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Scope of Study: This report presents a procedure which can be used in introducing Coordinate Geometry using a language that can be understood by a high school student. An attempt is made here to give the reader examples and also explain how the concept of set simplifies and clarifies the comprehension of locus, inequalities, equations, functions, ordered pairs, and other basic concepts.

ADVISER'S APPROVAL James H. Zant

A PROCEDURE FOR INTRODUCING COORDINATE GEOMETRY TO HIGH SCHOOL STUDENTS

By

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TO HIGH SCHOOL STUDENTS

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PREFACE

What type of presentation of Coordinate Geometry should be given in the high schools? This is a question which can be answered only after much experimentation, but one of the more logical methods is that which is presented in this report. With three years of teaching experience, it is the feeling of the writer that the students would appreciate and would understand this type of presentation.

In the contents I have tried to explain the concept of set and ordered pairs, which are particularly applicable to the development of the ideas of Coordinate Geometry.

I would like to express my appreciation to Dr. J. H. Zant, Mr. S. Douglas, Mr. R. Dean, Mr. M. Siebert, and Miss E. J. Kerby for the advice and suggestions they have given me in preparing this report.

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CHAPTER I

INTRODUCTION

This report is an introduction to Coordinate Geometry and is important preparation for the geometry work in analysis and other advanced courses in mathematics. One of the inadequacies in teaching elementary graphing in conventional courses is the very small number of types of problems which students can be expected to do. First, one teaches them to plot points, and here is the first teaching problem. In order to build in the students the reflex of "go over with the first component," one needs to have them actually plot quite a few points. Students of moderate or high ability often resent such a routine kind of job, but they come to class still not sure of whether to go up or to go over with the first component.

Next, one usually has students draw the loci of first-degree equations. What else can one do with beginning students? They can graph $\left\{ (x,y) \mid y = x^2 \right\}$, and make a few explorations into the loci of other non-linear equations, but soon the algebra is over their heads. To some extent, this is a problem with all Coordinate Geometry customarily taught in the beginning course: there is just not enough interesting content (7, 1).

Since the Cartesian plane is a cluster of points and the points are ordered pairs of numbers, it is necessary that each student understand the number system.

Everyone knows what numbers are. A person begins to learn about them as soon as he can speak, has many experiences with them as he grows older, and seldom passes a day of his life without using them in some way. The store of knowledge about them is so vast and so rapidly growing that only a few specialists have a clear picture of its entirety, yet the sentence with which this paragraph begins is hardly true if it is taken to mean that everyone can say precisely what numbers are.

There are many satisfactory theories describing the number system. Although they differ in their undefined terms, their definitions, and their axioms, they exhibit the same laws for the obvious reason that they all have been designed to formalize the properties of numbers as they are known and used.

The notion of set, involving essentially the mental operation of recognition, is deeply embedded in our intuition. This concept occurs in all branches of mathematics, as well as in the sciences and in life generally. For mathematics it is a unifying concept, which is simple and interesting. It will be used freely in this report in the firm belief that it has much to offer in teaching a number of standard topics in the high school program.

CHAPTER II

THE NUMBER SYSTEM

Natural Numbers

Certainly the first calculations we learn are concerned with the process of counting, and we all become acquainted at an early age with the counting numbers: 1, 2, 3, 4, 5, ... The numbers, 1, 2, 3, 4, 5, ..., are called the natural numbers because it is felt generally that they have, in some philosophical sense, a natural existence independent of man.

The natural numbers are, of course, an abstract concept, independent of the nomenclature used to represent them. The most common representation, 1, 2, 3, ..., also is called the decimal system because ten symbols or digits are employed to represent all numbers. The compactness of this excellent notation is due to its being a place or positional system. That is, in such a number as 543, the digit 5 represents 5 hundreds; the digit 4, 4 tens; the digit 3, 3 units. In contrast, the Roman numeral XX for 20 has two symbols x of equal value, ten. Thus the Hindu-Arabic system can be used to represent numbers of any size with ten symbols, whereas the nonpositional notation like the Roman numerals would require new symbols for reasonable brevity in the representation of larger and larger numbers.

The decimal system is said to have base ten. This unit probably originated in a anthropological sense from our having ten fingers for use in counting. However, ten is not the only possible base, for we could use any natural number greater than one.

With the advent of electronic digital computers, much use is made now of the base two. The base twelve has been suggested from time to time as a more practical base than ten.

We shall not attempt to define the natural numbers, but it is easy to see that, as a number system, they have some very special properties. Some rather obvious statements which one can make are:

A. There is a first, or least, counting number.

B. Each counting number gives rise to the next one merely by adding the least one.

C. Any collection of the counting numbers which has one or more members has a least member.

Notice that, by use of the first two of these statements, we can "generate" any counting number: We begin with the least one, 1, then add the least one to itself to get the next one, 2, then add the least one to 2 to obtain the next one, 3, and so forth until we arrive at the desired counting number. That is, the counting numbers are obtained by repeated addition of the least one.

Statement C is of a different sort. We use it in our daily conversation when we make statements of the following form: "The least populated state in the Union is Nevada." However, even in less definitive situations, we use this as a property of the counting numbers.

Algebraically, the counting numbers are not very rich. We can add any two counting numbers and obtain again a counting number, and the same is true of multiplication. However, we cannot make analogous statements about subtraction and division because, for example, $4 - 7$ and $4 \div 7$ are not counting numbers.

The Integers

A slightly richer number system algebraically is the collection of integers: $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$. Within this system we can add, subtract, and multiply. This system is essentially just a completion of the counting numbers.

We see meaning in Leopold Kronecker's often quoted remark, "God made the whole numbers; all the rest is the work of man (1, 215)."

Notice that o is a very special number algebraically. It is the only number with the property that, when added to any number of the system, we just obtain that number again; i.e., for every integer x, $x + o = x$. We call o the additive identity for this system. And if x is an integer, then the integer y with the property that $x + y = o$ is called the additive inverse of x. (Thus, -1 is the additive inverse of 1, and 7 is the additive inverse of -7. The number o is its own additive inverse. A more usual term for additive inverse is "negative.") (2, 8)

Now the sense in which the integers are a completion of the counting numbers is described as follows: the integers are obtained from the counting numbers by adjoining an additive identity and additive inverses. This means, algebraically, that we can find solutions in the integers of equations of the form $x + a = b$ no matter what the integers a and b are.

The integers enjoy two special properties:

1. $x \cdot o = o$ for every integer x and
2. $(-x)(-y) = xy$ for any integers x and y.

