BOUSSINESQ EQUATIONS WITH PARTIAL OR FRACTIONAL DISSIPATION

By

BEI XIAO

Bachelor of Science in Information and Computing Science
Henan University
Kaifeng, Henan, China
2012

Master of Science in Applied Mathematics
Oklahoma State University
Stillwater, Oklahoma, U.S.A.
2014

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Dissertation Approved:

Dr. Jiahong Wu

Dissertation Advisor

Dr. Weiwei Hu

Dr. JaEun Ku

Dr. Lan Zhu

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Name: BEI XIAO

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Abstract: The two-dimensional (2D) incompressible Boussinesq system is not only an important model in geophysics, but also retains some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. Especially, the inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows. Even though the global regularity of full dissipative Boussinesq equations is well known, the global regularity problem of inviscid case is still left open. First, we prove the global existence and uniqueness of 2D Boussinesq equations with partial dissipation in bounded main with Navier type boundary conditions. Secondly, we investigate Boussinesq equations with fractional dissipation on a d-dimensional periodic domain, and apply a re-developed tool of Littlewood-Paley decomposition to achieve global existence and uniqueness of weak solutions. Lastly, we focus on several variants of the 2D incompressible Euler equations. It is not known whether global well-posedness result would hold if there is only partially damping term for 2D Euler equation. Besides, in the vorticity equations, the partially damping term becomes a non-local operator $\mathcal{R}_1^2\omega$. Our numerical simulations show that by replacing $\mathcal{R}_1^2\omega$ with different operators (e.g. $\mathcal{R}_1\mathcal{R}_2\omega$), the solutions will behave quite differently.

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CHAPTER 1

Introduction

1.1 Navier-Stokes Equations

The Navier-Stokes equations govern the motion of viscous fluids. They could be used to monitor weather and ocean currents for scientific research, to model water flow in a pipe and air flow around a airplane for industry application, to simulate small-scale gaseous fluids, such as fire and smoke in video game design.

The incompressible Navier-Stokes equations can be written as

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f, \\ \nabla \cdot u = 0. \end{cases} \tag{1.1}$$

where u denotes the velocity field, p represents the pressure, ν is the kinematic viscosity and f denotes the external force(e.g. gravity). The first equation in (1.1) is derived from conservation of momentum and the second $\nabla \cdot u = 0$ represents the incompressibility, which comes from the conservation of mass. When $\nu = 0$, (1.1) is been reduced to Euler equations.

In 2000, to celebrate mathematics in the new millennium, the Clay Mathematics Institute listed seven most important open mathematics problems. Among these problems is the following:

Given a smooth initial velocity field u_0 with finite energy ($\|u_0\|_{L^2} < \infty$). Does the 3-dimensional Navier-Stokes (1.1) have global (in time) smooth solutions with finite energy?

1.2 Boussinesq Equations

In 2-dimensional case, by taking account of temperature/density variance, Navier-Stokes (1.1) becomes Boussinesq equations, which is given by

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + \theta e_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + (u \cdot \nabla)\theta = \kappa \Delta \theta. \end{cases} \tag{1.2}$$

where $u = (u_1(x_1, x_2), u_2(x_1, x_2))$ denotes 2-dimensional velocity field, $\theta = \theta(x_1, x_2)$ denotes the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, $p = p(x_1, x_2)$ represents the pressure, ν is the viscosity constant, κ is the thermal diffusivity constant and $e_2 = (0, 1)$ is the unit vector in vertical direction.

The two-dimensional (2D) incompressible Boussinesq system is not only an important model in geophysics, but also retains some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. Especially, the inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows. The Boussinesq system is one of the simplest models for which the global regularity of the inviscid equations remains open. The experience in studying the Navier-Stokes equations shows the dissipation terms ($\nu \Delta u$ and

$\kappa\Delta\theta$) play a crucial role in controlling the system from blowup. To reduce the effect from dissipation, it is natural to explore the global well-posedness problem on the Boussinesq equations with either partial or fractional dissipation.

In the recent years, the global well-posedness problem on the 2D Boussinesq equations has attracted considerable attention, and many significant progress [2, 3, 5, 6, 11, 18, 23] has been made but there are still some open cases left.

We focus mainly on finding the critical requirement of dissipation in order to achieve the existence or uniqueness of solutions in different spatial domains. That is given sufficiently smooth

$$u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x),$$

does the Boussinesq system have a global (in time) and unique solution?

1.2.1 Partial Disspative Boussinesq Equations

The first approach to reduce dissipation terms is based on the fact that Laplacian operator Δ can be written as the summation of double partial derivatives. That is for example, in 2D case,

$$\Delta f = \partial_{11} f + \partial_{22} f$$

with $\partial_{ii} f = \frac{\partial^2 f}{\partial x_i^2}$. We re-write (1.2) in the following form

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu_1 \partial_{11} u + \nu_2 \partial_{22} u + \theta e_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + (u \cdot \nabla)\theta = \kappa_1 \partial_{11}\theta + \kappa_2 \partial_{22}\theta. \end{cases} \tag{1.3}$$

When $\nu_1 = \nu_2 = \nu$ and $\kappa_1 = \kappa_2 = \kappa$, form (1.3) is the same as form (1.2). The advantage of the form (1.3) is that it allows us to have $\nu_1 \neq \nu_2$ or $\kappa_1 \neq \kappa_2$ so that we can consider the horizontal and vertical dissipation separately.

Many work have been done on the whole domain case $x \in \mathbb{R}^2$. Here is a brief summary:

(i) When $\nu_1 > 0, \nu_2 > 0, \kappa_1 > 0, \kappa_2 > 0$, (1.3) is the fully dissipative system. It is not hard to show any sufficiently smooth data ($(u_0, \theta_0) \in H^2(\mathbb{R}^2)$) leads to a unique global solution (see e.g. [24]).

(ii) When $\nu_1 = \nu_2 > 0, \kappa_1 = \kappa_2 = 0$, the global regularity problem of this kinetic dissipation only case was proven by Chae [6] ($(u_0, \theta_0) \in W^{2,q}(\mathbb{R}^2)$ with $q > 2$) and by Hou and Li [18] ($u_0 \in H^3(\mathbb{R}^2)$ and $\theta_0 \in H^2(\mathbb{R}^2)$).

(iii) When $\nu_1 = \nu_2 = 0, \kappa_1 = \kappa_2 > 0$, the existence and uniqueness of the global solution of this thermal diffusion only case was also established by Chae in [6] ($(u_0, \theta_0) \in W^{2,q}(\mathbb{R}^2)$ with $q > 2$).

(iv) When $\nu_1 > 0, \nu_2 = 0, \kappa_1 = 0, \kappa_2 = 0$, Danchin and Paicu [11] proved the result of this horizontal diffusion only case and their result has initial data that $u_0 \in H^2(\mathbb{R}^2)$ and $\theta_0 \in H^1(\mathbb{R}^2)$.

(v) When $\nu_1 = 0, \nu_2 = 0, \kappa_1 > 0, \kappa_2 = 0$, Danchin and Paicu in [11] also proved the result of this horizontal diffusion only case and the result has initial data that $(u_0, \theta_0) \in H^1(\mathbb{R}^2)$, $\omega_0 \in L^\infty(|RR^2)$ and $|\partial_1|^{1+s}\theta_0 \in L^2(\mathbb{R}^2)$ with $s \in (0, 1/2]$.

(vi) When $\nu_1 = 0, \nu_2 > 0, \kappa_1 = 0, \kappa_2 > 0$, Cao and Wu [5] obtained the global regularity results of this vertical dissipation case with initial data $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$.

Our work in chapter 3 will focus the case (iii) ($\kappa_1 = \kappa_2 = 0$) on bounded domain with the Navier type boundary conditions and established in [19] the well-posedness of the 2D Boussinesq equations with partial dissipation. It can be proved the global existence and uniqueness under minimal regularity assumptions on the initial data, and we also provide a direct and transparent approach that explicitly reveals the impacts of the Navier boundary conditions.

1.2.2 Fractional Disspative Boussinesq Equations

Another approach to reduce dissipation terms comes from the representation of Laplacian operator Δ in Fourier space. That is

$$\widehat{(-\Delta)f} = |\xi|^2 \hat{f}.$$

By replacing the power 2 with any positive real number, we have the definition of fractional Laplacian,

$$\widehat{\Lambda^\alpha f} = |\xi|^\alpha \hat{f}$$

Thus, we can re-write (1.2) in the following form

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + (u \cdot \nabla)\theta + \kappa \Lambda^\beta \theta = 0. \end{cases} \tag{1.4}$$

When $\alpha = \beta = 2$, (1.4) becomes the same as (1.2) with $\nu = \nu_1 = \nu_2$ and $\kappa = \kappa_1 = \kappa_2$.

Although the diffusion process is normally modeled by the standard Laplacian operator, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian arise. Flows in the middle atmosphere traveling upwards undergo changes due to the changes in atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled using the space fractional Laplacian.

Results of global regularity for (1.3) with different α and β are listed below:

(i) Subcritical case($\alpha + \beta > 1$): P. Constantin and V. Vicol [9]

$$\alpha \in (0,2), \quad \beta \in (0,2), \quad \beta > \frac{2}{2+\alpha}$$

W. Yang, Q. Jiu and J. Wu [33]

$$\alpha \in (0,1), \quad \beta \in (0,1), \quad \frac{\alpha}{2} + \beta > 1, \quad \beta \geq \frac{2}{3} + \frac{\alpha}{3}, \quad \beta > \frac{10-5\alpha}{10-4\alpha}$$

Z. Ye and X. Xu [34]

$$\alpha \in (0,1), \quad \beta \in (0,1), \quad \beta > \frac{2-\alpha}{2}, \quad \beta > \frac{1+\alpha}{2}$$

C. Miao and L. Xue [26]

$$\alpha \in (\frac{6-\sqrt{6}}{4}, 1), \quad \beta < \min\{\frac{7+2\sqrt{6}}{5}\alpha - 2, \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha}, 2-2\alpha\}$$

(ii) Critical case($\alpha + \beta = 1$): T. Hmidi, S. Keraani, and F. Rousset [16] [17]

$$\kappa = 0, \quad \alpha = 1. \quad \text{or} \quad \nu = 0, \quad \beta = 1$$

Q. Jiu, C. Miao, J. Wu, and Z. Zhang [20]

$$\alpha + \beta = 1, \alpha > \frac{23 - \sqrt{145}}{12} \approx 0.9132$$

A. Stefanov and J. Wu [27]

$$\alpha + \beta = 1, \alpha > \frac{\sqrt{1777} - 23}{24} \approx 0.798103$$

J. Wu, X. Xu and Z. Ye [31]

$$\alpha + \beta = 1, \alpha > \frac{10}{13} \approx 0.7692$$

F. Hadadifard and A. Stefanov [15]

$$\alpha + \beta = 1, \alpha > \frac{2}{3}$$

(iii) Supercritical case($\alpha + \beta < 1$): eventual regularity result of Q. Jiu, J. Wu, and W. Yang [21]

$$\alpha > \frac{23 - \sqrt{145}}{12} \approx 0.9132$$

In chapter 4, we consider (1.4) on a d-dimensional periodic domain, and a re-developed tool of Littlewood-Paley decomposition was applied to achieve global existence of L^2-weak solutions for any $\alpha > 0$ and the uniqueness of the weak solutions when $\alpha > \frac{1}{2} + \frac{d}{4}$ for $d \geq 2$. And this leads us to obtain the global L^2-stability of the hydrostatic balance for the 2D Boussinesq equations without thermal diffusion. Furthermore, we can also conclude the zero thermal diffusion limit with an explicit convergence rate for the weak solutions.

1.3 Variants of the 2D Euler Equation

Another area of this thesis is focused on several variants of the 2D incompressible Euler equation. It is well known that, we do have global well-posedness for the 2D Euler equations, as well as the Euler system with extra damping term. However, it is not even known whether the small data global well-posedness result would hold if there is only partially damping term (with only u_2 in corresponding velocity equation). Besides, during the derivation of vorticity equation from velocity equation, the partially damping term becomes a nonlocal operator $\mathcal{R}_1^2\omega$. We also discover that by replacing $\mathcal{R}_1^2\omega$ with different operators (e.g. $\mathcal{R}_1\mathcal{R}_2\omega$), the solutions will behave quite differently.

When we consider the periodic domain, it is very convenient to use spectral method in the numerical computation. But to get a better simulation of how solution behaves, it requires finer mesh and this is quite computational consuming. So, in order to reduce the computation time in an acceptable range, We used high performance computing and performed numerical simulation with supercomputer. On the one hand, the numerical experiment provides a great illumination of how long initial data will exist for equations, and this could guides us whether we could obtain well-posedness of that equations. On the other hand, the numerical simulations also show how functions, especially trigonometric functions, look like in Fourier space, this is quite beneficial in finding the blowup initial data in Fourier space together with analyzing the Fourier transform of certain equation.

CHAPTER 2

Preliminaries

2.1 Useful Inequalities

Lemma 2.1.1 (Hölder's inequality) *For $1 \le p \le q \le \infty$ and $1/p + 1/q = 1$,*

$$\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q} \tag{2.1}$$

where $\|\cdot\|_{L^p}$ denotes the standard L^p norm with

$$\|f\|_{L^p} = \begin{cases} \left(\int |f|^p\right)^{1/p}, & \text{for } 1 \le p < \infty, \\ esssup|f|, & \text{for } p = \infty. \end{cases}$$

More general form can be obtained by interpolation: for $\lambda \in (0,1)$,

$$\|f\|_{L^r} \le \|f\|_{L^p}^{\lambda} \|f\|_{L^q}^{1-\lambda}$$

for $f \in L^q$ with $p \le r \le q$ (Or $f \in L^p \cap L^q$) and $\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$.

Furthermore, similar application of Hölder gives Lyapunov's inequality: If $r = \lambda p + (1-\lambda)q$ with $\lambda \in (0,1)$, then

$$\|f\|_{L^r}^r \le \|f\|_{L^p}^{p\lambda} \|f\|_{L^q}^{q(1-\lambda)}$$

Lemma 2.1.2 (Minkowski inequality) *For $1 \le p \le \infty$,*

$$\left(\int \left|\int f(x,y)dx\right|^p dy\right)^{1/p} \le \int \left(\int |f(x,y)|^p dy\right)^{1/p} dx. \tag{2.2}$$

More generally, for $1 \le p \le q \le \infty$,

$$\|f\|_{L^p_x L^q_y} \le \|f\|_{L^q_y L^p_x}.$$

Lemma 2.1.3 (Young's inequality) *For* $a, b > 0$, $1 < p, q < \infty$ *and* $1/p + 1/q = 1$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Moreover, Young's inequality can be used to prove the following convolution inequality: for $1 \le p, q, r, \infty$,

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q} \tag{2.3}$$

with $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

Lemma 2.1.4 (Hausdorff-Young inequality) *If* $f \in L^p$ *for* $1 < p \le 2$, *the optimal bound is*

$$\|\hat{f}\|_{L^q} \le p^{1/2p} q^{-1/2q} \|f\|_{L^p} \tag{2.4}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.1.5 (Hardy-Littlewood maximal inequality) *Suppose* f *is locally integrable, the Hardy-Littlewood maximal operator* M *is defined by*

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

1. *Weak type estimate. If* $f \in L^1(\mathbb{R}^d)$, *then for every* $\alpha > 0$,

   $$m\{x : (Mf)(x) > \alpha\} \le \frac{A}{\alpha} \int_{\mathbb{R}^d} |f| dx,$$

   where A *is a constant which depends only on the dimension* n *(*$A = 5^n$ *will work).*

2. Strong type estimate. If $f \in L^p(\mathbb{R}^d)$, with $1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R}^d)$ and

$$\|Mf\|_p \leq A_p \|f\|_p,$$

where A_p depends on p and n.

Lemma 2.1.1 (Gronwall inequality) *Assume $a = a(t) \geq 0$ and $\int_0^T a(t)dt < \infty$. Assume $g \geq 0$ and $g \in L^1(0,T)$ ($\int_0^T g(t) < \infty$). Then, if $f \geq 0$ satisfies*

$$\frac{d}{dt} f \leq af + g$$

then

$$f(t) \leq e^{\int_0^t a(\tau)d\tau} f(0) + \int_0^t e^{\int_s^t a(\tau)d\tau} g(s)ds$$

for any $t \in [0,T)$. Especially,

$$f(t) \leq e^{\int_0^t a(\tau)d\tau}(f(0) + \int_0^t g(s)ds)$$

Lemma 2.1.2 (Osgood inequality) *Let $a > 0$ and $0 \leq t_0 < T$. Let ρ be a measurable function from $[t_0,T]$ to $[0,a]$. Let $\gamma(t) > 0$ be a locally integrable function on $[t_0,T]$. Let $\phi \geq 0$ be a continuous and non-decreasing function on $[0,a]$. Assume that ρ satisfies, for some constant c*

$$\rho(t) \leq c + \int_{t_0}^t \gamma(s)\phi(\rho(s))ds \quad \text{for a.e. } t \in [t_0,T]$$

Then, if $c > 0$, we have, for a.e. $t \in [t_0,T]$,

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leq \int_{t_0}^t \gamma(\tau)d\tau,$$

where

$$\mathcal{M}(x) = \int_x^a \frac{dr}{\phi(r)}.$$

If $c = 0$ and

$$\int_0^a \frac{dr}{\phi(r)} = \infty,$$

then $\rho(t) = 0$ a.e. $t \in [t_0, T]$.

