THE APPLICATION OF FOUR-VALUED BOOLEAN ALGEBRA TO SWITCHING CIRCUITS

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PREFACE

Although the study of switching circuits had previously drawn some attention, my interest was really sparked by attendance at a short course on the Design of Digital Control circuits sponsored by Bell Telephone Laboratories at Murray Hill, New Jersey, August 24 through September 4, 1953. The following year I had the privilege of teaching a course on this subject in the School of Electrical Engineering at Oklahoma Agricultural and Mechanical College. Occasionally, I saw reference to the use of diodes in switching circuits but no statement of how to include them in the mathematical treatment of the circuits. Mr Kirby B. Austin, Director of Research at Allied Control Company, once remarked to me that he had greatly reduced the number of relays needed in a particular switching problem by introducing diodes into the circuit. I decided that an interesting and useful study would be the development of a form of Boolean algebra with which to handle switching circuits containing diodes.

The mathematical basis of a four-valued Boolean algebra developed in this thesis will, I hope, be a useful tool for analyzing and simplifying switching circuits which include diodes. Although the use of this algebra or of matrices for very simple circuits may not be justifiable, it seems that the techniques offer a mathematical discipline which should be valuable for analyzing more

complex circuits.

I am indebted to members of the staff of Bell Telephone Laboratories who arranged for the short course on the Design of Digital Control Circuits and who inspired my interest in the subject. Sincere thanks are due Professor A. Naeter for making it possible for me to teach a course in switching circuits in his department. Work on a relay contract under the supervision of Professor Charles F. Cameron made it possible for me to learn a great deal about relays as elements of switching circuits. Special thanks are due my adviser, Dr. Herbert L. Jones, for his patience with my periods of inactivity and for his stimulation of the completion of this work.

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CHAPTER I

INTRODUCTION

In 1857, George Boole[1] published the original presentation of the algebra of logic. Since then, many mathematicians have extended his investigations. Susanne K. Langer[2] lists a number of publications on the subject. In 1938, Claude E. Shannon[3] applied Boolean algebra to relays and switching circuits. In addition to a number of journal articles, two books[4,5] have, to a great extent, been based on the application of mathematical logic to switching circuits.

This form of mathematics is ideal for an analysis of on-off devices: switches, relays, counters, multivibrators, and dial telephone systems. Any electrical circuit that has two possible states can be analyzed in terms of this algebra of logic, whether the two states are energized and deenergized, magnetized positively and

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[1]George Boole, An Investigation of the Laws of Thought (London, 1857).

[2]Susanne K. Langer, An Introduction to Symbolic Logic (2d ed., New York, 1953), pp. 356-360.

[3]Claude E. Shannon, "A Symbolic Analysis of Relay and Switching Circuits," AIEE Transactions, LVII (1938), 713-723.

[4]R. K. Richards, Arithmetic Operations in Digital Computers (New York, 1955).

[5]William Keister, A. E. Ritchie, and S. H. Washburn, The Design of Switching Circuits (New York, 1951).

magnetized negatively, or conducting and non-conducting.

In recent years an increasingly popular circuit element has been the diode, a device that conducts very well in one direction but has almost no conduction in the opposite direction. For many years diodes have been used as rectifiers to obtain unidirectional current from alternating-current sources. Highly efficient "solid-state" diodes of the germanium or silicon type that require no filament voltage have greatly stimulated the development of diode circuitry. Diodes have proved to be particularly useful in digital computer circuits.

When an attempt is made to apply ordinary Boolean algebra to diode switching circuits, a serious difficulty arises. The algebra allows two possible states, and a diode is either conducting or not conducting at any given instant, but a thorough analysis must somehow handle circuit elements that conduct well in both directions, elements that do not conduct in either direction, and diodes that conduct well in one direction but not in the other.

Mathematical analysis of diode circuits has recently been studied by a few authors. Lee and Chen[6] have applied a three-valued propositional logic introduced by Post[7] to switching circuits, but their work deals with three possible values of output voltage, for example, instead of treating the bilateral conduction of diodes as a special kind of circuit. Allowable voltage states are negative, zero, and positive,

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[6]C. Y. Lee and W. H. Chen, "Several-Valued Combinational Switching Circuits," AIEE Transactions, LXXV, pt. I (1956), 278-283.

[7]E. L. Post, "Introduction to a General Theory of Elementary Propositions," American Journal of Mathematics, XLIII (1921), 163-185.

so that a trinary arithmetic can be handled instead of the usual binary arithmetic. Yokelson and Ulrich[8] have shown how to solve for values of resistors and voltages to be used with diodes in logic circuits, but they have not applied an algebra of logic to the diodes. A rectifier algebra has been developed by Schaefer[9] using a $\vee$ b for "the more positive of a or b" and a $\wedge$ b for "the more negative of a or b." His algebra "is not the ordinary numerical algebra taught in high school, nor is it the Boolean algebra used for relay circuits. It is, rather, something of a union of the two."[10] The method is particularly useful when more than one voltage source is to be considered, but it does not handle combinations of diodes with ordinary on-off devices.

A study of the problem revealed that it would be useful to have an algebra adapted to regular switching circuits or diodes or both. In order to develop such an algebra, it seemed vital to understand the fundamental concepts of ordinary Boolean algebra more thoroughly than the average electrical engineer has done. Chapter II of this thesis covers the mathematical background necessary for a proper understanding. On this foundation, the third chapter develops the theory of the Boolean algebra of order two, that is, the two-valued logic with which ordinary switching circuits can be analyzed. In addition to the two standard binary operations universally used,

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[8]B. J. Yokelson and W. Ulrich, "Engineering Multistage Diode Logic Circuits," AIEE Transactions, LXXIV, pt. I (1955), 466-475.

[9]David H. Schaefer, "A Rectifier Algebra," AIEE Transactions, LXXIII, pt. I (1955), 679-682.

[10]Ibid., p. 679.

other binary relations are evaluated. Based on a few postulates and definitions, several theorems involving the various binary relations are tabulated.

Chapter IV develops the Boolean algebra of order four as the direct union of two two-valued Boolean algebras. Twenty operational tables are developed for the four-valued algebra. Except for the number of allowable digits, the new algebra does not negate the two-valued postulates and theorems. It is, rather, an extension of the old algebra to allow two more elements. Based on nine new dual postulates, in addition to the postulates and theorems of the two-valued algebra, a number of new theorems are tabulated.

A brief coverage of the application of two-valued algebra to conventional switching circuits is included as a basis for understanding the application of four-valued logic to circuits that include diodes, different kinds of voltage sources, and polarized relays. As an illustration of how present two-valued techniques can be expanded to include four allowable states, the use of matrices is discussed.

It appears to the author that the four-valued Boolean algebra herein developed is a sound mathematical basis for analysis of switching circuits which include unilateral devices. The technique should prove to be particularly valuable in the rapidly expanding fields of digital computers and automatic controls.

CHAPTER II

THE MATHEMATICAL BASIS OF BOOLEAN ALGEBRAS

Classes[1]

The words "class" and "set" are used interchangeably to refer to a collection of objects which have a common property. Any property defines a class, namely, the class of all objects which have that property. Conversely, any class determines a property by virtue of the fact that an object is said to have a particular property if and only if it belongs to the corresponding class. The objects are called the elements of the class. The combination of signs $x \in C$ means "x is an element of C" or "the element x is in the class C". Mathematically, one may consider as a valid set one which contains no elements whatsoever. This would be called the empty, null, or void set. When a class C is defined by an enumeration of all its elements (for example, a, b, and c), it is designated by enclosing the elements within braces as follows. $C = \{a,b,c\}$.

Classes may in turn be considered as elements of other classes. Any higher class is defined by pointing out the lower classes which belong to it. One restriction must be applied, however, in order to avoid logical contradictions. No class may belong to its own elements.

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[1]Garrett Birkhoff and Saunders Mac Lane, A Survey of Modern Algebra (rev. ed., New York, 1953), pp. 29-34.

Thus, the "class of all classes" is not considered valid. A set A is called a <u>subset</u>, or <u>subclass</u>, of set B if and only if every element of A is also an element of B. The set A is then said to be included or contained in set B. This relationship is expressed symbolically by $A \leqq B$ or $B \geqq A$. Class A is <u>equal</u> to class B if and only if A is a subset of B and B is a subset of A. If A is a subset of B but A is not equal to B, A is called a <u>proper subset</u> of B, designated $A < B$. The null set 0 is considered to be a subset of every set. The universal set I is the class which includes all subclasses and elements involved in a particular problem. If, for example, the elements under discussion were specific people, they might be classified into larger sets according to a particular property. The universal set would then include all people of all classes. The inclusion relation satisfies the following laws.

<u>Reflexive</u>: $A \leqq A$ for all A.

<u>Antisymmetric</u>: If $A \leqq B$ and $B \leqq A$, then $A = B$.

<u>Transitive</u>: If $A \leqq B$ and $B \leqq C$, then $A \leqq C$.

The <u>intersection</u> of two sets A and B is written $A \cap B = C$ and consists of all elements which are in both A and B but not in one of them alone. The symbol $\cap$ is frequently referred to as "cap". C is the greatest set included in both A and B.

The <u>union</u> of two sets E and F is written $E \cup F = G$ and consists of all elements which are in E or in F or in both E and F. The symbol $\cup$ is frequently referred to as "cup". G is the smallest set which contains all elements of E and all elements of F.

<u>Correspondence</u>

A correspondence $a \rightarrow b$ is a rule which prescribes for each element a of class A a corresponding element b of another class B. Two types of correspondence are possible, a many-to-one and a one-to-one. In a many-to-one correspondence there is at least one element b in the class B which corresponds to two or more elements a_1, a_2, etc., in A. Such a correspondence is designated with a single-headed arrow: $a \rightarrow b$. In a one-to-one correspondence, each element of B corresponds to one and only one element in A. For this relationship, a double-headed arrow is used: $a \twoheadrightarrow b$. Arrows are also used to show correspondences between sets: $A \rightarrow B$ or $A \twoheadrightarrow B$. For finite sets, a correspondence is sometimes designated by writing each element of the first set on one line, writing corresponding elements of the second set in appropriate positions underneath on a second line, and enclosing both lines within parentheses. Thus, for the letters and digits of a modern telephone dial, the many-to-one correspondence would be written as follows.

$$\begin{pmatrix} A & B & C & D & E & F & G & H & I & J & K & L & M & N & O & P & R & S & T & U & V & W & X & Y & Z \\ 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 & 0 \end{pmatrix}$$

When class B is included in class A, a many-to-one correspondence $A \rightarrow B$ is called a <u>single-valued transformation</u> and a one-to-one correspondence $A \twoheadrightarrow B$ is called a <u>one-to-one transformation</u>. A one-to-one transformation on a finite set is called a <u>permutation</u>.

<u>Relationship</u>

Elements of a set may be "related" to each other in many ways. For example, element a may equal element b of the same set. This relationship is universally expressed as $a = b$. In the set of integers, element a might be an integral divisor of b, written $a \mid b$. Such relationships between two elements are called <u>binary relations</u> because two elements are involved. These binary relations might, in general, be expressed by the symbol ρ. ρ is a binary relation for a class C if, for two elements a and b of C, either a stands in the relation ρ to b (in symbols, a ρ b) or a does not stand in the relation ρ to b.

An <u>equivalence</u> relation is a binary relation a ρ b which satisfies the following laws.

<u>Reflexive</u>: a ρ a for all a of the class C.

<u>Symmetric</u>: If a ρ b, then b ρ a for all elements a and b of C.

<u>Transitive</u>: If a ρ b and b ρ c, then a ρ c for all a, b, and c of C.

Obviously, the equivalence relation is satisfied by equality of numbers and by congruence of triangles. The sign $\sim$ is customarily used for the general equivalence relation.

<u>Binary Operations</u>

A binary operation on a class C is a rule which assigns to each pair of elements of C a unique element of C. For example, in the class of intergers, $2 + 3 = 5$ symbolically assigns the number 5 to the pair of numbers 2 and 3 when the binary operation is +. In $2 \cdot 3 = 6$, the binary operation is the product, symbolized by $\cdot$. Using the symbol $\circ$ for any binary operation, $a \circ b = c$ uniquely defines c. A set with such an equality relation between its elements is called an <u>operational system</u>. If the equivalent of any couple of an operational system S is itself an element of S, the system is said to be <u>closed</u>.

In general, the order of the elements involved in the binary operation will need to be observed, since $a \circ b$ might not be equal to $b \circ a$. A binary operational system in which $a \circ b = b \circ a$ is called <u>commutative</u>. The following axiom is assumed for the equality relation in an operational system.

Substitution Principle

In an operational equation, any one of the elements may be replaced by its equivalent, and any couple, triple, or n-tuple may be replaced by its equivalent. If $a_1 = a_2$ and $b_1 = b_2$, then $a_1 \circ b_1 = a_1 \circ b_2 = a_2 \circ b_1 = a_2 \circ b_2$. If $a \circ b = c$, then $c \circ d = (a \circ b) \circ d$. Since, by the substitution principle, a binary operational system can produce expressions like $(a \circ b) \circ c = e$, one may define a trinary operational system as one in which a rule assigns to each triple of elements a unique element. A set of rules for n elements would define an n-ary operational system.

A couple, triple, or n-tuple substituted for its equivalent must be enclosed within parentheses to indicate that the entire expression is acted upon in the same manner as its equivalent. Usually an operational sign is required between elements of a binary operational expression, although by custom the operational sign for multiplication is frequently omitted in algebraic expressions. Thus, $xy = x \cdot y$. Frequently a binary relation will be indicated by constructing an operational table. In such a table, the elements of a set are listed both as column headings and at the left of the rows. The binary relation is indicated at the upper left. For $x \circ y = z$, x is a row heading, y is a column heading, and z is the listing in the body of the table in the appropriate row and column.

TABLE I

A BINARY OPERATIONAL TABLE.

o	c_1	c_2	c_3	c_4
c_1	c_1	c_2	c_3	c_4
c_2	c_2	c_1	c_4	c_3
c_2	c_3	c_4	c_1	c_2
c_4	c_4	c_3	c_2	c_1

Table I indicates the following equivalences:

$$\begin{array}{llll}
c_1 \circ c_1 = c_1 & c_1 \circ c_2 = c_2 & c_1 \circ c_3 = c_3 & c_1 \circ c_4 = c_4 \\
c_2 \circ c_1 = c_2 & c_2 \circ c_2 = c_1 & c_2 \circ c_3 = c_4 & c_2 \circ c_4 = c_3 \\
c_3 \circ c_1 = c_3 & c_3 \circ c_2 = c_4 & c_3 \circ c_3 = c_1 & c_3 \circ c_4 = c_2 \\
c_4 \circ c_1 = c_4 & c_4 \circ c_2 = c_3 & c_4 \circ c_3 = c_2 & c_4 \circ c_4 = c_1
\end{array}$$

The main diagonal, as in a determinant, is a straight line drawn through the upper left element and the lower right element. Obviously, the array will be symmetrical about the main diagonal if and only if the system is commutative. An operational system S may be indicated by listing the binary operation and the elements in parentheses following the letter assigned to the system. Thus, the system tabulated above would be $S(o, c_1, c_2, c_3, c_4)$. If the binary system were definitely understood, the listing might omit the operational sign and be shown as $S(c_1, c_2, c_3, c_4)$.

