Zeros of Random Orthogonal Polynomials

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Abstract: Let $\{f_j\}$ be a sequence of orthonormal polynomials where the orthogonality relation is satisfied on either the real line (OPRL) or on the unit circle (OPUC). We study zero distribution of random linear combinations of the form

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z),$$

where $\{\eta_j\}$ are random variables. We give quantitative estimates on the zeros accumulating on the unit circle for a wide class of random polynomials P_n. When the coefficients $\{\eta_j\}$ are independent identically distributed (i.i.d.) real-valued standard Gaussian, we give asymptotics for the expected number of zeros of various classes of random sums P_n spanned by OPUC. For the case when the coefficients $\{\eta_j\}$ are i.i.d. complex-valued standard Gaussian coefficients, we derive a formula for the expected number of zeros of P_n. The formula is then applied to give asymptotics of the expected number of zeros of P_n when $\{f_j\}$ are from the Nevai class. We also compute the limiting value as $n \to \infty$ of the variance of the number of zeros of P_n in annuli that do not contain the unit circle for the case when $\{\eta_j\}$ are i.i.d. complex-valued standard Gaussian random variables, and $\{f_j\}$ are OPUC from the Nevai class. In the case of annuli that contain the unit circle, for a wide class of random variables $\{\eta_j\}$ and $\{f_j\}$ that are OPUC, we give quantitative results that show the variance of the number of zeros of P_n scaled by n^2 tends to zero as n tends to infinity. The work is concluded by providing formulas for the variance of the number of zeros of a random orthogonal power series, specifically when $\sum_{j=0}^{\infty} \eta_j f_j(z)$, with $\{\eta_j\}$ being i.i.d. complex-valued standard Gaussian, and $\{f_j\}$ are OPUC from the Szegő class.

TABLE OF CONTENTS

Chapter Page

Chapter Page

LIST OF SYMBOLS

$\mathbb{N}$	Set of natural numbers: $1, 2, 3, \ldots$
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{T}$	The unit circle
$\mathbb{D}$	The unit disk
$\mathbb{R}^d$	d-dimensional real vector space
$\mathbb{C}^d$	d-dimensional complex vector space
$\mathbb{S}^{d-1}$	unit sphere in $\mathbb{R}^d$
$f(n) = o(g(n))$	If $g(n) > 0$ and $f(n)/g(n) \to 0$ as $n \to \infty$
$f = \mathcal{O}(g)$	If there exists $C > 0$ such that $\lvert f \rvert \leq C$
$\mathrm{Re}(z)$	The real part of a complex number z
$\mathrm{Im}(z)$	The imaginary part of a complex number z

CHAPTER I

INTRODUCTION

1.1 Plan of this dissertation

In this dissertation zeros of random polynomials are studied. Chapter I covers a brief history of the subject and discusses the main results of the dissertation.

The second chapter contains results from a joint work with Pritsker [92]. The main results of the chapter give quantitative estimates on the zeros accumulating in various sets for a wide class of random polynomials. This wide class includes polynomials with random coefficients that may not have identical distributions, and such that the coefficients are dependent. The results are applied to random polynomials spanned by various deterministic polynomial bases.

Chapter III covers results and further extensions of two works; one that is joint work with Yattselev [122], and another that is solely by the author [124]. In the first section of the chapter, we give asymptotics for the expected number of zeros of various classes of random sums spanned by orthogonal polynomials on the unit circle (OPUC) with independent identically distributed (i.i.d.) real-valued standard Gaussian coefficients. The second section of the chapter considers the expected number of zeros of a random sum with i.i.d. complex-valued standard Gaussian coefficients spanned by a polynomial basis. Applications are directed to random sums spanned by OPRL or OPUC, and then asymptotics for the intensity function are derived.

In the fourth and final chapter, the variance of the number of zeros of a random sum is considered. We compute the limiting value as $n \to \infty$ of the variance of the number of zeros of a random sum with i.i.d. complex-valued standard Gaussian random variables spanned by OPUC that are from the Nevai class in annuli that do not contain the unit circle. In the case of annuli that contain the unit circle, asymptotics are provided for the variance of the number of zeros of a random sum under the assumption that the distribution for each of the random coefficients satisfies certain uniform bounds for the fractional and logarithmic moments, and the spanning functions are OPUC that either possess recurrence coefficients that are absolutely summable, or are such that they are regular in the sense of Ullman-Stahl-Totik. The chapter is concluded by giving a formula for the variance of the number of zeros of a random series with i.i.d. complex-valued standard Gaussian coefficients that are spanned by OPUC from the Szegő class.

1.2 A brief history of the study of random polynomials

The systematic study of the expected number of real zeros of polynomials

$$P_n(z) = \eta_n z^n + \eta_{n-1} z^{n-1} + \cdots + \eta_1 z + \eta_0 \tag{1.2.1}$$

with random coefficients $\{\eta_j\}$, called *random algebraic polynomials* (or *Kac polynomials*), dates back to the 1930's. Let $\mathbb{E}$ denote the mathematical expectation, $\mathbb{P}$ be the probability of an event, and let $N_n(S)$ denote the number of zeros of P_n in a set S. In 1932, Bloch and Pólya [11] showed that when $\{\eta_j\}$ are independent and identically distributed (i.i.d.) random variables such that $\eta_0 = 1$ almost surely (a.s.) (i.e. $\mathbb{P}(\eta_0 = 1) = 1$) and all other random variables that take values from the set $\{-1, 0, 1\}$ with equal probabilities, then

$$\mathbb{E}[N_n(\mathbb{R})] = \mathcal{O}(\sqrt{n}), \quad \text{as} \quad n \to \infty.$$

Starting in 1938 and spanning through 1948, Littlewood and Offord ([69], [70], [71], [72], [73]) produced upper and lower bounds for $N_n(\mathbb{R})$ of the random algebraic polynomial (1.2.1). Specifically, they showed that

$$\frac{\log n}{\log\log\log n} \ll N_n(R) \ll \log^2 n,$$

with probability $1 - o(1)$ as $n \to \infty$, when the random variables $\{\eta_j\}$ are i.i.d. with common distribution that is either real-valued standard Gaussian, Bernoulli, or uniform on $[-1, 1]$.

In 1943, Kac [61] produced an integral equation for $\mathbb{E}[N_n(\Omega)]$, with $\Omega \subset \mathbb{R}$ a measurable set, for the random algebraic polynomial P_n when the random variables $\{\eta_j\}$ are i.i.d. standard Gaussian. The formula Kac gave is

$$\mathbb{E}[N_n(\Omega)] = \frac{1}{\pi} \int_{\Omega} \frac{\sqrt{1 - h_n^2(x)}}{|1 - x^2|}\, dx, \quad h_n(x) = \frac{n x^{n-1}(1 - x^2)}{1 - x^{2n}}. \tag{1.2.2}$$

We note that independently in 1945, while studying random noise Rice [94] derived a similar formula for $\mathbb{E}[N_n(\mathbb{R})]$ in the Gaussian setting. After Kac established the above formula, he proceeded with the asymptotic

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{2 + o(1)}{\pi} \log n \quad \text{as} \quad n \to \infty. \tag{1.2.3}$$

The error term in the above asymptotic was later sharpened by Hammersley [47], Jamrom ([57], [58]), Wang [117], Edelman and Kostlan [27], and finally Wilkins [118] who showed that

$$\mathbb{E}[N_n(\mathbb{R})] \sim \frac{2}{\pi} \log n + \sum_{k=0}^{\infty} A_k n^{-k}$$

for some explicit constants $\{A_k\}$.

In [61] Kac conjectured that a similar formula as (1.2.2) should hold when the random variables are i.i.d. uniform on $[-1, 1]$ and the asymptotic (1.2.3) would follow from his original proof. Realizing that the same proof would not go through, in a follow up paper [62] Kac was able to produce the asymptotic (1.2.3) in this uniform distribution case by a different method.

Due to the work of Kac and Rice, formulas for the density function for the expected number of zeros of a random polynomial, called the *intensity function* (or the *first correlation function*), are known as *Kac-Rice formulas.*

When the random variables have a discrete distribution, one can formulate an explicit formula for the intensity function. However the formula takes a complicated shape and is not amenable to computations as done in the Gaussian case. Using a different approach of studying the number of sign changes on a fixed sequence of points to approximate the number of roots of P_n, Erdős and Offord [28] were able to show that when the random variables are i.i.d. from the Bernoulli distribution it follows that

$$N_n(\mathbb{R}) = \frac{2}{\pi}\log n + o\left((\log n)^{2/3}) \log\log n\right)$$

with probability $1 - o\left(1/\sqrt{\log\log n}\right)$. By refining the method given by Erdős and Offord, Ibragimov and Maslova ([51], [52]) proved that when the random variables $\{\eta_j\}$ are i.i.d. with mean zero and are from the domain of attraction of the normal law, for the random polynomial P_n defined by (1.2.1) we have

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{2}{\pi}\log n + o(\log n).$$

When the random variables from the domain of attraction of the normal law do not have mean zero, Ibragimov and Maslova [53] proved the above asymptotic holds with exactly half the expected number of zeros. In this case, the error term of $o(\log n)$ was recently sharpened to $\mathcal{O}(1)$ by Nguyen, Nguyen, and Vu [87].

In 1995, considering the case when $\{\eta_j\}$ are i.i.d. real-valued standard Gaussian, Shepp and Vanderbei [97] gave a formula for the expected number of zeros of P_n in (1.2.1) off the real line. In their work they were also able to obtain the limits

$$\lim_{n\to\infty} \rho_n^{\mathbb{C}}(z) = \frac{1}{\pi(1-|z|^2)^2}\sqrt{1 - \left|\frac{1-|z|^2}{1-z^2}\right|^2} \tag{1.2.4}$$

and

$$\lim_{n\to\infty} \rho_n^{\mathbb{R}}(x) = \frac{1}{\pi}\frac{1}{|1-x^2|}, \tag{1.2.5}$$

where $\rho_n^{\mathbb{C}}(z)$ is the intensity function for the number of purely complex zeros of the random polynomial, and $\rho_n^{\mathbb{R}}(x)$ is the intensity function for the number of real zeros of the random polynomial. Within the proof of computing the above limits, Shepp and Vanderbei showed that as $n \to \infty$, uniformly about $n - (2/\pi)\log n$ of zeros of P_n accumulate on the unit circle, and about $(2/\pi)\log n$ of real roots concentrate at ± 1. Ibragimov and Zeitouni [54] were able to generalize the work of Shepp and Vanderbei by giving the limit of the expected value of a scaled version of the expected number of zeros of the random algebraic polynomial P_n in a disk of radius r when the random variables $\{\eta_j\}$ are i.i.d. with common distribution that belongs to the domain of attraction of an α-stable law.

The formulas provided by Shepp and Vanderbei for the intensity functions for the number of real and complex zeros of the random algebraic polynomial has since forth been generalized Feldheim [39] and independently Vanderbei [115]. These general formulas give the intensity functions for random sums of the form

$$\sum_{k=0}^{n} \eta_k f_j(z),$$

where $\{\eta_k\}$ are i.i.d. real-valued standard Gaussian random variables, and $\{f_k\}$ are entire functions that are real-valued on the real line. In Chapter III Section 2 these formulas are stated in shape given by Vanderbei and applied to random orthogonal polynomials.

1.3 Equidistribution

We now give an overview of the results presented in Chapter II. For $\{Z_1, Z_2, \ldots, Z_n\}$ the set of complex zeros of the random polynomial P_n defined by (1.2.1) of degree n, these zeros give rise to the (normalized) *zero counting measure*

$$\tau_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{Z_k}, \tag{1.3.1}$$

which is a random unit Borel measure in $\mathbb{C}$. Using a classical result by Erdős and Turán [29] as a starting point, under mild conditions on the random variables $\{\eta_j\}$ Shparo and Shur [103] were able to show that as $n \to \infty$, for any $r < 1$

$$\frac{\nu_n(r)}{n} \xrightarrow{\mathbb{P}} 0, \quad \text{and} \quad \tau_n\left(A_r(\alpha,\beta)\right) \xrightarrow{\mathbb{P}} \frac{\beta-\alpha}{2\pi}, \tag{1.3.2}$$

where $\nu_n(r) = N_n(\{z : |z| < r\})$, with $\xrightarrow{\mathbb{P}}$ denoting convergence in probability, and $A_r(\alpha, \beta) = \{z \in \mathbb{C} : r < |z| < 1/r,\ \alpha \leq \arg z < \beta\}$. The results of Shparo and Shur in (1.3.2) (we note that similarly also by Arnold [4]) show that with a high probability, almost all of the roots of the random algebraic polynomial P_n are uniformly concentrated near the unit circle and that the arguments of the roots are asymptotically equidistributed.

The fact of equidistribution for the zeros of random polynomials can now be expressed via the convergence of τ_n in the weak* topology to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference, where $d\mu_{\mathbb{T}}(e^{it}) := dt/(2\pi)$. Recent papers on the global limiting distribution of zeros of random polynomials includes the works of Ibragimov and Zeitouni [54], Ibragimov and Zaporozhets [55], Hughes and Nikeghbali [50], and Kabluchko and Zaporozhets ([59], [60]). In particular, Ibragimov and Zaporozhets [55] arrived at the conclusion that if the coefficients of the random algebraic polynomial P_n are i.i.d. random variables, then the condition $\mathbb{E}[\log^+ |\eta_0|] < \infty$ is necessary and sufficient for the convergence $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$ to hold almost surely.

Under the assumption that the coefficients $\{\eta_k\}$ of a random polynomial are i.i.d. complex random variables with absolutely continuous distribution with $\mathbb{E}[|\eta_0|^t] = \mu < \infty$ for some $t > 0$, estimates for the rate at which $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$ were provided by Pritsker and Sola [89] using the largest order statistic $Y_n = \max_{k=0,\ldots,n} |\eta_k|$. In their work they also gave an equidistribution result that allows the condition that the random variables $\{\eta_j\}$ be i.i.d. can be dropped.

The main results of Chapter II are a joint work with Pritsker [92] which show how to remove many unnecessary restrictions and generalize the results of [89] in several directions. We first develop essentially the same theory as in [89] (but using a different approach) for the case of coefficients that are neither independent nor identically distributed, and whose distributions only satisfy certain uniform bounds for the fractional and logarithmic moments. We also generalized the results of [89] by considering random polynomials spanned by general bases. That is, we consider random sums of the form

$$P_n(z) = \sum_{k=0}^{n} \eta_k B_k(z),$$

where $B_k(z) = \sum_{j=0}^{k} b_{j,k} z^j$, with $b_{j,k} \in \mathbb{C}$ for all j and k, and $b_{k,k} \neq 0$ for all k, is a polynomial basis, i.e. a linearly independent set of polynomials. We apply the main theorem of Chapter II to obtain a quantitative result on the zero distribution of a certain class of random orthogonal polynomials, specifically when $\{B_k\}$ are polynomials that satisfy

$$\limsup_{k\to\infty} ||B_k||^{1/k} \leq 1, \quad \lim_{k\to\infty} |b_{k,k}|^{1/k} = 1.$$

Note that such conditions hold for various standard bases used for representing analytic functions in the disk. We also show how one can handle the discrete random coefficients by methods involving the highest order statistic Y_n, augmenting the ideas of [89]. Furthermore, since any real random variable is the limit of an increasing sequence of discrete random variables, we are able to extend the arguments to arbitrary random variables. Under the assumption that the coefficients satisfy uniform bounds on the first two moments, we further develop the highest order statistic approach to the case of dependent coefficients. This allows us to generalize Theorem 3.7 of [89].

It should be noted that the results of Chapter II have since been generalized by using potential theoretic techniques by Pritsker [90]. Among the results of [90] are the generalization of our result to orthogonal polynomials supported on general curves and supported on various sets in the plane.

1.4 The expected number of zeros

The next four subsections provide an outline for results contained in Chapter III and the surrounding history of the topics.

1.4.1 Random orthogonal polynomials

In 1971, Das [18] considered random polynomials of the form $\sum_{k=0}^{n} \eta_k p_k(z)$, where $\{p_k\}$ are Legendre polynomials, i.e. polynomials $\{p_k\}$ orthogonal with respect to an absolutely continuous measure μ that is supported on $[-1, 1]$ with $d\mu(x) = dx$, and the random variables $\{\eta_k\}$ are real-valued i.i.d. standard Gaussian. Das was able to show that the expected number of zeros of the random orthogonal polynomial in $(-1, 1)$ is asymptotic to $n/\sqrt{3}$. Generalizing these results, Farahmand ([32], [33], [34]) examined level crossings of random Legendre polynomials with coefficients that are allowed to have different distributions.

Das and Bhatt [19] extended the work in [18] to include the class of orthogonal polynomials on the real line (OPRL) to be the classical orthogonal polynomials, Jacobi (polynomials orthogonal with respect to μ on $[-1, 1]$ with $d\mu(x) = (1-x)^\alpha(1+x)^\beta dx$ for $\alpha, \beta > -1$), Laguerre (polynomials orthogonal with respect to μ on $[0, \infty)$ with $d\mu(x) = e^{-x}x^\alpha$ for $\alpha > -1$), and Hermite (polynomials orthogonal with respect to μ on $(-\infty, \infty)$ with $d\mu(x) = \exp(-x^2)dx$). They showed the same asymptotic held true for the zeros of the random orthogonal polynomial in $(-1, 1)$, however the results concerning the Hermite and Laguerre cases had some gaps.

The gaps [19] were fixed in 2015 by Lubinsky, Pritsker, and Xie [77] by considering a larger class of OPRL that had only mild assumptions on the measure and weight function. Using potential theory for their results, they showed that the same asymptotic holds for random sums spanned by the larger class of OPRL. These results were further generalized by Lubinsky, Pritsker, and Xie [78] to allow the OPRL to have support on the whole real line and arrived at the same asymptotic in this case.

Many examples and properties of OPRL and orthogonal polynomials on the unit circle (OPUC) are explored in the books by Szegő [111] and Simon [104]. One example of an OPUC that we have already mentioned are the monomials, that is z^n for $n \in$

$\mathbb{N} \cup \{0\}$. To see that this is indeed so, we note the orthogonality following relation

$$\int_{-\pi}^{\pi} e^{im\theta}\overline{e^{in\theta}}\,\frac{d\theta}{2\pi} = \delta_{m,n}, \quad m, n \in \mathbb{N} \cup \{0\}.$$

In Subsection 1.4.3 we will discuss the case when the coefficients of the random polynomial are i.i.d. complex-valued standard Gaussian. We remark that in the i.i.d. complex-valued standard Gaussian case, the paper of Shiffman and Zelditch [100] mentions a heuristic argument that provides the intensity function and its asymptotic for random polynomials spanned by OPUC associated to analytic weights in terms of the distributional Laplacian.

Other authors have studied the asymptotic zero distribution for random polynomials spanned by orthogonal polynomials with respect to various measures. There has also been work done in the higher dimensional analogs of these settings, see Shiffman and Zelditch ([99]-[102]), Bloom ([12], [13]), Bloom and Shiffman [15], Bloom and Levenberg [14], Bayraktar [6], and Pritsker ([90], [91]).

1.4.2 Random sums with real-valued i.i.d. standard Gaussian coefficients spanned by OPUC

The third chapter begins by considering random polynomials of the form

$$P_n(z) = \sum_{k=0}^{n} \eta_k \varphi_k(z), \tag{1.4.1}$$

where $\{\eta_k\}$ are real-valued i.i.d. standard Gaussian random variables, and $\{\varphi_k\}$ are OPUC. This part of the chapter contains results from a joint work with Yattselev [122]. Note that taking the functions of (1.4.1) to be OPUC that are real-valued on the real line complements the case considered by Lubinsky, Pritsker, and Xie ([77], [78]) where these spanning functions are OPRL. We use a version of Christoffel-Darboux formula suited for OPUC to simplify the intensity functions for the expected number of real and complex zeros of P_n. From these expressions, under the assumption that the spanning OPUC are from the Nevai class, we deduce the limiting value of these density functions away from the unit circle, hence generalizing the limits (1.2.4) and (1.2.5) provided by Shepp and Vanderbei. Under the mere assumption that the measure μ associated to the OPUC is doubling on subarcs of $\mathbb{T}$ centered at 1 and -1, we show that the expected number of real zeros of P_n is at most

$$(2/\pi)\log n + O(1),$$

and that the asymptotic equality holds when the corresponding recurrence coefficients associated to the OPUC decay no slower than $n^{-(3+\epsilon)/2}$, $\epsilon > 0$, thus extending the original work by Kac [61]. As with the complex Gaussian random variables case, our results show that the zeros are accumulating on the unit circle. Hence we conclude with providing results that estimate the expected number of complex zeros of P_n in shrinking neighborhoods of compact subsets of the unit circle.

1.4.3 Random polynomials with i.i.d. complex-valued standard Gaussian coefficients

We now consider the complex Kac polynomial $\sum_{k=0}^{n} \eta_k z^k$, where $\{\eta_k\}$ are i.i.d. complex-valued standard Gaussian random variables. That is, when $\eta_j = \alpha_j + i\beta_j$, where α_j and β_j are i.i.d. real-valued standard Gaussian for all $j \in \{0, 1, \ldots, n\}$. The classic result of Hammersley [47] (given later in different shapes by Arnold [4], Ledoan et. al. [41], Shiffman and Zelditch [100], and Farahmand ([31],[33])) says that the first intensity function for a complex Kac polynomial is given by

$$\rho_n(z) = \frac{1}{\pi}\frac{1 - |h_{n+1}(z)|^2}{(1 - |z|^2)^2}, \quad \text{where} \quad h_{n+1}(z) = \frac{(1 - |z|^2)(n+1)z^n}{1 - |z|^{2(n+1)}}. \tag{1.4.2}$$

We remark that in contrast to real-valued Gaussian case in which there is an intensity function for the purely complex zeros and a separate intensity function for the real zeros, in the setting of the random coefficients being complex-valued Gaussian, there is only an intensity function for purely complex zeros. This follows from the fact that when $\{\eta_k\}$ are i.i.d. complex-valued standard Gaussian, the common probability distribution for the random variables is absolutely continuous with respect to Lebesgue area measure (with density $e^{-|\eta|^2}/\pi$). Thus, since the Lebesgue area measure of a line segment is zero, the probability that the complex Kac polynomial has any real roots is zero. We refer the reader to pp. 142-143 of [32] for a complete discussion of this phenomenon.

As noted by Arnold [4], Farahmand ([31],[33])), Farahmand and Jahangiri [37], Ledoan et. al. [41], and Shiffman and Zelditch [100], one has

$$\lim_{n\to\infty} \rho_n(z) = \frac{1}{(1-|z|^2)}, \quad |z| \neq 1.$$

Furthermore, in [4] and [41] it was shown that

$$\mathbb{E}[N_n(D(0,r))] = \frac{r^2}{1-r^2}, \quad 0 < r < 1.$$

Ledoan et. al. [41] also proved the following scaling limit

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n(D(0,e^{-s/2n}))]}{n} = \frac{1-e^{-s}(1+s)}{s(1-e^{-s})}, \quad s > 0.$$

Taking the analysis of random polynomials with complex coefficients in a different direction, Farahmand [35] considered the spanning functions of the random polynomial to be $\{\cos j\theta\}_{j=0}^{n}$, which give what are called *random trigonometric polynomials*. For further results concerning random trigonometric polynomials with complex-valued Gaussian coefficients we refer the reader to the work of Farahmand and Grigorash [36], and for those with real-valued Gaussian coefficients the works of Dunnage [26], Das [20], Wilkins [119], Qualls [93], Sambandham [96], and Sambandham and Maruthachalam [82].

Others have derived formulas for the intensity function of Gaussian Analytic Functions (GAF) of the form $P(z) = \sum_{j=0}^{\infty} \eta_j f_j(z)$, where the f_j's are square summable analytic functions on a domain, and the η_j's are i.i.d. Gaussian random variables, in terms of the distributional Laplacian. For the case when the random variables are complex-valued i.i.d. Gaussian, in 2000 Hough, Krishnapur, Peres, and Virág (Section 2.4.2, pp. 24-29 of [49]) and Feldheim (Theorem 2, p. 6 of [39]) derived the intensity function of zeros. Feldheim also obtained the intensity function of zeros for a GAF in the same paper (Theorem 3, p. 7 of [39]) when the f_j's are real-valued on the real line and the η_j's are real-valued i.i.d. standard Gaussian random variables.

1.4.4 Random sums with i.i.d. complex-valued standard Gaussian coefficients spanned by orthogonal polynomials from the Nevai class

In the second section of Chapter III, we apply an extension of the intensity function (1.4.2) for the expected number of zeros of random sums of the form

$$P_n(z) = \sum_{k=0}^{n} \eta_k f_k(z), \tag{1.4.3}$$

where $\{\eta_k\}$ are i.i.d. complex-valued standard Gaussian, and $f_k(z)$ are polynomials of degree k, where $k \in \{0, 1, \ldots, n\}$. The proof of the of the formula that we apply is given in the Appendix. We note the formula we present was given independently by Ledoan [65] in the case of taking the spanning functions $\{f_k\}$ to be entire functions that are real-valued on the real line. However within the proof in [65], there are some justifications which are not fully clarified. Remaining with case of a polynomial basis, all these justifications can be made sound. Our main application of the formula for the expected number of zeros of P_n is to random orthogonal polynomials spanned by OPRL or OPUC.

Using the Christoffel-Darboux formula for OPRL and its analogue for OPUC, the intensity function for the expected number of zeros of P_n in these cases takes a very simple shape. From these expressions, under the mere assumption that the orthogonal polynomials are from the Nevai class, we give the limiting value of the intensity function away from their respective sets where the orthogonality holds. The limiting value of the intensity function concerning random orthogonal polynomials spanned by OPRL extends the work of Farahmand and Grigorash (Section 4 of [36]) in which the spanning functions of their random trigonometric polynomial can be modified to be the Chebyshev polynomials. Furthermore, our limiting value of the intensity function for random orthogonal polynomials spanned by OPUC generalizes the result given by Peres and Virág [88] (i.e. taking $n = 1$ of their Theorem 1) when the spanning functions were the monomials to that of a very general basis of OPUC. Our result further extends their work in that this limiting value also holds for the exterior of the unit circle.

In the case when $\{f_j\}$ are OPUC, the intensity function we arrive at shows that the zeros of P_n are clustering near the unit circle. To quantify this phenomenon, we give a result that estimates the expected number of complex zeros of P_n in shrinking neighborhoods of compact subsets of the unit circle.

1.5 The variance of the number of zeros of random polynomials

We now give an overview of the main results from Chapter IV. Before discussing these results, we mention some classical results on the variance of the number of zeros of random polynomials.

Let $\mathrm{Var}[N_n(\Omega)]$ denote the variance of the number of zeros of a random sum in a measurable set $\Omega \subset \mathbb{C}$. The first result concerning the variance of the number of real zeros of a random algebraic polynomial with i.i.d. real-valued standard Gaussian coefficients was an upper bound provided by Stevens [109] in 1965. Specifically, in this case he gave the upper bound

$$\mathrm{Var}[N_n(\mathbb{R})] < 32\mathbb{E}[N_n(\mathbb{R})] + 2.5 + (\log n)^2/\sqrt{n}, \quad \text{for} \quad n \geq 32.$$

Soon after, in 1968 Fairly [38] computed the exact variances in this case and in the case with the coefficients of the random algebraic polynomial take the values ± 1 with equal probabilities for polynomials of degree up to 11.

In 1974 Maslova [79] considered the case when the random algebraic polynomial has i.i.d. real-valued coefficients $\{\eta_k\}$ such that $\mathbb{P}[\eta_k = 0] = 0$, $\mathbb{E}[\eta_k] = 0$, and $\mathbb{E}[|\eta_k|^{2+s}| < \infty$ for some $s > 0$. For this case she established the asymptotic

$$\mathrm{Var}[N_n(\mathbb{R})] \sim \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right)\log n, \quad \text{as} \quad n \to \infty.$$

Note that if a random algebraic polynomial has coefficients $\{\eta_k\}$ that are i.i.d. real-valued standard Gaussian, then they satisfy the hypothesis needed for Maslova's asymptotic on the variance.

As the topics in the dissertation do not cover trigonometric random polynomials, we only note that asymptotics for the variance of the number of real zeros in $[0, 2\pi]$ has been well studied (cf. Boomolny, Bohigas, Leboeuf [17], Farahmand [35], Grandville and Wigman [46], and Su and Shao [110]). Similarly we only mention the works of Forrester and Honner [42], Hannay [48], Shiffman and Zeldtich [101], Bleher and Di [9], that concern asymptotics for variance of the number of zeros for weighted random polynomials, i.e. random polynomials of the form $\sum_{k=0}^{n} \eta_k c_k z^k$ where either $c_k = \binom{n}{k}^{1/2}$ or $c_k = 1/k!$.

1.5.1 The variance of the number of zeros of a random orthogonal polynomial

In 2016 Xie [121] examined the variance for the number of real zeros of random orthogonal polynomials spanned by OPRL. Specifically, she considered

$$P_n(x) = \sum_{k=0}^{n} \eta_k p_k(x),$$

where $\{\eta_k\}$ are i.i.d. real-valued standard Gaussian random variables, and $\{p_k\}$ are orthogonal polynomials with respect to a finite positive Borel measure μ supported on $[-1, 1]$ such that $d\mu(x) = w(x)dx$ with $w > 0$ a.e. on $[-1, 1]$. Under the further assumption that $w(\cos\theta)|\sin\theta|$, with $\theta \in [-\pi, \pi]$, satisfies the Lipschitz-Dini condition, i.e.

$$|w(\cos(\theta+\delta))|\sin(\theta+\delta)| - w(\cos\theta)|\sin\theta|| < L|\log\delta|^{-1-\lambda},$$

where $L, \lambda > 0$ are fixed numbers, Xie proved

$$\lim_{n\to\infty} \frac{\mathrm{Var}[N_n(\mathbb{R})]}{n^2} = 0.$$

Complementing the work of Xie, in the first section of Chapter 4 we study the variance of the number of zeros for

$$P_n(z) = \sum_{k=0}^{n} \eta_k \varphi_k(z),$$

where $\{\eta_k\}$ are complex-valued random variables, and $\{\varphi_k\}$ are OPUC. When $\{\eta_k\}$ are i.i.d. complex-valued standard Gaussian, assuming that $\{\varphi_k\}$ are from the Nevai class, we prove a formula for the limiting value of variance of the number of zeros in annuli that do not contain the unit circle. Under the assumption that the the distribution for each η_k satisfies certain uniform bounds for the fractional and logarithmic moments, for OPUC such that their associated recurrence coefficients are absolutely summable, or are regular in the sense of Ullman-Stahl-Totik, we give quantitative estimates that show that the variance of the number of zeros of P_n scaled by n^2 in annuli that contain the unit circle is $o(1)$ as $n \to \infty$.

1.5.2 The variance of the number of zeros of random orthogonal power series

Consider the GAF

$$f(z) = \sum_{k=0}^{\infty} \eta_k z^k,$$

where $\{\eta_k\}$ are i.i.d. complex valued standard Gaussian random variables. Let N_r be the number of zeros of f in a disk of radius $r < 1$ centered at the origin. Peres and Virág (Corollary 3. (iii) of [88]) have shown that for the random power series $f(z)$ we have

$$\mu_r = \mathbb{E}[N_r] = \frac{r^2}{1-r^2}, \quad \sigma_r^2 = \mathrm{Var}[N_r] = \frac{r^2}{1-r^4}, \tag{1.5.1}$$

and $(N_r - \mu_r)/\sigma_r$ converges in distribution to the standard normal as $r \uparrow 1$. For similar results as above concerning weighted GAF's, we refer the reader to Sodin and Tsirelson [105], Nazarov and Sodin [86], and Bleher, Shiffman, and Zelditch [10].

In Section 2 of Chapter 4, we generalize the basis of the random power series $f(z)$ to be OPUC from the Szegő class, meaning that the measure μ associated to the OPUC posses the property that $d\mu(\theta) = w(\theta)d\theta$ with

$$\int_{-\pi}^{\pi} |\log w(\theta)| d\theta$$

exists, and prove the analogs of (1.5.1) for this extension in annuli (further generalizing from disks) in the unit circle. As we will see, proving the analog of the central limit theorem given by Peres and Virág in this setting is still out of reach.

1.6 A remark on applications

The theory of random polynomials has many applications. For instance, they occur in approximation theory when the coefficients of a polynomial are computed from experimental data, and in which case, these coefficients are subject to random error. Random polynomials also arise in the study of difference and differential equations. In this application it is possible to obtain information about the needed solution of a differential equation by introducing random coefficients to the characteristic polynomial, then studying the zeros of this random characteristic polynomial. The characteristic polynomial of a random matrix can also viewed as a random polynomial, and finding or estimating the expected zeros of the random characteristic polynomial can give information about the eigenvalues of a random matrix. Other applications of random polynomials occur in the study of approximate solution of operator equations, polynomial regression equations estimated by the method of least squares, mathematical economics, statistical communication theory, ect. Recently in quantum chaos theory, studying the zeros of random polynomials that are spanned by other basis functions other than the monomials, i.e. replacing z^m with an entire function $f_m(z)$, have been very useful. In this setting, linear combinations of functions with random coefficients serve as a basic model for eigenfunctions of chaotic quantum systems (cf. P. Leboeuf and P. Shukla [64], P. Leboeuf [63]).

For a nice history of the early progress and applications in the subject of random polynomials, we refer the reader to the books by Bharucha-Reid and Sambandham [16] and by Farahmand [32].

CHAPTER II

EQUIDISTRIBUTION

Let $\eta_0, \eta_1, \ldots, \eta_n$ be complex valued random variables that are not necessarily independent nor identically distributed. Consider random polynomial $P_n(z) = \sum_{k=0}^{n} \eta_k z^k$. Let $Z(P_n) = \{Z_1, Z_2, \ldots, Z_n\}$ be the set of complex zeros of P_n. The set of zeros $Z(P_n)$ gives rise to the normalized zero counting measure

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{Z_k}.$$

Observe that with the normalization of $1/n$, the measure τ_n is a random unit Borel measure in $\mathbb{C}$.

In this chapter, which contains results from the joint work with Pritsker [92], our goal is to provide estimates on the expected rate of convergence of τ_n in the weak* topology to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference $\mathbb{T}$ defined by $d\mu_{\mathbb{T}} := dt/(2\pi)$. A standard way to study the deviation of τ_n from $\mu_{\mathbb{T}}$ is to consider the discrepancy of these measures in the annular sectors of the form

$$A_r(\alpha, \beta) = \{z \in \mathbb{C} : r < |z| < 1/r,\ \alpha \le \arg z < \beta\}, \quad 0 < r < 1.$$

We will give quantitative estimates on the rates of convergence for the expected discrepancy

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi}\right|\right],$$

and for the expected number of roots of P_n, denoted as $\mathbb{E}[n\tau_n(E)]$, in various sets $E \subset \mathbb{C}$. We also study random polynomials spanned by various deterministic bases.

2.1 Expected number of zeros of random polynomials

For $P_n(z) = \sum_{k=0}^n \eta_k z^k$, our first result generalizes Theorem 3.3 of [89] to allow that the random variables $\{\eta_k\}$ be neither independent nor identically distributed, but require only that their distributions satisfy certain uniform bounds for the fractional and logarithmic moments.

Theorem 2.1.1 *Suppose that the coefficients of $P_n(z) = \sum_{k=0}^n \eta_k z^k$ are complex random variables that satisfy:*

1. $\mathbb{E}[|\eta_k|^t] < \infty,\ k = 0, \ldots, n,$ *for a fixed* $t \in (0, 1]$
2. $\mathbb{E}[\log|\eta_0|] > -\infty$ *and* $\mathbb{E}[\log|\eta_n|] > -\infty$.

Then we have for all large $n \in \mathbb{N}$ that

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \le C_r \left[\frac{1}{n}\left(\frac{1}{t}\log\sum_{k=0}^n \mathbb{E}[|\eta_k|^t] - \frac{1}{2}\mathbb{E}[\log|\eta_0\eta_n|]\right)\right]^{1/2}, \tag{2.1.1}$$

where

$$C_r := \sqrt{\frac{2\pi}{\mathbf{k}}} + \frac{2}{1-r} \quad \text{with} \quad \mathbf{k} := \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2}$$

being Catalan's constant.

Introducing uniform bounds, we obtain the rates of convergence for the expected discrepancy as $n \to \infty$.

Corollary 2.1.1 *Let $P_n(z) = \sum_{k=0}^n \eta_{k,n} z^k,\ n \in \mathbb{N},$ be a sequence of random polynomials. If*

$$M := \sup\{\mathbb{E}[|\eta_{k,n}|^t] \mid k = 0, \ldots, n,\ n \in \mathbb{N}\} < \infty$$

and

$$L := \inf\{\mathbb{E}[\log|\eta_{k,n}|] \mid k = 0\,\&\,n,\ n \in \mathbb{N}\} > -\infty,$$

then

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \le C_r\left[\frac{1}{n}\left(\frac{\log(n+1)+\log M}{t} - L\right)\right]^{1/2} = O\left(\sqrt{\frac{\log n}{n}}\right)$$

as $n \to \infty$.

Observe that for $E \subset \mathbb{C}$, the quantity $n\tau_n(E)$ gives the number of zeros of P_n in E. Appealing to the arguments of [89], we now give quantitative results about the expected number of zeros of random polynomials in various sets $E \subset \mathbb{C}$. We first consider sets separated from $\mathbb{T}$, and in doing so we thus generalize Proposition 3.4 of [89].

Proposition 2.1.1 *Let* $E \subset \mathbb{C}$ *be a compact set such that* $E \cap \mathbb{T} = \emptyset$, *and set* $d := \mathit{dist}(E, \mathbb{T})$. *If* P_n *is as in Theorem 2.1.1, then the expected number of its zeros in* E *satisfies*

$$\mathbb{E}[n\tau_n(E)] \le \frac{d+1}{d}\left(\frac{2}{t}\log\left(\sum_{k=0}^{n}\mathbb{E}[|\eta_k|^t]\right) - \mathbb{E}[\log|\eta_0\eta_n|]\right).$$

Our next proposition gives a bound on the expected number of zeros in sets that have non-tangential contact with $\mathbb{T}$, and consequently generalizes Proposition 3.5 of [89].

Proposition 2.1.2 *If* E *is a polygon inscribed in* $\mathbb{T}$, *and the sequence* $\{P_n\}_{n=1}^{\infty}$ *is as in Corollary 2.1.1, then the expected number of zeros of* P_n *in* E *satisfies*

$$\mathbb{E}[n\tau_n(E)] = O\left(\sqrt{n\log n}\right) \quad \textit{as} \quad n \to \infty.$$

Finally, if an open set E intersects $\mathbb{T}$, then it must carry a positive fraction of zeros according to the normalized arclength measure on $\mathbb{T}$. This is illustrated below for the disks $D_r(w) = \{z \in \mathbb{C} : |z-w| < r\}$, with $w \in \mathbb{T}$, and gives the generalization of Proposition 3.6 of [89].

Proposition 2.1.3 *If* $w \in \mathbb{T}$ *and* $r < 2$, *and the sequence* $\{P_n\}_{n=1}^{\infty}$ *is as in Corollary 2.1.1, then the expected number of zeros of* P_n *in* $D_r(w)$ *satisfies*

$$\mathbb{E}[n\tau(D_r(w))] = \frac{2\arcsin(r/2)}{\pi}\, n + O\left(\sqrt{n\log n}\right) \quad \textit{as} \quad n \to \infty.$$

2.2 Random polynomials spanned by general bases

We now analyze the behavior of random polynomials spanned by general bases. Throughout this section, let $B_k(z) = \sum_{j=0}^{k} b_{j,k} z^j$, where $b_{j,k} \in \mathbb{C}$ for all j and k, and $b_{k,k} \neq 0$ for all k, be a polynomial basis, i.e. a linearly independent set of polynomials. Observe that deg $B_k = k$ for all $k \in \mathbb{N} \cup \{0\}$. We study the zero distribution of random polynomials

$$P_n(z) = \sum_{k=0}^{n} \eta_k B_k(z).$$

Throughout this section, we assume that

$$\limsup_{k\to\infty} \|B_k\|_\infty^{1/k} \le 1 \quad \text{and} \quad \lim_{k\to\infty} |b_{k,k}|^{1/k} = 1, \tag{2.2.1}$$

where $\|B_k\|_\infty := \sup_{\mathbb{T}} |B_k|$. Observe that

$$|b_{k,k}| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} B_k\left(e^{i\theta}\right) e^{-ik\theta} d\theta \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| B_k\left(e^{i\theta}\right) \right| d\theta \le \|B_k\|_\infty.$$

Hence (2.2.1) in fact implies $\lim_{k\to\infty} \|B_k\|_\infty^{1/k} = 1$. Conditions (2.2.1) hold for many standard bases used for representing analytic functions in the unit disk, e.g., for various sequences of orthogonal polynomials (cf. Stahl and Totik [98]). In the latter case, random polynomials spanned by such bases are called random orthogonal polynomials.

