COMPUTER SIMULATION OF WEB DYNAMICS

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ABSTRACT

Explicit finite difference schemes for modeling one-dimensional, transient web dynamics are designed, developed and tested. A sequence of numerical experiments are performed to ascertain the effects of various parameters on the stability and accuracy of the numerical results. Refinement of the numerical grid led to results which converged to analytical results for stable computations. For a particular web running speed and computational duration, there exists a time-step limit for stability. Longer computational time durations as well as higher web running speeds require reduced time-step size for stable numerical results. Upstream differencing of the Coriolis term to procure stability leads to numerical damping in the computations. A combination of upstream and central differencing yields stable results at larger time-steps than that required using full central differencing as well as less damping than that produced using full upstream differencing.

INTRODUCTION

The Problem

Materials manufactured in continuous strips or sheets are called webs. Many products may be referred to as webs, such as threads, strings, films, magnetic tapes, and various textile and paper products. The technology employed in the manufacture, processing and management of these webs is termed web handling. One of the interests of web handling industries is the subject of web dynamics, which is the study of the motion of a web that is subjected to any number of influences which drive the web to various types of movements. These influences might be directly controlled by web handlers or might be indirectly produced due to handling or other motion influences. Examples of influences which may be directly controlled include web running speed (the imposed axial motion of a web) and the web tension along its span, for example the tension imposed on a web being held between two rollers. Some influences which may not be directly controlled by handlers include aerodynamic effects due to the interaction of the moving web with the air around it, fluctuations in tension and web position due to roller irregularities or eccentricities and lateral tension variation since in general longitudinal tension is maximum at the centerline of the web and minimum at the sides.

Web flutter is a physical phenomenon which occurs in web handling under certain conditions. Its occurrence involves the interaction of web running speed, web tension and the surrounding air flow. In web handling, the subject of web flutter is of interest since the effects produced by its presence can be quite costly and time consuming to overcome. Problems encountered due to web flutter include breaks or wrinkling in the dryers of high-speed paper machines, register errors in offset printing presses and damage to coatings on polymer sheets. In industry today, high productivity requires high speed with low incidence of breaks, waste, or damage. To achieve this, it is necessary to be able to predict web flutter before it occurs; be able to avoid breaks due to too little or too much tension; and know how to manage airflow for optimum runability.

Research Objectives

The long term goal of numerical simulation (as applied to web dynamics) is to develop a computational algorithm which will model and predict the motion of a running web to a high degree of accuracy. The goal of the present research is to design, develop and test finite difference numerical schemes which provide a foundation for the attainment of the long term goal. The present scope is limited to the one-dimensional, transient web dynamic problem, as documented in the recent M.S. Thesis of Fox [1].

The main objective of the present paper is to develop and examine numerical solutions to the partial differential equations which describe one-dimensional web dynamics. The continuum problem is simulated by finite difference equations and these are numerically solved via a computational scheme. The paper describes a sequence of numerical experiments which were performed to ascertain the effect of various parameters on the stability and accuracy of the numerical results. The parameters included time-step size, spatial domain grid size and alternate finite difference formulations for specific terms in the partial differential equations. The effect of these parameters and conditions was evaluated and criteria for accurate web dynamic numerical simulations stipulated.

BACKGROUND

Numerical Methods

The subjects of web dynamics and oscillation wave behavior are governed by so-called hyperbolic partial differential equations (PDEs). The standard approach to solving them is to replace them with finite difference equations (FDEs) at all points of a grid system covering the one-dimensional or two-dimensional spatial domain. The numerical solution is then obtained by marching in time using the appropriate FDEs. Several advanced-level textbooks address various facets of the numerical simulation problem, concentrating on the PDEs [2], the numerical method generally [3] through [5], and the numerical solution of PDEs [6] through [10].

The Wave Equation

The wave equation, both one-dimensional and two-dimensional, has been studied exhaustively. Reference to the equation as well as its analytical solution can be found in numerous sources including most textbooks dealing with the theory of vibrations. Therefore, specific references will not be given here. However, the wave equation with an axial velocity term included has not been studied as extensively.

The linear theory of the vibration of a string with a running velocity was apparently first investigated by Skutch [11]. In his report, Skutch derived a formula for determining the configuration of the string at any time for given initial conditions by considering the superposition of two waves travelling in opposite directions. Later, a more involved examination was made by Sack [12]. In this report, the behavior of a uniform string with constant running speed and tension, pulled over two smooth supports was examined. This analysis, unlike that of Skutch was able to describe the behavior of a string under a sinusoidally alternating influence. In his paper Sack derived the running wave equation as well as a solution for the case of an imposed oscillation at one end of the string. Sack's paper also derived a solution for the equation with forced oscillations and linear damping.

Another fundamental examination of the running wave equation was performed by Archibald and Emslie [13]. Their paper dealt with the calculation of the natural frequencies of a string with axial velocity. They also looked at the case of the forced vibration of the string at an end.

Extending the investigation of the running wave equation, Swope and Ames [14] applied the methods of D'Alembert and of characteristics to the solution of the equation. In that report, the running wave equation was derived and named the "threadline equation". Application of the equation to both finite and infinite strings was examined and solutions generated using the methods previously stated. The report also explored vibrations of a string subjected to boundary excitation.