2.2 Sobolev Space and Sobolev Embedding

Definition 2.2.1 (Sobolev Space) *Let $1 \leq p \leq \infty$ and suppose f is locally integrable function such that for each multi-index α with $|\alpha| \leq k$, $D^\alpha f \in L^p$ exists in the weak sense. Then, Sobolev space $W^{k,p}$ consists of all such f with $\|f\|_{W^{k,p}} < \infty$ where*

$$\|f\|_{W^{k,p}} = \begin{cases} \left(\sum_{|\alpha|\leq k} \int |D^\alpha f|^p dx\right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \sum_{|\alpha|\leq k} esssup|D^\alpha f|, & \text{for } p = \infty. \end{cases}$$

For $p = 2$, we usually write $W^{k,2} = H^2$, which comes from the alternative definition of Sobolev space by using Fourier transform. That is, locally integrable function f in H^k if

$$\|f\|_{H^k} = \int (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi < \infty.$$

Lemma 2.2.1 *Let $1 \leq p < \infty$ and $k \geq 0$. Then the space $C_c^\infty(\mathbb{R}^d)$ of test functions is a dense subspace of $W^{k,p}(\mathbb{R}^d)$. However, at the endpoint $p = \infty$, the closure of $C_c^\infty(\mathbb{R}^d)$ in $W^{k,\infty}(\mathbb{R}^d)$ is $C_0^k(\mathbb{R}^d)$.*

Lemma 2.2.1 shows the closure of $C_c^\infty(\mathbb{R})$ in $L^\infty(\mathbb{R})$ is $C_0(\mathbb{R})$, we actually have stronger embedding, that $W^{1,1}(\mathbb{R})$ embeds continuously into $C_0(\mathbb{R})$. More generally,

Lemma 2.2.2 *For all $u \in W^{d,1}(\mathbb{R}^d)$,*

$$\|u\|_{C_0(\mathbb{R}^d)} \leq C(d)\|u\|_{W^{d,1}(\mathbb{R}^d)}.$$

Proposition 2.2.1 *Suppose p, r are finite, then $\mathcal{S}_0(\mathbb{R}^d)$ is dense in $\dot{B}^s_{p,r}(\mathbb{R}^d)$. When $r = \infty$, the closure of $\mathcal{S}_0$ for the Bosev norm $\dot{B}^s_{p,r}$ is the set in $\mathcal{S}'_h$ such that*

$$\lim_{j\to\pm\infty} 2^{js}\|\dot{\Delta}_j u\|_{L^p} = 0.$$

Lemma 2.2.3 (Gagliardo-Nirenberg-Sobolev ineqality) *[12] Assume $1 \le p < d$. Then, for all $u \in C^1_c(\mathbb{R}^d)$.*

$$\|u\|_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} \le C(p,d)\|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

The case $1 < p < d$ is due to Sobolev and case $p = 1$ to Gagliardo and Nirenberg. This proves $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{\frac{dp}{d-p}}(\mathbb{R}^d)$ when $1 \le p < d$.

Corollary 2.2.1 *[12] Suppose $1 \le p < d$ and assume $U \subset \mathbb{R}^d$ is open and bounded. Then, for $u \in W^{1,p}_0(U)$ we have the estimate*

$$\|u\|_{L^q(U)} \le C(p,q,d,U)\|\nabla u\|_{L^p(U)}$$

for each $q \in [1, \frac{dp}{d-p}]$.

Moreover, above embedding is compact when $1 \le q < \frac{dp}{d-p}$. That is,

Lemma 2.2.4 (Rellich-Kondrachov Compactness Theorem) *Suppose $1 \le p < d$ and assume $U \subset \mathbb{R}^d$ is open, bounded Lipschitz domain. Then,*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $q \in [1, \frac{dp}{d-p})$ (or $1 > 0$ and $\frac{d}{p} - 1 < \frac{d}{q} - 0$).

Lemma 2.2.5 *[1] For $p = d$ and $p \le q < \infty$,*

$$W^{m,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$$

Lemma 2.2.6 *If $u \in W^{1,d}(\mathbb{R}^d)$, then u is a function of bounded mean oscillation and*

$$\|u\|_{BMO} \leq C(d)\|Du\|_{L^d(\mathbb{R}^d)}.$$

This estimate is a corollary of the Poincare inequality.

Lemma 2.2.7 (Morrey's inequality) *[12] Assume $d < p \leq \infty$. Then, for all $u \in C^1(\mathbb{R}^d)$ with $\gamma = 1 - \frac{n}{p}$*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^d)} \leq C(p,d)\|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Corollary 2.2.2 *Assume $U \subset \mathbb{R}^d$ is open and bounded, and suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then, u has a version $u^* \in C^{0,\gamma}(\bar{U})$, with $\gamma = 1 - \frac{d}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C(p,d,U)\|u\|_{W^{1,p}(U)}.$$

Lemma 2.2.8 (Sobolev embedding with one derivative) *Let $1 \leq p \leq q \leq \infty$ be such that $\frac{d}{p} - 1 \leq \frac{d}{q} \leq \frac{d}{p}$ ($p \leq q \leq \frac{dp}{d-p}$), but not the endpoint cases $(p,q) = (d,\infty), (1, \frac{d}{d-1})$. Then $W^{1,p}(\mathbb{R}^d)$ embeds continuously into $L^q(\mathbb{R}^d)$.*

2.3 Bosev Space and the Littlewood-Paley Decomposition in $\mathbb{R}^d$

To introduce the Bosev spaces, we start with a few notation. $\mathcal{S}$ denotes the usual Schwartz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma \, dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^{\perp} = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials. For each $j \in \mathbb{Z}$, we write

$$A_j = \left\{ \xi \in \mathbb{R}^d : \ 2^{j-1} \le |\xi| < 2^{j+1} \right\}. \tag{2.5}$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 \quad , & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 \quad , & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0' \tag{2.6}$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f = 2^{jd} \int \Phi_0(2^j(x-y))\, f(y)\, dy, \qquad j \in \mathbb{Z}. \tag{2.7}$$

The homogeneous Littlewood-Paley decomposition (2.6) can then be written as

$$f = \sum_{j=-\infty}^{\infty} \mathring{\Delta}_j f, \qquad f \in \mathcal{S}_0'.$$

Definition 2.3.1 *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Bosev space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{2.8}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Bosev space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{2.9}$$

The inhomogeneous Littlewood-Paley decomposition (2.8) can then be written as

$$f = \sum_{j=-1}^{\infty} \Delta_j f, \qquad f \in \mathcal{S}'.$$

Definition 2.3.2 *The inhomogeneous Bosev space* $B^s_{p,q}$ *with* $1 \le p, q \le \infty$ *and* $s \in \mathbb{R}$ *consists of functions* $f \in \mathcal{S}'$ *satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

The Bosev spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \le p, q \le \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way. We will also use the space-time spaces introduced by Chemin-Lerner (see, e.g., [4]).

Definition 2.3.3 *For* $t > 0$, $s \in \mathbb{R}$ *and* $1 \le p, q, r \le \infty$, *the space-time spaces* $\widetilde{L}^r_t \mathring{B}^s_{p,q}$ *and* $\widetilde{L}^r_t B^s_{p,q}$ *are defined though the norms*

$$\|f\|_{\widetilde{L}^r_t \mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^r_t L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^r_t L^p}\|_{l^q}.$$

Here L^r_t is the abbreviation for $L^r(0,t)$. These spaces are related to the classical space-time spaces $L^r_t \mathring{B}^s_{p,q}$ and $L^r_t B^s_{p,q}$ via the Minkowski inequality, if $r \ge q$,

$$\widetilde{L}^r_t \mathring{B}^s_{p,q} \subseteq L^r_t \mathring{B}^s_{p,q}, \qquad \widetilde{L}^r_t B^s_{p,q} \subseteq L^r_t B^s_{p,q}$$

and, if $r < q$,

$$\widetilde{L}^r_t \mathring{B}^s_{p,q} \supset L^r_t \mathring{B}^s_{p,q}, \qquad \widetilde{L}^r_t B^s_{p,q} \supset L^r_t B^s_{p,q}$$

Many frequently used function spaces are special cases of Bosev spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition 2.3.1 *For any* $s \in \mathbb{R}$,

$$\mathring{H}^s \sim \mathring{B}^s_{2,2}, \quad H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ *and* $1 < q < \infty$,

$$\mathring{B}^s_{q,\min\{q,2\}} \hookrightarrow \mathring{W}^s_q \hookrightarrow \mathring{B}^s_{q,\max\{q,2\}}.$$

In particular, $\mathring{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \mathring{B}^0_{q,\max\{q,2\}}$.

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (2.9). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j and

$$S_j f(x) = 2^{dj} \Psi(2^j x) * f(x) = 2^{dj} \int \Psi(2^j (x-y))\, f(y)\, dy.$$

The operators Δ_j and S_j defined above satisfy the following properties:

$$\Delta_j \Delta_k f = 0 \quad \text{if } |k-j| \geq 2 \text{ and } \quad \Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |k-j| \geq 3.$$

Bernstein's inequalities is a useful tool on Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition 2.3.2 (Bernstein type inequalities) *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

1) If f satisfies

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \leq K2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

We shall also use Bonys notion of paraproducts to decompose a product into three parts

$$f\, g = T_f\, g + T_g\, f + R(f, g),$$

where

$$T_f\, g = \sum_j S_{j-1} f \Delta_j g, \qquad R(f, g) = \sum_j \sum_{k \geq j-1} \Delta_k f \widetilde{\Delta}_k g$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$.

CHAPTER 3

Boussinesq Equation with Partial Dissipation in Bounded Domain

3.1 Introduction

In this chapter, we will study the global in time regularity problem of the 2D Boussinesq equations with only kinematic dissipation (without thermal diffusion):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \, \Delta \mathbf{u} + \theta \mathbf{e}_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{3.1}$$

Our attention will be mainly focused on spatial domains $\Omega \subset \mathbb{R}^2$ that are bounded, connected and have smooth boundary, although the results presented here are also valid for $\Omega = \mathbb{R}^2$ and periodic domains, as explained later. We assume the velocity field $\mathbf{u}$ obeys the Navier boundary conditions. The Navier boundary conditions allow the fluid to slip along the boundary and require that the tangential component of the stress vector at the boundary be proportional to the tangential velocity. In the case of (3.1), the corresponding stress tensor $T = (T_{ij})$ is given by

$$T_{ij} = -\delta_{ij} p + 2\nu D_{ij}(\mathbf{u}), \qquad D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \text{or} \quad D(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

and, if $\mathbf{n}$ and τ are unit normal and tangent vectors to the boundary $\partial\Omega$, respectively, the proportionality is then represented by

$$\sum_{i,j=1,2} \tau_i \, T_{ij} \, n_j = \sigma \sum_{k=1,2} u_k \tau_k \quad \text{on} \quad \partial\Omega$$

for a constant σ. Due to the orthogonality of $\mathbf{n}$ and τ,

$$\sum_{i,j=1,2} \tau_i \, \delta_{ij} p \, n_j = 0.$$

The Navier boundary conditions for (3.1) then become

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad 2\,\mathbf{n} \cdot D(\mathbf{u}) \cdot \tau + \alpha \mathbf{u} \cdot \tau = 0 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

where $\alpha > 0$ is a constant.

In addition, (3.1) will be supplemented with the initial data

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \Omega. \tag{3.3}$$

The goal of this chapter is to prove the following theorem

Theorem 3.1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Let $\nu > 0$. Consider the initial and boundary value problem (IBVP) in (3.1) and (3.2) with $\alpha > 0$ being a constant and*

$$\mathbf{u}_0 \in H^1(\Omega), \quad \nabla \cdot \mathbf{u}_0 = 0$$

and

$$\theta_0 \in L^2(\Omega) \cap L^\infty(\Omega), \qquad \int_\Omega \theta_0(x)\, dx = 0.$$

Then the IBVP (3.1) and (3.2) has a unique global (in time) strong solution $(\mathbf{u}, \theta)$ *satisfying, for any* $T > 0$,

$$\begin{aligned}
&\mathbf{u} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;\dot{H}^2(\Omega)),\\
&\theta \in L^\infty(0,\infty;L^2(\Omega) \cap L^\infty(\Omega)),\\
&\int_\Omega \theta(x,t)\,dx = 0 \;\; \text{for any } t \in [0,\infty).
\end{aligned}$$

3.2 Integration by Parts Lemma for Navier Boundary Conditions

Due to the Navier boundary conditions, the integration by parts process in general generates boundary terms. The following two lemmas facilitate the integration by parts process. They are especially useful when we handle the dissipative term. We provide the proofs of Lemma 3.2.1 and Lemma 3.2.2. Some components can be found in [7, 22].

Lemma 3.2.1 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded and connected domain with sufficient smooth boundary, say* $\partial\Omega \in C^2$. *Let* κ *denote the curvature of* $\partial\Omega$. *As before,* τ *and* $\mathbf{n}$ *denote the unit tangential and out-normal vector along* $\partial\Omega$. *Assume* $u \in C^1(\overline{\Omega})$.

(1) Assume $\mathbf{u} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$. *Writing* $\tau \cdot \nabla\mathbf{u} \cdot \mathbf{n} = \tau_k \partial_k u_j n_j$ *with Einstein's summation convention, we have*

$$\tau \cdot \nabla\mathbf{u} \cdot \mathbf{n} + \kappa\, \mathbf{u} \cdot \tau = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

(2) Assume $\mathbf{u} \in C^1(\overline{\Omega})$ *satisfies the Navier boundary conditions*

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad 2\mathbf{n} \cdot D(\mathbf{u}) \cdot \tau + \alpha\, \mathbf{u} \cdot \tau = 0 \text{ on } \partial\Omega. \tag{3.5}$$

Then,

$$\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau+(\alpha-\kappa)\,\mathbf{u}\cdot\tau=0 \ \textit{on}\ \partial\Omega \tag{3.6}$$

and

$$\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau+\frac{\alpha}{2}(\mathbf{u}\cdot\tau)=\frac{\omega}{2}\ \textit{on}\ \partial\Omega. \tag{3.7}$$

Especially, $\omega=0$ *on* $\partial\Omega$ *if and only if* $\kappa=\frac{\alpha}{2}$.

Proofs of Lemma 3.2.1. Since $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$, the directional derivative of $\mathbf{u}\cdot\mathbf{n}$ along $\partial\Omega$ should also be zero, namely

$$\frac{d}{d\tau}(\mathbf{u}\cdot\mathbf{n})=0 \quad \text{on} \quad \partial\Omega.$$

The product rule then yields

$$\left(\frac{d}{d\tau}\mathbf{u}\right)\cdot\mathbf{n}+\mathbf{u}\cdot\left(\frac{d}{d\tau}\mathbf{n}\right)=0 \quad \text{or} \quad \tau\cdot\nabla\mathbf{u}\cdot\mathbf{n}+\mathbf{u}\cdot(\tau\cdot\nabla\mathbf{n})=0 \quad \text{on} \quad \partial\Omega.$$

Due to $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$,

$$\mathbf{u}=(\mathbf{u}\cdot\mathbf{n})\mathbf{n}+(\mathbf{u}\cdot\tau)\tau=(\mathbf{u}\cdot\tau)\tau \quad \text{on} \quad \partial\Omega.$$

Therefore, due to $\kappa=\tau\cdot\nabla\mathbf{n}\cdot\tau$,

$$\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n}+(\tau\cdot\nabla\mathbf{n}\cdot\tau)\,(\mathbf{u}\cdot\tau)=0 \quad \text{or} \quad \tau\cdot\nabla\mathbf{u}\cdot\mathbf{n}+\kappa\,\mathbf{u}\cdot\tau=0 \quad \text{on} \quad \partial\Omega.$$

To prove (3.6), we recall $2D(\mathbf{u})=\nabla\mathbf{u}+(\nabla\mathbf{u})^T$, and invoke (3.4) and (3.5)

$$\begin{aligned}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau &= 2\mathbf{n}\cdot D(\mathbf{u})\cdot\tau-\mathbf{n}\cdot(\nabla\mathbf{u})^T\cdot\tau\\ &= -\alpha\,(\mathbf{u}\cdot\tau)-\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n}\\ &= (\kappa-\alpha)\,(\mathbf{u}\cdot\tau).\end{aligned}$$

To prove (3.7), we write

$$\nabla \mathbf{u} = D(\mathbf{u}) + \frac{1}{2}\left(\nabla u - (\nabla u)^T\right) = D(\mathbf{u}) + \frac{1}{2}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} \mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau &= \mathbf{n}\cdot D(\mathbf{u})\cdot\tau + \mathbf{n}\cdot\frac{1}{2}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\cdot\tau \\ &= -\frac{\alpha}{2}(\mathbf{u}\cdot\tau) + \frac{\omega}{2}(-\tau_1 n_2 + n_1\tau_2) \\ &= -\frac{\alpha}{2}(\mathbf{u}\cdot\tau) + \frac{\omega}{2} \end{aligned}$$

due to $-\tau_1 n_2 + n_1\tau_2 = \tau_1^2 + \tau_2^2 = 1$. This completes the proof of Lemma 3.2.1. □

As we shall see in the subsequent sections, $\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau$ plays a crucial role in the handling of the dissipation and the identities stated here will be very handy. We alert that $\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n}$ differs from $\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau$ in general.

