PROPAGATION OF LONGITUDINAL TENSION IN A SLENDER MOVING WEB

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ABSTRACT

To date, most of the theoretical work on longitudinal web behavior has been directed at the problem of controlling average tension. Very little attention has been given to the subject of this paper - propagation of tension within a span.

The model presented here is based on the one-dimensional wave equation, modified for a moving medium. Boundary conditions are developed that, for the first time, incorporate tension and mass transfer on rolling supports. The P.D.E. is solved analytically using Laplace transforms.

A number of phenomena are described that will be of interest to process designers and troubleshooters. These can be used to explain existing tension problems, whose causes may have been unrecognized in the past, and to anticipate problems that will appear as line speeds are increased. Among these are:

1. Propagation of strain discontinuities when draw is increased suddenly.
2. Amplification of repetitive strain disturbances due to strain reflection and reinforcement.
3. Damping of solitary strain disturbances.
4. Alteration of longitudinal resonant frequencies by transport motion.

Another important use of the model is to serve as a necessary step toward more advanced models that include out-of-plane motion, viscoelasticity and aerodynamics.

The model is tested by comparing it to the currently accepted O.D.E. model. At large time scales, where propagation phenomena are imperceptible, the two models are in good agreement.

NOMENCLATURE

A_o	cross sectional area of web in relaxed state
A1, A2, A3	cross sectional area of the web at the entry to roller A, exit of roller A and entry to roller B
C	longitudinal velocity of sound in the web material
E	Young's modulus of web
L	length of unsupported span between rollers A and B
s	Laplace transform variable
t	time
T1, T2, T3	tension of the web at the entry to roller A, exit of roller A and entry to roller B
V1, V2, V3	velocity of the web at the entry to roller A, exit of roller A and entry to roller B
V_a	circumferential velocity of roller A
V_b	circumferential velocity of roller B
V_i	transport velocity of web
x	longitudinal position along web starting at roller A
σ_1, σ_2	density of web at the entry to roller A, and at the exit of roller A
σ_o	density of web in relaxed state
ε_I	initial strain in web
δv	amplitude of velocity change at roller B
ξ	displacement of a web particle from its rest position

INTRODUCTION

Current models of web tension do not include propagation effects. Web processes, so far, have not required it. Attention has focused on average tension and transverse (out-of- plane) oscillations. Very little attention has been given to longitudinal oscillations in moving solid materials.

For web handling it makes sense to consider longitudinal oscillations as one aspect of the broader subject of tension propagation. This paper presents such a model. It is based on the one-dimensional wave equation, modified for a moving medium. Boundary conditions are developed that, for the first time, incorporate tension transfer and mass transport on rolling supports. Use of the Laplace transform method to develop a closed-form solution facilitates analysis of a variety of forcing functions.

Solutions for step, single pulse, repetitive pulse and sine wave disturbances are presented along with a discussion of their implications for web processing.

BACKGROUND

An implicit assumption in deriving current models is that tension is uniform throughout the span at all times. A web being processed at a typical speed of 500 ft/min will take 1.2 seconds to move through a 10 foot span. Polymers have sound velocities ranging from 1 x 10^5 to 3 x 10^5 ft/min. Tension will, therefore, propagate through this span in approximately .002 to .006 seconds. At less than 0.5 % of the transport time, this seems insignificant. However, even at this line speed, there are ways propagation behavior can affect a process. Furthermore, as line speeds increase, propagation phenomena will become more important. At 10,000 ft/min paper lines are probably reaching this point now. The transport time for 10 feet at this speed would be only 0.06 seconds.

TRAVELING STRING STUDIES

There have been many investigations of transverse oscillations in moving strings. The moving material in these studies is often described as a "Traveling String". Use of the word string is not only due to simplifying assumptions. Many of the authors were interested in the problems of transporting yarn from spools into weaving processes. The earliest publication was by Skutch [1] in 1897. Subsequent papers by Sack, 1954, [2], Archibald and Emslie 1958, [3], Swope and Ames, 1963, [4], Ames, Lee and Zaiser, 1968, [5], Ames and Vicario, 1969 [6], Kim and Tabarrock, 1972 [7], and Fox and Lilley, 1991, [8] dealt with additional features of transverse oscillations, such as damping, large amplitude nonlinearities, nonconservative energy changes, and computational methods. Thurman and Mote [9], 1969, presented an analysis of band saw blades that included flexural rigidity. Miranker, 1960, [10], who was motivated by problems with magnetic tape transport, was the first to observe that energy changes were nonconservative. Yang and Mote, 1991, [11] introduced a method for active control of transverse oscillations in a moving string. A few of these papers included consideration of the longitudinal oscillations, that accompanied transverse oscillations - Ames, Lee and Zaiser; Ames and Vicario; Mote and Thurman. However, no attention has been given to longitudinal tension propagation as a principal feature of solid material transport.

THE PROBLEM

In the schematic of Figure 1, it is assumed that:

1. Both rollers are driven and their speeds may be controlled accurately.
2. Coulomb friction exists between the web and the rollers.
3. The web obeys the familiar capstan relationship [12] while it is on the roller.
4. The web is uniform in its relaxed state.
5. The web is elastic in the longitudinal direction (obeys Hooke's law).
6. The web is perfectly flexible in the transverse direction.

Possible inputs to the problem are:

1. V_a = Circumferential velocity of roller A
2. V_b = Circumferential velocity of roller B
3. ε_I = Strain at entry of roller A

The model has two independent variables, x and t. Variable x is the position along the span. Variable t is time. The dependent variable is ξ. It is the displacement of web particles from their relaxed positions. The span has length L, starting at the exit of roller A and ending at the entry of roller B. In Figure 1 the symbols T, V, and A refer, in the same order, to tension, velocity and cross sectional area.

There are a number of possible choices in setting up a particular problem. In a typical process line any one of three inputs, T1, V_a, or V_b could vary. However, in the next section it will become apparent that varying either of the first two variables leads to a nonlinear boundary condition requiring numerical methods for solution. Fortunately, varying V_b while holding T1 and V_a constant is a completely linear problem. This is the case that will be analyzed.

Although the principal topic of this paper is tension, most of the equations are formulated in terms of strain. Since Hooke's law is assumed, this creates no mathematical difficulties.

BOUNDARY CONDITIONS

Boundary conditions are needed to specify the particle velocities at the two ends of the span.