Within the collection of integers, we are able to add, subtract, and multiply; but we cannot always divide. Therefore, we are now interested in a system rich enough to allow all four of these basic algebraic operations.

The Rational Numbers

The system of rational numbers (the collection of all fractions $\frac{x}{y}$ where x and y are integers and $y \neq o$) is rich enough to allow all four of these basic algebraic operations (2, 11). We recall that equality, addition, and multiplication of rational numbers are given by

$$\frac{a}{b} = \frac{c}{d} \text{ if } ad = bc$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} .$$

In this system, we have essentially the integers, but we have a good deal more. In the integers, the integer 1 was the multiplicative identity, but only 1 and -1 have multiplicative inverses in that system. In the collection of rational numbers, the rational number 1 (i.e., $\frac{1}{1}$) is the multiplicative identity since

$$\frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b} \text{ for every rational } \frac{a}{b} \text{, and}$$

if

$$a \neq 0,$$

then

$$\frac{b}{a} \text{ is the multiplicative inverse of } \frac{a}{b}$$

since

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = \frac{1}{1} = 1.$$

The rational number 0, (i.e., $\frac{0}{1}$), cannot have a multiplicative inverse since

$$\frac{a}{b} \cdot 0 = \frac{a}{b} \cdot \frac{0}{1} = \frac{0}{b} = \frac{0}{1} = 0$$

for every rational $\frac{a}{b}$.

In the system of rational numbers, there is no "next" element as there was in the cases of natural numbers and the integers.

It is easy to see that 741 · 9321 is rational, since this number also can be written as $\frac{7419321}{10000}$, i.e., as a quotient of two integers.

It should be clear that all terminating or repeating decimals can be expressed as a rational number.

Consider 3.4222..., and we will see if it is rational. Let

$$x = 3.4222\ldots$$

$$10x = 34.222\ldots \qquad (1)$$

$$100x = 342.222\ldots \qquad (2)$$

Subtracting (1) from (2), we obtain

$$90x = 308$$

$$x = \frac{308}{90} = \frac{154}{45}\,.$$

This is the quotient of two integers and, hence, is rational.

The collections of all decimal expansions which do not terminate or repeat are called the <u>irrational numbers</u>; for example, $\pi = 3.14159\ldots$, $\sqrt{2} = 1{,}414\ldots$, $\sqrt{3} = 1.732\ldots$.

Therefore, the collection of all infinite decimal expansions forms the <u>Real Number System</u>.

Complex Numbers

If we restrict ourselves to the real numbers only, the symbols $\sqrt{-2}$, $\sqrt[4]{-16}$, $\sqrt[6]{-12}$, and so forth, are meaningless. However, because of the important part that such symbols can be made to play, not only in theoretical mathematics, but also in practical applications, it is desirable to give them a meaning.

The unit of the real numbers, the number 1, has two square roots, +1 and -1. In order to have a complete analogy with this situation, we define the two square roots of -1 to be +i and -i and agree always to replace the symbol $\sqrt{-1}$ by the symbol i, and i^2 by -1. The number i is called the imaginary unit.

Any number of the form a + bi, where a and b are real numbers, is a <u>complex number</u>. The a is the real part, and the bi is the imaginary part of the complex number; b is the coefficient of the imaginary part.

If $a = 0$ and $b \neq 0$ in the number a + bi, the number is a pure imaginary number. If $b = 0$, the number is a real number.

The complex number system need not be obtained by definition from the real number system. We shall not list such a case, however, but shall conclude with the following diagram, which exhibits a classification of the various kinds of numbers discussed in this chapter: (1, 217)

Natural numbers Zero Negative numbers	Integers Fractions	Rational numbers Irrational numbers	Real numbers Imaginary numbers	Complex numbers.

CHAPTER III

CONCEPTS AND SYMBOLISM OF SETS

A brief outline of the concepts, vocabulary, and symbols of the theory of sets follows. Illustrations will be given later of how these can be employed systematically over a long period of time to help clarify other mathematical ideas and lead to a more contemporary treatment of advance mathematics.

Set is an undefined concept. However, a set can be thought of as a collection of objects, physical or mental. A finite set can be designated by simply listing the objects that belong to it. Often it is possible, alternatively, to give a description that permits unequivocal determination as to whether any given object is or is not in the set.

Capital letters are used as the names of sets, and small letters are used as names for members of sets.

That any element k belongs to a set A is expressed by writing $k \in A$. That an element n does not belong to a set A is expressed by canceling the epsilon symbol with a vertical or slightly slanted line. The membership of a set is indicated by listing all elements and enclosing them in braces or by enclosing in braces a descriptive phase, e.g.

$$S = \{1, 3, 5, 7, 9\}$$

$$R = \{x, \text{ such that } x > 2\}.$$

This also can be written as $R = \{x \mid x > 2\}$ or $R = \{x \ni x > 2\}$. This is to be read: "R is the set of x's such that x > 2." The symbols, $\mid$, and $\ni$, are read: "such that."

If all the members of a set A are also members of a set B, A is called a subset of B. This situation is symbolized by: $A \subseteq B$, and read: "A is contained in B." If B has members which are not in A, then A is a proper subset of B. In symbols, $A \subset B$, read: "A is properly contained in B."

The intersection of two sets A and B is the set composed of those elements which are in A and also in B. The symbol for intersection is $\cap$. $A \cap B$ is read "the intersection of A and B" or "A cap B."

The union of two sets A and B is the set of those elements which are in A or in B or in both. The symbol for union is $\cup$. $A \cup B$ is read "the union of A and B" or "A cup B."

When in any discussion one or more sets are considered as subsets of one particular set, the one over-all set is called the universe, $\mathcal{U}$. The empty set, or $\emptyset$, is one that has no members, or one that is empty.

The empty set arises naturally in many mathematical and logical discussions. For example: The set of all one-digit odd prime numbers is a set containing three elements $\{3, 5, 7\}$. The set of all one-digit even prime numbers is a set containing only one element, $\{2\}$. The set of all two-digit odd primes is $\{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 97\}$. The set of all two-digit even primes is empty.

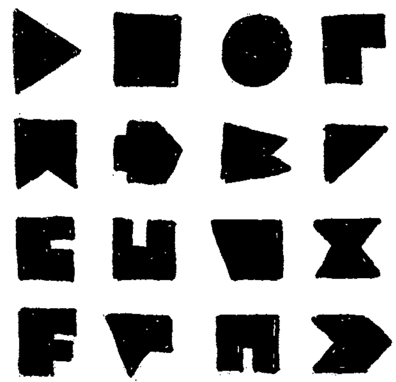
To say that the set of numbers satisfying a certain condition is empty is to say that no numbers satisfying that condition exist. Another simple example is: The set of all perfect squares ending in 7 is $\emptyset$.