Our main result of this section is the following:

Theorem 2.2.1 *For $P_n(z) = \sum_{k=0}^{n} \eta_k B_k(z)$, let $\{\eta_k\}_{k=0}^{n}$ be random variables satisfying $\mathbb{E}[|\eta_k|^t] < \infty,\ k = 0, \ldots, n$, for a fixed $t \in (0, 1]$, and set $D_n := \eta_n b_{n,n} \sum_{k=0}^{n} \eta_k b_{0,k}$. If $\mathbb{E}[\log |D_n|] > -\infty$ then we have for all large $n \in \mathbb{N}$ that*

$$\begin{aligned} &\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \\ &\le C_r \left[\frac{1}{n}\left(\frac{1}{t}\log\left(\sum_{k=0}^{n} \mathbb{E}[|\eta_k|^t]\right) + \log \max_{0\le k\le n} \|B_k\|_\infty - \frac{1}{2}\mathbb{E}[\log|D_n|]\right)\right]^{1/2}, \end{aligned} \tag{2.2.2}$$

where

$$C_r = \sqrt{\frac{2\pi}{\mathbf{k}}} + \frac{2}{1-r}.$$

In particular, if $\mathbb{E}[\log|\eta_n|] > -\infty$ *and* $\mathbb{E}[\log|\eta_0 + z|] \geq L > -\infty$ *for all* $z \in \mathbb{C}$, *then*

$$\mathbb{E}[\log|D_n|] \geq \log|b_{0,0}b_{n,n}| + \mathbb{E}[\log|\eta_n|] + L > -\infty, \tag{2.2.3}$$

and (2.2.2) *holds.*

An example of a typical basis satisfying (2.2.1) is given below by orthonormal polynomials on the unit circle (OPUC). Setting $B_k(z) = \varphi_k(z)$, $k = 0, 1, \ldots, n$, here the basis $\{\varphi_k\}_{k=0}^n$ is said to be OPUC if they are defined by a probability Borel measure μ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \varphi_k(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\, d\mu(e^{i\theta}) = \delta_{km}, \qquad \text{for all } k, m \in \mathbb{N} \cup \{0\}.$$

We apply Theorem 2.2.1 to obtain a quantitative result on the zero distribution of random orthogonal polynomials.

Corollary 2.2.1 *Let* $P_n(z) = \sum_{k=0}^n \eta_{k,n}\varphi_k(z)$, $n \in \mathbb{N}$, *be a sequence of random orthogonal polynomials. Suppose that the following uniform estimates for the coefficients hold true:*

$$\sup\{\mathbb{E}[|\eta_{k,n}|^t] \mid k = 0, \ldots, n;\ n \in \mathbb{N}\} < \infty, \quad t \in (0, 1], \tag{2.2.4}$$

and

$$\min\left(\inf_{n\in\mathbb{N}} \mathbb{E}[\log|\eta_{n,n}|], \inf_{n\in\mathbb{N}, z\in\mathbb{C}} \mathbb{E}[\log|\eta_{0,n} + z|]\right) > -\infty. \tag{2.2.5}$$

If the basis polynomials φ_k *are orthonormal with respect to a positive Borel measure* μ *supported on* $\mathbb{T} = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$, *such that the Radon-Nikodym derivative* $d\mu/d\theta > 0$ *for almost every* $\theta \in [0, 2\pi)$, *then* (2.2.1) *is satisfied and*

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi}\right|\right] = 0. \tag{2.2.6}$$

If $d\mu(\theta) = w(\theta)\,d\theta$*, where* $w(\theta) \geq c > 0,\ \theta \in [0, 2\pi)$*, then*

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] = O\left(\sqrt{\frac{\log n}{n}}\right) \quad n \to \infty. \tag{2.2.7}$$

Furthermore, if the measure of orthogonality μ *associated to* $\{\varphi_k\}$ *is regular in the sense of Ullman-Stahl-Totik, that is,*

$$\varepsilon_n := \frac{1}{n}\log|\kappa_n| \to 0, \quad \textit{as} \quad n \to \infty,$$

where κ_n *is the leading coefficient of* φ_n*, it follows that*

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] = \mathcal{O}\left(\max\left\{\sqrt{\frac{\log n}{n}}, \varepsilon_n^{1/4}\right\}\right) \quad n \to \infty. \tag{2.2.8}$$

It is clear that if the coefficients have identical distributions, then all uniform bounds in (2.2.4) and (2.2.5) reduce to those on the single coefficient η_0. One can relax conditions on the orthogonality measure μ while preserving the results, e.g., one can show that (2.2.7) also holds for the generalized Jacobi weights of the form $w(\theta) = v(\theta)\prod_{j=1}^{J}|\theta-\theta_j|^{\alpha_j}$, where $v(\theta) \geq c > 0,\ \theta \in [0, 2\pi)$. Note that the analogs of Propositions 2.1.2-2.1.3 for the random orthogonal polynomials follow from (2.2.7).

2.3 Discrete random coefficients

Let $\eta_0, \eta_1, \ldots$ be i.i.d. complex discrete random variables. We show that one can extend the ideas of [89] and prove essentially the same results in the discrete case. Furthermore, since any real random variable is the limit of an increasing sequence of discrete random variables, we are able to extend the arguments to arbitrary random variables. We assume as before that $\mathbb{E}[|\eta_0|^t] = \mu < \infty$ for a fixed real $t > 0$.

Proposition 2.3.1 *Let* $\eta_0, \eta_1, \ldots$ *be i.i.d. complex random variables, and let* $Y_n := \max\limits_{0\leq k\leq n}|\eta_k|$*. If* $\mu := \mathbb{E}[|\eta_0|^t] < \infty$*, where* $t > 0$*, then*

$$\mathbb{E}[\log Y_n] \leq \frac{\log(n+1) + \log\mu}{t}.$$

This result provides an immediate extension of Theorem 3.3 of [89] to arbitrary random variables (satisfying the moment assumption) by following the same proof. Indeed, we have that

$$\begin{aligned}\mathbb{E}[\log \|P_n\|_\infty] &= \mathbb{E}\left[\log\left(\sup_{z\in\mathbb{T}}\left|\sum_{k=0}^{n}\eta_k z^k\right|\right)\right] \\ &\le \mathbb{E}\left[\log\left(\sum_{k=0}^{n}|\eta_k|\right)\right] \\ &\le \mathbb{E}\left[\log\left((n+1)\max_{0\le k\le n}|\eta_k|\right)\right] \\ &= \log(n+1) + \mathbb{E}[\log Y_n].\end{aligned}$$

Thus referring to the proof of Theorem 3.3 of [89] and using our bound of $\mathbb{E}[\log Y_n]$ gives the result.

2.4 Dependent coefficients

We generalize Theorem 3.7 of [89] in this section, replacing the requirement that the first and the second moments of the absolute values of all coefficients be equal with the requirement they be uniformly bounded. More precisely, we assume that

$$\sup_k \mathbb{E}[|\eta_k|] =: M < \infty \quad \text{and} \quad \sup_k \mathrm{Var}[|\eta_k|] =: S^2 < \infty. \tag{2.4.1}$$

Following the ideas of Arnold and Groeneveld [5] (see also [21]), we show that

Proposition 2.4.1 *If* (2.4.1) *is satisfied, then we have for* $Y_n = \max_{0\le k\le n}|\eta_k|$ *that*

$$\mathbb{E}[Y_n] = O(\sqrt{n}) \quad \text{as } n\to\infty.$$

An analog of Theorem 3.7 from [89] is obtained along the same lines as before.

Theorem 2.4.1 *If the (possibly dependent) coefficients of* P_n *satisfy* (2.4.1) *as well as* $\mathbb{E}[\log|\eta_0|] > -\infty$ *and* $\mathbb{E}[\log|\eta_n|] > -\infty$*, then*

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \le C_r\sqrt{\frac{\frac{3}{2}\log(n+1) - \frac{1}{2}\mathbb{E}[\log|\eta_0|] - \frac{1}{2}\mathbb{E}[\log|\eta_n|] + O(1)}{n}}$$

as $n\to\infty$.

Clearly, this result has more restrictive assumptions than Theorem 2.1.1.

2.5 Proofs

2.5.1 Proofs for Section 2.1

Define the logarithmic Mahler measure (logarithm of geometric mean) of P_n by

$$m(P_n) = \frac{1}{2\pi} \int_0^{2\pi} \log |P_n(e^{i\theta})| d\theta.$$

It is immediate to see that $m(P_n) \le \log \|P_n\|_\infty$.

The majority of our results are obtained with help of the following modified version of the discrepancy theorem due to Erdős and Turán (cf. Proposition 2.1 of [89]):

Lemma 2.5.1 *Let* $P_n(z) = \sum_{k=0}^n c_k z^k,\ c_k \in \mathbb{C}$, *and assume* $c_0 c_n \neq 0$. *For any* $r \in (0,1)$ *and* $0 \le \alpha < \beta < 2\pi$, *we have*

$$\left| \tau_n \left(A_r(\alpha,\beta) \right) - \frac{\beta-\alpha}{2\pi} \right| \le \sqrt{\frac{2\pi}{\mathbf{k}}} \sqrt{\frac{1}{n} \log \frac{\|P_n\|_\infty}{\sqrt{|c_0 c_n|}}} + \frac{2}{n(1-r)}\, m\left(\frac{P_n}{\sqrt{|c_0 c_n|}} \right), \tag{2.5.1}$$

where $\mathbf{k} = \sum_{k=0}^\infty (-1)^k/(2k+1)^2$ *is Catalan's constant.*

This estimate shows how close the zero counting measure τ_n is to $\mu_{\mathbb{T}}$.

We will use the following lemma several times below.

Lemma 2.5.2 *If* $\eta_k,\ k=0,\ldots,n$, *are complex random variables satisfying* $\mathbb{E}[|\eta_k|^t] < \infty,\ k = 0,\ldots,n$, *for a fixed* $t \in (0,1]$, *then*

$$\mathbb{E}\left[\log \sum_{k=0}^n |\eta_k| \right] \le \frac{1}{t} \log \left(\sum_{k=0}^n \mathbb{E}[|\eta_k|^t] \right). \tag{2.5.2}$$

Proof. We first observe an elementary inequality. If $x_i \ge 0,\ i = 0,\ldots,n$, and $\sum_{i=0}^n x_i = 1$, then for any $t \in (0,1)$ we have that

$$\sum_{i=0}^n (x_i)^t \ge \sum_{i=0}^n x_i = 1.$$

Applying this inequality with $x_i = |\eta_i| / \sum_{k=0}^n |\eta_k|$, we obtain that

$$\left(\sum_{k=0}^{n} |\eta_k| \right)^t \le \sum_{k=0}^{n} |\eta_k|^t$$

and

$$\mathbb{E}\left[\log \sum_{k=0}^{n} |\eta_k| \right] \le \frac{1}{t} \mathbb{E}\left[\log \left(\sum_{k=0}^{n} |\eta_k|^t \right) \right]. \tag{2.5.3}$$

Jensen's inequality and linearity of expectation now give that

$$\mathbb{E}\left[\log \sum_{k=0}^{n} |\eta_k| \right] \le \frac{1}{t} \log \mathbb{E}\left[\sum_{k=0}^{n} |\eta_k|^t \right] = \frac{1}{t} \log \left(\sum_{k=0}^{n} \mathbb{E}[|\eta_k|^t] \right).$$

■

Proof of Theorem 2.1.1. Note that $m(Q_n) \le \log \|Q_n\|_\infty$ for all polynomials Q_n. Hence (2.5.1) and Jensen's inequality imply that

$$\begin{aligned} \mathbb{E}\left[\left| \tau_n\left(A_r(\alpha,\beta)\right) - \frac{\beta-\alpha}{2\pi} \right| \right] &\le \sqrt{\frac{2\pi}{\mathbf{k}}} \sqrt{\frac{1}{n} \mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|\eta_0\eta_n|}} \right] + \frac{2}{n(1-r)} \mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|\eta_0\eta_n|}} \right]} \\ &\le C_r \sqrt{\frac{1}{n} \mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|\eta_0\eta_n|}} \right]}, \end{aligned}$$

where the last inequality holds for all sufficiently large $n \in \mathbb{N}$. Since $\|P_n\|_\infty \le \sum_{k=0}^n |\eta_k|$, we use the linearity of expectation and (2.5.2) to estimate

$$\begin{aligned} \mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|\eta_0\eta_n|}} \right] &\le \mathbb{E}\left[\log \sum_{k=0}^{n} |\eta_k| \right] - \frac{1}{2}\mathbb{E}[\log|\eta_0\eta_n|] \\ &\le \frac{1}{t} \log \left(\sum_{k=0}^{n} \mathbb{E}[|\eta_k|^t] \right) - \frac{1}{2}\mathbb{E}[\log|\eta_0\eta_n|]. \end{aligned}$$

The latter upper bound is finite by our assumptions. ■

Proof of Corollary 2.1.1. The result follows immediately upon using the uniform bounds M and L in estimate (2.1.1). ■

Proof of Proposition 2.1.1. In was shown in [89] (see (5.3) in that paper) that

$$\tau_n(\mathbb{C} \setminus A_r(0,2\pi)) \le \frac{2}{n(1-r)} m\left(\frac{P_n}{\sqrt{|\eta_0\eta_n|}} \right).$$

Since $m(Q_n) \le \log \|Q_n\|_\infty$ for all polynomials Q_n, it follows that

$$\tau_n(\mathbb{C} \setminus A_r(0, 2\pi)) \le \frac{2}{n(1-r)} \log \left(\frac{\|P_n\|_\infty}{\sqrt{|\eta_0 \eta_n|}} \right).$$

Note that for $r = 1/(\text{dist}(E, \mathbb{T}) + 1)$, we have $E \subset \mathbb{C} \setminus A_r(0, 2\pi)$. Estimating $\|P_n\|_\infty$ as in the proof of Theorem 2.1.1, we obtain that

$$\begin{aligned}
\mathbb{E}[n\tau_n(E)] &\le \frac{2}{1-r} \mathbb{E} \left[\log \left(\frac{\|P_n\|_\infty}{\sqrt{|\eta_0 \eta_n|}} \right) \right] \\
&\le \frac{2}{1-r} \left(\frac{1}{t} \log \left(\sum_{k=0}^{n} \mathbb{E}[|\eta_k|^t] \right) - \frac{1}{2} \mathbb{E}[\log |\eta_0 \eta_n|] \right) \\
&= \frac{d+1}{d} \left(\frac{2}{t} \log \left(\sum_{k=0}^{n} \mathbb{E}[|\eta_k|^t] \right) - \mathbb{E}[\log |\eta_0 \eta_n|] \right).
\end{aligned}$$

■

Proof of Proposition 2.1.2. The proof of this proposition proceeds in the same manner as the proof of Proposition 3.5 in [89] by using our Corollary 2.1.1 along with Proposition 2.1.1 . ■

Proof of Proposition 2.1.3. As in the previous proof, this result follows in direct parallel to the proof of Proposition 3.6 of [89] while taking into account our bound in Proposition 2.1.2. ■

2.5.2 Proofs for Section 2.2

Proof of Theorem 2.2.1. We proceed with an argument similar to the proof of Theorem 2.1.1. Note that the leading coefficient of P_n is $\eta_n b_{n,n}$, and its constant term is $\sum_{k=0}^{n} \eta_k b_{0,k}$. Using the fact $m(Q_n) \le \log \|Q_n\|_\infty$ for all polynomials Q_n, we apply (2.5.1) and Jensen's inequality to obtain

$$\begin{aligned}
\mathbb{E}\left[\left| \tau_n\left(A_r(\alpha,\beta)\right) - \frac{\beta-\alpha}{2\pi} \right| \right] &\le \sqrt{\frac{2\pi}{\mathbf{k}}} \sqrt{\frac{1}{n}\, \mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|D_n|}} \right]} + \frac{2}{n(1-r)}\, \mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|D_n|}} \right] \\
&\le C_r \sqrt{\frac{1}{n}\, \mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|D_n|}} \right]}
\end{aligned}$$

for all sufficiently large $n \in \mathbb{N}$. It is clear that

$$\|P_n\|_\infty \le \max_{0\le k\le n} \|B_k\|_\infty \sum_{k=0}^{n} |\eta_k|.$$

Hence (2.5.1) yields

$$\begin{aligned}
\mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|D_n|}}\right] &\le \mathbb{E}\left[\log \sum_{k=0}^{n} |\eta_k|\right] + \log \max_{0\le k\le n} \|B_k\|_\infty - \frac{1}{2}\mathbb{E}[\log|D_n|] \\
&\le \frac{1}{t}\log\left(\sum_{k=0}^{n} \mathbb{E}[|\eta_k|^t]\right) + \log \max_{0\le k\le n} \|B_k\|_\infty - \frac{1}{2}\mathbb{E}[\log|D_n|].
\end{aligned}$$

Thus (2.2.2) follows as a combination of the above estimates.

We now proceed to the lower bound for the expectation of $\log|D_n|$ in (2.2.3) by estimating that

$$\begin{aligned}
\mathbb{E}[\log|D_n|] &= \mathbb{E}\left[\log\left|\eta_n b_{n,n} \sum_{k=0}^{n} \eta_k b_{0,k}\right|\right] \\
&= \mathbb{E}[\log|\eta_n|] + \log|b_{n,n}| + \mathbb{E}\left[\log\left|\sum_{k=0}^{n} \eta_k b_{0,k}\right|\right] \\
&= \mathbb{E}[\log|\eta_n|] + \log|b_{n,n}| + \log|b_{0,0}| + \mathbb{E}\left[\log\left|\eta_0 + \sum_{k=1}^{n} \eta_k \frac{b_{0,k}}{b_{0,0}}\right|\right] \\
&\ge \log|b_{0,0}b_{n,n}| + \mathbb{E}[\log|\eta_n|] + L,
\end{aligned}$$

where we used that $b_{0,0} \neq 0$ and $\mathbb{E}[\log|\eta_0 + z|] \ge L$ for all $z \in \mathbb{C}$. ■

Proof of Corollary 2.2.1. We apply (2.2.2) with (2.2.3). Writing

$$\varphi_k(z) = \kappa_{k,k} z^k + a_{k-1,k} z^{k-1} + a_{k-2,k} z^{k-2} + \cdots + a_{0,k}, \quad k \in \{0, 1, \ldots, n\}, \tag{2.5.4}$$

the uniform bounds on the expectations for the coefficients immediately give that

$$\frac{1}{tn}\log\left(\sum_{k=0}^{n} \mathbb{E}[|\eta_{k,n}|^t]\right) = O\left(\frac{\log n}{n}\right) \quad \text{and} \quad \frac{1}{2n}\mathbb{E}[\log|D_n|] \ge \frac{1}{n}\log|\kappa_{n,n}| + O\left(\frac{1}{n}\right).$$

The assumption $d\mu/d\theta > 0$ for a.e. θ implies (2.2.1), see Corollary 4.1.2 of [98], which in turn gives that

$$\lim_{n\to\infty} \frac{1}{n}\log|\kappa_{n,n}| = \lim_{n\to\infty} \frac{1}{n}\log \max_{0\le k\le n} \|\varphi_k\|_\infty = 0.$$

Hence (2.2.6) follows from (2.2.2). Recall that the leading coefficient $\kappa_{n,n}$ of the orthonormal polynomial φ_n gives the solution of the following extremal problem [98]:

$$|\kappa_{n,n}|^{-2} = \inf\left\{\int |Q_n|^2\, d\mu \ :\ Q_n \text{ is a monic polynomial of degree } n\right\}.$$

Using $Q_n(z) = z^n$, we obtain that

$$|\kappa_{n,n}|^2 \leq \int_{-\pi}^{\pi} |e^{in\theta}|^2 d\mu(\theta) = \int_{-\pi}^{\pi} d\mu(\theta) = \mu(\mathbb{T}).$$

Consequently

$$|\kappa_{n,n}| \geq (\mu(\mathbb{T}))^{-1/2} \text{ and } \frac{1}{n}\log|\kappa_{n,n}| \geq -\frac{1}{2n}\log\mu(\mathbb{T}).$$

We now show that $\log\|\varphi_n\|_\infty = O(\log n)$ as $n \to \infty$, provided $d\mu(\theta) = w(\theta)d\theta$ with $w(\theta) \geq c > 0,\ \theta \in [0, 2\pi)$. Indeed, the Cauchy-Schwarz inequality gives for the orthonormal polynomial (2.5.4) that

$$\begin{aligned}
\|\varphi_n\|_\infty &\leq |\kappa_{n,n}| + |a_{n-1,n}| + |a_{n-2,n}| + \cdots + |a_{0,n}| \\
&\leq \sqrt{n+1}\left(|\kappa_{n,n}|^2 + |a_{n-1,n}|^2 + |a_{n-2,n}|^2 + \cdots + |a_{0,n}|^2\right)^{1/2} \\
&= \sqrt{n+1}\left(\frac{1}{2\pi}\int_0^{2\pi} |\varphi_n(e^{i\theta})|^2\, d\theta\right)^{1/2} \\
&\leq \sqrt{\frac{n+1}{2\pi c}}\left(\int_0^{2\pi} |\varphi_n(e^{i\theta})|^2 w(\theta)\, d\theta\right)^{1/2} \\
&= \sqrt{\frac{n+1}{2\pi c}}.
\end{aligned}$$

This estimate completes the proof of (2.2.7).

Under the assumption the measure of orthogonality μ associated to $\{\varphi_k\}$ is regular in the sense of Ullman-Stahl-Totik, to establish (2.2.7) it suffices to show

$$\frac{1}{n}\log\max_{0\leq k\leq n}\|\varphi_k\|_\infty = \mathcal{O}(\sqrt{\varepsilon_n}), \tag{2.5.5}$$

where $\varepsilon_n = \log|\kappa_{n,n}|/n$.

Writing $\kappa_{k,k} = \kappa_k$, equation 1.5.22 of [104] gives

$$\kappa_k = \prod_{j=0}^{k-1}(1 - |\alpha_j|^2)^{-1/2}, \tag{2.5.6}$$

where $\{\alpha_j\} \subset \mathbb{D}$ are recurrence coefficients coming from the three term recurrence relation (c.f. Theorem 1.5.4 [104]):

$$\varphi_{j+1}(z) = \frac{z\varphi_j(z) - \bar{\alpha}_j \varphi_j^*(z)}{\sqrt{1 - |\alpha_j|^2}}, \quad j = 0, 1, \ldots,$$

with $\varphi_j^*(z) = z^j \overline{\varphi_j(1/\bar{z})}$. For the normalized OPUC, denoted as $\Phi_k(z)$, we have $\varphi_k(z) = \kappa_k \Phi_k(z)$, so that appealing to (2.5.6) and (1.5.17) of Theorem 1.5.3 of [104] yields

$$\begin{aligned}
\log \max_{0 \le k \le n} \|\varphi_k\|_\infty &= \log \max_{0 \le k \le n} \|\kappa_k \Phi_k(z)\|_\infty \\
&\le \log \left(|\kappa_n| \max_{0 \le k \le n} \|\Phi_k(z)\|_\infty \right) \\
&\le \log \left(|\kappa_n| \max_{0 \le k \le n} \exp \left(\sum_{j=0}^{k-1} |\alpha_j| \right) \right) \\
&\le \log \left(|\kappa_n| \exp \left(\sum_{j=0}^{n-1} |\alpha_j| \right) \right) \\
&= \log |\kappa_n| + \sum_{j=0}^{n-1} |\alpha_j| \\
&\le \log |\kappa_n| + \left(n \sum_{j=0}^{n-1} |\alpha_j|^2 \right)^{1/2},
\end{aligned}$$

where we have relied on the Cauchy-Schwarz inequality in the last inequality. To estimate the second term above, notice that since each $\alpha_j \in \mathbb{D}$, we have

$$\log \frac{1}{1 - |\alpha_j|^2} = \sum_{k=1}^{\infty} \frac{|\alpha_j|^{2k}}{k} > |\alpha_j|^2.$$

Thus

$$\begin{aligned}
\frac{1}{n} \log \max_{0 \le k \le n} \|\varphi_k\|_\infty &\le \frac{1}{n} \left(\log |\kappa_n| + \left(n \sum_{j=0}^{n-1} \log \frac{1}{1 - |\alpha_j|^2} \right)^{1/2} \right) \\
&= \frac{1}{n} \left(\log |\kappa_n| + (2n \log |\kappa_n|)^{1/2} \right) \\
&\le \mathcal{O}\left(\sqrt{\varepsilon_n} \right),
\end{aligned}$$

which completes the desired estimate to give (2.2.8). ■

2.5.3 Proofs for Section 2.3

Proof of Proposition 2.3.1. Assume that the discrete random variable $|\eta_0|$ takes values $\{x_k\}_{k=1}^{\infty}$ that are arranged in the increasing order, and note that the range of values for Y_n is the same. Let $a_k = \mathbb{P}(Y_n \leq x_k)$ and $b_k = \mathbb{P}(|\eta_0| \leq x_k)$, where $k \in \mathbb{N}$. It is clear that $\mathbb{P}(Y_n = x_k) = a_k - a_{k-1}$ and $\mathbb{P}(|A_0| = x_k) = b_k - b_{k-1},\ k \in \mathbb{N}$. Since the η_k's are independent and identically distributed, we have that

$$\begin{aligned}
a_k &= \mathbb{P}(Y_n \leq x_k) \\
&= \mathbb{P}(|\eta_0| \leq x_k, |\eta_1| \leq x_k, \ldots, |\eta_n| \leq x_k) \\
&= \mathbb{P}(|\eta_0| \leq x_k)\mathbb{P}(|\eta_1| \leq x_k) \cdots \mathbb{P}(|\eta_n| \leq x_k) \\
&= [\mathbb{P}(|\eta_0| \leq x_k)]^{n+1} \\
&= b_k^{n+1}
\end{aligned}$$

holds for all $k \in \mathbb{N}$. Thus

$$\begin{aligned}
\mathbb{E}[Y_n^t] := &\sum_{k=1}^{\infty} x_k^t \ \mathbb{P}(Y_n = x_k) \\
&= \sum_{k=1}^{\infty} x_k^t \ [a_k - a_{k-1}] \\
&= \sum_{k=1}^{\infty} x_k^t \ [b_k^{n+1} - b_{k-1}^{n+1}] \\
&= \sum_{k=1}^{\infty} x_k^t \ [b_k - b_{k-1}][b_k^n + b_k^{n-1} b_{k-1} + \cdots + b_{k-1}^n] \\
&\leq \sum_{k=1}^{\infty} x_k^t \ [b_k - b_{k-1}](n+1) b_k^n \\
&\leq (n+1) \sum_{k=1}^{\infty} x_k^t \ \mathbb{P}(|\eta_0| = x_k) \\
&= (n+1) \ \mathbb{E}[|\eta_0|^t].
\end{aligned}$$

By Jensen's inequality and the previous estimate, we have

$$\begin{aligned}\mathbb{E}[\log Y_n] &= \mathbb{E}\left[\frac{1}{t}\log Y_n^t\right] \\ &\le \frac{1}{t}\log \mathbb{E}[Y_n^t] \\ &\le \frac{1}{t}(\log((n+1)\ \mathbb{E}[|\eta_0|^t]) \\ &= \frac{1}{t}(\log(n+1) + \log\mu).\end{aligned}$$

We now show that this argument can be extended to arbitrary random variables $\{|\gamma_k|\}_{k=0}^n$. Consider the increasing sequences of simple (discrete) random variables $\{|\eta_{k,i}|\}_{i=1}^\infty$ such that $\lim_{i\to\infty}|\eta_{k,i}| = |\gamma_k|,\ k = 0,\ldots,n$. For $Y_{n,i} = \max_{0\le k\le n}|\eta_{k,i}|$ and $Z_n = \max_{0\le k\le n}|\gamma_k|$, one can see that

$$\lim_{i\to\infty} Y_{n,i}^t = Z_n^t \quad \text{and} \quad \lim_{i\to\infty}|\eta_{0,i}|^t = |\gamma_0|^t,$$

where $t > 0$. Moreover, the sequence of simple random variables $Y_{n,i}^t$ is increasing to Z_n^t, so that the Monotone Convergence Theorem gives

$$\lim_{i\to\infty}\mathbb{E}[Y_{n,i}^t] = \mathbb{E}[Z_n^t].$$

Using the already proven result for discrete random variables and passing to the limit as $i \to \infty$, we obtain that

$$\mathbb{E}[Z_n^t] \le (n+1)\mathbb{E}[|\gamma_0|^t].$$

Hence Jensen's inequality yields

$$\mathbb{E}[\log Z_n] \le \frac{1}{t}(\log(n+1) + \log\mathbb{E}[|\gamma_0|^t]),$$

as before. ■

2.5.4 Proofs for Section 2.4

The following lemma is due to Arnold and Groeneveld [5], and is also found in [21, p. 110]. We prove it in our setting for completeness.

Lemma 2.5.3 *Let X_i, $i = 0, 1, \ldots, n$, be possibly dependent random variables with $\mathbb{E}[X_i] = \mu_i$ and $Var[X_i] = \sigma_i^2$. Then for any real constants c_i, the ordered random variables $X_{0:n} \leq X_{1:n} \leq \cdots \leq X_{n:n}$ satisfy*

$$\left|\mathbb{E}\left[\sum_{i=0}^{n} c_i(X_{i:n} - \bar{\mu})\right]\right| \leq \left(\sum_{i=0}^{n}(c_i - \bar{c})^2 \sum_{i=0}^{n}[(\mu_i - \bar{\mu})^2 + \sigma_i^2]\right)^{1/2},$$

where $\bar{c} = n^{-1}\sum_{i=0}^{n} c_i$, $\bar{\mu} = n^{-1}\sum_{i=0}^{n} \mu_{i:n} = n^{-1}\sum_{i=0}^{n} \mu_i$, and $\mu_{i:n} = \mathbb{E}[X_{i:n}]$.

Proof. We use the Cauchy-Schwartz inequality in the following estimate:

$$\begin{aligned}\left|\sum_{i=0}^{n} c_i(X_{i:n} - \bar{\mu})\right| &= \left|\sum_{i=0}^{n}(c_i - \bar{c})(X_{i:n} - \bar{\mu})\right| \\ &\leq \left[\sum_{i=0}^{n}(c_i - \bar{c})^2 \sum_{i=0}^{n}(X_{i:n} - \bar{\mu})^2\right]^{1/2}.\end{aligned}$$

Observe that $|\mathbb{E}(Y)| \leq \mathbb{E}(|Y|)$ for any random variable Y, and that $\mathbb{E}(Z^{1/2}) \leq [\mathbb{E}(Z)]^{1/2}$ for $Z \geq 0$ by Jensen's inequality. Applying these facts while taking the expectation of the previous inequality gives

$$\begin{aligned}\left|\mathbb{E}\left[\sum_{i=0}^{n} c_i(X_{i:n} - \bar{\mu})\right]\right| &\leq \left[\sum_{i=0}^{n}(c_i - \bar{c})^2\right]^{1/2} \left[\mathbb{E}\left[\sum_{i=0}^{n}(X_{i:n} - \bar{\mu})^2\right]\right]^{1/2} \\ &= \left[\sum_{i=0}^{n}(c_i - \bar{c})^2\right]^{1/2} \left[\sum_{i=0}^{n} \mathbb{E}[X_{i:n}^2] - 2\mathbb{E}[X_{i:n}]\bar{\mu} - \bar{\mu}^2)\right]^{1/2} \\ &= \left[\sum_{i=0}^{n}(c_i - \bar{c})^2\right]^{1/2} \left[\sum_{i=0}^{n} \sigma_i^2 + (\mu_i - \bar{\mu})^2\right]^{1/2}.\end{aligned}$$

■

Proof of Proposition 2.4.1. To obtain bounds for $\mathbb{E}[Y_n] = \mu_{n:n} = \mathbb{E}[\eta_{n:n}]$, we apply the previous lemma while choosing $c_0 = c_1 = \cdots = c_{n-1} = 0$ and $c_n = 1$. This yields

$$\begin{aligned}
\mathbb{E}[\eta_{n:n}] - \bar{\mu} &\leq \left((n\bar{c}^2 + (1-\bar{c})^2) \sum_{i=0}^{n} (\mu_i^2 - 2\mu_i\bar{\mu} + \bar{\mu}^2 + \sigma_i^2) \right)^{1/2} \\
&= \left(\left(\frac{n}{(n+1)^2} + \left(1 - \frac{1}{n+1}\right)^2 \right) \sum_{i=0}^{n} (\mu_i^2 - 2\mu_i\bar{\mu} + \bar{\mu}^2 + \sigma_i^2) \right)^{1/2} \\
&\leq \left(\sum_{i=0}^{n} (M^2 + 2M^2 + M^2 + S^2) \right)^{1/2} \\
&= (4M^2 + S^2)^{1/2}(n+1)^{1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{E}[Y_n] = \mathbb{E}[\eta_{n:n}] &\leq \bar{\mu} + (4M^2 + S^2)^{1/2}(n+1)^{1/2} \\
&\leq M + (4M^2 + S^2)^{1/2}(n+1)^{1/2}.
\end{aligned}$$

■

Proof of Theorem 2.4.1. As in the proof of Theorem 2.1.1, we apply (2.5.1) and Jensen's inequality to obtain for all sufficiently large $n \in \mathbb{N}$ the following

$$\begin{aligned}
\mathbb{E}\left[\left| \tau_n\left(A_r(\alpha,\beta)\right) - \frac{\beta - \alpha}{2\pi} \right| \right] &\leq C_r \sqrt{\frac{1}{n}\,\mathbb{E}\left[\log \frac{\|P_n\|_\infty}{\sqrt{|\eta_0\eta_n|}} \right]} \\
&= C_r \sqrt{\frac{\mathbb{E}[\log \|P_n\|_\infty] - \frac{1}{2}\mathbb{E}[\log|\eta_0|] - \frac{1}{2}\mathbb{E}[\log|\eta_n|]}{n}}.
\end{aligned}$$

Observe that

$$\|P_n\|_\infty = \sup_{\mathbb{T}} \left| \sum_{k=0}^{n} \eta_k z^k \right| \leq \sum_{k=0}^{n} |\eta_k| \leq (n+1) \max_{0 \leq k \leq n} |\eta_k| = (n+1)Y_n.$$

Taking the logarithm and then the expectation of the above yields

$$\begin{aligned}\mathbb{E}[\log \|P_n\|_\infty] &\leq \mathbb{E}[\log(n+1) + \log Y_n] \\ &= \log(n+1) + \mathbb{E}[\log Y_n] \\ &\leq \log(n+1) + \log \mathbb{E}[Y_n],\end{aligned}$$

where the last inequality follows from Jensen's inequality. As $n \to \infty$, applying Proposition 2.4.1 gives

$$\begin{aligned}\log(n+1) + \log \mathbb{E}[Y_n] &\leq \log(n+1) + \log O(\sqrt{n}) \\ &= \log(n+1) + \frac{1}{2}\log n + O(1) \\ &< \frac{3}{2}\log(n+1) + O(1).\end{aligned}$$

Combining these bounds gives the result of Theorem 2.4.1. ∎

CHAPTER III

THE EXPECTED NUMBER OF ZEROS

Let $\{f_j\}_{j=0}^n$ be a sequence of orthonormal polynomials where the orthogonality relation is satisfied on either the real line (OPRL) or on the unit circle (OPUC). In this chapter we study zero distribution of random linear combinations of the form

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z),$$

where $\eta_0, \ldots, \eta_n$ are i.i.d. real-valued or complex-valued standard Gaussian random variables.

We first consider the case when $\{\eta_j\}$ are i.i.d. real-valued standard Gaussian random variables and $\{f_j\}$ are OPUC. These results are a joint work with Yattselev [122]. We use an analogue of the Christoffel-Darboux formula for OPRL suited for OPUC to simplify the density functions provided by Vanderbei for the expected number of real and complex of zeros P_n. From these expressions, under the assumption that the measure μ associated to the OPUC is from the Nevai class, we deduce the limiting value of the density functions away from the unit circle. Under the mere assumption that μ is doubling on subarcs of $\mathbb{T}$ centered at 1 and -1, we show that the expected number of real zeros of P_n is at most

$$(2/\pi) \log n + O(1),$$

and that the asymptotic equality holds when the corresponding recurrence coefficients decay no slower than $n^{-(3+\epsilon)/2}$, $\epsilon > 0$. The section is concluded by providing results that estimate the expected number of complex zeros of P_n in shrinking neighborhoods of compact subsets of $\mathbb{T}$.

For the case when $\{\eta_j\}$ are i.i.d. complex-valued standard Gaussian and $\{f_j\}$ are OPRL or OPUC, we apply a general formula by Peres and Virág [88] for the expected number of zeros of P_n. In the setting that our applications are in, e.g. $\{f_j\}$ is a polynomial basis, in the appendix following the method of Vanderbei [115] we give an alternate proof the formula for the expected number of zeros of P_n. Using the Christoffel-Darboux formula for OPRL and its analogue for OPUC, the density function for the expected number of zeros of P_n in these cases takes a very simple shape. When the orthogonal polynomials are from the Nevai class, we give the limiting value of the density function away from their respective sets where the orthogonality holds. In the case when $\{f_j\}$ are OPUC, the density function shows that the expected number of zeros of P_n are clustering near the unit circle. To quantify this phenomenon, we give a result that estimates the expected number of complex zeros of P_n in shrinking neighborhoods of compact subsets of the unit circle.

3.1 Expected number of zeros of random orthogonal polynomials with real-valued Gaussian coefficients

As previously mentioned, the results of this section were obtained as joint work with Yattselev [122]. This work generalizes the asymptotic (1.2.3) given by Kac for the expected number of real zeros and the limits (1.2.4) and (1.2.5) of these intensity functions by Shepp and Vanderbei for random orthogonal polynomials of the form

$$P_n(z) = \eta_0\varphi_0(z) + \eta_1\varphi_1(z) + \cdots + \eta_{n-1}\varphi_{n-1}(z), \tag{3.1.1}$$

where $\{\eta_j\}$ are real-valued i.i.d. standard Gaussian variables, and $\{\varphi_j\}$ are OPUC that are real-valued on the real line. We note that the key formulas in this section we arrive at are the result by using the analogue of the Christoffel-Darboux formula suited for OPUC. Hence, to simplify formulas the random orthogonal polynomial P_n is taken to have $n-1$ summands.

Taking the random variables $\{\eta_j\}$ to be real-valued i.i.d. standard Gaussian variables, the density function for the expected number of zeros of P_n, which is known as the intensity function, will have support on the real line and in the complex plane. To distinguish these two intensity functions of the random orthogonal polynomial P_n, we write

$$\mathbb{E}[N_n(\Omega)] = \int_{\Omega\cap\mathbb{R}} \rho_n^{(1,0)}(x)\ dx + \int_{\Omega} \rho_n^{(0,1)}(z)\ dx\ dy, \tag{3.1.2}$$

where $\Omega \subset \mathbb{C}$ is measurable, with $\rho_n^{(1,0)}(x)$ being the intensity function for the expected number of real zeros, and $\rho_n^{(0,1)}(z)$ is intensity function for the expected number of zeros in $\mathbb{C} \setminus \mathbb{R}$.

