# A NONLINEAR ORTHOTROPIC VISCOELASTIC WINDING MODEL

by

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## ABSTRACT

A realistic and adaptive viscoelastic model for prediction of transient wound roll stress distributions is presented. The web material is taken to be orthotropic with a nonlinear radial creep compliance dependent upon interlayer pressure. Viscoelastic behavior is represented by a generalized Maxwell model for creep written as a convolution integral. Numerical solutions to the resulting integral boundary value problem give both initial and transient stress distributions within the wound roll. The model is successfully compared to the exact solution for a simple case of isotropy as well as to published works on this topic. In contrasting the solutions, the advantages and adaptability of this nonlinear formulation will be readily seen.

## NOMENCLATURE

| | | |
|---|---|---|
| $E_0, E_1,...,E_n$ | = | moduli of the generalized Maxwell model, kPa |
| $E_c$ | = | core stiffness, kPa |
| $h$ | = | size of spatial grid, cm |
| $H(t)$ | = | Heaviside step function |
| $J_r$ | = | radial creep compliance, $kPa^{-1}$ |
| $J_\theta$ | = | circumferential creep compliance, $kPa^{-1}$ |
| $J_{r\theta}$ and $J_{\theta r}$ | = | Poisson creep compliances, $kPa^{-1}$ |
| $J_0, J_1,...,J_n$ | = | creep compliance coefficients, $kPa^{-1}$ |
| $n$ | = | number of relaxation times |
| $N$ | = | number of increments in spatial grid |
| $r$ | = | roll radius, cm |
| $r_{in}$ | = | inner roll radius, cm |
| $r_{out}$ | = | outer roll radius, cm |
| $s$ | = | Laplace transform operator |

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| | | |
|---|---|---|
| $t$ | $=$ | time, s |
| $T_w$ | $=$ | winding stress, kPa |
| $u$ | $=$ | radial displacement, cm |
| $\varepsilon_r$ | $=$ | radial strain |
| $\varepsilon_\theta$ | $=$ | circumferential strain |
| $\sigma_r$ | $=$ | radial stress, kPa |
| $\sigma_\theta$ | $=$ | circumferential stress, kPa |
| $\eta_1, \eta_2, ..., \eta_n$ | $=$ | viscosity, s-kPa |
| $(\Delta\sigma_r)_j$ , $(\Delta\sigma_\theta)_j$ | $=$ | change in $\sigma_\theta$ and $\sigma_\theta$ from time t=$t_{j-1}$ to t=$t_j$ |
| $\Delta\sigma_{r(i)}$ , $\Delta\sigma_{\theta(i)}$ | $=$ | $\Delta\sigma_r$ and $\Delta\sigma_\theta$ at the ith lap |
| $\overline{\sigma_r}, \overline{\sigma_\theta}, \overline{E}$ | $=$ | Laplace transform of $\sigma_r$, $\sigma_\theta$, and E |
| $\Delta t$ | $=$ | change in time, s |
| $\tau_1, \tau_2, ..., \tau_n$ | $=$ | relaxation times, s |

## INTRODUCTION

Wound structures, termed rolls, built by winding thin webs with little or no resistance to bending, such as paper, polymeric films, or magnetic tape, onto a compliant core develop both radial and circumferential stress distributions along their radii. Due to the viscoelastic nature of most polymers and many paper products, the stress distributions within many rolls are dependent upon both storage environment and time. Hence, rolls wound from viscoelastic material are prone to decreasing stress profiles and if stored for prolonged periods of time, may experience significant reduction in resident stresses. If the interlayer stresses are substantially reduced, the roll structure is subject to damage during transportation or subsequent rewind operations. Foreknowledge of transient stress distributions allow operating parameters or rewind schedules to be devised such that structural integrity of the wound roll can be maintained. This paper provides a simulation technique which may be used to predict transient wound roll stress distributions where optimum winding parameters and roll geometry may be chosen.

Detailed analysis of initial wound roll stresses began with the work of Gutterman [1]. Altmann [2] followed and presented an analytical solution to the winding problem by assuming homogeneous and anisotropic elastic web properties constant throughout the roll. Yagoda [3] was the first to accurately treat the inner boundary condition, which accounted for core deformation. Pfeiffer [4] included a nonlinear radial modulus, written as an exponential, and provided an approximate solution to the elastic case using an energy balance formulation which neglected core flexibility. Prediction of initial wound roll stresses was modernized by Hakiel [5] who presented a numerical method employing a pressure dependent radial modulus. Hakiel solved the resulting nonlinear second order differential equation using finite difference techniques by employing two boundary conditions including (1) that the deformation of the first layer be equal to the deformation of the core and (2) that the webline tension be equal to the wound-on-tension in the current outer layer of the winding roll. These boundary conditions are suitable for the condition of centerwinding, and Hakiel verified his method for that condition. In centerwinding, the roll is wound through the provision of torque applied to the core with no driven or undriven rolls impinged on the outer surface of the winding roll.

Transient analysis of wound roll stresses was first considered by Tramposch [6,7] who modeled viscoelastic web behavior with a Maxwell-Kelvin constitutive law. Tramposch's first work assumed roll isotropy, however he later allowed for the circumferential and radial modulus to differ by a multiplicative constant. Lin and Westmann [8] expanded the work in viscoelastic winding mechanics by accounting for winding time in the analysis. Lin and Westmann assumed roll isotropy and chose a generalized Maxwell model to describe the roll's viscoelastic behavior. Still, no author has coupled the nonlinear orthotropic attributes of the wound roll with a complex viscoelastic constitutive relation.

In this paper, a realistic orthotropic viscoelastic model for centerwound rolls is presented. Viscoelastic behavior of the wound structure is characterized by a generalized Maxwell model with allowance for pressure dependent radial behavior. Choice of a generalized Maxwell model enables accurate portrayal of most any material behavior since the number of retardation times is limited only by computational resources. Results are compared to an exact solution developed herein for a simplified isotropic case with one time constant. In addition, this work is successfully compared to the published works of Tramposch [6,7] and to that of Lin and Westmann [8].

