. For low, medium and high values of the nominal modulus, the stiffness profiles are chosen as seen in Fig.3. These are representative for typical edge rolls.

For an inhomogeneous web the constant K of the differential equation, Eq.(16), and the fourth boundary condition, Eq.(20), is reduced to

$$K^2 = \frac{TS_0}{\Omega S_0 - H^2} \qquad u'' = \frac{TH}{\Omega S_0 - H^2} \quad \text{at } x = L \tag{23}$$

The deflection at the downstream roller is then

$$u_L = -\frac{H}{S_0}\left[\frac{2 - 2\cosh(KL) + KL\sinh(KL)}{\cosh(KL) - 1}\right] \tag{24}$$

Results for paper without frozen-in strain, but with stiffness profiles as in Fig.3 are illustrated by Fig.4. We see that the deflection at the downstream roller increases with tension and decreases with nominal MD modulus. The deflection is positive, indicating that the web shifts towards the side with lower stiffness, i.e. the low tension side.

Paper with Frozen-in Strain (Camber)

We now focus on frozen-in strain. For web with profiles of frozen-in strain with constant profiles of stiffness, the constant K of the differential equation, Eq.(16), and the fourth boundary condition, Eq.(20), is reduced to

$$K^2 = \frac{T}{E_0 I} \qquad u'' = -\frac{m_i}{E_0 I} \quad \text{at } x = L \tag{25}$$

The deflection at the downstream roller is

$$u_L = \frac{m_i}{T}\left[\frac{2 - 2\cosh(KL) + KL\sinh(KL)}{\cosh(KL) - 1}\right] \tag{26}$$

which is somewhat equivalent to that of Swanson[2]. Swanson was not able to find an expression for m_i as he was applying elementary beam theory. However, he was correct in estimating that its value was somewhere between zero and a given upper value.

Quantitatively little is known about the profile of frozen-in strain in paper. From observation we may make a crude estimate. On occasions paper in a press is observed to have slack edges. This can not be explained by typical values for stiffness profiles. If the slackness is due to frozen-in strain and if the profile has the following form:

$$\varepsilon_i(y) = \Delta\varepsilon_i \frac{y}{W} \tag{27}$$

a reasonable estimate would be $\Delta\varepsilon_i = 0.001$ which is equivalent to a variation in frozen-in strain of 0.1% per widthwise meter of paper. Poorly documented measurements [3] indicate a value of $\Delta\varepsilon_i = 0.00045$ which falls within the same order of magnitude as the above estimate. Assuming the first estimate to be valid and neglecting any variations in stiffness profiles, calculations yield results as in Fig.5. The lateral deflection decreases as the tension increases, but the decrease is extremely weak. This is consistent with the experimental results of Swanson[2] who found no significant influence of web tension. The calculations also show that the paper deflects towards the slack side, i.e. the low tension side, which is confirmed by the same experiments and those of Shelton[4].

Inhomogeneous Paper with Frozen-in Strain

Real paper tend to be both inhomogeneous in terms of stiffness and have some frozen-in strain. Combining the effects from the two examples above, we get results as in Fig.6. Deflection increases with tension. That is due to the inhomogeneity in stiffness. The calculated deflection is equivalent to the added contribution of the deflection of a paper with frozen-in strain and paper with stiffness profile. Since the theory is linear this is consistent with the principle of superposition.

The principle of superposition is better illustrated when plotting the deflection as a function of MD-position between upstream and downstream roller for a paper with frozen-in strain, a paper with stiffness profile and a paper with both defects. This is seen in Fig.7. At the chosen tension level, 300N/m which is typical for unwinding of paper, the curves also show that frozen-in strain is most significant. Since the effect of frozen-in strain does not increase with tension and the effect of stiffness profile does, the situation is reversed at higher tensions. This is shown in Fig.8 where the curves represent deflection at a tension level of 800N/m. At that level, frozen-in strain is not more significant than stiffness profiles.

