DEFLECTION AND CRITICAL VELOCITY OF ROLLERS

by

J. J. Shelton

Oklahoma State University
Stillwater, Oklahoma, U.S.A.

ABSTRACT

A coordinate system with its origin at the center of the roller, and with the origin moving as the roller deflects, is used for solution of the fourth order differential equation of beam mechanics. The result is a continuous polynomial in terms of x, the distance from the center. The form of the equation is adaptable to further analyses such as finding the profile of a nip roller, predicting wrinkling of a web, or predicting the natural frequency of a live-shaft roller as demonstrated in this paper.

The natural frequency of a live shaft roller is predicted by substituting values of deflection, caused by the weight of the roller determined either experimentally or analytically as in this paper, into a relationship derived by Rayleigh's method.

NOMENCLATURE

D_m	mean diameter of roller shell
E	modulus of elasticity of roller shell
E_s	modulus of elasticity of stub shaft
f	loading from resultant web tension (force /unit width)
f_n	natural frequency (cycles per second)
f_1, f_2, f_3, f_4	functions of web width, roller length, and bearing spacing for determination of roller deflection (dimensionless)
g	acceleration of gravity (386 in./sec^2 or 9807 mm/sec^2)
I	moment of inertia of roller shell
I_s	moment of inertia of stub shaft
L	length between centers of bearings
L_R	length of roller face
m	mass of roller per unit length
q	resultant loading on roller (force/unit length)
t_R	thickness of roller shell

W	width of web
w	weight of roller per unit length
x, y	coordinates of roller
δ	deflection
δ_R	deflection of roller because of its own bending
δ_{Rmax}	bending deflection of the roller at its center
δ_s	deflection of roller because of deflection of stub shafts
ρ	density of material in roller shell
ω_n	natural frequency (radians/second)

DEFLECTION

Fourth-Order Differential Equation with Moving Coordinate Axes

Although there are several methods for calculating deflection, the following method results in the simplest equations for deflection, slope, curvature or moment, and shear in a roller shell. The equations are continuous within each span of continuous stiffness and loading, so that they can be used for design of nip rollers and for other analyses which require a knowledge of conditions along the length of the roller. Further, the correctness of a derivation can be verified by substituting the equations of boundary conditions back into the original differential equation.

A roller is nearly always symmetrical about its center, with the maximum-width web approximately centered on the roller. The tension distribution is generally unknown, but an assumption of a uniformly distributed tension is usually satisfactory for roller design. The load on a roller is usually the vector sum of the tensile forces in the entering and exiting web spans (equal for an idler with the tension in its steady state) and the weight of the roller. Additional loads may come from liquid inside the roller or from a nip roller.

The coordinate system and sign conventions are shown in Figure 1, which illustrates a live shaft roller. The governing equation ([1], page 2) for any span of the roller with continuous loading q and constant stiffness is:

$$\frac{d^4y}{dx^4} = \frac{q}{EI} \tag{1}$$

where positive loading is upward. The loadings shown in Figures 1, 2, and 4 are negative. Other relationships are

$$EI\frac{d^3y}{dx^3} = N \tag{2}$$

for normal shear force, and for internal moment:

$$EI\frac{d^2y}{dx^2} = M. \tag{3}$$

Deflection Caused by Tension. The loading f in Figure 2 is shown as the resultant load caused by tension. The uniformly distributed load w of the weight of the roller has a length L_R, and the different distributed weight of the stub shafts could be included in analysis of the portion between the end of the roller and the center of a bearing, but the contribution of this weight to deflection is usually negligible. A separate solution of equation (1) is required for each section between discontinuities of loading or stiffness. Separate solutions for tensile forces and weight forces are generally more useful than a combined solution.

The general solution of the differential equation for the roller loaded by the tensile load f is

$$y = \frac{f}{24EI} x^4 + C_1x^3 + C_2x^2 + C_3x + C_4 \tag{4}$$

for a positive (upward) loading. The first term would be negative for a downward loading. The constants C_1 through C_4 are determined by substitution of boundary conditions into equation (4) and its derivatives.

Boundary conditions at the center of the roller are that the deflection (as defined by the moving coordinate system) and the slope are zero. At the edge of the web, the moment and shear are obtained from a free body diagram:

$$y_o = 0, \tag{5a}$$

$$y'_o = 0, \tag{5b}$$

$$EIy''_{(W/2)} = \frac{fW}{4}(L-W), \tag{5c}$$

and

$$EIy'''_{(W/2)} = -\frac{fW}{2}. \tag{5d}$$

The resulting equations for deflection and its derivatives are

$$y = \frac{fW}{8EI}\left(L-\frac{W}{2}\right)x^2 - \frac{f}{24EI}x^4, \tag{6}$$

$$y' = \frac{fW}{4EI}\left(L-\frac{W}{2}\right)x - \frac{f}{6EI}x^3, \tag{7}$$

$$y'' = \frac{fW}{4EI}\left(L-\frac{W}{2}\right) - \frac{f}{2EI}x^2, \tag{8}$$

and

$$y''' = -\frac{f}{EI}x. \tag{9}$$

Evaluation of equations (6) and (7) at x = W/2 results in equations for the deflection of the center relative to the edge of the web and the slope of the roller at the edge of the web, results useful in analyzing distortion of the web:

$$y_{(W/2)} = \frac{fWL^3}{48EI}\left[\frac{3}{2}\left(\frac{W}{L_R}\right)^2\left(\frac{L_R}{L}\right)^2 - \frac{7}{8}\left(\frac{W}{L_R}\right)^3\left(\frac{L_R}{L}\right)^3\right], \tag{10}$$