## GOVERNING EQUATIONS

The process of winding has historically been viewed as the addition of pretensioned concentric hoops of web material onto a compliant core. The process is then one of accretion in which the stresses in the roll are incrementally changed as each layer is added. Assuming the wound roll is axisymmetric dictates the stresses, strains, and displacements are functions only of roll radius. The governing equations in polar coordinates are then the:

Equilibrium Equation:

$$r \frac{\partial \sigma_r}{\partial r} + \sigma_r - \sigma_\theta = 0 \qquad (1)$$

Strain Compatibility Equation:

$$r \frac{\partial \varepsilon_\theta}{\partial r} + \varepsilon_\theta - \varepsilon_r = 0 \qquad (2)$$

Strain-Displacement Relation:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad (3)$$

Viscoelastic Constitutive Equations:

$$\varepsilon_r = \int_0^t [J_r(t-t') \frac{\partial \sigma_r}{\partial t'} + J_{r\theta}(t-t') \frac{\partial \sigma_\theta}{\partial t'}] \, dt' \qquad (4)$$

$$\varepsilon_\theta = \int_0^t [J_\theta(t-t') \frac{\partial \sigma_\theta}{\partial t'} + J_{\theta r}(t-t') \frac{\partial \sigma_r}{\partial t'}] \, dt' \qquad (5)$$

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Where $J_r(t)$ and $J_\theta(t)$ are the radial and circumferential creep functions, respectively, and $J_{r\theta}(t)$ and $J_{\theta r}(t)$ are creep functions representing the Poisson coupling between the radial and circumferential directions. Model development continues by eliminating the circumferential stress, $\sigma_\theta$, by solving (1) for $\sigma_\theta$ and substituting the result into (4) and (5). This yields two equations, one for the radial strain and the other for the circumferential strain. These equations describing the radial strain, $\varepsilon_r$, and circumferential strain, $\varepsilon_\theta$, are substituted into the compatibility equation (2). This treatment leads to the following integral boundary value problem written as a second order partial differential equation in terms of radial stress, $\sigma_r$, the radius, r, and time, t.

$$\int_0^t [J_\theta(t-\tau)\frac{\partial}{\partial\tau}(r^2\frac{\partial^2\sigma_r}{\partial r^2}) + \{3J_\theta(t-\tau) + J_{\theta r}(t-\tau) - J_{r\theta}(t-\tau) + r\frac{\partial}{\partial r}J_\theta(t-\tau)\}\frac{\partial}{\partial\tau}(r\frac{\partial\sigma_r}{\partial r})$$

$$+ \{r\frac{\partial}{\partial r}(J_\theta(t-\tau) + J_{\theta r}(t-\tau)) + J_\theta(t-\tau) + J_{\theta r}(t-) - J_r(t-\tau) - J_{r\theta}(t-\tau)\}\frac{\partial\sigma_r}{\partial\tau}]d\tau = 0$$

(6)

## SOLUTION METHOD

The method of solution consists of first choosing the generalized Maxwell model to represent the viscoelastic behavior. With this choice, the creep compliance takes the following form:

$$J(t) = J_o + \sum_{i=1}^{N} J_i e^{-(t-\tau)/\tau_i}$$

(7)

The generalized Maxwell model provides for instantaneous deformation followed by time dependent deformation, termed creep, when subjected to a step change in stress. For an orthotropic material, the creep compliance in the radial and circumferential directions must be defined independently. Pfeiffer and Hakiel [4,5] have shown the instantaneous radial modulus to be a function of interlayer pressure. As a result, $J_0$ of the radial creep function must be a function of the radial stress. At this point, special treatment of the nonlinear behavior will not be required. Each integral in the boundary value problem can now be analyzed separately. All terms in equation (6) can be segregated into the two distinct forms defined in equations (8) and (9).

$$I = \int_0^t J(t-\tau)\frac{\partial f}{\partial\tau}d\tau$$

(8)

$$J = \int_0^t r\frac{\partial}{\partial r}[J(t-\tau)]\frac{\partial f}{\partial\tau}d\tau$$

(9)

It can be seen that each term in equation (6) takes on the appearance of either (8) or (9). First, integral $I$ is discretized over time in either constant or in some cases variable intervals by replacing the continuous function, $f$, with a step function. The integral $I$ can then be written at some time $t=t_j$ as follows:

$$I_j = J(t_j - t_{j-1})(\partial f / \partial \tau)_j \Delta t + J(t_j - t_{j-2})(\partial f / \partial \tau)_{j-1} \Delta t +$$

$$.... + J(t_j - 0)(\partial f / \partial \tau)_1 \Delta t \tag{10}$$

Substitution of the finite difference approximation

$$(\frac{\partial f}{\partial t})_j = \Delta f_j / \Delta t \tag{11}$$

where,

$$\Delta f_{t=j} = f_{t=j} - f_{t=j-1} \tag{12}$$

enables equation (10) to be rewritten as:

$$I_j = J(t_j - t_{j-1})\Delta f_j + \sum_{n=1}^{n=j-1} J(t_j - t_{n-1})\Delta f_n \tag{13}$$

Replacing the general form of the creep compliance with the chosen Maxwell model permits the above integral to be further simplified as follows:

$$I_j = (J_o + \sum_{i=1}^{N} J_i e^{-(t_j - t_{j-1})/\tau_i})\Delta f_j + J_o f_{j-1} + \sum_{i=1}^{N} J_i \alpha_{i,j} \tag{14}$$

where,

$$f_{j-1} = \sum_{k=1}^{j-1} \Delta f_k \tag{15}$$

and,

$$\alpha_{i,j} = e^{-2\Delta t/\tau_i}\Delta f_{j-1} + e^{-\Delta t/\tau_i}\alpha_{i,j-1} \tag{16}$$

Equation (16) is a recursive formula which allows the current value of $\alpha_{i,j}$ to be calculated from only the previous value. Applying a similar treatment to the integral in (9) yields another recursion parameter defined as $\beta_{i,j}$. As shown in (6), the partial derivative with respect to the radius operates on the creep functions $J_\theta$ and $J_{\theta r}$. It is then appropriate to question whether $J_\theta$ and $J_{\theta r}$ are functions of roll radius.