CONCLUSIONS

A model describing the lateral mechanics of an imperfect web has been derived. Calculations show that a web with CD profile in stiffness and/or frozen-in strain will shift towards the low tension side. The deflection increases with increasing tension and decreasing stiffness. The dependence upon tension is related to the effect of stiffness profile. A web with a constant profile in stiffness, but with a varying profile in frozen-in strain, has a very weak influence of tension.

ACKNOWLEDGMENTS

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[4] Shelton,J.J. "Effects of Web Camber on Handling". Proceedings of the Fourth International Conference on Web Handling. Oklahoma State University, 1997.

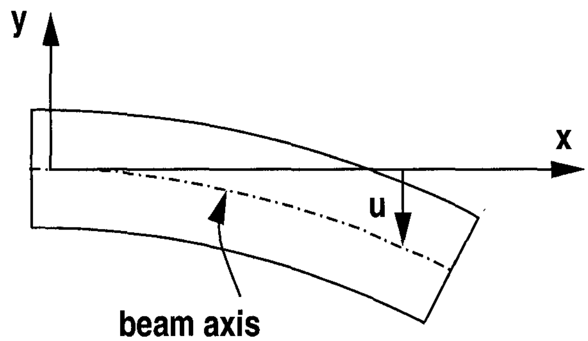

Figure 1: A web with deflection u.

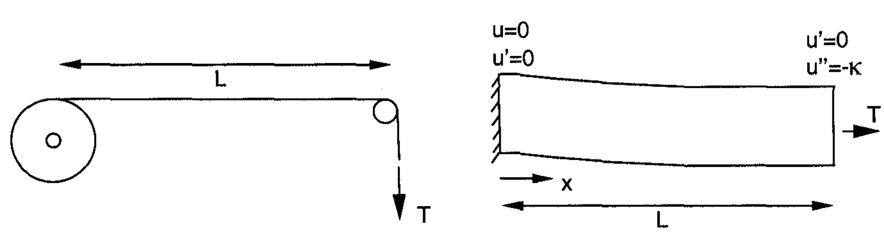

Figure 2: Unwinding of paper seen from the side (left) and from above with beam terminology.

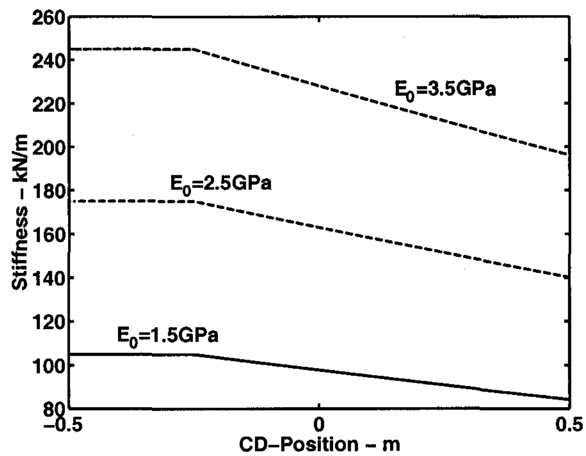

Figure 3: MD Stiffness as a function of CD-position.

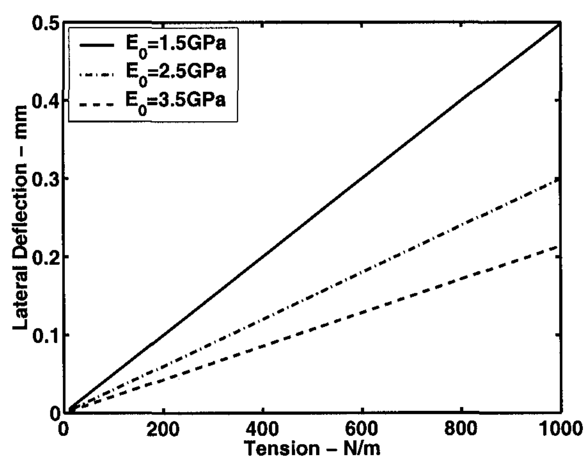

Figure 4: Lateral deflection at downstream roller of paper with stiffness profile as a function of web tension.