Lemma 3.2.2 *Assume Ω obeys the same conditions as in Lemma 3.2.1. Assume that $\mathbf{u}, \mathbf{v} \in C^2(\Omega)\cap C^1(\overline{\Omega})$ and they both satisfy the Navier boundary conditions, namely (3.5). Then*

$$\begin{aligned} \int_\Omega \Delta\mathbf{u}\cdot\mathbf{v}\,dx &= -2\int_\Omega D(\mathbf{u})\cdot D(\mathbf{v})\,dx - \int_{\partial\Omega}\alpha(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x) & (3.8) \\ &= -\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_{\partial\Omega}(\kappa-\alpha)(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x). & (3.9) \end{aligned}$$

In particular, when $\mathbf{u} = \mathbf{v}$, we have

$$\begin{aligned} \int_\Omega \Delta\mathbf{u}\cdot\mathbf{u}\,dx &= -2\int_\Omega |D(\mathbf{u})|^2\,dx - \int_{\partial\Omega}\alpha(\mathbf{u}\cdot\tau)^2\,dS(x) \\ &= -\int_\Omega |\nabla\mathbf{u}|^2\,dx + \int_{\partial\Omega}(\kappa-\alpha)(\mathbf{u}\cdot\tau)^2\,dS(x). \end{aligned}$$

Proofs of Lemma 3.2.2. Adopting Einstein's summation convention, we write

$$\begin{aligned}\int_\Omega \Delta\mathbf{u}\cdot\mathbf{v}\,dx &= \int_\Omega (\partial_k\partial_k u_j)v_j\,dx\\ &= \int_\Omega (\partial_k(\partial_k u_j v_j) - \partial_k u_j \partial_k v_j)\,dx\\ &= \int_{\partial\Omega} n_k\partial_k u_j v_j\,dx - \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx\\ &= \int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{v}\,dS(x) - \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx\end{aligned}$$

Due to $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, we write $\mathbf{v} = (\mathbf{v}\cdot\tau)\tau$ and obtain, by Lemma 3.2.1,

$$\begin{aligned}\int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{v}\,dS(x) &= \int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau(\mathbf{v}\cdot\tau)dS(x)\\ &= \int_{\partial\Omega} (\kappa-\alpha)(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)dS(x)\end{aligned}$$

Therefore, we have obtained (3.9),

$$\int_\Omega \Delta\mathbf{u}\cdot\mathbf{v}\,dx = -\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_{\partial\Omega}(\kappa-\alpha)(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x).$$

To prove (3.8), we write out the terms in $D(\mathbf{u})\cdot D(\mathbf{v})$,

$$\begin{aligned}2\int_\Omega D(\mathbf{u})\cdot D(\mathbf{v})\,dx &= \int_\Omega \left(\nabla\mathbf{u}\cdot\nabla\mathbf{v} + \nabla\mathbf{u}\cdot(\nabla\mathbf{v})^T\right)dx\\ &= \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_\Omega \partial_j u_k\partial_k v_j\,dx\\ &= \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_\Omega \partial_k(\partial_j u_k v_j)\,dx\\ &= \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_{\partial\Omega} n_k\partial_j u_k v_j\,dS(x)\\ &= \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}dx + \int_{\partial\Omega} \mathbf{v}\cdot\nabla\mathbf{u}\cdot\mathbf{n}\,dS(x).\end{aligned}$$

Writing $\mathbf{v} = (\mathbf{v}\cdot\tau)\tau$ and applying Lemma 3.2.1, we have

$$2\int_\Omega D(\mathbf{u})\cdot D(\mathbf{v})\,dx = \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx - \kappa\int_{\partial\Omega}(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x). \qquad (3.10)$$

Combining (3.9) and (3.10) yields (3.8). This completes the proof of of Lemma 3.2.2.
□

3.3 Existence

3.3.1 Global L^2 bound

This subsection proves the *a priori* bounds stated in the following proposition.

Proposition 3.3.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^2$. Assume the initial data $(\mathbf{u}_0, \theta_0)$ satisfies the conditions stated in Theorem 3.1.1. Let $(\mathbf{u}, \theta)$ be the corresponding solution of the IBVP (3.1), (3.2) and (3.3). Then $(\mathbf{u}, \theta)$ obeys the global bounds, for any $t > 0$,*

$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q} \quad \text{for any} \quad 2 \le q \le \infty,$$
$$\|\mathbf{u}(t)\|_{L^2} \le \|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2}, \qquad \int_0^t \int_\Omega |\nabla \mathbf{u}|^2 \, dx \, d\tau \le (\|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Proof of Proposition 3.3.1. For any $2 \le q < \infty$, we obtain by multiplying the equation of θ in (3.1) by $\theta|\theta|^{q-2}$,

$$\frac{1}{q}\frac{d}{dt}\|\theta\|_{L^q}^q = -\int_\Omega \theta|\theta|^{q-2}\mathbf{u} \cdot \nabla\theta \, dx.$$

Due to $\nabla \cdot \mathbf{u} = 0$, the divergence theorem and (3.2),

$$\int_\Omega \theta|\theta|^{q-2}\mathbf{u} \cdot \nabla\theta \, dx = \frac{1}{q}\int_{\partial\Omega} |\theta|^q \mathbf{u} \cdot \mathbf{n} \, dS(x) = 0.$$

As a consequence, for any $t > 0$,

$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q} \quad \text{and} \quad \|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}.$$

Taking the inner product of $\mathbf{u}$ with the equation of $\mathbf{u}$ in (3.1) yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 = \nu \int_\Omega \mathbf{u}\cdot\Delta\mathbf{u}\,dx + \int_\Omega \theta\, u_2\,dx, \tag{3.11}$$

where we have invoked the facts, due to $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$,

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\,\cdot\mathbf{u}\,dx = \frac{1}{2}\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})|\mathbf{u}|^2\,dS(x) = 0, \qquad \int_\Omega \mathbf{u}\cdot\nabla p\,dx = 0.$$

According to Lemma 3.2.2,

$$\int_\Omega \mathbf{u}\cdot\Delta\mathbf{u}\,dx = -2\int_\Omega |D(\mathbf{u})|^2\,dx - \int_{\partial\Omega} \alpha\,(\mathbf{u}\cdot\tau)^2\,dS(x).$$

Therefore,

$$\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + 4\nu\int_\Omega |D(\mathbf{u})|^2\,dx + 2\nu\int_{\partial\Omega} \alpha\,(\mathbf{u}\cdot\tau)^2\,dS(x) \le 2\,\|\theta\|_{L^2}\|\mathbf{u}\|_{L^2},$$

which, in particular, implies

$$\|\mathbf{u}(t)\|_{L^2} \le \|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2}.$$

Furthermore, for any $t > 0$,

$$\int_0^t\int_\Omega |D(\mathbf{u})|^2\,dxd\tau, \quad \int_0^t\int_{\partial\Omega} \alpha\,(\mathbf{u}\cdot\tau)^2\,dS(x) \le (\|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

By Lemma 3.2.2,

$$\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,dx\,d\tau \le (\|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

This completes the proof of Proposition 3.3.1. □

3.3.2 Global H^1 bound

This subsection establishes the global H^1-bound for $\mathbf{u}$.

Proposition 3.3.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Assume the initial data $(\mathbf{u}_0, \theta_0)$ satisfies the conditions stated in Theorem 3.1.1. Let $(\mathbf{u}, \theta)$ be the corresponding solution of the IBVP (3.1), (3.2) and (3.3). Then $(\mathbf{u}, \theta)$ obeys the global H^1 bounds, for any $t > 0$,*

$$\|\nabla \mathbf{u}(t)\|_{L^2}, \quad \int_0^t \|\mathbf{u}\|_{H^2(\Omega)}^2 \, d\tau, \quad \int_0^t \|p\|_{H^1(\Omega)}^2 \, d\tau \le C(t, \|\mathbf{u}_0\|_{H^1}, \|\theta_0\|_{L^2 \cap L^\infty}).$$

Proof. Recall the definition of the operator A defined in (3.17). Dotting the velocity equation in (3.1) by $A\mathbf{u}$ yields

$$\nu \|A\mathbf{u}\|_{L^2}^2 = -\int_\Omega \partial_t \mathbf{u} \cdot A\mathbf{u} \, dx - \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot A\mathbf{u} \, dx + \int_\Omega \theta e_2 \cdot A\mathbf{u} \, dx. \tag{3.12}$$

By the definition of A in (3.17),

$$-\int_\Omega \partial_t \mathbf{u} \cdot A\mathbf{u} \, dx = \int_\Omega \partial_t \mathbf{u} \cdot \Delta \mathbf{u} \, dx - \frac{1}{\nu} \int_\Omega \partial_t \mathbf{u} \cdot \nabla p \, dx.$$

Writing the dot product in terms of the components and adopting Einstein's summation convention, we have

$$\begin{aligned} \int_\Omega \partial_t \mathbf{u} \cdot \Delta \mathbf{u} \, dx &= \int_\Omega \partial_k (\partial_t u_j \, \partial_k u_j) \, dx - \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \mathbf{u}|^2 \, dx \\ &= -\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \int_{\partial\Omega} n_k \partial_k u_j \partial_t u_j \, dS(x). \end{aligned}$$

Since $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we can write

$$\mathbf{u} = (\mathbf{u} \cdot \tau)\tau \quad \text{on } \partial\Omega.$$

By Lemmas 3.2.1 and 3.2.2,

$$\begin{aligned}\int_\Omega \partial_t \mathbf{u}\cdot \Delta \mathbf{u}\, dx &= -\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_{L^2}^2 + \int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau\,\partial_t(\mathbf{u}\cdot\tau)\,dx \\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_{L^2}^2 + (\kappa-\alpha)\int_\Omega (\mathbf{u}\cdot\tau)\,\partial_t(\mathbf{u}\cdot\tau)\,dx \\ &= -\frac{1}{2}\frac{d}{dt}\left(\|\nabla \mathbf{u}\|_{L^2}^2 + (\alpha-\kappa)\|\mathbf{u}\cdot\tau\|_{L^2(\partial\Omega)}^2\right) \\ &= -\frac{1}{2}\frac{d}{dt}\left(2\|D(\mathbf{u})\|_{L^2}^2 + \alpha\,\|\mathbf{u}\cdot\tau\|_{L^2(\partial\Omega)}^2\right).\end{aligned}$$

By $\nabla\cdot\mathbf{u}=0$ in Ω and $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$,

$$\int_\Omega \partial_t\mathbf{u}\cdot\nabla p\,dx = \int_\Omega \nabla\cdot(p\,\partial_t\mathbf{u})\,dx = \int_{\partial\Omega} p\,\mathbf{n}\cdot\partial_t\mathbf{u}\,dS(x) = 0.$$

By Hölder's inequality,

$$\left|\int_\Omega \theta e_2\cdot A\mathbf{u}\,dx\right| \le \|\theta\|_{L^2}\,\|A\mathbf{u}\|_{L^2} \le \frac{\nu}{4}\|A\mathbf{u}\|_{L^2}^2 + C\,\|\theta_0\|_{L^2}^2.$$

By Hölder's inequality, Ladyzhenskaya's inequality and (3.18),

$$\begin{aligned}\left|\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot A\mathbf{u}\,dx\right| &\le \|A\mathbf{u}\|_{L^2}\,\|\mathbf{u}\|_{L^4}\,\|\nabla\mathbf{u}\|_{L^4} \\ &\le C\,\|A\mathbf{u}\|_{L^2}\,\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}\,\|\nabla(\nabla\mathbf{u})\|_{L^2}^{\frac{1}{2}} \\ &\le C\,\|A\mathbf{u}\|_{L^2}^{\frac{3}{2}}\,\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2} \\ &\le \frac{\nu}{4}\|A\mathbf{u}\|_{L^2}^2 + C\,\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^4.\end{aligned}$$

By Lemma 3.2.2,

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^2}^2 &= 2\|D(\mathbf{u})\|_{L^2}^2 + \kappa\,\|\mathbf{u}\cdot\tau\|_{L^2(\partial\Omega)}^2 \\ &\le \left(1+\frac{|\kappa|}{\alpha}\right)\left(2\|D(\mathbf{u})\|_{L^2}^2 + \alpha\,\|\mathbf{u}\cdot\tau\|_{L^2(\partial\Omega)}^2\right). \qquad (3.13)\end{aligned}$$

Inserting the estimates above in (3.12) and writing

$$Y(t) \equiv 2\|D(\mathbf{u})\|_{L^2}^2 + \alpha\,\|\mathbf{u}\cdot\tau\|_{L^2(\partial\Omega)}^2,$$

we obtain, after integrating in time,

$$Y(t) + \nu \int_0^t \|A\mathbf{u}(\tau)\|_{L^2}^2\,d\tau \le C\,\|\theta_0\|_{L^2}^2\,t + C\int_0^t \|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\,Y(\tau)\,d\tau.$$

Gronwall's inequality and the global bound in Proposition 3.3.1 imply, for any $t > 0$,

$$\|D(\mathbf{u})\|_{L^2}^2, \quad \|\mathbf{u}\cdot\tau\|_{L^2(\partial\Omega)}^2, \quad \int_0^t \|A\mathbf{u}(\tau)\|_{L^2}^2\,d\tau \le C(t, \|\mathbf{u}_0\|_{H^1}, \|\theta_0\|_{L^2\cap L^\infty}). \tag{3.14}$$

Then, (3.18) and (3.13) lead to the desired global bound in Proposition 3.3.2. □

3.4 Uniqueness

As we know, regularity estimates for solutions of the Navier-Stokes equations with the classical no-slip boundary condition rely on the Stokes operator associated with the no slip boundary condition. For the Stokes problem with Navier type boundary conditions, H. Beirão da Veiga in [29] established a general existence and regularity theory. A special consequence of his theory is provided in the following lemma.

Lemma 3.4.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Let $\alpha \ge 0$ be a constant and let $f \in L^2(\Omega)$. Consider the following Stokes problem with the Navier type boundary condition,*

$$\begin{cases} -\nu\,\Delta\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{u}\cdot\mathbf{n} = 0, \quad 2\,\mathbf{n}\cdot D(\mathbf{u})\cdot\tau + \alpha\mathbf{u}\cdot\tau = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.15}$$

Then (3.15) has a unique strong solution $(\mathbf{u}, p) \in H^2(\Omega) \times H^1(\Omega)$ *(p is unique up to an additive constant). Moreover, for a constant* $C = C(\Omega, \nu)$,

$$\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \le C\, \|f\|_{L^2(\Omega)}. \tag{3.16}$$

For notational convenience, we write

$$A\mathbf{u} \equiv -\Delta \mathbf{u} + \frac{1}{\nu} \nabla p. \tag{3.17}$$

(3.16) implies

$$\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \le C\, \|A\,\mathbf{u}\|_{L^2(\Omega)}. \tag{3.18}$$

This subsection proves the uniqueness part of Theorem 3.1.1. More precisely, we establish the following proposition.

Proposition 3.4.1 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded and connected domain with sufficient smooth boundary, say* $\partial\Omega \in C^{2,1}$. *Assume the initial and boundary conditions as stated in Theorem 3.1.1. Let* $(\mathbf{u}^{(1)}, \theta^{(1)})$ *and* $(\mathbf{u}^{(2)}, \theta^{(2)})$ *be two solutions of the IBVP (3.1), (3.2) and (3.3) satisfying (3.4). Then* $(\mathbf{u}^{(1)}, \theta^{(1)}) = (\mathbf{u}^{(2)}, \theta^{(2)})$.

We need the following existence and regularity result on solutions of the Poisson equation with a Neumann boundary condition. This result can be found in [25] or [28].

Lemma 3.4.2 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded and connected domain with sufficient smooth boundary, say* $\partial\Omega \in C^2$. *Let* $1 < p < \infty$. *Assume* $f \in L^p(\Omega)$ *satisfies*

$$\int_\Omega f(x)\, dx = 0.$$

Then the Poisson equation with a pure Neumann boundary condition

$$\Delta g = f \quad \text{in } \Omega, \qquad \frac{dg}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega$$

has a unique solution g (up to an additive constant) satisfying

$$\|g\|_{W^{2,p}(\Omega)} \le C(\Omega, p)\, \|f\|_{L^p(\Omega)}.$$

We now prove Proposition 3.4.1.