A more sophisticated investigation of a one-dimensional running web was performed by Ames, Lee and Zaiser [15]. In this report nonlinear models for two-dimensional and three-dimensional vibrations of a string were investigated. The equations of motion for a moving string consisting of continuity, momentum and mass-tension relations were presented. Both geometric and material nonlinearities were included. In addition, various phenomena, due to nonlinearities, were examined both analytically and experimentally. In a related paper by Ames and Vicario [16], the equations of motion were explored further. The equations were analyzed using different mathematical techniques. In this analysis, the case of small transverse oscillations was studied to ascertain the effects of the string running speed on the motion. Kim and Tabarrok [17] also derived the equations of motion for a running string. The equations were shown to be fully hyperbolic in nature and were solved using the method of characteristics. In addition, physical interpretations of the characteristic lines were discussed in detail. Many other papers relate to or investigate the movement of a one-dimensional running web, see [18] through [25] for example.

NUMERICAL PROCEDURE AND ANALYSIS

The Partial Differential Equations

The one-dimensional nonrunning web wave equation is

$$Z_{tt} = c^2 Z_{xx}$$

where $Z = Z(x, t)$ is the displacement, x is the distance along the web of length L, t is time, and $c^2 = T/m$ where T is the x-direction tension per unit width and m is the mass per unit area of web. In order to allow for x-direction running web motion, the total derivative with respect to time is used:

$$DZ/Dt = \partial Z/\partial t + U \, \partial Z/\partial x$$

where U is the running web velocity. The left hand side represents the Lagrangian derivative (as seen by a moving observer) for the moving web. The two terms on the right hand side represent the Eulerian derivative (as seen by a fixed observer) and the advective rate of change. Squaring the Lagrangian derivative operator and substituting in the first equation yields the one-dimensional running web wave equation:

$$Z_{tt} = c^2 Z_{xx} - U^2 Z_{xx} - 2 U Z_{xt}$$

where centripetal and Coriolis terms occur on the right.

Finite Difference Expressions

Using uniform time and spatial subdivisions and central finite difference expressions for the partial derivatives, the explicit finite difference expression for the nonrunning web becomes

$$Z_P' = 2 Z_P - Z_P^- + A (Z_E - 2 Z_P + Z_W)$$

where $A = (c \, dt/dx)^2$ with dt being the time increment and dx being the uniform spacing. Here, the superscripts prime ('), blank ( ) and minus ($^-$) denote values at the new, current and old time-levels (t + dt), t and (t - dt), respectively. The subscripts W, P and E denote the west, point and east sequence of three finite difference points in the ascending x-direction, with uniformly-spaced dx distance. The corresponding finite difference expression for the running web becomes

$$\begin{aligned} Z_P' = {} & 2 Z_P - Z_P^- + A (Z_E - 2 Z_P + Z_W) \\ & - A1 (Z_E - 2 Z_P + Z_W) \\ & - A1^{1/2} (Z_E - Z_W - Z_E^- + Z_W^-) \end{aligned}$$

where $A1 = (U \, dt/dx)^2$ and a backward central difference expression has been used for the cross-derivative term. For stability reasons, it may be necessary to replace the central difference expression with an upstream difference expression; then the last term in the previous equation becomes

$$-2 A1^{1/2} (Z_P - Z_W - Z_P^- + Z_W^-)$$

At the very first time-step there are no values for Z_E^- and Z_W^-. Therefore an assumption is made at this point to allow for these values on the first time-step. On the assumption that the web is not moving, the fictitious point technique ([3] and [4] for example) can be used to obtain the finite difference expression

$$Z_P' = (2\ Z_P + A\ (Z_E - 2\ Z_P + Z_W))\ /\ 2 + dt\ g(x)$$

where $g(x) = Z_t$ = initial velocity of web in z-direction. This finite difference expression is used on the very first time-step only.

Boundary Conditions

The finite difference expressions are solved sequentially in a time-marching fashion, starting from the specified initial position Z(x) of the one-dimensional web, picking up and using the boundary conditions at each step of the march. Three types of boundary conditions are incorporated into the computational algorithms developed in this study:

1. $Z = 0$ fixed position
2. $Z_x = 0$ zero derivative and the web displacement is calculated as time proceeds
3. Z = specified sinusoidal transient behavior, for example, Z = SIN(B*PI*c*t/L) where B is a specified constant and PI is the constant 3.14159

Stability Criteria

Three criteria must be adhered to in order to avoid instabilities and inaccuracies during the time-march. The **first** of these criteria is the time-step limitation developed analytically via von Neumann stability analysis, see [10] and [26] through [29] for example, or, more easily and equivalently, from the finite difference expression itself. The coefficient of the Z_P term on the right hand side of the update equation must be positive or at least zero. For the nonrunning web, this gives the maximum time-step requirement as

$$dt \leq dx/c$$

where dx and c are as defined previously. For the running web with a central difference expression for the cross-derivative term, the requirement becomes

$$dt \leq dx/k$$

where $k^2 = c^2 - U^2$. This is a less restrictive criterion than the nonrunning web case.