At the entry to roller B, the web will be in the stick zone, where coulomb friction and tension act to keep it from slipping. Therefore, at that boundary the speed will match the circumferential speed of the roller and the boundary condition is quite simple.

$$\frac{\partial \xi(L,t)}{\partial t} = V_b \qquad \text{Boundary Condition II} \tag{1}$$

Boundary condition I is more complicated. Web particles exiting the slip zone of roller A won't match the roller speed. They change velocity as the web detaches from the roller surface and responds to the tension in the span. An exact analysis of conditions within the slip zone is complicated by the nonlinear effect of friction between the web and roller. No attempt will be made to do this. Instead, the boundary will be assumed to be at the exit of the slip zone. Then, the principle of conservation of mass will be used to develop a relationship between strain and velocity at that point. Since the length of the slip zone is small compared to the total span, this should provide a reasonable approximation for Boundary I. At roller A:

$$A_1 \rho_1 V_1 = A_2 \rho_2 \frac{\partial \xi(0,t)}{\partial t} \tag{2}$$

Web particles at the entry to roller A will be in a stick zone. So:

$$V_1 = V_a \tag{3}$$

Expressions for A_1, ρ_1, A_2 and ρ_2 are determined as follows. Consider an increment of the web that in its relaxed state has a length l_0 cross sectional area, A_0 and density, ρ_0. When subjected to longitudinal stress, conservation of mass requires that the new values of area, A' and density, ρ' must conform to the following equation.

$$A' \rho' l_o (1+\varepsilon) = A_o \rho_o \, l_o \tag{4}$$

The symbol ε is longitudinal strain. For infinitesimal lengths, it is equivalent to the partial derivative of ξ with respect to x. Therefore by applying the principle of conservation of mass:

$$A_2\rho_2 = A_0\rho_0(1+\varepsilon)^{-1} = A_0\rho_0\left(1+\frac{\partial\xi}{\partial x}\right)^{-1} \quad and \quad A_1\rho_1 = A_0\rho_0(1+\varepsilon_1)^{-1} \qquad (5)$$

Boundary condition I can be defined by substituting equations (3) and (5) in (2).

$$(1+\varepsilon_1)^{-1}V_a = \left(1+\frac{\partial\xi(0,t)}{\partial x}\right)^{-1}\frac{\partial\xi(0,t)}{\partial t} \qquad \text{Boundary condition I} \qquad (6)$$

Equation (6) may seem unusual for a boundary condition. Nevertheless, it fits quite neatly into the subsequent analysis and produces results that correlate well with the results of the O.D.E. model. It is responsible for the mass transport that transfers tension from the previous span. This is an established feature of the O.D.E. model.

THE O.D.E. MODEL

Since the O.D.E. model has been confirmed by many years of use, it will be used to check the results of this analysis. Two versions are in use. One is nonlinear. It is used for cases such as startup of a process line where the web speed varies over a wide range. A linearized version is used for situations where the web speed changes by small amounts from a steady value. The linearized model will be used for the comparison. It is:

$$L\frac{d\varepsilon_2(t)}{dt} = \left[V_{a0}\varepsilon_1 - V_{b0}\varepsilon_2(t)\right] + \left(V_b(t) - V_a(t)\right) \qquad \text{O.D.E. model} \qquad (7)$$

where

$$\varepsilon_1 = \frac{T_1}{A_0E} \quad and \quad \varepsilon_2(t) = \frac{T_2(t)}{A_0E} \qquad (8)$$

V_{a0} and V_{b0} are constant, nominal values of roller speed. $V_a(t)$ and $V_b(t)$ are small perturbations from V_{a0} and V_{b0}. For the purpose of this study $V_a(t)$ and ε_1 are held constant, $V_{a0} = V_{b0} = V_i$. $V_b(t)$ will be assumed to be a step input of magnitude, δv. The solution with these conditions is:

$$\varepsilon_2(t) = \varepsilon_1 + \frac{\delta v}{V_i}\left(1 - e^{-\frac{V_i}{L}t}\right) \qquad \text{O.D.E. solution} \qquad (9)$$

At large time scales, where the propagation behavior of tension disturbances is invisible, the P.D.E. model should behave like equation (9).

THE P.D.E. MODEL

$$\frac{\partial^2\xi}{\partial t^2} = C^2\frac{\partial^2\xi}{\partial x^2} \qquad C^2 = \frac{E}{\rho} \qquad \text{P.D.E.} \qquad (10)$$

The one-dimensional wave equation (10) will be used to model the web. It is commonly seen in connection with the transverse oscillations of a fixed string or the longitudinal oscillation of a solid bar. If the strain never becomes compressive, a string under tension may be treated as a solid bar. The equation is based on two forces acting on particles of the string - inertial forces due to acceleration and elastic forces due to the spatial derivative of the strain. Its derivation can be found in most acoustics textbooks and will not be repeated here.

THE EULER DESCRIPTION

Equation (10) is in a form known as the Lagrange description. It applies to a situation in which the parameter being calculated is associated with a point that is allowed to move, under the influence of physical laws, relative to the observer. The alternative to the Lagrange description, (L. D.), is the Euler description, (E. D.). In the E. D. the equations describing the physics are modified so that the point of observation is held fixed as the material moves past. This is done by using the chain rule to explicitly separate the time and position derivatives. For example, if T represents temperature in a material that is moving along the x-axis with transport velocity, V, the relationship between the Lagrange and Euler derivatives is:

$$\frac{\partial T_L}{\partial t} = \frac{\partial T_e}{\partial t} + \frac{\partial T_e}{\partial x} V \qquad (11)$$

The Lagrange derivative is on the left. In fluid dynamics the term material derivative is used to emphasize that the Lagrange derivative is associated with a particular particle or piece of material. The first term on the right is the Euler time derivative. It does not apply to a particular portion of material and includes none of the variation associated with a change in position. The last term adds the position variation caused by transport motion.

THE EULER DESCRIPTION P.D.E.

Two changes will be made in the problem variables. First, each will be separated into a large steady value plus a small varying component. Second, an Euler description will be adopted.

The longitudinal velocity of web particles will be assumed to consist of the axial transport velocity, V_i plus a varying component.

$$\frac{\partial \xi_L}{\partial t} = V_i + \frac{\partial \xi'}{\partial t}$$

Corresponding to the two velocity components, there will be a constant component of strain, ε_1 plus a varying component. Therefore, the complete transformation of variables, including the Euler description, is:

$$\frac{\partial \xi_L}{\partial t} = V_i + \frac{\partial \xi'}{\partial t} = V_i + \frac{\partial \xi_E}{\partial t} + V_i \frac{\partial \xi_E}{\partial x} \quad , \quad \frac{\partial \xi_L}{\partial x} = \varepsilon_1 + \frac{\partial \xi_E}{\partial x} \qquad (12)$$

The "E" and "L" subscripts identify variables as Euler or Lagrange. The "E" subscript will be understood to encompass the separation of constant and varying components as well as the Euler description. This practice will be followed throughout the remainder of the paper.