Nonlinear behavior in the radial direction has been attributed to factors including entrapped air and the contact of asparities. There are no such mechanisms in the circumferential direction. This coupled with the fact that circumferential material properties presented in [1-8,11] are independent of radial pressure suggests that $J_\theta$ is also radially independent. The Poisson term accounting for the portion of circumferential stress caused by radial strain was measured by Willett and Poesch [11] to be approximately 0.07. Any variation of this term with roll radius would then be small. Hence, the derivative with respect to roll radius is insignificant and all recursion parameters, $\beta_{i,j}$, will be neglected. However, if thermal influences are to be accounted for and the thermal profile is radially dependent, these creep functions may vary with roll radius. Special attention must then be given to the differentials of the creep compliance. A detailed derivation of the recursion parameters occurring from the generalized Maxwell model was published by Zak [9] in analyzing thermoviscoelastic stresses in solid rocket propellants.

As a direct result of this particular method of solution, the terms $\alpha_{i,j}$ and $\beta_{i,j}$ contain all history effects of the viscoelastic material. Since all history effects are held within the recursion formulas, significant storage space in the form of computer memory is saved. Once the integrals in equation (6) are replaced by the discretized form, the boundary value problem collectively reduces to the following:

$$[F_1(r)r^2 \frac{\partial^2 \Delta\sigma_r}{\partial^2 r} + F_2(r)r \frac{\partial \Delta\sigma_r}{\partial r} + F_3(r)\Delta\sigma_r + F_4(r)]_{t=t_j} = 0 \qquad (17)$$

Where in the above second order differential equation, $\Delta\sigma_r$ is the change in radial stress from time $t=t_{j-1}$ to time $t=t_j$. An exact solution is not possible since the coefficients are functions of the radius. As a result, a numerical solution technique is employed. The wound roll is discretized radially and the derivatives replaced by finite difference operators. Central difference approximations for the first and second derivatives of the radial stress are:

$$\frac{\partial \sigma_r}{\partial r} \approx \frac{\Delta\sigma_{r(i+1)} - \Delta\sigma_{r(i-1)}}{2h} \qquad (18a)$$

$$\frac{\partial^2 \sigma_r}{\partial r^2} \approx \frac{\Delta\sigma_{r(i+1)} - 2\Delta\sigma_{r(i)} + \Delta\sigma_{r(i-1)}}{h^2} \qquad (18b)$$

Substituting these into equation (17) yields (19), which is written once for each internal layer within the roll. Writing equation (19) at each spatial location leads to a set of N-1 simultaneous algebraic equations with N+1 unknowns. When boundary conditions are applied, the tridiagonal system of equations is solved by Gaussian elimination for the change in radial stress at the current time step.

$$(F_1(r)\frac{r_i^2}{h^2} + F_2(r)\frac{r_i}{2h})\Delta\sigma_{r(i+1)} + (F_3(r) - 2F_1(r)\frac{r_i^2}{h^2})\Delta\sigma_{r(i)} +$$
$$(F_1(r)\frac{r_i^2}{h^2} - F_2(r)\frac{r_i}{2h})\Delta\sigma_{r(i-1)} + F_4(r) = 0 \qquad (19)$$

**Inner Boundary Condition**

The inner boundary condition provides for continuity of displacement at the core. This condition is one which many researchers have implemented in previous winding algorithms. However, this specific application is more delicate in that it contains an integral subject to temporal discretization. Taking the deformation of the core and the first lap of wound material to be equivalent requires:

$$\frac{(\sigma_r)_j}{E_c} = (\varepsilon_\theta)_j \qquad (20)$$

where $E_c$ is the core stiffness, and the subscript, j, refers to the current time, $t=t_j$. Replacing the circumferential strain by its integral form defined in (5) and substituting for the total stress, $(\sigma_r)_j$, the sum of the total stress at the previous time step, $(\sigma_r)_{j-1}$, and the change in stress from the previous to the current time step, $(\Delta\sigma_r)_j$, gives a form

which can be discretized temporally and radially. The result of this operation is shown below.

$$\frac{(\sigma_r)_{j-1}+(\Delta\sigma_r)_j}{E_c} = \int_0^t [J_\theta(t-t')\frac{\partial}{\partial t'}(r\frac{\partial \sigma_r}{\partial r})+J_\theta(t-t')\frac{\partial \sigma_r}{\partial t'}+J_{\theta r}(t-t')\frac{\partial \sigma_r}{\partial t'}]\,dt'$$

(21)

Once the right hand side is discretized and the derivatives replaced by their corresponding finite difference operators, an equation is developed in terms of the unknown change in radial stress, $\Delta\sigma_{r=1}$ and $\Delta\sigma_{r=2}$, at the current time, $t=t_j$. This equation will become the first row in the tridiagonal system of equations.

## Outer Boundary Condition

The outer boundary condition is developed by taking the strain in the outer layer to be constant and equal the winding stress, $T_w$, multiplied by the circumferential creep function, $J_\theta$ evaluated at time, $t=0$. In taking the strain in the outer layer as constant, it is assumed the deformation, u, at the outside radius caused by changes in the underlying structure is negligible. The outer boundary condition becomes (22) when the circumferential strain is replaced with its definition in (5).

$$T_w J_\theta(0) = \int_0^t [J_\theta(t-t')\frac{\partial \sigma_\theta}{\partial t'}+J_{\theta r}(t-t')\frac{\partial \sigma_\theta}{\partial t'}]\,dt'$$

(22)

The right hand side is discretized temporally using methods aforementioned. Equation (22) is then solved explicitly for the change in circumferential stress, $(\Delta\sigma_\theta)_j$. With the change in circumferential stress known, the corresponding change in radial stress beneath the outer layer can be determined by employing the hoop stress formula.

$$(\Delta\sigma_r)_j = \frac{(\Delta\sigma_\theta)_j h}{r_{out}}$$

(23)

Where, h, is the thickness of the radial segment and $r_{out}$ is the current outer radius of the roll. This condition provides the final constraint needed to solve the simultaneous set of equations.