The **second** criterion is akin to the famous Courant-Friedrichs-Lewy (CFL) criterion (see [27] through [29]) applicable to first-order hyperbolic partial differential equations with explicit finite difference simulations. It is that the time-step must not be larger than the time it takes for the web running velocity to go from point E to point P, for then values more distant than the immediate neighbors would affect Z_P' and these points do not appear in the finite difference expressions. The criterion is

$dt \leq dx/U$

where U is the web running velocity.

The **third** criterion is implemented in the presence of high cross velocities. In computational fluid dynamics and advection-diffusion-transient problems, mere central differencing of the advection terms generates a finite difference expression set whose solution is contrary to physical laws, when the cross velocity is high. The problem is discussed at length in [30] through [32], where it is asserted that the use of linear differencing practices under high cross-flow conditions is inappropriate, with excessive and dominant errors ensuing through its use. One remedy is to apply upwind (or upstream) differencing techniques to the offending terms. For the running web problem, this amounts to replacing the central difference expression for the cross-derivative term Z_{xt}

$$Z_{xt} = [(Z_E - Z_W) - (Z_E^- - Z_W^-)/(2\ dx\ dt)]$$

by the upstream difference expression

$$Z_{xt} = [(Z_P - Z_W) - (Z_P^- - Z_W^-)/(dx\ dt)]$$

In order to simplify the analysis and presentation of results in this report, the first criterion for the nonrunning web case will be used as the standard to which other time-step values are compared. Therefore, the maximum time-step in any computation will be dx/c, and any other time-step will be referenced to this value.

Problem Parameters

The web physical parameters which are used in study are:

Length L = 1 m (= 3.2808 ft.)
Tension T = 100 N/m (= 6.8520 lbf/ft)
Mass m = 0.01 kg/m^2 (= 0.002048 lbm/ft^2)

These give the parameter

$c = (T/m)^{1/2} = 100$ m/s (= 328.08 ft/s)

The web running velocity U is a variable usually in the range zero to

U = 90 m/s (= 295.27 ft/s)

Analytical Solution and Sample Calculations

For boundary conditions corresponding to type 1 from Section 3.3, the running web equation has an analytical solution (which can be readily established)

Z = SIN(PI*X/L) * COS[PI*K*t/L + PI*U*X/(c*L)] where $K = (c^2 - U^2)/c$ and $c = (T/m)^{1/2}$. The initial conditions for this case then follow as vertical displacement

$$Z(t=0) = SIN(PI*X/L) * COS[PI*U*X/(c*L)]$$

and vertical velocity

$$Z_t(t=0) = (PI*K/L) * [-SIN(PI*X/L)] * SIN[PI*K*t/L + PI*U*X/(c*L)]$$

Some helpful observations concerning the analytical solution just described are now presented. The non-running web (U = 0) has the analytic solution

$$Z = SIN(PI*X/L) * COS(PI*c*t/L)$$

with frequency f = c/(2L). With the problem parameters given earlier for the non-running web:

f = 50 oscillations per second
t = 0.02 seconds

where the periodic time (time for one oscillation) is given by t. The running web has frequency f = K/(2L) which is a reduced frequency tending to zero as the web running speed tends to c. With U = 10 m/s one calculates:

f = 49.5 oscillations per second
t = 0.0202 seconds

With U = 50 m/s one calculates:

f = 37.5 oscillations per second
t = 0.0267 seconds

Since the frequency is different for differing values of web running speed, at any specified time the number of oscillations undergone by two webs with different running speeds will not be the same. Therefore, a standard time reference with which to compare all solutions would simplify the analysis of the results by making the time-frame consistent between sets of results. Therefore, when the term oscillation is used in this paper, the meaning shall be taken as the oscillation of the nonrunning web (U = 0). For example, for the case of a web with a running speed of 50 m/s, 10 oscillations would correspond to a real time of 0.2 seconds (10 * period of nonrunning web (0.02 s)) even though the web would only experience 7.5 oscillations (0.2 seconds * 37.5 oscillations per second) when subjected to the analytic oscillatory problem.

RESULTS

All the results were obtained with the physical parameters of length L, tension T and mass m fixed as specified earlier. These given the parameter c = 100 m/s. When U (the web running velocity) is given, the important ratio U/c can be quickly deduced.

Computational Stability Using Central Differencing

The first finite differencing of the one-dimensional running wave equation was accomplished using central differencing for all derivative terms. In this Section, the stability of the finite differencing scheme was examined. The boundary conditions were those corresponding to fixed end conditions (Z = 0 at X = 0, L). A web running velocity of 50 m/s and a discretized grid of five equal intervals were used. The first run was attempted using the maximum time-step for stability as described earlier. This choice lead to instability in the numerical calculations. Reducing the time-step by up to a factor of 20 delays the onset of instability, especially at lower web running speeds.