Isomorphisms and Homomorphisms

An isomorphism exists between two binary operational systems $S_1(o, a_1, b_1, \ldots)$ and $S_2(\oplus, a_2, b_2, \ldots)$ if and only if there exists a one-to-one correspondence, $S_1 \leftrightarrow S_2$, between their elements such that $a_1 \circ b_1 = c_1$ implies $a_2 \oplus b_2 = c_2$, and vice versa.

A homomorphism exists between two binary operational systems $S_1(o, a_1, b_1, \ldots)$ and $S_2(\oplus, a_2, b_2, \ldots)$ if and only if there exists a many-to-one correspondence, $S_1 \rightarrow S_2$, from S_1 to the whole of S_2 such that $a_1 \circ b_1 = c_1$ implies $a_2 \oplus b_2 = c_2$, and vice versa.

If $S_2 \leqq S_1$, the isomorphism $S_1 \leftrightarrow S_2$ is called an automorphism and the homomorphism $S_1 \leftrightarrow S_2$ is called an endomorphism.

Groups

A group G is a binary operational system which satisfies the following axioms.

1. G is a non-empty set of elements a, b, c, . . . which is closed under a single-valued binary operation $x \circ y = z$, x, b, z, $\in$ G.

2. The binary operation satisfies the associative law. $x \circ (y \circ z) = (x \circ y) \circ z$ for all x, y, z.

3. With respect to the binary operation, there is an identity element $i \in G$ which satisfies the identity law. $x \circ i = i \circ x = x$ for all x.

4. For each element x in G there is an inverse element x^{-1} also in G which satisfies the inverse law. $x \circ x^{-1} = x^{-1} \circ x = i$ for each x and some element x^{-1} of G.

Groups are not necessarily commutative. A group whose binary operation satisfies the law $x \circ y = y \circ x$ is called a commutative or Abelian group.

The binary operational system of Table I, page 9, satisfies the four axioms and is, therefore, a group. In this group the identity element is c_1 and each element is its own inverse. The group is commutative.

It can be shown[2] that, in any group, $xa = b$ and $ay = b$ have the unique solutions $x = ba^{-1}$ and $y = a^{-1}b$. A binary associative operational system in which $xa = b$ and $ay = b$ are not always solvable is called a semigroup.

Groups of Transformation

On page 7 it was stated that when class B is included in class A, a many-to-one correspondence $A \rightarrow B$ is called a single-valued

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[2]Birkhoff and Mac Lane, p. 127.

transformation, and a one-to-one correspondence $A \leftrightarrow B$ is called a one-to-one transformation. For example, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}$ is a single-valued transformation, and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ is a one-to-one transformation. An operational system of transformations involves a concept of equality and a binary operation. Two transformations, t and u, are equal if and only if they have the same effect on every element of the set S on which they operate. $xt = xu$ for every $x \in S$. By definition, the binary operation (indicated as the product $t \circ u$ or tu) of two transformations is the result of performing them in succession, first t, then u. $x(tu) = (xt)u$.

It can be shown[3,4] that the algebra of transformations has the following properties. Multiplication of transformations obeys the associative law. $(tu)v = t(uv)$. The identity transformation t_i is the correspondence which leaves every element x of the set S unchanged. $xt_i = x$ for every $x \in S$. $t_i t = t t_i = t$ for all t. When the transformations are one-to-one, the inverse of t is that transformation t^{-1} which carries xt back into x. Then $xtt^{-1} = x$ for all x of S and $tt^{-1} = t^{-1}t = t_i$. The non-void set T of transformations is a group if the set is closed under multiplication and the inverse t^{-1} of every element t of T is in T.

Symmetries

Group theory is the foundation of a consequential algebra of symmetry. Geometrically, a symmetry implies a transformation by means of a rigid motion of translation, rotation, or reflection, which

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[3]Birkhoff and Mac Lane, pp. 120-123.

[4]Stephen A. Kiss, Transformations on Lattices and Structures of Logic (New York, 1947), pp. 28-30.

maintains distances between any two points and carries the figure so transformed into itself. For example, consider the square of Figure 1.

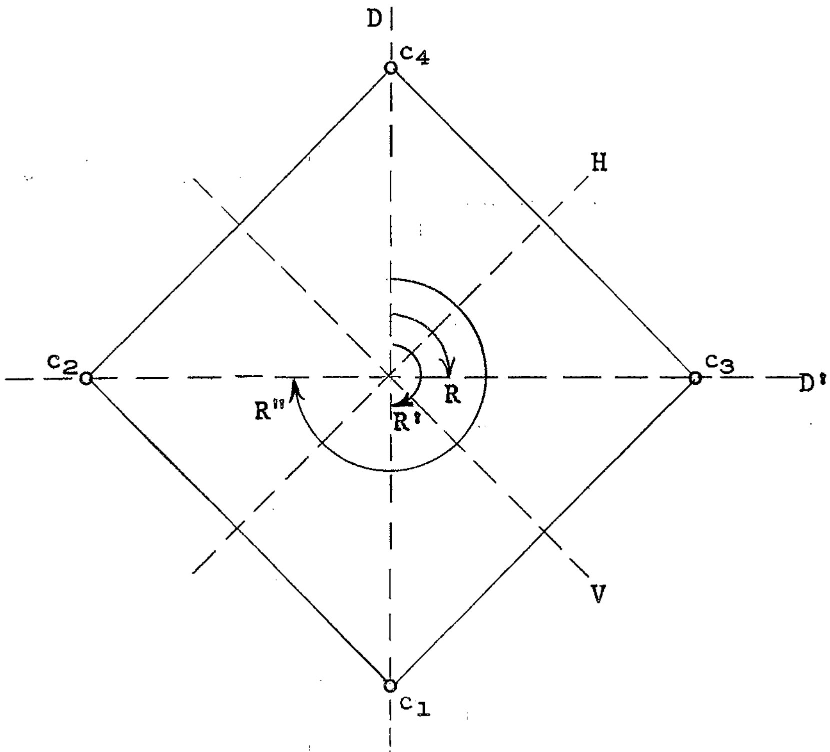

Fig. 1. Axes of Reflection and Angles of Rotation of a Square.

This square is shown in a position rotated forty-five degrees counter-clockwise from the position usually shown[5] in modern algebra textbooks. It was intentionally displaced so that the figure will show certain relations characteristic of lattice structures, which will be discussed later.

For Figure 1, the symmetries are as follows:

[5]Birkhoff and Mac Lane, p. 118.

I = the identity transformation = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$

H = a reflection in the H axis = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_1 & c_4 & c_3 \end{pmatrix}$

V = a reflection in the V axis = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_4 & c_1 & c_2 \end{pmatrix}$

R' = a 180° rotation about the center = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix}$

D = a reflection in the D diagonal = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_3 & c_2 & c_4 \end{pmatrix}$

R = a 90° clockwise rotation around the center = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_4 & c_1 & c_3 \end{pmatrix}$

R" = a 270° clockwise rotation around the center = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_1 & c_4 & c_2 \end{pmatrix}$

D' = a reflection in the D' diagonal = $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_2 & c_3 & c_1 \end{pmatrix}$

A multiplication table for the group of symmetrical transformations of the square is shown in Table II. This table can be checked

TABLE II

MULTIPLICATION TABLE FOR TRANSFORMATIONS OF A SQUARE.

∘	I	H	V	R'	D	R	R"	D'
I	I	H	V	R'	D	R	R"	D'
H	H	I	R'	V	R"	D'	D	R
V	V	R'	I	H	R	D	D'	R"
R'	R'	V	H	I	D'	R"	R	D
D	D	R	R"	D'	I	H	V	R'
R	R	D	D'	R"	V	R'	I	H
R"	R"	D'	D	R	H	I	R'	V
D'	D'	R"	R	D	R'	V	H	I

by listing for each transformation the two lines of corresponding elements within parentheses, then listing the resultant correspondence by tracing each element through its two transformations.

$$H \circ V = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_1 & c_4 & c_3 \end{pmatrix} \circ \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_4 & c_1 & c_2 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix} = R'$$

Most of the group properties can be read directly from Table II. For instance, the existence of an identity implies that some row must be a replica of the top heading and that the corresponding column must be a replica of the left heading. Obviously, I is the identity element. The possibility of solving the equation $ay = b$ indicates that the row opposite a must contain the entry b. Since the solution is unique, b must occur only once in this row. A group is commutative if and only if its operational table is symmetrical about the main diagonal. Obviously, the group shown in Table II is not commutative. The set $\{I,H,V,R'\}$, however, is commutative and is itself a group.

Subgroups

A non-empty subset K of a group G is a subgroup of G if the postulate of closure is satisfied; that is, if xy is in K whenever x and y are in K. Thus, the subset $\{I,H,V,R'\}$ is a commutative subgroup of the non-commutative group $\{I,H,V,R',D,R,R'',D'\}$. The intersection $K \cap L$ of two subgroups K and L of a group G is a subgroup of G. The union $K \cup L$ of two subgroups K and L of a group G is a subgroup of G.[6,7]

Isomorphism and Automorphism of Groups

Any two groups, or any two binary operational systems, $G_1(o,a_1,b_1,\ldots)$ and $G_2(\oplus,a_2,b_2,\ldots)$ are isomorphic if and only if there is a one-to-one correspondence $a_1 \leftrightarrow a_2$ between their elements such that $a_1 \circ b_1 = d_1$ implies $a_2 \oplus b_2 = d_2$. There is thus an

[6]Kiss, p. 33.
[7]Birkhoff and Mac Lane, p. 141.

isomorphism between the transformation subgroup $T(\circ,I,H,V,R')$ of Table II and the group $G(o,c_1,c_2,c_3,c_4)$ of Table I. $I \leftrightarrow c_1$, $H \leftrightarrow c_2$, $V \leftrightarrow c_3$, $R' \leftrightarrow c_4$.

An automorphism t_a of a group G is a one-to-one transformation on G such that $(x \circ y)t_a = (xt_a) \circ (yt_a)$ for all x and y in G. The automorphisms of any group G themselves form a group.

Rings

A group or a semigroup is a system having only a single binary operation defined between pairs of elements. A ring, however, has two binary operations, usually called addition and multiplication. A ring R is a set of elements $\{a,b,c,\ldots\}$ with two binary operations satisfying the following axioms.

1. The elements of R form a commutative group under addition. $x + y = y + x$.

2. The set R is closed under multiplication. $xy = z$, $z \in R$ for all x and y.

3. The set R is associative under multiplication. $x(yz) = (xy)z$.

4. Multiplication is distributive over addition. $x(y + z) = xy + xz$. $(x + y)z = xz + yz$.

5. The substitution principle is valid for multiplication. If $xy = z$, then $zw = xyw$.

If $xy = yx$ for every x and y of R, the ring is commutative. It can be shown[8] that rings also have the following properties. The associative law of addition holds. $x + (y + z) = (x + y) + z$. The equation $a + x = b$ is solvable for all a and b of R. There is a

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[8]Kiss, pp. 49-51.

unique zero element, the identity of addition, such that $x + 0 = x$ for all x. The product of any element and zero is zero. $x0 = 0$ for all x. There are negative elements such that $-x + x = x + (-x) = 0$ and $-(-x) = x$ for all x.

Division Rings

A division ring is a ring which has a unity element 1 and in which every non-zero element x has an inverse x^{-1} such that $xx^{-1} = x^{-1}x = 1$. It can be shown[9,10] that a division ring also has the following properties. A division ring has no divisors of zero. A division ring is a ring which has at least two elements, the non-zero elements of which form a group under multiplication. It is, therefore, closed under multiplication. The equations $ax = b$ and $ya = b$ are solvable whenever a is not equal to zero. The cancellation laws are valid; that is, there are unique solutions for $ax = b$ and $ya = b$ when a is not equal to zero. A division ring may or may not be commutative. A commutative division ring is called a field. A subset S of a ring R is said to be a subring of R if and only if S is a ring with respect to the operations of addition and multiplication in R.

Direct Unions

Direct unions are compound systems obtained from two or more operational systems. The direct product $P = G \times H$ of two groups $G(a_1,a_2,\ldots,a_m)$ and $H(b_1,b_2,\ldots,b_n)$ is an operational system of order mn, the elements of which are ordered couples (a_i,b_j) in which the $a_i \in G$ and $b_j \in H$. Two couples are equal if and only if their corre-

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[9]Kiss, pp. 52-56.
[10]Birkhoff and Mac Lane, pp. 126-127.

sponding components are equal. $(a_i,b_j) = (a_k,b_\ell)$ if and only if $a_i = a_k$ and $b_j = b_\ell$. The product of two couples is defined by the equation $(a_i,b_j)(a_k,b_\ell) = (a_ia_k,b_jb_\ell)$.

The direct product P of the groups G and H contains the subgroups G_s and H_s isomorphic with G and H, respectively. Every element of P is expressible as a permutable product of an element of G_s by an element of H_s. The system of P is closed under multiplication because G and H are closed. Because multiplication is associative in the groups G and H, the associative law is valid in P. The identity element of P is (i_1,i_2), where i_1 and i_2 are the identity elements of G and H, respectively. The inverse of (a_i,b_j) is (a_i^{-1},b_j^{-1}), because $(a_i^{-1},b_j^{-1})(a_i,b_j) = (i_1,i_2)$.

The direct sum $S = R_1 + R_2$ of two rings R_1 and R_2 is the set of all pairs (a,b) with a in R_1 and b in R_2. The two operations in S are $(a_1,b_1) + (a_2,b_2) = (a_1 + a_2,\ b_1 + b_2)$ and $(a_1,b_1)(a_2,b_2) = (a_1a_2,b_1b_2)$. S is itself a ring.

Partially Ordered Systems

The inclusion relation for sets stated on page 6 is a specific example of the broader concept of a partially ordered system which has the same general properties as the set inclusion. A partially ordered system P is any set with a binary relation $\leq$ between elements of the set, which satisfies the reflexive, anti-symmetric, and transitive laws. By $a < b$ is meant that a is included in b, but a is not equal to b. In other words, a is properly included in b.

Least and Greatest Elements

By a least element of a partially ordered system P is meant an element 0 satisfying the relation $0 \leqq p$ for all p of P. By a greatest element of P is meant an element u satisfying the relation $p \leqq u$ for all p of P. The element b is said to cover the element a if $a < b$ and if $a < x < b$ is possible for no $x \in P$; that is, if there is no element between a and b. The elements a and b are said to be linked to each other if and only if a covers b or b covers a. In a partially ordered system in which coverage and linkage relations exist, and in which there is a least element 0, an element is called an atom if and only if it covers zero.

Duality

The duality principle states that any theorem which is true in every partially ordered system remains true if the symbols $\leqq$ and $\geqq$ are interchanged throughout the statement of the theorem. A partially ordered system P is called self-dual if and only if its dual system P_d is obtained by a one-to-one transformation on the elements of P.