## Accretion of the Wound Roll

Accretion of initial pressures within the wound roll is accomplished using a similar method to that employed by Hakiel [5]. The stress state within a roll wound of N layers is taken to be the superposition of stress states resulting from the addition of each layer from 1 to N. Thus, the roll is considered to be composed of N subrolls or substructures. The first subroll consists of only one layer, the second consists of two layers, etc. The Nth and final subroll contains N layers. The stress state of each subroll consists only of that resulting from the addition of that subroll's outermost layer. The total stress of a single layer in the actual wound roll is found by superimposing the stress state corresponding to that layer from all subrolls. Pressure dependent material properties are evaluated at each spatial location from the total pressure existing in the wound roll at the time that material property is first needed.

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## EXACT SOLUTION

An exact solution will be developed by employing the common method of viscoelastic analogies, in which the viscoelastic problem is transformed by an integral operator to an associated elastic problem. Once the solution to the elastic problem is found, the expression is inverted back to time domain giving the solution to the viscoelastic problem. In this work the Laplace transform will be used exclusively with the frequency variable, s. For the viscoelastic problem both the radial and circumferential stress are functions of time. Let the Laplace transform of the radial stress be denoted $\bar{\sigma}_r$ and that of the circumferential stress be denoted $\bar{\sigma}_\theta$. The stress distribution for an axisymmetric hollow cylinder with homogeneous and isotropic material properties is given by Timoshenko [10]. Taking the Laplace transform of these stress distributions yields:

$$\bar{\sigma}_r = \frac{A}{r^2} + 2C \tag{24}$$

$$\bar{\sigma}_0 = -\frac{A}{r^2} + 2C \tag{25}$$

The parameters A and C are unknown and must be evaluated by applying appropriate boundary conditions sufficient to constrain the problem. Boundary conditions similar to those aforementioned will be used with some modification. When the strain is replaced by the transformed circumferential stress divided by the transformed modulus the inner boundary condition can be given as:

$$\frac{\bar{\sigma}_r}{E_c} = \frac{\bar{\sigma}_\theta}{\bar{E}} \qquad \text{at } r=r_{in} \tag{26}$$

where,

$$\bar{E} = \frac{\bar{\sigma}}{\bar{\varepsilon}} \tag{27}$$

The outer boundary condition remains conceptually identical to that given in (22) and (23), but the form slightly differs. Equating the hoop stress formula, $\sigma_r = T_w h/r$, with the core displacement relation, then solving for the transformed radial stress in terms of the winding tension yields (28).

$$\bar{\sigma}_r = \frac{T_w \bar{E} h}{E_0 r_{out}} \qquad \text{at } r=r_{out} \tag{28}$$

where $E_0$ is the value of the modulus at time, t=0. With these boundary conditions, the two unknown coefficients can be found algebraically.

To this point the constitutive law has remained arbitrary and represented only as $\bar{E}$. It is necessary to define this function so the it is consistent with the generalized Maxwell model. As shown in Figure 1, a generalized Maxwell model with one time constant consists of an elastic element connected in series with a parallel combination of a viscous and elastic element. This is equivalent to a series connection of an elastic

element with a Kelvin model. As a result, total the strain can be written as the sum of the Kelvin and elastic components.

$$\bar{\varepsilon} = \bar{\varepsilon}_K + \bar{\varepsilon}_E \tag{29}$$

Replacing the Kelvin and elastic components of strain with their stress-modulus equivalents gives:

$$\bar{\varepsilon} = \frac{\bar{\sigma}}{E_1 + \eta_1 s} + \frac{\bar{\sigma}}{E_0} \tag{30}$$

Using (27) allows the transformed modulus to be written in terms of specific material properties, $E_0$, $E_1$, and $\eta_1$ as shown in (31).

$$\bar{E} = \frac{E_1 + \eta_1 s}{1 + \dfrac{E_1}{E_0} + \dfrac{\eta_1 s}{E_0}} \tag{31}$$

As stated, each concentric layer is pretensioned at the winding stress when added to the roll structure. The layer is then assumed to have undergone a step change in stress. Making use of the Heaviside operator, the Laplace transform of the preload is found.

$$T_w H(t) \Rightarrow \frac{T_w}{s} \tag{32}$$

Making the substitution of $T_w/s$ for the winding tension and performing the inverse transform gives the stress distribution throughout the roll as a function of time. Of primary interest is the radial stress, which simplifies to that shown below.

$$\sigma_r(r,t) = \frac{2E_c^2 E_0 T_w r_{in}^2 r_{out} h (r_{out}^2 - r^2)(r_{in}^2 + r_{out}^2)}{r^2(r_{in}^2 - r_{out}^2)XY} e^{\frac{-tY}{\eta_1 X}} + \frac{E_0 T_w r_{out} h(r^2 - r_{in}^2)}{r^2(r_{in}^2 - r_{out}^2)(E_0 + E_1)} e^{\frac{-t(E_0 + E_1)}{\eta_1}}$$

$$+ \frac{E_1 T_w r_{out} h\{E_1[E_c(r_{in}^2 + r^2) + E_0(r^2 - r_{in}^2)] + E_o E_c(r_{in}^2 + r^2)\}}{r^2(E_o + E_1)Y}$$

$$\tag{33}$$

where,

$$X = E_c(r_{in}^2 + r_{out}^2) + E_0(r_{out}^2 - r_{in}^2) \tag{34}$$

$$Y = E_1 X + E_1 E_0 r_{in}(r_{in}^2 + r_{out}^2) \tag{35}$$

These expressions give the radial stress in an isotropic hollow viscoelastic cylinder supported elastically at the inner surface and subjected to a time dependent uniform external pressure. The wound roll stress distribution is found by superimposing stress states of all substructures.

## NUMERICAL SOLUTIONS AND RESULTS

Numerical results are first compared to the exact solution just presented. Comparison serves to shown the numerical model degenerates to the exact solution for

an isotropic viscoelastic material with one retardation time. Values for creep parameters, $J_0$ and $J_1$, and the retardation time ,$\tau_1$, were taken from Lin and Westmann [8] and are presented in Table 1(a). As both Figure 1 and equation (27) imply, strains are not effected by out of plane stresses. Consequently, values for all Poisson terms will be selected as zero. This approach is supported by Altmann [2] who suggests taking Poisson's ratio to be nearly zero.