Fundamental Examination of Effects of Time-step on Numerical Computations

In the previous Section it was found that reduction of time-step produced delayed onset of instability in the numerical results. In this Section, a fundamental examination of the effects of time-step on numerical results was performed for a nonrunning web with a grid of five intervals. The boundary conditions were fixed roller type (Z = 0 at X = 0, L). To ascertain the effects of time-step variation, a single perturbation was imposed at the center of an initially flat string. The analytic solution for this case corresponds to two waves of amplitudes half the initial perturbation amplitude travelling away from the center of the string. These waves are reflected with negative amplitudes at the endpoints and rejoin at the center. This pattern is repeated to obtain one analytical oscillation and the oscillations are then repeated indefinitely. The time-step reference used in this case is the theoretical maximum time-step for stability. The variable DTF (acronym for delta-time fraction) is a user specific number less than one that is used to reduce the theoretical maximum time step in an attempt to procure stability in the computations. It was found that a value of DTF equal to 1 yields the exact analytic solution. However, as the time-step is decreased, values differing from the analytic are observed. This demonstrates a phenomenon commonly termed over-stability, see [10] for example. Over-stability is the decrease of time-step past a value which is required for stability, causing a loss of accuracy in the numerical calculations.

Effect of Time-Step on the One-Dimensional Running Wave Equation

In the previous Section, a fundamental examination of the effects of time-step variation for nonrunning webs was presented. In this Section, effects of time-step on the numerical simulation of the one-dimensional running wave equation were examined. The analytical solution of the equation, as presented earlier, was used to evaluate the accuracy of the numerical computations by comparing numerical results to analytical results obtained using identical problem parameters. The boundary conditions corresponded to fixed roller conditions (Z = 0 at X = 0, L). In order to study the effects of time-step only, a web speed of 10 m/s was chosen, since it was shown in Section 4.1 that numerical computations at low web running speeds do not require the use of restrictive time-steps to obtain stability. In this case the string was divided into six intervals. Results with DTF = 1, 0.5, 0.25, 0.1 and 0.05 (discussed in [1]) indicate that for a DTF value of 1, the amplitude of the numerical oscillations is smaller than the analytic, but as the time-step is lowered, the numerical amplitude increases, becoming closer to the analytical amplitude, but becoming more out of phase. This observation is opposite of what one would expect if the phenomenon of over-stability was being encountered. In the previous Section, it was shown that over-stability leads to less accurate results as the time-step is decreased. This suggests that over-stability is not being encountered or at least not having a great effect on this numerical simulation. Although over-stability does not appear to be causing errors in the computations, results show that at lower time-steps, errors do exist which cause a phase shift in the numerical results with respect to the analytical results. Figure 1 illustrates results with DTF = 0.5 at the tenth oscillation.

Effect of Grid Refinement on Phase Shift Phenomenon.

We have just seen that reduction of the time step led to a phase shift of the numerical solution with respect to the analytical solution. A possible cause of this might be the discretization of the web. Discretization error is the error in the numerical solution due to the replacement of the continuous web by a discrete model of the web. In general, refining the mesh decreases the discretization error. However the refinement also incurs penalties in the form of round-off errors and increased computational time. Round-off errors are the errors introduced when the equations for the discretized model are not solved exactly.

In previous cases the string was divided into five or six intervals. In this Section, the grid was refined, and the effect of grid discretization with respect to the phase shift phenomenon as seen previously was studied. The numerical parameters and conditions were the same as in the last section except that the string was now discretized into 12 intervals. Again, a web velocity of 10 m/s was used. Numerical results using different time-steps, as well as the analytical result, show a slight phase shift (away from the analytic result) of the numerical solution as DTF is lowered. However, the phase shift obtained using a grid with 12 intervals is much smaller than that attained by using a six interval grid. This shows that greater accuracy is indeed obtained by refining the numerical grid.

Effect of Grid Refinement on Numerical Accuracy

In the last section it was noted that using more grid intervals in the numerical simulation seemed to increase the accuracy of the numerical results. In this Section the effect of the number of grid intervals on the numerical simulation of the one-dimensional running wave equation was examined. The parameters in this case were much like those taken before. The boundary conditions were those for fixed rollers ($Z = 0$ at $X = 0, L$), the web velocity was 10 m/s, and DTF was chosen to be 0.1. Numerical results were obtained under these conditions for different numbers of grid intervals. Figure 2 shows the results in the tenth oscillation. It is easily seen that the discretization of the grid has a great effect on the accuracy of the calculations. The solution with six grid intervals has a phase shift associated with it. However, merely doubling the number of grid intervals to 12, yields values which are very close to the analytical solution. A further increase of grid intervals to 24 now yields results which are almost exactly to the analytical solution. The solution with 48 grid intervals is also shown, but the increase in accuracy over the case with 24 grid intervals is insignificant. This analysis shows that for the aforementioned boundary conditions and a low web running speed, an excellent simulation of the threadline equation can be accomplished using 24 grid intervals and DTF = 0.1.

Effects of Time-Step Variation on a Refined Grid

In the previous section it was seen that using a refined grid, greatly improved the accuracy of the numerical simulation. In this Section, the effect of time-step variation with a refined grid was examined. Using the same boundary conditions (Z = 0 at X = 0, L), web running speed (U = 10 m/s) and an acceptable number of grid intervals (24 grid intervals) as in the previous case, test runs were performed using different time-steps. It was found that all values of DTF compare relatively well with the analytical result at the tenth oscillation. The least accurate numerical result is that for DTF = 1, while results for DTF less than or equal to 0.5 match the analytical result almost exactly. However for a given time-step, instability in the calculations did occur later, the onset of this being delayed as the value of DTF was reduced.