Substituting (12) into (10) produces the wave equation for a moving medium. It is the form appropriate for this problem.

$$\frac{\partial^2 \xi_E}{\partial t^2}+2V_i\frac{\partial^2 \xi_E}{\partial x \partial t}+\frac{\partial^2 \xi_E}{\partial x^2}V_i^2=C^2\frac{\partial^2 \xi_E}{\partial x^2} \qquad \text{The E.D. P.D.E.} \qquad (13)$$

Equation (13) was presented in one of the earliest traveling string papers by Sack [2]. In that instance it applied to transverse motion. It is important to keep in mind that any solution of (13) must be transformed back to a Lagrange description using equations (12) before comparing it to laboratory results.

CONVERSION OF THE BOUNDARY AND INITIAL CONDITIONS TO AN EULER DESCRIPTION

The equations will be presented first in their most natural form - the Lagrange description. Then, to be mathematically consistent, equations (12) must be used to transform them into an Euler description.

At time zero the web will be assumed to be running with uniform speed, V_i and strain ε_1. Thus, the circumferential velocity of roller A is V_i. The circumferential velocity of roller B will be V_i plus a forcing function, $f(t)$ beginning at time zero.

In each case the Lagrange description is on the left and the Euler on the right:

$$(1+\varepsilon_1)^{-1}\left(1+\frac{\partial \xi_L(0,t)}{\partial x}\right)V_i=\frac{\partial \xi_L(0,t)}{\partial t} \quad , \quad V_i\frac{-\varepsilon_1}{(1+\varepsilon_1)}\left(\frac{\partial \xi_E(0,t)}{\partial x}\right)=\frac{\partial \xi_E(0,t)}{\partial t} \qquad \text{B. C. I} \quad (14)$$

$$\frac{\partial \xi_L(L,t)}{\partial t}=V_i+f(t) \quad , \quad \frac{\partial \xi_E(L,t)}{\partial t}+V_i\frac{\partial \xi_E(L,t)}{\partial x}=f(t) \qquad \text{B. C. II} \quad (15)$$

$$\frac{\partial \xi_L(x,0)}{\partial t}=V_i \quad , \quad \frac{\partial \xi_E(x,0)}{\partial t}=0 \qquad \text{I. C. I} \quad (16)$$

$$\frac{\partial \xi_L(x,0)}{\partial x}=\varepsilon_1 \quad , \quad \frac{\partial \xi_E(x,0)}{\partial x}=0 \qquad \text{I. C. II} \quad (17)$$

$$\xi_L(x,0)=\varepsilon_1 x \quad , \quad \xi_E(x,0)=0 \qquad \text{I. C. III} \quad (18)$$

$$\xi_L(0,0)=0 \quad , \quad \xi_E(0,0)=0 \qquad \text{I. C. IV} \quad (19)$$

$$\xi_L(L,0)=\varepsilon_1 L \quad , \quad \xi_L(L,0)=0 \qquad \text{I. C. V} \quad (20)$$

THE SOLUTION

Laplace transforms will be used to integrate the P.D.E. The solution follows a procedure described by Churchill in his book "Operational Mathematics" [13].

The first step is to take the time transform of the P.D.E., (13).

$$s^2 L\xi_E(x,t) - s\frac{\partial \xi_E(x,0)}{\partial t} - s\xi_E(x,0) + 2V_i\left[\frac{\partial\left(sL\xi - \xi(x,0)\right)}{\partial x}\right] = \left(C^2 - V_i^2\right)L\frac{\partial^2 \xi_E(x,t)}{\partial x^2} \tag{21}$$

The next steps will be clearer if the following change of variable is made.

$$U(x,s) = L\xi_E(x,t) \tag{22}$$

The last term of (21) can be modified to facilitate the analysis. It can be shown that an interchange in the order of differentiation with respect to x and integration with respect to t leaves the value unchanged. Thus,

$$L\frac{\partial^2 \xi_E(x,t)}{\partial x^2} = \frac{\partial^2 L\xi_E(x,t)}{\partial x^2} = \frac{\partial^2 U}{\partial x^2} \tag{23}$$

Substituting (22), (23), (17) and (18) in (21) produces a relationship that can be treated as an ordinary differential equation in one variable.

$$s^2 U + 2V_i s\frac{\partial^2 U}{\partial x^2} = \left(C^2 - V_i^2\right)\frac{\partial^2 U}{\partial x^2} \tag{24}$$

The solution of (24) is:

$$U = c_1 e^{\alpha_1 sx} + c_2 e^{\alpha_2 sx} \qquad \alpha_1 = \frac{1}{C - V_i} \qquad \alpha_2 = \frac{1}{C + V_i} \tag{25}$$

For this problem, strain is more important than displacement. To obtain strain, equation (25) is differentiated with respect to x.

$$\frac{\partial L\xi_E(x,t)}{\partial x} = c_1 s\alpha_1 e^{\alpha_1 sx} + c_2 s\alpha_2 e^{\alpha_2 sx} \tag{26}$$

The inverse transform of (26) solves the problem provided c_1 and c_2 can be evaluated. They are found from the boundary conditions.

Taking the time transform of boundary condition I and substituting (25), (26) and (18) with $x = 0$ produces:

$$-V_i\frac{\varepsilon_1}{1+\varepsilon_1}\left(c_1\alpha_1 s + c_2\alpha_2 s\right) = s\left(c_1 + c_2\right) \tag{27}$$

Taking the time transform of boundary condition II and substituting (25), (26) and (18) with $x = L$ produces:

$$V_i\left(c_1\alpha_1 s e^{\alpha_1 sL} + c_2\alpha_2 s e^{\alpha_2 sL}\right) + s\left(c_1 e^{\alpha_1 sL} + c_2 e^{\alpha_2 sL}\right) = Lf(t) \tag{28}$$

Equations (27) and (28) can now be solved simultaneously for c_1 and c_2. Substituting these values along with expressions for α_1 and α_2 into (26) produces:

$$\frac{\partial \xi_E}{\partial x} = L^{-1}\left\{\frac{Lf(t)}{C}\left[e^{\frac{s(x-L)}{C-V_i}} + \beta e^{-\frac{sx}{C+V_i}-\frac{sL}{C-V_i}}\right]\left[1-\beta e^{-\frac{2sLC}{C^2-V_i^2}}\right]^{-1}\right\} \tag{29}$$

where

$$\beta = \frac{-V_i + (\varepsilon_1 + 1)C}{V_i + (\varepsilon_1 + 1)C} \tag{30}$$

The last bracketed term of (29) prevents a straightforward use of a table of transforms. In a similar situation Churchill [13] uses the following series expansion.