For comparison of solutions, a relation between coefficients of the creep compliance, $J(t)$, and the modulus, $\overline{E}(s)$, must be made. It can be shown that by applying the Laplace transform to the integral form of the constitutive equation, solving for the ratio of stress to strain and equating the result to the transformed modulus in (31), the parameters are related in the following manner.

$$E_0 = \frac{1}{J_0 - J_1} \qquad E_1 = \frac{1}{J_1} \qquad \eta = \frac{\tau}{J_1} \tag{36}$$

Figure 2 presents the results for the normalized radial stress as a function of roll radius at various times steps. In this degenerate isotropic case, it is possible to plot the radial stress normalized by the winding stress. However, when nonlinear material properties are used, normalization is not possible as solutions are no longer linearly dependent upon the winding stress. In this illustration, the decaying stress profile can be observed. In all cases shown, the numerical results agree very well with the exact solution.

Comparison with the results of Lin and Westmann [8] requires special consideration. Lin and Westmann require the circumferential stress at the outer boundary to be constant. As a result, the roll is viewed as a viscoelastic hollow cylinder subjected to an invariant external pressure and supported elastically at the inner surface. The response of the system just described should be clear. As the viscoelastic material becomes more fluid-like, the external pressure will be supported less by the roll structure and more by the elastic core. The circumferential stress in the outer layer of a wound roll is typically the maximum level of circumferential stress in the entire roll. Thus, the propensity for viscoelastic relaxation will be maximum at the outer layer. Although forcing the outer layer to remain constant is unrealistic for the case of a wound roll, the work of Lin and Westmann is an invaluable reference for comparison of results. A modification of the boundary condition presented in this work will then be required to allow this comparison.

Once the task of altering the outer boundary condition is accomplished, comparison with Lin and Westmann's results is simple since they also utilized the generalized Maxwell model. Again, the orthotropic viscoelastic model developed herein was reduced to the isotropic case as to make comparisons possible. Isotropic material properties shown in Table 1(b) and winding parameters used by Lin and Westmann are employed to construct this comparison. Using a total of seven retardation times and taking Poisson's ratio as a constant, the radial stress distribution was numerically obtained. Figures 3 and 4 present results which can be likened to Lin and Westmann's [8] Figure 6 and 7. The remarkable agreement of the numerical results to that of Lin and Westmann's is apparent. It can also be seen that by altering the outer boundary condition, the projected stress distribution now increases with time. Even so, this viscoelastic model has been proven to yield accurate results even when multiple retardation times are needed to represent material behavior.

At first an accurate comparison of solutions to that of Tramposch in [6] and [7] seems unlikely since they employed a Maxwell-Kelvin model, whereas a generalized Maxwell model was used in this work. At this point, the adaptivity of the generalized Maxwell model will be shown. It was found the generalized Maxwell model could represent those material behaviors depicted by the Maxwell-Kelvin model used by Tramposch. This requires converting the viscoelastic operators in Tramposch's equations (20) and (21) to the equivalent creep functions as shown in (37) and (38).

$$J(t) = g_1 + g_2(1-e^{-t}) + g_3\, t \tag{37}$$

$$J_\nu(t) = \frac{7g_1}{20} + \frac{g_2}{2}(1-e^{-t}) + \frac{g_3}{2}t \tag{38}$$

All terms, with the exception of the last, t, can be directly incorporated into the generalized Maxwell model. Fitting a curve consisting of five exponentials to the linear term in time, over an interval consistent with that used by Tramposch, gives a correlation coefficient of approximately one. Once these exponentials are united with equations (37) and (38), the behavior of the Maxwell-Kelvin model was accurately represented by the generalized Maxwell model. Material properties resulting from the above procedure are presented in Table 2. Figure 5 presents results for the normalized radial stress in an isotropic wound roll at normalized times, t=0, 1, 10, 100, and 1000. Note, that Tramposch's definition of the core flexibility, F, is inversely proportional to the core stiffness as defined in this work. For example a hub flexibility, F=0, implies the core is perfectly rigid and has infinite stiffness. As shown, the wound roll can approach a stress free state when stored for a sufficiently long period of time. These results can be directly compared to those found in Tramposch 's [6] Figure 3(a).

In Tramposch's second work [7], allowance was made for the circumferential and radial moduli to differ. In doing so, Tramposch defined a proportionality factor, k, which was restricted to values greater or equal to unity. Interpretation of the proportionality factor is given as $k=E_\theta/E_r$. Extension of this factor to the creep compliances, $J_r(t)$ and $J_\theta(t)$ is simply made and is the ratio of the radial to circumferential creep compliance, $k=J_r(t)/J_\theta(t)$. Parameter values used for this comparison are essentially the same as those shown in Table 2. The circumferential creep compliance will remain unchanged and the radial compliance will be equated to the circumferential compliance multiplied by the proportionality factor, k. Results of this comparison are given in Figure 6, which displays the initial radial and circumferential stress profiles for various values of k at normalized times, t=0 and t=10. These illustrations can be compared to Tramposch's [7] Figures (3a) and (3b). Both initial and transient stress distributions for various proportionality factors show excellent agreement with Tramposch's results. Here it is shown how the compressibility of the material in the radial direction directly affects the stress profiles. Larger values of k are shown to produce substantially higher stresses in the radial direction given all other winding parameters are held constant. This observation was also made by Pfeiffer and Hakiel [4,5].

To illustrate the importance of the nonlinear creep function on the wound roll stress distribution, an example is presented in Figure 7. The predicted radial stress distributions are shown for two cases (1) for anisotropic material properties with $J_r(t)/J_\theta(t)=10$, constant coefficients, and $J_\theta(t)$ described in Table 1(a) and (2) for a nonlinear radial creep function with material properties described in Figure 7. Pressure

dependent properties were chosen such that the ratio of $J_r(t)$ to $J_\theta(t)$ throughout the roll is approximately equal to ten. This figure shows the significance of the nonlinear term and it's necessity for accurate prediction of transient wound roll stress distributions.