These and other results show that for a particular web running speed and computational duration, there exists a time-step limit for stability. A summary of these results is presented in Table I. In the Table, an 'S' indicates stable computations for the corresponding running speed and time-step while 'U' indicates unstable computations. It must be noted that Table I is only valid for the first ten web oscillations. To understand the results more clearly, test runs such as those previously described were repeated for 20 oscillations. Table II contains a summary of those results. It is seen that in general, smaller time-steps are required to procure stability after 20 oscillations than ten oscillations.

It is very pleasing to note that all of the stable numerical simulations deducible from these two tables given results that compared very well to the analytical results. In fact, in most cases the numerical simulations virtually matched the analytical results exactly. The preceding analysis proves that the time-step must be reduced at higher web running speeds to generate stable numerical simulations. In addition, it was shown that longer computational time durations require reduced time-step size for stability. Also shown was that at moderate and high web running speeds (over 25 m/s), the time-step requirement becomes very restrictive. Accurate numerical simulations at these web running speeds would take a considerable amount of computer time to attain.

Procurement of Stability Using Upstream Differencing

In this Section, a possible solution to the instability problem demonstrated in the previous Sections was examined. Upstream differencing is commonly used in computational fluid dynamics when advection terms and high crossflow conditions exist. The cross derivative term $2*U*Z_{xt}$ in the one-dimensional running wave equation is much like an advection term, and in previous analysis it was shown that stability was a problem in numerical computations involving high web running velocities. Therefore, upstream differencing of the cross derivative term was attempted. The boundary conditions, web velocity and number of grid intervals were the same as taken earlier (Z = 0 at X = 0, L; U = 50 m/s; 6 grid intervals). In the first run, the maximum time-step for stability was used. It is seen in Figure 3 instability was still prevalent. However, reduction of the time-step by a factor of two yields results in which instability has been overcome, albeit with a damping amplitude in the oscillations. Smaller time-steps produced similar results with the use of upstream differencing for very high web velocities was also tested, with similar conclusions.

Examination of Numerical Damping

Since the one-dimensional running wave equation as defined previously does not contain damping terms, the amplitude damping encountered in previous test cases is a phenomenon which must be accounted for. In this section, an examination of numerical damping was performed. Damping was first encountered with the introduction of upstream differencing. This fact indicates the upstream differencing scheme to be a possible cause of the numerical damping. Therefore, full upstream differencing was used. The boundary conditions and number of grid intervals were the same as those used before ($Z = 0$ at $X = 0, L$; 5 grid intervals). A time-step reduction factor of two was used since results from Section 4.7 show this factor was sufficient for stability, even at high web running speeds. The web running velocity was taken as 0 m/s, 10 m/s, 50 m/s and 90 m/s and it was seen that increased damping occurs as the web velocity increases. This result suggests that upstream differencing of the Coriolis term to indeed be causing the erroneous damping.

Effects of Upstream Differencing on the Numerical Simulation

It has been shown that numerical stability could be obtained with larger time-steps by using upstream differencing in place of central differencing for the cross derivative term Z_{xt} . However, it was shown in the last section that the use of upstream differencing for stability procurement also led to a decrease in the accuracy of the numerical simulation in the form of damping: the oscillation magnitudes decreased as time progressed. In order to better understand these results, a detailed examination of the effects of upstream differencing on the numerical simulation was performed. Perhaps a combination of upstream and central differencing would produce more accurate results with less restrictive time-step sizes.

Test runs were performed to ascertain the types of results produced by such a combination of differencing schemes. The boundary conditions were the same as in previous sections ($Z = 0$ at $X = 0, L$). Also, a grid of 24 intervals and a time-step half the maximum time-step (DTF = 0.5) were used. In the computational algorithm an upstream difference weighting parameter (GAMA) is used to indicate the proportions of upstream and central differencing used in the numerical computations. GAMA may vary from zero to one where a value of GAMA equal to zero represents full central differencing and a value of GAMA equal to one represents full upstream differencing of the cross derivative term. Results [1] show that the use of upstream differencing has procured stability in the computations where central differencing has failed. It is also seen that a combination of central and upstream differencing will procure stability with less damping ensuing than if full upstream differencing were used. In fact, the less upstream differencing used in conjunction with the central differencing, the less damping produced in the numerical simulation.

In analyzing the previous results, it would seem that there must be a limit to the smallness of the amount of upstream differencing which will still produce stable results. It would also appear that results obtained using this limiting combination would possess the most accuracy. Additional test runs were performed to test for the existence and determination of this limit. A summary of the test runs is presented in Table III. An entry of 'U' in the table represents unstable computations at the corresponding web running speed and value of GAMA. An entry of 'S' in the Table represents stable computations with damping. It is seen that at higher web running speeds, higher values of GAMA are required for stable numerical results. That is to say, the higher the web running speed, the higher the proportion of upstream differencing required for stability procurement in the numerical computations.