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n \qquad 0 < z < 1 \tag{31}$$

β and $e^{-\frac{2sLC}{C^2-V_i^2}}$ are both less than 1. (There is no requirement to consider the transform variable, s as complex in this problem. Therefore, it can be considered as real and positive). Applying (31), to equation (29) converts it to a form that is easily inverted.

$$\frac{\partial \xi_E}{\partial x} = L^{-1}\left\{\frac{Lf(t)}{C}\left[e^{\frac{s(x-L)}{C-V_i}} + \beta e^{-\frac{sx}{C+V_i}-\frac{sL}{C-V_i}}\right]\sum_{n=0}^{m} e^{-\frac{2nLCs}{C^2-V_i^2}}\beta^n\right\} \tag{32}$$

The final step is to change variables by reapplying (12).

$$\frac{\partial \xi_L}{\partial x} = \varepsilon_1 + L^{-1}\left\{\frac{Lf(t)}{C}\left[e^{\frac{s(x-L)}{C-V_i}} + \beta e^{-\frac{sx}{C+V_i}-\frac{sL}{C-V_i}}\right]\sum_{n=0}^{m}\beta^n e^{-\frac{2nLCs}{C^2-V_i^2}}\right\} \tag{33}$$

Equation (33) will now be solved for a variety of driving functions.

STEP FUNCTION INPUT

The first driving function to be analyzed will be the unit step, $\Phi(t)$. The transform for a step that starts at $t = 0$ and adds a small increment, δv to the initial velocity V_i is:

$$Lf(t) = \frac{\delta v}{s} \tag{34}$$

Substituting (34) into(33) and inverting produces the following solution.

$$\frac{\partial \xi_L}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\left[\begin{array}{l}\beta^n \Phi\left(\dfrac{x - V_i t + Ct}{C - V_i} - \dfrac{L}{C - V_i} - \dfrac{2nLC}{C^2 - V_i^2}\right) + \\ \beta^{n+1}\Phi\left(\dfrac{-(x - V_i t) + Ct}{C + V_i} - \dfrac{L}{C - V_i} - \dfrac{2nLC}{C^2 - V_i^2}\right)\end{array}\right] \tag{35}$$

For a given value of t, this is a finite series because the time-shifted unit step functions, Φ, (Heaviside functions) are replaced by zero when their arguments become negative. So, a solution at some value of t requires $m+1$ terms where:

$$m = t\frac{C^2 - V_i^2}{2LC} \qquad \text{rounded up to the nearest integer} \tag{36}$$

INTERPRETATION OF EQUATION (35)

Although equation (35) is straightforward for purposes of calculation, it is hard to visualize. The diagram in Figure 2 may help.

The bars in the chart can be viewed as time-shifted step functions. There are two groups of bars corresponding to the two terms of equation of (35). The bottom bar in each group represents the step function for the summation index, $n = 0$. The next bar up is for $n = 1$, etc. The leftmost column shows the amplitude coefficient of each step. On the horizontal axis there are two scales. One is for time. The other is for position. The position axis starts at $x = L$ goes to $x = 0$ and then back to L in a repeating pattern. The time scale advances along with x at a rate consistent with the propagation velocity, $C+V_i$ or $C-V_i$. In the bottom third of the chart, equations show how the summation progresses with time. Each series is formed by adding up the terms indicated by the bars in its respective column. For example, during the time interval from $2T_1+T_2$ to $2T_1+2T_2$ the leading edge of the disturbance is in the process of moving from 0 to L. It has already gone through three previous cycles, advancing from L to 0, 0 to L, L to 0 again. In each cycle the strain grows by an amount shown by the last term in the series.

It will be noticed that the amplitude of the disturbance changes slightly on reflection at roller A, ($x = 0$). It changes by the ratio of β^{n+1}/β^n. At roller B, ($x = L$) the ratio is 1.

COMPARISON OF RESULTS WITH THE O.D.E. MODEL

Comparison of the P.D.E. model (35) with the O.D.E. model of equation (9) shows that on large time scales (large compared to the time for disturbances to propagate through the span) they behave alike. Graphs in Figures 3 and 4 illustrate an example. Parameters for both models are shown below (in the P.D.E. model $x=L$).

V_i = 10 m/sec   C = 1500 m/sec L = 1 m ε_1 = .0005 δv = .0001*V_i

Figure 3 shows the step response of the P.D.E. model. Figure 4 shows the percent difference between the two. On this time scale they are in very close agreement. The graph in Fig. 4 appears solid because the error makes a cycle once every 1.33 milliseconds. This is due the stair-like behavior of the P.D.E solution.

Figure 5 shows a portion of the P.D.E. model data (solid line) at higher resolution. It is superimposed on the O.D.E. data (dashed line). The stair-like shape of the P.D.E. graph is due to the propagation delay. The strain disturbance initiated at roller B travels through the web span toward roller A. The strain doesn't change at a particular point in the span until the disturbance reaches it. In this case the P.D.E. data is shown at $x = L$ and the step interval is 1.33 milliseconds.

Figure 6 is shows a different view of the solution. The abscissa is distance along the span instead of time. The disturbance is shown at four different times. It starts at x = 1 meter and progresses to the left until it reaches the end at x = 0 where it is reflected. It takes .671 milliseconds to travel this distance. The upper ramp at the left end has been reflected and is moving back to the source. It will take .662 milliseconds to make the return trip. This action continues with the strain rising in progressively smaller increments on each cycle until it reaches its steady state value of $\delta V_i/V_i + \varepsilon_1$. The velocity of the disturbance is equal to C-V_i traveling upstream and C+V_i downstream.

EXPONENTIAL BEHAVIOR OF P.D.E. (35)

While the example in the previous section is very strong evidence for agreement between the P.D.E. and the O.D.E., it is not proof. What is needed is to demonstrate mathematically that if C is allowed to become arbitrarily large, equation (35) becomes equivalent to (9).