The complement of a set with respect to some stated or understood universe is the set of elements in the universe which are not in the given set. That is, the complement of A, symbolized by A', is the set $\{x \mid x \notin A\}$.

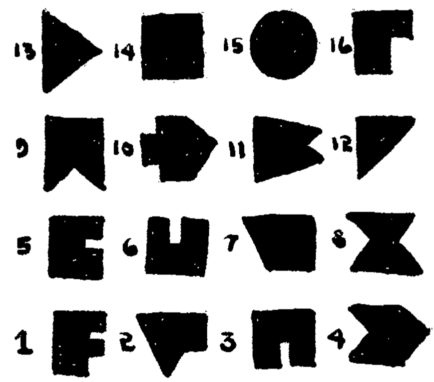
CHAPTER IV

SENTENCES, VARIABLES, AND ORDERED PAIRS

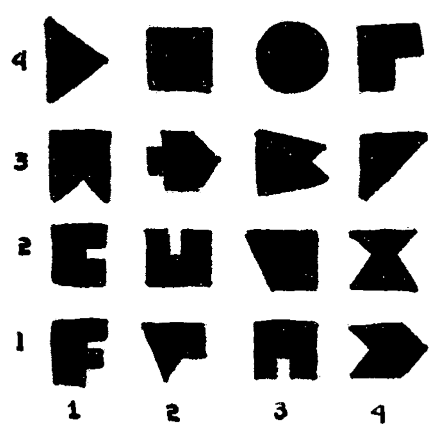
Consider the diagram below as a map of a section of your present school campus.



Your job is to make up 16 names for these rooms so that a new student could find his way around as quickly and easily as possible. Of course, you could give 16 different names like "Art Room," "Science Room," etc., and use those. Such names would tell the new student the courses that are taught in the rooms, but they would not help him learn where the rooms are located. A simpler set of names would be the integers from 1 through 16. It would be easier to use these names if you assigned them in order. For instance, like this:



Then, it would be very easy for the new student to learn where the rooms are located. If you told the new student to go to room 12, he would think: "Room 12"; there are four (4) rooms in each row, so the first row is 1 to 4, the second row is 5 to 8, and the third row is 9 to 12. So row 12 is the fourth room in the third row. Notice that he tells himself the location of room 12. He went from the single number 12 to two numbers: the fourth room in the third row. Since you think of the rooms in terms of "which row" and "which room in the row," you might just as well have named them that way in the first place, like this:



Then by giving someone two numbers, that is, a pair of numbers, you can tell him precisely which room you are talking about. But, note that if you just tell him, say "1 and 2," he will not know whether you mean "first row, second room" or "first room, second row." You could avoid this difficulty by always telling which part of your instructions gives the row and which part gives the room in the row. If everyone agreed to follow this convention, then it would be enough to say, for example, "Go to '3,5'" or, for another room, "Go to '5,3.'"

When you give directions in this manner, you are giving a pair of numbers in order, a first number and a second number. Such a pair of numbers is called an ordered pair of numbers. In mathematics, one customary way of naming an order pair of numbers is to put the names for the numbers in parentheses, with the name of the first number on the left and the name of the second number on the right, and separate them with a comma. As for an example, consider the order pair of numbers where 2 is the first number and 3 is the second number. We may write this example as (2,3). There are many other ways that ordered pairs may be named, but in this report, the writer will use the symbol, (x,y).

Sentences in One Variable

A sentence, such as "The state of Virginia is one of the original thirteen," may be changed into an open sentence by replacing the word "Virginia" by a blank, thus: "The state of ________ is one of the original thirteen."

This open sentence is neither true nor false, but it becomes true or false for any specific replacement of the blank from an appropriate list (or set) of names. Thus, if we replace the blank by "New York," the resulting statement is true; if we replace the blank by "Ohio," the statement is false. The appropriate set of names here is the list of the forty-eight states, and now there is a possibility that "Alaska" and "Hawaii" will be included in the universal set. If we choose as a replacement the name "China," the statement must be regarded as neither true nor false, since China is not the name of a state at all. Similarly, filling the blank by "Frigidaire," "Chevrolet," "television," or the name of anything other than the name of one of our states, yields a meaningless rather than merely a false sentence.

Usually, of course, we substitute a letter, called a variable, for the blank. Thus we would write, more conventionally: The state of $\underline{x}$ is one of the original thirteen.

In mathematics such open sentences are usually equations, or inequalities, rather than verbal expressions; thus,

$$x + 3 = 7 \qquad x > 2$$

The role of the variable in such sentences is clear: it is a placeholder for the name of some object in "the totality of things under discussion," the universe, or the universal set. As has been stated, it is customary to denote the universal set by **U**. If, for example, we are interested in the number of dots on the faces of one of a pair of dice, then

$$U = \{1, 2, 3, 4, 5, 6\}$$

It is most important to keep in mind the totality of things under discussion, or the universal set. This means, in high school mathematics, that we always must bear in mind the particular number system under consideration. In other words, what are the possible replacements for the variable? Once the universal set, U, is specified, a sentence such as $X + 4 = 7$ divides U into two sets: one set contains all replacements y or x that make $x + 4 = 7$ true; the other set contains all replacements for x that make $x + 4 = 7$ false. Thus, if U is the set of integers, the first set contains just one integer, 3; all other integers belong to the second set.

We can thus think of our sentence $x + 4 = 7$ as a set-selector: it selects from U (the set of numbers under consideration) just those numbers that make the sentence true when used as replacements for x. This selected set of numbers is called the solution set of the sentence.

For example, if U is the set of integers, the solution set of the sentence

$$x \geqq 3$$

is the set of integers greater than or equal to 3, namely:

$$\{3, 4, 5, \ldots\}$$

or

$$\{x \mid x \geq 3\}$$

which we read as: "the set of all x's such that $x \geq 3$."

Let us assume the usual one-to-one correspondence between the real numbers and the points on a line. Then, we can regard a sentence such as $x > 10$ as the selector of a set of points corresponding to its solution set. This set of points can be graphed on the line. In practice, the graph of the set of points is called the graph of the sentence for short.