We will rely on the generalizations of Kac's formula (1.2.2) for the intensity function on the real line, and of the formula given by Shepp and Vanderbei [97] for the intensity function off the real line. For these generalizations, we will replace the basis $\{z^j\}$ from the previous works with an arbitrary set of polynomials $\{f_j(z)\}$ that are real on the real line with $\deg f_j = j$, for $j \in \{0, 1, \dots\}$. That is, we will be considering random functions of the form

$$P_n(z) = \eta_0 f_0(z) + \eta_1 f_1(z) + \cdots + \eta_{n-1} f_{n-1}(z), \tag{3.1.3}$$

where $\{\eta_j\}$ are i.i.d. real-valued standard Gaussian variables. In this case it is well known (cf. Edelman and Kostlan [27], Das [18], Lubinsky, Pritsker, and Xie [77], and Vanderbei [115]) that

$$\rho_n^{(1,0)}(x) = \frac{1}{\pi} \frac{\sqrt{K_n(x,x)K_n^{(1,1)}(x,x) - K_n^{(1,0)}(x,x)^2}}{K_n(x,x)}, \tag{3.1.4}$$

and due to Vanderbei [115] we have

$$\begin{aligned} \rho_n^{(0,1)}(z) &= \frac{1}{\pi} \frac{K_n^{(1,1)}(z,z)}{\big(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\big)^{1/2}} \\ &- \frac{1}{\pi} \frac{K_n(z,z)\big(|K_n^{(1,0)}(z,z)|^2 + |K_n^{(1,0)}(z,\overline{z})|^2\big)}{\big(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\big)^{3/2}} \\ &+ \frac{2}{\pi} \frac{\mathrm{Re}\big(K_n(z,\overline{z})K_n^{(1,0)}(z,z)K_n^{(1,0)}(\overline{z},z)\big)}{\big(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\big)^{3/2}}, \end{aligned} \tag{3.1.5}$$

where

$$\begin{aligned} K_n(z,w) &= \textstyle\sum_{j=0}^{n-1} f_j(z)\overline{f_j(w)}, \\ K_n^{(1,0)}(z,w) &= \textstyle\sum_{j=0}^{n-1} f_j'(z)\overline{f_j(w)}, \\ K_n^{(1,1)}(z,w) &= \textstyle\sum_{j=0}^{n-1} f_j'(z)\overline{f_j'(w)}. \end{aligned} \tag{3.1.6}$$

In this section we will consider the case when $f_j = \varphi_j$, where $\{\varphi_j\}$ are OPUC. We remind the reader that the OPUC are orthogonal polynomials $\{\varphi_j\}$ defined by a probability Borel measure μ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\, d\mu(e^{i\theta}) = \delta_{nm}, \quad \text{for all } n, m \in \mathbb{N} \cup \{0\}. \tag{3.1.7}$$

Observe that when we restrict μ to be symmetric with respect to conjugation, the sequence $\{\varphi_j\}$ of OPUC will have real coefficients and consequently be real-valued on the real line.

The Three Term Recurrence Relation (Theorem 1.5.4 [104]) for a sequence $\{\varphi_n\}$ of OPUC says

$$\varphi_{n+1}(z) = \frac{z\varphi_n(z) - \bar{\alpha}_n\varphi_n^*(z)}{\sqrt{1-|\alpha_n|^2}}, \quad n = 0, 1, \ldots, \tag{3.1.8}$$

where the sequence of recurrence coefficients $\{\alpha_n\} \subset \mathbb{D}$, and $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})}$. Writing $\varphi_n(z) = \kappa_n\Phi_n(z)$, where Φ_n is monic, the above equivalence relation in this case for polynomials Φ_n can be written as

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha_n}\Phi_n^*(z), \quad n = 0, 1, \ldots, \tag{3.1.9}$$

where $\Phi_n^*(z) := z^n\overline{\Phi_n(1/\bar{z})}$.

We remark that under the assumption $\{\varphi_j\}$ are real-valued on the real-line, it follows that $\{\alpha_j\} \subset (-1, 1)$. Recall also that given $\{\alpha_j\} \subset (-1, 1)$, there exists a unique conjugate-symmetric probability measure μ whose monic orthogonal polynomials satisfy (3.1.9), see [104, Theorem 1.7.11]. Moreover, in this case it holds that

$$\kappa_n = \prod_{j=0}^{n-1} \left(1 - \alpha_j^2\right)^{-1/2}, \tag{3.1.10}$$

see [104, Equation (1.5.22)].

Taking the functions $f_j = \varphi_j$, for $j = 0, 1, \ldots, n-1$, to be OPUC that are real-valued on the real line complements the case considered by Lubinsky, Pritsker, and Xie ([77], [78]) where f_j, $j = 0, 1, \ldots, n-1$, were OPRL.

Theorem 3.1.1 *Let $\{\varphi_j\}$ be a sequence of polynomials satisfying* (3.1.7)*. Further, let P_n be a real random polynomial* (3.1.1) *with $\{\eta_j\}$ being i.i.d. real-valued standard Gaussian random variables. Then the intensity function $\rho_n^{(1,0)}$ from* (3.1.4) *can be written as*

$$\rho_n^{(1,0)}(x) = \frac{1}{\pi}\frac{\sqrt{1-h_n^2(x)}}{|1-x^2|}, \quad h_n(x) = \frac{(1-x^2)b_n'(x)}{1-b_n^2(x)}, \quad b_n(x) = \frac{\varphi_n(x)}{\varphi_n^*(x)}. \tag{3.1.11}$$

Clearly, when the recurrence coefficients are all zero, $\varphi_n(x) = x^n$ and respectively $b_n(x) = x^n$. That is, we recover the intensity function from the Kac formula (1.2.2).

Since the Blaschke products b_n necessarily satisfy $|b_n(z)| \le 1$ in $\overline{\mathbb{D}}$, they form a normal family there. Moreover, as

$$b_n(1/z) = \frac{\varphi_n(1/z)}{\varphi_n^*(1/z)} = \frac{\varphi_n(1/z)}{z^{-n}\overline{\varphi_n(\overline{z})}} = \frac{\varphi_n^*(z)}{\varphi_n(z)} = \frac{1}{b_n(z)},$$

where we have used that $\{\varphi_j\}$ are real-valued on the real line in the second to last equality, we see that

$$\rho_n^{(1,0)}(1/x) = x^2\rho_n^{(1,0)}(x). \tag{3.1.12}$$

The following corollary is immediate.

Corollary 3.1.1 *In the setting of Theorem 3.1.1, let $\mathcal{N} \subset \mathbb{N}$ be such that $b_n(z) \to b(z) \not\equiv 1$ as $\mathcal{N} \ni n \to \infty$ for some analytic function $b(z)$ in $\mathbb{D}$. Then*

$$\rho_n^{(1,0)}(x) \to \frac{1}{\pi}\frac{\sqrt{1-h^2(x)}}{1-x^2}, \quad h(x) = b'(x)\frac{1-x^2}{1-b^2(x)},$$

locally uniformly on $(-1,1)$ as $\mathcal{N} \ni n \to \infty$. In particular, if $\alpha_k \to 0$ as $k \to \infty$, then

$$\rho_n^{(1,0)}(x) \to \frac{1}{\pi}\frac{1}{|1-x^2|} \quad \text{as} \quad n \to \infty$$

uniformly on closed subsets of $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as $n\to\infty$.

The ratio asymptotics ([104], Theorem 1.7.4) for OPUC state that

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = z,$$

uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{D}$. Theorem 1.7.4 of [104] also shows that the above is equivalent to

$$\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi_n^*(z)} = 0, \tag{3.1.13}$$

where the convergence holds locally uniformly for $z \in \mathbb{D}$. When (3.1.13) holds for a sequence $\{\varphi_n\}$ of OPUC, we say that the sequence is from the Nevai Class. Thus the second claim of the corollary is a straightforward consequence of the fact that $b_n(z) = \varphi_n(z)/\varphi_n^*(z) \to 0$ locally uniformly in $\mathbb{D}$.

To prove an asymptotic result for the expected number of real zeros of the random orthogonal polynomial P_n defined in (3.1.1) similar to (1.2.3) done by Kac, we first need to relate the measure μ to the kernels $K_n(z,z)$, $K_n^{(1,0)}(z,z)$, $K_n^{(1,1)}(z,z)$. For the upper bound of $\mathbb{E}[N_n(\mathbb{R})]$, we will assume that the measure μ is doubling on subarcs of $\mathbb{T}$ centered at 1 and -1. Recall that a measure μ is called doubling on a subarc $T \subseteq \mathbb{T}$ if there exists a constant $L > 0$ such that

$$\mu(2I) \leq L\mu(I), \quad 2I \subseteq T,$$

for any subarc I, where $2I$ is a subarc of T with the same center as I and twice the arclength.

Theorem 3.1.2 *Let $P_n(z) = \sum_{k=0}^{n-1} \eta_k \varphi_k(z)$, where $\{\eta_k\}$ are i.i.d. real-valued standard Gaussian, and $\{\varphi_k\}$ are OPUC defined by a measure that is symmetric with respect to conjugation. Assume that there exist two subarcs of $\mathbb{T}$, centered at 1 and -1, on which μ is doubling. Then it holds that*

$$\mathbb{E}[N_n(\mathbb{R})] \leq \frac{2}{\pi} \log n + \mathcal{O}(1). \tag{3.1.14}$$

Moreover, if the quantities $|k^p \alpha_k|$ are uniformly bounded above for some $p > 3/2$, then

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{2 + o(1)}{\pi} \log n. \tag{3.1.15}$$

The assumption on α_k's in (3.1.15) implies for an absolute constant C, we have

$$\sum_{k=0}^{\infty} |\alpha_k| \leq C \sum_{k=0}^{\infty} \frac{1}{k^{3/2}} < \infty,$$

that is the α_k's are absolutely summable. This condition not only allows one to prove a lower bound of $\mathbb{E}[N_n(\mathbb{R})]$, but by Baxter's theorem [104, Theorem 5.2.1] this is known to be equivalent to μ being absolutely continuous with respect to the arclenth distribution on $\mathbb{T}$ and the Radon-Nikodym derivative is continuous and positive there. In particular, μ is doubling on $\mathbb{T}$.

Proposition 3.1.1 *In the setting of Theorem 3.1.1, assume that*

$$\mu = t\nu + (1-t)\delta_1, \quad t \in (0,1),$$

where ν is a conjugate-symmetric probability measure on the unit circle such that $|k^p \alpha_k(\nu)|$ are uniformly bounded above for some $p > 3/2$. Then (3.1.15) *holds while*

$$\alpha_{n-1} = \alpha_{n-1}(\nu) + \varphi_{n-1}(1;\nu)\varphi_n(1;\nu)\frac{\sqrt{1 - |\alpha_{n-1}(\nu)|^2}}{t(1-t)^{-1} + K_n(1,1;\nu)}, \tag{3.1.16}$$

where the quantities $\alpha_n(\nu), \varphi_n(z;\nu), K_n(z,w;\nu)$ are defined as before only with respect to the measure ν.

Formula (3.1.16) was derived in [120] and shows that $\alpha_n \sim 1/n$ as $n \to \infty$, that is, the recurrence coefficients do not obey the conditions of Theorem 3.1.2. Indeed, the coefficients $\alpha_k(\nu)$ are absolutely summable. Thus, there exists a constant $c > 1$ such that $c^{-1} \leq |\varphi_k(1;\nu)| \leq c$ for all k, see [104, Equation (1.5.16)] and (3.1.10). Hence, $n/c^2 \leq K_n(1,1;\nu) \leq nc^2$, which yields the claim.

Theorem 3.1.3 *In the setting of Theorem 3.1.1, assume that $\alpha_k \to 0$ as $k \to \infty$. Then we have*

$$\rho_n^{(0,1)}(z) \to \frac{1}{\pi(1-|z|^2)^2}\sqrt{1 - \left|\frac{1-|z|^2}{1-z^2}\right|^2}$$

locally uniformly in $\overline{\mathbb{C}} \setminus (\mathbb{T} \cup \mathbb{R})$ *as* $n \to \infty$.

It follows from Theorem 3.1.3 that the unit circle is attracting the zeros of P_n. To quantify this phenomena we will rely on a universality result by Levin and Lubinsky [68] which concerns OPUC that are regular in the sense of Ullman-Stahl-Totik. OPUC $\{\varphi_j\}$ are said to be regular in the sense of Ullman-Stahl-Totik if

$$\lim_{n\to\infty} \frac{\log|\kappa_n|}{n} = 0, \tag{3.1.17}$$

where κ_n is the leading coefficient of $\varphi_n(z)$. Observe that if one assumes that the recurrence coefficients associated to $\{\varphi_j\}$ satisfy $\alpha_j \to 0$ as $j \to \infty$, appealing to (3.1.10) we see that

$$\lim_{n\to\infty} \frac{\log|\kappa_n|}{n} = \lim_{n\to\infty} \frac{-\frac{1}{2}\sum_{j=0}^{n} \log|1-\alpha_j^2|}{n} = 0 \tag{3.1.18}$$

so that the measure μ is regular in the sense of Ullman-Stahl-Totik. Hence this class of OPUC contains the Nevai class.

Theorem 3.1.4 *In the setting of Theorem 3.1.1, assume that* $\alpha_k \to 0$ *as* $k \to \infty$. *Let* S *be a compact subset of* $\mathbb{T} \setminus \{\pm 1\}$. *Assume, in addition, that* μ *is absolutely continuous with respect to the arclength measure on an open set containing* S *and its Radon-Nikodym derivative is positive and continuous at each point of* S. *Given* $-\infty < \tau_1 < \tau_2 < \infty$, *it follows that*

$$\frac{1}{n}\mathbb{E}\big[N_n\big(\Omega(S,\tau_1,\tau_2)\big)\big] \to \frac{|S|}{2\pi}\left(\frac{H'(\tau_2)}{H(\tau_2)} - \frac{H'(\tau_1)}{H(\tau_1)}\right) \quad \textit{as} \quad n\to\infty, \tag{3.1.19}$$

where $\Omega(S,\tau_1,\tau_2) := \big\{rz : z \in S,\ r \in (1+\frac{\tau_1}{2n}, 1+\frac{\tau_2}{2n})\big\}$ *and* $H(\tau) := \dfrac{e^\tau - 1}{\tau}$.

It can be readily verified that H'/H is increasing on the real line with

$$\lim_{\tau\to-\infty} \frac{H'(\tau)}{H(\tau)} = 0 \quad \text{and} \quad \frac{H'(\tau)}{H(\tau)} = 1 - \frac{H'(-\tau)}{H(-\tau)}.$$

Thus, the zeros of P_n approaching S are expected to be contained in an annular band around S of width $n^{-1+\epsilon}$ for any $\epsilon > 0$.

3.2 Expected number of zeros for random orthogonal polynomials with complex Gaussian coefficients

Let us first start with a motivating example. Consider the complex Kac polynomial

$$p_n(z) = \sum_{k=0}^{n} \eta_k z^k, \tag{3.2.1}$$

with $\eta_j = \alpha_j + i\beta_j,\ j = 0, 1, \ldots, n,$ where $\{\alpha_j\}_{j=0}^n$ and $\{\beta_j\}_{j=0}^n$ are sequences of i.i.d. standard normal random variables. Define

$$A(s,t) := \{z \in \mathbb{C} : 0 \le s < |z| < t\}.$$

Using a classical result by Hammersely [47] that gives a formula for the expected number of zeros of p_n, we have the following:

Proposition 3.2.1 *For the complex Kac polynomial $p_n(z)$ we have*

$$\mathbb{E}[N_n(A(s,t))] = \frac{1}{1-t^2} - \frac{n+1}{1-t^{2n+2}} - \left(\frac{1}{1-s^2} - \frac{n+1}{1-s^{2n+2}}\right), \tag{3.2.2}$$

provided the annulus $A(s,t)$ does not contain the unit circle.

Corollary 3.2.1 *The complex Kac polynomial $p_n(z)$ possess the properties that the density function for the expected number of zeros is equal to*

$$\frac{n(n+2)}{12\pi}, \quad \textit{for} \quad |z| = 1, \tag{3.2.3}$$

and

$$\mathbb{E}[N_n(\mathbb{D})] = \frac{n}{2}. \tag{3.2.4}$$

We are interested in examining asymptotic analogues of the above results for random sums spanned a polynomial basis instead of remaining with the monomials as the basis. Before going further, we need an extension of Hammersely's formula for such random sums.

Let $\{f_j\}$ be a sequence of polynomials such that $\deg f_j = j$, for $j \in \{0, 1, \ldots, n\}$. Set

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z), \qquad z \in \mathbb{C}, \tag{3.2.5}$$

where n is a fixed integer, and $\eta_j = \alpha_j + i\beta_j$, $j = 0, 1, \ldots, n$, with $\{\alpha_j\}_{j=0}^n$ and $\{\beta_j\}_{j=0}^n$ being sequences of i.i.d. real-valued standard Gaussian random variables. For a Jordan region $\Omega \subset \mathbb{C}$, applying a general result of Peres and Virág [88] (c.f. Shiffman and Zeldith [100] and Ledoan [66] for alternate versions of the result), the formula for the expected number of zeros of P_n is given by

$$\mathbb{E}[N_n(\Omega)] = \int_\Omega \rho_n^{(1)}(x, y)\ dx\ dy,$$

with

$$\rho_n^{(1)}(x, y) = \rho_n^{(1)}(z) = \frac{K_n^{(1,1)}(z, z) K_n(z, z) - \left| K_n^{(0,1)}(z, z) \right|^2}{\pi \left(K_n(z, z) \right)^2}, \tag{3.2.6}$$

where

$$K_n(z, w) = \sum_{j=0}^{n} f_j(z) \overline{f_j(w)}, \qquad K_n^{(0,1)}(z, w) = \sum_{j=0}^{n} f_j(z) \overline{f_j'(w)}, \tag{3.2.7}$$

and

$$K_n^{(1,1)}(z, w) = \sum_{j=0}^{n} f_j'(z) \overline{f_j'(w)}. \tag{3.2.8}$$

Following the method of proof given by Vanderbei [115], in the Appendix we give an alternate proof of (3.2.6) for the setting in which our applications are in, that is when $\{f_j\}$ is a polynomial basis with $\deg f_j = j$, for $j \in \{0, 1, \ldots, n\}$.

We note that since all the functions that make up $\rho_n^{(1)}$ are real valued, the function $\rho_n^{(1)}$ is real valued. We also remark that since $\{f_j\}$ is a polynomial basis, we have $K_n(z, z) \geq |f_0(z)|^2 > 0$. Since Cauchy Schwarz gives

$$K_n^{(1,1)}(z, z) K_n(z, z) - \left| K_n^{(0,1)}(z, z) \right|^2 \geq 0,$$

the function $\rho_n^{(1)}$ is also in fact nonnegative. Furthermore, for $(a, b) \subset \mathbb{R}$, it is also known the $\mathbb{E}[N_n(a, b)] = 0$, so that $\rho_n^{(1)}$ does not have mass on the real line.

In the following results we will be considering the case when the spanning functions $\{f_j\}$ of (3.2.5) are OPRL or OPUC (cf. definition (3.1.7)). We say that a collection of polynomials $\{p_j\}_{j\geq 0}$ are orthogonal on the real line (OPRL) with respect to μ, with supp $\mu \subseteq \mathbb{R}$, if

$$\int p_n(x)p_m(x)d\mu(x) = \delta_{nm}, \quad \text{for all } n, m \in \mathbb{N}\cup\{0\}. \tag{3.2.9}$$

We note that when polynomials are orthogonal on the real line, they have real coefficients, and thus are real-valued on the real line.

For analogues of our results concerning random linear combinations of OPRL or OPUC with the random coefficients $\{\eta_j\}$ of P_n being real-valued standard i.i.d. Gaussian, we refer the reader to works of Das [18], Das and Bhatt [19], Lubinsky, Pritsker, and Xie ([77], [78] Theorems 2.2 and 2.3), and the previous section taken from the work of Yattselev and the author [122].

Using the Christoffel-Darboux formula we show that the intensity function from (3.2.6) greatly simplifies when the spanning functions are OPRL or OPUC.

Theorem 3.2.1 *Let $P_n(z) = \sum_{j=0}^n \eta_j f_j(z)$, where $\{\eta_j\}$ are complex-valued i.i.d. standard Gaussian random variables, and $\{f_j\}$ are orthogonal polynomials. Let $\rho_n^{(1)}$ be defined as in* (3.2.6)*.*

1. *When $f_j = p_j$, $j = 0, \ldots, n$, where the p_j's are OPRL, the intensity function simplifies as*

$$\rho_n^{(1)}(z) = \frac{1 - h_n(z)^2}{4\pi\left(\mathrm{Im}(z)\right)^2}, \quad h_n(z) = \frac{\mathrm{Im}(z)|c_n'(z)|}{\mathrm{Im}(c_n(z))}, \quad c_n(z) = \frac{p_{n+1}(z)}{p_n(z)}, \tag{3.2.10}$$

   for $z \in \mathbb{C}$.

2. *Let $f_j = \varphi_j$, $j = 0, \ldots, n$, where the φ_j's are OPUC. When $|z| \neq 1$, the intensity function reduces to*

$$\rho_n^{(1)}(z) = \frac{1 - |k_n(z)|^2}{\pi(1 - |z|^2)^2}, \quad k_n(z) = \frac{(1 - |z|^2)b_n'(z)}{1 - |b_n(z)|^2}, \quad b_n(z) = \frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)}. \tag{3.2.11}$$

   where $\varphi_n^*(z) = z^n\overline{\varphi_n\left(\frac{1}{\bar{z}}\right)}$.

Regarding (3.2.10), we note that

$$\operatorname{Im}(c_n(z)) = 0 \iff c_n(z) = \overline{c_n(z)} = c_n(\bar{z}) \iff z \in \mathbb{R}.$$

Thus as written in the shape above (which is written as such for purposes of computing the limit as $n \to \infty$), the intensity function $\rho_n^{(1)}$ in (3.2.10) appears to have singularities on the real axis due to the $\operatorname{Im}(z)$ and $\operatorname{Im}(c_n(z))$ in the denominators. However these singularises exist only due to the way the intensity function is written.

The restriction $|z| \neq 1$ in (3.2.11) of Theorem 3.2.1 is present due to the use and hence assumptions of the Christoffel-Darboux formula for OPUC. This restriction is from the fact that only when $|z| = 1$ do we have $|\varphi_{n+1}^*(z)| = |\varphi_{n+1}(z)|$ (i.e. $|b_n(z)| = 1$). Furthermore, it is known that all the zeros of $\varphi_{n+1}(z)$ lie in $\mathbb{D}$, and all the zeros of $\varphi_{n+1}^*(z)$ are outside of $\mathbb{D}$. Thus these two polynomials cannot vanish simultaneously.

Our limiting results of $\rho_n^{(1)}$ will be phrased in terms of assumptions on the recurrence coefficients of the orthogonal polynomials. When $\{\varphi_n\}$ are OPUC, we remind the reader of the recurrence relation (3.1.8), and the definition of the Nevai class (3.1.13). For a sequence $\{p_n\}$ of OPRL, the Three Term Recurrence Relation (Theorem 3.2.1 [111]) states

$$xp_n(z) = a_n p_{n+1}(z) + b_n p_n(z) + a_{n-1} p_{n-1}(z), \quad n = 1, 2, \ldots, \tag{3.2.12}$$

where the recurrence coefficient sequences $\{a_n\}$ and $\{b_n\}$ can be given explicitly in terms of the leading coefficient of p_n and p_{n-1}. Due to Nevai (Theorem 13 p. 33 [85], see also Totik p. 99 [113]), the condition that $a_n \to a$ and $b_n \to b$ as $n \to \infty$, with $a \geq 0$ and $b \in \mathbb{R}$, is equivalent to

$$\lim_{n\to\infty} \frac{p_{n+1}(z)}{p_n(z)} = \frac{z - b + \sqrt{(z-b)^2 - 4a^2}}{2}, \tag{3.2.13}$$

with the convergence being valid locally uniformly for $z \notin \operatorname{supp} \mu$. When (3.2.13) holds for a sequence $\{p_n\}$ of OPRL, we say that the sequence is in the Nevai Class. We note that this class is sometimes denoted as $M(a, b)$.

Corollary 3.2.2 *Let $P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z)$, where $\{\eta_j\}$ are complex-valued i.i.d. standard Gaussian random variables, and $\{f_j\}$ are orthogonal polynomials.*

1. *When $\{p_j\}$ are OPRL from the Nevai class, the intensity function $\rho_n^{(1)}$ from* (3.2.10) *for the random orthogonal polynomial satisfies*

$$\lim_{n\to\infty} \rho_n^{(1)}(z) = \frac{1}{4\pi\,(\mathrm{Im}(z))^2} - \frac{|z-b+\sqrt{(z-b)^2-4a^2}|^2}{4\pi|(z-b)^2-4a^2|(\mathrm{Im}(z+\sqrt{(z-b)^2-4a^2}\ ))^2}, \tag{3.2.14}$$

locally uniformly for all $z \notin$ supp μ.

2. *Let $\{\varphi_j\}$ be OPUC from the Nevai class. Then the intensity function $\rho_n^{(1)}$ in* (3.2.11) *for the random orthogonal polynomial possesses the property that*

$$\lim_{n\to\infty} \rho_n^{(1)}(z) = \frac{1}{\pi(1-|z|^2)^2}, \tag{3.2.15}$$

locally uniformly for all $z \in \mathbb{C} \setminus \mathbb{T}$.

When $a = 1/2$ and $b = 0$ in the definition of the Nevai class for the OPRL (3.2.13), it is known that this class contains contains the Chebyshev polynomials. The result of (3.2.14) extends the limiting value given by Farahmand and Grigorash (Section 4 of [36]) in which the spanning functions of their random trigonometric polynomial can be modified to be the Chebyshev polynomials.

We note that the result of (3.2.15) extends the limiting value of the first correlation function given by Peres and Virág [88] (i.e. taking $n = 1$ of their Theorem 1) when the spanning functions were the monomials to that of a very general basis of OPUC. The result further extends their work in that this limiting value also holds for the exterior of the unit circle.

Due to the simplicity and local uniform convergence in (3.2.15), we have the following:

Corollary 3.2.3 *Provided the annuli $A(s,t)$ does not contain the unit circle, for the random orthogonal polynomial spanned by $\{\varphi_j\}$ that are OPUC from the Nevai class,* *we have*

$$\lim_{n\to\infty} \mathbb{E}[N_n(A(s,t))] = \frac{t^2 - s^2}{(1-t^2)(1-s^2)}.$$

Observe that taking $s = 0$ and $t < 1$ in the above result we achieve

$$\lim_{n\to\infty} \mathbb{E}[N_n(D(0,t))] = \frac{t^2}{1-t^2},$$

where $D(0,t) = \{z \in \mathbb{C} : |z| < t\}$.

From (3.2.11) of Theorem 3.2.1 and (3.2.15) of Corollary 3.2.2 we see that the intensity function and its limiting value for the random orthogonal polynomial spanned by OPUC is singular on the unit circle. Assuming a little more on the measure μ associated to the OPUC we can quantify how the zeros approach the unit circle.

Theorem 3.2.2 *Let $P_n(z) = \sum_{j=0}^{n} \eta_j \varphi_j(z)$, where $\{\eta_j\}$ are complex-valued i.i.d. standard Gaussian random variables, and $\{\varphi_j\}$ are OPUC that are regular in the sense of Ullman-Stahl-Totik (cf. definition* (3.1.17)*). Let S be a compact subset of $\mathbb{T}$. Assume, in addition, that the measure μ associated to the sequence $\{\varphi_j\}$ is absolutely continuous with respect to the arclength measure on an open set containing S and its Radon-Nikodym derivative is positive and continuous at each point of S. Given $-\infty < \tau_1 < \tau_2 < \infty$, it follows that*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\big[N_n\big(\Omega(S,\tau_1,\tau_2)\big)\big] = \frac{|S|}{2\pi}\left(\frac{H'(\tau_2)}{H(\tau_2)} - \frac{H'(\tau_1)}{H(\tau_1)}\right), \tag{3.2.16}$$

where $\Omega(S,\tau_1,\tau_2) := \big\{rz : z \in S,\ r \in (1+\frac{\tau_1}{2n}, 1+\frac{\tau_2}{2n})\big\}$ *and* $H(\tau) := \dfrac{e^\tau - 1}{\tau}$.

Remarkably, both the cases of random orthogonal polynomials with real-valued or complex-valued coefficients yield the same asymptotic in (3.2.16). Due to the simplicity of the intensity function $\rho_n^{(1)}(z)$ in the complex-valued i.i.d. standard Gaussian case, we note that the result of Theorem 3.2.2 holds valid for a larger class of OPUC, namely OPUC that are Ullman-Stahl-Totik regular.

3.3 Proofs for Chapter 3

3.3.1 Proofs for Section 3.1

Proof of Theorem 3.1.1. According to the Christoffel-Darboux formula [104, Theorem 2.2.7], since our random sum has $n-1$ terms and the polynomials φ_n have real coefficients, it holds that

$$K_n(z,w) = \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(w)} = \frac{\varphi_n^*(z)\varphi_n^*(\overline{w}) - \varphi_n(z)\varphi_n(\overline{w})}{1 - z\overline{w}}. \tag{3.3.1}$$

Hence,

$$K_n^{(1,0)}(z,w) = \sum_{k=0}^{n-1} \varphi_k'(z)\overline{\varphi_k(w)} = \frac{(\varphi_n^*)'(z)\varphi_n^*(\overline{w}) - \varphi_n'(z)\varphi_n(\overline{w})}{1 - z\overline{w}} + \overline{w}\frac{K_n(z,w)}{1 - z\overline{w}} \tag{3.3.2}$$

and

$$\begin{aligned} K_n^{(1,1)}(z,w) &= \sum_{k=0}^{n-1} \varphi_k'(z)\overline{\varphi_k'(w)} \\ &= \frac{(\varphi_n^*)'(z)(\varphi_n^*)'(\overline{w}) - \varphi_n'(z)\varphi_n'(\overline{w})}{1 - z\overline{w}} + z\frac{(\varphi_n^*)'(z)\varphi_n^*(\overline{w}) - \varphi_n'(z)\varphi_n(\overline{w})}{(1 - z\overline{w})^2} \\ &\quad + \overline{w}\frac{\varphi_n^*(z)(\varphi_n^*)'(\overline{w}) - \varphi_n(z)\varphi_n'(\overline{w})}{(1 - z\overline{w})^2} + \frac{(1 + z\overline{w})K_n(z,w)}{(1 - z\overline{w})^2}. \end{aligned} \tag{3.3.3}$$

Thus,

$$K_n(x,x)K_n^{(1,1)}(x,x) - K_n^{(1,0)}(x,x)^2 = \frac{K_n^2(x,x)}{(1-x^2)^2} - \left(\frac{\varphi_n^*(x)\varphi_n'(x) - \varphi_n(x)(\varphi_n^*)'(x)}{1 - x^2}\right)^2.$$

Therefore, the claim of the theorem now follows from (3.1.4) since

$$\begin{aligned} \rho_n^{(1,0)}(x)^2 &= \frac{1}{\pi^2}\frac{K_n(x,x)K_n^{(1,1)}(x,x) - K_n^{(1,0)}(x,x)^2}{K_n(x,x)^2} \\ &= \frac{1}{\pi^2}\left[\frac{1}{(1-x^2)^2} - \left(\frac{b_n'(x)}{1 - b_n^2(x)}\right)^2\right] \\ &= \frac{1}{\pi^2}\frac{1 - h_n^2(x)}{(1-x^2)^2}, \end{aligned}$$

where

$$h_n(x) = b_n'(x)\frac{1 - x^2}{1 - b_n^2(x)}, \quad \text{with} \quad b_n(x) = \frac{\varphi_n(x)}{\varphi_n^*(x)}.$$

■

Proof of Theorem 3.1.2. In what follows, to avoid complicated schemes of labeling constants, we shall write

$$f_n(z) \lesssim g_n(z), \quad z \in K,\ n \in \mathbb{N} \quad \Leftrightarrow \quad f_n(z) \le C g_n(z), \quad z \in K,\ n \in \mathbb{N},$$

where the constant C depends possibly on K but not on z. Furthermore, we write

$$f_n(z) \asymp g_n(z) \quad \Leftrightarrow \quad f_n(z) \lesssim g_n(z) \lesssim f_n(z).$$

An Auxiliary Estimate: Recall that the n-th Christoffel function of μ is given by

$$\lambda_n(z;\mu) := \inf_{\deg(p)\le n-1} |p(z)|^{-2} \int |p|^2 d\mu = K_n^{-1}(z,z), \tag{3.3.4}$$

where the last equality is extremely well known, see for example [104, Equation (1.2.39)]. We will prove the following claim: *if the measure μ is doubling on a subarc $T \subset \mathbb{T}$, then it holds that*

$$\lambda_n\big(ze^{ia/n};\mu\big) \asymp \mu_n(z) := \int_{T(z,\frac{1}{n})} d\mu, \quad z \in T', \quad |a| \le 2, \tag{3.3.5}$$

uniformly with respect to z, a, n, where $T' \subset T$ is a subarc with endpoints different from those of T and $T(z,\delta)$ stands for the subarc of $\mathbb{T}$ centered at z of arclength 2δ. When μ is doubling on the whole circle $\mathbb{T}$ and $a = 0$, this claim is simply [81, Theorem 4.3]. The proof of the localized version (3.3.5) is quite similar to the one of [81, Theorem 4.3]. However, to improve readability, we adduce the full proof of this fact below.

Given an integer m that we shall fix later, put

$$\begin{aligned}
S_n(z,\eta) &:= \gamma_n \left(\sum_{k=0}^{n-1}\left(\frac{z}{\eta}\right)^k\right)^m \left(\sum_{k=0}^{n-1}\left(\frac{\eta}{z}\right)^k\right)^m \\
&= \gamma_n \left(\frac{1-(z/\eta)^n}{1-z/\eta}\right)^m \left(\frac{1-(\eta/z)^n}{1-\eta/z}\right)^m \\
&= \gamma_n \left(\frac{1}{z\eta}\right)^{m(n-1)} \left(\frac{z^n-\eta^n}{z-\eta}\right)^{2m} \\
&= \gamma_n \left(\frac{e^{ia}-e^{ib}}{e^{i(a+b)(n-1)/(2n)}\big(e^{ia/n}-e^{ib/n}\big)}\right)^{2m}, \quad z=e^{ia/n}, \quad \eta=e^{ib/n} \\
&= \gamma_n \left(\frac{e^{i(a-b)/2}-e^{-i(a-b)/2}}{e^{i(a-b)/(2n)}-e^{-i(a-b)/(2n)}}\right)^{2m} \\
&= \gamma_n \left(\frac{\sin(\frac{a-b}{2})}{\sin(\frac{a-b}{2n})}\right)^{2m}
\end{aligned}$$

where the normalizing constant γ_n is chosen so that $\int_{\mathbb{T}} S_n(z,\eta)|d\eta| = 1$. It is known that $\gamma_n \asymp n^{-2m+1}$. The last representation of $S_n(z,\eta)$ shows that

$$\operatorname{Re}\big(S_n\big(e^{ia/n},e^{ib/n}\big)\big) \gtrsim n. \tag{3.3.6}$$

locally uniformly for $|a-b|<2\pi$. Similarly, we can easily see from the third representation that

$$|S_n(z,\eta)| \lesssim \begin{cases} n, & |z-\eta| \le \frac{1}{n}, \\ n^{-2m+1}|z-\eta|^{-2m}, & |z-\eta| \ge \frac{1}{n}, \end{cases} \tag{3.3.7}$$

for $|n(|z|-1)|, |n(|\eta|-1)| \le A$, where the constant is uniform in $A>0$.

We start with an upper bound. Let $z \in T'$ and $|a| \le 2$. Since $S_n\big(ze^{ia},ze^{ia}\big) = \gamma_n n^{2m} \asymp n$, it is immediate from (3.3.4) that

$$\begin{aligned}
\lambda_n\big(ze^{ia};\mu\big) &\le |S_{\lfloor n/2m\rfloor}\big(ze^{ia},ze^{ia}\big)|^{-2}\int_{\mathbb{T}} |S_{\lfloor n/2m\rfloor}\big(ze^{ia},\eta\big)|^2 d\mu(\eta) \\
&\lesssim \frac{1}{n}\int_T |S_{\lfloor n/2m\rfloor}\big(ze^{ia},\eta\big)|^2\mu_n(\eta)|d\eta| + \frac{1}{n^2}\int_{\mathbb{T}\setminus T} |S_{\lfloor n/2m\rfloor}\big(ze^{ia},\eta\big)|^2 d\mu(\eta),
\end{aligned} \tag{3.3.8}$$

where the inequality on T follows from [2, Equation (4.25)]. It is known, see for example [81, Lemma 2.1(ix)], that the doubling property is equivalent to

$$\mu_n(\eta) \lesssim (1+n|z-\eta|)^s \mu_n(z), \tag{3.3.9}$$

uniformly $z, \eta \in T$, where the parameter s depends only on the constant L in the doubling inequality $\mu(2I) \leq L\mu(I)$. Choose $m > (s+1)/2$. Then (3.3.7) and (3.3.9) yield that the first integral in (3.3.8) can be estimates above by a constant times

$$\frac{\mu_n(z)}{n} \int_T |S_{\lfloor n/2m \rfloor}(ze^{ia}, \eta)|^2 (1 + n|z-\eta|)^s |d\eta| \\ \lesssim n\mu_n(z) \int_{T(z,\frac{1}{n})} |d\eta| + \frac{\mu_n(z)}{n^{4m-s-1}} \int_{T \setminus T(z,\frac{1}{n})} \frac{|d\eta|}{|z-\eta|^{4m-s}} \lesssim \mu_n(z). \quad (3.3.10)$$

To estimate the second integral in (3.3.8), let us point out that (3.3.9) is a consequence of the inequality

$$\int_I d\mu \lesssim \left(\frac{|I| + |J| + \operatorname{dist}(I, J)}{|J|} \right)^s \int_J d\mu, \quad I, J \subseteq T,$$

where the constant is independent of I, J, see [81, Lemma 2.1(viii)]. Therefore,

$$\mu_n(z) \gtrsim n^{-s}, \quad z \in T. \quad (3.3.11)$$

Thus, our choice of m, (3.3.7), and (3.3.11) imply that

$$\frac{1}{n^2} \int_{\mathbb{T} \setminus T} |S_{\lfloor n/2m \rfloor}(z, \eta)|^2 d\mu(\eta) \lesssim \frac{1}{n^{4m}} \int_{\mathbb{T} \setminus T} \frac{d\mu(\eta)}{|z-\eta|^{4m}} \lesssim \frac{1}{n^{4m}} \lesssim \frac{1}{n^{2s+2}} \lesssim \mu_n(z). \quad (3.3.12)$$

The upper bound in (3.3.5) follows now by plugging estimates (3.3.10) and (3.3.12) into (3.3.8).