The first step is to recognize that the index, n, of the P.D.E. solution is proportional to time. The relationship is:

$$n \cong \frac{t}{2L/c} \tag{37}$$

The only part of equation (35) that can account for the exponential behavior is β^n. So, for $V_i << C$ it is necessary to show that:

$$\beta^n = \left(\frac{-V_i+(\varepsilon_1+1)C}{V_i+(\varepsilon_1+1)C}\right)^n \cong e^{-(something)} \tag{38}$$

The first step toward (38) is to take the natural log of β^n.

$$\ln\left(\beta^n\right) = n\ln\left(\frac{\frac{V_i}{C}+1+\varepsilon_1}{\frac{V_i}{C}+1+\varepsilon_1}\right)^n = n\left[\ln\left(\frac{-V_i}{C}+1+\varepsilon_1\right) - \ln\left(\frac{V_i}{C}+1+\varepsilon_1\right)\right] \tag{39}$$

If z << 1, ln(1 + z) may be replaced by z. So, expression (39) leads to:

$$\beta^n \cong e^{-2n\frac{V_i}{C}} = e^{-\frac{V_i}{L}t} \tag{40}$$

Equation (40) can now be used with (35) to calculate the shape of the amplitude envelope.

$$\frac{\delta V_i}{C}\sum_{n=0}^{m}\beta^n(1+\beta) \cong \frac{\delta V_i}{C}\sum_{n=0}^{m}2\beta^n \cong 2\frac{\delta V_i}{C}\int_0^m e^{-2n\frac{V_i}{C}}dn = \frac{\delta V_i}{V_i}\left(1-e^{-\frac{V_i}{L}t}\right) \tag{41}$$

The first expression on the left is the envelope of the P.D.E. step response. Replacing β^n with (40) and approximating the summation with a continuous integration produces the O.D.E. equation.

RESPONSE TO A SINGLE PULSE

What happens when a single velocity pulse occurs at roller B? In the case of a fixed string without any damping, the pulse would travel back and forth between the supports indefinitely. Intuition suggests that the traveling string will be different, because material is flowing out of the span at roller B and being replaced with new material at roller A. To answer this question a single pulse of amplitude δv and length t_1 will be used in equation (33). The transform for this is:

$$Lf(t)=\delta v\frac{1-e^{-t_1 s}}{s} \tag{42}$$

Substituting in (33) and inverting:

$$\frac{\partial\xi}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\left[\begin{array}{l}\beta^n\left[\begin{array}{l}\Phi\left(\dfrac{x-V_it+Ct}{C-V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}\right)- \\ \Phi\left(\dfrac{x-V_it+Ct}{C-V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}-t_1\right)\end{array}\right] \\ +\beta^{n+1}\left[\begin{array}{l}\Phi\left(\dfrac{-(x-V_it)+Ct}{C+V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}\right)- \\ \Phi\left(\dfrac{-(x-V_it)+Ct}{C+V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}-t_1\right)\end{array}\right]\end{array}\right] \tag{43}$$

The results are illustrated in Figures 7 and 8, using parameters similar to those for the step input. The summation limit m is defined as before in equation (36). Figure 7 shows a single pulse at three different times - advancing from B to A and then back toward B. Figure 8 shows the pulse amplitude envelope over one second. Investigation of equation (43), shows that it decays exponentially with a time constant of L/V_i from an initial amplitude of $\delta v/C$. A good descriptive term for this decay is transport damping.

ENERGY TRANSFERS

Decay of the pulse in the previous example illustrates an important difference between wave propagation in a fixed string and propagation in a moving web. Energy in the material between the supports is not conserved. Miranker [10], was the first to point this out for transverse oscillations. Wickert and Mote [14] later showed that the energy transfer involved the supports. The amplitude decay described in the previous section makes it apparent that similar conclusions apply to longitudinal strain variations. A complete analysis of the energy transfers is beyond the scope of this paper. However, the following observations can be made.

Energy can be exchanged with the supports in the following way. When a pulse arrives at a roller there is a change in roller torque due to the tension change. And, since the roller is rotating, work is done, either on the roller or on the web. At A, the roller would take energy from the web because the direction of tension in the pulse and the direction of motion are the same. At B, the roller would add energy to the web because the direction of tension and the direction of motion are opposed. In addition to the exchanges with the supports there can also be a transfer of strain energy into the next span during the time the pulse is at roller B.

In addition to the transport damping, there will be viscoelastic damping in the web and friction losses at the rollers. Therefore, it is safe to assume that a single, short pulse will be attenuated quickly.

RESPONSE TO REPETITIVE PULSES

The Laplace transform for a repetitive pulse train of amplitude δv, period t_2 and pulse length, t_1 is:

$$Lf(t)=\frac{\delta v}{s}\frac{1-e^{-st_1}}{1-e^{-st_2}} \tag{44}$$

The denominator of the forcing function requires the same treatment as in (29). Using (31) a second time leads to a double summation.

$$\frac{\partial \xi_L}{\partial x}(x,t)=\varepsilon_1+\frac{\delta v}{C}\sum_{n=0}^{m}\beta^n\sum_{k=0}^{p}\left[\begin{array}{l}\left[\begin{array}{l}\Phi\left(\frac{x-V_it+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}-kt_2\right)-\\ \Phi\left(\frac{x-V_it+Ct}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}-kt_2-t_1\right)\end{array}\right]\\ +\beta\left[\begin{array}{l}\Phi\left(\frac{-(x-V_it)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}-kt_2\right)-\\ \Phi\left(\frac{-(x-V_it)+Ct}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^2}-kt_2-t_1\right)\end{array}\right]\end{array}\right] \tag{45}$$

The summation limit p will be:

$$p = \frac{t}{t_2} \qquad \text{rounded up} \tag{46}$$

A particularly interesting case arises when the repetition period is equal to the propagation delay, $L/(C-V_i)+L/(C+V_i)$. Then, each new pulse is met by the reflection of the one before. The amplitude reduction due to transport damping is greatly exceeded by the reinforcement of the new pulse. With such an input, the pulses grow exponentially to an amplitude of $\delta V_i/V_i$ at a time constant of L/V_i. The final amplitude is the same as if the pulse had been a step.

This clearly has implications for a web process. An eccentric or unbalanced roller could produce a disturbance once each revolution. If the web speed and span length are such that the period of the disturbance is an integer fraction (or if damping is low, an integer multiple) of $L/(C-V_i)+L/(C+V_i)$ the pulse may be amplified. Even a pulse that is attenuated by the viscoelastic damping of the web material may be amplified to many times that of a single pulse.