Diagrams of Partially Ordered Systems

The relations of a partially ordered system can be illustrated by geometrical diagrams. Each element of the system is represented by a small circle so placed that the circle for a is above that for b, with respect to a horizontal line, whenever $a > b$. Then a line is drawn between a and b if a covers b. This line represents the link between the two points. When b covers a, c covers b, d covers c, . . . , and n covers (n - 1), then the element a is said to be connected to n by an ascending chain of links, and n is connected to a by a descending chain of links. Since links are always drawn between the circles of two elements to indicate coverage and since one of the circles in a coverage relation must be higher than the other, no link is ever drawn horizontally. Two elements on a diagram are connected by a chain if and only if there is an inclusion relationship between them. A chain must be either ascending or descending; a combination of both rising and falling lines does not represent a chain.

Figure 2 shows a number of diagrams for partially ordered systems. In Figure 2(a), a is covered by b. In (b), d covers b and c; b and c cover a. In (b) there is an ascending chain from a to d through b, and another through c. In this same diagram, b does not cover c, nor does c cover b. The diagram of (c) could be obtained by rotating (b) 180°. Obviously, (c) is the dual of (b), since for (b) $a \leqq b$, $a \leqq c$, $b \leqq d$, $c \leqq d$, and for (c) $a \geqq b$, $a \geqq c$, $b \geqq d$, $c \geqq d$. One system can be obtained from the other by the transformation $t = \begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}$. The two are self-dual systems.

The system diagrammed in Figure 2(d) is not self-dual. Its dual would be obtained by rotating the diagram 180°, but since there is no transformation which will transform the diagram and its dual into each other, the system is not self-dual.

Figure 1, page 13, the diagram for the symmetries of the square, has the following coverages: c_4 covers c_2, c_4 covers c_3, c_2 covers c_1, and c_3 covers c_1.

Lower and Upper Bounds

By a lower bound of a subset X of a partially ordered system P is meant an element $a \in P$ satisfying the relation $a \leqq x$ for all $x \in X$. The greatest lower bound is a lower bound including all other lower bounds. The upper bound of a subset X of a partially ordered system P is an element $b \in P$ satisfying the relation $x \leqq b$ for all $x \in X$. The least upper bound is an upper bound included in all other upper

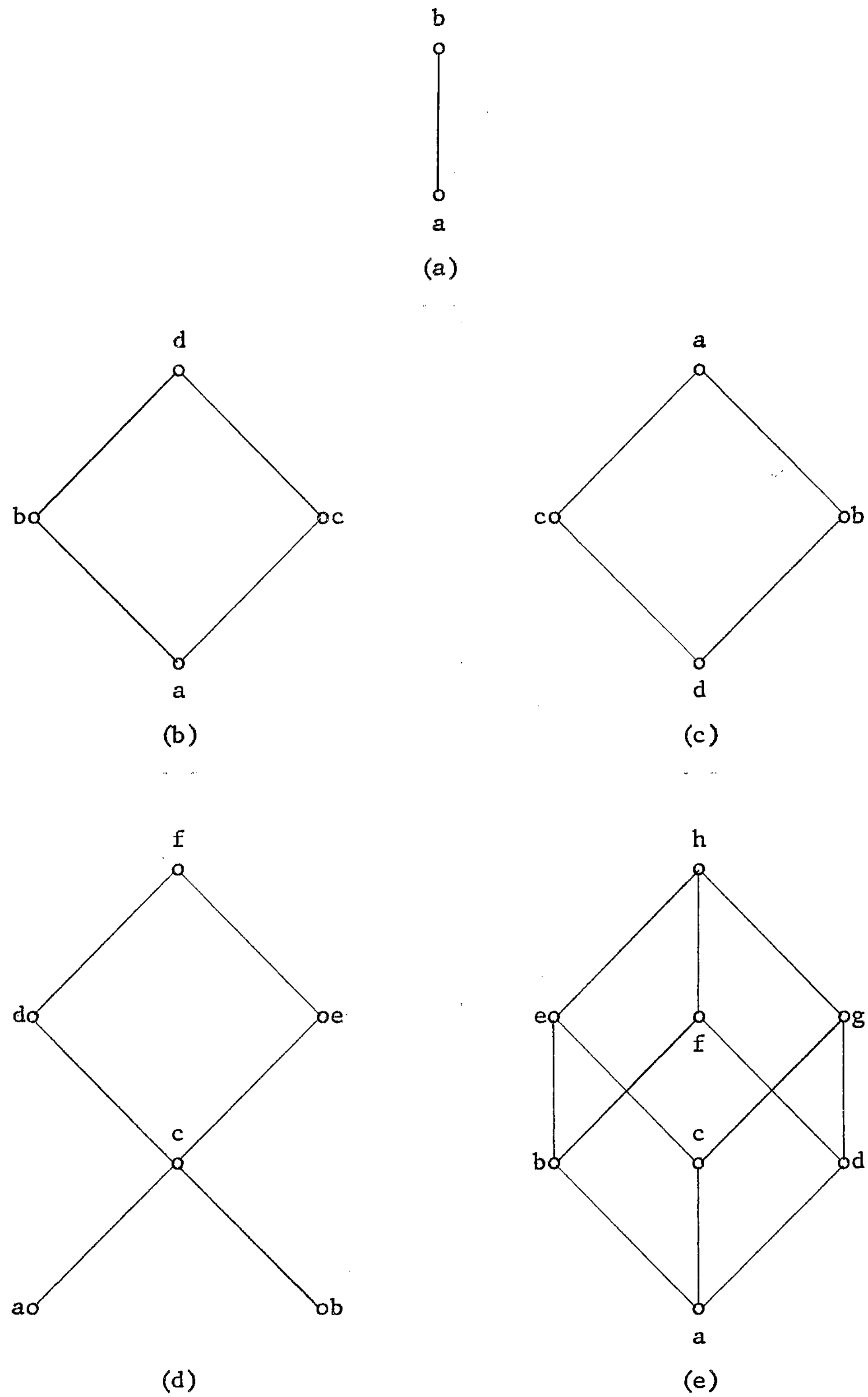

Fig. 2. Diagrams of Some Partially Ordered Systems.

bounds. To illustrate, Table III shows the subsets, lower bounds, and upper bounds of Figure 2(b).

TABLE III

SUBSETS AND LOWER AND UPPER BOUNDS OF THE PARTIALLY ORDERED SYSTEM SHOWN IN FIGURE 2 (b).

Subsets	Lower Bounds	Upper Bounds
{a}	a	a
{b}	b	b
{c}	c	c
{d}	d	d
{a,b}	a	b
{a,c}	a	c
{a,d}	a	d
{b,c}	a	d
{b,d}	b	d
{c,d}	c	d
{a,b,c}	a	d
{a,b,d}	a	d
{a,c,d}	a	d
{b,c,d}	a	d
{a,b,c,d}	a	d

Since, in each case shown in Table III, there is only one lower bound, it is the greatest lower bound. The upper bound is also the least upper bound. The fact that there is not always only one lower bound is shown in Table IV, which lists some of the subsets of Figure 2(e).

TABLE IV

SOME SUBSETS AND BOUNDS FOR THE SYSTEM SHOWN IN FIGURE 2(e).

Subsets	Lower Bounds	Greatest Lower Bounds	Upper Bounds	Least Upper Bounds
{a,b}	a	a	b,e,f,h	b
{b,c,d}	a	a	h	h
{e,g}	a,c	c	h	h
{b,d,f}	a	a	f,h	f
{c,e}	a,c	c	e,h	e

It will be observed that the system diagrammed in Figure 2(d) does not have a greatest lower bound. When a greatest lower bound exists, it is unique. If a least upper bound exists, it is unique.

Chains and Lengths of Chains

A partially ordered system P is called a simply ordered system, or chain, if and only if either $x \leqq y$ or $x \geqq y$ for all arbitrary elements x and y. The number of links in a chain is called the length of the chain. For example, any of the ascending chains from a to h of Figure 2(e) has three links. In this diagram there are six ascending chains from a to h. Designated by a succession of elements, they are (a,b,e,h), (a,b,f,h), (a,c,e,h), (a,c,g,h), (a,d,f,h) and (a,d,g,h). The Jordan-Dedekind chain condition[11] is that all chains between fixed points have the same length. Although this condition is satisfied for all diagrams of Figure 2, not all finite partially ordered systems satisfy it.

Lattices

A lattice is defined as a partially ordered system in which any two elements x and y have a greatest lower bound and a least upper bound. In any lattice, the greatest lower bound of elements a and b is denoted $a \cap b$, and the least upper bound is indicated by $a \cup b$. For lattices, $a \cap b$ is called the meet of a and b and coincides with the set-theoretical definition of the intersection $A \cap B$ of two sets, as stated on page 6. The set-theoretical union $A \cup B$ of two sets corresponds to the lattice-theoretical definition of $a \cup b$, called the join.

Two lattices L and L_d are dual if they are dual when considered

[11] Kiss, p. 78.

as partially ordered systems. Since every inclusion relation $x \leqq y$ in L becomes $x \geqq y$ in L_d, the $\cap$ operational table of L becomes the $\cup$ operational table of L_d, and the $\cup$ operational table of L becomes the $\cap$ operational table of L_d.

It can be shown[12,13] that all lattices obey the following operational laws.

Idempotent: $x \cap x = x$ and $x \cup x = x$

Commutative: $x \cap y = y \cap x$ and $x \cup y = y \cup x$

Associative: $(x \cap y) \cap z = x \cap (y \cap z)$ and $(x \cup y) \cup z = x \cup (y \cup z)$

Absorptive: $x \cap (x \cup y) = x$ and $x \cup (x \cap y) = x$

Self-Distributive: $x \cap (y \cap z) = (x \cap y) \cap (x \cap z)$ and

$x \cup (y \cup z) = (x \cup y) \cup (x \cup z)$

Semidistributive: $x \cap (y \cup z) \geqq (x \cap y) \cup (x \cap z)$ and

$x \cup (y \cap z) \leqq (x \cup y) \cap (x \cup z)$

4-Element Semidistributive: $(x \cup y) \cap (u \cup v) \geqq (x \cap u) \cup (y \cap v)$

and $(x \cap y) \cup (u \cap v) \leqq (x \cup u) \cap (y \cup v)$

Semimodular: If $z \leqq x$, then $x \cap (y \cup z) \geqq (x \cap y) \cup z$, and

if $z \geqq x$, then $x \cup (y \cap z) \leqq (x \cup y) \cap z$

A modular lattice is a lattice which satisfies the following modular laws.

If $z \leqq x$, then $x \cap (y \cup z) = (x \cap y) \cup z$, and

if $z \geqq x$, then $x \cup (y \cap z) = (x \cup y) \cap z$.

It will be noted that the modular laws are like the semimodular laws except for the presence of the = signs in the modular laws. A visual check to determine whether a finite lattice is modular

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[12]Birkhoff and Mac Lane, pp. 352-354.
[13]Kiss, pp. 81-83.

is to look at its diagram. If the lattice is modular, it satisfies the Jordan-Dedekind chain condition; that is, all chains connecting each pair of fixed points must have the same length. For example, Figure 3 shows the diagram of a non-modular lattice. The left chain from a to g contains four links; the right one, only three.

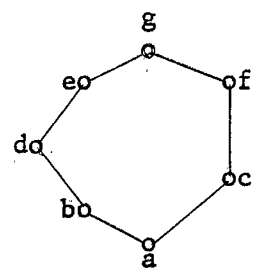

Fig. 3. Diagram of a Non-Modular Lattice.

Distributive Lattices

A distributive lattice is a lattice which satisfies the following dual distributive laws.

$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ and $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

A distributive lattice is always modular. A chain is a distributive lattice.

Complements

In set-theoretical considerations, the complement[14] $\bar{X}$ of X is the set of all elements not in X. $X \cap \bar{X} = 0$. $X \cup \bar{X} = U$, the universal set. In a lattice with a least element 0 and a greatest

[14]The symbols $\sim X$ and X' can be used for the complement of X.

element u, the complement $\bar{x}$ of an element x is an element such that $x \cap \bar{x} = 0$ and $x \cup \bar{x} = u$. The complement of $\bar{x}$ is x. $\bar{\bar{x}} = x$. This is known as the <u>involution law</u>. u is the complement of 0. In a distributive lattice, complements (when they exist) are unique and satisfy the involution law and the following two <u>dualization laws</u>.

$$\overline{(x \cap y)} = \bar{x} \cup \bar{y} \text{ and } \overline{(x \cup y)} = \bar{x} \cap \bar{y}$$

<u>Boolean Algebras</u>

A Boolean algebra is a distributive lattice which contains a least element 0 and a greatest element u, with $0 \leq x \leq u$ for all x and with a complement $\bar{x}$ for each element x.

<u>Isomorphisms and Automorphisms of Binary Operational Systems</u>

As indicated on page 10, an isomorphism exists between two binary operational systems $S_1(\circ, a_1, b_1, \ldots)$ and $S_2(\oplus, a_2, b_2, \ldots)$ if and only if there exists a one-to-one correspondence, $S_1 \leftrightarrow S_2$, between their elements such that $a_1 \circ b_1 = c_1$ implies $a_2 \oplus b_2 = c_2$. A one-to-one correspondence implies a one-to-one transformation t from S_1 to S_2. If $a_1 \circ b_1 = c_1$, then $(a_1 \circ b_1)t = c_1t$. That is, $a_1t \oplus b_1t = c_1t$, or $(a_1 \circ b_1)t = a_1t \oplus b_1t$. If t^{-1} denotes the inverse of t, the equation becomes $a_1 \circ b_1 = (a_1t \oplus b_1t)t^{-1}$.

Now assume that the elements of S_1 have the same symbols as those of S_2. That is, the two binary operational systems are $S_1(\circ, a, b, \ldots)$ and $S_2(\oplus, a, b, \ldots)$. This does not require that $a \leftrightarrow a$ and $b \leftrightarrow b$. The one-to-one correspondences may be $a \leftrightarrow p$, $b \leftrightarrow a$, $c \leftrightarrow h$, etc., where elements on the left are in S_1 and elements on the right are in S_2. Now $a \circ b = (at \oplus bt)t^{-1}$, where a and b are any elements of S_1 and are also elements of S_2. Substitution of ft^{-1} for a and gt^{-1} for b yields $ft^{-1} \circ gt^{-1}$. Multiplication of both sides of the equation by t results in $f \oplus g = (ft^{-1} \circ gt^{-1})t$.

It can be shown[15] that, given a set t_1, t_2, and t_3, . . . of one-to-one transformations on a binary operational system S with operational signs o_1, o_2, o_3, . . . of the transformed systems S_1, S_2, S_3, . . ., and given $t_1 t_2 = t_3$, the operational equation $a\ o_1\ b = c$ in S_1 is transformed into an operational equation in S_3 by transforming by t_2 all the terms, including o_1. $at_2\ o_3\ bt_2 = ct_2$. An automorphism is defined[16] as an isomorphism of a group with itself.