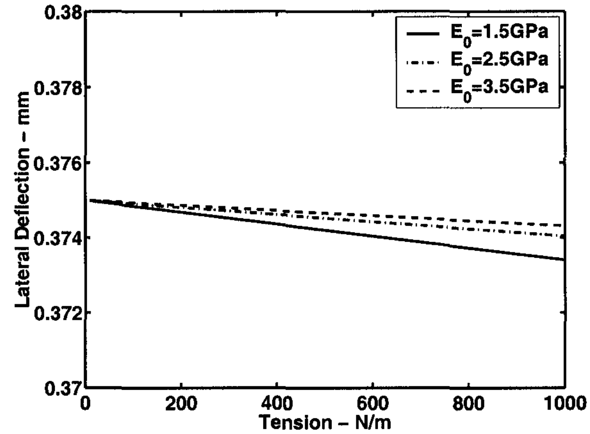

Figure 5: Lateral deflection at downstream roller of homogenous paper with frozen-in strain as a function of web tension.

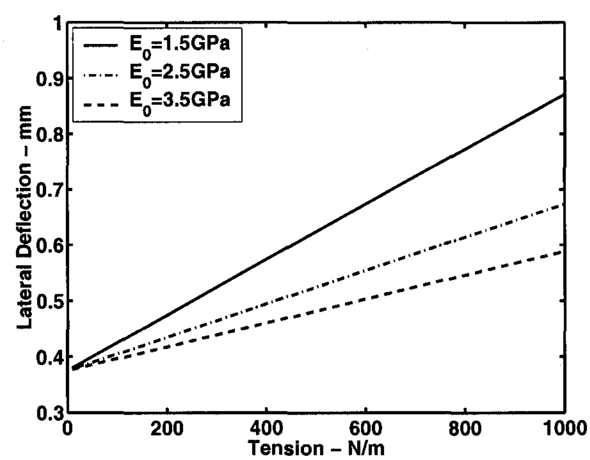

Figure 6: Lateral deflection at downstream roller of inhomogeneous paper with frozen-in strain as a function of web tension.

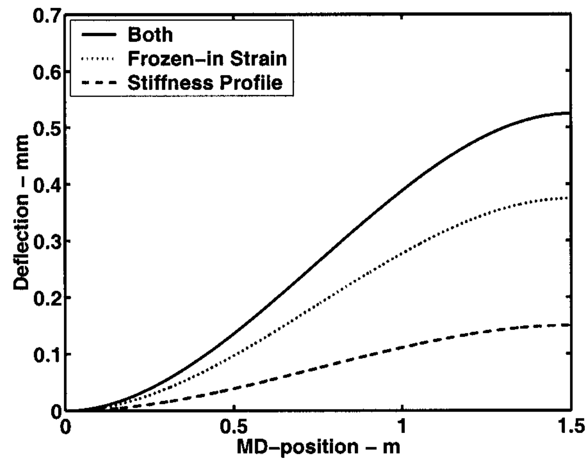

Figure 7: Deflection as a function of MD-position of paper with stiffness profile, frozen-in strain profile and both for $\mathrm{T} = 300\mathrm{N/m}$.

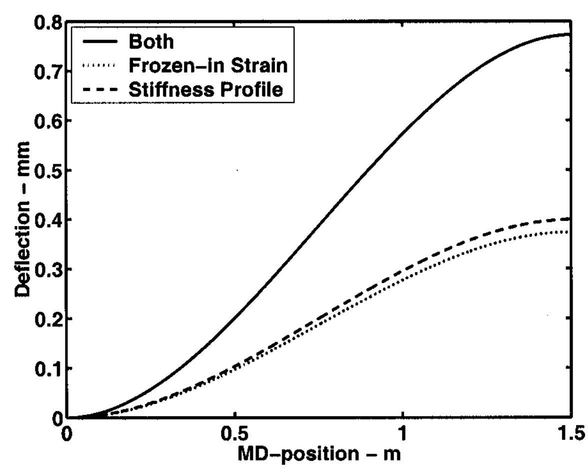

Figure 8: Deflection as a function of MD-position of paper with stiffness profile, frozen-in strain profile and both for $\mathrm{T} = 800\mathrm{N/m}$.