SINUSOIDAL INPUT

The solution for a sinusoidal input of amplitude, δV_i is:

$$\frac{\partial \xi_L}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\beta^n \left[\begin{array}{l} \sin\left[\omega\left(\dfrac{x-V_it+Ct}{C-V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}\right)\right]\bullet \\ \Phi\left(\dfrac{x-V_it+Ct}{C-V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}\right)+ \\ \beta\sin\left[\omega\left(\dfrac{-(x-V_it)+Ct}{C+V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}\right)\right]\bullet \\ \Phi\left(\dfrac{-(x-V_it)+Ct}{C+V_i}-\dfrac{L}{C-V_i}-\dfrac{2nLC}{C^2-V_i^2}\right) \end{array}\right] \tag{47}$$

The input will reinforce itself in the same manner as a repetitive pulse when:

$$\omega = 2\pi n\frac{\left(C^2-V_o^2\right)}{2LC} \qquad n = 1, 2, 3 \cdots \tag{48}$$

At these frequencies, the amplitude behaves in a manner similar to repetitive pulses. It grows exponentially to an amplitude of $\delta V_i/V_i$ with a time constant of L/V_i.

HIGH SPEED BEHAVIOR

As the transport speed, V_i approaches C, equation (48) approaches 0 for all n. Also, the upstream propagation velocity, $C-V_i$ approaches zero. This clearly indicates that something unusual happens at $V_i = C$. Could one see a standing wave of zero frequency? Study of the traveling string literature suggests that a more sophisticated nonlinear model is needed at these speeds. Furthermore, many other phenomena will become significant as speeds increase. The answer to this question should be postponed pending further study.

CONCLUSIONS

This highly idealized model has two principal uses. First, it is a necessary step toward more realistic models. Second, it provides a framework for understanding tension problems whose causes may have been unrecognized in the past.

Some of the shortcomings in the present model are:

1. There is no provision for the variation in mass per unit length in the span. The effect is small. But, it could be important, because conservation of mass is central to tension behavior. In the current model it is incorporated only in the upstream boundary equation. It is not hard to derive the P.D.E. But it is nonlinear and requires numerical methods for its solution. Lack of this feature will probably not affect the general behavior at low speeds.

2. There is no provision for viscoelasticity or damping. In actual webs, particularly polymers, viscoelasticity will have a strong effect on the shape, velocity (dispersion) and amplitude of disturbances. Metals, with no viscoelasticity and low damping, should conform more closely to the model.

3. No testing has been done. Laboratory work should be done to check the validity and limitations of this model before going on to more complex models.

Even though viscoelasticity, nonlinearities and friction will distort them, the following phenomena should be observable.

1. The strain pulse produced by a brief (of the order of $L/(C-V_i)+L/(C+V_i)$) speed difference between two rollers will be very small. The amplitude will be of the order of V_i/C times the amplitude that would be produced if the speed difference were present continuously.

2. The velocity of a disturbance is equal to $C-V_i$ traveling upstream and $C+V_i$ downstream. C may be a function of wavelength due to nonlinearities and viscoelasticity. But, for a single wavelength the relationship should hold.

3. When the difference in web speed between two rollers is increased rapidly to a new steady value, the initial strain will be only of the order of $\delta v/C$. This change will travel at the speed of sound to the nearest roller where it will almost double in size and be reflected back toward the source. At the source it will be reflected and increase again. This action will continue, with the strain rising in progressively smaller increments on each cycle, until it reaches a steady state value of $\delta v/V_i$. The rise will be approximately exponential with a time constant of L/V_i . At large time scales, where the steps are imperceptible, the behavior will match the O.D.E. model.

4. A single brief pulse (shorter than the time for the pulse to travel up the span and back again) will be damped by the transport motion. The pulse will be reflected back and forth. But, unlike a pulse in an ideal fixed string, it will decay a little each cycle until

it disappears. This "transport damping" along with viscoelasticity and friction will help remove energy from disturbances. An example of such a pulse is a sudden slip on a roller due to passage of a wrinkle or a splice.

5. A repetitive disturbance can be amplified if the period is an integer fraction *of* $L/(C-V_i)+L/(C+V_i)$. If damping is low it may be amplified even at integer multiples of the delay time. It will start with low amplitude and grow exponentially. The time constant will be L/V_i. The final amplitude will be the same as if the pulse had been a step. An example of a repetitive pulse source is the cyclic disturbance of an embossing roller.

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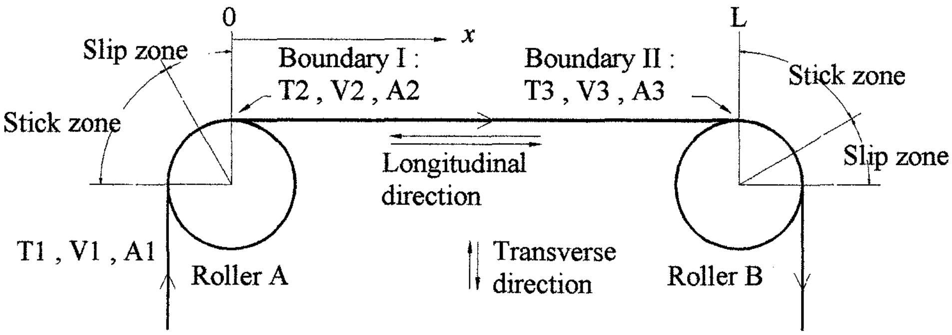