Sets of Ordered Pairs

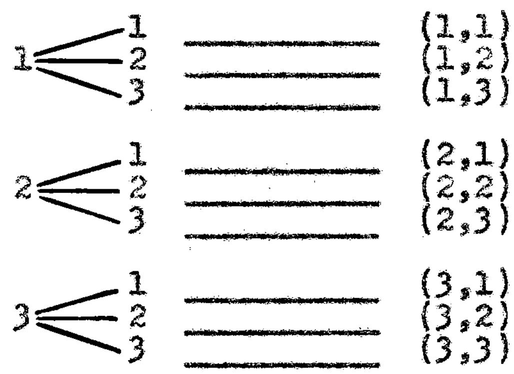
It is customary to denote the location of a point by ordered pairs of numbers; for example, (2, 3). As previously stated, the order in which the numbers are written is important. Thus, (2, 3) is not the same ordered pair as (3,2); their points are different. In general, we denote by (x,y) a pair of numbers x and y, considered in the order x first and y second. We call x and y the coordinates of the ordered pair (x,y).

We now wish to think about sets of ordered pairs. Any set of numbers can yield a set of ordered pairs in the following manner. Let U be a given set of numbers, say

$$U = \{1, 2, 3\}.$$

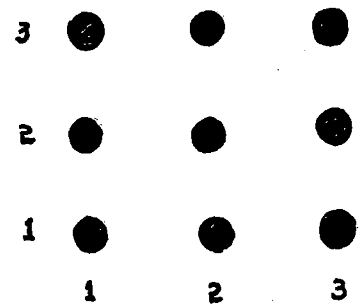
Then we can form the set of all ordered pairs whose coordinates belong to U. One way of doing it is as follows: form all possible ordered pairs with the first coordinate 1; then form all possible ordered pairs with the first coordinate 2; and so on.

In this operation, the following tree graph (6, 10) helps us to avoid missing any eligible ordered pairs:



The set of ordered pairs obtained in the foregoing manner is denoted by U x U (read "U cross U") and is called the Cartesian set of U. Thus, if $U = \{1, 2, 3\}$, then $U \times U = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

Considering each ordered pair as a position of a building in a city block, we can graph each ordered pair by selecting the first number as the "row" and the second number as the "building in the row." This graph is a square array consisting of nine points, as follows:



Similarly, if we consider the set J, where J is the set of integers, say

$$J = \{ \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \},$$

then J x J, as above, would be

. . . .

. . . .

. . . .

. . . (-4, 4), (-3, 4), (-2, 4), (-1, 4), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4) . . .

. . . (-4, 3), (-3, 3), (-2, 3), (-1, 3), (0, 3), (1, 3), (2, 3), (3, 3), (4, 3) . . .

. . . (-4, 2), (-3, 2), (-2, 2), (-1, 2), (0, 2), (1, 2), (2, 2), (3, 2), (4, 2) . . .

. . . (-4, 1), (-3, 1), (-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1) . . .

. . . (-4, 0), (-3, 0), (-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0), (3, 0), (4, 0) . . .

. . . (-4, -1), (-3, -1), (-2, -1), (-1, -1), (0, -1), (1, -1), (2, -1), (3, -1), (4, -1) . . .

. . . (-4, -2), (-3, -2), (-2, -2), (-1, -2), (0, -2), (1, -2), (2, -2), (3, -2), (4, -2) . . .

. . . (-4, -3), (-3, -3), (-2, -3), (-1, -3), (0, -3), (1, -3), (2, -3), (3, -3), (4, -3) . . .

. . . (-4, -4), (-3, -4), (-2, -4), (-1, -4), (0, -4), (1, -4), (2, -4), (3, -4), (4, -4) . . .

. . . .

. . . .

. . . .

One can see clearly the number of ordered pairs that we would have obtained by listing all the integers.

From previous work with graphs in algebra, one can see clearly the manyness of points in a plane. Each point in a plane corresponds to an ordered pair of numbers; the numbers in the pair are the coordinates of the point. The first number in the ordered pair is the first coordinate or the abscissa of the point corresponding to the ordered pair; the second number is the second coordinate or the ordinate of the point. The point is called the graph of the ordered pair of numbers.

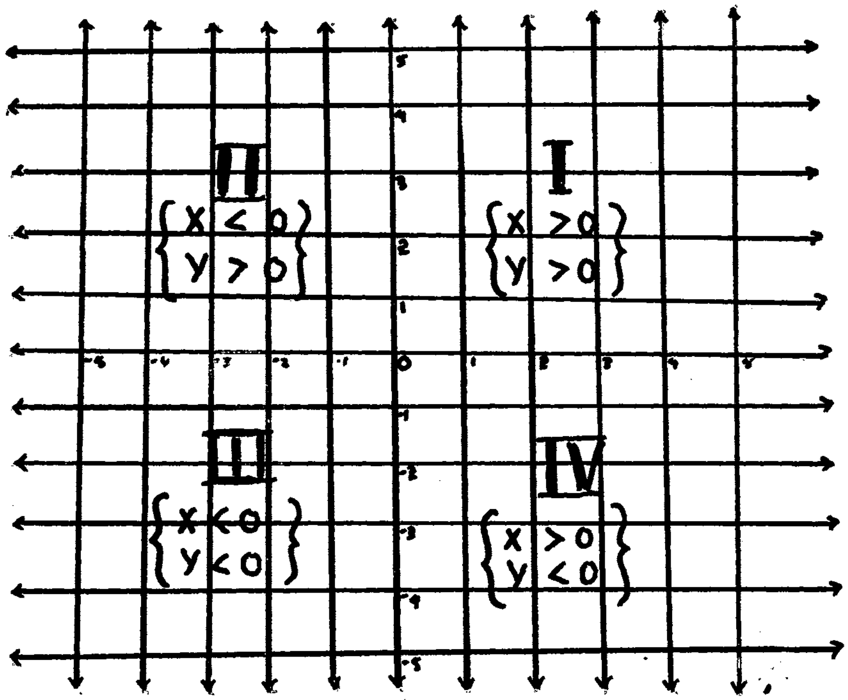
The set of points, each of which has second coordinate 0, is called the first coordinate axis or the x-axis. Points in this set are lined up horizontally in J x J. The set of points, each of which has first coordinate 0, is called the second coordinate axis or the y-axis. The points in this set are lined up vertically in J x J. Note that the two sets of points have one point in common. This point corresponds to the pair (0,0) and is called the origin.

The set of points, each of which has both the first and second coordinate greater than 0, is called the first quadrant. The set of points, each of which has the first coordinate less than 0 and second coordinate greater than 0, is called the second quadrant. The set of points, each of which has both the first and second coordinate less than 0, is called the third quadrant, and the set of points, each of which has the first coordinate greater than 0 and second coordinate less than 0, is called the fourth quadrant.