Proof of Proposition 3.4.1. Let $(\mathbf{u}^{(1)}, \theta^{(1)})$ and $(\mathbf{u}^{(2)}, \theta^{(2)})$ be two solutions of the IBVP (3.1), (3.2) and (3.3) satisfying (3.4). Define $h^{(1)}$ and $h^{(2)}$ by

$$\Delta h^{(1)} = \theta^{(1)} \quad \text{in } \Omega, \qquad \frac{dh^{(1)}}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{3.19}$$

$$\Delta h^{(2)} = \theta^{(2)} \quad \text{in } \Omega, \qquad \frac{dh^{(2)}}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega. \tag{3.20}$$

According to Lemma 3.4.2, $h^{(1)}$ and $h^{(2)}$ exist and are unique (up to additive constants). Denote by $p^{(1)}$ and $p^{(2)}$ the associated pressures. Then the differences

$$\tilde{\mathbf{u}} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \quad \tilde{p} = p^{(1)} - p^{(2)}, \quad \tilde{\theta} = \theta^{(1)} - \theta^{(2)}, \quad \tilde{h} = h^{(1)} - h^{(2)}, \quad \tilde{\theta} = \Delta\tilde{h},$$

satisfy

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + (\mathbf{u}^{(1)} \cdot \nabla)\tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} = -\nabla \tilde{p} + \nu \Delta \tilde{\mathbf{u}} + \Delta \tilde{h}\, \mathbf{e}_2, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, \\ \partial_t \Delta \tilde{h} + \mathbf{u}^{(1)} \cdot \nabla(\Delta \tilde{h}) + \tilde{\mathbf{u}} \cdot \nabla \theta^{(2)} = 0, \\ \tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x) = 0, \quad \tilde{\theta}(x,0) = \tilde{\theta}_0(x) = 0. \end{cases} \tag{3.21}$$

Dotting the first equation of (3.21) with $\tilde{\mathbf{u}}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}\|_{L^2}^2 = \nu \int_\Omega \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}\, dx - \int_\Omega \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \tilde{\mathbf{u}}\, dx + \int_\Omega \tilde{u}_2 \Delta \tilde{h}\, dx, \tag{3.22}$$

where, we have invoked the facts, due to $\mathbf{u}^{(1)} \cdot \mathbf{n} = 0$ and $\tilde{\mathbf{u}} \cdot \mathbf{n} = 0$,

$$\int_\Omega \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}\, dx = 0, \qquad -\int_\Omega \nabla \tilde{p} \cdot \tilde{\mathbf{u}}\, dx = 0.$$

According to Lemma 3.2.2,

$$\int_\Omega \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}\, dx = -2\int_\Omega |D(\tilde{\mathbf{u}})|^2\, dx - \int_{\partial\Omega} \alpha(\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x).$$

By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned} \left|\int_\Omega \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \tilde{\mathbf{u}}\, dx\right| &\leq \|\nabla \mathbf{u}^{(2)}\|_{L^2} \|\tilde{\mathbf{u}}\|_{L^4}^2 \\ &\leq C\, \|\nabla \mathbf{u}^{(2)}\|_{L^2} \|\tilde{\mathbf{u}}\|_{L^2} \|\nabla \tilde{\mathbf{u}}\|_{L^2}. \end{aligned}$$

By Young's inequality and (3.13)

$$\begin{aligned} &\left|\int_\Omega \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \tilde{\mathbf{u}}\, dx\right| \\ \leq\ & \frac{1}{4}\left(2\nu \int_\Omega |D(\tilde{\mathbf{u}})|^2\, dx + \nu \int_{\partial\Omega} \alpha(\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x)\right) + C\, \|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\tilde{\mathbf{u}}\|_{L^2}^2. \end{aligned}$$

By the divergence theorem and the definitions of h_1 in (3.19) and h_2 in (3.20),

$$\begin{aligned} \int_\Omega \tilde{u}_2\, \Delta \tilde{h}\, dx &= \int_{\partial\Omega} \frac{d\tilde{h}}{d\mathbf{n}}\, \tilde{u}_2\, dS(x) - \int_\Omega \nabla \tilde{h} \cdot \nabla \tilde{u}_2\, dx \\ &= -\int_\Omega \nabla \tilde{h} \cdot \nabla \tilde{u}_2\, dx. \end{aligned}$$

By Höler's inequality and (3.13),

$$\begin{aligned} \left|\int_\Omega \tilde{u}_2\, \Delta \tilde{h}\, dx\right| &\leq \|\nabla \tilde{\mathbf{u}}\|_{L^2}\, \|\nabla \tilde{h}\|_{L^2} \\ &\leq \frac{1}{4}\left(2\nu \int_\Omega |D(\tilde{\mathbf{u}})|^2\, dx + \nu \int_{\partial\Omega} \alpha(\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x)\right) + C\, \|\nabla \tilde{h}\|_{L^2}^2. \end{aligned}$$

Combining the estimates above with (3.22), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\tilde{\mathbf{u}}\|_{L^2}^2 + 2\nu \int_\Omega |D(\tilde{\mathbf{u}})|^2\,dx + \nu\,\alpha \int_{\partial\Omega} (\tilde{\mathbf{u}}\cdot\tau)^2\,dS(x) \\
\le\ & C\,\|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\tilde{\mathbf{u}}\|_{L^2}^2 + C\,\|\nabla \tilde{h}\|_{L^2}^2. \qquad (3.23)
\end{aligned}
$$

Multiplying the equation of $\tilde{h}$ by $\tilde{h}$ and integrating over Ω, we obtain

$$
\frac{d}{dt}\|\nabla\tilde{h}\|_{L^2}^2 = \int_\Omega \mathbf{u}^{(1)}\cdot\nabla(\Delta\tilde{h})\tilde{h}\,dx + \int_\Omega \tilde{\mathbf{u}}\cdot\nabla\theta^{(2)}\tilde{h}\,dx. \qquad (3.24)
$$

By integration by parts and Hölder's inequality,

$$
\begin{aligned}
\left|\int_\Omega \tilde{\mathbf{u}}\cdot\nabla\theta^{(2)}\tilde{h}\,dx\right| &= \left|\int_\Omega \theta^{(2)}\tilde{\mathbf{u}}\cdot\nabla\tilde{h}\,dx\right| \\
&\le \|\theta^{(2)}\|_{L^\infty}\left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\tilde{h}\|_{L^2}^2\right).
\end{aligned}
$$

The first term on the right of (3.24) is more difficult to handle. By integrating by parts and invoking the boundary conditions for $\mathbf{u}^{(1)}$ and $\tilde{h}$, we have

$$
\begin{aligned}
&\int_\Omega \mathbf{u}^{(1)}\cdot\nabla(\Delta\tilde{h})\tilde{h}\,dx \\
=\ & -\int_\Omega \Delta\tilde{h}\,\mathbf{u}^{(1)}\cdot\nabla\tilde{h}\,dx + \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{u}^{(1)}\,\Delta\tilde{h}\,\tilde{h}\,dS(x) \\
=\ & \int_\Omega \partial_k\partial_k\tilde{h}\,\mathbf{u}^{(1)}\cdot\nabla\tilde{h}\,dx \\
=\ & -\int_\Omega \partial_k\tilde{h}(\partial_k\mathbf{u}^{(1)}\cdot\nabla\tilde{h} + \mathbf{u}^{(1)}\cdot\nabla\partial_k\tilde{h})\,dx + \int_{\partial\Omega}\frac{d\tilde{h}}{d\mathbf{n}}\,\mathbf{u}^{(1)}\cdot\nabla\tilde{h}\,dS(x) \\
=\ & -\int_\Omega \partial_k\tilde{h}\partial_k\mathbf{u}^{(1)}\cdot\nabla\tilde{h}\,dx + \frac{1}{2}\int_{\partial\Omega}\mathbf{n}\cdot\mathbf{u}^{(1)}\,|\nabla\tilde{h}|^2\,dS(x) \\
=\ & -\int_\Omega \nabla\tilde{h}\cdot\nabla\mathbf{u}^{(1)}\cdot\nabla\tilde{h}\,dx. \qquad (3.25)
\end{aligned}
$$

We employ Yudovich's method to estimate the term on (4.13). For notational convenience, we denote it by I

$$
I = \int_\Omega \nabla\tilde{h}\cdot\nabla\mathbf{u}^{(1)}\cdot\nabla\tilde{h}\,dx.
$$

The Yudovich approach applies to the situation when the bound for

$$\|\nabla \mathbf{u}^{(1)}\|_{L^\infty} < \infty$$

is unknown, but any L^q bound of $\nabla \mathbf{u}^{(1)}$ does not grow faster than $O(q)$, namely

$$\sup_{q\geq 2} \frac{\|\nabla \mathbf{u}^{(1)}\|_{L^q}}{q} < \infty. \tag{3.26}$$

Recall that $\mathbf{u}^{(1)}$ is in (3.4), namely, for any $T > 0$,

$$\mathbf{u}^{(1)} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$

which allows us to verify (3.26). In fact, by Sobolev's embedding inequality, for any $2 \leq q < \infty$,

$$\begin{aligned} \|\nabla \mathbf{u}^{(1)}\|_{L^q(\Omega)} &\leq C(\Omega)\, q\, \|\nabla \mathbf{u}^{(1)}\|_{L^2(\Omega)} + C(\Omega)\, q\, \|\nabla\nabla \mathbf{u}^{(1)}\|_{L^2(\Omega)} \\ &\leq C(\Omega)\, q\, \|\nabla \mathbf{u}^{(1)}\|_{L^2(\Omega)} + C(\Omega)\, q\, \|\mathbf{u}^{(1)}\|_{\dot{H}^2(\Omega)}. \end{aligned}$$

That is,

$$\sup_{q\geq 2} \frac{\|\nabla \mathbf{u}^{(1)}\|_{L^q}}{q} \leq C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}. \tag{3.27}$$

For any $2 < q < \infty$, by Hölder's inequality,

$$\begin{aligned} |I| &\leq \|\nabla \tilde{h}\|_{L^2}\, \|\nabla \mathbf{u}^{(1)}\|_{L^q}\, \|\nabla \tilde{h}\|_{L^{\frac{2q}{q-2}}} \\ &\leq \|\nabla \tilde{h}\|_{L^2}\, \|\nabla \mathbf{u}^{(1)}\|_{L^q}\, \|\nabla \tilde{h}\|_{L^2}^{1-\frac{2}{q}} \|\nabla \tilde{h}\|_{L^\infty}^{\frac{2}{q}}. \end{aligned}$$

Since $\theta^{(1)}$ and $\theta^{(2)}$ are in the class (3.4), Lemma 3.4.2 states that, for any $2 < r < \infty$,

$$M \equiv \|\nabla \tilde{h}\|_{L^\infty} \leq C\, \|\nabla\nabla \tilde{h}\|_{L^r} \leq C \|\tilde{\theta}\|_{L^r} < \infty.$$

Therefore, by (3.27), for any $2 < q < \infty$,

$$\begin{aligned} |I| &\le C\, M^{\frac{2}{q}}\, \|\nabla \tilde{h}\|_{L^2}^{2(1-\frac{1}{q})}\, \|\nabla \mathbf{u}^{(1)}\|_{L^q} \\ &\le C\, q \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, M^{\frac{2}{q}}\, \|\nabla \tilde{h}\|_{L^2}^{2-\frac{2}{q}} \\ &= C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, \|\nabla \tilde{h}\|_{L^2}^2 \left(q M^{\frac{2}{q}} \|\nabla \tilde{h}\|_{L^2}^{-\frac{2}{q}} \right). \end{aligned}$$

By taking $q = 2\ln(M/\|\nabla \tilde{h}\|_{L^2})$, we obtain the minimizer of $qM^{\frac{2}{q}}\|\nabla \tilde{h}\|_{L^2}^{-\frac{2}{q}}$, namely

$$\min_{2 \le q < \infty} qM^{\frac{2}{q}} \|\nabla \tilde{h}\|_{L^2}^{-\frac{2}{q}} = 2e \left(\ln M - \ln \|\nabla \tilde{h}\|_{L^2} \right).$$

Consequently,

$$|I| \le C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, \|\nabla \tilde{h}\|_{L^2}^2\, (\ln M - \ln \|\nabla \tilde{h}\|_{L^2})$$

Inserting these bounds in (3.24) leads to

$$\begin{aligned} &\frac{d}{dt}\|\nabla \tilde{h}\|_{L^2}^2 \\ &\quad \le \|\theta^{(2)}\|_{L^\infty} \left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla \tilde{h}\|_{L^2}^2 \right) + C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, \|\nabla \tilde{h}\|_{L^2}^2\, (\ln M - \ln \|\nabla \tilde{h}\|_{L^2}), \end{aligned}$$

which, together with (3.23), yields

$$\begin{aligned} &\frac{d}{dt}\left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla \tilde{h}\|_{L^2}^2 \right) + 2\nu \int_\Omega |D(\mathbf{u})|^2\, dx + \nu\, \alpha \int_{\partial\Omega} (\mathbf{u}\cdot\tau)^2\, dS(x) \\ \le\quad & C\, \|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\tilde{\mathbf{u}}\|_{L^2}^2 + C\, \|\nabla \tilde{h}\|_{L^2}^2 + \|\theta^{(2)}\|_{L^\infty} \left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla \tilde{h}\|_{L^2}^2 \right) \\ & + C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, \|\nabla \tilde{h}\|_{L^2}^2\, (\ln M - \ln \|\nabla \tilde{h}\|_{L^2}). \end{aligned}$$

Especially, $Y(t) \equiv \delta + \|\tilde{u}\|_{L^2}^2 + \|\nabla \tilde{h}\|_{L^2}^2$ with any small $\delta > 0$ satisfies

$$\frac{d}{dt} Y \le C(1 + \|\nabla u^{(2)}\|_{L^2}^2 + \|\theta^{(2)}\|_{L^\infty}) Y\, +\, C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, Y\, (\ln M - \ln Y).$$

where we have used the fact $z \to z(\ln M - \ln z)$ is an increasing function for $0 < z < M/e$. It then follows from the Osgood inequality in Lemma 2.1.2, for any $t > 0$,

$$Y(t) \equiv 0.$$

Then letting $\delta \to 0$ yields the desired uniqueness. This completes the proof of Proposition 3.4.1. $\square$

CHAPTER 4

Boussinesq Equation with Fractional Dissipation in Periodic Domain

4.1 Introduction

The motivation to study fractional dissipation can be traced back in the research of fractional Navier Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p + (-\Delta)^\alpha u = 0, \quad x \in \mathbb{R}^d \\ \nabla \cdot u = 0. \end{cases} \tag{4.1}$$

It is well known that when $\alpha > \frac{1}{2} + \frac{d}{4}$, this generalized Navier-Stokes equation has global regularity. Moreover, the study of fractional dissipation in Boussinesq system also has its physical application. Although the diffusion process is normally modeled by the standard Laplacian operator, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian arise.

On the other hand, the Littlewood-Paley decomposition and related Besov space techniques for functions defined on the whole space $\mathbb{R}^d$ have become crucial tools in the study of many PDEs. So, we want to develop a unified approach for PDEs defined on periodic domains $\mathbb{T}^b$ and PDEs on the whole space $\mathbb{R}^d$. Because $S_N f(x) = \sum_{|k|\leq N} \hat{f}(k) e^{-ik\cdot x}$ may not converge to f for $f \in L^p(\Pi^d)$ with $p \neq 2$, the classical Fourier series expansion is not convenient for our purpose. We need a new cutoff of

Fourier series for functions on $\mathbb{T}^d$ instead of the normal circular cutoff as in classical Littlewood-Paley decomposition for functions on $\mathbb{R}^d$.

After taking into account of the convergence and boundedness properties as well as the easiness of being split into dyadic Fourier blocks of these partial sums, we choose the square cutoff to define the dyadic Fourier blocks. More precisely, we define the following localized Fourier projection operators as

$$\Delta_0 f(x) = \sum_{k\in A_0} \hat{f}(k)e^{ik\cdot x},$$
$$\Delta_j f(x) = \sum_{k\in A_j\setminus A_{j-1}} \hat{f}(k)e^{ik\cdot x}, \quad j\geq 1,\, j\in\mathbb{N},$$

where A_j's are the 2^j-sized blocks of d-dimensional integer lattice points,

$$A_j = \{k = (k_1, k_2, \cdots, k_d) \in \mathbb{Z}^d : |k_m| \leq 2^j,\, m = 1, 2, \cdots, d\}.$$

Since, for any $f \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$,

$$S_j f(x) := \sum_{m=0}^{j-1} \Delta_m f(x) = \sum_{k\in A_{j-1}} \hat{f}(k)e^{ik\cdot x}$$

converges to f in $L^p(\mathbb{T}^d)$ for $1 < p < \infty$ and a.e. for $p = \infty$, we can write the Littlewood-Paley decomposition of f as

$$f(x) = \sum_{k=0}^{\infty} \Delta_k f(x).$$

By defining the operator Δ_j as above, the important properties such as Bernstein type inequalities, in the whole space case can also be proved valid in the periodic case. By applying the tools developed here for periodic functions, we could obtain the global existence and uniqueness of weak solution to the following fractional d-dimensional Boussinesq system

$$
\begin{cases}
\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nu(-\Delta)^\alpha \mathbf{u} - \nabla P + \theta\,\mathbf{e}_d, \quad \mathbf{x}\in\mathbb{T}^d,\ t>0,\\
\partial_t\theta + \mathbf{u}\cdot\nabla\theta = \gamma\, u_d, \quad \mathbf{x}\in\mathbb{T}^d,\ t>0,\\
\nabla\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\mathbb{T}^d,\ t>0,\\
(\mathbf{u},\theta)|_{t=0} = (\mathbf{u}_0,\theta_0), \quad \mathbf{x}\in\mathbb{T}^d,
\end{cases}
\tag{4.2}
$$

By applying the tools developed in section 4.2 for periodic functions, we establish two main results for (4.2). The first is the global existence and uniqueness of weak solutions of (4.2) with initial data $\mathbf{u}_0 \in L^2(\mathbb{R}^d), \theta_0 \in L^2(\mathbb{R}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$. Our key point here is the uniqueness of solutions in a very weak setting for a partially dissipated system. The precise result is stated in the following theorem.

Theorem 4.1.1 *Consider the d-D Boussinesq equations in (4.2) in the periodic domain $\mathbb{T}^d$.*

(1) Let $\alpha > 0$ and $(\mathbf{u}_0,\theta_0)\in L^2(\mathbb{T}^d)$ with $\nabla\cdot\mathbf{u}_0 = 0$. Let $T>0$ be arbitrarily fixed. Then (4.2) has a global weak solution $(\mathbf{u},\theta)$ on $[0,T]$ satisfying

$$
\mathbf{u}\in C_w([0,T];L^2)\cap L^2(0,T;\dot H^\alpha), \quad \theta\in C_w([0,T];L^2)\cap L^\infty(0,T;L^2),
$$

where $C_w([0,T];L^2)$ denotes the standard time continuous functions in the weak L^2-sense.

(2) Let $\alpha\geq\frac{1}{2}+\frac{d}{4}$. Assume $\mathbf{u}_0\in L^2(\mathbb{T}^d)$ and $\theta_0\in L^2(\mathbb{T}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{T}^d)$ with $\nabla\cdot\mathbf{u}_0=0$. Then (4.2) has a unique and global weak solution $(\mathbf{u},\theta)$ satisfying

$$
\begin{aligned}
&\mathbf{u}\in C([0,T];L^2)\cap L^2(0,T;\dot H^\alpha), \quad \mathbf{u}\in \widetilde{L}^1(0,T;B^{1+\frac{d}{2}}_{2,2}),\\
&\theta\in C_w([0,T];L^2)\cap L^\infty(0,T;L^2\cap L^{\frac{4d}{d+2}}),
\end{aligned}
$$

where the definition of $\widetilde{L}^1(0,T;\ B^{1+\frac{d}{2}}_{2,2})$ *can be found in Section 2. Especially,* $\mathbf{u}$ *satisfies*

$$\sup_{q\geq 2} \frac{1}{\sqrt{q}} \int_0^T \|\nabla \mathbf{u}(t)\|_{L^q}\, dt < \infty.$$

4.2 Littlewood-Paley Decomposition for Periodic Functions

The purpose of this section is to introduce the concepts of Fourier dyadic blocks and the Littlewood-Paley decomposition for periodic functions, and develop associated tools that are useful for the study of solutions of PDEs in $\mathbb{T}^d$ with $d \geq 2$.