Link-Preserving Transformations of Lattices

A link-preserving transformation on a lattice L is a one-to-one transformation t on L such that if any two elements a and b are linked in L, then their corresponding elements at and bt are linked in the transformed lattice L_t. The symmetries of the square are examples of link-preserving transformations. For the lattice of Figure 1, page 13, the link-preserving transformations are the same as the symmetries of the square.

$$I = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \quad H = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_1 & c_4 & c_3 \end{pmatrix} \quad V = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_4 & c_1 & c_2 \end{pmatrix}$$

$$R' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix} \quad D = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_3 & c_2 & c_4 \end{pmatrix} \quad R = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_4 & c_1 & c_3 \end{pmatrix}$$

$$R'' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_1 & c_4 & c_2 \end{pmatrix} \quad D' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_2 & c_3 & c_1 \end{pmatrix}$$

A lattice L is transformed into a dual lattice L_d by a duality transformation t_d. A duality transformation t_d changes every relation $a \leq b$ of L into a relation $a \geq b$ in L_d, transforming an ascending chain into a descending chain, and vice versa. The duality transformations for the lattice of Figure 1

[15]Kiss, p. 94.
[16]Birkhoff and Mac Lane, p. 147.

are R' and D'.

Direct Unions of Lattices

The direct union $A \times B$ of two partially ordered systems A and B is the system C whose elements are the ordered couples (a,b), where a is any element of A and b is any element of B. An equality and four ordering relations are defined as follows.

$(a_1,b_1) = (a_2,b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

$(a_1,b_1) \leqq (a_2,b_2)$ if and only if $a_1 \leqq a_2$ and $b_1 \leqq b_2$.

$(a_1,b_1) \prec (a_2,b_2)$ if and only if $a_1 \leqq a_2$ and $b_1 \geqq b_2$.

$(a_1,b_1) \succ (a_2,b_2)$ if and only if $a_1 \geqq a_2$ and $b_1 \leqq b_2$.

$(a_1,b_1) \geqq (a_2,b_2)$ if and only if $a_1 \geqq a_2$ and $b_1 \geqq b_2$.

For each of the four relations ρ in the direct union of two partially ordered systems there is a dual relation ρ_d such that if $c_1 \ \rho \ c_2$, then $c_2 \ \rho_d \ c_1$. Thus, $\leqq$ and $\geqq$ are dual relations, and $\prec$ and $\succ$ are dual relations. The elements $c_1, c_2, \ldots$ of the direct union $c = A \times B$ of two partially ordered systems A and B obey the three laws for the inclusion relation stated on page 6.

The direct union $C = A \times B$ of two lattices A and B is a lattice with four single-valued binary relations, $\leqq$, $\prec$, $\succ$, and $\geqq$. An operation is carried out component by component, in the $\cap$ system of the components if the $\leqq$ relation is involved, in the $\cup$ system if $\geqq$ is involved. There is a one-to-one correspondence between the relations and the operations. The four operations are designated in Table V.

For example, if $a_1 \leqq a_2$ in lattice A and $b_1 \leqq b_2$ in lattice B, the operational tables for A and B are shown in Table VI. The elements of C are defined as ordered couples of the elements of A and B. $c_1 = (a_1,b_1)$, $c_2 = (a_1,b_2)$, $c_3 = (a_2,b_1)$, $c_4 = (a_2,b_2)$. The table for the $\circ$ operation of C shown in Table VII can then be determined as indicated by the following calculations.

TABLE V

RELATIONS AND OPERATIONS FOR THE DIRECT UNION OF TWO LATTICES.

Relation in C		Relations between elements of A and between elements of B		Corresponding operation in C		Corresponding operations between elements of A and between elements of B
$\leqq$	=	$(\leqq,\leqq)$	$\leftrightarrow$	$\circ$	=	$(\cap,\cap)$
$\prec$	=	$(\leqq,\geqq)$	$\leftrightarrow$	$\perp$	=	$(\cap,\cup)$
$\succ$	=	$(\geqq,\leqq)$	$\leftrightarrow$	$\top$	=	$(\cup,\cap)$
$\geqq$	=	$(\geqq,\geqq)$	$\leftrightarrow$	$+$	=	$(\cup,\cup)$

TABLE VI

OPERATIONAL TABLES FOR LATTICES A AND B.

A:

$\cap$	a_1	a_2
a_1	a_1	a_1
a_2	a_1	a_2

$\cup$	a_1	a_2
a_1	a_1	a_2
a_2	a_2	a_2

(a)

B:

$\cap$	b_1	b_2
b_1	b_1	b_1
b_2	b_1	b_2

$\cup$	b_1	b_2
b_1	b_1	b_2
b_2	b_2	b_2

(b)

TABLE VII

THE $\circ$ OPERATIONAL TABLE FOR $C = A \times B$.

$\circ$	c_1	c_2	c_3	c_4
c_1	c_1	c_1	c_1	c_1
c_2	c_1	c_2	c_1	c_2
c_3	c_1	c_1	c_3	c_3
c_4	c_1	c_2	c_3	c_4

$c_1 \circ c_1 = (a_1,b_1) \circ (a_1,b_1) = (a_1 \cap a_1,\ b_1 \cap b_1) = (a_1,b_1) = c_1$

$c_1 \circ c_2 = (a_1,b_1) \circ (a_1\ b_2) = (a_1 \cap a_1,\ b_1 \cap b_2) = (a_1,b_1) = c_1$

$c_1 \circ c_3 = (a_1,b_1) \circ (a_2,b_1) = (a_1 \cap a_2,\ b_1 \cap b_1) = (a_1,b_1) = c_1$

$$c_1 \circ c_4 = (a_1,b_1) \circ (a_2,b_2) = (a_1 \cap a_2,\ b_1 \cap b_2) = (a_1,b_1) = c_1$$

$$c_2 \circ c_4 = (a_1,b_2) \circ (a_1,b_2) = (a_1 \cap a_1,\ b_2 \cap b_2) = (a_1,b_2) = c_2$$

$$c_2 \circ c_3 = (a_1,b_2) \circ (a_2,b_1) = (a_1 \cap a_2,\ b_2 \cap b_1) = (a_1,b_1) = c_1$$

$$c_2 \circ c_4 = (a_1,b_2) \circ (a_2,b_2) = (a_1 \cap a_2,\ b_2 \cap b_2) = (a_1,b_2) = c_2$$

$$c_3 \circ c_3 = (a_2,b_1) \circ (a_2,b_1) = (a_2 \cap a_2,\ b_1 \cap b_1) = (a_2,b_1) = c_3$$

$$c_3 \circ c_4 = (a_2,b_1) \circ (a_2,b_2) = (a_2 \cap a_2,\ b_1 \cap b_2) = (a_2,b_1) = c_3$$

$$c_4 \circ c_4 = (a_2,b_2) \circ (a_2,b_2) = (a_2 \cap a_2,\ b_2 \cap b_2) = (a_2,b_1) = c_4$$

It can be shown[17] that all four of the operational systems of the union of two lattices are idempotent, associative, and commutative. Since they are commutative, $c_4 \circ c_1 = c_1 \circ c_4$, for example, so $c_4 \circ c_1$ need not be calculated if $c_1 \circ c_4$ is already known.

The entries for the operational tables for the three other operations of $C = A \times B$ can be calculated in a similar manner. They are shown in Table VIII.

TABLE VIII

THE ⊥, T, AND + OPERATIONAL TABLES FOR $C = A \times B$.

(a)

⊥	c_1	c_2	c_3	c_4
c_1	c_1	c_2	c_1	c_2
c_2	c_2	c_2	c_2	c_2
c_3	c_1	c_2	c_3	c_4
c_4	c_2	c_2	c_4	c_4

(b)

T	c_1	c_2	c_3	c_4
c_1	c_1	c_1	c_3	c_3
c_2	c_1	c_2	c_3	c_4
c_3	c_3	c_3	c_3	c_3
c_4	c_3	c_4	c_3	c_4

(c)

+	c_1	c_2	c_3	c_4
c_1	c_1	c_2	c_3	c_4
c_2	c_2	c_2	c_4	c_4
c_3	c_3	c_4	c_3	c_4
c_4	c_4	c_4	c_4	c_4

Diagrams for A and B are shown in Figure 4(a) and (b), respectively; relations for C are indicated in (c) through (f).

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[17] Kiss, p. 104.

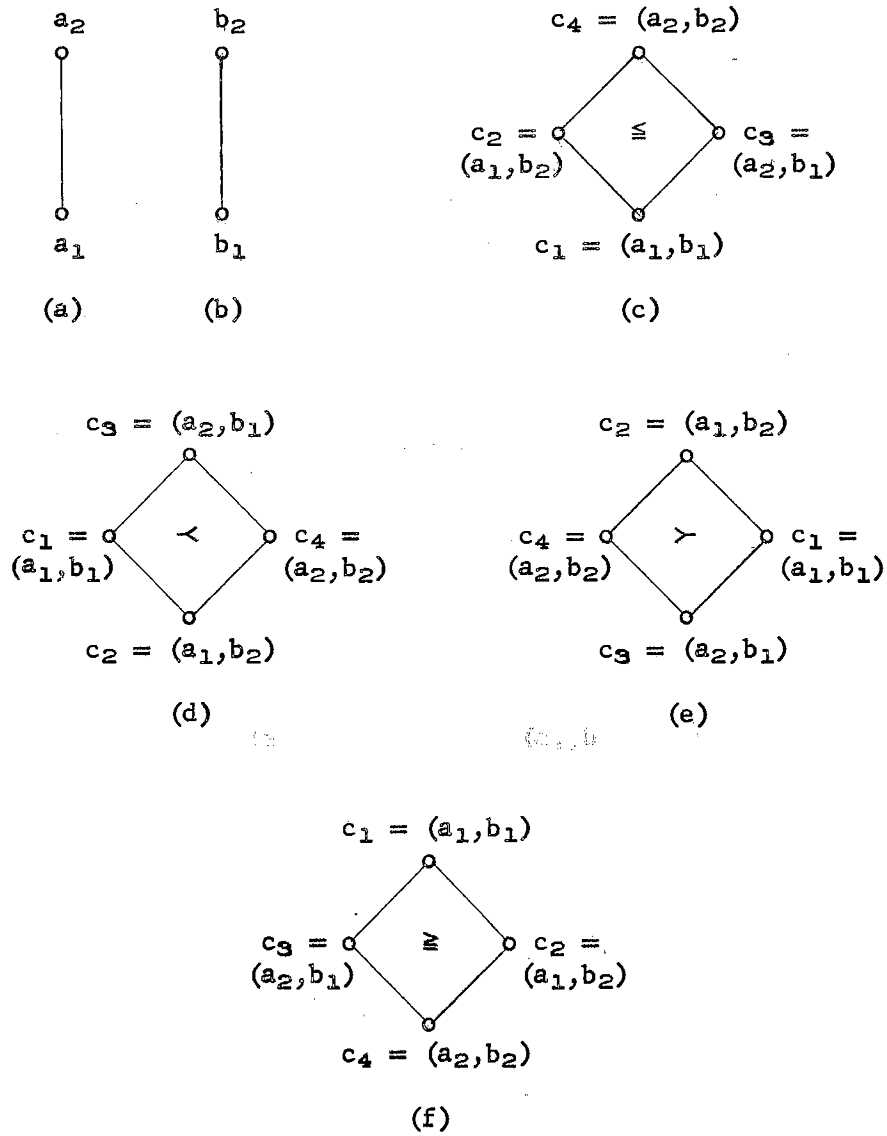

Fig. 4. Lattice Diagrams for A, B, and $C = A \times B$.

The four ordering relations of C are all shown in Figure 4 (c), provided different directions are assigned for ascending chains. For the $\leq$ relation, the ascending direction is, as usual, from bottom to top. For $\prec$, it is from left to right, but right to left for $\succ$. For $\geq$, it is from top to bottom.

Obviously, the four distinct relations for C are transformations of each other. All are link-preserving transformations. The direct union of two transformations t_1 and t_2 on the elements of two systems A and B is the couple (t_1,t_2) having for components a transformation on each of the two systems. The couple operates on $C = A \times B$. For the systems of A and B, let $t_1 = I = \begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix}$ or $\begin{pmatrix} b_1 & b_2 \\ b_1 & b_2 \end{pmatrix}$ and let $t_2 = R' = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$ or $\begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix}$. Then the following transformations apply to C.

$$(t_1,t_1) = \left(\begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_1 & b_2 \end{pmatrix} \right) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} = I$$

$$(t_1,t_2) = \left(\begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix} \right) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_1 & c_4 & c_3 \end{pmatrix} = H$$

$$(t_2,t_1) = \left(\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_1 & b_2 \end{pmatrix} \right) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_4 & c_1 & c_2 \end{pmatrix} = V$$

$$(t_2,t_2) = \left(\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix} \right) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix} = R'$$

The direct union (t_1,t_2) of transformations on two partially ordered systems A and B is called a <u>principal transformation</u> on $C = A \times B$ if and only if each of the transformations t_1 and t_2 is either the identity transformation or a duality transformation. The transformations above are principal transformations, since R' is a duality transformation. As indicated on page 14, four other transformations are possible: D, R, R'', and D'. These can not be obtained as the couple of two identity or dual transformations on A and B.

CHAPTER III

THE BOOLEAN ALGEBRA OF ORDER TWO

The most common Boolean algebra is that of order two. This algebra contains only two elements and is, therefore, a natural development for the algebra of logic in which a statement is either true or false; no other values seem possible. Mathematically, however, more elaborate systems can be developed to include four, eight, or more elements.

Basic Theory of Two-Valued Boolean Algebra

This algebra is the set of two elements. Since the null set 0 is considered to be a subset of every set, let one of the elements be designated by 0 and the other by 1. Thus, the Boolean algebra of order two, $B = \{0,1\}$, is a completely ordered system. Since 0 is a subset of 1 but is not equal to 1, it is a proper subset. Thus, $0 < 1$ and $1 > 0$, but the more general $\leq$ and $\geq$ will be used.

Two binary operations $\cap$ and $\cup$ can be applied to the two elements of B, assigning to each pair (0,0), (0,1), (1,0), and (1,1) a unique element of B. The element chosen for each pair might appear to be quite arbitrary, but the choices are determined by the characteristics of the system.

B is a partially ordered system since it satisfies the reflexive, antisymmetric, and transitive laws. It contains a least element 0 and a greatest element 1. 1 covers 0. 0 is the greatest lower bound, and 1 is the least upper bound of the set. B thus satisfies the conditions given on page 23 for a lattice. Moreover, B is a chain, as defined on page 23. It is a distributive lattice, as explained on page 25. Since the meet $x \cap y$ of two elements of a lattice is their greatest lower bound, $0 \cap 1 = 1 \cap 0 = 0$. Since the join $x \cup y$ of two elements of a lattice is their least upper bound, $0 \cup 1 = 1 \cup 0 = 1$. The greatest lower bound of 0 and 0 is 0; therefore, $0 \cap 0 = 0$. Similarly, $1 \cap 1 = 1$. The least upper bound of 0 and 0 is 0; of 1 and 1, 1. Therefore, $0 \cup 0 = 0$, and $1 \cup 1 = 1$. These values are indicated in Table IX.

TABLE IX

THE $\cap$ AND $\cup$ OPERATIONAL TABLES FOR THE BOOLEAN ALGEBRA OF ORDER TWO.