Predictions With an Oscillating Boundary

All computational experiments so far have been performed using fixed end boundary conditions (Z = 0 at X = 0, L). In this section, various production runs were performed using an oscillating boundary condition at the upstream roller (Z = SIN(B*PI*c*t/L) where B is a computational user input parameter which varies the oscillation frequency). The solutions described in this section are presented as tentative results of a practical problem. The initial conditions for the web were zero vertical displacement (Z(x, t=0) = 0) and zero vertical velocity (Z_t(x, t=0) = 0). For a real web under these conditions, one would expect the boundary excitation to begin vibrating downstream points of the web and continue until some form of steady state oscillating motion is achieved.

The first runs were performed for various web running speeds using a numerical grid with 24 intervals and time-steps from Tables I and II which were expected to render stable computations at those running speeds. In general, it was found that the time-step sizes required for stability were smaller than those shown in Tables I and II, especially for high frequency forced oscillations. Additional runs were performed using combined upstream and central differencing as in the last section. These results showed that larger values of GAMA were required for stability than those shown in Table III. In other words, at very low oscillation frequencies, the required stability parameters were nearly sufficient, but, at higher oscillation frequencies, more restrictive parameters were required for stability. This would suggest that stability parameters are dependent on the boundary oscillation frequency as well as the web running speed.

In order to examine stable numerical results, production runs were performed for a web running speed of 30 m/s using full upstream differencing and a time-step corresponding to DTF = 0.5. It was found that by the 50th oscillation, results had reached steady state oscillating motion. It was also seen that the frequency of the numerical oscillations compare exactly to the frequency of the oscillating boundary. In each case of boundary oscillation frequencies given by the parameter B from 0.1 to 10, it was found that the numerical oscillation frequency matches the boundary oscillation frequency exactly. In Figures 4, 5 and 6, the web position at different times throughout one steady state numerical oscillation is shown with left hand boundary oscillation frequencies corresponding to B = 0.5, 1 and 10. The first of these shows displacements which are expected from a relatively slow oscillation rate at the upstream roller contact point. The second one illustrates propagation and amplification of waves along the web - the case with upstream roller oscillation frequency equal to the natural frequency of the web. The final figure illustrates propagation of the oscillation along the web with multiple waves.

CLOSURE

The present research was concerned with the design, development and testing of finite difference numerical schemes for one-dimensional web dynamics. Effects of grid refinement and time-step size on numerical stability and accuracy were examined. In addition, analysis of central and upstream finite differencing techniques was presented. A major accomplishment of this research was the development of a computational algorithm which models the one-dimensional running wave equation. Using central differencing for the spatial derivative terms, numerical experiments showed that a denser grid yields more accurate results. In fact, as the grid is refined, the solution converges to the analytical solution if the computations are stable. It was found that accurate numerical results could be obtained using a grid with 24 intervals. The results of this study showed that time-step size greatly affected the stability of the numerical computations. For a particular web running speed and computational duration, there exists a time-step limit for stability. It was found that longer computational time durations as well as higher web running speeds require reduced time-step size for stable numerical results. High web running speed simulations require very restrictive time-step sizes requiring excessive computational times.

To alleviate the time-step restriction required for high web running speeds or long computational durations, the use of upstream differencing for the Coriolis term in the one-dimensional running wave equation was investigated. It was found that this differencing scheme did procure stability in the computations, allowing a less restrictive time-step than that required for central differencing. Full upstream differencing produces an additional discretization error which leads to damping in the numerical computations, which is incorrect. However, it was found that a combination of upstream and central differencing yields stability at larger time-steps than that required using full central differencing as well as less damping than that produced using full upstream differencing.

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TABLE I

EFFECT OF WEB RUNNING SPEED AND TIME-STEP FRACTION ON COMPUTATIONAL STABILITY AFTER TEN OSCILLATIONS (24 GRID INTERVALS, CENTRAL DIFFERENCING)

	U/c					
DTF	0.05	0.10	0.15	0.20	0.25	0.30
1.00	S	U	U	U	U	U
0.50	S	S	U	U	U	U
0.40	S	S	S	U	U	U
0.25	S	S	S	S	U	U
0.20	S	S	S	S	S	U
0.10	S	S	S	S	S	S
0.05	S	S	S	S	S	S

TABLE II

EFFECT OF WEB RUNNING SPEED AND TIME-STEP FRACTION ON COMPUTATIONAL STABILITY AFTER 20 OSCILLATIONS (24 GRID INTERVALS, CENTRAL DIFFERENCING)

	U/c					
DTF	0.05	0.10	0.15	0.20	0.25	0.30
1.00	S	U	U	U	U	U
0.50	S	S	U	U	U	U
0.40	S	S	U	U	U	U
0.25	S	S	S	U	U	U
0.20	S	S	S	U	U	U
0.10	S	S	S	S	U	U
0.05	S	S	S	S	S	U

TABLE III

EFFECT OF WEB RUNNING SPEED AND UPSTREAM DIFFERENCING ON COMPUTATIONAL STABILITY (24 GRID INTERVALS, DTF = 0.5)