It only remains to prove the lower bound in (3.3.5). Let $z \in T'$ and $|a| \leq 2$. Define

$$Q_n(w) := w^{m(\lfloor n/2m \rfloor - 1)} \int_{\mathbb{T}} S_{\lfloor n/2m \rfloor}(w, \eta) \big(n\mu_n(\eta)\big)^{1/2} |d\eta|,$$

which is a polynomial of degree at most $n - 2m$. We get from (3.3.6) and (3.3.9) that

$$\begin{aligned} \left|Q_n\big(ze^{ia/n}\big)\right| &\gtrsim \left| \int_{T(z,\frac{1}{n})} \operatorname{Re}\left(S_{\lfloor n/2m \rfloor}\big(ze^{ia/n}, \eta\big)\right) \big(n\mu_n(\eta)\big)^{1/2} |d\eta| \right| \\ &= \left| \int_{-1}^{1} \frac{1}{n} \operatorname{Re}\left(S_{\lfloor n/2m \rfloor}\big(e^{ia/n}, e^{it/n}\big)\right) \big(n\mu_n\big(ze^{it/n}\big)\big)^{1/2} dt \right| \\ &\gtrsim \int_{-1}^{1} \big(n\mu_n\big(ze^{it/n}\big)\big)^{1/2} dt \gtrsim \big(n\mu_n(z)\big)^{1/2}. \end{aligned} \quad (3.3.13)$$

As $S_n(w,\eta)$ is positive for $w,\eta \in \mathbb{T}$, it follows from the normalization of S_n and Jensen's inequality that

$$|Q_n(z)|^2 \le \left(\int_T + \int_{\mathbb{T}\setminus T}\right) S_{\lfloor n/2m \rfloor}(z,\eta) n\mu_n(\eta)|d\eta|.$$

Similarly to (3.3.10), the first integral above can be estimated as follows:

$$\int_T S_{\lfloor n/2m \rfloor}(z,\eta) n\mu_n(\eta)|d\eta| \lesssim n\mu_n(z)\int_T (1+n|z-\eta|)^s S_{\lfloor n/2m \rfloor}(z,\eta)|d\eta| \lesssim n\mu_n(z),$$

where the first estimate follows from (3.3.9) and the second one from (3.3.7). Moreover, we also have that

$$\int_{\mathbb{T}\setminus T} S_{\lfloor n/2m \rfloor}(z,\eta) n\mu_n(\eta)|d\eta| \lesssim n\int_{\mathbb{T}\setminus T} S_{\lfloor n/2m \rfloor}(z,\eta)|d\eta| \lesssim \frac{1}{n^{2(m-1)}} \le \frac{n}{n^s} \lesssim n\mu_n(z),$$

where we again used (3.3.7) as well as (3.3.11). Altogether, we get that

$$|Q_n(z)|^2 \lesssim n\mu_n(z), \quad z \in T'. \tag{3.3.14}$$

Let T_n be a polynomial of degree at most $n-1$ normalized to have value 1 at $ze^{ia/n}$. It follows from (3.3.13) and (3.3.14) that

$$\begin{aligned}\int |T_n|^2 d\mu \ge \int_{T'} |T_n|^2 d\mu \gtrsim \int_{T'} |T_n(\eta)|^2 n\mu_n(\eta)|d\eta| \gtrsim \int_{T'} |(T_nQ_n)(\eta)|^2|d\eta| \\ = \big|Q_n\big(ze^{ia/n}\big)\big|^2 \int_{T'} \frac{|(T_nQ_n)(\eta)|^2}{|Q_n(ze^{ia/n})|^2}|d\eta| \gtrsim n\mu_n(z)\int_{T'} \frac{|(T_nQ_n)(\eta)|^2}{|Q_n(ze^{ia/n})|^2}|d\eta|,\end{aligned}$$

where we also used [2, Equation (4.25)] for the second inequality. Since the polynomial in the last integral above is normalized to have value 1 at $ze^{ia/n}$ and is of degree at most $2n$, (3.3.4) yields that

$$\lambda_n\big(ze^{ia/n};\mu\big) \gtrsim n\mu_n(z)\lambda_{2n}\big(ze^{ia/n};\sigma\big), \tag{3.3.15}$$

where σ is the arclength measure on T'. Now, we know from [116, Theorem 2.4] that

$$n\lambda_{2n}(z;\sigma) = nK_{2n}^{-1}(z,z;\sigma) \gtrsim 1, \quad z \in T', \tag{3.3.16}$$

with the constant independent of z and n, where $K_{2n}(z,w;\sigma)$ is defined exactly as in (3.1.6) with $f_i(z)=\varphi_i(z;\sigma)$ and $\varphi_i(z;\sigma)$ being i-th orthonormal polynomial with respect to σ. Then since

$$|K_{2n}(z,w;\sigma)| \leq \sqrt{K_{2n}(z,z;\sigma)}\sqrt{K_{2n}(w,w;\sigma)},$$

which is simply an application of the Cauchy-Schwarz inequality, and by applying Bernstein-Walsh inequality to each variable of $K_{2n}(z,w;\sigma)$ as it was done in [68, Lemma 6.2], we get from (3.3.16) that

$$\left|K_{2n}\left(ue^{ib/n}, ve^{ic/n};\sigma\right)\right| \lesssim n, \quad u,v\in T', \quad |b|,|c|\leq 2.$$

The lower bound in (3.3.5) now follows by restricting the above inequality to the diagonal and plugging it in (3.3.15).

Upper Estimate: It follows from (3.1.2), (3.1.11), and (3.1.12) that

$$\begin{aligned}\mathbb{E}[N_n(\mathbb{R})] &= 2\int_{-1}^{1}\rho_n^{(1,0)}(x)dx\\ &\leq \frac{2}{\pi}\log n + \mathcal{O}(1) + 2\left(\int_{-1}^{-1+1/n} + \int_{1-1/n}^{1}\right)\rho_n^{(1,0)}(x)dx.\end{aligned}$$

We would like to show that the last two integrals are bounded above by an absolute constant. We shall show this only for the integral on $[1-\frac{1}{n},1]$, the case of the other one being completely identical. It holds that

$$\int_{1-1/n}^{1}\rho_n^{(1,0)}(x)dx \leq \int_{1-1/n}^{1}\sqrt{\frac{K_n^{(1,1)}(x,x)}{K_n(x,x)}}\frac{dx}{\pi} = \int_0^1\sqrt{\frac{K_n^{(1,1)}(1-\frac{y}{n},1-\frac{y}{n})}{n^2K_n(1-\frac{y}{n},1-\frac{y}{n})}}\frac{dy}{\pi}. \tag{3.3.17}$$

which is an easy consequence of (3.1.4). As μ is doubling in some neighborhood of 1, it follows from the Cauchy-Schwarz inequality, (3.3.4), and the lower bound in (3.3.5) that

$$\left|K_n\left(1+\frac{u}{n},1+\frac{\bar{v}}{n}\right)\right| \leq K_n^{1/2}\left(1+\frac{u}{n},1+\frac{u}{n}\right)K_n^{1/2}\left(1+\frac{\bar{v}}{n},1+\frac{\bar{v}}{n}\right) \lesssim \mu_n^{-1}(1)$$

for $|u|,|v|\leq 3/2$. Consequently, the Cauchy integral formula for derivatives of holomorphic functions gives us

$$\left|K_n^{(1,1)}\left(1+\frac{u}{n},1+\frac{\bar v}{n}\right)\right| = \left|\frac{1}{(2\pi i)^2}\int_{|\eta|=3/2}\int_{|\xi|=3/2}\frac{K_n(1+\frac{\eta}{n},1+\frac{\bar\xi}{n})}{(\frac{\eta}{n}-\frac{u}{n})^2(\frac{\bar\xi}{n}-\frac{\bar v}{n})^2}\frac{d\eta}{n}\frac{d\bar\xi}{n}\right| \lesssim \frac{n^2}{\mu_n(1)}$$

for $|u|,|v|\leq 1$. The desired claim now follows from the above inequality combined with the upper estimate in (3.3.5).

Lower Estimate: Under the current assumptions the measure μ is doubling on the whole circle (see the explanation after the statement of the theorem) and therefore we only need to prove the lower estimate. To this end, observe that

$$\mathbb{E}[N_n(\mathbb{R})] > \frac{2}{\pi}\int_{-1+\frac{\log n}{n}}^{1-\frac{\log n}{n}}\frac{\sqrt{1-h_n^2(x)}}{1-x^2}dx \geq -\frac{2}{\pi}\sqrt{1-M_n^2}\log\left(\frac{\log n}{n}\right),$$

where M_n is the maximum of $|h_n(x)|$ on the interval of integration above. Thus, to prove (3.1.15) it is enough to show that $M_n=o(1)$ as $n\to\infty$.

By the conditions of the theorem the sequence of the recurrence coefficients is absolutely summable. Hence, it follows from [104, Theorem 1.5.3] that

$$1\lesssim|\Phi_n^*|\lesssim 1 \tag{3.3.18}$$

uniformly on $\overline{\mathbb{D}}$. As $|\Phi_n|=|\Phi_n^*|$ on $\mathbb{T}$, it also follows from the Bernstein-Walsh inequality that

$$|\Phi_n(z)|\lesssim|z|^n,\quad |z|\geq 1. \tag{3.3.19}$$

We now claim that

$$\Phi_n^*(z)=1-z\sum_{k=0}^{n-1}\alpha_k\Phi_k(z). \tag{3.3.20}$$

Observe that (3.1.9) gives

$$\Phi_{n+1}^*(z)=\Phi_n^*(z)-\alpha_n z\Phi_n(z).$$

As $\Phi_0(z)=1=\Phi_0^*(z)$, we have

$$\Phi_1^*(z)=1-\alpha_0 z\Phi_0(z),$$

and

$$\Phi_2^*(z) = \Phi_1^*(z) - \alpha_1 z \Phi_1(z) = 1 - z(\alpha_0 \Phi_0(z) + \alpha_1 \Phi_1(z)).$$

Hence we have a basis for induction. Assume that

$$\Phi_{n-1}^*(z) = 1 - z \sum_{k=0}^{n-2} \alpha_k \Phi_k(z)$$

holds true. Then

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) - \alpha_{n-1}\Phi_{n-1}(z) = 1 - z \sum_{k=0}^{n-1} \alpha_k \Phi_k(z),$$

so that (3.3.20) is valid by Math Induction.

From (3.3.20), (3.3.19), and the absolute summability of α_k's that

$$|\Phi_n^*(z)| \lesssim \sum_{k=0}^{n-1} |\alpha_k||z|^{k+1} = \left(\sum_{k=0}^{m-1} + \sum_{k=m}^{n-1} \right) |\alpha_k||z|^{k+1} \lesssim |z|^m + \Lambda_m |z|^n \tag{3.3.21}$$

for $|z| \geq 1$, where $\Lambda_m := \sum_{k=m}^{\infty} |\alpha_k|$. That is, it holds that

$$|\Phi_n(z)| = |z^n \Phi_n^*(1/z)| \lesssim \Lambda_m + |z|^{n-m}, \quad |z| \leq 1. \tag{3.3.22}$$

Combining the above inequality with the lower bound in (3.3.18), we see that

$$|b_n(z)| = \left| \frac{\varphi_n(z)}{\varphi_n^*(z)} \right| = \left| \frac{\Phi_n(z)}{\Phi_n^*(z)} \right| \lesssim \Lambda_m + |z|^{n-m}, \quad |z| \leq 1. \tag{3.3.23}$$

It further follows from the Cauchy integral formula for the derivatives that

$$\left| \big(z b_{n-1}(z) \big)' \right| \leq \int_{|\zeta|=r} \frac{|\zeta b_{n-1}(\zeta)|}{|\zeta - z|^2} \frac{|d\zeta|}{2\pi} \leq r^2 \frac{\Lambda_m + r^{n-m-1}}{r^2 - |z|^2}, \quad |z| < r. \tag{3.3.24}$$

On the other hand, the Bernstein inequality for polynomials on the disk of radius r, (3.3.18), and (3.3.22) yield that

$$\max_{|z| \leq r} \left| \big(z b_{n-1}(z) \big)' \right| \lesssim n \left(\Lambda_m + r^{n-m-1} \right). \tag{3.3.25}$$

Now, take m to be the integer part of $n/\log n$ and recall that $\Lambda_m \lesssim m^{1-p}$ according to the condition placed on the recurrence coefficients. Thus, inequalities (3.3.24) and (3.3.25), both applied with $r = 1 - \log n/n$, give

$$\left| \big(z b_n(z) \big)' \right| \lesssim \big(\log n \big)^{p-1} \begin{cases} n^{3/2-p}, & |z| \leq 1 - n^{-1/2}, \\ n^{2-p}, & 1 - n^{-1/2} < |z| \leq 1 - \frac{\log n}{n}. \end{cases} \tag{3.3.26}$$

It also follows from (3.3.23) that for $x \in (-1,1)$ we have that

$$\frac{1-x^2}{1-(xb_{n-1}(x))^2} \lesssim \begin{cases} 1, & |x| \le 1-n^{-1/2}, \\ n^{-1/2}, & 1-n^{-1/2} < |x| \le 1-\frac{\log n}{n}. \end{cases}$$

Now observe that from the recurrence relation (3.1.9) it follows that

$$h_n(z) = (1-z^2)\frac{(zb_{n-1}(z))'}{1-(zb_{n-1}(z))^2}. \tag{3.3.27}$$

Therefore, we deduce from (3.3.27) that $M_n \lesssim (\log n)^{p-1} n^{3/2-p} = o(1)$ as desired.

■

Proof of Proposition 3.1.1. Notice that the upper bound (3.1.14) remains valid in this case. We prove the lower bound as in the previous section by showing that the maximum of $|h_{n+1}(x)|$ on $[-1+\log n/n, 1-\log n/n]$ behaves like $o(1)$ as $n \to \infty$.

It was discovered by Geronimus [44], see also [120], that

$$\Phi_n(z) = \Phi_n(z;\nu) - \frac{\Phi_n(1;\nu)K_n(z,1;\nu)}{t(1-t)^{-1} + K_n(1,1;\nu)}. \tag{3.3.28}$$

As $\Phi_n(1;\nu) = \kappa_n^{-1}\varphi_n(1;\nu)$ and $\varphi_n(1;\nu)\varphi_n^*(1;\nu) = \varphi_n^2(1;\nu)$, using the Christoffel-Darboux formula (3.3.1) yields

$$\begin{aligned} \Phi_n(1;\nu)K_n(z,1;\nu) &= \kappa_n^{-1}\varphi_n(1;\nu)\left(\frac{\varphi_n^*(z;\nu)\varphi_n^*(1;\nu) - \varphi_n(z;\nu)\varphi_n(1;\nu)}{1-z}\right) \\ &= \varphi_n^2(1;\nu)\left(\frac{\Phi_n^*(z;\nu) - \Phi_n(z;\nu)}{1-z}\right). \end{aligned}$$

Thus (3.3.28) can be written as

$$\Phi_n(z) = \Phi_n(z;\nu) - \beta_n \frac{\Phi_n^*(z;\nu) - \Phi_n(z;\nu)}{1-z}$$

where we have set

$$\beta_n := \frac{\varphi_n^2(1;\nu)}{t(1-t)^{-1} + K_n(1,1;\nu)}.$$

Consequently

$$\Phi_n^*(z) = \Phi_n^*(z;\nu) - z\beta_n \frac{\Phi_n^*(z;\nu) - \Phi_n(z;\nu)}{1-z}.$$

Put $s_n(z) := zb_n(z;\nu)$. Then it holds that

$$zb_n(z) = \frac{(1-z)s_n(z) - \beta_n(z - s_n(z))}{1 - z(1+\beta_n) + \beta_n s_n(z)} = 1 - \frac{(1-z)(1-s_n(z))}{1 - z(1+\beta_n) + \beta_n s_n(z)}.$$

Therefore,

$$1 - \big(zb_n(z)\big)^2 = (1-z)^2(1-s_n(z))\frac{1 + s_n(z) + 2(s_n(z) - z)\frac{\beta_n}{1-z}}{(1 - z(1+\beta_n) + \beta_n s_n(z))^2}$$

and

$$\big(zb_n(z)\big)' = \frac{(1-z)^2(1+\beta_n)s_n'(z) - \beta_n((1+s_n(z)(s_n(z)-2))}{(1 - z(1+\beta_n) + \beta_n s_n(z))^2}.$$

Thus, we get from (3.3.27) that

$$h_{n+1}(z) = \frac{(1-z^2)(1+\beta_n)s_n'(z) - \beta_n\frac{1+z}{1-z}(1+s_n(z)(s_n(z)-2))}{(1-s_n(z))(1+s_n(z)+\frac{2\beta_n}{1-z}(s_n(z)-z))}. \tag{3.3.29}$$

It follows from the explanation given after the statement of the proposition that $\beta_n \asymp 1/n$ and therefore

$$\beta_n\frac{1+x}{1-x}|1+s_n(x)(s_n(x)-2)| \lesssim \frac{\beta_n}{1-x} \lesssim \frac{1}{\log n}, \quad -1 \leq x \leq 1 - \frac{\log n}{n},$$

where we used the fact that $|s_n(z)| \leq 1$ in $\mathbb{D}$. It also follows from (3.3.26) that

$$(1-x^2)(1+\beta_n)|s_n'(x)| \lesssim (\log n)^{p-1}n^{3/2-p}, \quad |x| \leq 1 - \frac{\log n}{n}.$$

Hence, the numerator of (3.3.29) is of order $o(1)$ as $n \to \infty$ on the interval of interest. Similarly, we see from (3.3.23) that the denominator behaves like $1+o(1)$ there, which finishes the proof of the proposition. ∎

Proof of Theorem 3.1.3. Let us modify expression (3.1.5) for $\rho_n^{(0,1)}$ to make it more amenable to the asymptotic analysis. Write

$$\pi\left(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\right)^{3/2}\rho_n^{(0,1)}(z) := S_1(z) + S_2(z) + S_3(z), \tag{3.3.30}$$

where from (3.1.5) we have

$$S_1(z) := K_n^{(1,1)}(z,z)\left(K_n(z,z)^2 - |K_n(z,\bar{z})|^2\right),$$

$$S_2(z) := -K_n(z,z)\left(\left|K_n^{(1,0)}(z,z)\right|^2 + \left|K_n^{(1,0)}(z,\bar{z})\right|^2\right),$$

$$S_3(z) := 2\mathrm{Re}\left(K_n(z,\bar{z})K_n^{(1,0)}(z,z)K_n^{(1,0)}(\bar{z},z)\right).$$

For brevity, put

$$S_n(z,w) := (\varphi_n^*)'(z)\varphi_n^*(\overline{w}) - \varphi_n'(z)\varphi_n(\overline{w}).$$

Then it follows from (3.3.1)–(3.3.3) that $S_1(z)$ is equal to

$$\frac{1+|z|^2}{(1-|z|^2)^2} K_n(z,z)\left(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\right) \tag{3.3.31}$$

$$+ \quad 2\frac{\mathrm{Re}(zS_n(z,z))}{(1-|z|^2)^2}\left(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\right) \tag{3.3.32}$$

$$+ \quad \frac{|(\varphi_n^*)'(z)|^2 - |\varphi_n'(z)|^2}{1-|z|^2}\left(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\right), \tag{3.3.33}$$

$S_2(z)$ is equal to

$$-|z|^2 K_n(z,z)\left(\frac{K_n(z,z)^2}{(1-|z|^2)^2} + \frac{|K_n(z,\overline{z})|^2}{|1-z^2|^2}\right) \tag{3.3.34}$$

$$- \quad 2\frac{K_n(z,z)^2\mathrm{Re}(zS_n(z,z))}{(1-|z|^2)^2} - 2\frac{K_n(z,z)\mathrm{Re}\big(zK_n(z,\overline{z})\overline{S_n(z,\overline{z})}\big)}{|1-z^2|^2} \tag{3.3.35}$$

$$- \quad \frac{K_n(z,z)|S_n(z,z)|^2}{(1-|z|^2)^2} - \frac{K_n(z,z)|S_n(z,\overline{z})|^2}{|1-z^2|^2}, \tag{3.3.36}$$

and $S_3(z)$ is equal to

$$\frac{K_n(z,z)|K_n(z,\overline{z})|^2}{1-|z|^2}\left(\frac{1-|z|^4}{|1-z^2|^2} - 1\right) \tag{3.3.37}$$

$$+ \quad 2\frac{|K_n(z,\overline{z})|^2}{1-|z|^2}\mathrm{Re}\left(\frac{\overline{z}S_n(z,z)}{1-\overline{z}^2}\right) + 2\frac{K_n(z,z)}{1-|z|^2}\mathrm{Re}\left(\frac{\overline{z}K_n(z,\overline{z})\overline{S_n(z,\overline{z})}}{1-\overline{z}^2}\right) \tag{3.3.38}$$

$$+ \quad \frac{2}{1-|z|^2}\mathrm{Re}\left(\frac{K_n(z,\overline{z})S_n(z,z)\overline{S_n(z,\overline{z})}}{1-\overline{z}^2}\right), \tag{3.3.39}$$

where we used the identity $2\mathrm{Re}(z^2) = 1 + |z|^4 - |1-z^2|^2$ in (3.3.37). Then we can rewrite (3.3.30) as

$$\pi\left(K_n(z,z)^2 - |K_n(z,\overline{z})|^2\right)^{3/2}\rho_n^{(0,1)}(z) := \Sigma_{n,1}(z) + \Sigma_{n,2}(z) + \Sigma_{n,3}(z), \tag{3.3.40}$$

where $\Sigma_{n,1}$ is the sum of (3.3.31), (3.3.34), and (3.3.37), $\Sigma_{n,2}$ is the sum of (3.3.32), (3.3.35), and (3.3.38), and $\Sigma_{n,3}$ is the sum of (3.3.33), (3.3.36), and (3.3.39). One can readily verify that

$$\Sigma_{n,1}(z) = \frac{K_n(z,z)^3}{(1-|z|^2)^2} - K_n(z,z)|K_n(z,\overline{z})|^2\left(\frac{2}{(1-|z|^2)^2} - \frac{1}{|1-z^2|^2}\right). \tag{3.3.41}$$

Furthermore, one can check that the sum of (3.3.32) and the first summands of (3.3.35) and (3.3.38) is equal to

$$2\frac{\mathrm{Re}\big((\overline{z}-z)\overline{K_n(z,\overline{z})}(\varphi_n^*(z)^2-\varphi_n(z)^2)S_n(z,z)\big)}{(1-|z|^2)^2|1-z^2|^2},$$

while the sum of the last summands of (3.3.35) and (3.3.38) is equal to

$$2\frac{\mathrm{Re}\big((z-\overline{z})\overline{K_n(z,\overline{z})}(|\varphi_n^*(z)|^2-|\varphi_n(z)|^2)S_n(z,\overline{z})\big)}{(1-|z|^2)^2|1-z^2|^2}.$$

By adding up the last two expressions and simplifying, we get that

$$\Sigma_{n,2}(z)=\frac{2\mathrm{Re}\left((z-\bar{z})K_n(z,\bar{z})(\overline{(\varphi_n^*)'(z)\varphi_n(z)}-\overline{\varphi_n'(z)\varphi_n^*(z)})(\varphi_n(z)\overline{\varphi_n^*(z)}-\varphi_n^*(z)\overline{\varphi_n(z)})\right)}{(1-|z|^2)^2|1-z^2|^2}. \tag{3.3.42}$$

To compute $\Sigma_{n,3}$, notice that the sum of the first summands of (3.3.33) and (3.3.36) is equal to

$$-\frac{K_n(z,z)|(\varphi_n^*)'(z)\varphi_n(z)-\varphi_n'(z)\varphi_n^*(z)|^2}{(1-|z|^2)^2}.$$

The remaining summand of (3.3.33) is equal to

$$\big(|\varphi_n'(z)|^2-|(\varphi_n^*)'(z)|^2\big)\frac{|\varphi_n(z)|^4+|\varphi_n^*(z)|^4-2\mathrm{Re}\big(\varphi_n^*(z)^2\overline{\varphi_n(z)^2}\big)}{(1-|z|^2)|1-z^2|^2},$$

while the remaining summand of (3.3.36) is equal to

$$|\varphi_n'(z)|^2\frac{|\varphi_n(z)|^4-|\varphi_n(z)\varphi_n^*(z)|^2}{(1-|z|^2)|1-z^2|^2}-|(\varphi_n^*)'(z)|^2\frac{|\varphi_n^*(z)|^4-|\varphi_n(z)\varphi_n^*(z)|^2}{(1-|z|^2)|1-z^2|^2} \\ +2K_n(z,z)\frac{\mathrm{Re}\big((\varphi_n^*)'(z)\varphi_n^*(z)\overline{\varphi_n'(z)\varphi_n(z)}\big)}{|1-z^2|^2}.$$

Moreover, (3.3.39) can be rewritten as

$$|\varphi_n'(z)|^2\frac{2\mathrm{Re}\big(\varphi_n^*(z)^2\overline{\varphi_n(z)^2}\big)-2|\varphi_n(z)|^4}{(1-|z|^2)|1-z^2|^2}-|(\varphi_n^*)'(z)|^2\frac{2\mathrm{Re}\big(\varphi_n^*(z)^2\overline{\varphi_n(z)^2}\big)-2|\varphi_n^*(z)|^4}{(1-|z|^2)|1-z^2|^2} \\ -2K_n(z,z)\frac{\mathrm{Re}\big((\varphi_n^*)'(z)\varphi_n^*(z)\overline{\varphi_n'(z)\varphi_n(z)}+(\varphi_n^*)'(z)\varphi_n(z)\overline{\varphi_n'(z)\varphi_n^*(z)}\big)}{|1-z^2|^2}.$$

By adding the last four expressions together we get that

$$\Sigma_{n,3}(z)=K_n(z,z)|(\varphi_n^*)'(z)\varphi_n(z)-\varphi_n'(z)\varphi_n^*(z)|^2\left(\frac{1}{|1-z^2|^2}-\frac{1}{(1-|z|^2)^2}\right). \tag{3.3.43}$$

Notice that

$$\frac{(\varphi_n^*)'(z)\varphi_n(z) - \varphi_n'(z)\varphi_n^*(z)}{\phi_n^2(z)} = \begin{cases} -b_n'(z), & |z| < 1, \\ (b_n^{-1})'(z), & |z| > 1, \end{cases} \tag{3.3.44}$$

where

$$\phi_n(z) := \begin{cases} \varphi_n^*(z), & |z| < 1, \\ \varphi_n(z), & |z| > 1. \end{cases}$$

Finally, the assumption $\alpha_k \to 0$ as $k \to \infty$ implies that

$$\begin{cases} b_n(z) \to 0, & \text{locally uniformly in } |z| < 1, \\ b_n^{-1}(z) \to 0, & \text{locally uniformly in } |z| > 1, \end{cases} \tag{3.3.45}$$

as $n \to \infty$ according to [104, Theorem 1.7.4], and since $\varphi_n(z)$ are real-valued on the real line so that $b_n^{-1}(z) = b_n(1/z)$. By recalling (3.3.1) and pugging (3.3.44) into (3.3.42), (3.3.43) and using (3.3.45), we get that

$$\frac{(\Sigma_{n,2} + \Sigma_{n,3})(z)}{|\phi_n(z)|^6} \to 0 \tag{3.3.46}$$

as $n \to \infty$ locally uniformly in $\overline{\mathbb{C}} \setminus \mathbb{T}$. Similarly, we get that

$$\frac{\Sigma_{n,1}(z)}{|\phi_n(z)|^6} \to \frac{1}{|1-|z|^2|}\left(\frac{1}{(1-|z|^2)^2} - \frac{1}{|1-z^2|^2}\right)^2 \tag{3.3.47}$$

as $n \to \infty$ locally uniformly in $\overline{\mathbb{C}} \setminus \mathbb{T}$. Finally, since

$$\frac{K_n(z,z)^2 - |K_n(z,\overline{z})|^2}{|\phi_n(z)|^4} \to \left(\frac{1}{(1-|z|^2)^2} - \frac{1}{|1-z^2|^2}\right)$$

as $n \to \infty$ locally uniformly in $\overline{\mathbb{C}} \setminus \mathbb{T}$, the claim of the theorem follows from (3.3.47), (3.3.46), and (3.3.40). ■

The final proof of this section will rely on a universality result by Levin and Lubinsky [68]. For convenience of the reader, the result we will use is the following:

Theorem 3.3.1 (Theorem 6.3 Levin and Lubinsky [68]) *Let μ be a finite positive Borel measure on $[-\pi, \pi)$ that is Ullman-Stahl-Totik regular. Let $J \subset (-\pi, \pi)$ be* *compact, and such that μ is absolutely continuous in an open interval containing J. Assume moreover, that $w = \mu'$ is positive and continuous at each point of J. Then uniformly for a, b in compact subsets of the plane and $z = e^{i\theta}$, $\theta \in J$ and we have*

$$\lim_{n\to\infty} \frac{K_n\left(z\left(1+\frac{i2\pi a}{n}\right), z\left(1+\frac{i2\pi \bar{b}}{n}\right)\right)}{K_n(z,z)} = e^{i\pi(a-b)} \frac{\sin \pi(a-b)}{\pi(a-b)}.$$

Changing the variables by $a = u/(2\pi i)$ and $\bar{b} = \bar{v}/(2\pi i)$, the conclusion of the above result can be restated as

$$\lim_{n\to\infty} \frac{K_n\left(z\left(1+\frac{u}{n}\right), z\left(1+\frac{\bar{v}}{n}\right)\right)}{K_n(z,z)} = \frac{e^{u+v}-1}{u+v} := H(u+v).$$

Proof of Theorem 3.1.4. It follows from (3.1.2) that

$$\begin{aligned}\frac{1}{n}\mathbb{E}\big[N_n\big(\Omega(S,\tau_1,\tau_2)\big)\big] &= \frac{1}{n}\iint_{\Omega(S,\tau_1,\tau_2)} \rho_n^{(0,1)}(z)dA \\ &= \frac{1}{n}\int_S \int_{1+\frac{\tau_1}{2n}}^{1+\frac{\tau_2}{2n}} \rho_n^{(0,1)}(zr)rdr|dz| \\ &= \frac{1}{2n^2}\int_S \int_{\tau_1}^{\tau_2} \rho_n^{(0,1)}\left(z\left(1+\frac{\tau}{2n}\right)\right)\left(1+\frac{\tau}{2n}\right)d\tau|dz|.\end{aligned}$$

Since

$$\frac{1}{2}\int_S |dz| = \frac{|S|}{2},$$

and as $n \to \infty$ we have $1 + \tau/(2n) \to 1$ uniformly for τ on compact subsets of the real line, to complete the the proof it suffices to show

$$\lim_{n\to\infty} \frac{1}{n^2}\rho_n^{(0,1)}\left(z\left(1+\frac{\tau}{2n}\right)\right) = \frac{1}{\pi}\left(\frac{H'(\tau)}{H(\tau)}\right)' \tag{3.3.48}$$

uniformly for $z \in S$ and τ on compact subsets of the real line.

Under the assumption that $\alpha_k \to 0$ as $k \to \infty$, the measure μ is regular in the sense of Ullman-Stahl-Totik, see (3.1.10) and (3.1.17). Therefore, Theorem 3.3.1 (taken from [68, Theorem 6.3]) is applicable on S and hence

$$\lim_{n\to\infty} K_n(z_{n,u}, z_{n,\bar{v}})K_n^{-1}(z,z) = H(u+v) \tag{3.3.49}$$

uniformly for $z \in S$ and u, v on compact subsets of $\mathbb{C}$, where $z_{n,a} := z(1 + a/n)$. Moreover, we have that

$$\frac{\partial^{i+j}}{\partial u^i \partial v^j} K_n(z_{n,u}, z_{n,\overline{v}}) = \frac{z^{i-j}}{n^{i+j}} K_n^{(i,j)}(z_{n,u}, z_{n,\overline{v}})$$

for any non-negative integers i, j. Thus Cauchy's integral formula and the uniform convergence of (3.3.49) give

$$\lim_{n\to\infty} \frac{z^{i-j}}{n^{i+j}} \frac{K_n^{(i,j)}(z_{n,u}, z_{n,\overline{v}})}{K_n(z,z)} = H^{(i+j)}(u+v) \tag{3.3.50}$$

uniformly for $z \in S$ and u, v on compact subsets of $\mathbb{C}$.

In another connection, since $\alpha_i \to 0$ as $i \to \infty$, [83, Theorem 4] states that

$$\lim_{n\to\infty} \max_{z\in\mathbb{T}} |\varphi_n(z)|^2 K_n^{-1}(z,z) = 0.$$

By compactness, the set S can be covered by finitely many closed subarcs $I_j \subset \mathbb{T} \setminus \{\pm 1\}$ such that μ' is continuous and positive on each I_j. Since $\cup_j I_j$ is separated from ± 1, the Christoffel-Darboux formula (3.3.1) and the above limit yield that

$$\lim_{n\to\infty} \max_{z\in\cup_j I_j} K_n(z,\overline{z}) K_n^{-1}(z,z) = 0.$$

Since $K_n(z,\overline{z})$ is a polynomial of degree $2n-2$, it follows from Lemma 6.1 in [68] applied on each I_j separately, that

$$|K_n(z_{n,a}, \overline{z}_{n,a})| \lesssim \max_{z\in\cup_j I_j} |K_n(z,\overline{z})|$$

uniformly in n and a on compact subsets of $\mathbb{C}$. Thus, it holds that

$$\lim_{n\to\infty} |K_n(z_{n,a}, \overline{z}_{n,a})| K_n^{-1}(z,z) = 0 \tag{3.3.51}$$

uniformly for $z \in S$ and a on compact subsets of $\mathbb{C}$. Moreover, since

$$K_n^{(1,0)}(z_{n,a}, \overline{z}_{n,a}) = \frac{n}{2z} \frac{\partial}{\partial a} K_n(z_{n,a}, \overline{z}_{n,a}),$$

it follows from (3.3.51) and Cauchy's integral formula that

$$\lim_{n\to\infty} n^{-1} |K_n^{(1,0)}(z_{n,a}, \overline{z}_{n,a})| K_n^{-1}(z,z) = 0 \tag{3.3.52}$$

uniformly for $z \in S$ and a on compact subsets of $\mathbb{C}$.

The desired claim (3.3.48) now is an immediate consequence of (3.1.5) and (3.3.49)–(3.3.52). ■

3.3.2 Proofs for Section 3.2

Proof of Proposition 3.2.1. The classic result of Hammersley [47] says that the first intensity function for a complex Kac polynomial $\sum_{k=0}^{n} \eta_k z^k$ is

$$\rho_n^{(1)}(z) = \frac{1}{\pi}\frac{1-|h_{n+1}(z)|^2}{(1-|z|^2)^2}, \quad \text{where} \quad h_{n+1}(z) = \frac{(1-|z|^2)(n+1)z^n}{1-|z|^{2(n+1)}}.$$

Thus for $A(s,t)$ not containing the unit circle it follows that

$$\begin{aligned}
\mathbb{E}[N_n(A(s,t))] &= \int_{A(s,t)} \rho_n^{(1)}(z)dA(z) \\
&= \frac{1}{\pi}\int_{A(s,t)} \left(\frac{1}{(1-|z|^2)^2} - \frac{(n+1)^2|z|^{2n}}{(1-|z|^{2(n+1)})^2}\right) dA(z) \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_s^t \left(\frac{1}{(1-r^2)^2} - \frac{(n+1)^2 r^{2n}}{(1-r^{2(n+1)})^2}\right) r\ dr\ d\theta \\
&= \frac{1}{1-t^2} - \frac{n+1}{1-t^{2n+2}} - \left(\frac{1}{1-s^2} - \frac{n+1}{1-s^{2n+2}}\right).
\end{aligned}$$

■

Proof of Corollary 3.2.1. The conclusion of (3.2.3) first follows from writing the first intensity function in the form

$$\pi\rho_n^{(1)}(z) = \frac{K_n(z,z)K_n^{(1,1)}(z,z) - |K_n^{(0,1)}(z,z)|^2}{K_n(z,z)^2}, \tag{3.3.53}$$

where

$$K_n(z,z) = \sum_{k=0}^{n} |z|^{2k}, \quad K_n^{(0,1)}(z,z) = \sum_{k=0}^{n} kz^k\overline{z}^{k-1}, \quad K_n^{(1,1)}(z,z) = \sum_{k=0}^{n} k^2|z|^{2k-2}. \tag{3.3.54}$$

When $|z| = 1$, using standard summation formulas the kernels take the shape

$$K_n(z,z) = n+1, \quad K_n^{(0,1)}(z,z) = \frac{1}{\overline{z}}\frac{n(n+1)}{2}, \quad K_n^{(1,1)}(z,z) = \frac{n(n+1)(2n+1)}{6}. \tag{3.3.55}$$

Inserting the above evaluations of the kernels in first intensity function (3.3.53) and simplifying then gives (3.2.3).

For the result in (3.2.4), with setting $s = 0$ and $0 < t < 1$ in (3.2.2) we have

$$\mathbb{E}[N_n(D(0,t))] = \frac{t^2(1 - t^{2n}(1 + n(1 - t^2)))}{(1 - t^2)(1 - t^{2n+2})}.$$

Appealing to L'Hopital's rule twice we see that

$$\lim_{t \to 1} \frac{t^2(1 - t^{2n}(1 + n(1 - t^2)))}{(1 - t^2)(1 - t^{2n+2})} = \frac{n}{2},$$

which yields the desired result. ■

We will now specify $\{f_j\}$ to be either OPRL or OPUC, and then use the Christoffel-Darboux formula and its analogue for OPUC (3.3.1) to simplify the kernels $K_n(z, z)$, $K_n^{(0,1)}(z, z)$, and $K_n^{(1,1)}(z, z)$ which make up the intensity function ρ_n from (3.2.6). For convenience of the reader, we state the Christoffel-Darboux formula for OPRL (Theorem 3.2.2, p. 43 of [111]): for $z, w \in \mathbb{C}$ and $\{p_j\}_{j \geq 0}$ OPRL, with k_j being the leading coefficient of p_j, we have

$$\sum_{j=0}^{n} p_j(z)p_j(w) = \frac{k_n}{k_{n+1}} \cdot \frac{p_{n+1}(z)p_n(w) - p_n(z)p_{n+1}(w)}{z - w}, \quad z \neq w. \tag{3.3.56}$$

Furthermore, on the diagonal $z = w$ it takes the form

$$\sum_{j=0}^{n} (p_j(z))^2 = \frac{k_n}{k_{n+1}} \cdot (p'_{n+1}(z)p_n(z) - p'_n(z)p_{n+1}(z)). \tag{3.3.57}$$

Before obtaining our representations of the kernels, let us note that since the polynomials $\{p_j\}$ are orthogonal on the real line, and since we are assuming that the recurrence coefficients $\{\alpha_j\}$ associated to $\{\varphi_j\}$ satisfy $\{\alpha_j\} \subset (-1, 1)$, both classes of orthogonal polynomials have real coefficients. Thus when using conjugation we have that $\overline{p_j(z)} = p_j(\bar{z})$ and $\overline{\varphi_j(z)} = \varphi_j(\bar{z})$ for all $j = 0, 1, \ldots,$ and all $z \in \mathbb{C}$.