$\cap$	0	1
0	0	0
1	0	1

(a)

$\cup$	0	1
0	0	1
1	1	1

(b)

The diagram for the lattice of B is given in Figure 5 (a). Since there is only one chain joining the two elements, the lattice is modular. By definition of the complement on page 26, 0 is the complement of 1, and 1 is the complement of 0.

(a) (b)

Fig. 5. Diagrams of Two-Element Lattices.

The two link-preserving transformations are $I = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $R' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. I is the identity transformation. R' transforms the lattice of B, shown in Figure 5 (a), into the lattice shown in Figure 5 (b). The R' transformation is the negative transformation of logic. Thus, $R'x = -x$ for each $x \in B$. The inverse t^{-1} of a transformation t satisfies the equation $tt^{-1} = t_1$, the identity transformation. Thus $R'^{-1} = R'$. Since $R'R' = I$, $-(-x) = x$. The negative of an element is its complement. $-x = \bar{x}$.

In logic, the intersection or meet $\cap$ is called a conjunction and is frequently represented by the ampersand[1] & or by the multiplication sign[2] $\circ$, which is frequently omitted between the two elements it connects. $x \circ y = xy = x \cap y = x \,\&\, y =$ "x and y" . The union or join $\cup$ is called a disjunction and is alternately represented[1,2] by $\vee$ or by the + sign[3]. $x + y = x \cup y = x \vee y =$ "x or y or both". In most electrical engineering literature the $\circ$ and + signs are preferred.

<u>Other Binary Operations</u>

Special symbols have been used[4] for various combinations which can be expressed with the negation and the + and $\circ$ signs. The "zero conditional" $x \underset{0}{\leftarrow} y$ is used for $\bar{x} \circ y$. The dual "one conditional" $x \underset{1}{\rightarrow} y = \bar{x} + y$. The "reversed zero conditional" $x \underset{0}{\leftarrow} y$ $x \circ \bar{y}$, and the "reversed one conditional" $x \underset{1}{\leftarrow} y$ equals $x + \bar{y}$.

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[1]D. Hilbert and W. Ackermann, <u>Grundzüge der theoretischen Logic</u> (2d ed., Berlin, 1938).

[2]A. N. Whitehead and B. Russell, <u>Principia Mathematica</u> (2d ed., vols. I, II, III, Cambridge, 1925-27).

[3]C. I. Lewis and C. H. Langford, <u>Symbolic Logic</u> (New York, 1932).

[4]Stephen A. Kiss, <u>Transformations on Lattices and Structures of Logic</u> (New York, 1947), pp. 124-125.

$x \underset{0}{\leftarrow} y = x \circ \bar{y} = \bar{\bar{x}} \circ \bar{y} = \bar{x} \underset{0}{\rightarrow} \bar{y}$. $x \underset{1}{\leftarrow} y = x + \bar{y} = \bar{\bar{x}} + y = \bar{x} \underset{1}{\rightarrow} y$.
The "zero biconditional" $x \underset{0}{\leftrightarrow} y$, also written as $x \underline{\times} y$, can be expressed as $(x \circ \bar{y}) + (\bar{x} \circ y)$. Its dual, the "one biconditional" $x \underset{1}{\leftrightarrow} y = x \bar{\times} y = (x + \bar{y}) \circ (\bar{x} + y)$. The "stroke systems" of Sheffer[5] are the negations of the $\circ$ and $+$ systems. The "one stroke", $x \mid y = -(x \circ y) = (\overline{x \circ y})$, means "not both x and y", a proposition which is false if and only if both x and y are true. The "two stroke", $x \parallel y = -(x + y) = (\overline{x + y})$, means "neither x nor y", a proposition which is true if and only if both x and y are false. Table X lists the operational tables for the various binary relations.

Dual and Negatives

A distinction should be made between the dual of an expression and the negative (or complement) of an expression. Following the discussion on pages 26 and 27, the two binary systems for B, the Boolean algebra of order two, are $B_1(\circ,0,1)$ and $B_2(+,0,1)$. A one-to-one correspondence between the systems implies a one-to-one transformation t from B_1 to B_2. If $x \circ y = z$, then $(x \circ y)t = zt$. That is, $xt + yt = zt$, or $(x \circ y)t = xt + yt$. Similarly, $(x + y)t^{-1} = xt^{-1} \circ yt^{-1}$.

If $t = R' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $t^{-1} = R'$. $0R' = \bar{0} = 1$. $1R' = \bar{1} = 0$. $(x \circ y)R' = xR' + yR'$. $(\overline{x \circ y}) = \bar{x} + \bar{y} = x \mid y$. Thus the negative of $(x \circ y)$ is $(x \mid y)$. As indicated in Table X, each value of the function $x \mid y$ is the negative of the corresponding value for $x \circ y$. If, however, every element of the operational table

[5]H. M. Sheffer, "A Set of Five Independent Postulates for Boolean Algebras," Transactions of the American Mathematical Society, XIV (1913), 481-488.

TABLE X

OPERATIONAL TABLES FOR VARIOUS BINARY RELATIONS OF BOOLEAN ALGEBRA OF ORDER TWO.

$x \circ y$

$\circ$	0	1
0	0	0
1	0	1

(a)

$x + y$

$+$	0	1
0	0	1
1	1	1

(b)

$x \underset{0}{\rightarrow} y = \bar{x} \circ y$

$\underset{0}{\rightarrow}$	0	1
0	0	1
1	0	0

(c)

$x \underset{1}{\rightarrow} y = \bar{x} + y$

$\underset{1}{\rightarrow}$	0	1
0	1	1
1	0	1

(d)

$x \underset{0}{\leftarrow} y = x \circ \bar{y}$

$\underset{0}{\leftarrow}$	0	1
0	0	0
1	1	0

(e)

$x \underset{1}{\leftarrow} y = x + \bar{y}$

$\underset{1}{\leftarrow}$	0	1
0	1	0
1	1	1

(f)

$x \times y = (x \circ \bar{y}) + (\bar{x} \circ y)$

$\times$	0	1
0	0	1
1	1	0

(g)

$x \bar{\times} y = (x + \bar{y}) \circ (\bar{x} + y)$

$\bar{\times}$	0	1
0	1	0
1	0	1

(h)

$x \mid y = \overline{(x \circ y)} = \bar{x} + \bar{y}$

$\mid$	0	1
0	1	1
1	1	0

(i)

$x \parallel y = (x + y) = \bar{x} \circ \bar{y}$

$\parallel$	0	1
0	1	0
1	0	0

(j)

for $x \circ y$ were changed, including column and row headings as well as the value of the function, Table XI(a) would result. If this table is rearranged to put column and row headings in their customary locations, the resultant Table XI(b) is seen to be the table for the + relation.

TABLE XI

RESULT OF CHANGING EVERY ELEMENT OF TABLE X(a), INCLUDING COLUMN AND ROW HEADINGS.

	1	0
1	1	1
0	1	0

(a)

	0	1
0	0	1
1	1	1

(b)

The expression $x + y$ is called the dual of $x \circ y$. Note that, in the dual expression, any literal element is not replaced by its negative, but a 0 is replaced by 1 and vice versa. $x \circ 1 = x$ would have as its dual $x + 0 = x$. $0 \circ y = 0$, and $1 + y = 1$. If 0 and 1 were not interchanged in going from an expression to its dual, incorrect equations would result.

The rule, then, is that to write the dual of a Boolean algebra expression, + and $\circ$ must be interchanged and 0 and 1 must be interchanged in the original expression. To write the negative, or inverse, of an expression, take its dual and then change each variable to its inverse. Table XII shows dual and negative relations.

Postulates and Theorems for the + and $\circ$ Operations.

Table XIII lists postulates and definitions for two-valued Boolean algebra, using only the + and $\circ$ binary operations. Once

TABLE XII

DUAL AND NEGATIVE EXPRESSIONS.

Expression	Dual	Negative
$x \circ y$	$x + y$	$x \mid y$
$x \underset{0}{\rightarrow} y$	$x \underset{1}{\rightarrow} y$	$x \underset{1}{\leftarrow} y$
$x \underset{0}{\leftarrow} y$	$x \underset{1}{\leftarrow} y$	$x \underset{1}{\rightarrow} y$
$x \times y$	$x \bar{\times} y$	$x \bar{\times} y$
$x \times y$	$x \times y$	$x \times y$
$x \mid y$	$x \parallel y$	$x \circ y$
$x \underset{1}{\rightarrow} y$	$x \underset{0}{\rightarrow} y$	$x \underset{0}{\leftarrow} y$
$x \underset{1}{\leftarrow} y$	$x \underset{0}{\leftarrow} y$	$x \underset{0}{\rightarrow} y$
$x \bar{\times} y$	$x \times y$	$x \times y$
$x \parallel y$	$x \mid y$	$x + y$

TABLE XIII

POSTULATES AND DEFINITIONS FOR TWO-VALUED BOOLEAN ALGEBRA

[P1] $x = 0$ or $x = 1$ for all $x \in B$

[P2a] $\bar{x} = 1$ if $x = 0$	[P2b] $\bar{x} = 0$ if $x = 1$
[P3a] $x + 0 = 0 + x = x$	[P3b] $x \circ 1 = 1 \circ x = x$
[P4a] $x + 1 = 1 + x = 1$	[P4b] $x \circ 0 = 0 \circ x = 0$

these postulates and definitions are established, the theorems of Table XIV can be proved. It will be noticed that most of the listings in Tables XIII and XIV are in pairs. As indicated on page 19, any theorem that is true in a partially ordered system remains true if the symbols ≦ and ≧ are interchanged throughout the statement of the theorem. That is, if a particular equality is proved to be true, its dual expression is also true.

A method of proof expecially useful for two-valued Boolean algebra is proof by perfect induction[6]. The procedure is to test the theorem by means of the postulates for all possible values of the variables. This is not difficult, since each variable has only two possible values. Even if n variables are involved, only 2^n possible combinations need to be tested. T7a, for example, can be proved by first letting $x = 1$. Then, by P2b, $\bar{x} = 0$. Substitution of these values for x and $\bar{x}$ gives the expression $1 + 0$, which, by either P3a or P4a, must equal 1. Next, let $x = 0$. By P2a, $\bar{x} = 1$. The resultant expression $0 + 1$ must equal 1, again by P3a or P4a. Since the theorem has been shown to be true for all possible values of x, the theorem must always be true.

Theorem T7a could have been proved by reference to the theory on page 34 where the $\cup$ operational table, Table IX(b), was developed, since the $\cup$ and + symbols are used interchangeably for the union or join.

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[6]William Keister, A. E. Ritchie, and S. H. Washburn, The Design of Switching Circuits (New York, 1951), p. 72.

TABLE XIV

THEOREMS FOR TWO-VALUED BOOLEAN ALGEBRA

[T1] $(\bar{\bar{x}}) = x$

[T2a] $0 + 0 = 0$ [T2b] $1 \circ 1 = 1$

[T3a] $1 + 0 = 1$ [T3b] $0 \circ 1 = 0$

[T4a] $1 + 1 = 1$ [T4b] $0 \circ 0 = 0$

[T5a] $0 + 1 = 1$ [T5b] $1 \circ 0 = 0$

[T6a] $x + x = x$ [T6b] $x \circ x = x$

[T7a] $x + \bar{x} = 1$ [T7b] $x \circ \bar{x} = 0$

[T8a] $x + y = y + x$ [T8b] $x \circ y = y \circ x$

[T9a] $(x + y) + z = x + (y + z)$ [T9b] $(x \circ y) \circ z = x \circ (y \circ z)$

[T10a] $(x \circ y) + z = (x + z) \circ (y + z)$

[T10b] $(x + y) \circ z = (x \circ z) + (y \circ z)$

[T11a] $\overline{(x + y)} = \bar{x} \circ \bar{y}$ [T11b] $\overline{(x \circ y)} = \bar{x} + \bar{y}$

[T12a] $\overline{(x + y + z)} = \bar{x} \circ \bar{y} \circ \bar{z}$ [T12b] $\overline{(x \circ y \circ z)} = \bar{x} + \bar{y} + \bar{z}$

[T13a] $x + (x \circ y) = x$ [T13b] $x \circ (x + y) = x$

[T14a] $(x \circ \bar{y}) + y = x + y$ [T14b] $(x + \bar{y}) \circ y = x \circ y$

[T15a] $(x \circ y) + (\bar{x} \circ z) = (x + z) \circ (\bar{x} + y)$

[T15b] $(x + y) \circ (\bar{x} + z) = (x \circ z) + (\bar{x} \circ y)$

[T16a] $(x \circ y) + (y \circ z) + (z \circ \bar{x}) = (x \circ y) + (z \circ \bar{x})$

[T16b] $(x + y) \circ (y + z) \circ (z + \bar{x}) = (x + y) \circ (z + \bar{x})$

[T17a] $(\bar{x} \circ y) + (\bar{y} \circ z) + (\bar{z} \circ x) = (x + y + z) \circ (\bar{x} + \bar{y} + \bar{z})$

[T17b] $(\bar{x} + y) \circ (\bar{y} + z) \circ (\bar{z} + x) = (x \circ y \circ z) + (\bar{x} \circ \bar{y} \circ \bar{z})$

[T18a] $(x \circ \bar{y}) + (y \circ \bar{z}) + (z \circ \bar{x}) = (x + y + z) \circ (\bar{x} + \bar{y} + \bar{z})$

[T18b] $(x + \bar{y}) \circ (y + \bar{z}) \circ (z + \bar{x}) = (x \circ y \circ z) + (\bar{x} \circ \bar{y} \circ \bar{z})$

Previously proved theorems may, of course, be used in proof of other theorems. Thus, if one were to prove theorem T13b by perfect induction, as indicated in Table XV, he might use T2 through T5 in arriving at values for the fourth, fifth, and sixth columns, instead of going back to basic postulates.

TABLE XV

PROOF BY PERFECT INDUCTION THAT $(x + \bar{y}) \circ y = x \circ y$.

x	y	y	$x + y$	$(x + y) \circ y$	$x \circ y$
0	0	1	1	0	0
0	1	0	0	0	0
1	0	1	1	0	0
1	1	0	1	1	1

<u>Postulates and Theorems for Other Binary Operations</u>.

As indicated on pages 35 and 36, the other binary operations for two-valued Boolean algebra can be expressed in terms of the + and $\circ$ operations and negatives. These relations are shown in Table XVI. With the equations of Tables XIII, XIV, and XVI, it is possible to prove a great many theorems, some of which are shown in Table XVII.

TABLE XVI

POSTULATES FOR OTHER BINARY OPERATIONS FOR TWO-VALUED BOOLEAN ALGEBRA.