	U/c				
GAMA	0.10	0.20	0.30	0.40	0.50
0.00	U	U	U	U	U
0.05	S	U	U	U	U
0.10	S	S	U	U	U
0.15	S	S	S	U	U
0.20	S	S	S	S	U
0.25	S	S	S	S	S
0.30	S	S	S	S	S

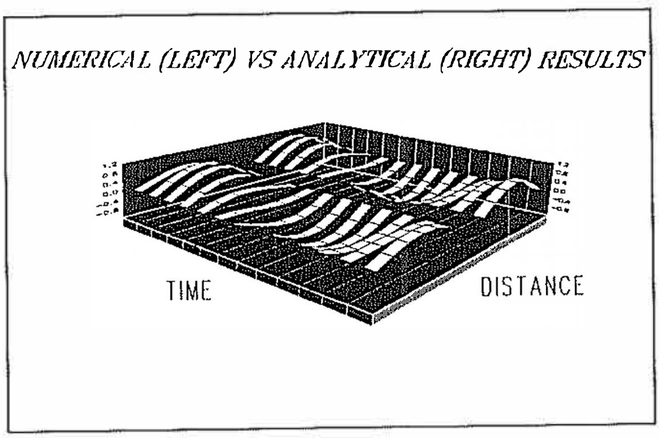

Figure 1. Numerical vs. Analytical Solutions of 1-D Running Web - 3-D View (Six Grid Intervals, DTF = 0.5, Central Differencing, 10th Oscillation, Running Speed = 10 m/s)

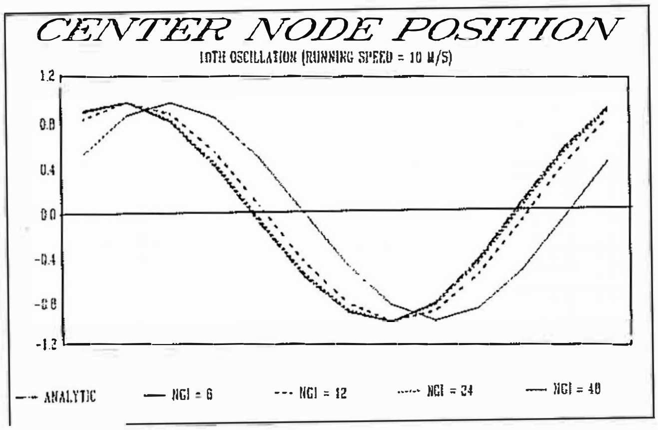

Figure 2. Numerical vs. Analytical Solutions Using Different Grid Refinements (DTF = 0.1, Central Differencing, 10th Oscillation, Running Speed = 10 m/s)

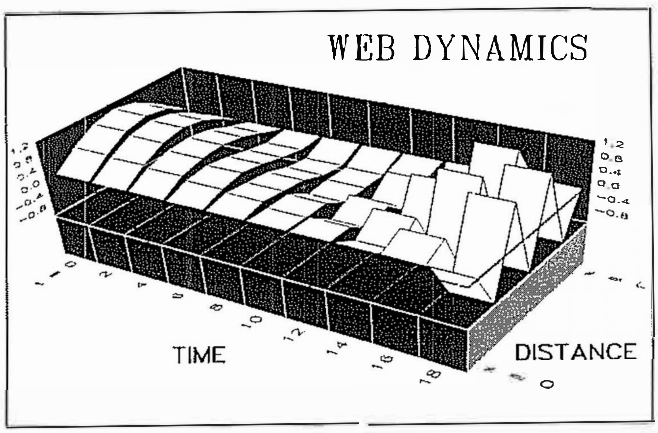

Figure 3. Failure of Full Upstream Differencing of Z_{xt} Term to Procure Numerical Stability Using a Time-step Fraction of 1 - 3-D View (Six Grid Intervals, Time = t/1000, Running Speed = 50 m/s)

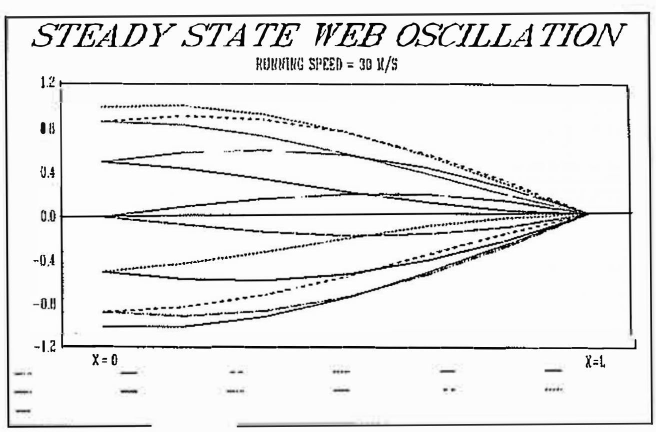

Figure 4. Numerical Solution of a Single Steady State Oscillation of a Web with Left Hand Boundary Oscillation Corresponding to B = 0.5 (24 Grid Intervals, DTF = 0.5, Upstream Differencing)