where the function inside the brackets is plotted as f_3 in Figure 3, and

$$y'_{(W/2)} = \frac{fW^3}{8EI}\left(\frac{L}{W} - \frac{2}{3}\right) \tag{11}$$

The deflection curve of the roller beyond the edge of the web ($W/2 < x < L_R/2$) is similar to equation (4) except that the first term does not exist, because there is no distributed force f in this span. The constants C_1 through C_4 assume different values, based on the boundary conditions

$$y_{(W/2)} = \frac{fW^4}{48EI}\left(\frac{3}{2}\frac{L}{W} - \frac{7}{8}\right), \tag{12a}$$

$$y'_{(W/2)} = \frac{fW^3}{8EI}\left(\frac{L}{W} - \frac{2}{3}\right) \tag{12b}$$

$$EIy''_{(L_R/2)} = \frac{fW}{4}\left(L - L_R\right) \tag{12c}$$

and

$$EIy'''_{(L_R/2)} = -\frac{fW}{2}. \tag{12d}$$

The resulting equations for deflection and its derivatives for this span between the edge of the web and the end of the roller face are

$$y = -\frac{fW}{12EI}x^3 + \frac{fWL}{8EI}x^2 - \frac{fW^3}{48EI}x + \frac{fW^4}{384EI}, \tag{13}$$

$$y' = -\frac{fW}{4EI}x^2 + \frac{fWL}{4EI}x - \frac{fW^3}{48EI}, \tag{14}$$

$$y'' = -\frac{fW}{2EI}x + \frac{fWL}{4EI}, \tag{15}$$

and

$$y''' = -\frac{fW}{2EI}. \tag{16}$$

Evaluation of equations (13) and (14) at $x = L_R/2$ results in equations for the deflection of the center relative to the end of the roller face and the slope of the roller at the end of the roller face:

$$y_{(L_R/2)} = \frac{fWL^3}{48EI}\left[-\frac{1}{2}\left(\frac{L_R}{L}\right)^3 + \frac{3}{2}\left(\frac{L_R}{L}\right)^2 - \frac{1}{2}\left(\frac{W}{L_R}\right)^2\left(\frac{L_R}{L}\right)^3 + \frac{1}{8}\left(\frac{W}{L_R}\right)^3\left(\frac{L_R}{L}\right)^3\right] \tag{17}$$

and

$$y'_{(L_R/2)} = \frac{fWL^2}{8EI}\left[-\frac{1}{2}\left(\frac{L_R}{L}\right)^2 + \frac{L_R}{L} - \frac{1}{6}\left(\frac{W}{L_R}\right)^2\left(\frac{L_R}{L}\right)^2\right]. \qquad (18)$$

The total deflection caused by the resultant uniformly distributed tensile force f is the deflection of equation (17), plus the deflection caused by the angle of the stub shafts as given by equation (18) multiplied by $(L-L_R)/2$, plus the deflection of the stub shafts as cantilever beams. The latter deflection is equal to fW/2, the force on the end of the cantilever (at the center of the bearing), multiplied by the cube of the length of the cantilever, $(L-L_R)/2$, and divided by 3 E_sI_s:

$$y_{(L/2)} = y_{(L_R/2)} + (L-L_R)y'_{(L_R/2)}/2 + \frac{fW}{6E_sI_s}\left(\frac{L-L_R}{2}\right)^3. \qquad (19)$$

Substitution of equations (17) and (18) into (19) results in

$$y_{(L/2)} = \frac{fWL^3}{48EI}\left[\left(\frac{L_R}{L}\right)^3 - \frac{1}{2}\left(\frac{W}{L_R}\right)^2\left(\frac{L_R}{L}\right)^2 + \frac{1}{8}\left(\frac{W}{L_R}\right)^3\left(\frac{L_R}{L}\right)^3 + 3\left(\frac{L_R}{L}\right) - 3\left(\frac{L_R}{L}\right)^2\right]$$

$$+ \frac{fWL^3}{48E_sI_s}\left[1-3\left(\frac{L_R}{L}\right) + 3\left(\frac{L_R}{L}\right)^2 - \left(\frac{L_R}{L}\right)^3\right]. \qquad (20)$$

Deflection Caused by Weight of the Roller. Deflection caused by the weight of the roller may also be important for evaluating distortion of a web; further, this deflection is required for calculating the natural frequency by the method of this paper.

The weight of the stub shafts and the heads are neglected in this analysis. This assumption is valid for most rigidly mounted rollers even if the numerical comparison of weights does not seem to justify the assumption, because the energy associated with deflection near the support bearings is low in comparison to the energy at points near the center of the roller.

The weight per unit length of the roller, w, can be substituted for f and L_R for W in equations (5a) through (9), resulting in the equation for deflection of the center of the roller relative to its end because of its weight:

$$y_{(L_R/2)} = \frac{wL_R^4}{48EI}\left(\frac{3}{2}\frac{L}{L_R} - \frac{7}{8}\right), \qquad (21)$$

and the slope of the end of the roller is

$$y'_{(L_R/2)} = \frac{wL_R^3}{8EI}\left(\frac{L}{L_R} - \frac{2}{3}\right). \qquad (22)$$

The total deflection caused by the weight of the roller is the deflection of equation (21) plus the deflection caused by the angle of the stub shafts as given by equation (22) multiplied by $(L-L_R)/2$, plus the cantilever deflection of the stub shafts. The first two of these components of deflection are dependent on the stiffness EI of the roller, while the third is dependent on the stiffness E_sI_s of the shafts. After combination of the first two components, collection of terms, and algebraic manipulation, these components of deflection of the center of the roller relative to the bearings caused by roller weight can be expressed as

$$\delta_{Rmax} = \frac{wL_RL^3}{48EI}\left[3\left(\frac{L_R}{L}\right) - \frac{7}{2}\left(\frac{L_R}{L}\right)^2 + \frac{9}{8}\left(\frac{L_R}{L}\right)^3\right]. \tag{23}$$

Equation (23) is the deflection of the center of the live shaft roller shown in Figure 2(B) because of the weight of the roller, if the usually negligible deflection caused by the weight of the stub shafts is neglected, and assuming the shafts to be infinitely stiff.