[P5a]	$x \underset{0}{\rightarrow} y = \bar{x} \cdot y$	[P5b]	$\bar{x} \underset{1}{\rightarrow} y = x + y$
[P6a]	$x \underset{0}{\leftarrow} y = x \circ \bar{y}$	[P6b]	$x \underset{1}{\leftarrow} y = x + \bar{y}$
[P7a]	$x \times y = (x \circ \bar{y}) + (\bar{x} \circ y)$	[P7b]	$x \bar{\times} y = (x + \bar{y}) \cdot (\bar{x} + y)$
[P8a]	$x \mid y = (\overline{x \circ y})$	[P8b]	$x \parallel y = (x + y)$

If one has a good working knowledge of the + and $\circ$ operations and the definitions of the other operations in terms of + and $\circ$, he will probably prefer to prove the theorems of Table XVII in terms of these two operations. For example, T35a states $(x \underset{o}{\rightarrow} y) \underset{o}{\rightarrow} (x \underset{o}{\rightarrow} z) = x \underset{o}{\rightarrow} (y \underset{o}{\rightarrow} z)$. Formal proof of this theorem could proceed as indicated in Table XVIII. Proof by perfect induction is shown in Table XIX, using values from the operational Table X(c), which are the values given in the (a) parts of T20 through T23.

It is not necessary to prove all theorems in terms of the + and $\circ$ operations. Table XX shows a proof of T41a using previous theorems dealing solely with the $\underset{o}{\rightarrow}$ operation.

TABLE XVII

THEOREMS FOR OTHER BINARY OPERATIONS OF TWO-VALUED BOOLEAN ALGEBRA.

[T19a]	$\overline{(x \underset{0}{\rightarrow} y)} = x \leftarrow y$	[T19b]	$\overline{(x \underset{1}{\rightarrow} y)} = x \underset{0}{\leftarrow} y$
[T20a]	$0 \underset{0}{\rightarrow} 0 = 0$	[T20b]	$1 \underset{1}{\rightarrow} 1 = 1$
[T21a]	$0 \underset{0}{\rightarrow} 1 = 1$	[T21b]	$1 \underset{1}{\rightarrow} 0 = 0$
[T22a]	$1 \underset{0}{\rightarrow} 0 = 0$	[T22b]	$0 \underset{1}{\rightarrow} 1 = 1$
[T23a]	$1 \underset{0}{\rightarrow} 1 = 0$	[T23b]	$0 \underset{1}{\rightarrow} 0 = 1$
[T24a]	$x \underset{0}{\rightarrow} 0 = 0$	[T24b]	$x \underset{1}{\rightarrow} 1 = 1$
[T25a]	$x \underset{0}{\rightarrow} 1 = \bar{x}$	[T25b]	$x \underset{1}{\rightarrow} 0 = \bar{x}$
[T26a]	$x \underset{0}{\rightarrow} x = 0$	[T26b]	$x \underset{1}{\rightarrow} x = 1$
[T27a]	$x \underset{0}{\rightarrow} \bar{x} = \bar{x}$	[T27b]	$x \underset{1}{\rightarrow} \bar{x} = \bar{x}$
[T28a]	$\bar{x} \underset{0}{\rightarrow} x = x$	[T28b]	$\bar{x} \underset{1}{\rightarrow} x = x$
[T29a]	$x \underset{0}{\rightarrow} y = y \underset{0}{\leftarrow} x$	[T29b]	$x \underset{1}{\rightarrow} y = y \underset{1}{\leftarrow} x$
[T30a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\rightarrow} z = (x + \bar{y}) \circ z$	[T30b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\rightarrow} z = (x \circ \bar{y}) + z$
[T31a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\rightarrow} z = (x \circ z) + (\bar{y} \circ z)$		
[T31b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\rightarrow} z = (x + z) \circ (\bar{y} + z)$		
[T32a]	$x \underset{0}{\rightarrow} (y \underset{0}{\rightarrow} z) = \bar{x} \circ \bar{y} \circ z$	[T32b]	$x \underset{1}{\rightarrow} (y \underset{1}{\rightarrow} z) = \bar{x} + \bar{y} + z$
[T33a]	$x \underset{0}{\rightarrow} (y \underset{0}{\rightarrow} z) = y \underset{0}{\rightarrow} (x \underset{0}{\rightarrow} z)$	[T33b]	$x \underset{1}{\rightarrow} (y \underset{1}{\rightarrow} z) = y \underset{1}{\rightarrow} (x \underset{1}{\rightarrow} z)$
[T34a]	$x \underset{0}{\rightarrow} (y \underset{0}{\rightarrow} z) = (x + y) \underset{0}{\rightarrow} z$	[T34b]	$x \underset{1}{\rightarrow} (y \underset{1}{\rightarrow} z) = (x \circ y) \underset{1}{\rightarrow} z$
[T35a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\rightarrow} (x \underset{0}{\rightarrow} z) = x \underset{0}{\rightarrow} (y \underset{0}{\rightarrow} z)$		
[T35b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\rightarrow} (x \underset{1}{\rightarrow} z) = x \underset{1}{\rightarrow} (y \underset{1}{\rightarrow} z)$		
[T36a]	$\bar{x} \underset{0}{\rightarrow} \bar{y} = x \underset{0}{\leftarrow} y$	[T36b]	$\bar{x} \underset{1}{\rightarrow} \bar{y} = x \underset{1}{\leftarrow} y$
[T37a]	$(x \underset{0}{\rightarrow} \bar{y}) \underset{0}{\rightarrow} \bar{z} = (x \parallel y) \parallel z$	[T37b]	$(x \underset{1}{\rightarrow} \bar{y}) \underset{1}{\rightarrow} \bar{z} = (x \mid y) \mid z$
[T38a]	$x \underset{0}{\rightarrow} (y \underset{0}{\rightarrow} \bar{z}) = x \underset{0}{\rightarrow} (\bar{y} \underset{0}{\leftarrow} z)$	[T38b]	$x \underset{1}{\rightarrow} (y \underset{1}{\rightarrow} \bar{z}) = x \underset{1}{\rightarrow} (\bar{y} \underset{1}{\leftarrow} z)$
[T39a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\rightarrow} x = x$	[T39b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\rightarrow} x = x$
[T40a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\rightarrow} y = x \circ y$	[T40b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\rightarrow} y = x + y$
[T41a]	$x \underset{0}{\rightarrow} (y \underset{0}{\rightarrow} x) = 0$	[T41b]	$x \underset{1}{\rightarrow} (y \underset{1}{\rightarrow} x) = 1$

TABLE XVII (Continued).

[T42a]	$x \underset{0}{\rightarrow} (x \underset{0}{\rightarrow} y) = \bar{x} \circ y$	[T42b]	$x \underset{1}{\rightarrow} (x \underset{1}{\rightarrow} y) = \bar{x} + y$
[T43a]	$x \underset{0}{\rightarrow} (y \underset{0}{\leftarrow} z) = \bar{x} \circ y \circ \bar{z}$	[T43b]	$x \underset{1}{\rightarrow} (y \underset{1}{\leftarrow} z) = \bar{x} + y + \bar{z}$
[T44a]	$(x \underset{0}{\leftarrow} y) \underset{0}{\rightarrow} z = (\bar{x} + y) \circ z$	[T44b]	$(x \underset{1}{\leftarrow} y) \underset{1}{\rightarrow} z = (\bar{x} \circ y) + z$
[T45a]	$(x \underset{1}{\rightarrow} y) \underset{0}{\rightarrow} z = (z \underset{1}{\rightarrow} y) \underset{0}{\rightarrow} x$	[T45b]	$(x \underset{0}{\rightarrow} y) \underset{1}{\rightarrow} z = (z \underset{0}{\rightarrow} y) \underset{1}{\rightarrow} x$
[T46a]	$(x \underset{0}{\leftarrow} y) \underset{0}{\rightarrow} z = z \underset{0}{\leftarrow} (y \underset{0}{\rightarrow} x)$	[T46b]	$(x \underset{1}{\leftarrow} y) \underset{1}{\rightarrow} z = z \underset{1}{\leftarrow} (y \underset{1}{\rightarrow} x)$
[T47a]	$(\bar{x} \underset{0}{\leftarrow} y) \underset{0}{\rightarrow} z = (x \underset{0}{\rightarrow} \bar{y}) \underset{0}{\rightarrow} z$	[T47b]	$(\bar{x} \underset{1}{\leftarrow} y) \underset{1}{\rightarrow} z = (x \underset{1}{\rightarrow} \bar{y}) \underset{1}{\rightarrow} z$
[T48a]	$x \underset{0}{\rightarrow} (x \circ y) = 0$	[T48b]	$x \underset{1}{\rightarrow} (x + y) = 1$
[T49a]	$(x \circ y) \underset{0}{\rightarrow} (y \circ x) = 0$	[T49b]	$(x + y) \underset{1}{\rightarrow} (y + x) = 1$
[T50a]	$\overline{(x \underset{0}{\leftarrow} y)} = x \underset{1}{\rightarrow} y$	[T50b]	$\overline{(x \underset{1}{\leftarrow} y)} = x \underset{0}{\rightarrow} y$
[T51a]	$0 \underset{0}{\leftarrow} 0 = 0$	[T51b]	$1 \underset{1}{\leftarrow} 1 = 1$
[T52a]	$0 \underset{0}{\leftarrow} 1 = 0$	[T52b]	$1 \underset{1}{\leftarrow} 0 = 1$
[T53a]	$1 \underset{0}{\leftarrow} 0 = 1$	[T53b]	$0 \underset{1}{\leftarrow} 1 = 0$
[T54a]	$1 \underset{0}{\leftarrow} 1 = 0$	[T54b]	$0 \underset{1}{\leftarrow} 0 = 1$
[T55a]	$x \underset{0}{\leftarrow} x = 0$	[T55b]	$x \underset{1}{\leftarrow} x = 1$
[T56a]	$x \underset{0}{\leftarrow} \bar{x} = x$	[T56b]	$x \underset{1}{\leftarrow} \bar{x} = x$
[T57a]	$\bar{x} \underset{0}{\leftarrow} x = \bar{x}$	[T57b]	$\bar{x} \underset{1}{\leftarrow} x = \bar{x}$
[T58a]	$x \underset{0}{\leftarrow} y = y \underset{0}{\rightarrow} x$	[T58b]	$x \underset{1}{\leftarrow} y = y \underset{1}{\rightarrow} x$
[T59a]	$(x \underset{0}{\leftarrow} y) \underset{0}{\leftarrow} z = x \circ \bar{y} \circ \bar{z}$	[T59b]	$(x \underset{1}{\leftarrow} y) \underset{1}{\leftarrow} z = x + \bar{y} + \bar{z}$
[T60a]	$x \underset{0}{\leftarrow} (y \underset{0}{\leftarrow} z) = x \circ (\bar{y} + z)$	[T60b]	$x \underset{1}{\leftarrow} (y \underset{1}{\leftarrow} z) = x + (\bar{y} \circ z)$
[T61a]	$x \underset{0}{\leftarrow} (y \underset{0}{\leftarrow} z) = x \circ \bar{y} + x \circ z$		
[T61b]	$x \underset{1}{\leftarrow} (y \underset{1}{\leftarrow} z) = (x + \bar{y}) \circ (x + z)$		
[T62a]	$\bar{x} \underset{0}{\leftarrow} \bar{y} = x \underset{0}{\rightarrow} y$	[T62b]	$\bar{x} \underset{1}{\leftarrow} \bar{y} = x \underset{1}{\rightarrow} y$
[T63a]	$\bar{x} \underset{0}{\leftarrow} (\bar{y} \underset{0}{\leftarrow} z) = x \parallel (y \parallel z)$	[T63b]	$\bar{x} \underset{1}{\leftarrow} (\bar{y} \underset{1}{\leftarrow} z) = x \mid (y \mid z)$
[T64a]	$(x \underset{0}{\leftarrow} y) \underset{0}{\leftarrow} x = 0$	[T64b]	$(x \underset{1}{\leftarrow} y) \underset{1}{\leftarrow} x = 1$
[T65a]	$(x \underset{0}{\leftarrow} y) \underset{0}{\leftarrow} y = x \circ \bar{y}$	[T65b]	$(x \underset{1}{\leftarrow} y) \underset{1}{\leftarrow} = x + \bar{y}$
[T66a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\leftarrow} z = \bar{x} \circ y \circ \bar{z}$	[T66b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\leftarrow} z = \bar{x} + y + \bar{z}$

TABLE XVII (Continued).

[T67a]	$x \underset{0}{\leftarrow} (x \underset{0}{\rightarrow} z) = x \circ (y + \bar{z})$	[T67b]	$x \underset{1}{\leftarrow} (y \underset{1}{\rightarrow} z) = x + (y \circ \bar{z})$
[T68a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\leftarrow} z = x \underset{0}{\rightarrow} (y \underset{0}{\leftarrow} z)$	[T68b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\leftarrow} z = x \underset{1}{\rightarrow} (y \underset{1}{\leftarrow} z)$
[T69a]	$(x \underset{0}{\rightarrow} y) \underset{0}{\leftarrow} z = z \underset{0}{\rightarrow} (y \underset{0}{\leftarrow} x)$	[T69b]	$(x \underset{1}{\rightarrow} y) \underset{1}{\leftarrow} z = z \underset{1}{\rightarrow} (y \underset{1}{\leftarrow} x)$
[T70a]	$x \underset{0}{\leftarrow} (y \underset{0}{\rightarrow} \bar{z}) = x \underset{0}{\leftarrow} (\bar{y} \underset{0}{\leftarrow} z)$	[T70b]	$x \underset{1}{\leftarrow} (y \underset{1}{\rightarrow} \bar{z}) = x \underset{1}{\leftarrow} (\bar{y} \underset{1}{\leftarrow} z)$
[T71a]	$\overline{(x \times y)} = x \bar{\times} y$	[T71b]	$\overline{(x \bar{\times} y)} = x \times y$
[T72a]	$0 \times 0 = 0$	[T72b]	$1 \bar{\times} 1 = 1$
[T73a]	$0 \times 1 = 1$	[T73b]	$1 \bar{\times} 0 = 0$
[T74a]	$1 \times 0 = 1$	[T74b]	$0 \bar{\times} 1 = 0$
[T75a]	$1 \times 1 = 0$	[T75b]	$0 \bar{\times} 0 = 1$
[T76a]	$x \times x = 0$	[T76b]	$x \bar{\times} x = 1$
[T77a]	$x \times \bar{x} = 1$	[T77b]	$x \bar{\times} \bar{x} = 0$
[T78a]	$\bar{x} \times x = 1$	[T78b]	$\bar{x} \bar{\times} x = 0$
[T79a]	$x \times y = (x + y) \circ (\bar{x} + \bar{y})$	[T79b]	$x \bar{\times} y = (x \circ y) + (\bar{x} \circ \bar{y})$
[T80a]	$x \times y = y \times x$	[T80b]	$x \bar{\times} y = y \bar{\times} x$
[T81a]	$x \times (y \times z) = (x \times y) \times z$	[T81b]	$x \bar{\times} (y \bar{\times} z) = (x \bar{\times} y) \bar{\times} z$
[T82a]	$(x \times y) \times z = (x \circ y \circ z) + (x \circ \bar{y} \circ \bar{z}) + (\bar{x} \circ y \circ \bar{z}) + (\bar{x} \circ \bar{y} \circ z)$		
[T82b]	$(x \bar{\times} y) \bar{\times} z = (x + y + z) \circ (x + \bar{y} + \bar{z}) \circ (\bar{x} + y + \bar{z}) \circ (\bar{x} + \bar{y} + z)$		
[T83a]	$(x \times y) \times z = (x + y + z) \circ (x + \bar{y} + \bar{z}) \circ (\bar{x} + y + \bar{z}) \circ (\bar{x} + \bar{y} + z)$		
[T83b]	$(x \bar{\times} y) \bar{\times} z = (x \circ y \circ z) + (x \circ \bar{y} \circ \bar{z}) + (\bar{x} \circ y \circ \bar{z}) + (\bar{x} \circ \bar{y} \circ z)$		
[T84a]	$(x \times y) \times z = (x \bar{\times} y) \bar{\times} z$	[T84b]	$(x \bar{\times} y) \bar{\times} z = (x \times y) \times z$
[T85a]	$\bar{x} \times \bar{y} = x \times y$	[T85b]	$\bar{x} \bar{\times} \bar{y} = x \bar{\times} y$
[T86a]	$(x \times y) \times x = y$	[T86b]	$(x \bar{\times} y) \bar{\times} x = y$
[T87a]	$(x \times y) \times y = x$	[T87b]	$(x \bar{\times} y) \bar{\times} y = x$
[T88a]	$(x \times y) \times \bar{x} = \bar{y}$	[T88b]	$(x \bar{\times} y) \bar{\times} \bar{x} = \bar{y}$
[T89a]	$(x \times y) \times \bar{y} = \bar{x}$	[T89b]	$(x \bar{\times} y) \bar{\times} \bar{y} = \bar{x}$

TABLE XVII (Continued).