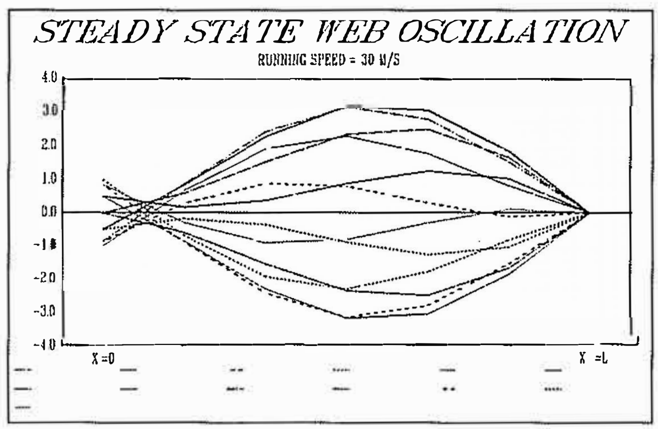

Figure 5. Numerical Solution of a Single Steady State Oscillation of a Web with Left Hand Boundary Oscillation Corresponding to B = 1.0 (24 Grid Intervals, DTF = 0.5, Upstream Differencing)

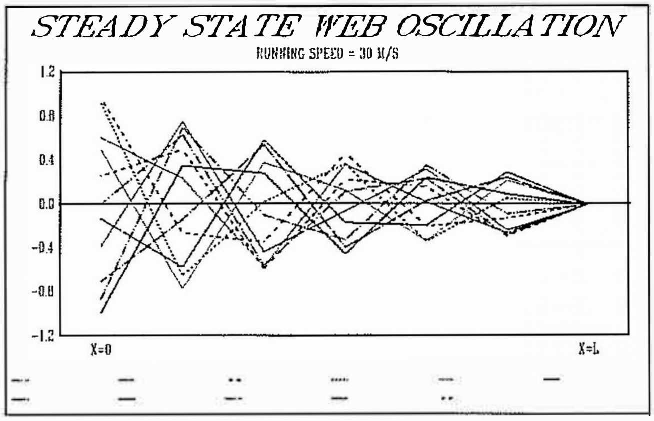

Figure 6. Numerical Solution of a Single Steady State Oscillation of a Web with Left Hand Boundary Oscillation Corresponding to B = 10.0 (24 Grid Intervals, DTF = 0.5, Upstream Differencing)

COMPUTER SIMULATION OF WEB DYNAMICS
D. G. Lilley

What 3-D graphics package did you use to make the slides?
Robert Walton, Eastman Kodak Company

This was Boeing Graphics 3-D. They're actually incorporated now in a package called Enable. It's the big integrated package that the government people use as their standard. Originally when I got this package it was Boeing Graphics 3D. It is very versatile and runs on microcomputers and costs about $300. It's a very useful package.

Was experimental validation of this work performed?
Robert Walton, Eastman Kodak Company

Not specifically. Now there was even better validation with analytic answers, performed for the cases which had analytic answers. Including one dimension of running web, for example, when it is displaced in a certain initial displacement and then released at a certain velocity. There is an analytic answer that we can use for the running one dimensional web equation. That one is used as mentioned in the paper, and has been confirmed by myself, Dr. Moretti, and Dr. Chang. We reconfirmed it in two or three different ways.

Concerning the two additional terms, how important is the Coriolis term? At what velocities can it be neglected?
Ron Swanson, 3M

That - I'm going to have to wait for some practical data from Dr. Moretti. I would like to validate that with some experimental data as well. I don't have an answer at this point unfortunately. (Moretti) *"We did it both ways and in some other works like Dr. Chang's thesis the effect is shown. It's not an insignificant effect it's traditional in the literature to leave the Coriolis term out because it leads to all the problems Dr. Lilley ran into, but it is not unimportant. It does make a difference and it does lead for example to the asymmetry of the solutions that you observed and that we observe in the field. So we did compare this solution also with some of our lab work and NO - you can't neglect the Coriolis term. I wish we could."*.

How important is the web velocity term? At what velocities can it be neglected?
Ron Swanson, 3M

Well again we haven't done any definite work to make that discrimination. Obviously a very slow running web for most displacements would oscillate in a manner which is not very dependent on the web velocity. I think you are asking what is the value of U divided by C, the natural frequency given by square root of tension over mass. You're asking what is the value of U/C at which the running becomes important. I don't have any personal opinion right now on that since we have not yet performed the appropriate analysis. (Moretti) *"U/C is much much smaller than 1. Again we did look at that in some of the threadline work and because there is a theoretical "C" that is obtained with the mass of the web, and then when you add aerodynamics the actual critical velocity may be from 40 to 25 percent of that so it is possible under some airloading circumstances to reach instability at 25% of theoretical "C" which means you have to be one order of magnitude down from that.""*

What kind of air flow effects to you think you will have to add to model the onset of web flutter?
Oral question from Dr. Frank Chambers

Well that's the perfect example of the synthesis of experimental work and theoretical work. Dr. Moretti is pushing forward very dramatically in that area and looking indeed at whether, for example, air drag vs. air loading are the most important of those extra terms required. I would not like to speculate too much prior to developing that work. (Moretti) *"This whole project started as a combined theoretical-experimental project. My job was to obtain mass quotations from simple situations and then feed them to Dr. Lilley who would use them to solve complex situations. We hope to do that in the near future."*