Let $d \geq 2$. the partial sum can be defined in many ways. Two of the most natural ones are the partial sum with square-cutoff and the partial sum with circular cutoff, namely

$$S_N f = \sum_{|k_1|\leq N,\cdots,|k_d|\leq N} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} = D_N * f \tag{4.3}$$

and

$$\widetilde{S}_N f = \sum_{|\mathbf{k}|=\sqrt{k_1^2+\cdots+k_d^2}\leq N} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} = \widetilde{D}_N * f, \tag{4.4}$$

where D_N denotes the d-dimensional square Dirichlet kernel and $\widetilde{D}_N$ the circular Dirichlet kernel,

$$D_N(\mathbf{x}) = \sum_{|k_1|\leq N,\cdots,|k_d|\leq N} e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \widetilde{D}_N(\mathbf{x}) = \sum_{|\mathbf{k}|\leq N} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

These two partial sums have different convergence properties, as stated in the following lemmas (see, e.g., [14, 30]). The partial sum defined via the square-cutoff is bounded on any L^p for $1 < p < \infty$ and converges to the original function in L^p.

Lemma 4.2.1 *Let $d \geq 1$. The partial sum with the square cutoff $S_N f$ satisfies, for any $f \in L^p(\mathbb{T}^d)$ with $1 < p < \infty$,*

$$\|S_N f\|_{L^p} \leq C_p \|f\|_{L^p} \quad \textit{and} \quad \|S_N f - f\|_{L^p} \to 0 \quad \textit{as } N \to \infty. \tag{4.5}$$

(4.5) is false for $p = 1$ and for $p = \infty$. In addition, if $f \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$, then

$$S_N f \quad \to \quad f, \qquad \textit{a.e.} \quad \textit{as } N \to \infty.$$

The partial sum defined via the circular cutoff in $\mathbb{T}^d$ with $d \geq 2$ is not bounded on L^p except for $p = 2$ and is not known to converge to the original function except for $p = 2$.

Lemma 4.2.2 *Let $d \geq 2$ and $f \in L^2(\mathbb{T}^d)$. Then*

$$\|\widetilde{S}_N f\|_{L^2} \leq \|f\|_{L^2} \quad \textit{and} \quad \|\widetilde{S}_N f - f\|_{L^2} \to 0 \quad \textit{as } N \to \infty. \tag{4.6}$$

(4.6) is false if we change L^2 to L^p with $p \neq 2$.

The circular cutoff defined in (4.4) is not suitable since it does not have the desired property

$$\|\widetilde{S}_N f\|_{L^p} \leq C_p \|f\|_{L^p} \tag{4.7}$$

for all $1 < p < \infty$. (4.7) only holds for $p = 2$ according to Lemma 4.2.2.

So, we choose the square-cutoff defined in (4.3). We introduce a few notation first. For an integer $j \geq 0$, we set A_j to be the 2^j-sized block of d-dimensional integer lattice points,

$$A_j = \{\mathbf{k} = (k_1, k_2, \cdots, k_d) \in \mathbb{Z}^d : \, |k_m| \leq 2^j, \; m = 1, 2, \cdots, d\}.$$

We define the following localized Fourier projection operators as

$$\Delta_0 f(\mathbf{x}) = \sum_{\mathbf{k}\in A_0} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\Delta_j f(\mathbf{x}) = \sum_{\mathbf{k}\in A_j\setminus A_{j-1}} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad j\ge 1,\ j\in\mathbb{N}. \tag{4.8}$$

For notational convenience, we also write $\Delta_j = 0$ for $j < 0$. With a slight abuse of notation, we set

$$S_j f(\mathbf{x}) = \sum_{m=0}^{j-1} \Delta_m f(\mathbf{x}) = \sum_{\mathbf{k}\in A_{j-1}} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4.9}$$

In terms of these operators, we can write the Littlewood-Paley decomposition, for any $f \in L^p(\mathbb{T}^d)$ with $1 < p \le \infty$,

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \Delta_k f(\mathbf{x}). \tag{4.10}$$

The following lemma presents useful basic properties of the operators defined above.

Lemma 4.2.3 *Let $j \ge 0$ be an integer. Let Δ_j and S_j be defined as in (4.8) and (4.9). Then the following properties hold.*

(a) If $f \in L^p(\mathbb{T}^d)$ with $1 < p \le \infty$, then

$$\|\Delta_j f\|_{L^p} \le C\,\|f\|_{L^p}, \qquad \|S_j f\|_{L^p} \le C\,\|f\|_{L^p},$$

where C's are constants depending on p and d only.

(b) Let $j \ge 0$ and $k \ge 0$ be integers. Assume $f \in L^p(\mathbb{T}^d)$ with $1 < p \le \infty$. Then

$$\Delta_j \Delta_k f = 0 \quad \textit{if} \quad j \ne k.$$

(c) Let $j \geq 0$ and $m \geq 0$ be integers. Assume $f, g \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$. Then

$$\Delta_j(S_{m-d} f \Delta_m g) = 0 \quad \textit{if } |m-j| \geq d+1.$$

The operators Δ_j defined for periodic functions share many properties with those for the the whole space Δ_j. One crucial property is the following Bernstein type inequalities. The proof of these properties appears to be more difficult than that for the whole space case.

Proposition 4.2.1 *Let $\sigma \geq 0$ and $1 \leq q \leq p \leq \infty$.*

(1) There exists a constant $C > 0$ such that

$$\|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T}^d)} \leq C\, 2^{\sigma j + jd(\frac{1}{q} - \frac{1}{p})}\, \|\Delta_j f\|_{L^q(\mathbb{T}^d)} \tag{4.11}$$

and

$$\|S_j f\|_{L^p(\mathbb{T}^d)} \leq C\, 2^{jd(\frac{1}{q} - \frac{1}{p})}\, \|S_j f\|_{L^q(\mathbb{T}^d)}. \tag{4.12}$$

(2) Let $1 \leq p \leq \infty$. There exist constants $0 < C_1 < C_2$ (depending on p) such that, for any integer $j \geq 0$,

$$C_1\, 2^{\sigma j} \|\Delta_j f\|_{L^p(\mathbb{T}^d)} \leq \|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T}^d)} \leq C_2\, 2^{\sigma j} \|\Delta_j f\|_{L^p(\mathbb{T}^d)}. \tag{4.13}$$

We can also define the Besov type space $B^s_{p,q}(\mathbb{T}^d)$ via the operators Δ_j defined above in the same fashion as in the whole space case. Let $\mathcal{S}$ denotes the usual Schwartz class and $\mathcal{S}'$ the distributions.

Definition 4.2.1 *Let $f \in \mathcal{S}'$. The Besov space $B^s_{p,q}(\mathbb{T}^d)$ with $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'(\mathbb{T}^d)$ satisfying*

$$\|f\|_{B^s_{p,q}(\mathbb{T}^d)} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

We can also define the space-time spaces for periodic functions. This type of functional settings was introduced by Chemin-Lerner for functions defined on the whole space (see, e.g., [4]).

Definition 4.2.2 *For $t > 0$, $s \in \mathbb{R}$ and $1 \leq p, q, r \leq \infty$, the space-time space $\widetilde{L}^r_t B^s_{p,q}$ is defined through the norm*

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^r_t L^p}\|_{l^q}.$$

These Besov spaces defined above are closely related to some of the standard spaces and share similar properties with their whole space counterparts.

Lemma 4.2.4 *Let $s \in \mathbb{R}$.*

(1) For $1 \leq p \leq \infty$ and $q_1 \leq q_2$, $B^s_{p,q_1}(\mathbb{T}^d) \subset B^s_{p,q_2}(\mathbb{T}^d)$.

(2) $H^s(\mathbb{T}^d)$ can be identified with $B^s_{2,2}(\mathbb{T}^d)$,

$$H^s(\mathbb{T}^d) \sim B^s_{2,2}(\mathbb{T}^d).$$

4.3 Proof of Main Theorem

This section proves Theorem 4.1.1. A crucial smoothing estimate is obtained using the Littlewood-Paley decomposition and Besov space techniques introduced in the previous section.

4.3.1 Global existence of weak solutions when $\alpha > 0$

We start with the definition of weak solutions of (4.2) with any $\alpha > 0$.

Definition 4.3.1 *Consider (4.2) with $\alpha > 0$, $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$ and $\nabla \cdot \mathbf{u}_0 = 0$. Let $T > 0$ be arbitrarily fixed. A pair $(\mathbf{u}, \theta)$ satisfying*

$$\mathbf{u} \in C_w([0,T]; L^2) \cap L^2(0,T; \mathring{H}^\alpha), \quad \nabla \cdot \mathbf{u} = 0,$$
$$\theta \in C_w([0,T]; L^2) \cap L^\infty(0,T; L^2)$$

is weak solution of (4.2) on $[0,T]$ if (a) and (b) below hold:

(a) For any $\phi \in C_0^\infty(\mathbb{T}^d \times [0,T))$ with $\nabla \cdot \phi = 0$ and for any $t \leq T$,

$$\begin{aligned}
&-\int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \partial_t \phi \, d\mathbf{x} \, d\tau + \int_{\mathbb{T}^d} \mathbf{u}(\mathbf{x},t) \cdot \phi(\mathbf{x},t) \, d\mathbf{x} - \int_{\mathbb{T}^d} \mathbf{u}_0(\mathbf{x}) \cdot \phi(\mathbf{x},0) \, d\mathbf{x} \\
&-\int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla \phi \mathbf{u} \, d\mathbf{x} \, d\tau + \int_0^t \int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} \mathbf{u} \cdot (-\Delta)^{\alpha/2} \phi \, d\mathbf{x} \, d\tau \\
&= \int_0^t \int_{\mathbb{T}^d} \theta \boldsymbol{e}_d \cdot \phi \, d\mathbf{x} \, d\tau. \qquad (4.14)
\end{aligned}$$

(b) For any $\psi \in C_0^\infty(\mathbb{T}^d \times [0,T))$ and $t \leq T$,

$$\begin{aligned}
&-\int_0^t \int_{\mathbb{T}^d} \partial_t \psi \theta \, d\mathbf{x} \, d\tau + \int_{\mathbb{T}^d} \theta(\mathbf{x},t) \psi(\mathbf{x},t) \, d\mathbf{x} - \int_{\mathbb{T}^d} \theta_0(\mathbf{x}) \psi(\mathbf{x},0) \, d\mathbf{x} \\
&= \int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla \psi \theta \, d\mathbf{x} \, d\tau + \gamma \int_0^t \int_{\mathbb{T}^d} u_d \, \psi \, d\mathbf{x} d\tau. \qquad (4.15)
\end{aligned}$$

For any $\alpha > 0$ and $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$, (4.2) always has a global weak solution. In the special case when $\theta \equiv 0$, this result assesses the global existence of weak solutions of the generalized Navier-Stokes equations with any $\alpha > 0$ and $\mathbf{u}_0 \in L^2(\mathbb{T}^d)$. And a similar type of argument can be used to prove global existence of weak solutions of (4.2). The result can be stated in the following Proposition.

Proposition 4.3.1 *Consider (4.2) with $\alpha > 0$, $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$ and $\nabla \cdot \mathbf{u}_0 = 0$. Let $T > 0$ be arbitrarily fixed. Then (4.2) has a global weak solution $(\mathbf{u}, \theta)$ as given in Definition 4.3.1 satisfying*

$$\|\mathbf{u}(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^2}^2 \, d\tau \le e^{C\,t} (\|\mathbf{u}_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2).$$

4.3.2 Uniqueness of weak solutions when $\alpha \ge \frac{1}{2} + \frac{d}{4}$

Next we establish a smoothing estimate for the weak solution shown in Proposition 4.3.1. Before starting the proof, we need the following two properties.

Proposition 4.3.2 *Let $d \ge 2$. Consider (4.2) with $\alpha \ge \frac{1}{2} + \frac{d}{4}$. Assume $(\mathbf{u}_0, \theta_0)$ satisfies*

$$\mathbf{u}_0 \in L^2(\mathbb{T}^d), \quad \nabla \cdot \mathbf{u}_0 = 0, \qquad \theta_0 \in L^2(\mathbb{T}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{T}^d).$$

Let $(\mathbf{u}, \theta)$ be the corresponding global weak solution of (4.2). Then, for any $0 < t \le T$,

$$\|\mathbf{u}\|_{\widetilde{L}^1_t B^{1+\frac{d}{2}}_{2,2}} \le C(t, \|\mathbf{u}_0\|_{L^2}, \|\theta_0\|_{L^2}). \tag{4.16}$$

As a special consequence,

$$\sup_{q \ge 2} \int_0^t \frac{\|\nabla \mathbf{u}(\tau)\|_{L^q}}{\sqrt{q}} d\tau \le C(t, \|\mathbf{u}_0\|_{L^2}, \|\theta_0\|_{L^2}). \tag{4.17}$$

Lemma 4.3.1 *Assume $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^2)$ with $\nabla \cdot \mathbf{u}_0 = 0$. Consider the 2D Boussinesq equation in (4.2) with $\alpha \ge 1$. Let $(\mathbf{u}, \theta)$ be the corresponding weak solution. Then $\mathbf{u}$ satisfies*

$$\int_0^T \|\mathbf{u}(t)\|_{L^\infty}^2 \, d\tau \le C(t, \|\mathbf{u}_0\|_{L^2}, \|\theta_0\|_{L^2}). \tag{4.18}$$

Now, we prove the second part of Theorem 4.1.1.

Proof. Due to Proposition 4.3.1 and Proposition 4.3.2, it suffices to show the uniqueness of the weak solutions of (4.2). Suppose (4.2) has two weak solutions $(\mathbf{u}^{(1)}, \theta^{(1)})$ and $(\mathbf{u}^{(2)}, \theta^{(2)})$ with the same initial data $(\mathbf{u}_0, \theta_0)$. We show that $(\mathbf{u}^{(1)}, \theta^{(1)})$ and $(\mathbf{u}^{(2)}, \theta^{(2)})$ must coincide. To do so, we consider the difference $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ with

$$\widetilde{\mathbf{u}} := \mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \qquad \widetilde{\theta} := \theta^{(1)} - \theta^{(2)}.$$

Let $P^{(1)}$ and $P^{(2)}$ be the corresponding pressure terms and $\widetilde{P} := P^{(1)} - P^{(2)}$. In addition, we introduce the lower regularity quantities $h^{(1)}$ and $h^{(2)}$ satisfying

$$-\Delta h^{(1)} = \theta^{(1)}, \qquad -\Delta h^{(2)} = \theta^{(2)}$$

and set

$$\widetilde{h} = h^{(1)} - h^{(2)}.$$

It follows from (4.2) that $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ satisfies

$$\begin{cases} \partial_t \widetilde{\mathbf{u}} + \mathbf{u}^{(1)} \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} + \nu(-\Delta)^\alpha \widetilde{\mathbf{u}} + \nabla \widetilde{P} = \widetilde{\theta} \mathbf{e}_d, \\ \partial_t \widetilde{\theta} + \mathbf{u}^{(1)} \cdot \nabla \widetilde{\theta} + \widetilde{\mathbf{u}} \cdot \nabla \theta^{(2)} = \gamma \widetilde{u}_d, \\ \nabla \cdot \widetilde{\mathbf{u}} = 0, \\ (\widetilde{\mathbf{u}}, \widetilde{\theta})|_{t=0} = 0. \end{cases} \tag{4.19}$$

Dotting the first equation of (4.19) by $\widetilde{\mathbf{u}}$ and integrating by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \nu\|\Lambda^\alpha \widetilde{\mathbf{u}}\|_{L^2}^2 &= -\int \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \widetilde{\mathbf{u}}\, d\mathbf{x} + \int \widetilde{\theta} \cdot (\mathbf{e}_d \cdot \widetilde{\mathbf{u}})\, d\mathbf{x} \\ &:= K_1 + K_2, \end{aligned} \tag{4.20}$$

where we have invoked the fact that, for $\alpha \geq \frac{1}{2} + \frac{d}{4}$,

$$\int_{\mathbb{T}^d} \mathbf{u}^{(1)} \cdot \nabla \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}}\, d\mathbf{x} = 0,$$

due to $\nabla \cdot \mathbf{u}^{(1)} = 0$, $\nabla \cdot \widetilde{\mathbf{u}} = 0$ and

$$\int_0^T \int_{\mathbb{T}^d} |\mathbf{u}^{(1)} \cdot \nabla \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}}| d\mathbf{x} dt \le \int_0^T \|\mathbf{u}^{(1)}(\tau)\|_{L^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \widetilde{\mathbf{u}}(\tau)\|_{L^2}^2 \, d\tau < \infty.$$

By Hölder's and Sobolev's inequalities, for $d = 2$,

$$\begin{aligned} |K_1| &\le \|\widetilde{\mathbf{u}}\|_{L^4}^2 \|\nabla \mathbf{u}^{(2)}\|_{L^2} \\ &\le \|\widetilde{\mathbf{u}}\|_{L^2} \|\nabla \widetilde{\mathbf{u}}\|_{L^2} \|\nabla \mathbf{u}^{(2)}\|_{L^2} \\ &\le \frac{\nu}{16} \|\nabla \widetilde{\mathbf{u}}\|_{L^2}^2 + C \|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\widetilde{\mathbf{u}}\|_{L^2}^2. \end{aligned} \tag{4.21}$$