The third component of deflection of the roller because of its weight is the deflection of the entire roller as if it were a rigid body supported on its cantilevered stub shafts. The force on the end of the cantilever (at the center of the bearing) is $wL_R/2$, and the length of the shaft is $(L-L_R)/2$. The well-known equation for the deflection caused by compliance of the shafts therefore becomes

$$\delta_s = \frac{1}{3E_sI_s}\left(\frac{wL_R}{2}\right)\left(\frac{L-L_R}{2}\right)^3,$$

which, after expansion and algebraic manipulation, can be expressed as

$$\delta_s = \frac{wL_RL^3}{48E_sI_s}\left[1-3\frac{L_R}{L} + 3\left(\frac{L_R}{L}\right)^2 - \left(\frac{L_R}{L}\right)^3\right]. \tag{24}$$

For a dead shaft roller supported on bearings at the ends of the roller ($L_R = L$) as shown in Figure 2(A), equation (23) simplifies to

$$\delta_{Rmax} = \frac{wL^4}{48EI}\left(\frac{5}{8}\right), \tag{25}$$

which shows the total deflection caused by the uniformly distributed weight to be 5/8 as great as if the total weight wL were concentrated at the center.

<u>Summary of Equations for Deflection at Specific Points</u>

Deflection equations are here presented as a parameter which determines the deflection of a simply supported beam with a concentrated force in the center, multiplied by a modifying function for the specific condition of design and loading. These modifying functions are plotted in Figure 3 for the purpose of a quick reference or a check of calculations.

Uniformly Distributed Resultant Tension with Width of Web Less Than Roller Face. For the live shaft roller of Figure 1, equation (20) for the deflection at the center relative to the bearings can be written as

$$\delta = \frac{fWL^3}{48EI}\left[f_1 + \frac{EI}{E_s I_s} f_4 \right], \tag{26}$$

where

$$f_1 = \left(\frac{L_R}{L}\right)^3 - \frac{1}{2}\left(\frac{W}{L_R}\right)^2\left(\frac{L_R}{L}\right)^2 + \frac{1}{8}\left(\frac{W}{L_R}\right)^3\left(\frac{L_R}{L}\right)^3 + 3\left(\frac{L_R}{L}\right) - 3\left(\frac{L_R}{L}\right)^2 \tag{27}$$

and

$$f_4 = 1 - 3\left(\frac{L_R}{L}\right) + 3\left(\frac{L_R}{L}\right)^2 - \left(\frac{L_R}{L}\right)^3. \tag{28}$$

For the usual dead shaft roller with L, the spacing of the bearings, equal to L_R, the face length of the roller, f_1 simplifies to

$$f_1)_{L=L_R} = 1 - \frac{1}{2}\left(\frac{W}{L_R}\right)^2 + \frac{1}{8}\left(\frac{W}{L_R}\right)^3. \tag{29}$$

The minimum value of this special case of f_1 is 5/8 for a web width equal to the face length. The maximum value is unity for the hypothetical case of a concentrated tension in the center of the web, for which case the deflection is given by the factor outside the brackets in equation (26).

For the live shaft roller of Figure 1, the deflection of the center relative to the end of the roller, equation (17), can be written as

$$y_{(L_R/2)} = \frac{fWL^3}{48EI} f_2, \tag{30}$$

where

$$f_2 = \frac{1}{8}\left(\frac{W}{L_R}\right)^3\left(\frac{L_R}{L}\right)^3 - \frac{1}{2}\left(\frac{W}{L_R}\right)^2\left(\frac{L_R}{L}\right)^3 + \frac{3}{2}\left(\frac{L_R}{L}\right)^2 - \frac{1}{2}\left(\frac{L_R}{L}\right)^3. \tag{31}$$

For a dead shaft roller with $L = L_R$, f_2 of equation (31) becomes identical to the same special case of f_1 in equation (29).

__Uniform Force from Weight of Roller.__ Equation (23) can be written as

$$\delta_{Rmax} = \frac{wL_R L^3}{48EI} f_{1a}, \tag{32}$$

where

$$f_{1a} = \frac{9}{8}\left(\frac{L_R}{L}\right)^3 - \frac{7}{2}\left(\frac{L_R}{L}\right)^2 + 3\frac{L_R}{L}. \tag{33}$$

In equation (32), w is the weight per unit length of the roller shell including the cover; therefore, wL_R is the total weight of the roller shell. The weights of the heads (hubs) and stub shafts are neglected, as explained previously. The function f_{1a} is the same as f_1 in equation (27) and Figure 3 if $W/L_R = 1$.

The cantilever deflection of the stub shafts, δ_s of equation (24), can be expressed as

$$\delta_s = \frac{wL_R L^3}{48E_s I_s} f_4 , \tag{34}$$

where f_4 is given by equation (28) and is plotted in Figure 3. Therefore, the deflection (because of weight) of the center of the roller relative to the bearings is

$$\delta = \frac{wL_R L^3}{48EI} \left[f_{1a} + \frac{EI}{E_s I_s} f_4 \right] . \tag{35}$$

Equations (32) and (34) are used for calculation of the natural frequency in the second section of this paper.