J. E. Olsen – Norwegian Pulp and Paper Research Institute, Norway

Name & Affiliation	Question
K. Shin – KonKuk University	Your first equation has the dimension of strain. In the third term on the right hand side of the equation does Y have the units of length? Could you explain the units of Y and κ?
Name & Affiliation	**Answer**
J. Olson – Norwegian Pulp and Paper Research Institute	κ is 1 over radius and Y has units of length.
Name & Affiliation	**Question**
K. Shin – KonKuk University	In the equation in your paper, is Y also displacement?
Name & Affiliation	**Answer**
J. Olson – Norwegian Pulp and Paper Research Institute	Y is not displacement, Y is just the internal cross-width position, *U* is displacement, or lateral deflection.
Name & Affiliation	**Question**
K. Shin – KonKuk University	X is also displacement in equation 1 in your paper?
Name & Affiliation	**Answer**
J. Olson – Norwegian Pulp and Paper Research Institute	X is the machine directional coordinate.
Name & Affiliation	**Question**
M. Kurki – Metso Paper Inc.	You developed an analysis based on MD, machine directional elastic properties. How about cross directional properties? How significant they are because they are also creating their own CD directional tension, which is zero on the edges, and it is creating different situation compared to the middle of this web span or the edges of the web span. What is the general affect of CD directional material properties?
Name & Affiliation	**Answer**
J. Olson – Norwegian Pulp and Paper Research Institute	The CD properties are not accounted for in this theory. It is not included in these lateral dynamic theories.
Name & Affiliation	**Comment**
D. Pfeiffer – JDP Innovations	I can only underscore for you the importance of looking at elastic modulus variations when you're considering registration problems in printing, because they are very important and they do happen due to mysterious causes. I remember one test done over 25 years ago, where many rolls of paper were measured to see why a registration

	problem was occurring. With all the recorded values that we sensed, including wound off tension through the rolls and basis weight, caliper, and everything else, no reason for the registration error was found. We found that the one property that we didn't measure online, which was modulus in the machine direction, had a surge in it and the mill finally traced it back to a surge in the height of a stock chest, as it was not being held constant. So, you can look for various problems in printing registration when the modulus changes suddenly. You are going to have a similar problem, as you have already pointed out, when the MD modulus changes or is non-uniform across the web. But, you have the combination of the MD modulus and the CD modulus being variable.
Name & Affiliation	**Question**
L. Eriksson – Stora Enso Research	I understand this is a linear elastic modulus, but do you have limits due to buckling and wrinkling on your model?
Name & Affiliation	**Answer**
J. Olson – Norwegian Pulp and Paper Research Institute	No, my model assumes that you have a straight web across the width, so whole portions of the web are in tension, if that is not the case you will have some serious problems with your boundary conditions.
Name & Affiliation	**Question**
C. Bronkhorst – Weyerhaeuser	This question focuses on the material properties and is probably unique to the group of people here representing the paper industry. Not only do you have an inhomogenuity of Young's modulus and machine and cross machine direction, but through the thickness you have significant inhomogenuity of Young's modulus and I note in your theory you assume that you have no Z direction differential in Young's modulus through the thickness. Do you think that aspect will have a significant influence on the conclusions that you draw from this theory?
Name & Affiliation	**Answer**
J. Olson – Norwegian Pulp and Paper Research Institute	For an in-plane problem, this does not have much effect. For problems in which the web distorts out-of-plane, it will be important.