Proof of (3.2.10) *in Theorem 3.2.1.* For $z \neq w$, taking derivatives of (3.3.56) yields

$$\sum_{j=0}^{n} p_j(z)p_j'(w) = \frac{k_n}{k_{n+1}}\left(\frac{p_{n+1}(z)p_n'(w) - p_n(z)p_{n+1}'(w)}{z-w}\right.$$
$$\left. + \frac{p_{n+1}(z)p_n(w) - p_n(z)p_{n+1}(w)}{(z-w)^2}\right)$$
$$= \frac{k_n}{k_{n+1}} \cdot \frac{p_{n+1}(z)p_n'(w) - p_n(z)p_{n+1}'(w)}{z-w} + \frac{\sum_{j=0}^{n} p_j(z)p_j(w)}{z-w}, \quad (3.3.58)$$

and

$$\sum_{j=0}^{n} p_j'(z)p_j'(w) = \frac{k_n}{k_{n+1}}\left(\frac{p_{n+1}'(z)p_n'(w) - p_n'(z)p_{n+1}'(w)}{z-w}\right.$$
$$- \frac{p_{n+1}(z)p_n'(w) - p_n(z)p_{n+1}'(w)}{(z-w)^2}$$
$$+ \frac{p_{n+1}'(z)p_n(w) - p_n'(z)p_{n+1}(w)}{(z-w)^2}$$
$$\left. - \frac{2\left(p_{n+1}(z)p_n(w) - p_n(z)p_{n+1}(w)\right)}{(z-w)^2}\right)$$
$$= \frac{k_n}{k_{n+1}} \cdot \frac{p_{n+1}'(z)p_n'(w) - p_n'(z)p_{n+1}'(w)}{z-w}$$
$$- \frac{\sum_{j=0}^{n} p_j(z)p_j'(w)}{z-w} + \frac{\sum_{j=0}^{n} p_j'(z)p_j(w)}{z-w}. \quad (3.3.59)$$

Setting $w = \bar{z}$ in (3.3.56), (3.3.58), and (3.3.59), since the coefficients of $\{p_j\}$ are real it follows that

$$K_n(z,z) = \sum_{j=0}^{n} p_j(z)\overline{p_j(z)} = \frac{k_n}{k_{n+1}} \cdot \frac{p_{n+1}(z)p_n(\bar{z}) - p_n(z)p_{n+1}(\bar{z})}{2i\mathrm{Im}(z)}, \quad (3.3.60)$$

$$K_n^{(0,1)}(z,z) = \sum_{j=0}^{n} p_j(z)\overline{p_j'(z)} = \frac{k_n}{k_{n+1}} \cdot \frac{p_{n+1}(z)p_n'(\bar{z}) - p_n(z)p_{n+1}'(\bar{z})}{2i\mathrm{Im}(z)} + \frac{K_n(z,z)}{2i\mathrm{Im}(z)}, \quad (3.3.61)$$

$$K_n^{(1,1)}(z,z) = \sum_{j=0}^{n} p_j'(z)\overline{p_j'(z)} = \frac{k_n}{k_{n+1}} \cdot \frac{\mathrm{Im}(p_{n+1}'(z)p_n'(\bar{z}))}{\mathrm{Im}(z)} - \frac{K_n^{(0,1)}(z,z)}{2i\mathrm{Im}(z)} + \frac{\overline{K_n^{(0,1)}(z,z)}}{2i\mathrm{Im}(z)}. \quad (3.3.62)$$

For our representation of $K_n(z,\bar{z})$ we simply use (3.3.57) and again that the coefficients of $\{p_j\}$ are real to achieve

$$K_n(z,\bar z)=\sum_{j=0}^{n}p_j(z)\overline{p_j(\bar z)}=\sum_{j=0}^{n}p_j(z)p_j(z)=\frac{k_n}{k_{n+1}}\left(p_{n+1}'(z)p_n(z)-p_n'(z)p_{n+1}(z)\right). \tag{3.3.63}$$

Using our derived expressions (3.3.60), (3.3.61), (3.3.62), and (3.3.63), the numerator of the intensity function ρ_n from (3.2.6) simplifies as

$$K_n^{(1,1)}(z,z)K_n(z,z)-|K_n^{(0,1)}(z,z)|^2=\frac{\left(K_n(z,z)\right)^2-\left|K_n(z,\bar z)\right|^2}{4\left(\mathrm{Im}(z)\right)^2}.$$

Therefore, using the expression for the numerator above and recalling the relations (3.3.60) and (3.3.63), we see that the intensity function given by (3.2.6) is

$$\begin{aligned}\rho_n^{(1)}(z)&=\frac{K_n^{(1,1)}(z,z)K_n(z,z)-|K_n^{(0,1)}(z,z)|^2}{\pi\left(K_n(z,z)\right)^2}\\&=\frac{1}{4\pi\left(\mathrm{Im}(z)\right)^2}\left(1-\frac{\left|K_n(z,\bar z)\right|^2}{\left(K_n(z,z)\right)^2}\right)\\&=\frac{1}{4\pi\left(\mathrm{Im}(z)\right)^2}\left(1-\frac{(2i\mathrm{Im}(z))^2\left|p_{n+1}'(z)p_n(z)-p_n'(z)p_{n+1}(z)\right|^2}{\left(p_{n+1}(z)p_n(\bar z)-p_n(z)p_{n+1}(\bar z)\right)^2}\right)\\&=\frac{1}{4\pi\left(\mathrm{Im}(z)\right)^2}\left(1-\frac{(2i\mathrm{Im}(z))^2\left|\left(\frac{p_{n+1}(z)}{p_n(z)}\right)'\right|^2}{\left(\frac{p_{n+1}(z)}{p_n(z)}-\frac{p_{n+1}(\bar z)}{p_n(\bar z)}\right)^2}\right)\\&=\frac{1-h_n(z)^2}{4\pi\left(\mathrm{Im}(z)\right)^2},\end{aligned}$$

where

$$h_n(z)=\frac{\mathrm{Im}(z)|c_n'(z)|}{\mathrm{Im}(c_n(z))},\qquad c_n(z)=\frac{p_{n+1}(z)}{p_n(z)},$$

which gives the result of (3.2.10) in Theorem 3.2.1. ∎

Proof of (3.2.11) *in Theorem 3.2.1.* Applying the analogue of the Christoffel-Darboux formula (3.3.1), and making derivations analogously as done for the kernels for OPRL, our representations of $K_n(z,z)$, $K_n^{(0,1)}(z,z)$, and $K_n^{(1,1)}(z,z)$ are as follows:

$$K_n(z,z)=\sum_{j=0}^{n}\varphi_j(z)\overline{\varphi_j(z)}=\frac{\left|\varphi_{n+1}^*(z)\right|^2-\left|\varphi_{n+1}(z)\right|^2}{1-|z|^2}, \tag{3.3.64}$$

$$K_n^{(0,1)}(z,z) = \sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j'(z)} = \frac{\overline{\varphi_{n+1}^{*\,\prime}(z)}\varphi_{n+1}^{*}(z) - \overline{\varphi_{n+1}'(z)}\varphi_{n+1}(z)}{1-|z|^2} + \frac{zK_n(z,z)}{1-|z|^2}, \tag{3.3.65}$$

and

$$\begin{aligned} K_n^{(1,1)}(z,z) &= \sum_{j=0}^{n} |\varphi_j'(z)|^2 \\ &= \frac{|\varphi_{n+1}^{*\,\prime}(z)|^2 - |\varphi_{n+1}'(z)|^2}{1-|z|^2} + \frac{\bar{z}K_n^{(0,1)}(z,z) + z\overline{K_n^{(0,1)}(z,z)} + K_n(z,z)}{1-|z|^2}. \end{aligned} \tag{3.3.66}$$

Using (3.3.64), (3.3.65), and (3.3.66), the numerator of the intensity function ρ_n from (3.2.6) reduces to

$$K_n^{(1,1)}(z,z)K_n(z,z) - |K_n^{(0,1)}(z,z)|^2 = \frac{(K_n(z,z))^2}{(1-|z|^2)^2} - \frac{\left|\varphi_{n+1}^{*}(z)\varphi_{n+1}'(z) - \varphi_{n+1}^{*\,\prime}(z)\varphi_{n+1}(z)\right|^2}{(1-|z|^2)^2}.$$

From the above numerator and (3.3.64), the intensity function becomes

$$\begin{aligned} \rho_n^{(1)}(z) &= \frac{K_n^{(1,1)}(z,z)K_n(z,z) - \left|K_n^{(0,1)}(z,z)\right|^2}{\pi\,(K_n(z,z))^2} \\ &= \frac{1}{\pi\,(1-|z|^2)^2}\left(1 - \frac{\left|\varphi_{n+1}^{*}(z)\varphi_{n+1}'(z) - \varphi_{n+1}^{*\,\prime}(z)\varphi_{n+1}(z)\right|^2}{(K_n(z,z))^2}\right) \\ &= \frac{1}{\pi\,(1-|z|^2)^2}\left(1 - \frac{(1-|z|^2)^2\left|\varphi_{n+1}^{*}(z)\varphi_{n+1}'(z) - \varphi_{n+1}^{*\,\prime}(z)\varphi_{n+1}(z)\right|^2}{\left(|\varphi_{n+1}(z)|^2 - |\varphi_{n+1}^{*}(z)|^2\right)^2}\right) \end{aligned} \tag{3.3.67}$$

$$\begin{aligned} &= \frac{1}{\pi\,(1-|z|^2)^2}\left(1 - \frac{(1-|z|^2)^2\left|\left(\frac{\varphi_{n+1}(z)}{\varphi_{n+1}^{*}(z)}\right)'\right|^2}{\left(\left|\frac{\varphi_{n+1}(z)}{\varphi_{n+1}^{*}(z)}\right|^2 - 1\right)^2}\right) \\ &= \frac{1-|k_n(z)|^2}{\pi(1-|z|^2)^2}, \end{aligned}$$

where

$$k_n(z) = \frac{(1-|z|^2)b_n'(z)}{1-|b_n(z)|^2}, \quad b_n(z) = \frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)},$$

and hence completes the proof of (3.2.11) in Theorem 3.2.1. ■

Proof of (3.2.14) *in Corollary 3.2.2.* Since the convergence of (3.2.13) is uniform on compact subsets away from the support of μ, for $z \notin \text{supp } \mu$ we can differentiate to yield

$$\begin{aligned}
\lim_{n\to\infty} c_n'(z) &= \lim_{n\to\infty} \left(\frac{p_{n+1}(z)}{p_n(z)}\right)' \\
&= \frac{d}{dz}\left(\frac{z-b+\sqrt{(z-b)^2-4a^2}}{2}\right) \\
&= \frac{z-b+\sqrt{(z-b)^2-4a^2}}{2\sqrt{(z-b)^2-4a^2}}. \qquad (3.3.68)
\end{aligned}$$

Also from (3.2.13) we see that

$$\begin{aligned}
\lim_{n\to\infty} \text{Im}(c_n(z)) &= \lim_{n\to\infty} \frac{\frac{p_{n+1}(z)}{p_n(z)} - \frac{p_{n+1}(\bar z)}{p_n(\bar z)}}{2i} \\
&= \frac{z+\sqrt{(z-b)^2-4a^2} - (\bar z + \sqrt{(\bar z-b)^2-4a^2}\ )}{4i} \qquad (3.3.69)
\end{aligned}$$

Combining (3.3.68) and (3.3.69) gives

$$\lim_{n\to\infty} h_n(z)^2 = \lim_{n\to\infty} \frac{(\text{Im}(z))^2\,|c_n'(z)|^2}{(\text{Im}(c_n(z)))^2} = \frac{(\text{Im}(z))^2\,|z-b+\sqrt{(z-b)^2-4a^2}|^2}{|(z-b)^2-4a^2|(\text{Im}(z+\sqrt{(z-b)^2-4a^2}\ ))^2}.$$

Therefore, using the representation of the intensity function in (3.2.10) of Theorem 3.2.1, from the above limit we see that

$$\begin{aligned}
\lim_{n\to\infty} \rho_n^{(1)}(z) &= \lim_{n\to\infty} \frac{1-h_n^2(z)}{4\pi\,(\text{Im}(z))^2} \\
&= \frac{1}{4\pi\,(\text{Im}(z))^2} - \frac{|z-b+\sqrt{(z-b)^2-4a^2}|^2}{4\pi|(z-b)^2-4a^2|(\text{Im}(z+\sqrt{(z-b)^2-4a^2}\ ))^2},
\end{aligned}$$

locally uniformly for $z \notin \text{supp } \mu$, and thus completes the proof. ■

Proof of (3.2.15) *in Corollary 3.2.2.* Under the assumption that $\{\varphi_j\}$ are OPUC in the Nevai class, (3.1.13) gives

$$\lim_{n\to\infty} b_n(z) = \lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)} = 0, \qquad (3.3.70)$$

uniformly on compact subsets of $\mathbb{D}$. Since the convergence is locally uniform in $\mathbb{D}$, within $\mathbb{D}$ we can differentiate to achieve

$$\lim_{n\to\infty} b_n'(z) = \lim_{n\to\infty} \frac{d}{dz}\left(\frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)}\right) = 0. \tag{3.3.71}$$

Thus combining (3.3.70) and (3.3.71) we see that

$$\lim_{n\to\infty} k_n(z) = \lim_{n\to\infty} \frac{(1-|z|^2)b_n'(z)}{1-|b_n(z)|^2} = 0. \tag{3.3.72}$$

This gives that the intensity function in Theorem 3.2.1 represented by (3.2.11) satisfies

$$\lim_{n\to\infty} \rho_n(z) = \lim_{n\to\infty} \frac{1-|k_n(z)|^2}{\pi(1-|z|^2)^2} = \frac{1}{\pi(1-|z|^2)^2}$$

locally uniformly on $\mathbb{D}$.

To see that the same limit holds in the exterior of the disk, as noted by Igor Pritsker, observe that for $w = \bar{z}^{-1} \in \mathbb{D}$

$$\frac{\varphi_n^*(z)}{\varphi_n(z)} = \frac{z^n\overline{\varphi_n(1/\bar{z})}}{\varphi_n(z)} = \frac{\bar{w}^{-n}\overline{\varphi(w)}}{\varphi(1/\bar{w})} = \overline{\left(\frac{\varphi_n(w)}{\varphi_n^*(w)}\right)}.$$

Thus, under the assumption that $\{\varphi_j\}$ are from the Nevai class, for $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$ we have

$$\lim_{n\to\infty} \frac{\varphi_n^*(z)}{\varphi_n(z)} = 0. \tag{3.3.73}$$

Notice that from (3.3.67) we can factor in a different manor to achieve

$$\begin{aligned}
\rho_n^{(1)}(z) &= \frac{1}{\pi(1-|z|^2)^2}\left(1 - \frac{(1-|z|^2)^2|\varphi_{n+1}^*(z)\varphi_{n+1}'(z) - \varphi_{n+1}^{*\,\prime}(z)\varphi_{n+1}(z)|^2}{(|\varphi_{n+1}(z)|^2 - |\varphi_{n+1}^*(z)|^2)^2}\right) \\
&= \frac{1}{\pi(1-|z|^2)^2}\left(1 - \frac{(1-|z|^2)^2\left|\left(\frac{\varphi_{n+1}^*(z)}{\varphi_{n+1}(z)}\right)'\right|^2}{\left(\left|\frac{\varphi_{n+1}^*(z)}{\varphi_{n+1}(z)}\right|^2 - 1\right)^2}\right) \\
&= \frac{1-|l_n(z)|^2}{\pi(1-|z|^2)^2},
\end{aligned}$$

where

$$l_n(z) = \frac{(1-|z|^2)d_n'(z)}{1-|d_n(z)|^2}, \quad d_n(z) = \frac{\varphi_{n+1}^*(z)}{\varphi_{n+1}(z)}.$$

Using (3.3.73) and continuing analogously as done for the case in the unit disk, it follows that $l_n(z) \to 0$ locally uniformly for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \rho_n^{(1)}(z) = \lim_{n\to\infty} \frac{1 - |l_n(z)|^2}{\pi(1-|z|^2)^2} = \frac{1}{\pi(1-|z|^2)^2} \tag{3.3.74}$$

uniformly on compact subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$, and hence gives our desired result. ■

Proof of Corollary 3.2.3. Since the convergence of (3.2.15) in Corollary 3.2.2 is locally uniform on annuli that do not contain the unit circle, we can pass the limit through the integral over

$$A(s,t) = \{z \in \mathbb{C} \setminus \mathbb{T} : 0 \le s < |z| < t\}.$$

We remark that formally we need to consider a closed annulus the does not contain the unit circle. However, since the measure associated to the integral is Lebesgue area measure which is absolutely continuous, and the limiting values above are continuous functions away from the unit circle, we have that the boundary of the closed annulus has measure zero. Hence we just consider the open annulus $A(s,t)$. Thus we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{A(s,t)} \rho_n^{(1)}(z)\ dz &= \int_{A(s,t)} \lim_{n\to\infty} \rho_n^{(1)}(z)\ dz \\
&= \frac{1}{\pi} \int_{A(s,t)} \frac{1}{(1-|z|^2)^2}\ dz \\
&= \frac{1}{\pi} \int_0^{2\pi} \int_s^t \frac{r}{(1-r^2)^2}\ dr\ d\theta \\
&= \frac{1}{1-t^2} - \frac{1}{1-s^2} \\
&= \frac{t^2-s^2}{(1-t^2)(1-s^2)}.
\end{aligned}$$

■

Proof of Theorem 3.2.2. It follows from definition of the intensity function that

$$\begin{aligned}
\frac{1}{n}\mathbb{E}\big[N_n\big(\Omega(S,\tau_1,\tau_2)\big)\big] &= \frac{1}{n}\iint_{\Omega(S,\tau_1,\tau_2)} \rho_n^{(1)}(z)\ dA \\
&= \frac{1}{n}\int_S \int_{1+\frac{\tau_1}{2n}}^{1+\frac{\tau_2}{2n}} \rho_n^{(1)}(zr) r dr |dz| \\
&= \frac{1}{2n^2}\int_S \int_{\tau_1}^{\tau_2} \rho_n^{(1)}\left(z\left(1+\frac{\tau}{2n}\right)\right)\left(1+\frac{\tau}{2n}\right)\ d\tau |dz|.
\end{aligned}$$

Since

$$\frac{1}{2}\int_S |dz| = \frac{|S|}{2},$$

and as $n\to\infty$ we have $1+\tau/(2n)\to 1$ uniformly for τ on compact subsets of the real line, to complete the the proof it suffices to show

$$\lim_{n\to\infty}\frac{1}{n^2}\rho_n^{(1)}\left(z\left(1+\frac{\tau}{2n}\right)\right) = \frac{1}{\pi}\left(\frac{H'(\tau)}{H(\tau)}\right)' \tag{3.3.75}$$

uniformly for $z\in S$ and τ on compact subsets of the real line.

To this end, using the representation (3.2.6) of $\rho_n^{(1)}$ and the two limits (3.3.49) and (3.3.50), we see that

$$\begin{aligned}
&\frac{1}{n^2}\rho_n^{(1)}\left(z\left(1+\frac{\tau}{2n}\right)\right) \\
&\quad = \frac{1}{n^2\pi}\frac{K_n(z_{n,\tau/2}, z_{n,\bar\tau/2})K_n^{(1,1)}(z_{n,\tau/2}, z_{n,\bar\tau/2}) - |K_n^{(0,1)}(z_{n,\tau/2}, z_{n,\bar\tau/2})|^2}{K_n(z_{n,\tau/2}, z_{n,\bar\tau/2})^2} \\
&\quad = \frac{1}{\pi}\frac{\frac{K_n(z_{n,\tau/2}, z_{n,\bar\tau/2})K_n^{(1,1)}(z_{n,\tau/2}, z_{n,\bar\tau/2})}{n^2K_n(z,z)^2} - \frac{|K_n^{(0,1)}(z_{n,\tau/2}, z_{n,\bar\tau/2})|^2}{n^2K_n(z,z)^2}}{\frac{K_n(z_{n,\tau/2}, z_{n,\bar\tau/2})^2}{K_n(z,z)^2}} \\
&\quad \to \frac{1}{\pi}\frac{H(\tau)H''(\tau) - H'(\tau)^2}{H(\tau)^2} \qquad (n\to\infty) \\
&\quad = \frac{1}{\pi}\left(\frac{H'(\tau)}{H(\tau)}\right)',
\end{aligned}$$

and thus completes the proof. ■

CHAPTER IV

VARIANCE OF THE NUMBER OF ZEROS OF RANDOM SUMS

In this chapter we study the variance of the number of zeros for

$$P_n(z) = \sum_{k=0}^{n} \eta_k \varphi_k(z) \quad \text{and} \quad P(z) = \sum_{k=0}^{\infty} \eta_k \varphi_k(z),$$

where $\{\eta_k\}$ are complex-valued random variables, and $\{\varphi\}$ are OPUC. For measurable $\Omega \subset \mathbb{C}$, the variance of the number of zeros of P_n in Ω will be denoted as

$$\text{Var}[N_n(\Omega)] := \mathbb{E}[N_n(\Omega)^2] - \mathbb{E}[N_n(\Omega)]^2, \tag{4.0.1}$$

and respectively for P as

$$\text{Var}[N(\Omega)] := \mathbb{E}[N(\Omega)^2] - \mathbb{E}[N(\Omega)]^2. \tag{4.0.2}$$

When $\{\eta_k\}$ are complex-valued random variables whose distributions only satisfy certain uniform bounds for the fractional and logarithmic moments, and the recurrence coefficients $\{\alpha_k\}$ associated to $\{\varphi_k\}$ are absolutely summable, or such that the measure of orthogonality μ associated to $\{\varphi_k\}$ is regular in the sense of Ullman-Stahl-Totik (UST), we give quantitative estimates that show the variance of the number of zeros scaled by n^2 of P_n in annuli that intersect the unit circle is $o(1)$ as $n \to \infty$. When $\{\eta_k\}$ are i.i.d. complex-valued standard Gaussian, and $\{\varphi_k\}$ are OPUC associated to a conjugate symmetric measure μ from the Nevai class, we give the limit of the variance of the number of zeros of P_n in annuli that do not contain the unit circle.

Setting $\{\eta_k\}$ to be i.i.d. complex-valued standard Gaussian, we take $\{\varphi_k\}$ to be from the Szegő class to ensure the almost sure convergence of the random series P

within the unit disk. In this case we compute the variance of the number of zeros of P in annuli contained in the unit disk.

We note the hierarchy of the classes of OPUC we are considering as spanning functions:

$$\{\{\varphi_k\} : \text{ the associated recurrence coefficients } \{\alpha_k\} \text{ are absolutely summable}\}$$
$$\subset \{\{\varphi_k\} \text{ from the Szegő class}\}$$
$$\subset \{\{\varphi_k\} \text{ from the Nevai class}\}$$
$$\subset \{\{\varphi_k\} : \text{ the associated measure } \mu \text{ of orthogonality is regular in the sense of UST}\}.$$

In the case that $\{\eta_k\}$ are i.i.d. complex-valued standard Gaussian, we will study the variance of the number of zeros via examining the second correlation function. We denote the second correlation function for $P_n(z)$ as $\rho_n^{(2)}(z,w)$, and $\rho^{(2)}(z,w)$ for $P(z)$. To see the connection between the variance of the number of zeros and the second correlation function, observe that for a measurable set $\Omega \subset \mathbb{C}$ it follows that

$$\begin{aligned}
\mathrm{Var}[N_n(\Omega)] &= \mathbb{E}[(N_n(\Omega))^2] - (\mathbb{E}[N_n(\Omega)])^2 \\
&= \mathbb{E}[N_n(\Omega)] - \mathbb{E}[N_n(\Omega)] + \mathbb{E}[(N_n(\Omega))^2] - (\mathbb{E}[N_n(\Omega)])^2 \\
&= \mathbb{E}[N_n(\Omega)] + \mathbb{E}[N_n(\Omega)(N_n(\Omega)-1)] - (\mathbb{E}[N_n(\Omega)])^2 \\
&= \int_\Omega \rho_n^{(1)}(z)\ dA(z) + \int_\Omega \int_\Omega \rho_n^{(2)}(z,w)\ dA(z)\ dA(w) \\
&\quad - \int_\Omega \int_\Omega \rho_n^{(1)}(z)\rho_n^{(1)}(w)\ dA(z)\ dA(w), \qquad (4.0.3)
\end{aligned}$$

where the equality

$$\mathbb{E}[N_n(\Omega)(N_n(\Omega)-1)] = \int_\Omega \int_\Omega \rho_n^{(2)}(z,w)\ dA(z)\ dA(w)$$

is a known result. Replacing $N_n(\Omega)$ by $N(\Omega)$ in the above, similarly we have a relation for $\rho^{(2)}(z,w)$.

4.1 The variance of the number of zeros for random orthogonal polynomials

Let us first consider the simplest type of OPUC, $\{z^k\}$, as a spanning basis. In this case we have the complex Kac polynomial $P_n(z) = \sum_{k=0}^n \eta_{k,n} z^k$, where $\{\eta_{k,n}\}$ are complex-valued random variables. Let

$$A_r(\alpha, \beta) = \{z \in \mathbb{C} : \alpha \leq \arg z < \beta \leq 2\pi,\ 1/r < |z| < r,\ 0 < r < 1\}.$$

Set

$$M := \sup\{\mathbb{E}[|\eta_{k,n}|^t] \mid k = 0, 1, \ldots, n,\ n \in \mathbb{N}\} < \infty, \quad t \in (0, 1] \tag{4.1.1}$$

and

$$L := \inf\{\mathbb{E}[\log |\eta_{k,n}|] \mid k = 0 \ \&\ n,\ n \in \mathbb{N}\} > -\infty. \tag{4.1.2}$$

As an application of Corollary 2.1.1 we obtain the following result concerning the scaled variance of the number of zeros of the complex Kac polynomial.

Theorem 4.1.1 *For $P_n(z) = \sum_{k=0}^n \eta_{k,n} z^k$ where $\{\eta_{k,n}\}$ are complex valued random variables satisfying conditions* (4.1.1) *and* (4.1.2)*, we have*

$$\frac{\mathrm{Var}[N_n(A_r(\alpha, \beta))]}{n^2} = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right), \quad \text{as} \quad n \to \infty. \tag{4.1.3}$$

Consider now the random orthogonal polynomial $P_n(z) = \sum_{k=0}^n \eta_{k,n} \varphi_k(z)$ where $\{\varphi_k\}$ are OPUC, and $\{\eta_{k,n}\}$ are complex valued random variables such that

$$\sup\{\mathbb{E}[|\eta_{k,n}|^t] \mid k = 0, 1, \ldots, n,\ n \in \mathbb{N}\} < \infty, \quad t \in (0, 1] \tag{4.1.4}$$

and

$$\min\left(\inf_{n \in \mathbb{N}} \mathbb{E}[\log |\eta_{n,n}|],\ \inf_{n \in \mathbb{N}, z \in \mathbb{C}} \mathbb{E}[\log |\eta_{0,n} + z|]\right) > -\infty. \tag{4.1.5}$$

Applying the results of (2.2.7) and (2.2.8) of Corollary 2.2.1 we are able achieve the following:

Theorem 4.1.2 *Let $P_n(z) = \sum_{k=0}^{n} \eta_{k,n}\varphi_k(z)$ where $\{\eta_{k,n}\}$ are complex valued random variables satisfying conditions* (4.1.4) *and* (4.1.5).

1. *When the recurrence coefficients $\{\alpha_k\}$ associated to $\{\varphi_k\}$ are absolutely summable, we have*
$$\frac{\mathrm{Var}[N_n(A_r(\alpha,\beta))]}{n^2} = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right), \quad as \quad n\to\infty. \tag{4.1.6}$$

2. *Assume the measure of orthogonality μ associated to $\{\varphi_k\}$ is regular in the sense of Ullman-Stahl-Totik, that is,*
$$\varepsilon_n := \frac{1}{n}\log|\kappa_n| \to 0, \;\; as\; n\to\infty, \tag{4.1.7}$$
where κ_n is the leading coefficient of φ_n. Then
$$\frac{\mathrm{Var}[N_n(A_r(\alpha,\beta))]}{n^2} = \mathcal{O}\left(\max\left\{\sqrt{\frac{\log n}{n}}, \varepsilon_n^{1/4}\right\}\right) \quad as \quad n\to\infty, \tag{4.1.8}$$
where ε_n is given by (4.1.7).

We remark that the result of (2.2.7) of Corollary 2.2.1 was stated for $\{\varphi_k\}$ that are orthonormal with respect to a positive Borel measure μ supported on the unit circle, such that $d\mu(\theta) = w(\theta)d\theta$, where $w(\theta) \geq c > 0$, $\theta \in [0, 2\pi)$. Due to Baxter's Theorem (see [104], Theorem 5.2.1) this assumption is equivalent to $\sum_{k=0}^{\infty}|\alpha_k| < \infty$. We also note that since
$$\sum_{k=0}^{\infty}|\alpha_k|^2 \leq \left(\sum_{k=0}^{\infty}|\alpha_k|\right)^2 < \infty,$$
relying on Theorem 2.7.15 of [104] which states that the condition $\sum_{k=0}^{\infty}|\alpha_k|^2 < \infty$ is equivalent to $\{\varphi_k\}$ being in the Szegő class, we see that the result of (4.1.6) holds for a subset of OPUC from the Szegő class, and consequently a subset of OPUC from the Nevai class.

For the random sum $f_n(z) = \sum_{k=0}^{n} \eta_k p_j(z)$, where $\{\eta_j\}$ are complex valued i.i.d. Gaussian random variables and $\{p_j(z)\}$ are a polynomial basis with the degree of $p_j(z)$ equal to j, Corollary 3.4.2 of [49] gives the following formulas for the correlation functions:

$$\rho_n^{(m)}(z_1, \ldots, z_m) = \frac{\mathrm{Perm}(C - B^* A^{-1} B)}{\pi^m \mathrm{Det(A)}}, \tag{4.1.9}$$

where Perm$(\cdot)$ denotes the permanent of a matrix, B^* is the conjugate transpose of the matrix B, and

$$A = \left[\mathbb{E}\left[p_n(z_i)\overline{p_n(z_j)}\right]\right]_{0\le i,j\le m} = \left[\sum_{k=0}^{n} p_k(z_i)\overline{p_k(z_j)}\right]_{0\le i,j\le m} := [K_n(z_i, z_j)]_{0\le i,j\le m},$$
$$B = \left[\mathbb{E}\left[p_n(z_i)\overline{p_n'(z_j)}\right]\right]_{0\le i,j\le m} = \left[\sum_{k=0}^{n} p_k(z_i)\overline{p_k'(z_j)}\right]_{0\le i,j\le m} := [K_n^{(0,1)}(z_i, z_j)]_{0\le i,j\le m},$$
$$C = \left[\mathbb{E}\left[p_n'(z_i)\overline{p_n'(z_j)}\right]\right]_{0\le i,j\le m} = \left[\sum_{k=0}^{n} p_k'(z_i)\overline{p_k'(z_j)}\right]_{0\le i,j\le m} := [K_n^{(1,1)}(z_i, z_j)]_{0\le i,j\le m},$$

with the second equality in each row above following from the property that the random variables $\{\eta_j\}$ have mean zero and variance one. We remark that for the second correlation function with $m = 2$ in the above and $z_1 = z$ and $z_2 = w$, in this case we have

$$\begin{aligned}
\det A &= K_n(z,z)K_n(w,w) - K_n(z,w)K_n(w,z)\\
&= \sum_{j=0}^{n} |p_j(z)|^2 \sum_{j=0}^{n} |p_j(w)|^2 - \left|\sum_{j=0}^{n} p_j(z)\overline{p_j(w)}\right|^2\\
&= \sum_{k=1}^{n} |p_0(z)p_k(w) - p_0(w)p_k(z)|^2 + \sum_{k=2}^{n} |p_1(z)p_k(w) - p_1(w)p_k(z)|^2\\
&\quad + \cdots + \sum_{k=n-2}^{n} |p_{n-3}(z)p_k(w) - p_{n-3}(w)p_k(z)|^2 + |p_n(z)p_{n-1}(w) - p_n(w)p_{n-1}(z)|^2\\
&\ge |p_0(z)p_1(w) - p_0(w)p_1(z)|^2.
\end{aligned}$$

As $\{p_j\}$ is a polynomial basis with $\deg p_j = j$ for all $j \in \mathbb{N} \cup \{0\}$, we have $p_0(z) = c$ and $p_1(z) = az + b$, for some constants a, b, c, with $a, c \ne 0$. Thus

$$|p_0(z)p_1(w) - p_0(w)p_1(z)|^2 = |c(aw+b) - c(az+b)|^2 = |ca(w-z)|^2.$$

Hence we see that $\det A > 0$ for all $z \neq w$. As $\rho_n^{(2)}(z,z) = 0$, we see that the representation

$$\rho_n^{(2)}(z,w) = \frac{\text{Perm}(C - B^* A^{-1} B)}{\pi^m \text{Det(A)}}$$

is valid everywhere for all random polynomials spanned by a polynomial basis.

Expanding the permanent and the determinant in the definition of second correlation function, one sees that $\rho_n^{(2)}(z,w)$ can be written as

$$\pi^2 \rho_n^{(2)}(z,w) = f_n(z,w) f_n(w,z) + g_n(z,w) g_n(w,z), \tag{4.1.10}$$

where

$$\begin{aligned} f_n(z,w) = & \frac{K_n^{(1,1)}(z,z)}{(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2)^{1/2}} \\ & + \frac{2\text{Re}\left(K_n(z,w)\overline{K_n^{(0,1)}(z,z)}K_n^{(0,1)}(w,z)\right)}{(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2)^{3/2}} \\ & - \frac{K_n(w,w)|K_n^{(0,1)}(z,z)|^2 + K_n(z,z)|K_n^{(0,1)}(w,z)|^2}{(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2)^{3/2}}, \end{aligned}$$

and

$$\begin{aligned} g_n(z,w) = & \frac{K_n^{(1,1)}(z,w)}{(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2)^{1/2}} \\ & + \frac{K_n(z,w)\overline{K_n^{(0,1)}(z,z)}K_n^{(0,1)}(w,w) + \overline{K_n(z,w)K_n^{(0,1)}(w,z)}K_n^{(0,1)}(z,w)}{(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2)^{3/2}} \\ & - \frac{K_n(w,w)\overline{K_n^{(0,1)}(z,z)}K_n^{(0,1)}(z,w) + K_n(z,z)\overline{K_n^{(0,1)}(w,z)}K_n^{(0,1)}(w,w)}{(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2)^{3/2}}. \end{aligned}$$

As an observation, we note that

$$\begin{aligned} f_n(z,\overline{z}) = & \frac{K_n^{(1,1)}(z,z)}{(K_n(z,z)K_n(z,z) - |K_n(z,\overline{z})|^2)^{1/2}} \\ & + \frac{2\text{Re}\left(K_n(z,\overline{z})K_n^{(1,0)}(z,z)\overline{K_n^{(1,0)}(z,\overline{z})}\right)}{(K_n(z,z)K_n(z,z) - |K_n(z,\overline{z})|^2)^{3/2}} \\ & - \frac{K_n(z,z)|K_n^{(1,0)}(z,z)|^2 + K_n(z,z)|K_n^{(1,0)}(z,\overline{z})|^2}{(K_n(z,z)K_n(z,z) - |K_n(z,\overline{z})|^2)^{3/2}} \\ = & \pi \rho_n^{(0,1)}(z), \end{aligned}$$

where $\rho_n^{(0,1)}(z)$ was given by Vanderbei [115] as the first intensity function for the expected number of purely complex zeros for a random sum $g_n(z) = \sum_{k=0}^{n} \xi_k p_k(z)$ with $\{\xi_k\}$ being i.i.d. real-valued standard Gaussian random variables.

For our remaining results of this section we consider random orthogonal polynomials of the form

$$P_n(z) = \sum_{j=0}^{n} \eta_j \varphi_j(z), \quad z \in \mathbb{C}, \tag{4.1.11}$$

where $\{\eta_j\}$ are i.i.d. complex-valued standard Gaussian random variables, and $\{\varphi_j\}$ are orthogonal polynomials from the Nevai class that are real valued on the real line.

While the formula for the second correlation function associated to the random orthogonal polynomial (4.1.11) is rather complicated, its limit as n tends to infinity has a striking simplicity.

Theorem 4.1.3 *When z and w are both in the unit disk or both in the the exterior of the unit disk, the second correlation function for the random orthogonal polynomial* (4.1.11) *satisfies*

$$\lim_{n\to\infty} \rho_n^{(2)}(z,w) = \frac{1}{\pi^2}\left(\frac{1}{(1-|z|^2)^2(1-|w|^2)^2} - \frac{1}{|1-z\overline{w}|^4}\right), \tag{4.1.12}$$

where the above convergence takes place locally uniformly.

Our next theorem gives the limiting value of the variance of the number of zeros of the random orthogonal polynomial (4.1.11) in an annulus

$$A(s,t) = \{z \in \mathbb{C} : 0 \le s < |z| < t\},$$

that does not contain the unit disk.

Theorem 4.1.4 *The random orthogonal polynomial* (4.1.11) *possesses the property that*

$$\lim_{n\to\infty} \mathrm{Var}[N_n(A(s,t))] = \begin{cases} \dfrac{(t^2-s^2)[1-s^2(t^4(2+s^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)}, & A(s,t) \subsetneq \mathbb{D}, \\[2ex] \dfrac{(t^2-s^2)[1-t^2(s^4(2+t^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)}, & A(s,t) \subsetneq \mathbb{C}\setminus\overline{\mathbb{D}}. \end{cases}$$

We note that taking $s = 0$ and $t < 1$ in the above theorem, we achieve that the random orthogonal polynomial (4.1.11) gives

$$\lim_{n\to\infty} \mathrm{Var}[N(D(0,t))] = \frac{t^2}{1-t^4},$$

where $D(0,t) = \{z \in \mathbb{C} : |z| < t\}$.

Given our results concerning random orthogonal polynomials $P_n(z) = \sum_{k=0}^{n} \eta_k \varphi_k$, where $\{\eta_k\}$ are complex-valued random variables, and $\{\varphi_k\}$ are OPUC, we end this section on some conjectures that the author intends to work on in the future.

Conjecture (Due to Igor Pritsker): For suitable assumptions on $\{\eta_k\}$ and $\{\varphi_k\}$, and and c a positive constant, we have

$$\lim_{n\to\infty} \frac{\mathrm{Var}[N_n(\mathbb{D})]}{n} = c.$$

Conjecture: For suitable assumptions on $\{\eta_k\}$ and $\{\varphi_k\}$, it follows that

$$\frac{N_n(\mathbb{D}) - \mathbb{E}[N_n(\mathbb{D})]}{\sqrt{\mathrm{Var}[N_n(\mathbb{D})]}} \xrightarrow{d} N(0,1), \quad \text{as} \quad n \to \infty.$$

4.2 The variance of the number of zeros for a random power series

Let

$$f(z) = \sum_{k=0}^{\infty} \eta_k z^k,$$

$D(0,r) := \{z \in \mathbb{C} : |z| < r < 1\}$, and $N(D(0,r))$ be the number of zeros of f in $D(0,r)$. The goal of this section is to extend the results given by Peres and Virág (Corollary 3. (iii) of [88]) which state that for the random power series $f(z)$, it follows that

$$\mu_r = \mathbb{E}[N(D(0,r))] = \frac{r^2}{1-r^2}, \quad \text{and} \quad \sigma_r^2 = \mathrm{Var}[N(D(0,r))] = \frac{r^2}{1-r^4}. \qquad (4.2.1)$$

We note that Peres and Virág also showed that $(N_r - \mu_r)/\sigma_r$ converges in distribution to the standard normal as $r \uparrow 1$.

We will generalize the basis of the random power series $f(z)$ to be OPUC from the Szegő class and prove the analogs of (4.2.1) for this extension in annuli (further generalizing from disks) in the unit circle. Before defining the Szegő class, recall that the OPUC associated to non-negative 2π-periodic weight $W(\theta)$ that is Lebesgue integrable on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} W(\theta)\, d\theta > 0,$$

are polynomials $\{\varphi_k\}$ that satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}W(e^{i\theta})\, d\theta = \delta_{nm}, \quad n, m \in \mathbb{N} \cup \{0\}.$$

We will assume that the weight function $W(\theta)$ is an even function to ensure that that coefficients of $\{\varphi_k\}$ are real. We say that a collection of OPUC $\{\varphi_k\}$ are from the Szegő class if their associated weight function $W(\theta)$ possesses the property that

$$\int_{-\pi}^{\pi} |\log W(\theta)|\, d\theta$$

is finite. Note that in case considered by Peres and Virág, the weight function for the monomials is $W(\theta) = 1$ which is clearly in the Szegő class.

Our object of study in this section is the random orthogonal series

$$P(z) = \sum_{k=0}^{\infty} \eta_k \varphi_k(z), \tag{4.2.2}$$

where $\{\varphi_k\}$ are OPUC associated to an even weight function from the Szegő class. When $z \in \mathbb{D}$, due to Szegő (see equation (12.3.17) p. 303 in [111]) we know that $\sum_{k=0}^{\infty} |\varphi_k(z)|^2 < \infty$, where the convergence takes place locally uniform. Hence by the random variables $\{\eta_k\}$ being i.i.d. mean zero with variance one, by Lemma 2.2.3 of [49], the random power series (4.2.2) converges almost surely locally uniformly on the unit disk and thus defines a holomorphic function. This gives that the random orthogonal series (4.2.2) is of the class of functions known as Gaussian Analytic Functions (GAF).

To prove the next two theorems, we rely on a formula for the n-point correlation function provided by Hough, Krishnapur, Peres, and B. Virág ([49],Corollary 3.4.2.) which we state at (4.3.65). In the cases when $n = 1$ and $n = 2$, their formula gives a representation for the first intensity function and second intensity function. We denote the number of zeros of $P(z)$ in a measurable region $\Omega \subset \mathbb{D}$ by $N(\Omega)$, and the first and second correlation functions by $\rho^{(1)}(z)$ and $\rho^{(2)}(z, w)$ respectively.

Theorem 4.2.1 *When $z \in \mathbb{D}$, the first correlation function for the random orthogonal series* (4.2.2) *is given by*

$$\rho^{(1)}(z) = \frac{1}{\pi(1 - |z|^2)^2}. \tag{4.2.3}$$

Theorem 4.2.2 *The second correlation function for the random orthogonal series* (4.2.2) *satisfies*

$$\rho^{(2)}(z, w) = \frac{1}{\pi^2}\left(\frac{1}{(1 - |z|^2)^2(1 - |w|^2)^2} - \frac{1}{|1 - z\overline{w}|^4}\right). \tag{4.2.4}$$

Remark: For $K(z, w) := \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}$, to appeal to Corollary 3.4.2 of [49], it is required that when $z \neq w$, we have

$$K(z, z)K(w, w) - |K(z, w)|^2 \neq 0.$$

To see that this is indeed so, since equation (12.3.17) of page 303 in [111], gives

$$K(z, w) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(w)} = \frac{1}{D(z)\overline{D(w)}(1 - z\overline{w})}, \tag{4.2.5}$$

where

$$D(\xi) = D(W; \xi) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log W(\theta)\, \frac{1 + \xi e^{-i\theta}}{1 - \xi e^{-i\theta}}\, d\theta\right\}, \tag{4.2.6}$$

it follows that

$$\begin{aligned}K(z,z)K(w,w) - |K(z,w)|^2 &= \frac{1}{|D(z)|^2(1-|z|^2)}\frac{1}{|D(w)|^2(1-|w|^2)} \\ &\quad - \frac{1}{|D(z)D(w)|^2|1-z\overline{w}|^2} \\ &= \frac{|z|^2+|w|^2-\overline{z}w-z\overline{w}}{|D(z)D(w)|^2(1-|z|^2)(1-|w|^2)|1-z\overline{w}|^2} \\ &= \frac{|z-w|^2}{|D(z)D(w)|^2(1-|z|^2)(1-|w|^2)|1-z\overline{w}|^2},\end{aligned}$$

which is zero precisely when $z = w$.