Design of Crowned Nip Rollers

The continuous equations derived in this paper, where applicable, may be used for determining the profile required for imposing a uniformly distributed force to a nip, as in a calender, laminator, surface treater, roller-type coater, "wringer" rollers for liquid removal, and other web processes. If the assumed boundary conditions are not applicable, such as for a dead shaft roller with its bearing spacing less than the face of the roller, the method of derivation with the fourth-order differential equation and four boundary conditions for each continuous section is recommended. With this method, the solution is straightforward and verifiable for correctness.

The profile of a nip roller is correct for only one level of loading, which must generally be determined by experience or experimentation. Some nip processes, notably calenders, extend the range of satisfactory operation by imposing a moment on the ends of one of the nip rollers by means of hydraulic cylinders. However, such correction of the profile is not exact, because the terms of the polynomial of the variable x are not the same for the desired uniform distribution as for the pure moment on the ends. Another level of compensation for this discrepancy is sometimes implemented in the form of an "M" profile of the roller of the order of 0.001 inch (0.03 mm), but again such a profile is only exact for one value of loading and roller-bending moment. The basic crowning profile or compensation for the mismatch of polynomials can be determined by the methods of this report.

A common arrangement of rollers comprising a nip is a large, fixed, uncovered, driven roller, with a smaller rubber-covered unpowered nip roller located by pivoted arms and movable by means of air cylinders into and out of contact with the large roller. The large roller is generally cylindrical for economic reasons, while the small roller can be profiled to achieve a uniformly distributed loading.

In the above arrangement, the usual practice is to significantly wrap the smaller, moving, crowned nip roller with the web. The advantage of this arrangement is that the nip roller turns if the nip is open, so that there is no acceleration required of the nip roller when the nip closes. Disadvantages compared to slightly (10 degrees or so) wrapping the larger roller are: (1) Tension may significantly deflect the small roller, causing the crowning to be incorrect except at one value of tension. (2) Alignment of the moving roller is difficult, especially in the open-nip condition. (3) Crowning of the roller which is wrapped by the web tends to create wrinkles. (4) Wrinkling is less likely in a web feeding on to a large, low-friction roller than a small, high-friction roller.

A crowned nip roller is designed by assuming the nip loading to be uniformly distributed at the desired level, calculating the resulting deflection (the sum of the deflections of both rollers if both deflect significantly), then crowning one roller according to this calculated deflection function. If the crowned roller is rubber covered, the metal face may have to be crowned the same amount as the finished rubber face in order for this method to be correct because of the strong influence of thickness of the cover on the stiffness of the cover. The influence of the diameter on the stiffness of the cover is probably negligible for the small variation in diameter.

If the thickness of the web is negligible in comparison to the deflection of the rubber cover, the crowned nip roller can be designed for uniform loading across the entire face, and this loading will be maintained for any width of the web. For this case, equation (6) can be used to calculate the deflection curve, with W considered the face width of the roller (L_R in Figure 1). On the other hand, if the thickness of the web is not negligible, a design is exact for only one width of web. The design could again be done with equation (6) if the web is thick enough to space the rollers apart at all points outside the edges of the web, so that loading by the nip occurs only across the width W.

Discussion of Analysis of Deflection

The assumption throughout this paper of simple supports (no constraint imposed by the bearings against angular deflection) might be questioned by one unfamiliar with web handling machinery. Total constraint could be approached only by heavily preloaded bearings, such as double tapered roller bearings, at each support. The turning torque of such bearings would usually be prohibitive for an idler roller; furthermore, one end of a roller must generally be allowed to float axially to allow for differential expansion of the roller and its mounting. Constraint of this floating bearing would be extremely complicated. Therefore, most support bearings for rollers are completely self aligning within the small angles of misalignment encountered in operation.

The basic beam theory behind this analysis of deflection does not consider the component of deflection caused by shear stresses; therefore, the prediction of deflection will be low for a roller which is large and short, such as in calenders and throughout many metals processing lines. The complexity of incorporating the effect of shear stresses is rarely justified unless the diameter of the roller is greater than 20% of its length, if the roller shell is metal. (Hopkins [3] lists design factors and equations which show the deflection from shear stresses to be 6 percent of the deflection from beam bending for a uniformly distributed load on a simply supported thin-walled metal cylinder if D_m/L_R is 0.1, 24 percent for D_m/L_R of 0.2, and equal bending and shear deflections if D_m/L_R is 0.41.) Roller shells of fiber composites, however, have much greater relative shear deflections because of low moduli in shear. The deflection because of shear stresses in a solid metal shaft is rarely worth consideration, because if the shaft is short enough for this deflection to be relatively significant, it is likely to be so short that the total shaft deflection is negligible in comparison to the roller deflection. (From Hopkins [3], design factors and equations show that the deflection from shear stresses in a round cantilevered solid metal shaft to be 14 percent of the deflection from beam bending if D/L_R is 0.5, and 25 percent for D/L_R of 0.67.)

The analysis of a live shaft roller does not consider angular deflection of the stub shafts in addition to the angle of the ends of the roller shell as caused by flexibility of the heads (hubs) of the roller. Heads are usually designed to be quite stiff, by such means as thick heads, double heads, or stiff flanges with large bolt circles on removable stub shafts. If the deflection of the heads is significant, this effect could be readily incorporated into the equations for deflection. (See [4], Case 21, page 368, for the angular deflection of a flat circular plate with a fixed edge, loaded by a central moment.)