[T90a] $x \times (y \bar{\times} z) = (x \bar{\times} y) \times z$ [T90b] $x \bar{\times} (y \times z) = (x \times y) \bar{\times} z$

[T91a] $(x \bar{\times} y) \times z = (x \circ y \circ \bar{z}) + (x \circ \bar{y} \circ z) + (\bar{x} \circ y \circ z) + (\bar{x} \circ \bar{y} \circ \bar{z})$

[T91b] $(x \times y) \bar{\times} z = (x + y + \bar{z}) \circ (x + \bar{y} + z) \circ (\bar{x} + y + z) \circ (\bar{x} + \bar{y} + \bar{z})$

[T92a] $(x \times y) \times z = (x + y + \bar{z}) \circ (x + \bar{y} + z) \circ (\bar{x} + y + z) \circ (\bar{x} + \bar{y} + \bar{z})$

[T92b] $(x \times y) \bar{\times} z = (x \circ y \circ \bar{z}) + (x \circ \bar{y} \circ z) + (\bar{x} \circ y \circ z) + (\bar{x} \circ \bar{y} \circ \bar{z})$

[T93a] $(x \circ y) \times (x \circ z) = x \circ (y \times z)$

[T93b] $(x + y) \bar{\times} (x + z) = x + (y \bar{\times} z)$

[T94a] $(x \circ y) \times (x \times y) = x + y$ [T94b] $(x + y) \bar{\times} (x \bar{\times} y) = x \circ y$

[T95a] $(x + y + z) \times (x + y + z) = 0$

[T95b] $(x \circ y \circ z) \bar{\times} (x \circ y \circ z) = 1$

[T96a] $[\ x + (y \circ z)\ ] \times [\ (x + y) \circ (x + z)\ ] = 0$

[T96b] $[\ x \circ (y + z)\ ] \bar{\times} [\ (x \circ y) + (x \circ z)\ ] = 1$

[T97a] $(x \underset{0}{\rightarrow} y) \times (x \underset{0}{\rightarrow} y) = x \underset{0}{\rightarrow} (y \times z)$

[T97b] $(x \underset{1}{\rightarrow} y) \bar{\times} (x \underset{1}{\rightarrow} z) = x \underset{1}{\rightarrow} (y \bar{\times} z)$

[T98a] $(x \underset{0}{\rightarrow} z) \times (y \underset{0}{\rightarrow} z) = (x \bar{\times} y) \underset{0}{\rightarrow} z$

[T98b] $(x \underset{1}{\rightarrow} z) \bar{\times} (y \underset{1}{\rightarrow} z) = (x \times y) \underset{1}{\rightarrow} z$

[T99a] $[\ x \underset{0}{\rightarrow} (y \underset{0}{\rightarrow} z)\ ] \times [\ (x + y) \underset{0}{\rightarrow} z\ ] = 0$

[T99b] $[\ x \underset{1}{\rightarrow} (y \underset{1}{\rightarrow} z)\ ] \times [\ (x \circ y) \underset{1}{\rightarrow} z\ ] = 1$

[T100a] $(x \overset{\rightarrow}{0} \bar{y}) \times z = (x \circ y \circ z) + (x \circ \bar{y} \circ z) + (\bar{x} \circ y \circ z) + (\bar{x} \circ \bar{y} \circ \bar{z})$

[T100b] $(x \overset{\rightarrow}{0} \bar{y}) \bar{\times} z = (x + y + z) \circ (x + \bar{y} + z) \circ (\bar{x} + y + z) \circ (\bar{x} + \bar{y} + \bar{z})$

[T101a] $\overline{(x \mid y)} = x \circ y$ [T101b] $\overline{(x \parallel y)} = x + y$

[T102a] $0 \mid 0 = 1$ [T102b] $1 \parallel 1 = 0$

[T103a] $0 \mid 1 = 1$ [T103b] $1 \parallel 0 = 0$

[T104a] $1 \mid 0 = 1$ [T104b] $0 \parallel 1 = 0$

[T105a] $1 \mid 1 = 0$ [T105b] $0 \parallel 0 = 1$

TABLE XVII (Continued).

[T106a] $x \mid x = \bar{x}$ [T106b] $x \parallel x = \bar{x}$

[T107a] $x \mid \bar{x} = 1$ [T107b] $x \parallel \bar{x} = 0$

[T108a] $x \mid y = y \mid x$ [T108b] $x \parallel y = y \parallel x$

[T109a] $(x \mid y) \mid z = (x \circ y) + \bar{z}$ [T109b] $(x \parallel y) \parallel z = (x + y) \circ \bar{z}$

[T110a] $x \mid (y \mid z) = \bar{x} + (y \circ z)$ [T110b] $x \parallel (y \parallel z) = \bar{x} \circ (y + z)$

[T111a] $\bar{x} \mid \bar{y} = x + y$ [T111b] $\bar{x} \parallel \bar{y} = x \circ y$

[T112a] $(x \mid y) \mid (x \mid y) = x \circ y$ [T112b] $(x \parallel y) \parallel (x \parallel y) = x + y$

[T113a] $(x \mid x) \mid (y \mid y) = x + y$ [T113b] $(x \parallel x) \parallel (y \parallel y) = x \circ y$

[T114a] $x \mid [y \mid (y \mid y)] = \bar{x}$ [T114b] $x \parallel [y \parallel (y \parallel y)] = \bar{x}$

[T115a] $[x \mid (y \mid z)] \mid [x \mid (y \mid z)] = x \circ (\bar{y} + \bar{z})$

[T115b] $[x \parallel (y \parallel z)] \parallel [x \parallel (y \parallel z)] = x + (\bar{y} \circ \bar{z})$

[T116a] $[x \mid (y \mid z)] \mid [x \mid (y \mid z)] = [(y \mid y) \mid x] \mid [(z \mid z) \mid x]$

[T116b] $[x \parallel (y \parallel z)] \parallel [x \parallel (y \parallel z)] = [(y \parallel y) \parallel x] \parallel [(z \parallel z) \parallel x]$

[T117a] $(x \parallel y) \mid z = x + y + \bar{z}$ [T117b] $(x \mid y) \parallel z = x \circ y \circ \bar{z}$

[T118a] $x \mid (y \mid z) = \bar{x} + y + z$ [T118b] $x \parallel (y \mid z) = \bar{x} \circ y \circ z$

[T119a] $(x \parallel y) \mid \bar{z} = \bar{x} \mid (y \parallel z)$ [T119b] $(x \mid y) \parallel \bar{z} = \bar{x} \parallel (y \mid z)$

[T120a] $(x \underset{0}{\rightarrow} y) \circ (x \underset{0}{\rightarrow} z) = x \underset{0}{\rightarrow} (y \circ z)$

[T120b] $(x \underset{1}{\rightarrow} y) + (x \underset{1}{\rightarrow} z) = x \underset{1}{\rightarrow} (y + z)$

[T121a] $(x \underset{0}{\rightarrow} z) \circ (y \underset{0}{\rightarrow} z) = (x + y) \underset{0}{\rightarrow} z$

[T121b] $(x \underset{1}{\rightarrow} z) + (y \underset{1}{\rightarrow} z) = (x \circ y) \underset{1}{\rightarrow} z$

[T122a] $(x \underset{0}{\rightarrow} y) + (x \underset{0}{\rightarrow} z) = x \underset{0}{\rightarrow} (y + z)$

[T122b] $(x \underset{1}{\rightarrow} y) \circ (x \underset{1}{\rightarrow} z) = x \underset{1}{\rightarrow} (y \circ z)$

[T123a] $(x \underset{0}{\rightarrow} z) + (y \underset{0}{\rightarrow} z) = (x \circ y) \underset{0}{\rightarrow} z$

[T123b] $(x \underset{1}{\rightarrow} z) \circ (y \underset{1}{\rightarrow} z) = (x + y) \underset{1}{\rightarrow} z$

[T124a] $(x \underset{0}{\rightarrow} y) + (y \underset{0}{\rightarrow} x) = x \times y$ T124b $(x \underset{1}{\rightarrow} y) \circ (y \underset{1}{\rightarrow} x) = x \bar{\times} y$

TABLE XVIII

PROOF OF THEOREM T35a BY USE OF POSTULATES AND THEOREMS.

$$
\begin{aligned}
(x \underset{o}{\rightarrow} y) \underset{o}{\rightarrow} (x \underset{o}{\rightarrow} z) &= (\bar{x} \circ y) \underset{o}{\rightarrow} (x \underset{o}{\rightarrow} z) && \text{[P5a]}\\
&= (\bar{x} \circ y) \underset{o}{\rightarrow} (\bar{x} \circ z) && \text{[P5a]}\\
&= \overline{(\bar{x} \circ y)} \circ (\bar{x} \circ z) && \text{[P5a]}\\
&= (\bar{\bar{x}} + \bar{y}) \circ (\bar{x} \circ z) && \text{[T11b]}\\
&= (x + \bar{y}) \circ (\bar{x} \circ z) && \text{[T1]}\\
&= x \circ (\bar{x} \circ z) + \bar{y} \circ (\bar{x} \circ z) && \text{[T10b]}\\
&= (x \circ \bar{x}) \circ z + (\bar{y} \circ \bar{x}) \circ z && \text{[T9b]}\\
&= 0 \circ z + (\bar{y} \circ \bar{x}) \circ z && \text{[T7b]}\\
&= 0 + (\bar{y} \circ \bar{x}) \circ z && \text{[P4b]}\\
&= (\bar{y} \circ \bar{x}) \circ z && \text{[P3a]}\\
&= (\bar{x} \circ \bar{y}) \circ z && \text{[T8b]}\\
&= \bar{x} \circ (\bar{y} \circ z) && \text{[T9b]}\\
&= x \underset{o}{\rightarrow} (\bar{y} \circ z) && \text{[P5a]}\\
&= x \underset{o}{\rightarrow} (y \underset{o}{\rightarrow} z) && \text{[P5a]}
\end{aligned}
$$

TABLE XIX

PROOF OF THEOREM T35a BY PERFECT INDUCTION.

x	y	z	$x \underset{o}{\rightarrow} y$	$x \underset{o}{\rightarrow} z$	$y \underset{o}{\rightarrow} z$	$(x \underset{o}{\rightarrow} y) \underset{o}{\rightarrow} (x \underset{o}{\rightarrow} z)$	$x \underset{o}{\rightarrow} (y \underset{o}{\rightarrow} z)$
0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	1
0	1	0	1	0	0	0	0
0	1	1	1	1	0	0	0
1	0	0	0	0	0	0	0
1	0	1	0	0	1	0	0
1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0

TABLE XX

PROOF OF THEOREM T14a BY USE OF PREVIOUSLY PROVED THEOREMS.

$x \underset{o}{\rightarrow} (y \underset{o}{\rightarrow} x)$	$= y \underset{o}{\rightarrow} (x \underset{o}{\rightarrow} x)$	[T33a]
	$= y \underset{o}{\rightarrow} (0)$	[T26a]
	$= 0$	[T24a]

CHAPTER IV

THE BOOLEAN ALGEBRA OF ORDER FOUR

The Direct Union of Two Two-Valued Boolean Algebras

On pages 28 and 29, it has been shown that the direct union $A \times B$ of two lattices A and B is a lattice C with four binary operations $\circ$, $\perp$, $\top$, and $+$, as shown in Table V on page 29. The elements of $C = A \times B$ are defined as ordered couples of A and B. If $A = \{0,1\}$ and $B = \{0,1\}$, the elements of C are $c_1 = (0,0)$, $c_2 = (0,1)$, $c_3 = (1,0)$, and $c_4 = (1,1)$. Table VII is the $\circ$ operational table for C, and Table VIII(c) is the + operational table for C. It will be noted that if c_2 and c_3 are omitted from these tables, the results would be those shown in Table XXI. A comparison of Table XXI(a) and (b) with Table

TABLE XXI

THE $\circ$ AND + OPERATIONAL TABLES FOR ELEMENTS c_1 AND c_4 OF $C = A \times B$.

$\circ$	c_1	c_4
c_1	c_1	c_1
c_4	c_1	c_4

(a)

$+$	c_1	c_4
c_1	c_1	c_4
c_4	c_4	c_4

(b)

X(a) and (b) shows a one-to-one correspondence $\begin{pmatrix} c_1 & c_4 \\ 0 & 1 \end{pmatrix}$, as might be expected since c_1 is the couple (0,0) and $c_4 = (1,1)$. c_1 may therefore be represented by the symbol 0 and c_4 by 1. Let θ represent $c_2 = (0,1)$ and ϕ represent $c_3 = (1,0)$. Then the four operational tables for C become those shown in Table XXII. The direct union of two two-valued Boolean algebras is frequently represented as B^2, with B standing for the Boolean algebra of order two.

TABLE XXII

THE $\circ$, $+$, $\perp$, AND $\top$ OPERATIONAL TABLES FOR $B^2 = \{0,\theta,\phi,1\}$.