As a further remark, when $z = w$ in the formula for the second correlation function given in Corollary 3.4.2. of [49], in the case when the spanning functions of the random orthogonal series are OPUC from the Szegő class, it follows that $\lim_{w\to z}\rho^{(2)}(z,w) = 0$.

Corollary 4.2.1 *For the random orthogonal series* (4.2.2)*, when* $\Omega \subset \mathbb{D}$ *is a measurable set we have*

$$\text{Var}[N(\Omega)] = \frac{1}{\pi}\int_\Omega \frac{1}{(1-|z|^2)^2}\,dz - \frac{1}{\pi^2}\int_\Omega\int_\Omega \frac{1}{|1-z\bar{w}|^4}\,dz\,dw. \tag{4.2.7}$$

Since the integrals on the right hand side of (4.2.7) were computed in Theorem 4.1.4, we immediately obtain our next result.

Corollary 4.2.2 *The random orthogonal series* (4.2.2) *possesses the property that when* $t < 1$ *we have*

$$\mathbb{E}[N(A(s,t))] = \frac{t^2-s^2}{(1-t^2)(1-s^2)}, \tag{4.2.8}$$

and

$$\text{Var}[N(A(s,t))] = \frac{(t^2-s^2)[1-s^2(t^4(2+s^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)}. \tag{4.2.9}$$

As a consequence of the above corollary, taking $s = 0$ in the above we achieve

$$\mathbb{E}[N(D(0,t))] = \frac{t^2}{1-t^2} \quad \text{and} \quad \text{Var}[N(D(0,t))] = \frac{t^2}{1-t^4}, \tag{4.2.10}$$

thus giving the extension of the formulas for the expectation and variance provided in Corollary 3 (iii) of [88] to hold for random orthogonal series spanned OPUC from the Szegő class.

Given our explicit formulas for the first and second correlation functions in Corollary 4.2.2, we make the following

Conjecture : For the random orthogonal series (4.2.2) we have

$$\frac{N(A(s,t)) - \mathbb{E}[N(A(s,t))]}{\sqrt{\mathrm{Var}[N(A(s,t))]}} \to N(0,1), \quad \text{as } t \uparrow 1, \tag{4.2.11}$$

where the convergence takes place in a distributional sense.

We now give some insights for making such a conjecture. From Theorems 4.2.1 and 4.2.2, for the correlation functions for the random orthogonal series (4.2.2) we know that

$$\rho^{(1)}(z_1) = \frac{1}{\pi(1-|z_1|^2)^2} = \mathbb{K}(z_1, z_1)$$

and

$$\begin{aligned}\rho^{(2)}(z_1, z_2) &= \frac{1}{\pi^2}\left(\frac{1}{(1-|z_1|^2)^2(1-|z_2|^2)^2} - \frac{1}{|1-z_1\overline{z_2}|^4}\right) \\ &= \det\begin{bmatrix}\mathbb{K}(z_1,z_1) & \mathbb{K}(z_1,z_2) \\ \mathbb{K}(z_2,z_1) & \mathbb{K}(z_2,z_2)\end{bmatrix},\end{aligned}$$

where $\mathbb{K}(z,w) = (\pi)^{-1}\sum_{k=0}^{\infty}(k+1)z^k\overline{w}^k$ is the Bergman kernel. Note that the Bergman kernel satisfies the Hermitian symmetry property $\overline{\mathbb{K}(z,w)} = \mathbb{K}(w,z)$. If one could conclude that

$$\rho^{(n)}(z_1, \dots, z_n) = \det[\mathbb{K}(z_i, z_j)]_{0 \le i,j \le n}, \tag{4.2.12}$$

we would know that we have a determinantal random process. Furthermore by each of the annuli $A(s,t)$ having compact closure in the disk, and as $t \to 1$ it follows that

$$\lim_{t\uparrow 1} \mathrm{Var}[N(A(s,t))] = \lim_{t\uparrow 1} \frac{(t^2-s^2)[1-s^2(t^4(2+s^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)} = +\infty, \tag{4.2.13}$$

appealing to a result of Soshnikov (c.f. Theorem on page 497 of [106], which is specified to our needed situation above in [107] p. 174), the result of the conjecture would follow. Thus the conjecture actually rest on the conjecture that (4.2.12) holds true.

4.3 Proofs for Chapter 4

4.3.1 Proofs for Section 4.1

Proof of Theorem 4.1.1. Under the assumptions of the theorem, by Corollary 2.1.1 we have

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right). \tag{4.3.1}$$

Then

$$\begin{aligned}
&\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta))^2 - \left(\frac{\beta-\alpha}{2\pi}\right)^2\right|\right] \\
&\qquad = \mathbb{E}\left[\left|\left(\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right)\left(\tau_n(A_r(\alpha,\beta) + \frac{\beta-\alpha}{2\pi}\right)\right|\right] \\
&\qquad \leq 2\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \\
&\qquad = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right).
\end{aligned} \tag{4.3.2}$$

Thus

$$\mathbb{E}\left[\tau_n(A_r(\alpha,\beta))^2\right] \leq \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right) + \left(\frac{\beta-\alpha}{2\pi}\right)^2.$$

Therefore

$$
\begin{aligned}
\frac{\operatorname{Var}[N_n(A_r(\alpha,\beta))]}{n^2} &= \frac{1}{n^2}\mathbb{E}[N_n(A_r(\alpha,\beta))^2] - \frac{1}{n^2}\left(\mathbb{E}[N_n(A_r(\alpha,\beta))]\right)^2 \qquad (4.3.3)\\
&= \mathbb{E}[\tau_n(A_r(\alpha,\beta))^2] - \mathbb{E}[\tau_n(A_r(\alpha,\beta))]^2\\
&\le \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right) + \left(\frac{\beta-\alpha}{2\pi}\right)^2 - \mathbb{E}[\tau_n(A_r(\alpha,\beta))]^2\\
&= \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right)\\
&\quad + \left(\frac{\beta-\alpha}{2\pi} - \mathbb{E}[\tau_n(A_r(\alpha,\beta))]\right)\left(\frac{\beta-\alpha}{2\pi} + \mathbb{E}[\tau_n(A_r(\alpha,\beta))]\right)\\
&\le \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right) + \mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right]\left(\frac{\beta-\alpha}{2\pi}+1\right)\\
&= \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right) + \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right)\\
&= \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right),
\end{aligned}
$$

which gives the result of the theorem. ■

Proof of Theorem 4.1.2. The hypothesis of the theorem allows us to apply (2.2.7) of Corollary 2.2.1

$$
\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right), \quad n\to\infty, \qquad (4.3.4)
$$

when the recurrence coefficients are absolutely summable, and (2.2.8) of Corollary 2.2.1

$$
\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] = \mathcal{O}\left(\max\left\{\sqrt{\frac{\log n}{n}}, \varepsilon_n^{1/4}\right\}\right),
$$

when the measure of orthogonality μ is regular in the sense of Ullman-Stahl-Totik. Given the above bounds, repeating the calculations (4.3.2) and (4.3.3) one obtains the desired result. ■

Proof of Theorem 4.1.3. For a collection of OPUC $\{\varphi_j\}_{j\ge 0}$, the Christoffel-Darboux formula (c.f. Theorem 2.2.7, p. 124 of [104]) states that for $z, w \in \mathbb{C}$ with $\bar{w}z \ne 1$, we have

$$K_n(z,w) = \sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j(w)} = \frac{\overline{\varphi_{n+1}^*(w)}\varphi_{n+1}^*(z) - \overline{\varphi_{n+1}(w)}\varphi_{n+1}(z)}{1-\bar{w}z}, \tag{4.3.5}$$

where $\varphi_n^*(z) = z^n\overline{\varphi_n\left(\frac{1}{\bar{z}}\right)}$.

Before obtaining our representations of the kernels, let us note that since we are assuming that the OPUC are real valued on the real line, when using conjugation we have that $\overline{\varphi_j(z)} = \varphi_j(\bar{z})$ for all $j = 0, 1, \ldots,$ and all $z \in \mathbb{C}$.

Taking the derivative of (4.3.5) with respect to $\overline{w}$ yields

$$K_n^{(0,1)}(z,w) = \sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j'(w)} = \frac{S_n(z,w)}{1-z\overline{w}} + \frac{zK_n(z,w)}{1-z\overline{w}}, \tag{4.3.6}$$

with

$$S_n(z,w) = \overline{(\varphi_{n+1}^*)'(w)}\varphi_{n+1}^*(z) - \overline{\varphi_{n+1}'(w)}\varphi_{n+1}(z). \tag{4.3.7}$$

Differentiating (4.3.6) with respect to z gives

$$\begin{aligned} K_n^{(1,1)}(z,w) &= \sum_{j=0}^{n} \varphi_j'(z)\overline{\varphi_j'(w)} \\ &= \frac{R_n(z,w)(1-z\overline{w}) + z\overline{S_n(w,z)} + \overline{w}S_n(z,w) + (1+z\overline{w})K_n(z,w)}{(1-z\overline{w})^2}, \end{aligned} \tag{4.3.8}$$

with

$$R_n(z,w) = \overline{(\varphi_{n+1}^*)'(w)}(\varphi_{n+1}^*)'(z) - \overline{\varphi_{n+1}'(w)}\varphi_{n+1}'(z). \tag{4.3.9}$$

Let us rewrite (4.1.10) as

$$\pi^2\rho_n^{(2)}(z,w) = \frac{\tilde{f}_n(z,w)\tilde{f}_n(w,z) + \tilde{g}_n(z,w)\tilde{g}_n(w,z)}{(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2)^3} \tag{4.3.10}$$

where

$$\tilde{f}_n(z,w) = K_n^{(1,1)}(z,z)(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2) \tag{4.3.11}$$

$$+ 2\mathrm{Re}\left(K_n(z,w)\overline{K_n^{(0,1)}(z,z)}K_n^{(0,1)}(w,z)\right) \tag{4.3.12}$$

$$- K_n(w,w)|K_n^{(0,1)}(z,z)|^2 + K_n(z,z)|K_n^{(0,1)}(w,z)|^2, \tag{4.3.13}$$

and

$$\tilde{g}_n(z,w) = K_n^{(1,1)}(z,w)(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2) \tag{4.3.14}$$

$$+ K_n(z,w)\overline{K_n^{(0,1)}(z,z)}K_n^{(0,1)}(w,w) + \overline{K_n(z,w)K_n^{(0,1)}(w,z)}K_n^{(0,1)}(z,w) \tag{4.3.15}$$

$$- K_n(w,w)\overline{K_n^{(0,1)}(z,z)}K_n^{(0,1)}(z,w) + K_n(z,z)\overline{K_n^{(0,1)}(w,z)}K_n^{(0,1)}(w,w). \tag{4.3.16}$$

We now introduce the notation that $b_n(z) := \varphi_n(z)/\varphi_n^*(z)$. Observe that since the OPUC $\{\varphi_n\}$ have real coefficients, we have $b_n^{-1}(z) = b_n(1/z)$. Thus the condition of the OPUC being from the Nevai class can be restated as

$$\begin{cases} b_n(z) \to 0, & \text{locally uniformly in } |x| < 1, \\ b_n^{-1}(z) \to 0, & \text{locally uniformly in } |x| > 1. \end{cases} \tag{4.3.17}$$

Consequently

$$\frac{\varphi_n'(z)\varphi_n^*(z) - \varphi_n(z)(\varphi_n^*)'(z)}{\phi_n^2(z)} = \begin{cases} b_n'(z) \to 0, & \text{locally uniformly in } |x| < 1, \\ -(b_n^{-1})'(z) \to 0, & \text{locally uniformly in } |x| > 1, \end{cases} \tag{4.3.18}$$

where

$$\phi_n(z) := \begin{cases} \varphi_n^*(z), & |x| < 1, \\ \varphi_n(z), & |x| > 1. \end{cases}$$

Hence appealing to the above and (4.3.5), we see that as $n \to \infty$ the denominator of $\pi\rho_n^{(2)}(z,w)$ is

$$\begin{aligned}(K_n(z,z)&K_n(w,w) - |K_n(z,w)|^2)^3 \\ &= |\phi_{n+1}(z)\phi_{n+1}(w)|^6 \left[\left(\frac{1}{(1-|z|^2)(1-|w|^2)} - \frac{1}{|1-z\overline{w}|^2}\right)^3 + o(1)\right]\end{aligned} \tag{4.3.19}$$

We now find the asymptotic for $\tilde{f}_n(z,w)$. Set

$$S_1(z,w) = (4.3.11), \quad S_2(z,w) = (4.3.12), \quad S_3(z,w) = (4.3.13).$$

Using (4.3.6) and (4.3.8) we see that

$$S_1(z,w) = \frac{(1+|z|^2)K_n(z,z)}{(1-|z|^2)^2}(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2) \tag{4.3.20}$$

$$+ \frac{2\mathrm{Re}\,(\overline{z}S_n(z,z))}{(1-|z|^2)^2}(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2) \tag{4.3.21}$$

$$+ \frac{R_n(z,z)}{1-|z|^2}(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2), \tag{4.3.22}$$

$$S_2(z,w) = -\frac{|w|^2K_n(z,z)|K_n(z,w)|^2}{|1-z\overline{w}|^2} - \frac{|z|^2K_n(z,z)^2K_n(w,w)}{(1-|z|^2)^2} \tag{4.3.23}$$

$$- \frac{2K_n(z,z)\mathrm{Re}\,(\overline{w}S_n(w,z)K_n(z,w))}{|1-z\overline{w}|^2} - \frac{2K_n(z,z)K_n(w,w)\mathrm{Re}\,(\overline{z}S_n(z,z))}{(1-|z|^2)^2} \tag{4.3.24}$$

$$- \frac{K_n(z,z)|S_n(w,z)|^2}{|1-z\overline{w}|^2} - \frac{K_n(w,w)|S_n(z,z)|^2}{(1-|z|^2)^2}, \tag{4.3.25}$$

$$S_3(z,w) = \frac{K_n(z,z)|K_n(z,w)|^2}{1-|z|^2}\left(\frac{1-|zw|^2}{|1-z\overline{w}|^2} - 1\right) \tag{4.3.26}$$

$$+ \frac{2|K_n(z,w)|^2}{1-|z|^2}\mathrm{Re}\left(\frac{\overline{w}S_n(z,z)}{1-z\overline{w}}\right) \tag{4.3.27}$$

$$+ \frac{2K_n(z,z)}{1-|z|^2}\mathrm{Re}\left(\frac{\overline{z}K_n(z,w)S_n(w,z)}{1-\overline{z}w}\right) \tag{4.3.28}$$

$$+ \frac{2}{1-|z|^2}\mathrm{Re}\left(\frac{K_n(z,w)\overline{S_n(z,z)}S_n(w,z)}{1-\overline{z}w}\right), \tag{4.3.29}$$

where we have made use of the identity $2\mathrm{Re}(z\overline{w}) = 1 + |zw|^2 - |1-z\overline{w}|^2$ to get the expression in parentheses of (4.3.26) in the shape it is in.

We now define

$$\Sigma_{n,1}(z,w) := (4.3.20) + (4.3.23) + (4.3.26)$$

$$\Sigma_{n,2}(z,w) := (4.3.21) + (4.3.24) + (4.3.27) + (4.3.28)$$

$$\Sigma_{n,3}(z,w) := (4.3.22) + (4.3.25) + (4.3.29).$$

Simplifying and then appealing to (4.3.5), (4.3.17), and (4.3.18), we see that

$$\begin{aligned}\Sigma_{n,1}(z,w) &= \frac{K_n(z,z)^2 K_n(w,w)}{(1-|z|^2)^2} - \frac{K_n(z,z)|K_n(z,w)|^2}{1-|z|^2}\left(\frac{2}{1-|z|^2} + \frac{|w|^2-1}{|1-z\overline{w}|^2}\right) \\ &= |\phi_{n+1}(z)|^4|\phi_{n+1}(w)|^2\Bigg[\frac{1}{(1-|z|^2)^4(1-|w|^2)} \\ &\qquad - \frac{1}{(1-|z|^2)|1-z\overline{w}|^2}\left(\frac{2}{1-|z|^2} + \frac{|w|^2-1}{|1-z\overline{w}|^2}\right) + o(1)\Bigg] \\ &= |\phi_{n+1}(z)|^4|\phi_{n+1}(w)|^2\left(\frac{|z-w|^4}{(1-|z|^2)^4(1-|w|^2)|1-z\overline{w}|^4} + o(1)\right). \qquad (4.3.30)\end{aligned}$$

Turning now to the asymptotic for $\Sigma_{n,2}(z,w)$, observe the first summand of (4.3.21) and second summand of (4.3.24) cancel algebraically, and that the sum of the second summand in (4.3.21) and the first summand of (4.3.27) simplify to

$$2\frac{|K_n(z,w)|^2}{(1-|z|^2)^2}\mathrm{Re}\left(\frac{(\overline{w}-\overline{z})S_n(z,z)}{1-z\overline{w}}\right). \qquad (4.3.31)$$

The sum of the first summand of (4.3.24) and (4.3.28) collect to give

$$\frac{2K_n(z,z)}{(1-|z|^2)|1-z\overline{w}|^2}\mathrm{Re}\left((\overline{z}-\overline{w})K_n(z,w)S_n(w,z)\right). \qquad (4.3.32)$$

Combining (4.3.31) with (4.3.32) then appealing to (4.3.5), (4.3.6), (4.3.8) to simplify the expressions, and finally using the asymptotics of (4.3.17) and (4.3.18) on sees

$$\begin{aligned}\Sigma_{n,2}(z,w) &= \frac{1}{(1-|z|^2)^2|1-z\overline{w}|^2} \\ &\quad \cdot 2\mathrm{Re}\Big[(\overline{w}-\overline{z})K_n(z,w)\left(\overline{\varphi_{n+1}(z)(\varphi_{n+1}^*)'(z)} - \overline{\varphi_{n+1}'(z)\varphi_{n+1}^*(z)}\right) \\ &\qquad \cdot \left(\varphi_{n+1}(z)\varphi_{n+1}^*(w) - \varphi_{n+1}^*(z)\varphi_{n+1}(w)\right)\Big] \\ &= o(|\phi_{n+1}(z)|^4|\phi_{n+1}(w)|^2). \qquad (4.3.33)\end{aligned}$$

For the sum $\Sigma_{n,3}(z,w)$, first notice using (4.3.5) the first summand of (4.3.22) and the second summand of (4.3.25) sum to

$$-\frac{K_n(w,w)}{(1-|z|^2)^2}|(\varphi_{n+1}^*)'(z)\varphi_{n+1}(z) - \varphi_{n+1}^*(z)\varphi_{n+1}'(z)|^2. \qquad (4.3.34)$$

Appealing to (4.3.5), the remaining summand of (4.3.22) simplifies as

$$\begin{aligned}&(|\varphi_{n+1}'(z)|^2-|(\varphi_{n+1}^*)'(z)|^2)\\&\cdot\frac{|\varphi_{n+1}^*(z)\varphi_{n+1}^*(w)|^2+|\varphi_{n+1}(z)\varphi_{n+1}(w)|^2-2\mathrm{Re}\left(\varphi_{n+1}^*(z)\overline{\varphi_{n+1}^*(w)\varphi_{n+1}(z)}\varphi_{n+1}(w)\right)}{(1-|z|^2)|1-z\overline{w}|^2},\end{aligned}\tag{4.3.35}$$

the remaining summand of (4.3.25) reduces to

$$\begin{aligned}&\frac{2K_n(z,z)\mathrm{Re}\left(\overline{(\varphi_{n+1}^*)'(z)}\varphi_{n+1}^*(w)\varphi_{n+1}'(z)\overline{\varphi_{n+1}(w)}\right)}{|1-z\overline{w}|^2}\\&\quad-\frac{1}{(1-|z|^2)|1-z\overline{w}|^2}\Big[|(\varphi_{n+1}^*)'(z)|^2\left(|\varphi_{n+1}^*(z)\varphi_{n+1}^*(w)|^2-|\varphi_{n+1}(z)\varphi_{n+1}^*(w)\right)\\&\qquad\qquad+|\varphi_{n+1}(z)|^2\left(|\varphi_{n+1}^*(z)\varphi_{n+1}(w)|^2-|\varphi_{n+1}(z)\varphi_{n+1}(w)\right)\Big],\end{aligned}\tag{4.3.36}$$

and (4.3.29) can be written as

$$\begin{aligned}&\frac{2|(\varphi_{n+1}^*)'(z)|^2}{(1-|z|^2)|1-z\overline{w}|^2}\left(|\varphi_{n+1}^*(z)\varphi_{n+1}^*(w)|^2-\mathrm{Re}\left(\overline{\varphi_{n+1}^*(z)}\varphi_{n+1}(z)\varphi_{n+1}^*(w)\overline{\varphi_{n+1}(w)}\right)\right)\\&-\frac{2|\varphi_{n+1}'(z)|^2}{(1-|z|^2)|1-z\overline{w}|^2}\left(|\varphi_{n+1}(z)\varphi_{n+1}(w)|^2-\mathrm{Re}\left(\overline{\varphi_{n+1}^*(z)}\varphi_{n+1}(z)\varphi_{n+1}^*(w)\overline{\varphi_{n+1}(w)}\right)\right)\\&-\frac{2\left(|\varphi_{n+1}^*(w)|^2-|\varphi_{n+1}(w)|^2\right)\mathrm{Re}\left(\overline{(\varphi_{n+1}^*)'(z)}\varphi_{n+1}^*(z)\varphi_{n+1}'(z)\overline{\varphi_{n+1}(z)}\right)}{(1-|z|^2)|1-z\overline{w}|^2}\\&-\frac{2K_n(z,z)\mathrm{Re}\left(\overline{(\varphi_{n+1}^*)'(z)}\varphi_{n+1}^*(w)\varphi_{n+1}'(z)\overline{\varphi_{n+1}(w)}\right)}{|1-z\overline{w}|^2}.\end{aligned}\tag{4.3.37}$$

Simplifying the sum of expressions (4.3.35), (4.3.36), and (4.3.37), then combining with (4.3.34) we achieve

$$\begin{aligned}\Sigma_{n,3}(z,w)&=\frac{K_n(w,w)|(\varphi_{n+1}^*)'(z)\varphi_{n+1}(z)-\varphi_{n+1}^*(z)\varphi_{n+1}'(z)|^2}{1-|z|^2}\left(\frac{1-|w|^2}{|1-z\overline{w}|^2}-\frac{1}{1-|z|^2}\right)\\&=o(|\phi_{n+1}(z)|^4|\phi_{n+1}(w)|^2).\end{aligned}\tag{4.3.38}$$

Thus the sum of (4.3.30), (4.3.33), and (4.3.38), combine to give

$$\tilde{f}_n(z,w)=|\phi_{n+1}(z)|^4|\phi_{n+1}(w)|^2\left(\frac{|z-w|^4}{(1-|z|^2)^4(1-|w|^2)|1-z\overline{w}|^4}+o(1)\right).\tag{4.3.39}$$

The calculations for $\tilde{g}_n(z,w)$ are done in a similar fashion as for $\tilde{f}(z,w)$. Due the complication nature of the computations, we see fit to include the complete derivation of $\tilde{g}_n(z,w)$. Let

$$S_4(z,w) = (4.3.14), \quad S_5(z,w) = (4.3.15), \quad S_6(z,w) = (4.3.16).$$

From (4.3.6) and (4.3.8) it follows that

$$S_4(z,w) = \frac{(1+z\overline{w})K_n(z,w)}{(1-z\overline{w})^2}(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2) \tag{4.3.40}$$

$$+ \frac{z\overline{S_n(w,z)} + \overline{w}S_n(z,w)}{(1-z\overline{w})^2}(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2) \tag{4.3.41}$$

$$+ \frac{R_n(z,w)}{1-z\overline{w}}(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2), \tag{4.3.42}$$

$$S_5(z,w) = -\frac{|z|^2K_n(z,z)K_n(w,w)K_n(z,w)}{(1-|z|^2)(1-z\overline{w})} - \frac{|w|^2K_n(z,z)K_n(w,w)K_n(z,w)}{(1-|w|^2)(1-z\overline{w})} \tag{4.3.43}$$

$$- \frac{K_n(z,w)}{1-z\overline{w}}\left(\frac{zK_n(w,w)\overline{S_n(z,z)}}{1-|z|^2} + \frac{\overline{w}K_n(z,z)S_n(w,w)}{1-|w|^2}\right) \tag{4.3.44}$$

$$- \frac{K_n(z,z)K_n(w,w)}{1-z\overline{w}}\left(\frac{\overline{z}S_n(z,w)}{1-|z|^2} + \frac{w\overline{S_n(w,z)}}{1-|w|^2}\right) \tag{4.3.45}$$

$$- \frac{K_n(w,w)\overline{S_n(z,z)}S_n(z,w)}{(1-|z|^2)(1-z\overline{w})} - \frac{K_n(z,z)S_n(w,w)\overline{S_n(w,z)}}{(1-|w|^2)(1-z\overline{w})}, \tag{4.3.46}$$

$$S_6(z,w) = \frac{\overline{z}wK_n(z,z)K_n(w,w)K_n(z,w)}{(1-|z|^2)(1-|w|^2)} + \frac{z\overline{w}K(z,w)|K(z,w)|^2}{(1-z\overline{w})^2} \tag{4.3.47}$$

$$+ K(z,w)\left(\frac{wK_n(w,w)\overline{S_n(z,z)} + \overline{z}K_n(z,z)S_n(w,w)}{(1-|z|^2)(1-|w|^2)}\right) \tag{4.3.48}$$

$$+ K(z,w)\left(\frac{\overline{wK_n(z,w)}S_n(z,w) + z\overline{K_n(z,w)S_n(w,z)}}{(1-z\overline{w})^2}\right) \tag{4.3.49}$$

$$+ \frac{K_n(z,w)\overline{S_n(z,z)}S_n(w,w)}{(1-|z|^2)(1-|w|^2)} + \frac{\overline{K_n(z,w)S_n(w,z)}S_n(z,w)}{(1-z\overline{w})^2}. \tag{4.3.50}$$

We now set

$$\Sigma_{n,4}(z,w) := (4.3.40) + (4.3.43) + (4.3.47)$$

$$\Sigma_{n,5}(z,w) := (4.3.41) + (4.3.44) + (4.3.45) + (4.3.48) + (4.3.49)$$

$$\Sigma_{n,6}(z,w) := (4.3.42) + (4.3.46) + (4.3.50).$$

Simplifying then using the relations (4.3.5), (4.3.17), and (4.3.18) yields

$$\begin{aligned}
\Sigma_{n,4}(z,w) &= K_n(z,z)K_n(w,w)K_n(z,w)\left(\frac{1+\overline{z}(w-2z)+\overline{w}(z-2w+w|z|^2)}{(1-|z|^2)(1-|w|^2)(1-z\overline{w})^2}\right) \\
&\quad - \frac{K_n(z,w)|K_n(z,w)|^2}{(1-z\overline{w})^2} \\
&= |\phi_{n+1}(z)|^2|\phi_{n+1}(w)|^2\phi_{n+1}(z)\overline{\phi_{n+1}(w)} \\
&\quad \cdot \left(\frac{1+\overline{z}(w-2z)+\overline{w}(z-2w+w|z|^2)}{(1-|z|^2)^2(1-|w|^2)^2(1-z\overline{w})^3} - \frac{1}{(1-z\overline{w})^4(1-\overline{z}w)} + o(1)\right) \\
&= |\phi_{n+1}(z)|^2|\phi_{n+1}(w)|^2\phi_{n+1}(z)\overline{\phi_{n+1}(w)} \\
&\quad \cdot \left(\frac{-|z-w|^4}{(1-|z|^2)^2(1-|w|^2)^2(1-z\overline{w})^4(1-\overline{z}w)} + o(1)\right). \qquad (4.3.51)
\end{aligned}$$

To begin calculating $\Sigma_{n,5}$, first we combine (4.3.41), (4.3.45), and (4.3.49), to give

$$\frac{K_n(z,z)K_n(w,w)}{(1-z\overline{w})^2}\left(\frac{(z-w)\overline{S_n(w,z)}}{1-|w|^2} - \frac{(\overline{z}-\overline{w})S_n(z,w)}{1-|z|^2}\right). \qquad (4.3.52)$$

Now combing (4.3.44) with (4.3.48) yields

$$\begin{aligned}
&\frac{K_n(z,w)K_n(w,w)\overline{S_n(z,z)}}{1-|z|^2}\left(\frac{w}{1-|w|^2} - \frac{z}{1-z\overline{w}}\right) \\
&\quad + \frac{K_n(z,w)K_n(z,z)S_n(w,w)}{1-|w|^2}\left(\frac{\overline{z}}{1-|z|^2} - \frac{\overline{w}}{1-z\overline{w}}\right). \qquad (4.3.53)
\end{aligned}$$

Summing (4.3.52) with (4.3.53) then using (4.3.5), (4.3.6), (4.3.8) to simplify the expressions, and finally appealing the asymptotics of (4.3.17) and (4.3.18) on sees

$$\begin{aligned}
\Sigma_{n,5}(z,w) = {} & \frac{1}{(1-|z|^2)(1-|w|^2)(1-z\overline{w})^2} \\
& \cdot \Big[(z-w)K_n(w,w)(\varphi_{n+1}(z)(\varphi^*_{n+1})'(z) - \varphi^*_{n+1}(z)\varphi'_{n+1}(z)) \\
& \quad \cdot (\overline{\varphi^*_{n+1}(z)\varphi_{n+1}(w)} - \overline{\varphi_{n+1}(z)\varphi^*_{n+1}(w)}) \\
& \quad + (\overline{z}-\overline{w})K_n(z,z)(\overline{\varphi_{n+1}(w)(\varphi^*_{n+1})'(w)} - \overline{\varphi^*_{n+1}(w)\varphi'_{n+1}(w)}) \\
& \quad \cdot (\varphi^*_{n+1}(z)\varphi_{n+1}(w) - \varphi_{n+1}(z)\varphi^*_{n+1}(w))\Big] \\
= {} & o(|\phi_{n+1}(z)|^2|\phi_{n+1}(w)|^2\phi_{n+1}(z)\overline{\phi_{n+1}(w)}). \qquad (4.3.54)
\end{aligned}$$

Turning now to $\Sigma_{n,6}(z,w)$, combining the first summands of (4.3.42) and (4.3.46) and using (4.3.5), (4.3.6), (4.3.8) to further simplify gives

$$\begin{aligned}
& \frac{K_n(w,w)}{(1-|z|^2)(1-z\overline{w})}(((\varphi^*_{n+1})'(z)\varphi_{n+1}(z) - \varphi^*_{n+1}(z)\varphi'_{n+1}(z)) \\
& \qquad \cdot (\overline{\varphi'_{n+1}(w)\varphi^*_{n+1}(z)} - \overline{\varphi_{n+1}(z)(\varphi^*_{n+1})'(w)}). \qquad (4.3.55)
\end{aligned}$$

From the relations (4.3.5), (4.3.6), and (4.3.8), the sum of second summands of (4.3.42) and (4.3.50) reduces to

$$\begin{aligned}
& \frac{\overline{K_n(z,w)}}{(1-z\overline{w})^2}(\varphi_{n+1}(z)(\varphi^*_{n+1})'(z) - \varphi'_{n+1}(z)\varphi^*_{n+1}(z)) \\
& \qquad \cdot (\overline{(\varphi^*_{n+1})'(w)\varphi_{n+1}(w)} - \overline{\varphi^*_{n+1}(w)\varphi'_{n+1}(w)}). \qquad (4.3.56)
\end{aligned}$$

Using (4.3.5), (4.3.6), and (4.3.8), second summand of (4.3.46) with first summand of (4.3.50) combine to yield

$$\begin{aligned}
& \frac{(1-\overline{z}w)\overline{K_n(z,w)}}{(1-|z|^2)(1-|w|^2)(1-z\overline{w})}(\varphi_{n+1}(z)(\varphi^*_{n+1})'(z) - \varphi^*_{n+1}(z)\varphi'_{n+1}(z)) \\
& \qquad \cdot (\overline{\varphi^*_{n+1}(w)\varphi'_{n+1}(w)} - \overline{\varphi_{n+1}(w)(\varphi^*_{n+1})'(w)}). \qquad (4.3.57)
\end{aligned}$$

Combining (4.3.55), (4.3.56), and (4.3.57), then again appealing (4.3.5), (4.3.6), (4.3.8) to simplify the expressions, and using the asymptotics of (4.3.17) and (4.3.18) it follows that

$$\begin{aligned}\Sigma_{n,6}(z,w) = \frac{\overline{K_n(z,w)}}{1-z\overline{w}}&((\varphi_{n+1}^*)'(z)\varphi_{n+1}(z) - \varphi_{n+1}^*(z)\varphi_{n+1}'(z))\\ &\cdot (\overline{\varphi_{n+1}^*(w)\varphi_{n+1}'(w)} - \overline{\varphi_{n+1}(w)(\varphi_{n+1}^*)'(w)})\\ &\cdot \left(\frac{1-\overline{z}w}{(1-|z|^2)(1-|w|^2)} - \frac{1}{1-z\overline{w}}\right)\\ &= o(|\phi_{n+1}(z)|^2|\phi_{n+1}(w)|^2\phi_{n+1}(z)\overline{\phi_{n+1}(w)}). \end{aligned} \tag{4.3.58}$$

Summing (4.3.51), (4.3.54), and (4.3.58), we see that

$$\begin{aligned}\tilde{g}_n(z,w) = &|\phi_{n+1}(z)\phi_{n+1}(w)|^2\phi_{n+1}(z)\overline{\phi_{n+1}(w)}\\ &\cdot\left(\frac{-|z-w|^4}{(1-|z|^2)^2(1-|w|^2)^2(1-z\overline{w})^4(1-\overline{z}w)} + o(1)\right). \end{aligned} \tag{4.3.59}$$

Combining (4.3.10), (4.3.19), (4.3.39), and (4.3.59), we therefore achieve

$$\begin{aligned}\pi^2\rho_n^{(2)}(z,w) = &\left(|\phi_{n+1}(z)\phi_{n+1}(w)|^6\left(\left(\frac{1}{(1-|z|^2)(1-|w|^2)} - \frac{1}{|1-z\overline{w}|^2}\right)^3 + o(1)\right)\right)^{-1}\\ &\cdot\Bigg[|\phi_{n+1}(z)|^4|\phi_{n+1}(w)|^2\left(\frac{|z-w|^4}{(1-|z|^2)^4(1-|w|^2)|1-z\overline{w}|^4} + o(1)\right)\\ &\quad\cdot|\phi_{n+1}(w)|^4|\phi_{n+1}(z)|^2\left(\frac{|z-w|^4}{(1-|w|^2)^4(1-|z|^2)|1-z\overline{w}|^4} + o(1)\right)\\ &\quad+|\phi_{n+1}(z)\phi_{n+1}(w)|^2\phi_{n+1}(z)\overline{\phi_{n+1}(w)}\\ &\quad\cdot\left(\frac{-|z-w|^4}{(1-|z|^2)^2(1-|w|^2)^2(1-z\overline{w})^4(1-\overline{z}w)} + o(1)\right)\\ &\quad\cdot|\phi_{n+1}(w)\phi_{n+1}(z)|^2\phi_{n+1}(w)\overline{\phi_{n+1}(z)}\\ &\quad\cdot\left(\frac{-|z-w|^4}{(1-|z|^2)^2(1-|w|^2)^2(1-\overline{z}w)^4(1-z\overline{w})} + o(1)\right)\Bigg]\\ = &\frac{1}{(1-|z|^2)^2(1-|w|^2)^2} - \frac{1}{|1-z\overline{w}|^4} + o(1), \quad \text{as} \quad n\to\infty. \end{aligned}$$

■

Proof of Theorem 4.1.4. Recall that for the random orthogonal polynomial (4.1.11) we have

$$\lim_{n\to\infty}\rho_n^{(1)}(z) = \frac{1}{\pi(1-|z|^2)^2} \tag{4.3.60}$$

locally uniformly for all $z \in \mathbb{C} \setminus \mathbb{T}$. Appealing to (4.0.3) gives

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{Var}[N_n(A(s,t))] &= \frac{1}{\pi}\int_{A(s,t)} \frac{1}{(1-|z|^2)^2} dA(z) \\
&+ \frac{1}{\pi^2}\int_{A(s,t)}\int_{A(s,t)} \left(\frac{1}{(1-|z|^2)^2(1-|w|^2)^2} - \frac{1}{|1-z\overline{w}|^4}\right) dA(z)dA(w) \\
&- \frac{1}{\pi^2}\int_{A(s,t)}\int_{A(s,t)} \frac{1}{(1-|z|^2)^2(1-|w|^2)^2} dA(z)dA(w) \\
&= \frac{1}{\pi}\int_{A(s,t)} \frac{1}{(1-|z|^2)^2} dA(z) \qquad (4.3.61) \\
&- \frac{1}{\pi^2}\int_{A(s,t)}\int_{A(s,t)} \frac{1}{(1-z\bar{w})^2(1-\bar{z}w)^2} dA(z)dA(w), \qquad (4.3.62)
\end{aligned}$$

where we have used that the convergence of (4.3.60) and (4.1.12) are locally uniform on annuli that do not contain the unit circle so that we can pass the limit over the integration. We remark that formally we need to consider a closed annulus the does not contain the unit circle in the above. However, since the measure associated to the integrals is Lebesgue area measure which is absolutely continuous, and the limiting values above are continuous functions away from the unit circle, we have that the boundary of the closed annulus has measure zero. Hence we just consider the open annulus $A(s,t)$.