Example of Roller Deflection. A roller manufacturer lists two equations for calculating deflection, and shows sample calculations for a live shaft roller, with L_R = 189 in., L = 219 in., $E = 3.0(10)^7$ $lb_f/in.^2$, and I = 4787 $in.^4$ (24 inch O.D., 22 inch I.D. steel roller). The total load is 22,800 lb_f, presumably the vector sum of the tensile forces and the weight. The width of the web is not given, but the equations are for the assumption that the web width is equal to the roller face.

The first equation is $\delta = 5\ FL_R^3/384\ EI$, where F is specified as the total load (pounds). This equation is for a dead shaft roller with the load uniformly distributed over the face and with the bearings spaced at the ends of the roller shell. The calculated deflection is 0.014 inch.

The second equation, $\delta = FL_R^2\ (12L\text{-}7L_R)/384\ EI$, is equivalent to equation (10) if $W = L_R$. This equation does not consider the cantilever deflection of the stub shafts or the effect of the angle of the ends of the roller body. This equation gives a deflection of 0.019 inch.

The size of the 15-inch-long stub shafts was not mentioned, but if they were four-inch diameter solid steel, equation (20) gives a deflection of 0.0586 in., more than triple the higher deflection from the manufacturer's equations.

NATURAL FREQUENCY AND CRITICAL VELOCITY

Dead Shaft Rollers

The bearings in a dead shaft roller are usually near the ends of the roller shell, and usually do not constrain the roller against bending; therefore, the natural frequency is well documented in many vibration texts as a uniform beam with simple supports at its ends. This beam or roller is a distributed mass/spring vibration system, and therefore has an infinite number of natural frequencies. The mode shape at the first natural frequency is a half sine wave, and the next n-1 natural frequencies occur at 4, 9, 16, ..., n^2 times the frequency of the fundamental, with mode shapes of 1, $1\frac{1}{2}$, 2, ..., n/2 sine waves.

The first natural frequency occurs at ([2], page 421)

$$f_n = \frac{\pi}{2L^2}\sqrt{\frac{EI}{\rho A}} \quad \text{cycles/second.} \qquad (36)$$

If t_R, the thickness of the wall of the roller, is small in comparison to the mean diameter D_m, $I = (\pi/8)\, D_m^3 t_R$ and $A = \pi D_m t_R$, so that equation (36) becomes

$$f_n = \frac{\pi D_m}{4L^2}\sqrt{\frac{E}{2\rho}}\,. \qquad (37)$$

Because E/ρ is the same, for practical purposes, for carbon steel as for aluminum, the natural frequencies of a thin-walled aluminum roller and a steel roller of the same length and diameter are equal. The natural frequency is also unaffected by the thickness of the wall, for a given mean diameter.

Live Shaft Rollers

A live shaft roller usually has stub shafts which are much less stiff than the body of the roller, even if these shafts are solid steel. If the shafts and body were equal in stiffness and mass per unit length, the problem would be the same as the dead shaft roller previously analyzed. If the deflection of the shell were negligible in comparison to the deflection of the stub shafts, the problem would be the simple case of a concentrated mass on parallel springs. However, neither of these two simple models is commonly valid; instead, the body of the roller vibrates as a distributed mass/spring system while also vibrating as a mass on a spring. Avoidance of the first natural frequency is desirable and usually practical (by increasing the diameter of the roller and increasing the stiffness of the stub shafts) except for very long, high speed rollers such as in slitters in paper mills.

Rayleigh's Method. This method approximates the lowest natural frequency by equating the maximum kinetic and potential energies, based on a reasonable assumption for the deflection curve caused by the weight. The method has successfully predicted natural frequencies for a vast variety of problems.

The "exact" deflection curve for a live shaft roller as previously derived proved too unwieldy for a closed-form solution by Rayleigh's method and a practical presentation of the results.

The deflection curve (because of weight) of the roller body extended to include the stub shafts at the angle of the ends of the roller shell (as if the shafts were infinitely stiff) is assumed to be a parabola:

$$\delta_R = \delta_{Rmax}\left(1 - \frac{4}{L^2}x^2\right). \qquad (38)$$

Note that the coordinates shown in Figure 4 are different from those used for the derivations of deflection.

The total deflection δ is the sum of the above roller deflection δ_R and the cantilever deflection δ_s of the stub shafts, as shown in Figure 4:

$$\delta = \delta_{Rmax}\left(1 - \frac{4}{L^2}x^2\right) + \delta_s\,. \qquad (39)$$

The deflections δ_{Rmax} and δ_s can be calculated with equations (32) and (34).

The natural frequency by Rayleigh's method is:

$$\omega_n^2 = \frac{\int_0^{L/2} w\delta dx}{\int_0^{L/2} m\delta^2 dx}. \tag{40}$$

The masses of the stub shafts are neglected; however, all points on the roller undergo the full amount of the deflection of the stub shafts.