For $d \ge 3$,

$$\begin{aligned} |K_1| &\le \|\widetilde{\mathbf{u}}\|_{L^2} \|\nabla \mathbf{u}^{(2)}\|_{\frac{4d}{d+2}} \|\widetilde{\mathbf{u}}\|_{L^{\frac{4d}{d-2}}} \\ &\le C \|\widetilde{\mathbf{u}}\|_{L^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \mathbf{u}^{(2)}\|_{L^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \widetilde{\mathbf{u}}\|_{L^2} \\ &\le \frac{\nu}{16} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \widetilde{\mathbf{u}}\|_{L^2}^2 + C \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \mathbf{u}^{(2)}\|_{L^2}^2 \|\widetilde{\mathbf{u}}\|_{L^2}^2. \end{aligned} \tag{4.22}$$

By integration by parts and an interpolation inequality,

$$\begin{aligned} |K_2| &= \left| \int_{\mathbb{T}^d} (-\Delta \widetilde{h})(\mathbf{e}_d \cdot \widetilde{\mathbf{u}}) \, d\mathbf{x} \right| \\ &\le \|\nabla \widetilde{h}\|_{L^2} \|\nabla \widetilde{\mathbf{u}}\|_{L^2} \\ &\le C \|\nabla \widetilde{h}\|_{L^2} \|\widetilde{\mathbf{u}}\|_{L^2}^{\frac{d-2}{d+2}} \|\Lambda^{\frac{d+2}{4}} \widetilde{\mathbf{u}}\|_{L^2}^{\frac{4}{d+2}} \\ &\le C \|\nabla \widetilde{h}\|_{L^2} (\|\widetilde{\mathbf{u}}\|_{L^2} + \|\Lambda^{\frac{d+2}{4}} \widetilde{\mathbf{u}}\|_{L^2}) \\ &\le \frac{\nu}{16} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \widetilde{\mathbf{u}}\|_{L^2}^2 + C \|\nabla \widetilde{h}\|_{L^2} (\|\widetilde{\mathbf{u}}\|_{L^2} + \|\nabla \widetilde{h}\|_{L^2}). \end{aligned} \tag{4.23}$$

Dotting the second equation in (4.19) with $\widetilde{h}$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \widetilde{h}\|_{L^2}^2 &= \int_{\mathbb{T}^d} \mathbf{u}^{(1)} \cdot \nabla \widetilde{\theta} \, \widetilde{h} \, d\mathbf{x} + \int_{\mathbb{T}^d} \widetilde{\mathbf{u}} \cdot \nabla \theta^{(2)} \widetilde{h} \, d\mathbf{x} + \gamma \int \widetilde{u}_d \, \widetilde{h} \, d\mathbf{x} \\ &:= K_3 + K_4 + K_5. \end{aligned} \tag{4.24}$$

We estimate K_4 first. The case with $d=2$ is treated differently from $d\ge 3$. For $d=2$, by Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
|K_4| &\le \|\theta^{(2)}\|_{L^2}\,\|\widetilde{\mathbf{u}}\|_{L^{2p}}\,\|\nabla\widetilde{h}\|_{L^q}\\
&\le C\,\sqrt{p}\|\widetilde{\mathbf{u}}\|_{L^2}^{1/p}\,\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^{1-1/p}\|\theta_0\|_{L^2}\,\|\nabla\widetilde{h}\|_{L^2}^{1-\frac{1}{p}}\|\Delta\widetilde{h}\|_{L^2}^{\frac{1}{p}}\\
&\le C\,\sqrt{p}\,(\|\widetilde{\mathbf{u}}\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}\|_{L^2})\,\|\nabla\widetilde{h}\|_{L^2}^{1-\frac{1}{p}}\|\widetilde{\theta}\|_{L^2}^{\frac{1}{p}}\\
&\le \frac{\nu}{16}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\mathbf{u}}\|_{L^2}^2+C\,p\,M^{\frac{2}{p}}\,\|\nabla\widetilde{h}\|_{L^2}^{2(1-\frac{1}{p})},
\end{aligned}
\tag{4.25}
$$

where $1<p,q<\infty$ satisfy

$$\frac{1}{p}+\frac{2}{q}=1$$

and we have used the fact that

$$\|\Delta\widetilde{h}\|_{L^2}\le\|\theta^{(1)}\|_{L^2}+\|\theta^{(2)}\|_{L^2}\le C(T,\|\mathbf{u}_0\|_{L^2},\|\theta_0\|_{L^2}):=\sqrt{M}.$$

For $d\ge 3$, by integration by parts, Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
|K_4| &= \int_{\mathbb{T}^d}|\theta^{(2)}\widetilde{\mathbf{u}}\cdot\nabla\widetilde{h}|\,d\mathbf{x}\\
&\le \|\theta^{(2)}\|_{L^{\frac{4d}{d+2}}}\|\nabla\widetilde{h}\|_{L^2}\|\widetilde{\mathbf{u}}\|_{L^{\frac{4d}{d-2}}}\\
&\le \|\theta_0\|_{L^{\frac{4d}{d+2}}}\|\nabla\widetilde{h}\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}\\
&\le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}^2+\,C\,\|\theta_0\|_{L^{\frac{4d}{d+2}}}^2\|\nabla\widetilde{h}\|_{L^2}^2.
\end{aligned}
\tag{4.26}
$$

Recalling $\widetilde{\theta}=-\Delta\widetilde{h}$ and integrating by parts, we have

$$
\begin{aligned}
K_3 &= -\int_{\mathbb{T}^d}\mathbf{u}^{(1)}\cdot\nabla\Delta\widetilde{h}\,\widetilde{h}\,d\mathbf{x}\\
&= \int_{\mathbb{T}^d}\partial_{x_k}u_j^{(1)}\partial_{x_j}\partial_{x_k}\widetilde{h}\,\widetilde{h}\,d\mathbf{x}+\int_{\mathbb{T}^d}u_j^{(1)}\partial_{x_j}\partial_{x_k}\widetilde{h}\,\partial_{x_k}\widetilde{h}\,d\mathbf{x}\\
&= -\int_{\mathbb{T}^d}\partial_{x_k}u_j^{(1)}\partial_{x_k}\widetilde{h}\,\partial_{x_j}\widetilde{h}\,d\mathbf{x},
\end{aligned}
$$

where the repeated indices are summed and we have used $\nabla\cdot\mathbf{u}^{(1)}=0$. By Hölder's inequality, for $p>\frac{d}{2}$ and $\frac{1}{p}+\frac{2}{q}=1$,

$$\begin{aligned}
|K_3| &\le C\|\nabla\mathbf{u}^{(1)}\|_{L^p}\|\nabla\widetilde{h}\|_{L^q}^2\\
&\le C\|\nabla\mathbf{u}^{(1)}\|_{L^p}\|\nabla\widetilde{h}\|_{L^2}^{2-\frac{d}{p}}\|\widetilde{\theta}\|_{L^2}^{\frac{d}{p}}\\
&\le C\,\|\nabla\mathbf{u}^{(1)}\|_{L^p}\,M^{\frac{d}{2p}}\,\|\nabla\widetilde{h}\|_{L^2}^{2-\frac{d}{p}}. \qquad (4.27)
\end{aligned}$$

Clearly, K_5 can be similarly estimated as K_3 and the bound is the same. Adding (4.20) and (4.24) and collecting the estimates in (4.21), (4.22), (4.23), (4.25), (4.26) and (4.27), we find that, for $\delta>0$,

$$G_\delta(t):=\|\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\|\nabla\widetilde{h}(t)\|_{L^2}^2+\delta$$

obeys the differential inequality, when $d=2$,

$$\frac{d}{dt}G_\delta(t)\le C\left(1+\|\Lambda\mathbf{u}^{(2)}\|_{L^2}^2\right)\,G_\delta(t)+C\left(1+\frac{\|\nabla\mathbf{u}^{(1)}\|_{L^p}}{p}\right)p\,M^{\frac{1}{p}}\,G_\delta(t)^{1-\frac{1}{p}} \qquad (4.28)$$

and, for $d\ge 3$,

$$\frac{d}{dt}G_\delta(t)\le C\left(1+\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}^{(2)}\|_{L^2}^2\right)\,G_\delta(t)+C\,\frac{\|\nabla\mathbf{u}^{(1)}\|_{L^p}}{p}\,p\,M^{\frac{d}{2p}}\,G_\delta(t)^{1-\frac{d}{2p}} \qquad (4.29)$$

Optimizing the quantities $p\,M^{\frac{1}{p}}\,G_\delta(t)^{1-\frac{1}{p}}$ and $p\,M^{\frac{d}{2p}}\,G_\delta(t)^{1-\frac{d}{2p}}$ with respect to p, we obtain

$$\begin{aligned}
&p\,M^{\frac{1}{p}}\,G_\delta(t)^{1-\frac{1}{p}}\le e\,G_\delta\,(\ln M-\ln G_\delta),\\
&p\,M^{\frac{d}{2p}}\,G_\delta(t)^{1-\frac{d}{2p}}\le \frac{d}{2}e\,G_\delta\,(\ln M-\ln G_\delta).
\end{aligned}$$

Incorporating these bounds in (4.28) and (4.29), we find that both (4.28) and (4.29) are reduced to the following form

$$G_\delta(t)\le G_\delta(0)+C\int_0^t\gamma(s)\,\phi(G_\delta(s))\,ds,$$

where

$$\gamma(t) = C + C\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}^{(2)}\|_{L^2}^2 + C\,\frac{\|\nabla\mathbf{u}^{(1)}\|_{L^p}}{p}, \quad \phi(r) = r + r(\ln M - \ln r).$$

It follows from Proposition 4.16 that

$$\int_0^T \gamma(t)\,d\tau < \infty.$$

Let

$$\begin{aligned}\Omega(x) &= \int_x^1 \frac{dr}{\phi(r)} = \int_x^1 \frac{dr}{r + r(\ln M - \ln r)} \\ &= \ln(1 + \ln M - \ln x) - \ln(1 + \ln M).\end{aligned}$$

It then follows from Lemma 2.1.2 that

$$-\Omega(G_\delta(t)) + \Omega(G_\delta(0)) \le \int_0^t \gamma(s)\,ds.$$

Therefore,

$$-\ln(1 + \ln M - \ln G_\delta(t)) + \ln(1 + \ln M - \ln G_\delta(0)) \le \int_0^t \gamma(s)\,ds.$$

Therefore, for $\widetilde{C}(t) = \int_0^t \gamma(s)\,ds$,

$$G_\delta(t) \le (eM)^{1-e^{-\widetilde{C}(t)}}\, G_\delta(0)^{e^{-\widetilde{C}(t)}}.$$

Letting $\delta \to 0$ and noting that $G_0(0) = 0$, we obtain

$$G_0(t) := \|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\nabla\widetilde{h}(t)\|_{L^2}^2 = 0.$$

This completes the proof of Theorem 4.1.1. $\square$

CHAPTER 5

Variants of 2D Euler Equation

5.1 Introduction

In fuild dynamics, the 2D incompressible Euler equation models the inviscid incompressible flows. Yudovich [35] proved the existence of a unique solution to this system provided the vorticity is bounded. After adding a damping term u in velocity equation, we could even have exponential decay of velocity. However, this result can not be simply extended if there is only a partial damping term u_2 in velocity equation of the second component, that is consider

$$\begin{cases} \partial_t u_1 + u \cdot \nabla u_1 = -\partial_1 p, \quad x \in \mathbb{R}^2 \\ \partial_t u_2 + u \cdot \nabla u_2 + u_2 = -\partial_2 p, \quad x \in \mathbb{R}^2 \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{5.1}$$

The L^2 estimates shows

$$\int_0^\infty \|u_2(\tau)\|_{L^2}^2 d\tau < \infty$$

and we can further show that $\|u_2(t)\|_{L^2}^2$ is uniformly continuous in t, and this could conclude that

$$\|u_2(t)\|_{L^2} \to 0, \quad \text{as } t \to \infty.$$

Next, we estimate the H^2-norm on two component of velocity equation with invoking various estimates and we find

$$\frac{d}{dt}\|u_2\|_{H^s}^2 + \|u_2\|_{H^s}^2 \leq C\|u\|_{H^s}\|u_2\|_{H^s}^2,$$

and

$$\frac{d}{dt}\|u_1\|_{H^s}^2 \leq C\|u_2\|_{H^s}\|u_1\|_{H^s}^2.$$

By taking the curl of the velocity equations of (5.1), the vorticity equation is given by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \mathcal{R}_1^2 \omega, \\ u = \nabla^{\perp}\psi, \quad \Delta\psi = \omega, \end{cases} \tag{5.2}$$

where $\mathcal{R}_1 = \partial_{x_1}(-\Delta)^{-1/2}$ denotes the standard Riesz transform and $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$.

Furthermore, we are also interested in changing $\mathcal{R}_1^2\omega$ to be another non-local zero-degree singular integral operator $\mathcal{R}_1\mathcal{R}_2\omega$, that is consider

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \mathcal{R}_1\mathcal{R}_2 \omega, \\ u = \nabla^{\perp}\psi, \quad \Delta\psi = \omega, \end{cases} \tag{5.3}$$

The motivation to consider these two models (5.2) and (5.3) has three folds: (1) Several global existence and regularity problems on the Boussinesq equations and on the magneto-hydrodynamic (MHD) equations can be reduced to one of the variants listed here. Understanding these models would help solve those problems, (2) We intend to solve the global existence and regularity problem for the Euler and the

Euler variants when we do not know that the vorticity is bounded a priori, (2) The numerical simulation shows the solutions of these two variants can behave quite differently. We have performed extensive numerical simulations. We present some of the numerical solutions corresponding to several representative initial data from three different types. Those types of data have previously been used in the simulations of solutions to the surface quasi-geostrophic equation in the work of Constantin, Majda and Tabak [8].

1. Type I initial data:

$$\omega_0(x) = \sin(x_1)\sin(x_2) + \cos(x_2) \tag{5.4}$$

   Type I data represents the simplest type of smooth initial data with a hyperbolic saddle.

2. Type II initial data:

$$\omega_0(x) = -\cos(2x_1)\cos(x_2) - \sin(x_1)\sin(x_2) \tag{5.5}$$

   Type II data contains an elliptic center.

3. Type III initial data:

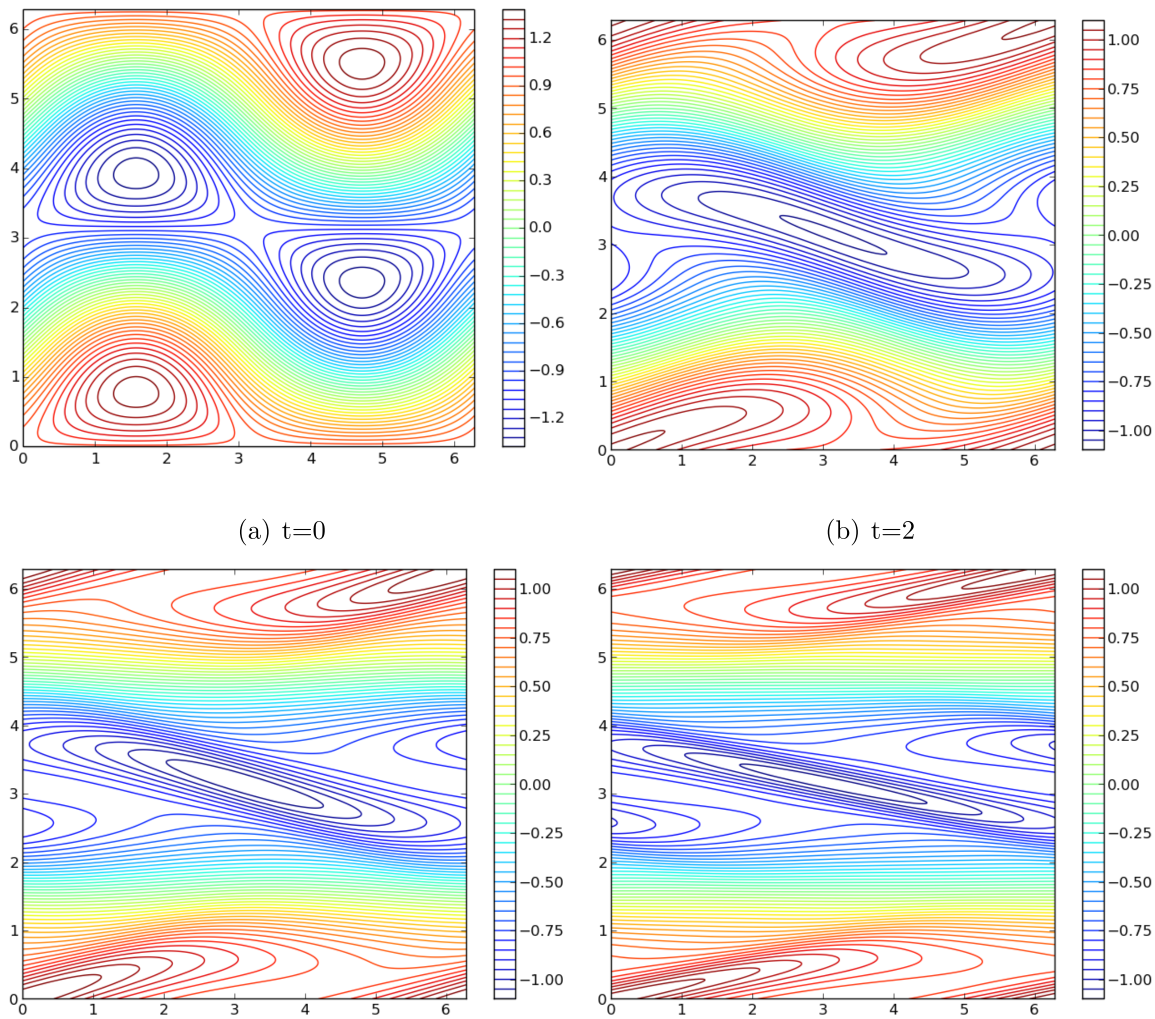
$$\omega_0(x) = \cos(2x_1)\cos(x_2) + \sin(x_1)\sin(x_2) + \cos(2x_1)\sin(3x_2) \tag{5.6}$$

   Type III data represents a general large-scale flow.