The origin and the coordinate axes are useful in locating graphs of ordered pairs. For example, suppose you are trying to find the graph of (3,4). First, put your finger on the origin. Then, move along the first coordinate axis until you locate the point corresponding to (3,0). You are in the column which contains, also, the graph of every ordered pair with first number 3. Therefore, this column contains the graph of (3,4). Now, return to the origin, and move your finger along the second coordinate axis until you locate the point corresponding to (0,4). You are in the row which contains, also, the graph of every ordered pair with second number 4. Therefore, this row contains the graph of (3,4). So let A = (3,x) and B = (x,4); therefore, $A \cap B$ (read "A intersects B") is the desired point.

The process of locating a point when you are given its coordinates often is called <u>plotting a point</u>.

Now, since the set of ordered pairs that gives rise to our first coordinate axis consists of all ordered pairs whose second number is zero, we can delete the zero and only consider the first number. In the second coordinate axis, delete the zero, and only consider the second number [excluding (0,0)]. We can represent the origin by a single 0. Therefore, the diagram below will suffice for the graphing of all ordered pairs in J x J.



Notice that each line in the diagram above, both vertical and horizontal, contains all the integers. Therefore, if we associate a line with each of the integers, the diagram above is capable of representing all possible ordered pairs in J x J.

Up to this point we have been working with a plane and with points which have integers as coordinates. It is natural to ask if it is possible to work with points which have coordinates belonging to the entire set of real numbers (directed real numbers). For example, can you plot the graph of $(3\frac{1}{2} - 16)$ or $(+97, -\sqrt{92})$? The answer is "yes." The space between each integer is completely filled with other points. These points consist of the rational numbers (fractions) and the irrational numbers. We could say that the spaces consist of all infinite decimal expansions.

Now we are able to say that there is a one-to-one correspondence between the real numbers and the points in a coordinate plane. Such a coordinate plane is called a Cartesian coordinate plane or, simply, a coordinate plane (7, 23).

As in the case of a coordinate plane with indefinitely many points, it is impossible to draw an accurate diagram of such a plane. Even if you draw a diagram of only part of a coordinate plane, you could not show all the points in that part of the plane because all you would have on your paper is a completely black region. So, when we make a diagram of part of a coordinate plane, we show only some of the points in that part. We show just a few sets of points such that each set consists of points having either the same first coordinate or the same second coordinate. Such sets of points are shown as straight lines in the diagram; these straight lines are sometimes called grid lines.

Sentences in Two Variables

Sentences in two variables are common in high school mathematics. For example:

$$y = x + 4 \qquad y > x \qquad x^2 + y^2 = 1$$

Such sentences are neither true nor false. However, they are true of, or false of, certain replacements for x and y. Thus, if $x = 1$ and $y = 5$, the sentence $y = x + 4$ is true; if $x = 5$ and $y = 1$, the sentence $y = x + 4$ is false. Such sentences are true of, or false of, ordered pairs of numbers (6, 11). An ordered pair such as (1,5) that, after replacement (take 1 for x and 5 for y), makes the sentence true, is called a <u>solution</u> of the sentence. The set of all such ordered pairs is the <u>solution set</u> of the sentence (6, 11).

The solution set for a given sentence depends on the totality of ordered pairs under consideration. For example, if the solution set must be found in the Cartesian set U x U, where

$$U = \{1, 2, 3\}$$

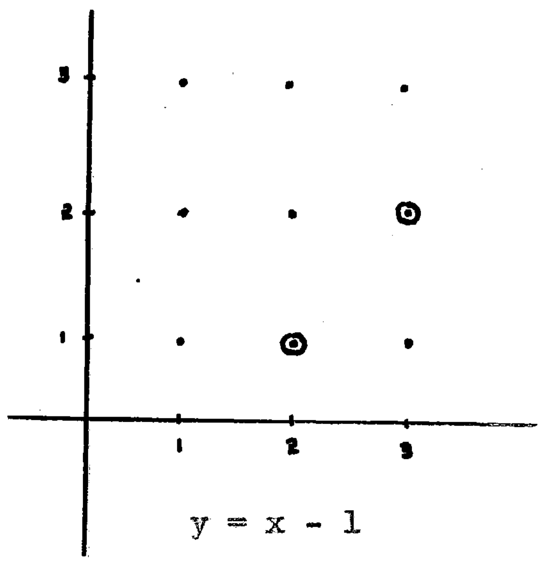
then the solution set for the sentence $y > x - 1$ is

$$\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

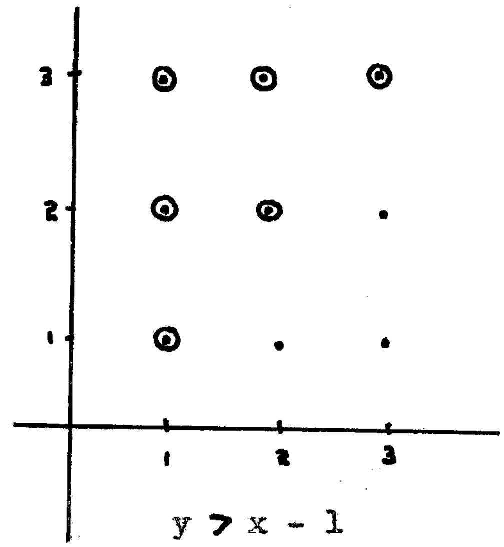
and the solution set for the sentence $y = x - 1$ is

$$\{(2,1), (3,2)\}$$

The graphs of these solution sets are shown below.