Substituting equation (39) into equation (40):

$$\omega_n^2 = \frac{w\int_0^{L/2} \left[\delta_{Rmax} + \delta_s - \frac{4}{L^2}\delta_{Rmax}\, x^2 \right] dx}{m\int_0^{L/2}\left[(\delta_{Rmax}+\delta_s)^2 - (\delta_{Rmax}+\delta_s)\left(\frac{8}{L^2}\delta_{Rmax}\right)x^2 + \frac{16}{L^4}\delta_{Rmax}^2\, x^4\right]dx}.$$

Integrating:

$$\omega_n^2 = \frac{w\left[(\delta_{Rmax}+\delta_s)\, x - \frac{4}{3L^2}\delta_{Rmax} x^3 \right]_0^{L/2}}{m\left[(\delta_{Rmax}+\delta_s)^2 x - (\delta_{Rmax}+\delta_s)\left(\frac{8}{3L^2}\delta_{Rmax}\right)x^3 + \frac{16}{5L^4}\delta_{Rmax}^2 x^5\right]_0^{L/2}}.$$

Substituting limits:

$$\omega_n^2 = \frac{w\left[(\delta_{Rmax}+\delta_s)\frac{L}{2} - \frac{L}{6}\delta_{Rmax}\right]}{m\left[\left(\delta_{Rmax}^2+2\delta_{Rmax}\delta_s+\delta_s^2\right)\frac{L}{2} - \delta_{Rmax}^2\frac{L}{3} - \delta_{Rmax}\delta_s\frac{L}{3} + \frac{L}{10}\delta_{Rmax}^2\right]}.$$

m = w/g, the L's in the numerator and denominator cancel, and fractions are cleared by multiplying the numerator and denominator by 30:

$$\omega_n^2 = \frac{10\delta_{Rmax}+15\delta_s}{8\delta_{Rmax}^2+20\delta_{Rmax}\delta_s+15\delta_s^2}\, g. \tag{41}$$

In metric units, g = 9807 mm/sec^2; therefore, for deflections in millimeters:

$$f_n = 15.76\sqrt{\frac{10\delta_{Rmax}+15\delta_s}{8\delta_{Rmax}^2+20\delta_{Rmax}\delta_s+15\delta_s^2}}\ \text{cps} \tag{42}$$

or for units of inches (g = 386 in./sec^2):

$$f_n = 3.13\sqrt{\frac{10\delta_{Rmax}+15\delta_s}{8\delta_{Rmax}^2+20\delta_{Rmax}\delta_s+15\delta_s^2}}\ \text{cps}. \tag{42a}$$

The deflections δ_{Rmax} and δ_s (because of weight) are calculated from equations (32) and (34).

For the special case of $\delta_s = 0$ (δ_{Rmax} in inches):

$$f_n = 3.50 \sqrt{\frac{1}{\delta_{Rmax}}} \text{ cps.} \quad (43)$$

Equation (43) can be checked against the "exact" derivation of equation (36) in English units for a uniformly distributed mass between bearings (dead shaft roller) for which $\delta_{Rmax} = 5wL^4/384EI$, and the difference is shown to be 1.0 percent:

$f_n = 30.6 \sqrt{EI/wL^4}$ from Rayleigh method, or

$f_n = 30.9 \sqrt{EI/wL^4}$ from "exact" method.

For the limiting case of $\delta_{Rmax} = 0$ (negligible roller deflection in comparison to stub-shaft deflection) with δ_s in inches:

$$f_n = 3.13 \sqrt{1/\delta_s} \text{ cps.} \quad (44)$$

This result is the same as that obtained from the analysis of a single mass on a massless spring. Confirmation of equation (42a) is therefore excellent for the two extreme cases. Another special case is if $\delta_s = \delta_{Rmax}$ (δ_{Rmax} in inches):

$$f_n = 2.39 \sqrt{1/\delta_{Rmax}} \quad (45)$$

or 68 percent of the natural frequency for $\delta_s = 0$. It is therefore important to consider the deflection of the stub shafts, if this deflection is significant in comparison to the deflection of the roller, in calculating the natural frequency.

If the overall deflection δ of the center of the roller because of its weight is determined experimentally without a knowledge of the components δ_{Rmax} and δ_s, the natural frequency can vary by a factor no greater than 1.12 by an incorrect assumption of the components of the deflection. This factor is equation (43) for zero shaft deflection divided by equation (44) for zero roller deflection, with δ substituted for δ_{Rmax} and δ_s, respectively.

Critical Velocity and Natural Frequency

This paper has used vibration texts as background for the dynamic analysis, and has therefore used the term "natural frequency" for the condition of a large magnification of the deflection of the roller. The physical phenomenon for a roller is different from that of the usual textbook vibration problem, although occurring at the same frequency as the natural frequency of vibration. The usual vibration problem is a reversing deflection, whereas the critical velocity of a roller is the condition at which the deflection of the roller, because of the inertia of the thick part of a nonuniform wall (even though balanced by masses at the ends), would become excessive. A roller running near its critical-velocity condition would therefore be whirling in a constant, highly deflected state. The internal damping of the material in the usual vibration problem is absent, but the web (when present) would help to limit the deflection.

A condition which more closely corresponds to the usual problems in vibration texts is the pulsating deflection (under the influences of web tension and gravity) because of the variation in the moment of inertia (especially for the common case of a cylindrical outside and an out-of-round and eccentric inside) as the roller rotates. This varying deflection is magnified as the natural frequency is approached. The internal damping of the roller material, particularly if it is covered with an elastomer, provides some damping for this type of vibration.

The above cyclical deflection type of vibration would occur at the frequency of roller rotation in the common case of an eccentric bore of a roller, in which a curved tube is machined only on the outside to make the outside cylindrical. If a flattened tube were machined only on the outside, vibration would occur at twice the frequency of rotation. (Two maximum and two minimum moments of inertia would occur per revolution.) Evidently, this latter condition has been labeled a "half-critical" vibration, because a resonance was observed at one half of the calculated or experimentally determined critical frequency. The term "half critical" is a misnomer, because the problem arises from a disturbance at twice the frequency of rotation instead of a resonance at half the calculated natural frequency.