Numerical simulations will be presented in section 5.2 and 5.3, and section 5.4 will discuss about the results that we have.

5.2 Numerical Simulations of Equation (5.2)

This section presents numerical simulations of equation (5.2) with three initial data.



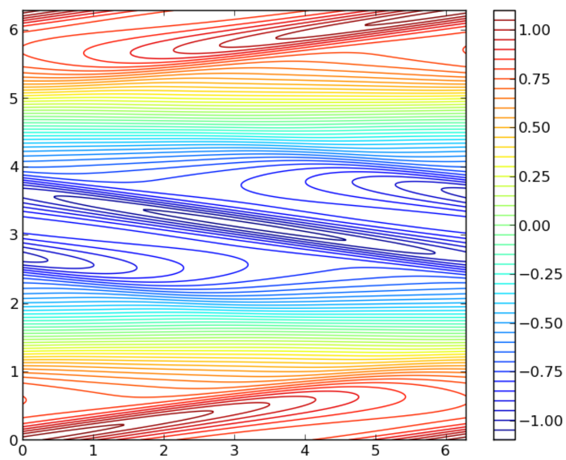
(a) t=0

(b) t=2

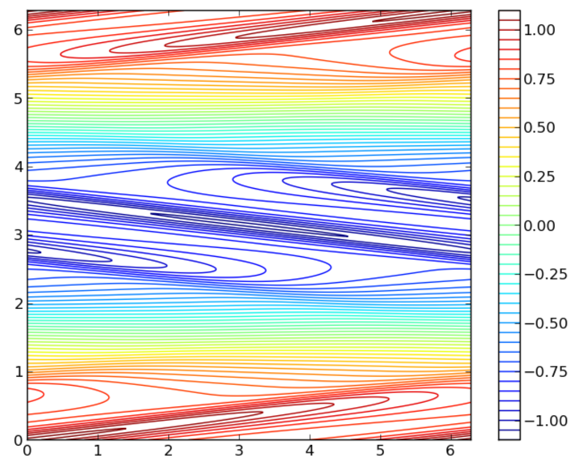
(c) t=4

(d) t=6

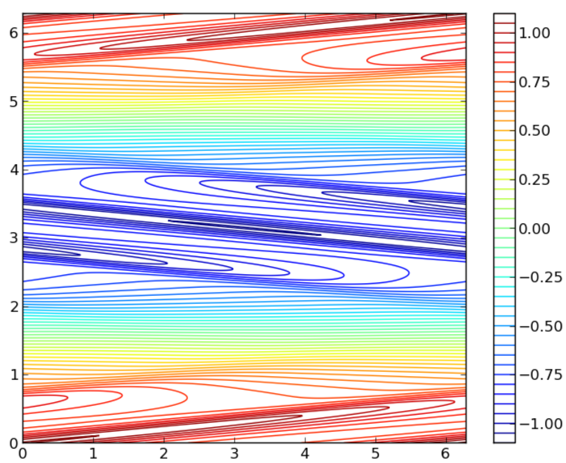
Figure 5.1: Contours of ω in equation (5.2) with Type I data at t = 0, 2, 4, 6



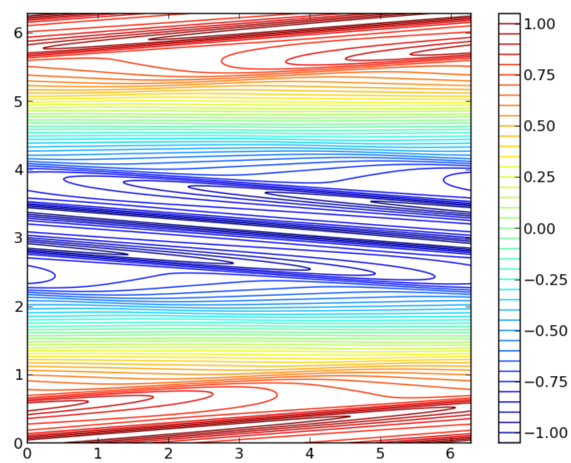
(a) t=8



(b) t=10

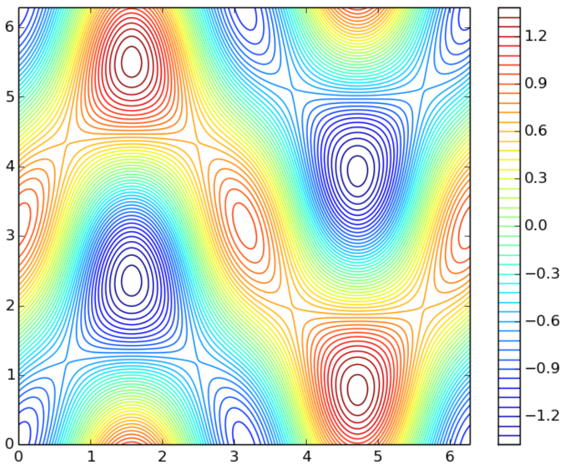


(c) t=12

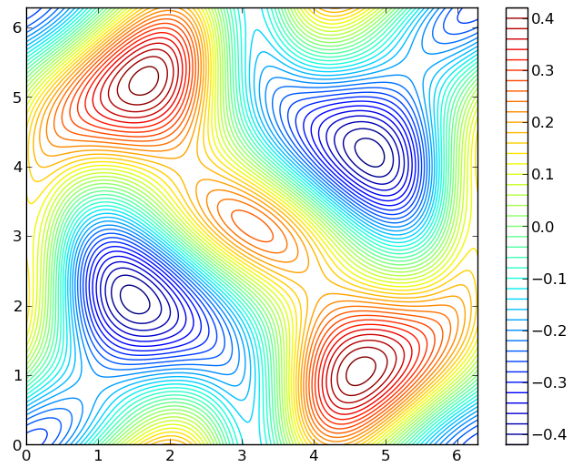


(d) t=14

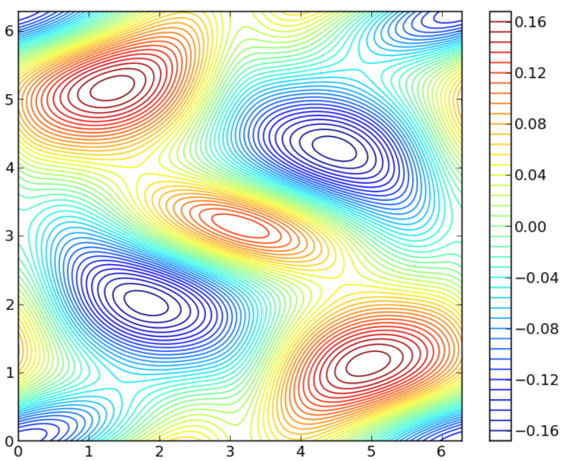
Figure 5.2: Contours of ω in equation (5.2) with Type I data at t = 8, 10, 12, 14



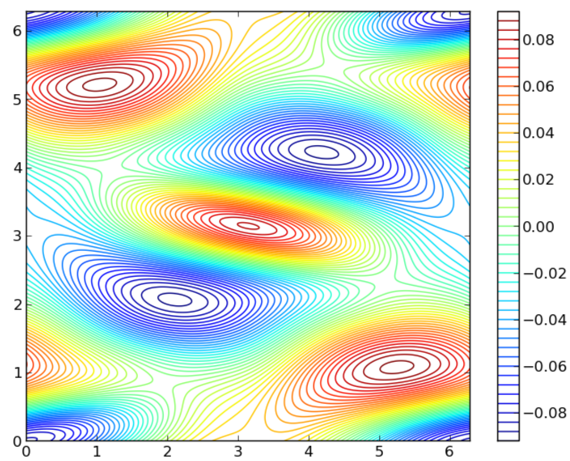
(a) t=0



(b) t=2

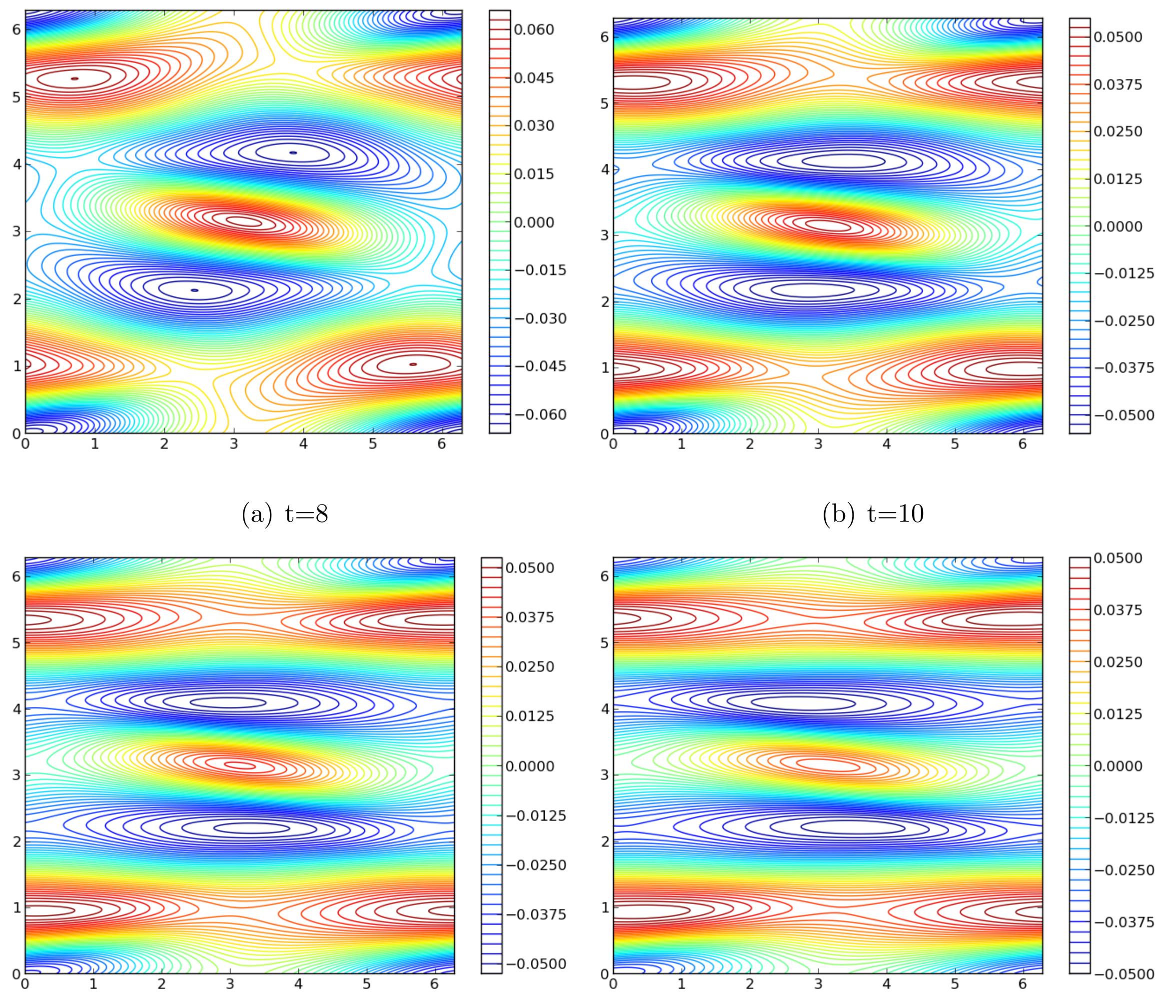


(c) t=4



(d) t=6

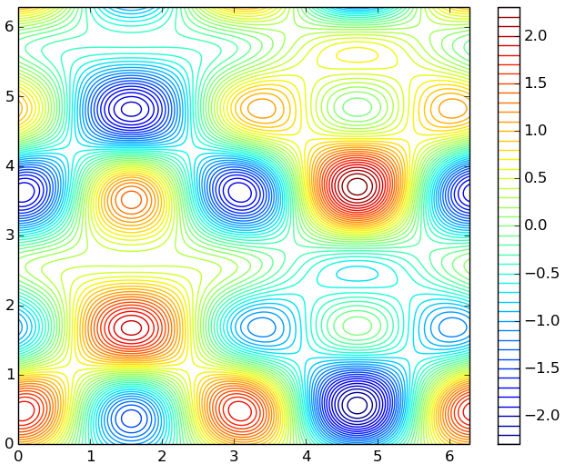
Figure 5.3: Contours of ω in equation (5.2) with Type II data at t = 0, 2, 4, 6



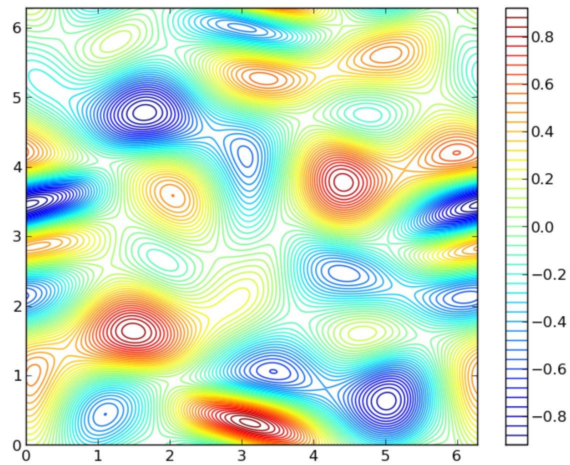
(a) t=8 (b) t=10

(c) t=12 (d) t=14

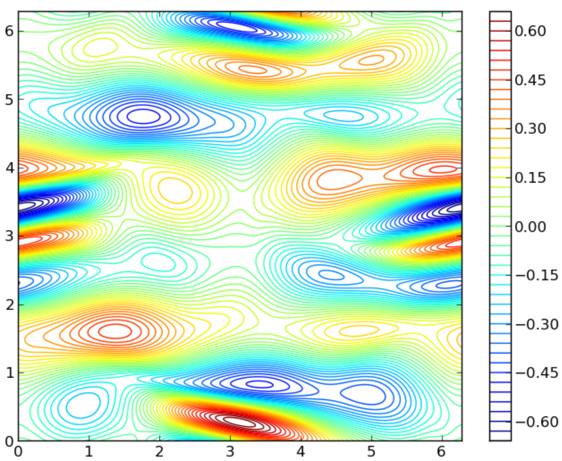
Figure 5.4: Contours of ω in equation (5.2) with Type II data at t = 8, 10, 12, 14