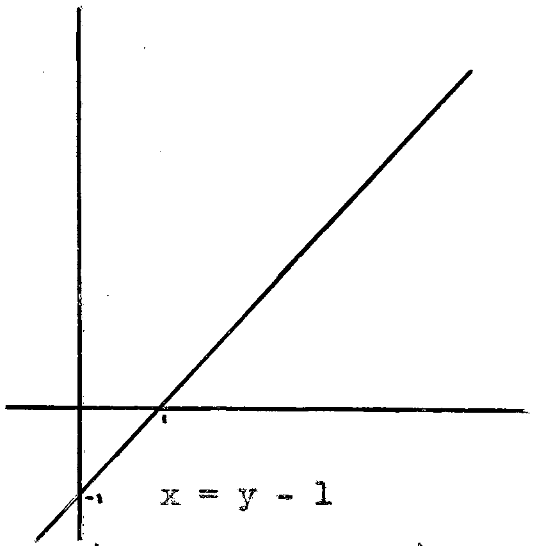
y = x - 1



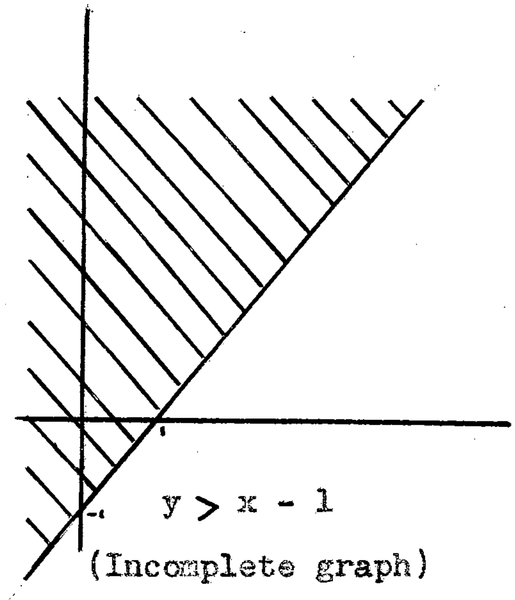
y > x - 1

On the other hand, if U is the set of real numbers, that is, "$R \cup R'$," where R is the set of rational numbers and R' is the set of irrational numbers, the union of the two sets consists of the entire set of real numbers. If U = R x R', then U x U is represented by the entire coordinate plane. The solution sets of y = x - 1 and y > x - 1 are infinite sets of ordered pairs. The graph of y = x - 1 is a straight line, and the graph of y > x - 1 is the "half-plane" above the straight line that represents the solution set of y > x - 1.

The figures below indicate the graphs of these infinite solution sets.



x = y - 1
(Incomplete graph)



y > x - 1
(Incomplete graph)

If we take U as the set of real numbers, we can regard a sentence in two variables, such as $y = x - 1$, as a set-selector. What is selected is a set of ordered pairs (the solution set), and this set of ordered pairs is a subset of U x U.

Using the set-builder notation, we can write:

$$\{(x,y) \mid y = x - 1\}$$

(read: "the set of ordered pairs (x,y) such that $y = x - 1$").

The graph of a sentence in two variables is, as we have seen, the graph of its solution set. We simply plot the set of ordered pairs in the usual way, but this viewpoint broadens the concept of a graph and admits a much wider variety of graphs in the earlier stages of mathematics.

Closely related is the notion of a locus, defined thus:

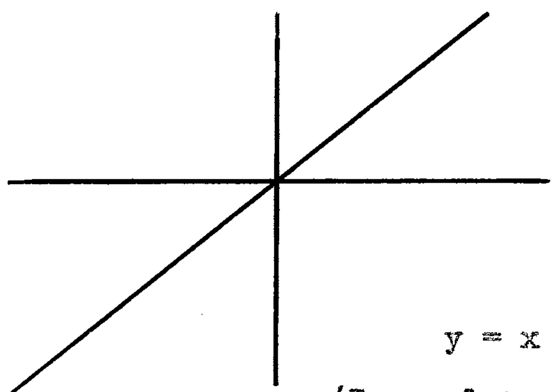
A locus is the set of those points, and only those points, that satisfy a given condition (3, 188).

The condition may be, but does not necessarily have to be, an equation or inequality with two variables. In case it is, the solution set of the equation or unequality corresponds to the set of points that makes up the locus. Thus, the graph of the equation or inequality pictures the locus, which is commonly called the locus of the equation or inequality. In other words, the locus of an equation or inequality is the set of points that corresponds to the solution set of the equation or inequality.

CHAPTER V

RELATIONS AND FUNCTIONS

To this point, we have regarded a sentence in two variables as a set-selector. For example, the sentence $y = x$ selects from the Cartesian Set U x U (represented by the coordinate plane) a subset of ordered pairs with equal coordinates (represented by a straight line as shown).



$y = x$

(Incomplete graph)

Now here is another way of looking at things: we may think of a sentence in two variables as expressing a relation that holds, or fails to hold, for the coordinates x any y of an ordered pair (x,y) belonging to U x U. Thus,

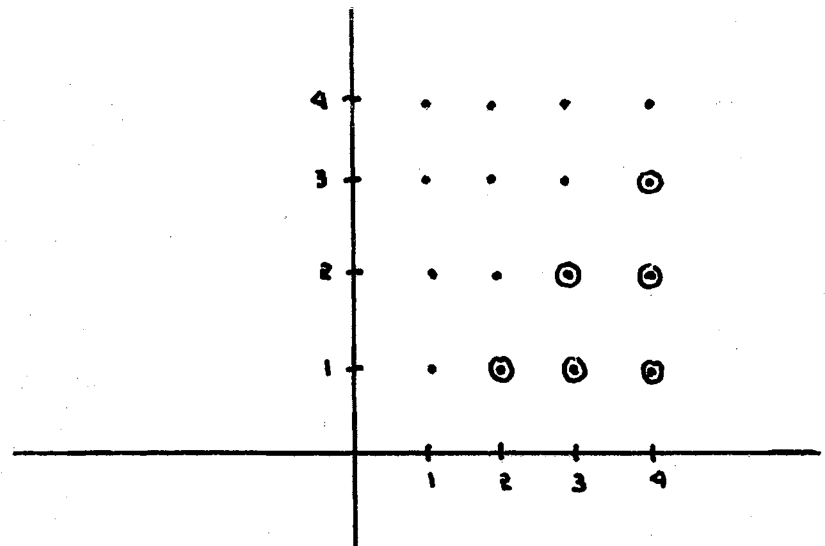
$$y < x$$

selects from U x U a subset of just those ordered pairs (x,y) of U x U for which the relation $y < x$ holds.

On the other hand, if

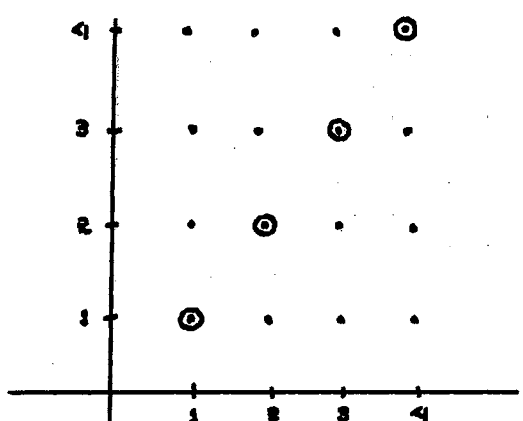
$$U = \{1, 2, 3, 4\}$$

then the set of ordered pairs U x U is represented by an array of 16 points as shown. The sentence y < x expresses a relation that holds for just six ordered pairs of U x U.