Recall that (4.3.61) simply integrates as

$$\frac{1}{\pi}\int_{A(s,t)} \frac{1}{(1-|z|^2)^2}\ dz = \frac{t^2-s^2}{(1-t^2)(1-s^2)}. \qquad (4.3.63)$$

We will compute (4.3.62) separately depending on whether or not the annulus is contained within the unit disk. We first consider the case when the annulus is contained with in the unit disk. Since for $x, y \in \mathbb{D}$ we have

$$\frac{1}{(1-xy)^2} = \sum_{n=0}^{\infty}(n+1)(xy)^n,$$

switching to polar coordinates with $z = re^{i\theta}$ and $w = ue^{i\phi}$, the double integral (4.3.62) becomes

$$\frac{1}{\pi^2}\int_0^{2\pi}\int_s^t\int_0^{2\pi}\int_s^t \left(\sum_{k=0}^{\infty}(k+1)(ru)^k e^{ik(\theta-\phi)}\sum_{n=0}^{\infty}(n+1)(ru)^n e^{in(\phi-\theta)}\right) rudrd\theta dud\phi. \qquad (4.3.64)$$

By the convergence of the above series being locally uniform in the unit disk, we can interchange the infinite sums and integrals. After expanding the product of sums, any sum with $n \neq k$ will give

$$\int_0^{2\pi} e^{i(k-n)\theta} d\theta = 0,$$

and similarly for the terms with ϕ. Thus only the terms $n = k$ will survive the integration. This yields (4.3.64) is now

$$\begin{aligned} 4\int_s^t \int_s^t \sum_{k=0}^{\infty}(k+1)^2(ru)^{2k+1} drdu &= 2\int_s^t \sum_{k=0}^{\infty}(k+1)u^{2k+1}(t^{2k+2} - s^{2k+2})du \\ &= \sum_{k=0}^{\infty}(t^{2k+2} - s^{2k+2})^2 \\ &= \frac{t^4}{1-t^4} - 2\frac{(st)^2}{1-(st)^2} + \frac{s^4}{1-s^4} \\ &= \frac{(t^2-s^2)^2(1+(st)^2)}{(1-t^4)(1-s^4)(1-(st)^2)}. \end{aligned}$$

Therefore, combining the above with (4.3.63) we see that for $A(s,t) \subsetneq \mathbb{D}$

$$\begin{aligned} \mathrm{Var}[N_n(A(s,t))] &= \frac{1}{\pi}\int_{A(s,t)} \frac{1}{(1-|z|^2)^2}\, dz \\ &\quad - \frac{1}{\pi^2}\int_{A(s,t)}\int_{A(s,t)} \frac{1}{(1-z\bar{w})^2(1-\bar{z}w)^2}\, dz\, dw \\ &= \frac{t^2-s^2}{(1-t^2)(1-s^2)} - \left(\frac{(t^2-s^2)^2(1+(st)^2)}{(1-t^4)(1-s^4)(1-(st)^2)}\right) \\ &= \frac{(t^2-s^2)[1-s^2(t^4(2+s^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)}. \end{aligned}$$

When $A(s,t) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, notice that for $x, y \in \mathbb{C} \setminus \overline{\mathbb{D}}$ it follows that

$$\frac{1}{(1-xy)^2} = \frac{1}{(xy)^2(1-(xy)^{-1})^2} = \frac{1}{(xy)^2}\sum_{n=0}^{\infty}(n+1)(xy)^{-n}.$$

Thus setting $x = z = re^{i\theta}$ and $y = w = ue^{i\phi}$ we have

$$\frac{1}{(1-z\overline{w})^2}\frac{1}{(1-\overline{z}w)^2} = \frac{1}{(ru)^4}\sum_{k=0}^{\infty}(k+1)(ru)^{-k}e^{ik(\phi-\theta)}\sum_{n=0}^{\infty}(n+1)(ru)^{-n}e^{in(\theta-\phi)}.$$

As in the case in the unit disk, due to the orthogonality of the exponential function, within the integral of (4.3.62) only the terms when $n = k$ will give a nonzero integration to hence yield that (4.3.62) is

$$
\begin{aligned}
4\int_s^t \int_s^t \sum_{k=0}^{\infty}(k+1)^2(ru)^{-2k-3}drdu &= -2\int_s^t \sum_{k=0}^{\infty}(k+1)u^{-2k-3}(t^{-2k-2}-s^{-2k-2})du \\
&= \sum_{k=0}^{\infty}(t^{-2k-2}-s^{-2k-2})^2 \\
&= \frac{t^4}{1-t^4} - 2\frac{(st)^2}{1-(st)^2} + \frac{s^4}{1-s^4} \\
&= \frac{-(t^2-s^2)^2(1+(st)^2)}{(1-t^4)(1-s^4)(1-(st)^2)}.
\end{aligned}
$$

Combining the above with (4.3.63), we conclude that for $A(s,t) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ it follows that

$$
\begin{aligned}
\mathrm{Var}[N_n(A(s,t))] &= \frac{1}{\pi}\int_{A(s,t)} \frac{1}{(1-|z|^2)^2}\, dz \\
&\quad - \frac{1}{\pi^2}\int_{A(s,t)}\int_{A(s,t)} \frac{1}{(1-z\bar{w})^2(1-\bar{z}w)^2}\, dz\, dw \\
&= \frac{t^2-s^2}{(1-t^2)(1-s^2)} - \left(\frac{-(t^2-s^2)^2(1+(st)^2)}{(1-t^4)(1-s^4)(1-(st)^2)}\right) \\
&= \frac{t^2-s^2}{(1-t^2)(1-s^2)} + \frac{(t^2-s^2)^2(1+(st)^2)}{(1-t^4)(1-s^4)(1-(st)^2)} \\
&= \frac{(t^2-s^2)[1-t^2(s^4(2+t^2)-2)]}{(1-t^4)(1-s^4)(1-(st)^2)}.
\end{aligned}
$$

■

4.3.2 Proofs for Section 4.2

As in the case random sums with a finite index set, for GAF's we have that for a measurable set $\Omega \subset \mathbb{D}$,

$$\begin{aligned}\mathrm{Var}[N(\Omega)] &= \mathbb{E}[(N(\Omega))^2] - (\mathbb{E}[N(\Omega)])^2 \\ &= \mathbb{E}[N(\Omega)] - \mathbb{E}[N(\Omega)] + \mathbb{E}[(N(\Omega))^2] - (\mathbb{E}[N(\Omega)])^2 \\ &= \mathbb{E}[N(\Omega)] + \mathbb{E}[N(\Omega)(N(\Omega)-1)] - (\mathbb{E}[N(\Omega)])^2 \\ &= \int_\Omega \rho^{(1)}(z)\ dz + \int_\Omega\int_\Omega \rho^{(2)}(z,w)\ dz\ dw - \int_\Omega\int_\Omega \rho^{(1)}(z)\rho^{(1)}(w)\ dz\ dw,\end{aligned}$$

where the equality

$$\mathbb{E}[N(\Omega)(N(\Omega)-1)] = \int_\Omega\int_\Omega \rho^{(2)}(z,w)\ dz\ dw$$

is a known result.

Due to Hough, Krishnapur, Peres, and Virág (c.f. Corollary 3.4.2. on page 40 of [49], or Proposition 8 of [88]) there are general formulas for correlation functions of GAF's which state that

$$\rho^{(n)}(z_1,\dots,z_n) = \frac{\mathrm{Perm}(C - BA^{-1}B^*)}{\pi^n \mathrm{Det(A)}}, \tag{4.3.65}$$

where

$$\begin{aligned}A &= \left[\mathbb{E}[p(z_i)\overline{p(z_j)}]\right]_{1\le i,j\le n} := [K(z_i,z_j)]_{1\le i,j\le n}, \\ B &= \left[\mathbb{E}[p'(z_i)\overline{p(z_j)}]\right]_{1\le i,j\le n} := [K^{(1,0)}(z_i,z_j)]_{1\le i,j\le n}, \\ C &= \left[\mathbb{E}[p'(z_i)\overline{p'(z_j)}]\right]_{1\le i,j\le n} := [K^{(1,1)}(z_i,z_j)]_{1\le i,j\le n},\end{aligned}$$

with $\mathrm{Perm}(\cdot)$ denoting the permanent of a matrix, and B^* the conjugate transpose of the matrix B.

In the case for the random power series (4.2.2), due to random variables being i.i.d. of mean zero and variance one, and along with Equation (12.3.17) of page 303 in [111], the kernels that arises in the matrix A are of the form

$$K(z,w)=\sum_{k=0}^{\infty}\varphi_k(z)\overline{\varphi_k(w)}=\frac{1}{D(z)\overline{D(w)}(1-z\overline{w})}, \tag{4.3.66}$$

where

$$D(\xi)=D(W;\xi)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\log W(\theta)\ \frac{1+\xi e^{-i\theta}}{1-\xi e^{-i\theta}}\ d\theta\right\}. \tag{4.3.67}$$

We note that under the assumption that the weight function $W(\theta)$ is from the Szegő class, the function $D(\xi)$ is analytic and non-vanishing in $\mathbb{D}$ (see Theorem 2.4.1. (i) p. 144 of [104]). The other kernels for the matrices B and C in correlation functions are the following derivatives of (4.3.66):

$$K^{(1,0)}(z,w)=\sum_{k=0}^{\infty}\varphi_k'(z)\overline{\varphi_k(w)}=\left(\frac{\overline{w}}{1-z\overline{w}}-\frac{D'(z)}{D(z)}\right)K(z,w), \tag{4.3.68}$$

and

$$K^{(1,1)}(z,w)=\sum_{k=0}^{\infty}\varphi_k'(z)\overline{\varphi_k'(w)}=\left(\frac{z}{1-z\overline{w}}-\frac{\overline{D'(w)}}{\overline{D(w)}}\right)K^{(1,0)}(z,w)+\frac{K(z,w)}{(1-z\overline{w})^2}. \tag{4.3.69}$$

Proof of Theorem 4.2.1. In the case when $n=1$, the matrices which make up the formula (4.3.65) are $A=K(z,z)$, $B=K^{(1,0)}(z,z)$, and $C=K^{(1,1)}(z,z)$. Thus (4.3.65) can be written as

$$\begin{aligned}\rho^{(1)}(z)&=\frac{K^{(1,1)}(z,z)-K^{(1,0)}(z,z)(K(z,z))^{-1}\overline{K^{(1,0)}(z,z)}}{\pi K(z,z)}\\&=\frac{K(z,z)K^{(1,1)}(z,z)-|K^{(1,0)}(z,z)|^2}{\pi K(z,z)^2}.\end{aligned} \tag{4.3.70}$$

Evaluating (4.3.66), (4.3.68), and (4.3.69) on the diagonal, after simplifying it follows that

$$K(z,z)K^{(1,1)}(z,z)-|K^{(1,0)}(z,z)|^2=\frac{1}{|D(z)|^4(1-|z|^2)^4}=\frac{1}{(1-|z|^2)^2}K(z,z)^2. \tag{4.3.71}$$

Combining (4.3.70) with (4.3.71) we achieve the result of the theorem. ■

Proof of Theorem 4.2.2. To ease notation, we will write $z_1 = z$ and $z_2 = w$. The matrices which make up the formula for the second correlation function given by (4.3.65) are

$$A = \begin{bmatrix} K(z,z) & K(z,w) \\ K(w,z) & K(w,w) \end{bmatrix}, \quad B = \begin{bmatrix} K^{(1,0)}(z,z) & K^{(1,0)}(z,w) \\ K^{(1,0)}(w,z) & K^{(1,0)}(w,w) \end{bmatrix},$$

$$C = \begin{bmatrix} K^{(1,1)}(z,z) & K^{(1,1)}(z,w) \\ K^{(1,1)}(w,z) & K^{(1,1)}(w,w) \end{bmatrix}.$$

Setting

$$\widetilde{A} = \begin{bmatrix} K(w,w) & -K(z,w) \\ -K(w,z) & K(z,z) \end{bmatrix},$$

equation (4.3.65) can be written as

$$\rho^{(2)}(z,w) = \frac{\mathrm{Perm}(\mathrm{Det}(A)C - B\widetilde{A}B^*)}{\pi^2(\mathrm{Det(A)})^3}. \tag{4.3.72}$$

After the matrix multiplication, we see that

$$\mathrm{Perm}(\mathrm{Det}(A)C - B\widetilde{A}B^*) = f(z,w)f(w,z) + g(z,w)g(w,z), \tag{4.3.73}$$

where

$$\begin{aligned} f(z,w) = {} & (K(z,z)K(w,w) - |K(z,w)|^2)K^{(1,1)}(z,z) - |K^{(1,0)}(z,z)|^2 K(w,w) \\ & + 2\mathrm{Re}\left[K^{(1,0)}(z,w)K(w,z)\overline{K^{(1,0)}(z,z)}\right] - |K^{(1,0)}(z,w)|^2 K(z,z), \end{aligned} \tag{4.3.74}$$

and

$$\begin{aligned} g(z,w) = {} & (K(z,z)K(w,w) - |K(z,w)|^2)K^{(1,1)}(z,w) - K(w,w)K^{(1,0)}(z,z)\overline{K^{(1,0)}(w,z)} \\ & + K(w,z)K^{(1,0)}(z,w)\overline{K^{(1,0)}(w,z)} + K(z,w)K^{(1,0)}(z,z)\overline{K^{(1,0)}(w,w)} \\ & - K(z,z)K^{(1,0)}(z,w)\overline{K^{(1,0)}(w,w)}. \end{aligned} \tag{4.3.75}$$

Substituting the relations (4.3.66), (4.3.68), and (4.3.69), we see that the terms which make up $f(z,w)$ are

$$\begin{aligned}(K(z,z)K(w,w) - |K(z,w)|^2)K^{(1,1)}(z,z)& \\ &= K(z,z)\left(K(z,z)K(w,w) - |K(z,w)|^2\right) \\ &\quad \cdot \left[\frac{1+|z|^2}{(1-|z|^2)^2} - \frac{2\mathrm{Re}\,(zD'(z)/D(z))}{1-|z|^2} + \left|\frac{D'(z)}{D(z)}\right|^2\right], \end{aligned} \tag{4.3.76}$$

$$\begin{aligned}-|K^{(1,0)}(z,z)|^2 K(w,w) &= -(K(z,z))^2 K(w,w) \\ &\quad \cdot \left[\frac{|z|^2}{(1-|z|^2)^2} - \frac{2\mathrm{Re}\,(zD'(z)/D(z))}{1-|z|^2} + \left|\frac{D'(z)}{D(z)}\right|^2\right], \end{aligned} \tag{4.3.77}$$

$$\begin{aligned}&2\mathrm{Re}\left[K^{(1,0)}(z,w)K(w,z)\overline{K^{(1,0)}(z,z)}\right] \\ &\quad = 2K(z,z)|K(z,w)|^2 \mathrm{Re}\left[\frac{z\overline{w}}{(1-|z|^2)(1-z\overline{w})} - \frac{\overline{w}\overline{D'(z)}}{\overline{D(z)}(1-z\overline{w})} - \frac{zD'(z)}{D(z)(1-|z|^2)}\right] \\ &\qquad + 2K(z,z)|K(z,w)|^2 \left|\frac{D'(z)}{D(z)}\right|^2, \end{aligned} \tag{4.3.78}$$

and

$$\begin{aligned}-|K^{(1,0)}(z,w)|^2 K(z,z) &= -K(z,z)|K(z,w)|^2 \\ &\quad \cdot \left(\frac{|w|^2}{|1-z\overline{w}|^2} - 2\mathrm{Re}\left[\frac{\overline{w}D'(z)}{D(z)(1-z\overline{w})}\right] + \left|\frac{D'(z)}{D(z)}\right|^2\right). \end{aligned} \tag{4.3.79}$$

Combining (4.3.76), (4.3.77), (4.3.78), (4.3.79), we see that from equation (4.3.74) we have

$$\begin{aligned}f(z,w) &= \frac{(K(z,z))^2 K(w,w)}{(1-|z|^2)^2} \\ &\quad - K(z,z)|K(z,w)|^2 \left[\frac{1+|z|^2}{(1-|z|^2)^2} - \frac{2\mathrm{Re}\,(z\overline{w}/(1-z\overline{w}))}{1-|z|^2} + \frac{|w|^2}{|1-z\overline{w}|^2}\right] \\ &= \frac{|z-w|^4}{|D(z)|^4|D(w)|^2(1-|w|^2)(1-|z|^2)^4|1-z\overline{w}|^4}. \end{aligned} \tag{4.3.80}$$

Using the relations (4.3.66), (4.3.68), and (4.3.69), the terms within $g(z,w)$ are

$$\begin{aligned}
&(K(z,z)K(w,w) - |K(z,w)|^2)K^{(1,1)}(z,w) \\
&\qquad = K(z,w)\left(K(z,z)K(w,w) - |K(z,w)|^2\right) \\
&\qquad\quad \cdot \left[\frac{1+z\overline{w}}{(1-z\overline{w})^2} - \frac{1}{1-z\overline{w}}\left(\frac{zD'(z)}{D(z)} + \frac{\overline{w}\overline{D'(w)}}{\overline{D(w)}}\right) + \frac{D'(z)\overline{D'(w)}}{D(z)\overline{D(w)}}\right],
\end{aligned} \tag{4.3.81}$$

$$\begin{aligned}
&- K(w,w)K^{(1,0)}(z,z)\overline{K^{(1,0)}(w,z)} \\
&= -K(z,z)K(w,w)K(z,w) \\
&\quad \cdot \left[\frac{|z|^2}{(1-|z|^2)(1-z\overline{w})} - \frac{\overline{z}\overline{D'(w)}}{(1-|z|^2)\overline{D(w)}} - \frac{zD'(z)}{(1-z\overline{w})D(z)} + \frac{D'(z)\overline{D'(w)}}{D(z)\overline{D(w)}}\right],
\end{aligned} \tag{4.3.82}$$

$$\begin{aligned}
&K(w,z)K^{(1,0)}(z,w)\overline{K^{(1,0)}(w,z)} \\
&= K(z,w)|K(z,w)|^2\left[\frac{z\overline{w}}{(1-z\overline{w})^2} - \frac{1}{1-z\overline{w}}\left(\frac{\overline{wD'(w)}}{\overline{D(w)}} + \frac{zD'(z)}{D(z)}\right) + \frac{D'(z)\overline{D'(w)}}{D(z)\overline{D(w)}}\right],
\end{aligned} \tag{4.3.83}$$

$$\begin{aligned}
&K(z,w)K^{(1,0)}(z,z)\overline{K^{(1,0)}(w,w)} \\
&= K(z,z)K(w,w)K(z,w) \\
&\quad \cdot \left[\frac{\overline{z}w}{(1-|z|^2)(1-|w|^2)} - \frac{\overline{zD'(w)}}{(1-|z|^2)\overline{D(w)}} - \frac{wD'(z)}{(1-|w|^2)D(z)}\frac{D'(z)\overline{D'(w)}}{D(z)\overline{D(w)}}\right]
\end{aligned} \tag{4.3.84}$$

and

$$\begin{aligned}
&- K(z,z)K^{(1,0)}(z,w)\overline{K^{(1,0)}(w,w)} \\
&= -K(z,z)K(w,w)K(z,w) \\
&\quad \cdot \left[\frac{|w|^2}{(1-|w|^2)(1-z\overline{w})} - \frac{\overline{wD'(w)}}{(1-z\overline{w})\overline{D(w)}} - \frac{wD'(z)}{(1-|w|^2)D(z)} + \frac{D'(z)\overline{D'(w)}}{D(z)\overline{D(w)}}\right].
\end{aligned} \tag{4.3.85}$$

Combining (4.3.81), (4.3.82), (4.3.83), (4.3.84), (4.3.85), we see that

$$\begin{aligned}
g(z,w) = &-\frac{K(z,w)|K(z,w)|^2}{(1-z\overline{w})^2} \\
&+ K(z,z)K(w,w)K(z,w) \cdot \Bigg[\frac{1+z\overline{w}}{(1-z\overline{w})^2} - \frac{|z|^2}{(1-|z|^2)(1-z\overline{w})} \\
&\qquad + \frac{\overline{z}w}{(1-|z|^2)(1-|w|^2)} - \frac{|w|^2}{(1-|w|^2)(1-z\overline{w})}\Bigg] \\
= &\frac{-|z-w|^4}{D(z)\overline{D(w)}|D(z)D(w)|^2(1-|w|^2)^2(1-|z|^2)^2(1-z\overline{w})^4(1-\overline{z}w)}. \qquad (4.3.86)
\end{aligned}$$

Thus from (4.3.80) and (4.3.86) we achieve

$$\begin{aligned}
\operatorname{Perm}(\operatorname{Det}(A)C - B\widetilde{A}B^*) &= f(z,w)f(w,z) + g(z,w)g(w,z) \\
&= \frac{|z-w|^8}{|D(z)D(w)|^6(1-|z|^2)^5(1-|w|^2)^5|1-z\overline{w}|^8} \\
&\quad + \frac{|z-w|^8}{|D(z)D(w)|^6(1-|z|^2)^4(1-|w|^2)^4|1-z\overline{w}|^{10}} \\
&= \left(\frac{1}{(1-|z|^2)^2(1-|w|^2)^2} - \frac{1}{|1-z\overline{w}|^4}\right) \\
&\quad \cdot \left(\frac{1}{|D(z)D(w)|^2(1-|z|^2)(1-|w|^2)} \right. \\
&\qquad \left. - \frac{1}{|D(z)D(w)|^2|1-z\overline{w}|^2}\right)^3 \\
&= \left(\frac{1}{(1-|z|^2)^2(1-|w|^2)^2} - \frac{1}{|1-z\overline{w}|^4}\right)(\operatorname{Det}(A))^3.
\end{aligned}$$

Therefore, combining the above with (4.3.72) we arrive at the desired expression for the second correlation function. ■

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APPENDICES

Derivation of the Intensity Function for a Random Polynomial with I.I.D. Complex-Valued Standard Gaussian Random Variables Spanned by a Polynomial Basis

In this Appendix we derive the intensity function for the expected number of zeros of a random polynomial with complex-valued i.i.d. standard Gaussian random variables spanned by a polynomial basis given in Chapter 3 (3.2.6). We note that the formula was also derived by Shiffman and Zelditch [100], Ledoan [66] (assuming the spanning functions are entire functions that are real-valued on the real line), and Peres and Virá [88] (assuming the spanning functions were entire functions). The proof we give follows the method given by Vanderbei [115] for the case when the random variables are real-valued i.i.d. standard Gaussian, and the spanning functions are entire functions that are real-valued on the real line. Assuming that the spanning functions of random polynomial are a polynomial basis, we are able to give an alternate proof than given in [100], [66], and [88] that fully justifies all steps (some of which that were previously deemed ”tedious, but doable” with no indication on how one would proceed).

To be specific of the result, let $\{f_j\}$ be a sequence of polynomials such that $\deg f_j = j$ and f_j, for $j \in \{0, 1, \ldots, n\}$. Set

$$P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z), \qquad z \in \mathbb{C}, \tag{A.1}$$

where n is a fixed integer, and $\eta_j = \alpha_j + i\beta_j$, $j = 0, 1, \ldots, n$, with $\{\alpha_j\}_{j=0}^{n}$ and $\{\beta_j\}_{j=0}^{n}$ being sequences of i.i.d. real-valued standard Gaussian random variables.

Theorem 0.0.1 *Let $P_n(z)$ be the random sum* (A.1) *with complex-valued i.i.d. Gaussian coefficients. For each Jordan region Ω, the intensity function $\rho_n^{(1)}(z)$ satisfies*

$$\mathbb{E}[N_n(\Omega)] = \int_\Omega \rho_n^{(1)}(x,y)\ dx\ dy,$$

with

$$\rho_n^{(1)}(x,y) = \rho_n^{(1)}(z) = \frac{K_n^{(1,1)}(z,z)K_n(z,z) - \left|K_n^{(0,1)}(z,z)\right|^2}{\pi\left(K_n(z,z)\right)^2}, \tag{A.2}$$

where

$$K_n(z,w) = \sum_{j=0}^{n} f_j(z)\overline{f_j(w)}, \qquad K_n^{(0,1)}(z,w) = \sum_{j=0}^{n} f_j(z)\overline{f_j'(w)}, \tag{A.3}$$

and

$$K_n^{(1,1)}(z,w) = \sum_{j=0}^{n} f_j'(z)\overline{f_j'(w)}. \tag{A.4}$$

The proof of this theorem consists of two main lemmas that relate the expected number of zeros of the random sum P_n to the kernels given in (A.3) and (A.4). To fully prove the first of these two lemmas, there are four additional lemmas required. Three of these gives a known result that completely explains what the complex Jacobian is in our situation. The other additional lemma shows the justification of the use of the complex form of the Fubini-Tonelli Theorem. Once these lemmas are established, the proof of Theorem 0.0.1 is presented.

As previously mentioned, to prove Theorem 0.0.1 we follow the method of proof given by Vanderbei in [115]. We note that many of the parts of the proof of this theorem are nearly identical to that given by Vanderbei in [115]. The one major difference in our proof is that since the random variables in the random sum are complex-valued, we will work with their real and imaginary parts, which in turn will make the covariance matrix we compute different than the one in the result by Vanderbei. The other major difference in our proof is that since we are restricting to a polynomial basis $\{f_j\}$ with $\deg f_j = j,\ j = 0, \ldots, n$, we are able to fully justify some parts of Vanderbei's method.

Our first needed lemma is the following:

Lemma 0.0.1 *For each Jordan region $\Omega \subset \mathbb{C}$, it follows that*

$$\mathbb{E}[N_n(\Omega)] = \frac{1}{2\pi i} \int_{\partial\Omega} \mathbb{E}\left[\frac{P_n'(z)}{P_n(z)}\right] \, dz. \tag{A.5}$$

Proof. Using the argument principle in [114] on page 79, we have an explicit formula for the random variable $N_n(\Omega)$ given by

$$N_n(\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{P_n'(z)}{P_n(z)} \, dz.$$

Taking the expectation and then interchanging the expectation and the contour integral via the complex form of the Fubini-Tonelli Theorem (Theorem 8.8, p. 164 of [95]) we obtain the desired result. A full proof of the justification of the exchange of the contour integral and the expectation is in the next two lemmas. ■

To justify the use of the Fubini-Tonelli we will use a change of variables that relates the zeros of the random sum P_n to the random variables $\{\eta_j\}$. In our setting, this change of variables will require knowing the complex Jacobian. The next three of these lemmas is to identify this Jacobian. The first lemma is given in [67]. For convenience of the reader provide a detailed proof of this lemma.

Lemma 0.0.2 *Let $U \subset \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}^n$ be a holomorphic function. Let $D_{\mathbb{R}}f(a)$ denote the real Jacobian matrix of f and*

$$Df(a) = \left[\frac{\partial f_j}{\partial z_k}(a)\right]_{1 \le j,k \le n},$$

which is the holomorphic derivative (or complex Jacobian Matrix) of f. Then

$$|\det Df(a)|^2 = \det D_{\mathbb{R}}f(a) \tag{A.6}$$

Proof. Observe that by $U \subset \mathbb{C}^n$ and $f : U \to \mathbb{C}^n$, the matrix $Df(a)$ will be an $n \times n$ complex matrix. Since f is holomorphic each $f_j(z_k) = u(z_k) + iv(z_k)$, where $z_k = x_k + iy_k$, for $j = 1, \ldots, n$ satisfies the Cauchy Riemann Equations. Hence we can write the entries in the matrix $Df(a)$ as

$$\frac{\partial f_j}{\partial z_k}(a) = \frac{\partial u_j}{\partial x_k}(a) + i\frac{\partial v_j}{\partial x_k}(a).$$

We can also write each entry $\frac{\partial f_j}{\partial z_k}(a)$ of $Df(a)$ as a 2×2 matrix of the form

$$\frac{\partial f_j}{\partial z_k}(a) = \begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

Using the above expression for each $j, k = 1, \ldots, n$, rewrite the matrix $Df(a)$ as $2n \times 2n$ real matrix $D_{\mathbb{R}} f(a)$. Let us now make a change of basis from $(x + iy, x + iy)$ to $(x + iy, x - iy)$ and denote the matrix relative to this basis change as M. Note that under this change of basis, the $2n \times 2n$ matrix M will take the following form:

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ -i & 0 & \cdots & 0 & i & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & -i & \cdots & 0 & 0 & i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -i & 0 & 0 & \cdots & i \end{pmatrix},$$

with inverse

$$M^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{i}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & -\frac{i}{2} \end{pmatrix}.$$

Furthermore, this change of basis matrix M gives us that

$$M^{-1} D_{\mathbb{R}} f(a) M = \begin{pmatrix} Df(a) & 0 \\ 0 & \overline{Df(a)} \end{pmatrix}.$$

Therefore

$$\begin{aligned}
\det(D_{\mathbb{R}}f(a)) &= \det(M)^{-1}\det(M)\det(D_{\mathbb{R}}f(a)) \\
&= \det(M^{-1})\det(D_{\mathbb{R}}f(a))\det(M) \\
&= \det(M^{-1}D_{\mathbb{R}}f(a)M) \\
&= \det(Df(a)\overline{Df(a)}) \\
&= \det(Df(a))\det(\overline{Df(a)}) \\
&= \det(Df(a))\overline{\det(Df(a))} \\
&= |\det(Df(a))|^2 .
\end{aligned}$$

■

We now use the previous lemma to compute the determinant of the real Jacobian matrix for a function that maps roots of a monic polynomial to it's coefficients. For this, let

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_{n-1}, a_{n-2}, \ldots, a_0$ are complex numbers, and z is a complex variable. Let $z_1, z_2, \ldots, z_n$ be the zeros of $p(z)$. By Vieta's formulas, we have the relationship between the coefficients and zeros is given by the following:

$$\begin{aligned}
-a_{n-1} &= z_1 + z_2 + \cdots + z_n, \\
a_{n-2} &= z_1z_2 + z_1z_3 + \cdots + z_2z_3 + z_2z_4 + \cdots + z_{n-1}z_n \\
&\vdots \\
(-1)^n a_0 &= z_1z_2\cdots z_n.
\end{aligned}$$

Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be the map

$$T(z_1, \ldots, z_n) = (a_{n-1}, \ldots, a_0).$$

Lemma 0.0.3 *For $p(z)$ and T as above, the determinant of the real Jacobian matrix for T is*

$$\prod_{1\le i<j\le n} |z_i - z_j|^2. \tag{A.7}$$

Proof. From Vieta's Equations we see that the determinant of the complex Jacobian matrix of T is

$$\det DT = \det \begin{pmatrix} -1 & -1 & \cdots & -1 \\ \sum_{i=2}^{n} z_i & \sum_{\substack{i=1\\ i\neq 2}}^{n} z_i & \cdots & \sum_{i=1}^{n-1} z_i \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n \prod_{i=2}^{n} z_i & (-1)^n \prod_{\substack{i=1\\ i\neq 2}}^{n} z_i & \cdots & (-1)^n \prod_{i=1}^{n-1} z_i \end{pmatrix}$$

$$= (-1)^{n+1}\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i=2}^{n} z_i & \sum_{\substack{i=1\\ i\neq 2}}^{n} z_i & \cdots & \sum_{i=1}^{n-1} z_i \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{i=2}^{n} z_i & \prod_{\substack{i=1\\ i\neq 2}}^{n} z_i & \cdots & \prod_{i=1}^{n-1} z_i \end{pmatrix}$$

$$:= (-1)^{n+1} \det A.$$

If two zeros z_i and z_j, where $i \neq j$, of $p(z)$ are such that $z_i = z_j$, then two columns in the above matrix will be equal. When this happens, the determinant above will be zero. Since z_i and z_j were two arbitrary zeros of $p(z)$, we have that $\prod_{1\le i<j\le n}(z_i - z_j)$ is a factor of $\det DT$.

When one computes the determinant of the matrix A by expanding the determinant along the first row, each term in the computed determinant we will have powers of the product of z_i's, $i = 1, \ldots, n$, such that the sum of these powers in each term is $\sum_{k=1}^{n}(k-1) = \frac{1}{2}n(n-1)$. That is, $\det A$, and consequently $\det DT$, will be a homogenous polynomial in the product of the z_i's, $i = 1, \ldots, n$, of degree $\frac{1}{2}n(n-1)$.

If one expands the product $\prod_{1\le i<j\le n}(z_i - z_j)$ we see that each term in this expanded product will also be a product of the z_i's, $i = 1, \dots, n$, with powers that sum to $\frac{1}{2}n(n-1)$. Hence $\prod_{1\le i<j\le n}(z_i - z_j)$ is also a homogenous polynomial of degree $\frac{1}{2}n(n-1)$.

Since both $\det A$ and $\prod_{1\le i<j\le n}(z_i - z_j)$ are homogenous polynomials of degree $\frac{1}{2}n(n-1)$, to show that $\det A = \prod_{1\le i<j\le n}(z_i - z_j)$ we must show the each coefficient of these polynomials are the same. Note that when expanding $\det A$ and $\prod_{1\le i<j\le n}(z_i - z_j)$, all terms that are of the form $\prod_{j=1}^{n} z_j$ cancel out. Hence term in each of the homogenous polynomials is only a product of $n-1$ distinct z_j's.

Recall that for a matrix $B = (b_{ij})_{1\le i,j\le n}$ we have the formula

$$\det B = \sum_{\substack{\sigma \text{ a permutations} \\ \text{of } \{1,2,\dots,n\}}} \operatorname{sign}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}.$$

Thus expanding out $\det A$ the coefficients of the homogenous polynomial are $(-1)^k$, $k = 0, \dots, n$. Similarly expanding out $\prod_{1\le i<j\le n}(z_i - z_j)$ the coefficients are $(-1)^k$, $k = 0, \dots, n$. Consequently each term in $\det A$ and $\prod_{1\le i<j\le n}(z_i - z_j)$ have the same coefficient. Hence

$$\det A = \prod_{1\le i<j\le n}(z_i - z_j),$$

and which gives us

$$\det DT = (-1)^{n+1} \det A = (-1)^{n+1} \prod_{1\le i<j\le n}(z_i - z_j).$$

Therefore, using Lemma 0.0.2 we obtain

$$\det D_{\mathbb{R}}T = |\det DT|^2 = \left|(-1)^{n+1} \prod_{1\le i<j\le n}(z_i - z_j)\right|^2 = \prod_{1\le i<j\le n} |z_i - z_j|^2.$$

■

We will now apply the previous lemmas to general polynomials in z. To this end, let

$$q(z) = \eta_n z^n + \eta_{n-1} z^{n-1} + \cdots + \eta_1 z + \eta_0,$$

where $\eta_n, \eta_{n-1}, \dots, \eta_0$ are complex numbers, z is a complex variable, and $\zeta_1, \zeta_2, \dots, \zeta_n$ are the zeros of $q(z)$. As before, using Vieta's formulas we have the relationship between the coefficients and zeros given by the following:

$$\begin{aligned}
-\frac{\eta_{n-1}}{\eta_n} &= \zeta_1 + \zeta_2 + \cdots + \zeta_n, \\
\frac{\eta_{n-2}}{\eta_n} &= \zeta_1\zeta_2 + \zeta_1\zeta_3 + \cdots + \zeta_2\zeta_3 + \zeta_2\zeta_4 + \cdots + \zeta_{n-1}\zeta_n \\
&\vdots \\
(-1)^n \frac{\eta_0}{\eta_n} &= \zeta_1\zeta_2 \cdots \zeta_n.
\end{aligned}$$

Let $\phi : \mathbb{C}^n \to \mathbb{C}^n$ be the map

$$\phi(\zeta_1, \dots, \zeta_n) = \left(\frac{\eta_{n-1}}{\eta_n}, \dots, \frac{\eta_0}{\eta_n} \right). \tag{A.8}$$

Lemma 0.0.4 *For $q(z)$ and ϕ as above, the determinant of the real Jacobian matrix of ϕ is*

$$|\eta_n|^{2n} \prod_{1 \le i < j \le n} |\zeta_i - \zeta_j|^2. \tag{A.9}$$

Proof. Using the Vieta's Equations, the determinant of the complex Jacobian matrix for ϕ is

$$\det D\phi = \det \begin{pmatrix} -\eta_n & -\eta_n & \cdots & -\eta_n \\ \eta_n \sum_{i=2}^{n} \zeta_i & \eta_n \sum_{\substack{i=1 \\ i\neq 2}}^{n} \zeta_i & \cdots & \eta_n \sum_{i=1}^{n-1} \zeta_i \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n \eta_n \prod_{i=2}^{n} \zeta_i & (-1)^n \eta_n \prod_{\substack{i=1 \\ i\neq 2}}^{n} \zeta_i & \cdots & (-1)^n \eta_n \prod_{i=1}^{n-1} \zeta_i \end{pmatrix}$$

$$= (-1)^{n+1} \eta_n^n \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i=2}^{n} \zeta_i & \sum_{\substack{i=1 \\ i\neq 2}}^{n} \zeta_i & \cdots & \sum_{i=1}^{n-1} \zeta_i \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{i=2}^{n} \zeta_i & \prod_{\substack{i=1 \\ i\neq 2}}^{n} \zeta_i & \cdots & \prod_{i=1}^{n-1} \zeta_i \end{pmatrix}.$$

Observe that the determinant above is exactly the determinant that was calculated in Lemma 0.0.3. Therefore using Lemma 0.0.2 and then Lemma 0.0.3 to compute the determinant of the above matrix, we obtain

$$\det D_{\mathbb{R}}\phi = \left| (-1)^{n+1} \eta_n^n \prod_{1 \le i < j \le n} (\zeta_i - \zeta_j) \right|^2 = |\eta_n|^{2n} \prod_{1 \le i < j \le n} |\zeta_i - \zeta_j|^2.$$

■

Justification of the use of Fubini-Tonelli. Since P_n and P_n' both depend on the random variables $\eta_0, \eta_1, \ldots, \eta_n$, we write $P_n(z, \vec{\eta})$ and $P_n'(z, \vec{\eta})$ where $\vec{\eta} := (\eta_0, \ldots, \eta_n)$. We denote the joint density function for the random variables $\eta_0, \ldots, \eta_n$ by

$$f(\eta_0, \eta_1, \ldots, \eta_n) = f(\vec{\eta}) = \frac{\exp\left(-\sum_{k=0}^{n} |\eta_k|^2\right)}{\pi^{n+1}}.$$

To use the Fubini-Tonelli Theorem we must show $\left| \frac{P_n'(z,\vec{\eta})}{P_n(z,\vec{\eta})} \right|$ is measurable and that

$$\int_{\mathbb{R}^{2(n+1)}} \int_{\partial\Omega} \left| \frac{P_n'(z, \vec{\eta})}{P_n(z, \vec{\eta})} \right| \; |dz| \; f(\vec{\eta}) \; dV < \infty. \tag{A.10}$$

Observe that since $\mathbb{E}[N_n(\Omega)]$ is a local property, it suffices to prove (A.10) when Ω is a disk $D := \{z \in \mathbb{C} : |z| < R\}$.

We will first prove (A.10) when $P_n(z)$ is just a random algebraic polynomial with complex coefficients, and then generalize to random polynomials spanned by polynomials $f_j(z)$ where $\deg f_j(z) = j$ for $j = 0, 1, \ldots, n$. To this end, let

$$P_n(z) = \eta_n z^n + \eta_{n-1} z^{n-1} + \cdots + \eta_0,$$

and let $\{\zeta_k\}_{k=1}^n = \{\zeta_k(\vec{\eta})\}_{k=1}^n$ be the zeros of $P_n(z)$. Then

$$\left|\frac{P_n'(z)}{P_n(z)}\right| = \left|\sum_{k=1}^n \frac{1}{z-\zeta_k}\right| \leq \sum_{k=1}^n \left|\frac{1}{z-\zeta_k}\right|.$$

From the above we see that if we can show that all the quotients $\left|\frac{1}{z-\zeta_k}\right|$, $k = 1, \ldots, n$ satisfy

$$\int_{\mathbb{R}^{2(n+1)}} \int_{\partial D} \left|\frac{1}{z-\zeta_k(\vec{\eta})}\right| \ |dz| \ f(\vec{\eta}) \ dV < \infty, \tag{A.11}$$

appealing to the linearity of the integration and the triangle inequality we will have (A.10).

For notational sake, we write dV_k to stand for the volume measure in $\mathbb{C}^k$. The integration will be done as follows:

$$\begin{aligned}
&\int_{\mathbb{C}^{(n+1)}} \int_{\partial D} \left|\frac{1}{z-\zeta_k(\eta_0,\ldots,\eta_n)}\right| \ |dz| \ f(\eta_0,\ldots,\eta_n) \ dV_{n+1} \\
&= \int_{\mathbb{C}} \int_{\mathbb{C}^n} \int_{\partial D} \left|\frac{1}{z-\zeta_k(\eta_0,\ldots,\eta_n)}\right| \ |dz| \ f(\eta_0,\ldots,\eta_{n-1}) \ dV_n \ f(\eta_n) \ d\eta_n,
\end{aligned}$$

where by abuse of notation we mean that

$$f(\eta_0,\ldots,\eta_{n-1}) = \frac{\exp\left(-\sum_{k=0}^{n-1}|\eta_k|^2\right)}{\pi^n} \quad \text{and} \quad f(\eta_n) = \frac{\exp(-|\eta_n|^2)}{\pi}.$$

Fixing the outer variable η_n, by Lemma 0.0.4 we have an almost everywhere differentiable change of variables $\phi : \mathbb{C}^n \to \mathbb{C}^n$ given at (A.8) from the set of roots to the set of coefficients with Jacobian determinant

$$|\eta_n|^{2n} \prod_{1\leq i<j\leq n} |\zeta_i - \zeta_j|^2 := J(\phi).$$

Let us first consider the case when an arbitrary zero $\zeta_k(\vec{\eta})$ of $P_n(z)$ lies on the contour ∂D. Define $A := \{\text{coefficients of } P_n(z) : \zeta_k(\vec{\eta}) \in \partial D\}$. We will show that $V_{n+1}(A) = 0$. Since

$$\phi(\{\text{roots of } P_n(z)\}) = \{\text{coefficients of } P_n(z)\},$$

we have

$$\phi(\{\text{roots of } P_n(z) : \zeta_k(\vec{\eta}) \in \partial D\}) = A.$$

Using the change of variables it follows that

$$\begin{aligned} V_n(A) &= \int_{\phi(\{\text{roots of } P_n(z):\zeta_k(\vec{\eta})\in\partial D\})} dV_n \\ &= \int_{\{\text{roots of } P_n(z):\zeta_k(\vec{\eta})\in\partial D\}} J(\phi)\ dV_n \\ &\leq \int_{\prod_{k=1}^n \partial D(0,R)} J(\phi)\ dV_n \\ &= 0, \end{aligned}$$

where the last equality follows since $\prod_{k=1}^n \partial D(0,R)$ is direct product of n circles which has $\mathbb{C}^{n-1}$-dimension and is a strictly less dimensional set than what is being integrated over in the $\mathbb{C}^n$-dimensional volume V_n. Hence we have $V_n(A) = 0$, and consequently $V_{n+1}(A) = 0$ as needed. Therefore the set A is negligible for (A.11) to hold, so that $|P_n'(z)/P_n(z)|$ is measurable.