Figure 1 - Schematic of a Web Span

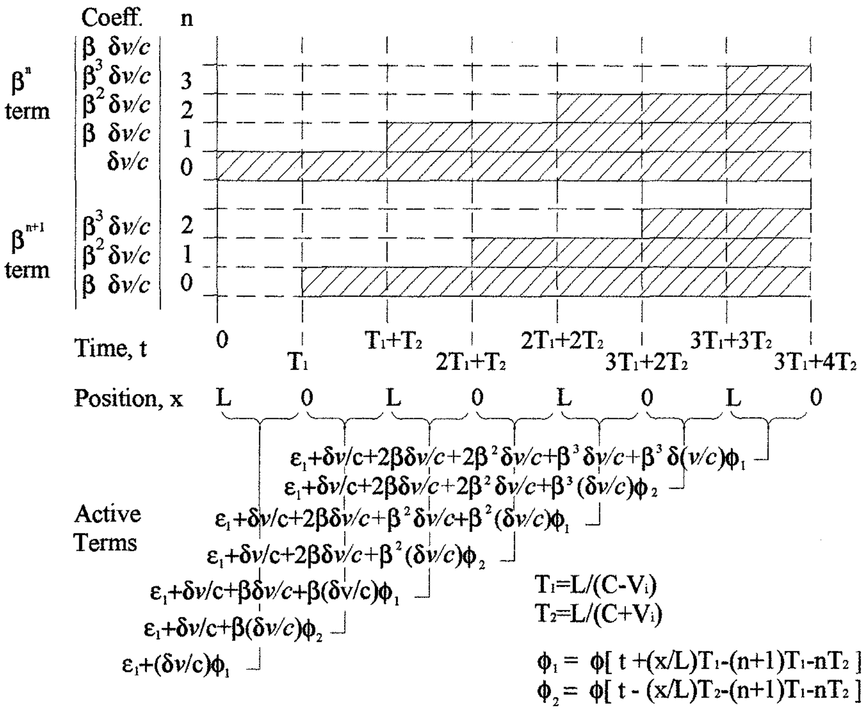

Figure 2 - This diagram illustrates equation (35), response to a step. The horizontal axis can be interpreted as either time or position for a disturbance starting at time, 0 and position, L. The vertical axis shows the contributions of the first few terms of the solution. As time progresses more terms are added in successively smaller increments. The top group of bars represent the trip from roller B to roller A. The bottom group represent the trip from A to B.

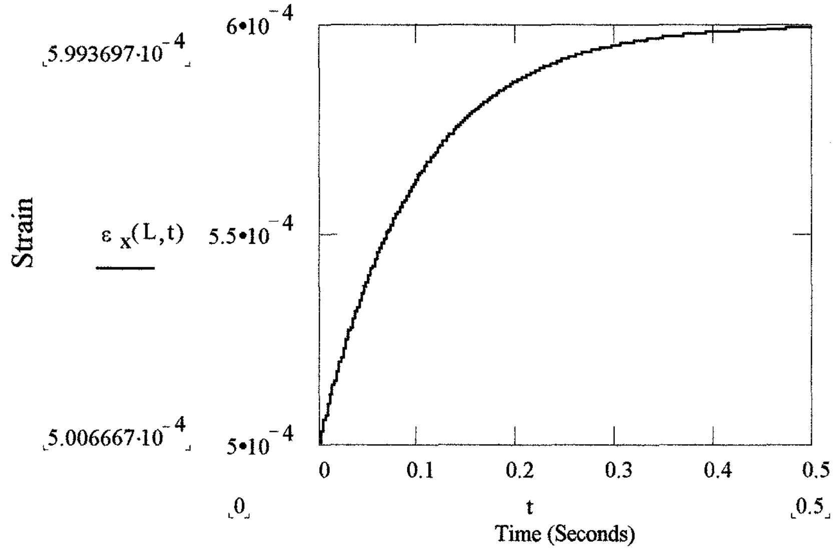

Figure 3 - Step response of the P.D.E. at $x=L$, $V_i = 10$ m/sec, $C = 1500$ m/sec, $L = 1$ m, $\varepsilon_1 = .0005$, $\delta v = .0001 * V_i$

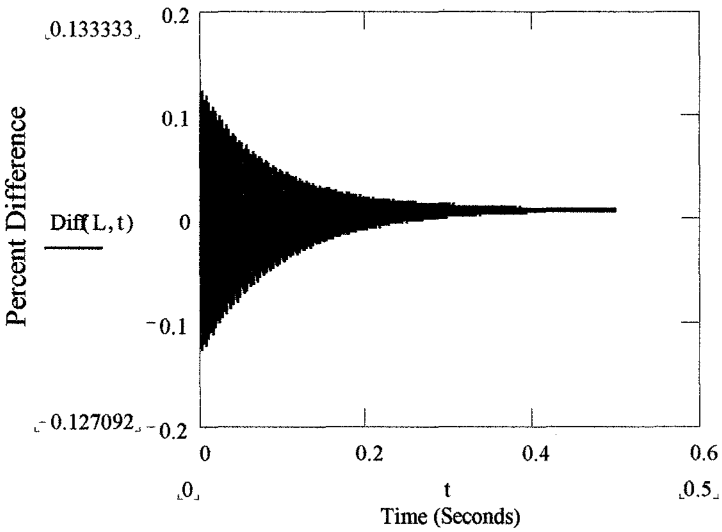

Figure 4 - Percent difference between the P.D.E. solution of Figure 3 and the O.D.E. model for the same conditions. The graph looks solid because of the closely spaced steps in the P.D.E. solution.

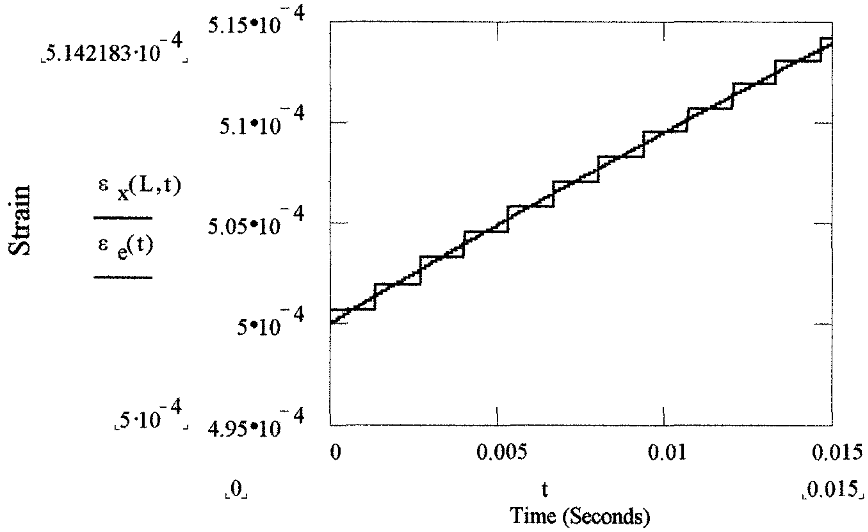

Figure 5 - High-resolution view of the first .015 seconds of the P.D.E solution of Figure 3, (solid line) and O.D.E. (dashed line) solutions.