## CONCLUSIONS

The model presented here can be used to predict transient stress profiles in wound rolls constructed of orthotropic viscoelastic material with nonlinear moduli. One limitation of this work is thermal influences are not accounted for. As many viscoelastic creep functions are thermally dependent, a thermoviscoelastic model would yield more accurate predictions of the stress state in wound rolls which are subjected to changing environments.

Numerical results presented have been shown to agree very well with exact and accepted solutions. Model implementation must be predeceased by determination of various material properties. We suggest measuring the instantaneous radial and tangential moduli and associated creep functions separately and then combining the results. Also, it is suggested that all Poisson terms be taken as zero, based upon suggestions by Altmann [2] and measured values provided by Willett and Poesch [11].

## ACKNOWLEDGEMENTS

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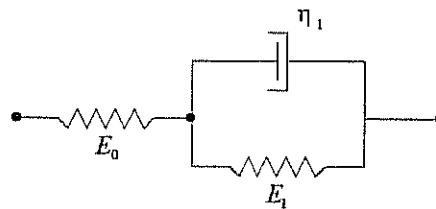
Figure 1. Mechanical Representation of the Generalized Maxwell
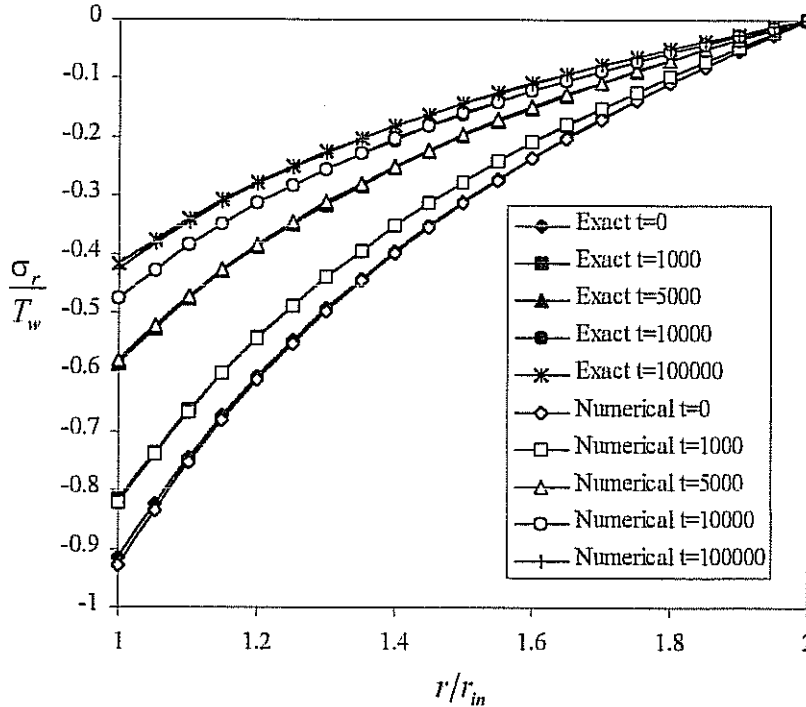Model with One Time Constant.



Figure 2. Comparison of Numerical and Exact Solutions at Various Times
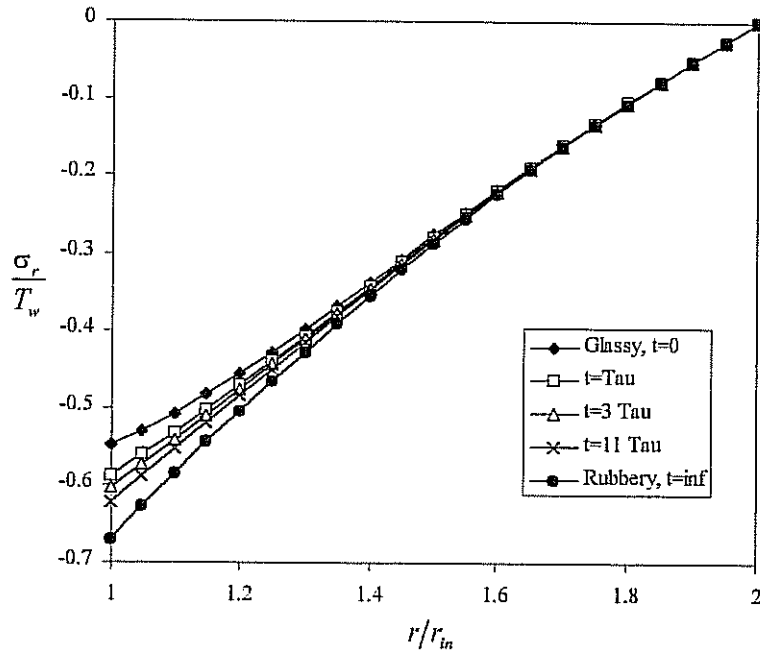using $E_o=\infty$, $h/r_{in}=0.02$ and $\Delta t=t/10$ sec.

Figure 3. Predicted Radial Stress Profile using $E_c$=2.128 GPa, $h/r_{in}$=0.02 and $\Delta t=t/10$ sec. For Comparison with Lin and Westmann [8].
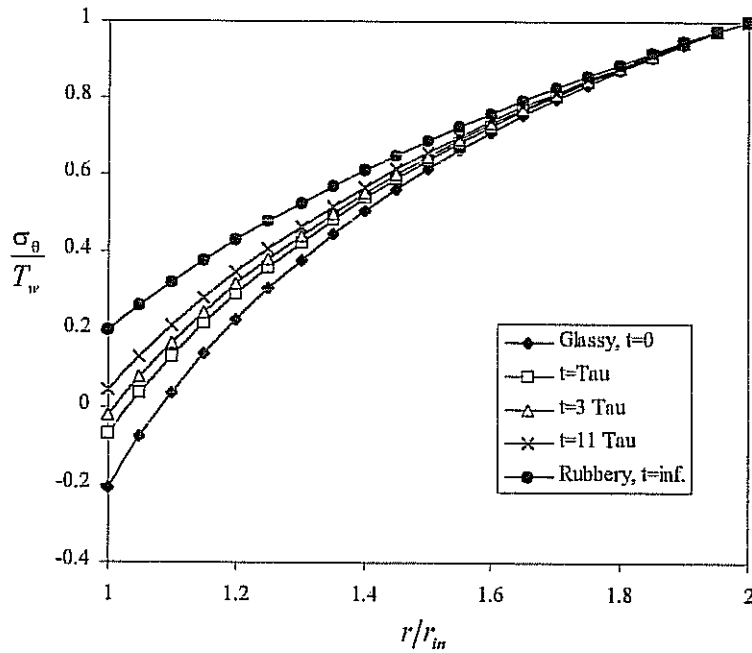