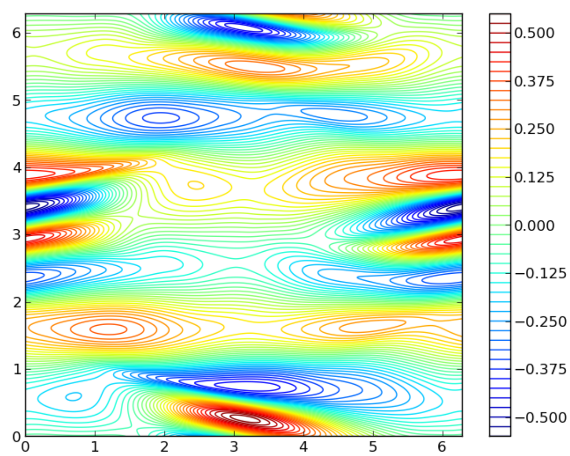
(a) t=0



(b) t=2

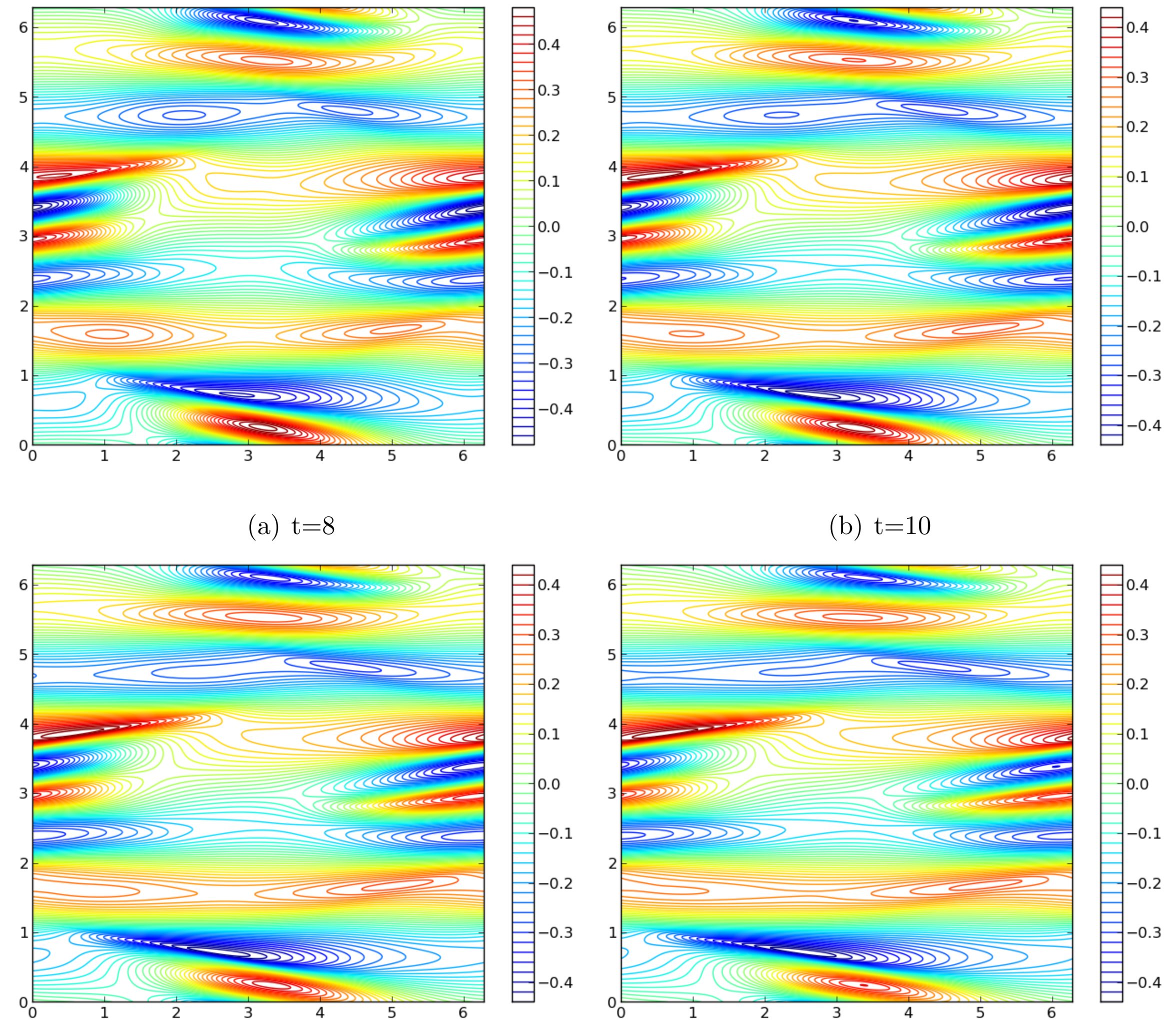


(c) t=4



(d) t=6

Figure 5.5: Contours of ω in equation (5.2) with Type III data at t = 0, 2, 4, 6



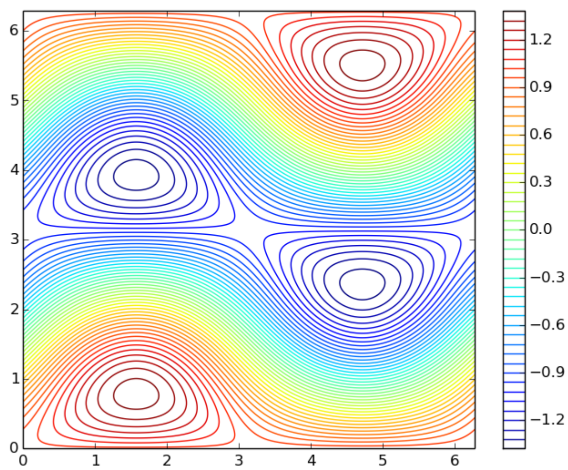
(a) t=8 (b) t=10

(c) t=12 (d) t=14

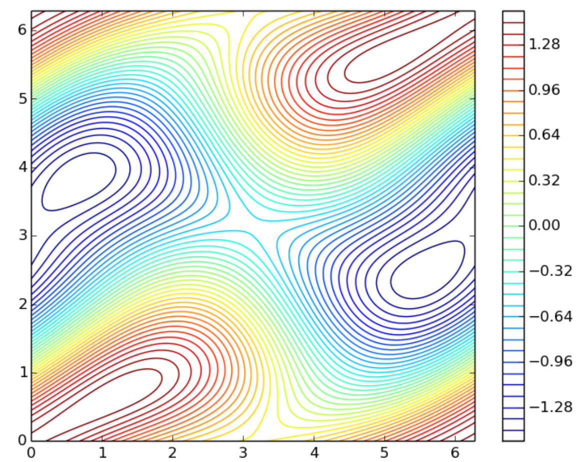
Figure 5.6: Contours of ω in equation (5.2) with Type III data at t = 8, 10, 12, 14

5.3 Numerical Simulations of Equation (5.3)

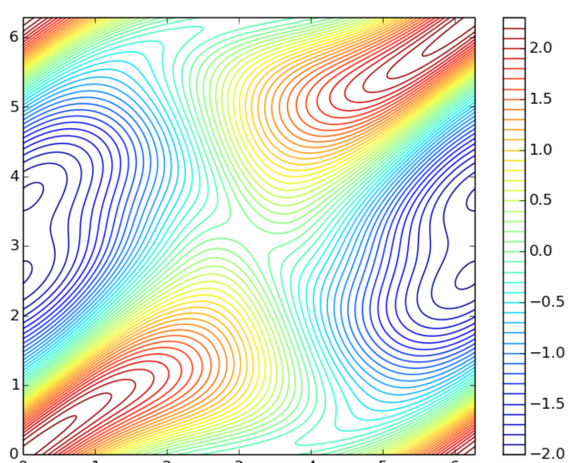
As a comparison, this section presents results of equation (5.3) with three initial data. Type 4 equation increase fast so that there is no result at $t = 8$ for all of our intial data.



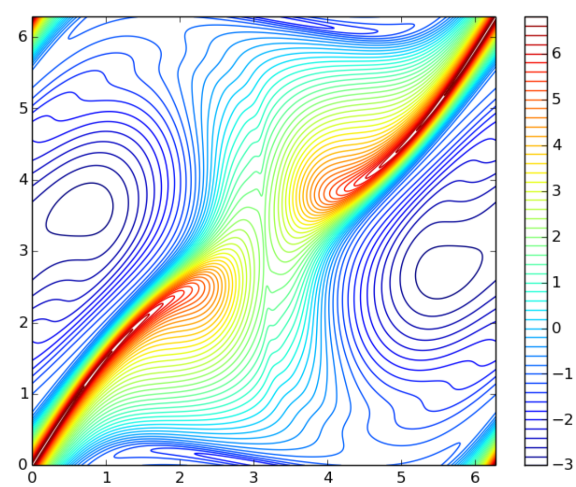
(a) t=0



(b) t=2

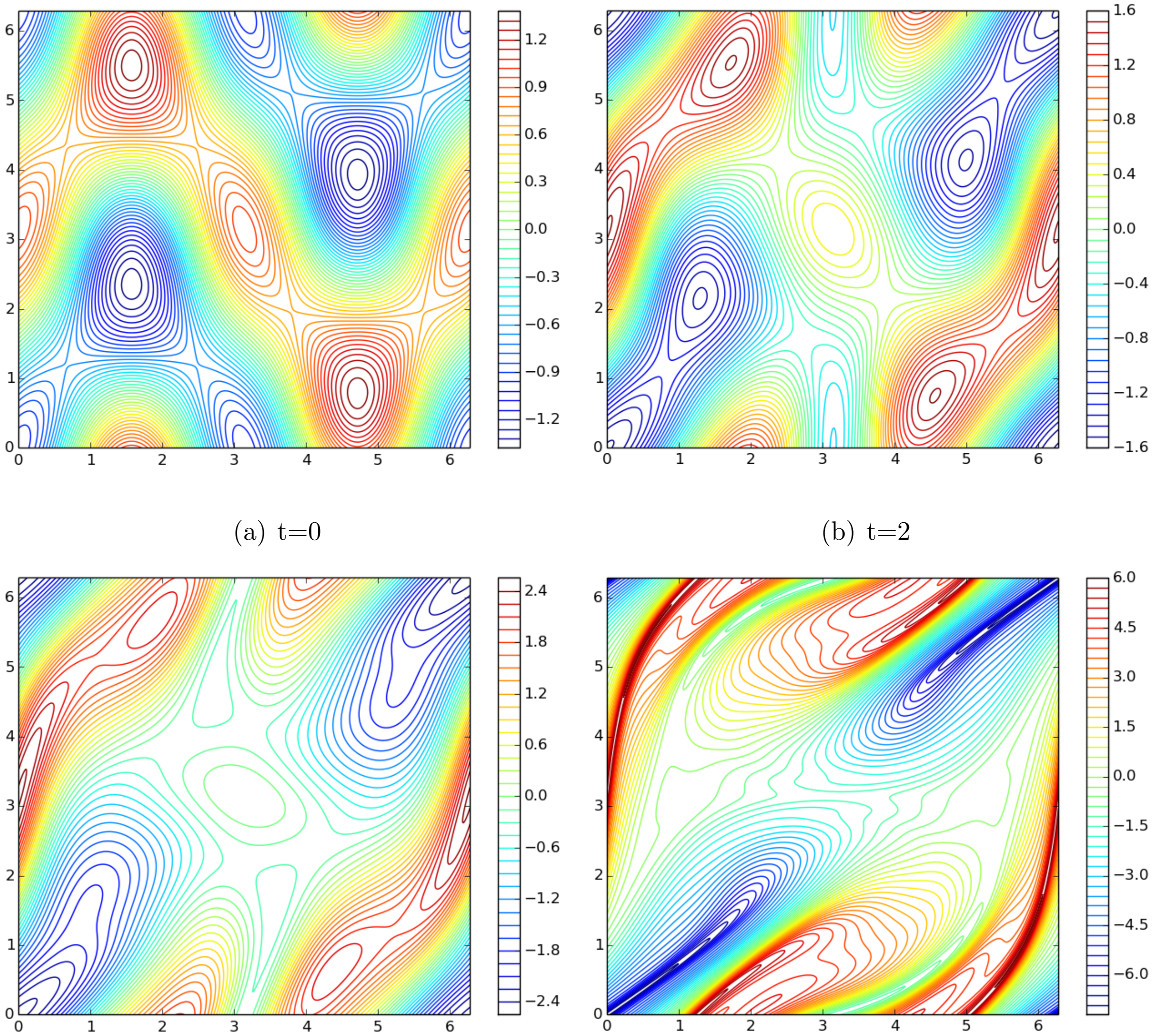


(c) t=4



(d) t=6

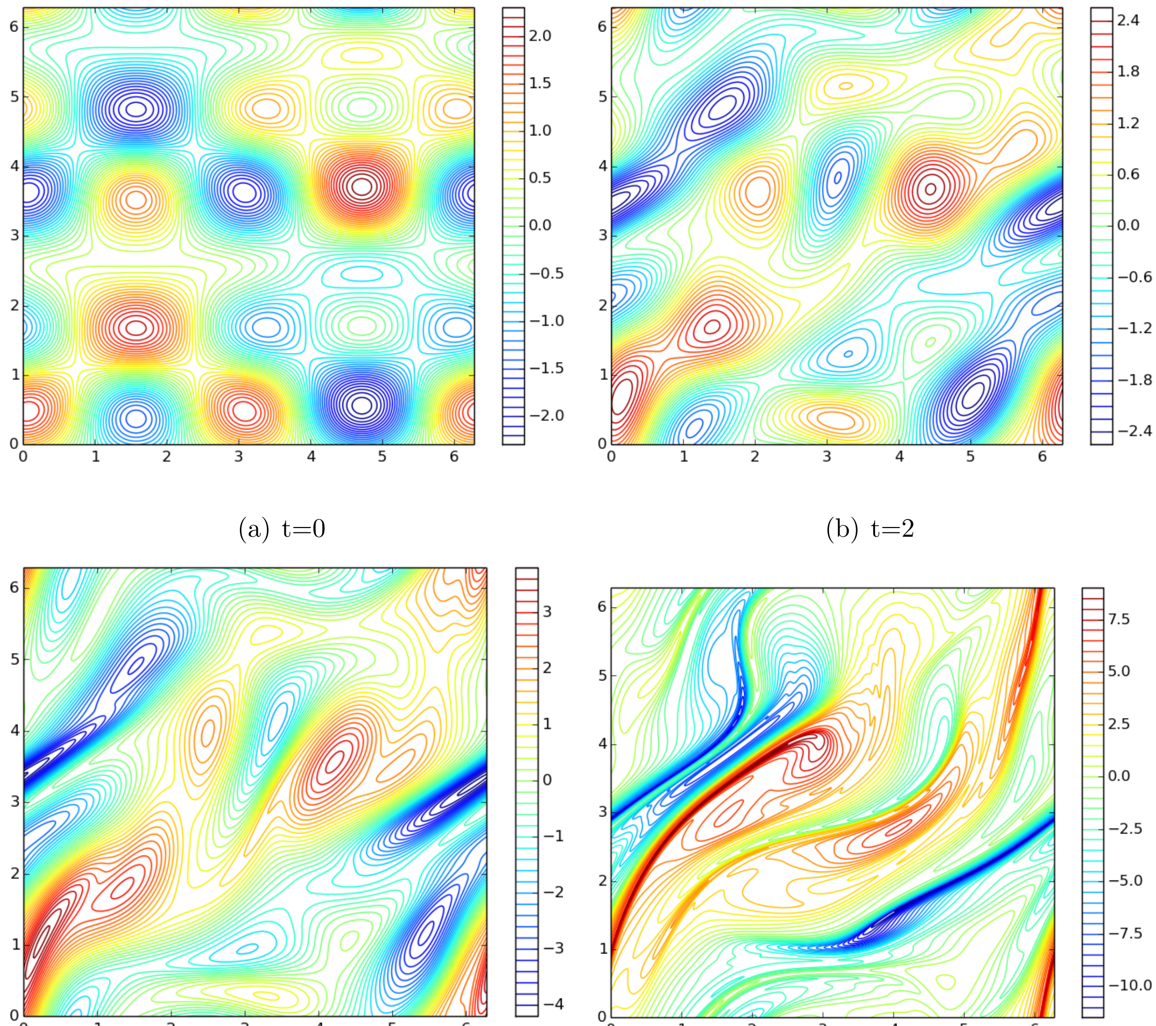
Figure 5.7: Contours of ω in equation (5.3) with Type I data at t = 0, 1, 2, 4



(a) t=0 (b) t=2

(c) t=4 (d) t=6

Figure 5.8: Contours of ω in equation (5.3) with Type II data at t = 0, 1, 2, 4



(a) t=0

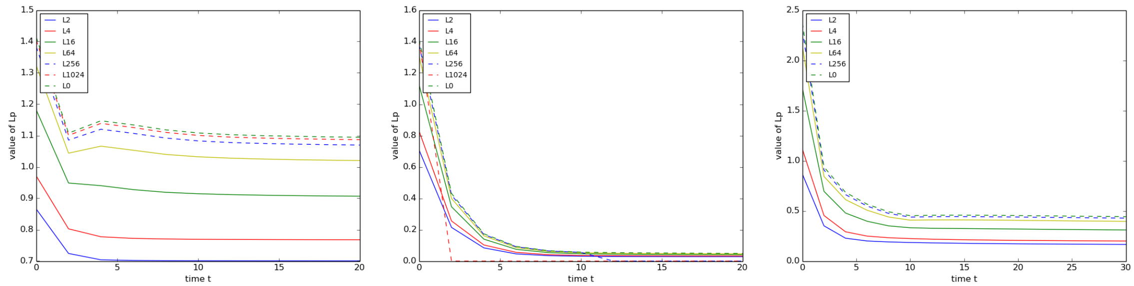
(b) t=2

(c) t=4

(d) t=6

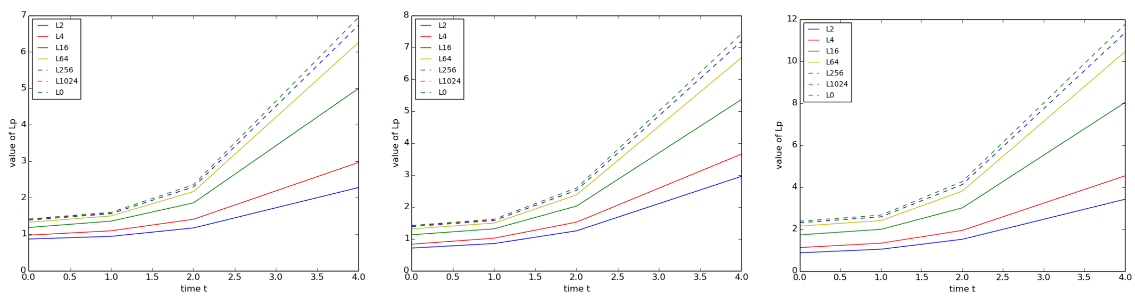
Figure 5.9: Contours of ω in Type 4 equation (5.3) with Type III data at t = 0, 1, 2, 4

5.4 Graph of L^p norms



(a) Type I data (b) Type II data (c) Type III data

Figure 5.10: Different L_p norms of equation (5.2)



(a) Type I data (b) Type II data (c) Type III data

Figure 5.11: Different L_p norms of equation (5.3)

The numerical simulations in section 5.2 and Figure 5.10 exhibit the same pattern that the L^∞-norm of the vorticity solving (5.2) decreases and the vertical velocity vanishes which makes the flow become horizontal.

However, in section 5.3, those numerical simulations and and Figure 5.11 show the solutions of (5.3) has the pattern that L^∞-norm of the vorticity increases and finite-time singularities do appear to occur during the simulations. This will be the future work to provide a rigorous proof with certain class of initial data such that the solutions blows up.

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VITA

BEI XIAO

Candidate for the Degree of

Doctor of Philosophy

Dissertation: BOUSSINESQ EQUATIONS WITH PARTIAL OR FRACTIONAL DISSIPATION

Major Field: APPLIED MATHEMATICS

Biographical:

Education:
Completed the requirements for the Doctor of Philosophy in Applied Mathematics at Oklahoma State University, Stillwater, Oklahoma in May, 2019.
Completed the requirements for the Master of Science in Applied Mathematics at Oklahoma State University, Stillwater, Oklahoma in December, 2014.
Completed the requirements for the Bachelor of Science in Information and Computing Science at Henan University, Kaifeng, Henan, China in June, 2012.

Experience:
Employed by Department of Mathematics, Oklahoma State University, as a graduate assistant from August 2012 to May 2019.

Professional Memberships:
American Mathematical Society (AMS).
Mathematics Association of America (MAA).