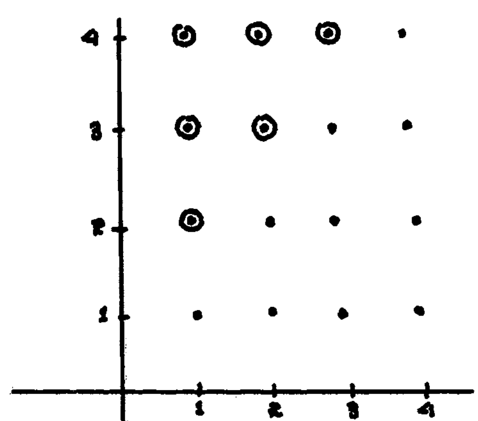


$\{(x,y) \mid y < x\}$

In like manner, we can graph the subset of U x U for which the relations expressed by y = x and y > x, respectively, hold.

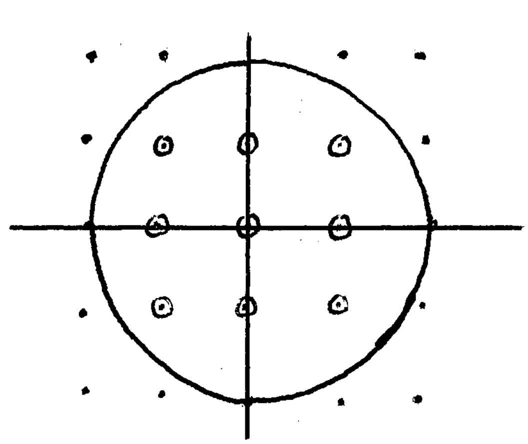


$\{(x,y) \mid y = x\}$



$\{(x,y) \mid y > x\}$

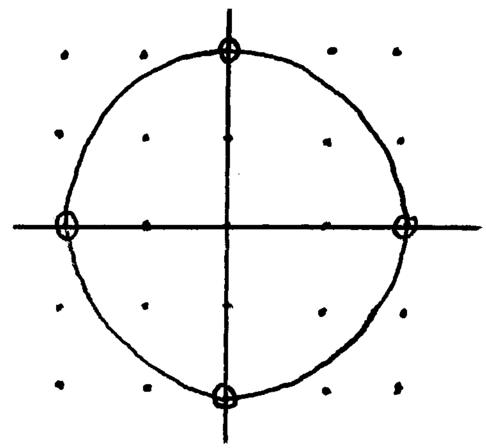
Since relations are sets of ordered pairs, they may be graphed. Here are some examples.



Example 1:

$U = \{-2, -1, 0, 1, 2\}$

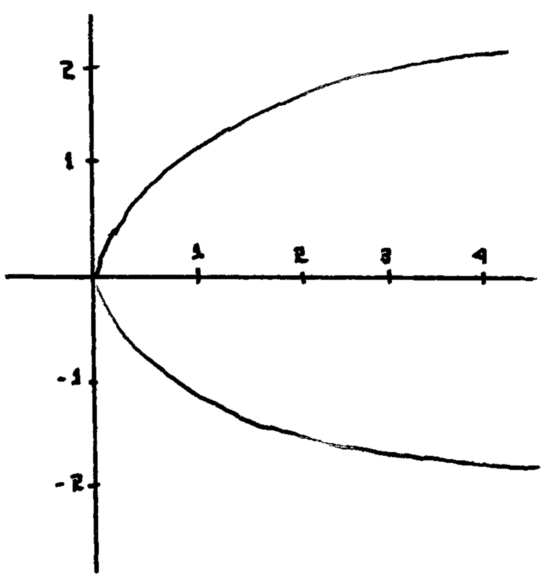
$\mathbb{R}_1 = \{(x,y) \mid x^2 + y^2 < 4\}$



Example 2:

$U = \{-2, -1, 0, 1, 2\}$

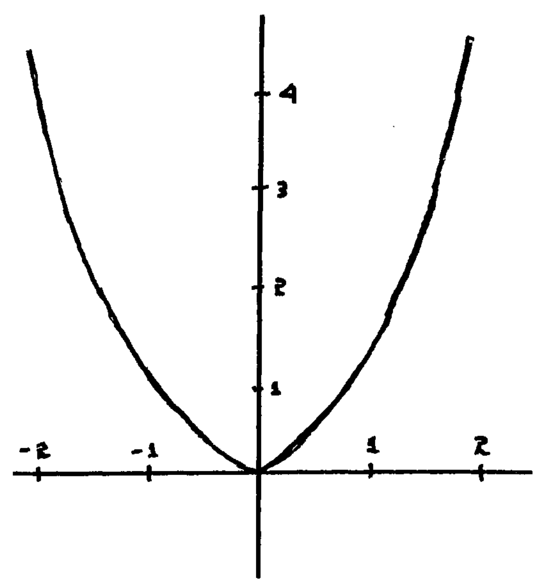
$\mathbb{R}_2 = \{(x,y) \mid x^2 + y^2 = 4\}$



Example 3:

$U = R^{\#}$

$\mathbb{R}_3 = \{(x,y) \mid y^2 = x\}$



Example 4:

$U = R^{\#}$

$\mathbb{R}_4 = \{(x,y) \mid y = x^2\}$

Suppose that we have a relation $\mathbb{R}$ in U. Then $\mathbb{R}$ is a set of ordered pairs (x,y). The subset in U for which x is a placeholder is called the <u>domain of the relation</u>; the subset in U for which y is a placeholder is called the <u>range of the relation</u>. As examples, refer to the four relations $\mathbb{R}_1$ through $\mathbb{R}_4$ graphed above. For convenience, we tabulate.

Relation	Domain	Range
$\mathbb{R}_1$	$\{-1, 0\ 1\}$	$\{-1, 0, 1\}$
$\mathbb{R}_2$	$\{-2, 0, 2\}$	$\{-2, 0, 2\}$
$\mathbb{R}_3$	The set of positive real numbers and 0.	The set of real numbers.
$\mathbb{R}_4$	The set of real numbers.	The set of positive real numbers and 0.

Note that $\mathbb{R}_1$ and $\mathbb{R}_2$ give rise to graphs containing a finite number of isolated points; the range and the domain are said to be <u>discrete</u>. However, relations $\mathbb{R}_3$ and $\mathbb{R}_4$ give rise to smooth graphs with no points missing; that is, no gaps; the range and the domain in such cases are said to be <u>continuous</u>.