The natural frequency of a given roller can be easily related to the velocity of the web. Simplified equations which have been used in industry include

(a) $V_{cr} = 49.1\ D/\delta^{1/2}$, and (b) $V_{cr} = 55.3\ D/\delta^{1/2}$,

for an answer in fpm if D is the outside diameter in inches and δ is the total deflection (because of weight) at the center in inches. The constants in these equations are 5π multiples of those in equations (44) and (43), respectively, for conversion from inches to feet and seconds to minutes, and to lineal travel per revolution. Equation (a) is for behavior as if all of the deflection occurred in the stub shafts, with no bending of the roller, or as if all of the mass were concentrated in the center. Equation (b) is for an evenly distributed mass as in a dead shaft roller, or for negligible deflection of the stub shafts. The equations, however, differ by a factor of only 1.12 if δ is the total deflection at the center of the roller.

This study shows that simple approximations commonly used in calculating the natural frequency or critical velocity are reasonably accurate if the total deflection of the center of the roller under the influence of its own weight is correct.

Example of Critical Velocity. The roller which was used as an example for deflection has a shell weight of 3865 lb_f, or $w = 20.4\ lb_f/in$. The deflection because of weight is 0.00237 inch according to the manufacturer's first equation, and 0.00327 inch according to the second. The critical velocity according to the more conservative industry equation (a) above, and using the deflection of 0.00327 inch, is 20,600 fpm. If the deflection of 0.00237 inch is used in equation (b), the critical velocity is calculated as 27,300 fpm.

By the method of this paper, for the above roller, equation (32) gives $\delta_{Rmax} =$ 0.00414 inch, while equation (34) gives $\delta_s = 0.00576$ inch. From equation (42a), $f_n =$

33.6 cps. The critical velocity is 12,700 fpm, or 62 percent and 47 percent of the above two calculations which utilized less accurate methods.

This example shows that critical velocity calculations may be quite inaccurate if common equations for deflection are used.

ACKNOWLEDGMENTS

Appreciation for sponsorship of projects which contributed background for this paper is extended to Bruce Feiertag, Fife Corporation (1959-1975); Randy Clark, Fife Corporation, Instrument System Division (1984): Jack Beery, Mead Imaging (1986-1988); Jack Beery, Arthur D. Little, Inc. (1990-1991); Ken Thompson, Mobil Chemical, Films Division (1993); and Ken Hopcus, Fife Corporation (1993). The Web Handling Research Center of Oklahoma State University supported the final writeup. Thanks is also extended to Richard L. Lowery of Oklahoma State University for helping to solve the mystery of the "half critical" vibration.

REFERENCES

1. Timoshenko, S.P., and J.M. Gere. Theory of Elastic Stability, Second Edition. McGraw-Hill Book Company, Inc., New York, N.Y. (1961).
2. Timoshenko, S.P., D.H. Young, and W. Weaver, Jr. Vibration Problems in Engineering, Fourth Edition. John Wiley & Sons, New York, N.Y. (1974).
3. Hopkins, R.B. Design Analysis of Shafts and Beams. McGraw-Hill Book Company, Inc., New York, N.Y. (1970).
4. Roark, R.J., and W.C. Young. Formulas for Stress and Strain, Fifth Edition. McGraw-Hill Book Company, Inc., New York, N.Y. (1975).