We now turn to the case when the zeros of $P_n(z)$ do not lie on the contour ∂D that is being integrated over. When $|z - \zeta_k(\vec{\eta})| \geq 1$ for $z \in \partial D$, trivially we have $\left|\frac{1}{z-\zeta_k(\vec{\eta})}\right| \leq 1$, so that

$$\int_{\mathbb{R}^{2(n+1)}} \int_{\partial D} \left|\frac{1}{z - \zeta_k(\vec{\eta})}\right|\ |dz|\ f(\vec{\eta})\ dV \leq 2\pi R < \infty.$$

Consider the case when $0 < |z - \zeta_k(\vec{\eta})| < 1$, and for now let us assume that $\zeta_k \in D$. Let us further assume that ζ_k is on the line segment $(0, R)$. Then $\left|\frac{1}{z-\zeta_k(\vec{\eta})}\right|$ the largest when $\arg z \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and for all z such that $\arg z \in [\frac{\pi}{2}, -\frac{\pi}{2}]$ we have $|z - \zeta_k(\vec{\eta})| \geq R$. Thus

$$\int_{\partial D} \left| \frac{1}{z - \zeta_k(\vec{\eta})} \right| \; |dz| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \frac{1}{z - \zeta_k(\vec{\eta})} \right| \; |dz| + \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left| \frac{1}{z - \zeta_k(\vec{\eta})} \right| \; |dz|$$
$$\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \frac{1}{z - \zeta_k(\vec{\eta})} \right| \; |dz| + \pi$$
$$= \int_{0}^{\frac{\pi}{2}} \frac{2R}{\sqrt{(R\cos t - \zeta_k(\vec{\eta}))^2 + (R \sin t)^2}} \; dt + \pi, \tag{A.12}$$

where have used that the integrand is an even function of t to obtain the last equality.

Given that $\zeta_k \in (0, R)$ and $|R - \zeta_k| < 1$, somewhere on the interval $[0, \frac{\pi}{2}]$ is $R - \zeta_k$. Hence for the integral above we split the integration as follows

$$\int_{0}^{\frac{\pi}{2}} \frac{2R}{\sqrt{(R\cos t - \zeta_k(\vec{\eta}))^2 + (R \sin t)^2}} \; dt = \int_{0}^{R-\zeta_k} + \int_{R-\zeta_k}^{\frac{\pi}{2}} := I_1 + I_2.$$

Since $0 \leq (R\cos t - \zeta_k(\vec{\eta}))^2$ and $0 \leq \frac{2}{\pi} t \leq \sin t$ when $t \in [0, \frac{\pi}{2}]$, for I_2 we have the bound

$$I_2 \leq \int_{R-\zeta_k}^{\frac{\pi}{2}} \frac{2R}{\sqrt{(R \sin t)^2}} \; dt \leq \int_{R-\zeta_k}^{\frac{\pi}{2}} \frac{2}{\frac{2}{\pi} t} \; dt = \pi \log\left(\frac{\pi}{2}\right) + \pi \log\left(\frac{1}{R - \zeta_k}\right).$$

For I_1, using $(\cos t)^2 + (\sin t)^2 = 1$ and $\cos t \leq 1$, yields

$$I_1 = \int_{0}^{R-\zeta_k} \frac{2R}{\sqrt{R^2 - 2R\zeta_k \cos t + \zeta_k^2}} \; dt \leq \int_{0}^{R-\zeta_k} \frac{2R}{\sqrt{R^2 - 2R\zeta_k + \zeta_k^2}} \; dt = 2R.$$

Combining the estimates for I_1 and I_2 and using (A.12) gives us

$$\int_{\partial D} \left| \frac{1}{z - \zeta_k(\vec{\eta})} \right| \; |dz| \leq C + \pi \log\left(\frac{1}{R - \zeta_k}\right),$$

where $C = 2R + \pi \log\left(\frac{\pi}{2}\right) + \pi$.

Hence to show (A.11) we need to show that

$$\int_{\mathbb{R}^{2(n+1)}} \log\left(\frac{1}{R - \zeta_k}\right) \; f(\vec{\eta}) \; dV < \infty.$$

Before showing the above bound, we note that all the estimates were under the assumption that $\zeta_k \in (0, R)$. Since that argument given is rotationally invariant, we can have ζ_k anywhere in the disk D if we replace in our estimates the $R - \zeta_k$ with $R - |\zeta_k|$. Furthermore, since we have assumed that $0 < |z - \zeta_k| < 1$ with $z \in \partial D$, it is possible that ζ_k is outside the disk D. Replacing in our argument $|R - |\zeta_k||$ where $R - |\zeta_k|$ is, we now have a bound that will work for all $0 < |z - \zeta_k| < 1$. Taking this into account and using the change of variables with the map ϕ from the set of zeros to the set of coefficients yields

$$\begin{aligned}&\int_{\mathbb{R}^{2(n+1)}} \log\left(\frac{1}{|R - |\zeta_k||}\right)\ f(\vec{\eta})\ dV \\ &\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^{2n}} \log\left(\frac{1}{|R - |\zeta_k||}\right)\ f(\phi(\eta_0, \ldots, \eta_{n-1}))|\eta_n|^{2n} \prod_{1\le i<j\le n} |\zeta_i - \zeta_j|^2\ dV_n\ f(\eta_n)\ dV_1.\end{aligned}$$

We denote

$$\begin{aligned} f(\phi(\eta_0, \ldots, \eta_{n-1})) &= f\left(-\eta_n \sum_{i=1}^n \zeta_i,\ \eta_n \sum_{i<j} \zeta_i\zeta_j,\ \ldots,\ (-1)^n \eta_n \prod_{i=1}^n \zeta_i\right) \\ &= \exp\left(-\left|\eta_n \sum_{i=1}^n \zeta_i\right|^2 - \left|\eta_n \sum_{i<j}\zeta_i\zeta_j\right|^2 - \cdots - \left|\eta_n \prod_{i=1}^n \zeta_i\right|^2\right) / \pi^n \\ &:= g(\zeta). \end{aligned}$$

To facilitate the computation, observe that since we have assumed that $0 < |z - \zeta_k| < 1$ with $z \in \partial D$, we have $|\zeta_k| < R + 1$. Thus

$$\prod_{\substack{i=1\\ i\neq k}}^n |\zeta_k - \zeta_i|^2 \le \prod_{\substack{i=1\\ i\neq k}}^n (|\zeta_k| + |\zeta_i|)^2 < \prod_{\substack{i=1\\ i\neq k}}^n (R + 1 + |\zeta_i|)^2.$$

The integral which we need to bound has integration with respect to ζ_k and posses a logarithmic singularity when $|\zeta_k| = R$. After moving the above mentioned product outside the integration with respect to ζ_k, the integral that we need to show is bounded is

$$\int_{\mathbb{R}^2} \log\left(\frac{1}{|R - |\zeta_k||}\right) g(\zeta)\ dV_1.$$

We will use polar integration with r being the radial variable to work with the above integral. When $r \ge R + 1$ we have $|z - \zeta_k| \ge 1$ for $z \in \partial D$. Since this case has already be handled, the limits of integration with respect to r that we need to take care of will be from 0 to $R+1$. Taking this into account and using polar integration we have

$$
\begin{aligned}
\int_{\{\zeta_k:\ 0<|z-\zeta_k|<1,\ z\in\partial D\}} & \log\left(\frac{1}{|R-|\zeta_k||}\right) g(\zeta)\ dV_1 \\
&= \int_0^{2\pi}\int_0^{R+1} \log\left(\frac{1}{|R-r|}\right) g(r,\theta) r\ d\theta\ dr \\
&= \int_0^{2\pi}\int_0^{R} \log\left(\frac{1}{|R-r|}\right) g(r,\theta) r\ d\theta\ dr \\
&\quad + \int_0^{2\pi}\int_R^{R+1} \log\left(\frac{1}{|R-r|}\right) g(r,\theta) r\ d\theta\ dr \\
&:= I_3 + I_4,
\end{aligned}
$$

where $g(r,\theta)$ is the same as $g(\zeta)$ except replacing ζ_k by $re^{i\theta}$ where it shows up in the expression for $g(\zeta)$ and not altering the other zeros ζ_i, $i=1,\ldots,n$, $i\neq k$.

For the I_3, observe that

$$\int_0^R \log\left(\frac{1}{|R-r|}\right)\ dr = \int_0^R \log\left(\frac{1}{R-r}\right)\ dr = R - R\log R < \infty.$$

Since the function $g(r,\theta)r$ is a bounded function, it follows that

$$
\begin{aligned}
I_3 &= \int_0^{2\pi}\int_0^{R} \log\left(\frac{1}{|R-r|}\right) g(r) r\ d\theta\ dr \\
&\le 2\pi R \max_{\{\zeta_k\in\overline{D}\}} g(\zeta) \int_0^R \log\left(\frac{1}{R-r}\right)\ dr \\
&= 2\pi \max_{\{\zeta_k\in\overline{D}\}} g(\zeta)(R^2 - R^2\log R).
\end{aligned}
$$

In I_4 we have

$$\int_R^{R+1} \log\left(\frac{1}{|R-r|}\right)\ dr = \int_R^{R+1} \log\left(\frac{1}{r-R}\right)\ dr = 1.$$

Again since the function $g(r,\theta)r$ is bounded function, we have

$$\begin{aligned} I_4 &= \int_0^{2\pi}\int_R^{R+1} \log\left(\frac{1}{|R-r|}\right) g(r)r \; d\theta \; dr \\ &\leq 2\pi(R+1) \max_{\{\zeta_k \in \overline{D(0,R+1)}\setminus D\}} g(\zeta) \int_R^{R+1} \log\left(\frac{1}{r-R}\right) \; dr \\ &= 2\pi \max_{\{\zeta_k \in \overline{D(0,R+1)}\setminus D\}} g(\zeta)(R+1). \end{aligned}$$

Therefore we have that I_3 and I_4 are convergent integrals. Taking into account the bound on the Jacobian that was obtained, and setting

$$K_1 := 2\pi(R^2 - R^2 \log R) \quad \text{and} \quad K_2 := 2\pi(R+1),$$

to complete this case of the proof we need that

$$K_1 \int_{\mathbb{R}^2}\int_{\mathbb{R}^{2n-2}} |\eta_n|^{2n} \prod_{\substack{1\leq i<j\leq n \\ i,j\neq k}} |\zeta_i - \zeta_j|^2 \prod_{\substack{i=1 \\ i\neq k}}^{n} (R+1+|\zeta_i|)^2 \max_{\{\zeta_k \in \overline{D}\}} g(\zeta) f(\eta_n) \; dV_{n-1} \; dV_1$$

and

$$K_2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^{2n-2}} |\eta_n|^{2n} \prod_{\substack{1\leq i<j\leq n \\ i,j\neq k}} |\zeta_i - \zeta_j|^2 \prod_{\substack{i=1 \\ i\neq k}}^{n} (R{+}1{+}|\zeta_i|)^2 \max_{\{\zeta_k \in \overline{D(0,R+1)}\setminus D\}} g(\zeta) f(\eta_n) \; dV_{n-1} \; dV_1$$

are finite.

Notice that both

$$\max_{\{\zeta_k \in \overline{D}\}} g(\zeta) \quad \text{and} \quad \max_{\{\zeta_k \in \overline{D(0,R+1)}\setminus D\}} g(\zeta)$$

decay at infinity exponentially with each $|\zeta_i| \to \infty$, $i = 1, \dots, n$. Hence the above two integrals are convergent. Therefore when the zero ζ_k is not on the contour, the bound (A.11) and consequently the bound (A.10) holds true.

This completes the justification of the use of the Fubini-Tonelli Theorem when the random sum is a random algebraic polynomial.

Now assume that

$$P_n(z) = \sum_{j=0}^{n} \nu_j f_j(z),$$

where the ν_j's are complex Gaussian random variables and the f_j's are polynomials such that

$$f_j(z) = a_{j,j}z^j + a_{j,j-1}z^{j-1} + \cdots + a_{j,0}$$

with $a_{j,i} \in \mathbb{C}$ for $i = 0, 1, \ldots, j$ and $j = 0, 1 \ldots, n$. Then

$$\begin{aligned} P_n(z) = \nu_n a_{n,n} z^n & \\ &+ (\nu_n a_{n,n-1} + \nu_{n-1} a_{n-1,n-1}) z^{n-1} \\ &+ (\nu_n a_{n,n-2} + \nu_{n-1} a_{n-1,n-2} + \nu_{n-2} a_{n-2,n-2}) z^{n-2} \\ &\vdots \\ &+ \nu_n a_{n,0} + \nu_{n-1} a_{n-1,0} + \cdots + \nu_0 a_{0,0}. \end{aligned}$$

Let us now make the following change of variables

$$\begin{aligned} \eta_n &= \nu_n a_{n,n} \\ \eta_{n-1} &= \nu_n a_{n,n-1} + \nu_{n-1} a_{n-1,n-1} \\ \eta_{n-2} &= \nu_n a_{n,n-2} + \nu_{n-1} a_{n-1,n-2} + \nu_{n-2} a_{n-2,n-2} \\ &\vdots \\ \eta_0 &= \nu_n a_{n,0} + \nu_{n-1} a_{n-1,0} + \cdots + \nu_0 a_{0,0}, \end{aligned}$$

and call the map which makes this change of variables ψ. That is for

$$\vec{\nu} = (\nu_n a_{n,n}, \nu_n a_{n,n-1} + \nu_{n-1} a_{n-1,n-1}, \ldots, \nu_n a_{n,0} + \nu_{n-1} a_{n-1,0} + \cdots + \nu_0 a_{0,0}),$$

we have $\psi(\vec{\nu}) = (\eta_n, \ldots, \eta_0) = \vec{\eta}$.

The real Jacobian determinant for this change of variables is

$$|\det D\psi|^2 = \left| \det \begin{pmatrix} a_{n,n} & 0 & 0 & \cdots & 0 \\ a_{n,n-1} & a_{n-1,n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,0} & a_{n-1,0} & a_{n-2,0} & \cdots & a_{0,0} \end{pmatrix} \right|^2 = \prod_{j=0}^{n} |a_{j,j}|^2.$$

Since $P_n(z)$ depends on $\nu_n, \ldots, \nu_0$, we write $P_n(z)$ as $P_n(z, \vec{\nu})$. Using the change of variables formula we have

$$\begin{aligned}
\int_{\psi(\mathbb{R}^{2(n+1)})} \int_{\partial\Omega} \left| \frac{P_n'(z, \vec{\nu})}{P_n(z, \vec{\nu})} \right| \ |dz| \ f(\vec{\nu}) \ dV \\
&= \int_{\mathbb{R}^{2(n+1)}} \int_{\partial\Omega} \left| \frac{P_n'(z, \psi(\vec{\nu}))}{P_n(z, \psi(\vec{\nu}))} \right| \ |dz| \ f(\psi(\vec{\nu})) \ \prod_{j=0}^{n} |a_{j,j}|^2 \ dV \\
&= \prod_{j=0}^{n} |a_{j,j}|^2 \int_{\mathbb{R}^{2(n+1)}} \int_{\partial\Omega} \left| \frac{P_n'(z, \vec{\eta})}{P_n(z, \vec{\eta})} \right| \ |dz| \ f(\vec{\eta}) \ \ dV < \infty,
\end{aligned}$$

since $\prod_{j=0}^{n} |a_{j,j}|^2 < \infty$, and by our previous calculations.

Also with a calculation analogously as done to show that $|P_n'(z)/P_n(z)|$ is measurable when $P_n(z)$ was assumed to be a random algebraic polynomial, using the change of variable map ψ, we have that $|P_n'(z, \vec{\nu})/P_n(z, \vec{\nu})|$ is measurable.

Therefore by $|P_n'(z, \vec{\nu})/P_n(z, \vec{\nu})|$ being measurable, and by the above bound, we are justified in using the complex version of the Fubini-Tonelli theorem to exchange the expectation and the contour integral when

$$P_n(z) = \sum_{j=0}^{n} \nu_j p_j(z).$$

■

To prove Theorem 0.0.1 we need one more lemma.

Lemma 0.0.5 *The following holds:*

$$\mathbb{E}\left[\frac{P_n'(z)}{P_n(z)}\right] = \frac{\overline{K_n^{(0,1)}(z,z)}}{K_n(z,z)}.$$

Proof. Since $P_n(z) = \sum_{j=0}^{n} \eta_j f_j(z)$ and $P_n^{'}(z) = \sum_{j=0}^{n} \eta_j {f_j}'(z)$, with $\eta_j = \alpha_j + i\beta_j$, are both complex Gaussian random variables, we will work their real and imaginary parts

$$\text{Re } P_n(z) = \sum_{j=0}^{n} (\alpha_j a_j - \beta_j b_j) := \xi_1, \quad \text{Im } P_n(z) = \sum_{j=0}^{n} (\alpha_j b_j + \beta_j a_j) := \xi_2,$$

$$\text{Re } P_n'(z) = \sum_{j=0}^{n} (\alpha_j c_j - \beta_j d_j) := \xi_3, \quad \text{Im } P_n'(z) = \sum_{j=0}^{n} (\alpha_j d_j + \beta_j c_j) := \xi_4,$$

where

$$a_j = \text{Re } f_j(z), \quad b_j = \text{Im } f_j(z), \quad c_j = \text{Re } f_j^{'}(z), \quad d_j = \text{Im } f_j^{'}(z).$$

Following Vanderbei's method proof, we will now be forming the covariance matrix of the vector $\xi := (\xi_1, \xi_2, \xi_3, \xi_4)^T$. Before doing so, observe that since the random variables $\{\alpha_j\}_{j=0}^n$ and $\{\beta_j\}_{j=0}^n$ are independent identically distributed $N(0,1)$, for $j = 0, 1, \ldots, n$ we have that $\mathbb{E}[\alpha_j] = \mathbb{E}[\beta_j] = 0$ and $\mathbb{E}[\alpha_j^2] = \mathbb{E}[\beta_j^2] = 1$. Thus by the expectation being linear, it follows that $\mathbb{E}[\xi_1] = \mathbb{E}[\xi_2] = \mathbb{E}[\xi_3] = \mathbb{E}[\xi_4] = 0$. Consequently each entry in the covariance matrix for ξ is of the form $\mathbb{E}[(\xi_i - \mathbb{E}[\xi_i])(\xi_k - \mathbb{E}[\xi_k])] = \mathbb{E}[\xi_i \xi_k]$, where $i, k = 1, \ldots, 4$. Using these observations, by definition of the covariance matrix we see that

$$\text{Cov}[\xi] = \mathbb{E}\left[\xi \xi^T\right] = \begin{bmatrix} \mathbb{E}[\xi_1^2] & \mathbb{E}[\xi_1\xi_2] & \mathbb{E}[\xi_1\xi_3] & \mathbb{E}[\xi_1\xi_4] \\ \mathbb{E}[\xi_2\xi_1] & \mathbb{E}[\xi_2^2] & \mathbb{E}[\xi_2\xi_3] & \mathbb{E}[\xi_2\xi_4] \\ \mathbb{E}[\xi_3\xi_1] & \mathbb{E}[\xi_3\xi_2] & \mathbb{E}[\xi_3^2] & \mathbb{E}[\xi_3\xi_4] \\ \mathbb{E}[\xi_4\xi_1] & \mathbb{E}[\xi_4\xi_2] & \mathbb{E}[\xi_4\xi_3] & \mathbb{E}[\xi_4^2] \end{bmatrix}. \tag{A.13}$$

Observe that the matrix $\text{Cov}[\xi]$ is a symmetric matrix. If the matrix is positive definite, by a known result in matrix theory (c.f. Theorem 5.2-3 p. 88 of [45]), the matrix can be represented as LL^T, where L is a lower triangular matrix. We will first show that $\text{Cov}[\xi]$ is real-valued. To this end, recall that the random variables $\{\alpha_j\}_{j=0}^n$ and $\{\beta_j\}_{j=0}^n$ are independent identically distributed $N(0,1)$; consequently $\mathbb{E}[\omega\nu] = \mathbb{E}[\omega]\mathbb{E}[\nu]$ for ω and ν being any two distinct elements among the α_j's and the β_j's. Using these properties and the definitions of the sums $K_n(z,z)$, $K_n^{(0,1)}(z,z)$, and $K_n^{(1,1)}(z,z)$ yields

$$\begin{aligned}
\mathbb{E}[\xi_1^2] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j a_j - \beta_j b_j\right)\left(\sum_{j=0}^{n}(\alpha_j a_j - \beta_j b_j\right)\right] \\
&= \sum_{j=0}^{n}(a_j^2 + b_j^2) \\
&= \sum_{j=0}^{n}\left[(\mathrm{Re}(f_j(z)))^2 + (\mathrm{Im}(f_j(z))^2\right] \\
&= \sum_{j=0}^{n}|f_j(z)|^2 \\
&= K_n(z,z),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\xi_2\xi_1] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j b_j + \beta_j a_j\right)\left(\sum_{j=0}^{n}(\alpha_j a_j - \beta_j b_j\right)\right] \\
&= \sum_{j=0}^{n}(a_j b_j - a_j b_j) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\xi_3\xi_1] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j c_j - \beta_j d_j\right)\left(\sum_{j=0}^{n}(\alpha_j a_j - \beta_j b_j\right)\right] \\
&= \sum_{j=0}^{n}(a_j c_j + b_j d_j) \\
&= \sum_{j=0}^{n}\left[\mathrm{Re}(f_j(z))\mathrm{Re}(f'(z)) + \mathrm{Im}(f_j(z))\mathrm{Im}(f'(z))\right] \\
&= \sum_{j=0}^{n}\frac{\overline{f_j(z)}f_j'(z) + f_j(z)\overline{f_j'(z)}}{2} = \frac{\overline{K_n^{(0,1)}(z,z)} + K_n^{(0,1)}(z,z)}{2} \\
&= \mathrm{Re}(\overline{K_n^{(0,1)}(z,z)}),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\xi_4\xi_1] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j d_j + \beta_j c_j\right)\left(\sum_{j=0}^{n}(\alpha_j a_j - \beta_j b_j\right)\right] \\
&= \sum_{j=0}^{n}(a_j d_j - b_j c_j) \\
&= \sum_{j=0}^{n}\left[\mathrm{Re}(f_j(z))\mathrm{Im}(f'(z)) - \mathrm{Im}(f_j(z))\mathrm{Re}(f'(z))\right] \\
&= \sum_{j=0}^{n}\frac{\overline{f_j(z)}f_j'(z) - f_j(z)\overline{f_j'(z)}}{2i} = \frac{\overline{K_n^{(0,1)}(z,z)} - K_n^{(0,1)}(z,z)}{2i} \\
&= \mathrm{Im}(\overline{K_n^{(0,1)}(z,z)}),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\xi_2^2] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j b_j + \beta_j a_j\right)\left(\sum_{j=0}^{n}(\alpha_j b_j + \beta_j a_j\right)\right] \\
&= \sum_{j=0}^{n}(b_j^2 + a_j^2) \\
&= \sum_{j=0}^{n}\left[(\mathrm{Im}(f_j(z)))^2 + (\mathrm{Re}(f_j(z))^2\right] \\
&= \sum_{j=0}^{n}|f_j(z)|^2 \\
&= K_n(z,z),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\xi_3\xi_2] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j c_j - \beta_j d_j\right)\left(\sum_{j=0}^{n}(\alpha_j b_j + \beta_j a_j\right)\right] \\
&= \sum_{j=0}^{n}(b_j c_j - a_j d_j) \\
&= -\mathbb{E}[\xi_4\xi_1] \\
&= -\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)}),
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[\xi_4\xi_2] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j d_j + \beta_j c_j\right)\left(\sum_{j=0}^{n}(\alpha_j b_j + \beta_j a_j\right)\right] \\
&= \sum_{j=0}^{n}(b_j d_j + a_j c_j) \\
&= \mathbb{E}[\xi_3\xi_1] \\
&= \operatorname{Re}(\overline{K_n^{(0,1)}(z,z)}),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\xi_3^2] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j c_j - \beta_j d_j\right)^2\right] \\
&= \sum_{j=0}^{n}(c_j^2 + d_j^2) \\
&= \sum_{j=0}^{n}\left[(\operatorname{Re}(f_j'(z)))^2 + (\operatorname{Im}(f_j'(z))^2\right] \\
&= \sum_{j=0}^{n}|f_j'(z)|^2 \\
&= K_n^{(1,1)}(z,z),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\xi_3\xi_4] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j d_j + \beta_j c_j\right)\left(\sum_{j=0}^{n}(\alpha_j c_j - \beta_j d_j\right)\right] \\
&= \sum_{j=0}^{n}(d_j c_j - c_j d_j) \\
&= 0,
\end{aligned}
$$

and

$$\begin{aligned}
\mathbb{E}[\xi_4^2] &= \mathbb{E}\left[\left(\sum_{j=0}^{n}(\alpha_j d_j + \beta_j c_j\right)^2\right] \\
&= \sum_{j=0}^{n}(d_j^2 + c_j^2) \\
&= \sum_{j=0}^{n}\left[(\mathrm{Im}(f_j'(z)))^2 + (\mathrm{Re}(f_j'(z))^2\right] \\
&= \sum_{j=0}^{n}|f_j'(z)|^2 \\
&= K_n^{(1,1)}(z,z).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Cov}[\xi] &= \mathbb{E}\left[\xi\xi^T\right] \\
&= \begin{bmatrix}
K_n(z,z) & 0 & \mathrm{Re}(\overline{K_n^{(0,1)}(z,z)}) & \mathrm{Im}(\overline{K_n^{(0,1)}(z,z)}) \\
0 & K_n(z,z) & -\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)}) & \mathrm{Re}(\overline{K_n^{(0,1)}(z,z)}) \\
\mathrm{Re}(\overline{K_n^{(0,1)}(z,z)}) & -\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)}) & K_n^{(1,1)}(z,z) & 0 \\
\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)}) & \mathrm{Re}(\overline{K_n^{(0,1)}(z,z)}) & 0 & K_n^{(1,1)}(z,z)
\end{bmatrix},
\end{aligned} \tag{A.14}$$

which is real-valued. Appealing to Theorem 12.4 of [56], we can conclude that this matrix is positive semidefinite. Thus to complete the argument that $\mathrm{Cov}[\xi]$ is positive definite, it suffices to show for $v = [v_1\ v_2\ v_3 v_4]^T$, we have $v^T\mathrm{Cov}[\xi]v = 0$ implies that

$v = 0$. To simplify this computation, observe the identity: $v^T \mathrm{Cov}[\xi] v = \mathrm{Var}(v^T \xi)$. As

$$\begin{aligned} v^T \xi &= v_1 \mathrm{Re}(P_n(z)) + v_2 \mathrm{Im}(P_n(z)) + v_3 \mathrm{Re}(P_n'(z)) + v_4 \mathrm{Im}(P_n'(z)) \\ &= \sum_{j=0}^{n} v_1(\alpha_j a_j - \beta_j b_j) + \sum_{j=0}^{n} v_2(\alpha_j b_j + \beta_j a_j) \\ &\quad + \sum_{j=0}^{n} v_3(\alpha_j c_j - \beta_j d_j) + \sum_{j=0}^{n} v_4(\alpha_j d_j + \beta_j c_j) \\ &= \sum_{j=0}^{n} (v_1 a_j + v_2 b_j + v_3 c_j + v_4 d_j)\alpha_j + \sum_{j=0}^{n} (-v_1 b_j + v_2 a_j - v_3 d_j + v_4 c_j)\beta_j, \end{aligned}$$

we have

$$\begin{aligned} \mathrm{Var}(v^T \xi) &= \mathbb{E}[(v^T \xi)^2] - (\mathbb{E}[v^T \xi])^2 \\ &= \sum_{j=0}^{n} (v_1 a_j + v_2 b_j + v_3 c_j + v_4 d_j)^2 + \sum_{j=0}^{n} (-v_1 b_j + v_2 a_j - v_3 d_j + v_4 c_j)^2. \end{aligned}$$

Hence $v^T \mathrm{Cov}[\xi] v = \mathrm{Var}(v^T \xi) = 0$ if and only if for all $j = 0, \ldots, n$ we have

$$(v_1 a_j + v_2 b_j + v_3 c_j + v_4 d_j)^2 = 0, \quad \text{and} \quad (-v_1 b_j + v_2 a_j - v_3 d_j + v_4 c_j)^2 = 0. \quad \text{(A.15)}$$

Recall that the polynomial basis is such that $\{f_j(z)\} = \{a_j(z) + ib_j(z)\}$, $\{f_j'(z)\} = \{c_j(z) + id_j(z)\}$, and $\deg f_j = j$, for all $j \in \{0, 1, \ldots, n\}$. Observe that since $f_0(z) = a_0(z) + ib_0(z) = a_0 + ib_0$ is constant, it follows that $f_0'(z) = c_0(z) + id_0(z) = 0$, so that $c_0 = d_0 = 0$. Thus for term $j = 0$ in (A.15) we have

$$0 = (v_1 a_0 + v_2 b_0)^2 + (-v_1 b_0 + v_2 a_0)^2 = (v_1 a_0)^2 + (v_2 b_0)^2 + (v_1 b_0)^2 + (v_2 a_0)^2.$$

Since we have a polynomial basis both a_0 and b_0 cannot both be zero. Thus we achieve that $v_1 = v_2 = 0$. Using this result and looking at the term $j = 1$ in (A.15), we similarly we see that

$$0 = (v_3 c_1 + v_4 d_1)^2 + (-v_3 d_1 + v_4 c_1)^2 = (v_3 c_1)^2 + (v_4 d_1)^2 + (v_3 d_1)^2 + (v_4 c_1)^2.$$

As $\deg f_1 = 1$, we have $\deg f_1' = 0$, so that c_1 and d_1 are constants, that cannot both of which be zero. Hence from the above we see that $v_3 = v_4 = 0$.

Therefore we achieve that when $v^T \mathrm{Cov}[\xi] v = 0$, it follows that $v = 0$. This along with the mentioned result in [56] gives that $\mathrm{Cov}[\xi]$ is positive definite. Invoking Theorem 5.2-3 p. 88 of [45], the matrix $\mathrm{Cov}[\xi]$ can be represented as LL^T, where L is a lower triangular matrix. Given this fact, we now represent the correlated Gaussian random variables $\xi_1, \xi_2, \xi_3, \xi_4$ in terms of four independent standard normals by finding a lower triangular matrix $L = [l_{pq}]$, $p, q = 1, 2, 3, 4$, with the property that $\xi \overset{d}{=} L\zeta$, where the notation $\overset{d}{=}$ denotes equality in distribution, with $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T$ being a vector of four independent standard normal random variables. We note that ζ is ensured to be a vector of standard normal variables given that the matrix $\mathrm{Cov}[\xi]$ is non-vanishing, which coupled with the vector ξ be a standard normal vector, gives that ζ as linear combination of standard normal vectors, hence yielding a standard normal vector. Since

$$\mathrm{Cov}[\xi] = \mathbb{E}\left[\xi\xi^T\right] = \mathbb{E}\left[L\zeta\zeta^T L^T\right] = LL^T, \tag{A.16}$$

we see that L is the Cholesky factor for the covariance matrix.

By $\xi \overset{d}{=} L\zeta$, and the fact that L is lower triangular, we have

$$\frac{P_n'(z)}{P_n(z)} = \frac{\xi_3 + i\xi_4}{\xi_1 + i\xi_2} \overset{d}{=} \frac{(l_{31} + il_{41})\zeta_1 + (l_{32} + il_{42})\zeta_2 + (l_{33} + il_{43})\zeta_3 + il_{44}\zeta_4}{(l_{11} + il_{21})\zeta_1 + il_{22}\zeta_2}.$$

So with $\alpha = l_{31} + il_{41}, \beta = l_{32} + il_{42}, \gamma = l_{11} + il_{21}$, and $\delta = il_{22}$, using the independence of the ζ_i's, it follows that

$$\mathbb{E}\left[\frac{P_n'(z)}{P_n(z)}\right] = \mathbb{E}\left[\frac{\alpha\zeta_1 + \beta\zeta_2}{\gamma\zeta_1 + \delta\zeta_2}\right].$$

If we now split up the numerator of the above and use the property that ζ_1 and ζ_2 are exchangeable, we can write the expectation as

$$F(z) = \mathbb{E}\left[\frac{P_n'(z)}{P_n(z)}\right] = \frac{\alpha}{\delta} f(\gamma/\delta) + \frac{\beta}{\gamma} f(\delta/\gamma),$$

where $f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ by $f(w) = \mathbb{E}\left[\frac{\zeta_1}{w\zeta_1 + \zeta_2}\right]$. Using the definition of the expectation in our Gaussian setting, and appealing to polar integration we see that

$$\begin{aligned} f(w) &= \mathbb{E}\left[\frac{\zeta_1}{w\zeta_1+\zeta_2}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{\rho\cos\theta}{w\rho\cos\theta+\rho\sin\theta}e^{-\rho^2/2}\rho d\rho d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{w+\tan\theta} \\ &= \begin{cases} \frac{1}{w+i} & \text{if } \mathrm{Im}(w)>0, \\ \frac{1}{w-i} & \text{if } \mathrm{Im}(w)<0. \end{cases} \end{aligned}$$

We need to evaluate f at γ/δ and δ/γ. Since in general l_{11} and l_{22} are nonnegative, we have that $\gamma/\delta = l_{21}/l_{22} - il_{11}/l_{22}$ has negative imaginary part while $\delta/\gamma = l_{21}l_{22}/(l_{11}^2 + l_{21}^2) + il_{11}l_{22}/(l_{11}^2+l_{21}^2)$ has positive imaginary part. Thus

$$\begin{aligned} F(z) &= \frac{\alpha}{\delta}f(\gamma/\delta)+\frac{\beta}{\gamma}f(\delta/\gamma) \\ &= \frac{\alpha}{\delta}\frac{1}{\frac{\gamma}{\delta}-i}+\frac{\beta}{\gamma}\frac{1}{\frac{\delta}{\gamma}+i} \\ &= \frac{i\alpha+\beta}{i\gamma+\delta} \\ &= \frac{l_{32}-l_{41}+i(l_{31}+l_{42})}{-l_{21}+i(l_{11}+l_{22})}. \end{aligned} \tag{A.17}$$

From the above we see that we need explicit formulas for the elements of the Cholesky factor L. Using (A.13) and (A.16) we obtain

$$\begin{aligned} &\mathbb{E}[\xi_1^2] = l_{11}^2, \\ &\mathbb{E}[\xi_2\xi_1] = l_{21}l_{11}, \qquad \mathbb{E}[\xi_2^2] = l_{21}^2 + l_{22}^2, \\ &\mathbb{E}[\xi_3\xi_1] = l_{31}l_{11}, \qquad \mathbb{E}[\xi_3\xi_2] = l_{31}l_{21} + l_{32}l_{22}, \\ &\mathbb{E}[\xi_4\xi_1] = l_{41}l_{11}, \qquad \mathbb{E}[\xi_4\xi_2] = l_{41}l_{21} + l_{42}l_{22}. \end{aligned}$$

Using values in (A.14) and solving for the needed entries of the Cholesky factor

L it follows that

$$l_{11} = \sqrt{K_n(z,z)}\ ,$$

$$l_{21} = 0\ , \qquad\qquad l_{22} = \sqrt{K_n(z,z)}\ ,$$

$$l_{31} = \frac{\mathrm{Re}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}}\ ,\ \ l_{32} = \frac{-\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}}\ ,$$

$$l_{41} = \frac{\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}}\ ,\ \ l_{42} = \frac{\mathrm{Re}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}}\ .$$

Therefore substituting these expressions into (A.17) and simplifying gives

$$\begin{aligned}
F(z) &= \frac{l_{32} - l_{41} + i(l_{31} + l_{42})}{-l_{21} + i(l_{11} + l_{22})} \\
&= \frac{\frac{-\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}} - \frac{\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}} + i\left(\frac{\mathrm{Re}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}} + \frac{\mathrm{Re}(\overline{K_n^{(0,1)}(z,z)})}{\sqrt{K_n(z,z)}}\right)}{0 + i\left(\sqrt{K_n(z,z)} + \sqrt{K_n(z,z)}\right)} \\
&= \frac{\mathrm{Re}(\overline{K_n^{(0,1)}(z,z)}) + i\mathrm{Im}(\overline{K_n^{(0,1)}(z,z)})}{K_n(z,z)} \\
&= \frac{\overline{K_n^{(0,1)}(z,z)}}{K_n(z,z)}.
\end{aligned}$$

■

We are now ready to give the proof of Theorem 0.0.1.

Proof of Theorem 0.0.1. Let us first observe that since $K_n(z,z)$ is the sum of the modulus squared of a polynomial basis $f_j(z)$, $j = 0, 1, \ldots, n$, it follows that $K_n(z,z) \neq 0$. For a Jordan region $\Omega \subset \mathbb{C}$, by Green's Theorem we have

$$\mathbb{E}[N_n(\Omega)] = \frac{1}{2\pi i}\int_{\partial\Omega} F(z)\ dz = \frac{1}{\pi}\int_{\Omega} \frac{\partial F(z,\overline{z})}{\partial \overline{z}}\, dx\, dy,$$

where are writing $F(z,\bar{z})$ to emphasize that F is a function of both z and $\bar{z}$.

Let the symbol $\overline{\partial}$ denote partial derivatives with respect to $\overline{z}$. Our goal is to simplify $\frac{1}{\pi}\overline{\partial}F(z,\bar{z})$ to $h_n(z)$, where by Lemma 0.0.5,

$$F(z,\bar{z}) = \frac{\overline{K_n^{(0,1)}(z,z)}}{K_n(z,z)}.$$

Using the Quotient Rule we see that

$$\overline{\partial}F(z,\bar{z}) = \frac{\overline{\partial}\,\overline{K_n^{(0,1)}(z,z)}K_n(z,z) - \overline{K_n^{(0,1)}(z,z)}\overline{\partial}K_n(z,z)}{(K_n(z,z))^2}. \tag{A.18}$$

We now will verify the following relations:

$$\overline{\partial}K_n(z,z) = K_n^{(0,1)}(z,z) \text{ and } \overline{\partial}\,\overline{K_n^{(0,1)}(z,z)} = K_n^{(1,1)}(z,z). \tag{A.19}$$

Indeed,

$$\begin{aligned}
\overline{\partial}K_n(z,z) &= \sum_{j=0}^{n} \overline{\partial}\left(f_j(z)\overline{f_j(z)}\right) \\
&= \sum_{j=0}^{n} f_j(z)\overline{\partial}\left(\overline{f_j(z)}\right) \\
&= \sum_{j=0}^{n} f_j(z)\overline{f_j'(z)} \\
&= K_n^{(0,1)}(z,z),
\end{aligned}$$

and

$$\begin{aligned}
\overline{\partial}\,\overline{K_n^{(0,1)}(z,z)} &= \sum_{j=0}^{n} \overline{\partial}\left(\overline{f_j(z)}f_j'(z)\right) \\
&= \sum_{j=0}^{n} f_j'(z)\overline{\partial}\left(\overline{f_j(z)}\right) \\
&= \sum_{j=0}^{n} f_j'(z)\overline{f_j'(z)} \\
&= K_n^{(1,1)}(z,z).
\end{aligned}$$

Since

$$\overline{K_n^{(0,1)}(z,z)}K_n^{(0,1)}(z,z) = \left|K_n^{(0,1)}(z,z)\right|^2,$$

substituting the relations (A.19) into (A.18) yields

$$\overline{\partial} F(z, \bar{z}) = \frac{K_n^{(1,1)}(z,z) K_n(z,z) - |K_n^{(0,1)}(z,z)|^2}{\left(K_n(z,z)\right)^2}.$$

Therefore

$$\rho_n^{(1)}(z) = \frac{1}{\pi}\, \overline{\partial} F(z, \bar{z}) = \frac{K_n^{(1,1)}(z,z) K_n(z,z) - |K_n^{(0,1)}(z,z)|^2}{\pi \left(K_n(z,z)\right)^2}.$$

■

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