Functions

Let us re-examine the relations $\mathbb{R}_1$ through $\mathbb{R}_4$. Note that $\mathbb{R}_4$ enjoys a special property: for each x in the domain, there is one and only one y such that (x,y) belongs to the relation. Such a relation is called a function. Hence, a function is a special kind of relation.

A function in U is a set of ordered pairs (x,y) belonging to U x U and having for each x one and only one y (6, 29).

Graphically, this means that a relation, $\mathbb{R}$, is a function if, and only if, no vertical line meets the graph of $\mathbb{R}$ at more than one point.

Let us consider the sets, B = $\{1, 3, 4\}$ and C = $\{6, 7, 9\}$; then form B x C = $\{(1,6), (1,7), (1,9), (3,6), (3,7), (3,9), (4,6), (4,7), (4,9)\}$. We shall use the set B x C to define in a very simple case a very important concept of mathematics; however, we will not delve into the concept to any extent. This concept mentioned previously is that of a function. A function from B to C is a subset of B x C with the one limitation that the elements of B appear once but not more than once as the first element of a pair.

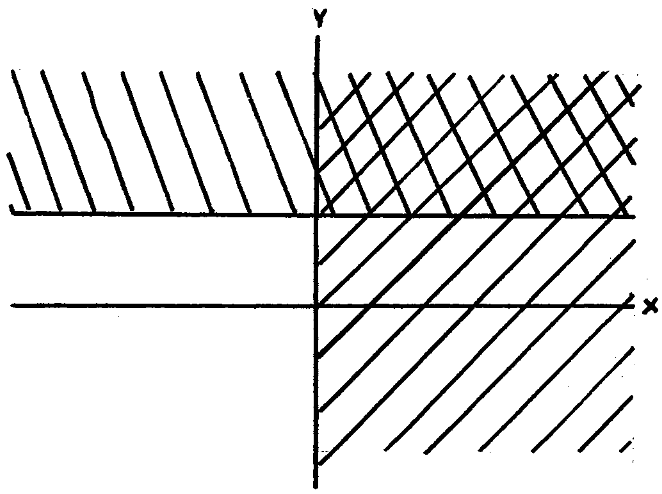
The notion of function involves a domain, a range, and a rule. The language of sets helps to clarify the meanings of these three words. The domain is the set of all first coordinates of the ordered pairs that make up the function; the range is the set of all second coordinates of the ordered pairs that make up the function; and the rule assigns a unique second coordinate in the range to a given first coordinate in the domain (6, 31).

CHAPTER VI

CONCLUSION

The ideas discussed in the previous chapters are particularly applicable to the development of the ideas of modern mathematics, especially those of locus. A locus is defined as the set of points which satisfy certain conditions. The conditions always can be considered as conditions placed on the coordinates (ordered pairs) of the points. These conditions can be expressed as open sentences, equations, and inequalities.

Open and closed half-planes are the loci of expressions like $x > 0$, $y \geq 1$, respectively. If we consider the set of points satisfying both of these conditions, we obtain the set of points in the doubly shaded area shown below.



$x > 0,\ y \geqq 1$

(Incomplete graph)

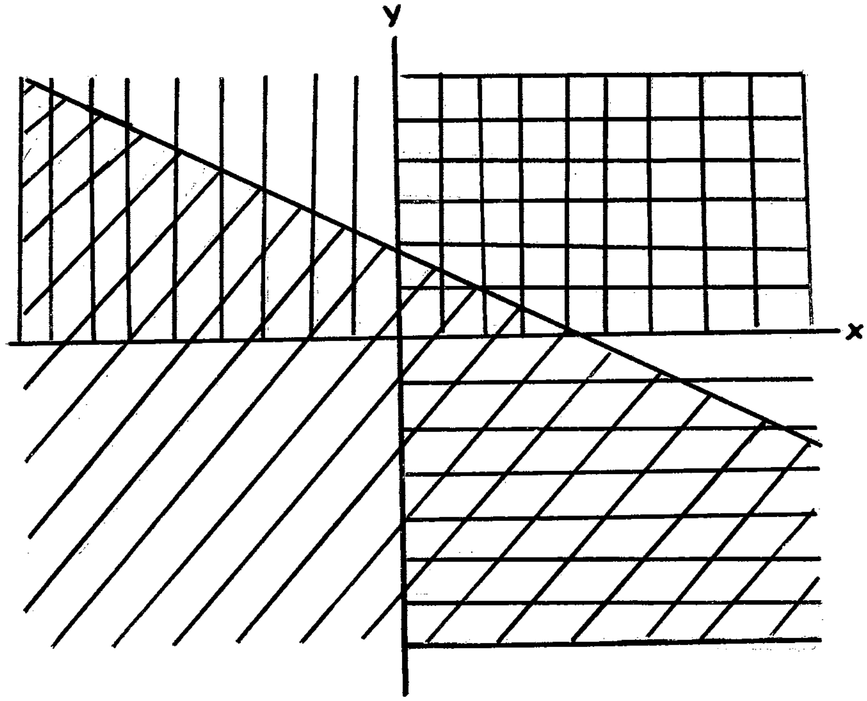
The doubly shaded portion is, of course, the intersection of the set of those points whose coordinates satisfy $x > 0$ and the set of those points whose coordinates satisfy $y \geqq 1$.

A particularly interesting extension of this work is given if we consider three open sentences, say:

$$x \geqq 0$$

$$y \geqq 0$$

$$x + 2y \leqq 6$$



$x \geqq 0,\ y \geqq 0,\ x + 2y \leqq 6$

(Incomplete graph)

The set of points shown by the horizontal shading is the solution set of $x \geqq 0$ (including the points on the axis); the set of points shown by vertical shading is the solution set of $y \geqq 0$ (including points on the axis); and the set of points shown with the diagonal shading is the solution set of $x + 2y \leqq 6$.

The intersection of the three sets is, of course, the triangle (triply shaded region), including its sides and its interior.

The complement of this intersection is the exterior of the triangle.

Extending this approach to the circle, it is easy to see that $x^2 + y^2 = 25$ would select the set of points on the circumference of a circle with center at the origin and a radius of 5. The sentence $x^2 + y^2 \leq 25$ would select the points of the disk with center at the origin and radius 5 (i.e., the set of points on and within the circle). The sentence $x^2 + y^2 > 25$ would be the set of points outside of the circle.

We could continue with this approach and develop and solve many geometrical problems, but that is not the purpose of this report.

One should see clearly the many roads in which a teacher can travel in preparing students for a course in Coordinate Geometry.

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