Figure 4. Predicted Circumferential Stress Profile using $E_o$=2.128 GPa, $h/r_{in}$=0.02 and $\Delta t=t/10$ sec. For Comparison with Lin and Westmann [8].
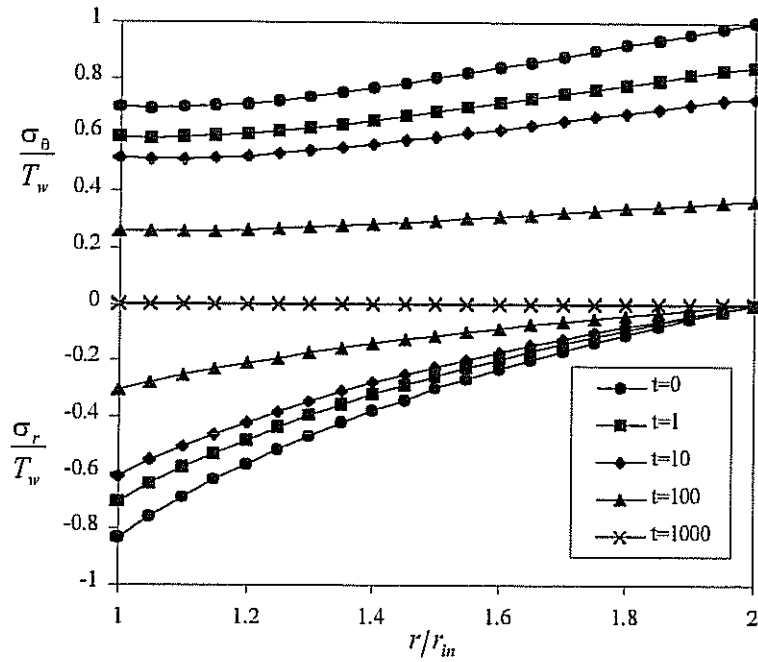
73

Figure 5. Numerical Solution using $E_o=\infty$, $h/r_{in}=0.02$ and $\Delta t=t/10$. For Comparison with Tramposch's Isotropic Results [6].
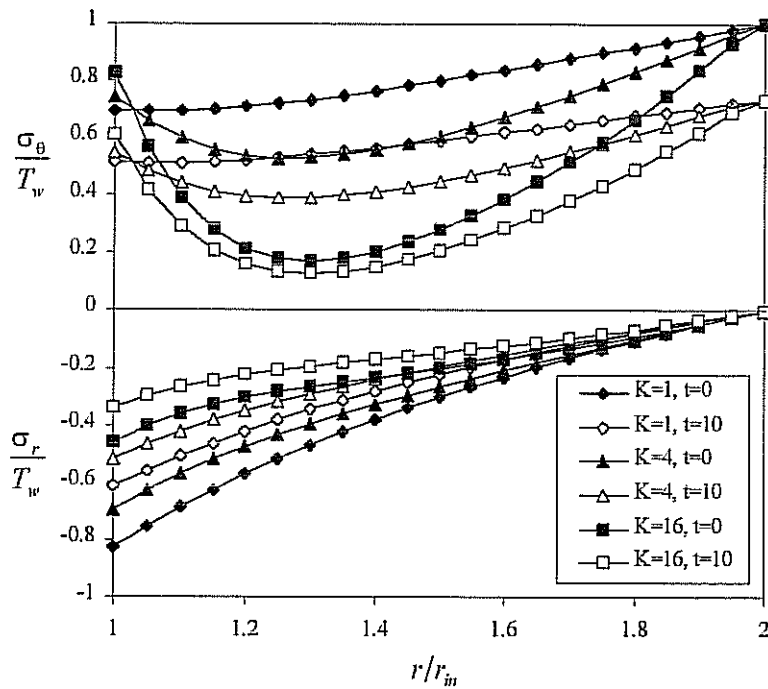


Figure 6. Numerical Solution using $E_o=\infty$, $h/r_{in}=0.02$ and $\Delta t=t/10$. For Comparison with Tramposch's Anisotropic Results [7].
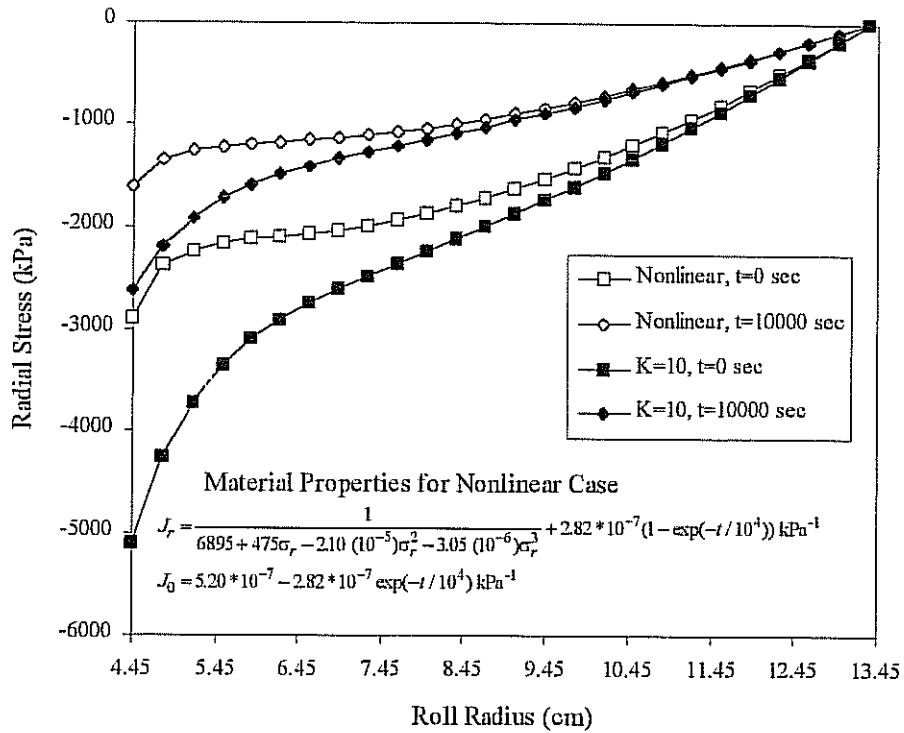
Figure 7. Comparison of Predicted Radial Stress Distributions Using (1) Nonlinear Material Properties and (2) Anisotropic Material Properties with Constant Coefficients. Winding Parameters $T_w$=6.895 MPa, $E_c$=∞, $h/r_{in}$=0.04. and $\Delta t$=t/10 sec.