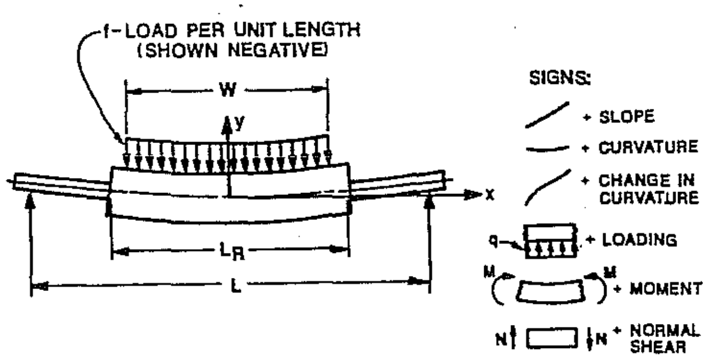

Fig. 1. Coordinate System and Sign Conventions

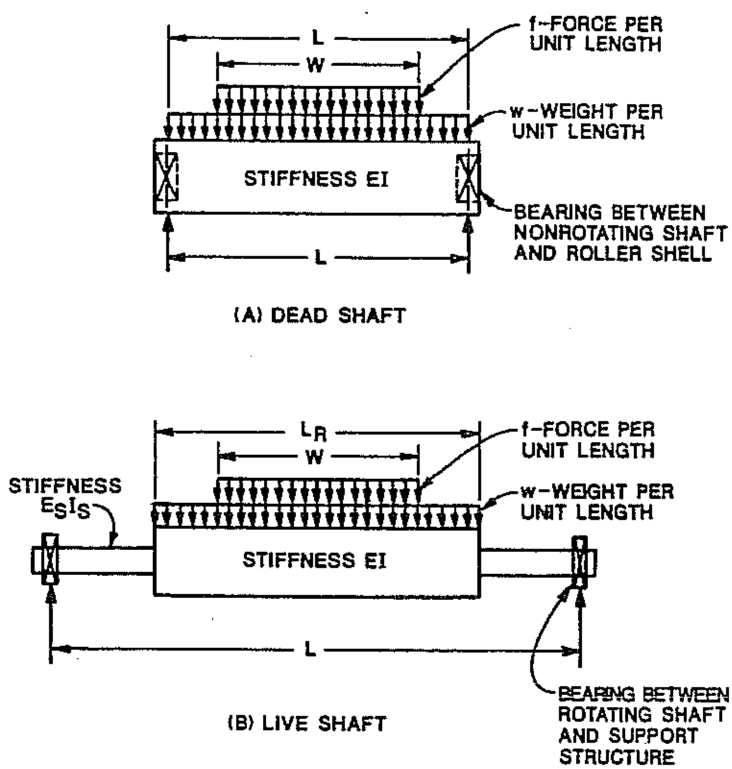

Fig. 2. Idealized Configurations of Rollers

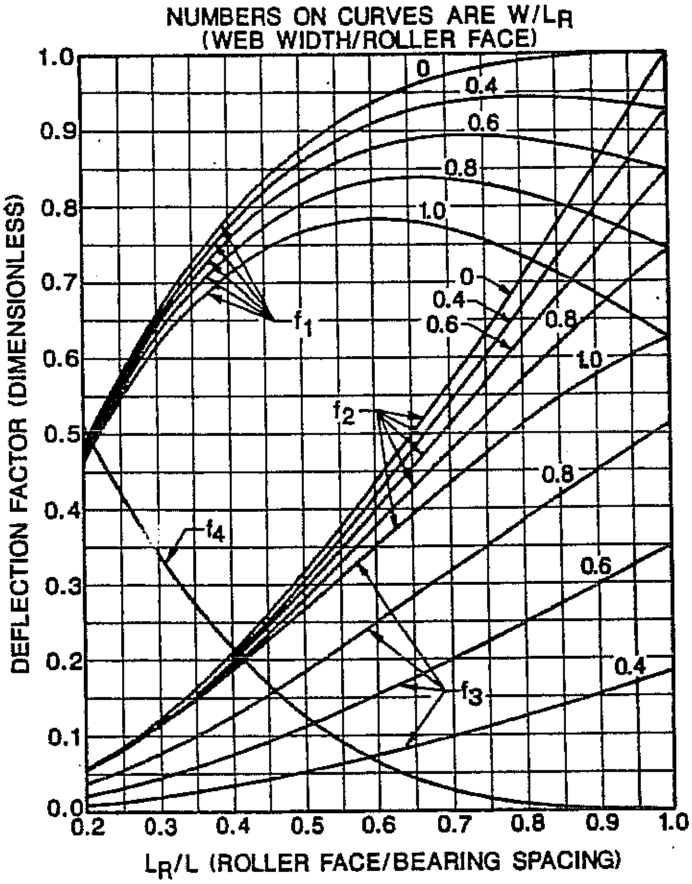

Fig. 3. Deflection Factors

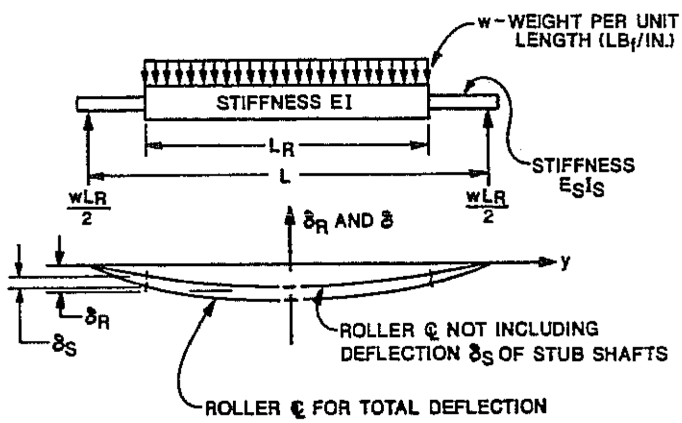

Fig. 4. Deflection Mode Assumed for Rayleigh's Method

Shelton, J.J.
Deflection and Critical Velocity of Rollers
6/21/95 Session 6 9:50 - 10:15 a.m.

Question - Your comment as to the lack of available published formulas, may be correct; I'm sure suppliers of equipment other than Beloit have more than adequate formulas but they just don't publish them.

Answer - One of the critical-velocity formulas reportedly came indirectly from Beloit.

Question - We do have formulas which more than adequately handle the situation which you described. A case which you did not cover is a roller with a massive head joining the shaft and the shell. Depending on the design of the roll and the dimensions of the assembly, sometimes the mass of these heads has a profound effect on the natural frequency. When you start going through your energy calculations for computing natural frequency, this is not insignificant.

Answer - Yes, I am sure that there are rollers with massive heads and long stub shafts for which the mass of the hubs would be significant. The deflection of the stub shafts, because of the weight of these hubs, would add to the deflection as caused by the weight of the roller shell as discussed in the paper.

Question - You just basically go through the procedure, plugging in numbers for deflections into the natural frequency formula?

Answer - I am not answering all your questions, but believe that this paper contains information which you will find useful. If you are trying to fine tune your calculation of natural frequency, this equation derived from Rayleigh's method will be useful. If you don't need great accuracy, you are not far off with published equations for natural frequency, if you know the deflection of the roller accurately. I have seen some very poor published equations for deflection of rollers.

Question - You talk about a stub shaft. You mean the shaft ends at the beginning of the roller, that is not a true shaft, upon which the roller is mounted?

Answer - The term "stub shaft" applies only to a live-shaft roller.

Question - It is a live shaft, but do you mean that it is not continuous?

Answer - The shaft in a live-shaft roller usually does not extend through the roller.

Question - Well that is going to make a difference.

Answer - That is why I call it a "stub shaft". A dead shaft roller may need to have its shaft analyzed separately. You don't want it flopping around.

Question - The shaft of a dead-shaft roller would be a beam with two points of loading and two supports?

Answer - Yes.

Thank you.