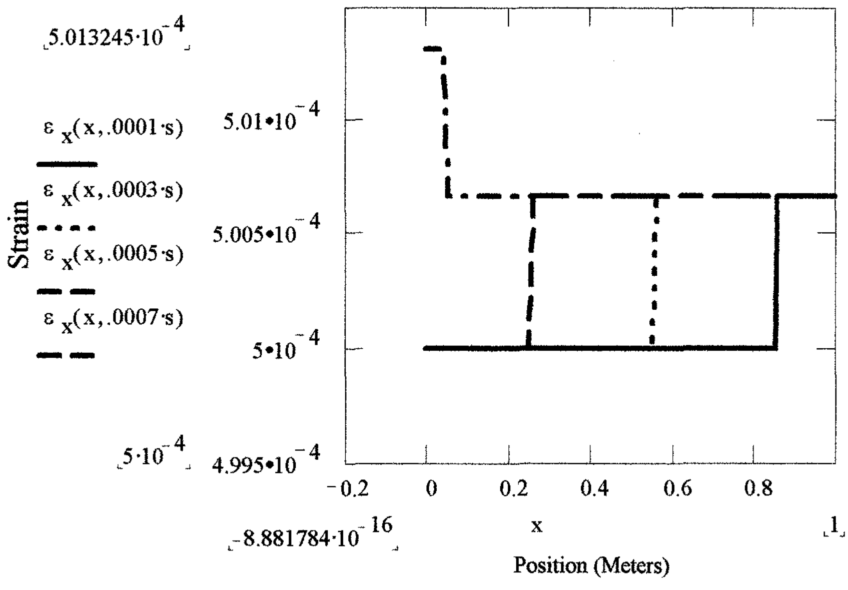

Figure 6 - Progress of a step-disturbance along the span. Strain is shown at .1, .3, .5 and .7 milliseconds. It starts at $x = 1$ meter and progresses to the left until it reaches the end at $x = 0$ where it is reflected. It takes .671 milliseconds to travel this distance. The upper step at the left end has been reflected and is moving back to the source. It will take .662 milliseconds to make the return trip. Conditions are the same as for Figure 3.

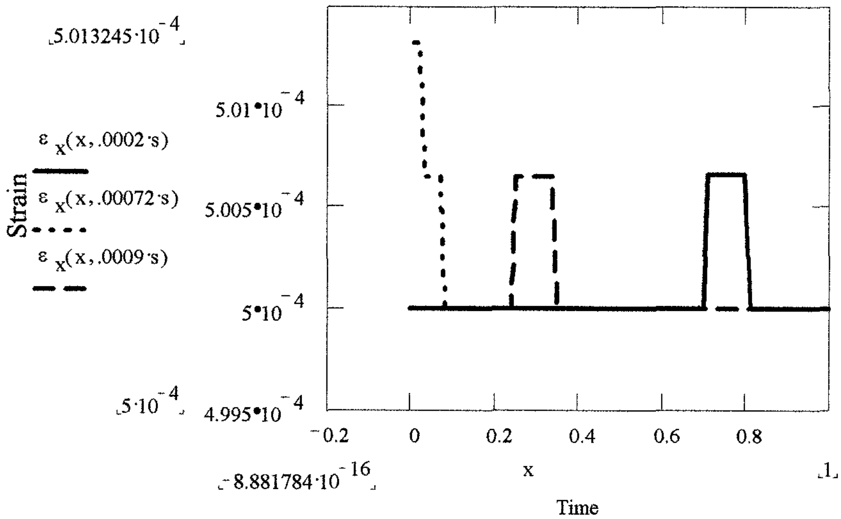

Figure 7 - A single pulse at three different times. It advances in the same manner as the step input. The first pulse on the right has just left roller B. The pulse on the left is the same pulse being reflected from roller A. The pulse in the middle shows it after it is reflected and is returning to B. V_i = 10 m/sec, C = 1500 m/sec, L = 1 m, ε_1 = .0005, δv = .0001*V_i , t_1 = .000067 sec $t_2 = L/(C-V_i)+L/(C+V_i)$ = .00133 *sec.*

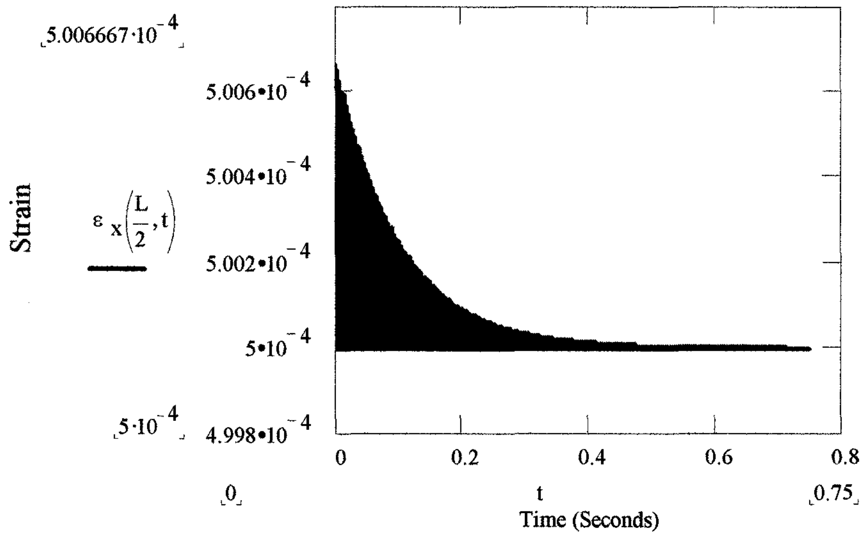

Figure 8 - Decay envelope of a single pulse over a period of 1 second. It decays exponentially from an initial amplitude of $\delta v/C$. The time constant is L/V_i. Conditions are the same as for Figure 7.

Jerald. L. Brown
Propagation of Longitudinal Tension in a Slender Moving Web
6/8/99 Session 3 1:45 - 2:10 p.m.

Question - Jim Dobbs, 3M
Are you taking into account the torque needed to overcome friction and inertia in the rollers?

Answer – J.L. Brown
No. It is assumed that the rollers are driven with perfect velocity control and there are no friction losses.

Question - Jim Dobbs, 3M
When the tension pulse is reflected off the opposite roll, some strain transport must occur. Are you modeling that or not? Or, haven't you got there yet?

Answer – J.L. Brown
That should be taken into account in the model, I made the assumption that on the entry into the roller the web simply takes on the circumferential speed of the roller. At the exit of the roller, I assume there is a slip zone. At roller A, something I didn't mention, is that the since the web speed coming off that roller isn't the same as the surface velocity of the roller; I used conservation of mass and assumed that the mass flow into the slip zone was the same as the mass flow out. From this, I got a relationship between strain and velocity at the upstream roller. To that extent, the effects of the roller are taken into account. At the downstream roller there is strain transfer. And the model is taking into account the strain that is transported out of the span.

Question - Jim Dobbs, 3M
You do assume that the roller turns at the incoming web speed?

Answer – J.L. Brown
At the downstream roller - yes.

Question - Jim Dobbs, 3M
So, this is not modeling rollers on which the web is slack.

Answer – J.L. Brown
No. Absolutely not. It does not assume any slippage on the rollers. To do that you get into some real analytical difficulties.