Table 1. Isotropic Material Properties Taken
From Lin and Westmann [8].

| | $J_r = J_\theta$ | $J_{r\theta} = J_{\theta r}$ | |
|---|---|---|---|
| $J_0$ | 5.21697E-07 | -1.56509E-07 | (1/kPa) |
| $J_1$ | -2.82132E-07 | 8.46396E-08 | |
| $\tau_1$ | 1.0E+04 | 1.0E+04 | (sec) |
| | (a) N=1 | | |
| $J_0$ | 5.20247E-07 | -1.56074E-07 | (1/kPa) |
| $J_1$ | -1.31723E-10 | 3.95169E-11 | |
| $J_2$ | -2.45149E-09 | 7.35446E-10 | |
| $J_3$ | -1.52125E-08 | 4.56374E-09 | |
| $J_4$ | -2.50471E-08 | 7.51414E-09 | |
| $J_5$ | -5.66933E-08 | 1.70080E-08 | |
| $J_6$ | -9.05888E-08 | 2.71766E-08 | |
| $J_7$ | -9.35722E-08 | 2.80716E-08 | |
| $\tau_1$ | 1.0E-01 | 1.0E-01 | (sec) |
| $\tau_2$ | 1.0E+00 | 1.0E+00 | |
| $\tau_3$ | 1.0E+01 | 1.0E+01 | |
| $\tau_4$ | 1.0E+02 | 1.0E+02 | |
| $\tau_5$ | 1.0E+03 | 1.0E+03 | |
| $\tau_6$ | 1.0E+04 | 1.0E+04 | |
| $\tau_7$ | 1.0E+05 | 1.0E+05 | |
| | (b) N=7 | | |

Table 2. Isotropic Mechanical Properties used
for Comparison with Tramposch [6].

| | $J_r = J_\theta$ | $J_{r\theta} = J_{\theta r}$ | |
|---|---|---|---|
| $J_0$ | 2.90065E-04 | -1.10225E-04 | (1/kPa) |
| $J_1$ | -5.80131E-05 | 2.90065E-05 | |
| $J_2$ | -6.68644E-05 | 3.34322E-05 | |
| $J_3$ | 4.73443E-03 | -2.38896E-03 | |
| $J_4$ | -2.89276E-01 | 1.44638E-01 | |
| $J_5$ | 2.84565E-01 | -1.42282E-01 | |
| $\tau_1$ | 1.0E+00 | 1.0E+00 | (sec) |
| $\tau_2$ | 1.0E+03 | 1.0E+03 | |
| $\tau_3$ | 1.0E+04 | 1.0E+04 | |
| $\tau_4$ | 1.0E+05 | 1.0E+05 | |
| $\tau_5$ | 1.0E+06 | 1.0E+06 | |
| | N=5 | | |

Qualls, W.R.; Good, J.K.
A Nonlinear Orthotropic Viscoelastic Winding Model
6/19/95    Session 1    11:33 - 12:00 a.m.

Question - I wanted to ask you a practical question say for instance concerning a paper,. If you were to apply this model on practical level how many material properties would you have to determine, what are those properties, and how would you measure them.

Answer - Well, I would begin with the machine direction creep function which is very simple measured by taking a sample, applying a fixed load to the end and measuring displacement or strain through time. For the radial creep function, I propose separating elastic and viscoelastic properties, you can later combine these, quit efficiently. For instance, we have methods for measuring elastic properties. I say we keep those methods, so when we apply this load, we have an instantaneous elastic deformation. Remove that from your creep data and only look at the transient data. You can then add that transient proportion back to your elastic modulus. Same thing for the radial; you can measure the radial creep function by applying a known pressure to a stack, measure the deformation of the stack, remove the instantaneous portion, and then combine your radial modulus data to that. Have I answered your question.

Yes. OK

Question -- What about the Poisson terms?

Answer - Right off the bat I would ignore the Poisson terms. Or, you can make a great deal of simplification if you use Maxwell's relationship. But for simplicity, I would either assume them to be either zero or constant.

Question - When you derive the model you started with the creek compliance equation with a elastic component and viscous component. Is there any reason you don't include a pure viscous component?

Answer - The Viscous component?

Question - In what terms could you discuss this component relating to a plastic material?

Answer - For a plastic material, well I guess that is one of the limitations of this study, I'm not quit sure what your meaning for one, so I'll admit that. Of course any constitutive law like a generalized Maxwell Model that contains dash pots is modeling viscous behavior. So beyond that, this development is not meant for plasticity; so that would be a limitation. Dr. Good would you agree? You might have an additional comment.

Comment - Although the viscosity term is not seen instantaneously it is accounted for through time with the generalized Maxwell model.

Answer - As time goes on in this storage condition you have discussed the component, you have continued formation. Will, you can actually model this behavior with your constitutive model, your constitutive law is as good as your time span used in conducting your material test.

Thank you.