$\circ$	0	θ	ϕ	1
0	0	0	0	0
θ	0	θ	0	θ
ϕ	0	0	ϕ	ϕ
1	0	θ	ϕ	1

(a)

$+$	0	θ	ϕ	1
0	0	θ	ϕ	1
θ	θ	θ	1	1
ϕ	ϕ	1	ϕ	1
1	1	1	1	1

(b)

$\perp$	0	θ	ϕ	1
0	0	θ	0	θ
θ	θ	θ	θ	θ
ϕ	0	θ	ϕ	1
1	θ	θ	1	1

(c)

$\top$	0	θ	ϕ	1
0	0	0	ϕ	ϕ
θ	0	θ	ϕ	1
ϕ	ϕ	ϕ	ϕ	ϕ
1	ϕ	1	ϕ	1

(d)

As indicated on page 29, an operation is carried out component by component. The operation $\circ$ in the four-valued system represents $\circ$ in both of the two-valued systems. $\perp$ represents $\cdot$ in the first two-valued lattice, $+$ in the second. $\top$ indicates $+$ in the first, $\circ$ in the second, whereas $+$ in the four-valued system implies $+$ in each of the two-valued systems. The following calculations show how the $\perp$ table can be determined.

$$0 \perp 0 = (0,0) \perp (0,0) = (0 \cdot 0,\ 0 + 0) = (0,0) = 0$$

$$0 \perp \theta = (0,0) \perp (0,1) = (0 \circ 0,\ 0 + 1) = (0,1) = \theta$$

$$0 \perp \phi = (0,0) \perp (1,0) = (0 \cdot 1,\ 0 + 0) = (0,0) = 0$$

$$0 \perp 1 = (0,0) \perp (1,1) = (0 \circ 1,\ 0 + 1) = (0,1) = \theta$$

$$\theta \perp 0 = (0,1) \perp (0,0) = (0 \circ 0,\ 1 + 0) = (0,1) = \theta$$

$$\theta \perp \theta = (0,1) \perp (0,1) = (0 \circ 0,\ 1 + 1) = (0,1) = \theta$$

$$\theta \perp \phi = (0,1) \perp (1,0) = (0 \circ 1,\ 1 + 0) = (0,1) = \theta$$

$$\theta \perp 1 = (0,1) \perp (1,1) = (0 \circ 1,\ 1 + 1) = (0,1) = \theta$$

$$\phi \perp 0 = (1,0) \perp (0,0) = (1 \circ 0,\ 0 + 0) = (0,0) = 0$$

$$\phi \perp \theta = (1,0) \perp (0,1) = (1 \circ 0,\ 0 + 1) = (0,1) = \theta$$

$$\phi \perp \phi = (1,0) \perp (1,0) = (1 \circ 1,\ 0 + 0) = (1,0) = \phi$$

$$\phi \perp 1 = (1,0) \perp (1,1) = (1 \circ 1,\ 0 + 1) = (1,1) = 1$$

$$1 \perp 0 = (1,1) \perp (0,0) = (1 \circ 0,\ 1 + 0) = (0,1) = \theta$$

$$1 \perp \theta = (1,1) \perp (0,1) = (1 \circ 0.\ 1 + 1) = (0,1) = \theta$$

$$1 \perp \phi = (1,1) \perp (1,0) = (1 \circ 1,\ 1 + 0) = (1,1) = 1$$

$$1 \perp 1 = (1,1) \perp (1,1) = (1 \circ 1,\ 1 + 1) = (1,1) = 1$$

<u>Lattice</u> <u>Considerations</u>

Figure 6 shows lattice diagrams of B, the two-valued Boolean

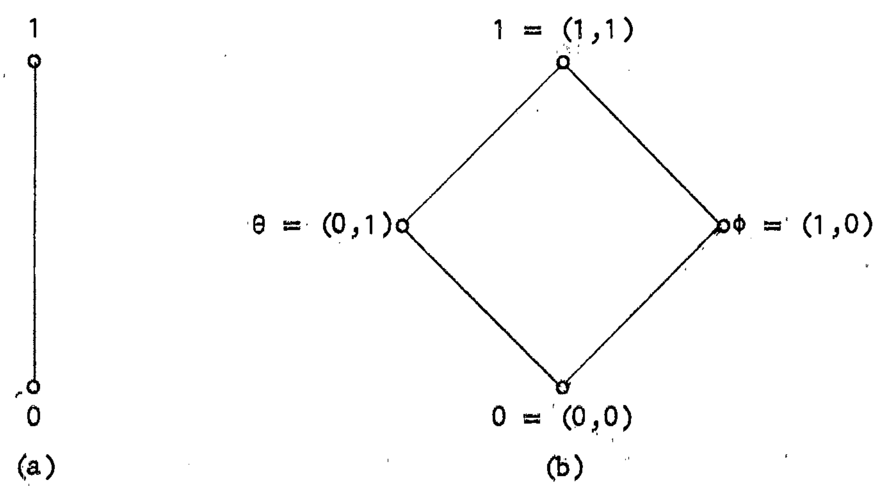

Fig. 6. Diagrams of Two- and Four-Valued Lattices.

algebra, and of B^2, the four-valued Boolean algebra. Since B^2 is a lattice, it obeys the idempotent, commutative, associative, absorptive, self-distributive, and semidistributive laws stated on page 24. Since B^2 satisfies the Jordan-Dedekind chain condition, it is a modular lattice and follows the modular laws stated on page 24.

<u>Transformations of B^2</u>

Using the symbols 0, θ, ϕ, and 1 for c_1, c_2, c_3, and c_4, respectively, the link-preserving transformations of B^2 indicated on page 27 are as shown in Table XXIII. The duality transformations are R' and

TABLE XXIII

LINK-PRESERVING TRANSFORMATIONS FOR B^2.

$$I = \begin{pmatrix} 0\theta\phi1 \\ 0\theta\phi1 \end{pmatrix} \qquad H = \begin{pmatrix} 0\theta\phi1 \\ \theta01\phi \end{pmatrix} \qquad V = \begin{pmatrix} 0\theta\phi1 \\ \phi10\theta \end{pmatrix} \qquad R' = \begin{pmatrix} 0\theta\phi1 \\ 1\phi\theta0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0\theta\phi1 \\ 0\phi\theta1 \end{pmatrix} \qquad R = \begin{pmatrix} 0\theta\phi1 \\ \theta10\phi \end{pmatrix} \qquad R'' = \begin{pmatrix} 0\theta\phi1 \\ \phi01\theta \end{pmatrix} \qquad D' = \begin{pmatrix} 0\theta\phi1 \\ 1\theta\phi0 \end{pmatrix}$$

D'. As indicated on page 32, the principal transformations are I, H, V, and R'. Table II, page 14, shows that I, H, V, R', D, and D' are self-inverse; that is, $t \cdot t = I$. R and R", however, are not self-inverse.

Obviously, $xI = x$, since I is the identity transformation. Consistent with the use of R' as the negative transformation for B, the Boolean algebra of order two, R' can be considered the negative transformation for B^2. $\bar{0} = 1$, $\bar{\theta} = \phi$, $\bar{\phi} = \theta$, $\bar{1} = 0$. On page 36, the zero biconditional $x \times y$ was defined for B as $(x \cdot \bar{y}) + (\bar{x} \cdot y)$. Since the $\cdot$ and + operations as well as negation exist in B^2, the same definition for $x \times y$ can be used for B^2. Using this definition and the $\cdot$ and + operational tables shown in Table XXII(a) and (b), the values of $x \times y$ for B^2 can be determined. These values are indicated in Table XXIV. Close examination of Tables XXIII and XXIV will reveal the following equalities: $xI = x \times 0 = x \circ 1 + \bar{x} \circ 0 = x$; $xH = x \times \theta = x \circ \phi + \bar{x} \circ \theta$; $xV = x \times \phi = x \circ \theta + \bar{x} \circ \phi$; and $xR' = x \times 1 = x \circ 0 + \bar{x} \circ 1 = \bar{x}$.

TABLE XXIV

THE $\times$ OPERATIONAL TABLE FOR B^2.

$\times$	0	θ	ϕ	1
0	0	θ	ϕ	1
θ	θ	0	1	ϕ
ϕ	ϕ	1	0	θ
1	1	ϕ	θ	0

Relationship of B and B^2

The union of two lattices, Boolean algebras of order two, into the lattice of B^2 certainly does not invalidate the postulates and theorems previously presented for the two-valued Boolean algebra, B. Postulate P1 still applies to B, but two new elements θ and ϕ must be introduced for B^2. Both B and B^2 have 1 as the least upper bound and 0 as the greatest lower bound. In both systems, therefore, postulates P3a, P3b, P4a, and P4b must be true. Comparison of Figure 6(a), the lattice diagram for B, with Figure 6(b), the one for B^2, shows that 1 covers 0 in both cases. The R' transformation, rotation of Figure 6(b) 180°, interchanges 0 and 1 in either B or B^2. P2a and P2b are true for either B or B^2, but for B^2 there must be added the negatives for the additional elements. $\bar{x} = \phi$ if $x = \theta$, and $\bar{x} = \theta$ if $x = \phi$. All theorems of Table XIV can be proved true for B^2 as well as for B. Operational tables for B^2 (e.g., Table XXIV) are true for B if only the elements 0 and 1 are considered. Relationships involving θ and ϕ do not, of course, apply to B.

Binary Relations and Transformations

As indicated on pages 26 and 27 , if there is an isomorphism between two groups $S_1 = (o, a, b, \ldots)$ and $S_2 = (\oplus, a, b, \ldots)$ with the same elements $a, b, \ldots$, $a \text{ o } b = (at \oplus bt)t^{-1}$, or $f \oplus g = (ft^{-1} \text{ o } gt^{-1})t$. If t is a principal transformation in B^2, it is self-inverse, and for every ordering relation o of B^2 there is a relation ρ such that $x \rho y = [(xt) \text{ o } (yt)]t$.

For example, let o be the $\circ$ operation. $x \rho_1 y = [(xt) \circ (yt)]t$. Now if t is the identity transformation I, $xt_1 = xI = x \times y = x$. $x \rho_{11} y = (xI \circ yI)I = (x \circ y)I = x \circ y$. Therefore, $\rho_{11} = \circ$. If $t = H$, $xt_2 = xH = x \times \theta = (x \circ \phi) + (\bar{x} \circ \theta)$. The following steps show the detailed development of a resultant expression. Notice that, for simplicity, the $\circ$ symbol is omitted between two elements it connects. Thus, $xy = x \circ y$.

$$
\begin{aligned}
x \rho_{12} y &= (xH \circ yH)H \\
&= [(x\phi + \bar{x}\theta) \circ (y\phi + \bar{y}\theta)]H \\
&= [(x\phi)(y\phi + \bar{y}\theta) + (\bar{x}\theta)(y\phi + \bar{y}\theta)]H && \text{[T10b]} \\
&= [(x\phi)(y\phi) + (x\phi)(\bar{y}\theta) + (\bar{x}\theta)(y\phi) + (\bar{x}\theta)(\bar{y}\theta)]H && \text{[T10b]} \\
&= [(x\phi)(\phi y) + (x\phi)(\theta\bar{y}) + (\bar{x}\theta)(\phi y) + (\bar{x}\theta)(\theta\bar{y})]H && \text{[T8b]} \\
&= [(x\phi\phi)y + (x\phi\theta)\bar{y} + (\bar{x}\theta\phi)y + (\bar{x}\theta\theta)\bar{y}]H && \text{[T9b]} \\
&= [(x\phi)y + (x0)\bar{y} + (\bar{x}0)y + (\bar{x}\theta)\bar{y}]H && \text{[Table XXII(a)]} \\
&= [(x\phi)y + 0 + 0 + (\bar{x}\theta)\bar{y}]H && \text{[Table XXII(a)]} \\
&= [(x\phi)y + (\bar{x}\theta)\bar{y}]H && \text{[Table XXII(b)]} \\
&= [xy\phi + \bar{x}\bar{y}\theta]H && \text{[T9b]} \\
&= (xy\phi + \bar{x}\bar{y}\theta)\phi + \overline{(xy\phi + \bar{x}\bar{y}\theta)}\theta && \text{[Table XXIV]} \\
&= (xy\phi + \bar{x}\bar{y}\theta)\phi + \overline{(xy\phi)} \circ \overline{(\bar{x}\bar{y}\theta)}\theta && \text{[T11a]} \\
&= (xy\phi + \bar{x}\bar{y}\theta)\phi + (\bar{x} + \bar{y} + \bar{\phi})(\bar{\bar{x}} + \bar{\bar{y}} + \bar{\theta})\theta && \text{[T12b]} \\
&= (xy\phi + \bar{x}\bar{y}\theta)\phi + (\bar{x} + \bar{y} + \bar{\phi})(x + y + \bar{\theta})\theta && \text{[T1]} \\
&= (xy\phi + \bar{x}\bar{y}\theta)\phi + (\bar{x} + \bar{y} + \theta)(x + y + \phi)\theta && \text{[Table XXIII for R']}
\end{aligned}
$$

$$= (xy\phi\phi + \bar{x}\bar{y}\theta\phi) + (\bar{x} + \bar{y} + \theta)(x + y + \phi)\theta \qquad \text{[T10b]}$$

$$= (xy\phi + \bar{x}\bar{y}0) + (\bar{x} + \bar{y} + \theta)(x + y + \phi)\theta \qquad \text{[Table XXII(a)]}$$

$$= xy\phi + 0 + (\bar{x} + \bar{y} + \theta)(x + y + \phi)\theta \qquad \text{[Table XXII(a)]}$$

$$= xy\phi + (\bar{x} + \bar{y} + \theta)(x + y + \phi)\theta \qquad \text{[Table XXII(a)]}$$

$$= xy\phi + (\bar{x} + \bar{y} + \theta)[\theta(x + y + \phi)] \qquad \text{[T8b]}$$

$$= xy\phi + [(\bar{x} + \bar{y} + \theta)\theta](x + y + \phi) \qquad \text{[T9b]}$$

$$= xy\phi + \theta(\bar{x} + \bar{y} + \theta)(x + y + \phi) \qquad \text{[T8b]}$$

$$= xy\phi + \theta(x + y + \phi) \qquad \text{[T13b]}$$

$$= xy\phi + (x + y + \phi)\theta \qquad \text{[T8b]}$$

$$= xy\phi + x\theta + y\theta + \phi\theta \qquad \text{[T10b]}$$

$$= xy\phi + x\theta + y\theta + 0 \qquad \text{[Table XXII(a)]}$$

$$= xy\phi + x\theta + y\theta \qquad \text{[Table XXII(b)]}$$

$$= xy\phi + xy\phi + x\theta + y\theta \qquad \text{[T6a]}$$

$$= (x\theta + xy\phi) + (y\theta + xy\phi) \qquad \text{[T8a]}$$

$$= x(\theta + y\phi) + y(\theta + x\phi) \qquad \text{[T10b]}$$

$$= x(y + \theta) + y(x + \theta) \qquad \text{[T14a]}$$

$$= xy + x\theta + yx + y\theta \qquad \text{[T10b]}$$

$$= xy + x\theta + xy + y\theta \qquad \text{[T8b]}$$

$$= xy + xy + x\theta + y\theta \qquad \text{[T8a]}$$

$$= xy + x\theta + y\theta \qquad \text{[T6a]}$$

$$= xy + (x + y)\theta \qquad \text{[T10b]}$$

By perfect induction, the expression $xy + (x + y)\theta$ can be shown to be equal to $x \perp y$, the operational table for which is Table XXII(c). Other expressions can be similarly derived by letting o be $\circ$, $\times$, $\underset{o}{\rightarrow}$, $\underset{o}{\leftarrow}$, or $|$, with t as any of the four principal transformations. The results obtained from the twenty possible combinations are listed in Table XXV. Ten of the expressions are identical to ones already

TABLE XXV

EXPRESSIONS DERIVED FROM $x \rho y = [(xt) \circ (yt)]t$.

o	t	$[(xt) \circ (yt)]t$	Operational Expression
$\circ$	I	$x \circ y$	$x \circ y$
$\circ$	H	$xy + (x + y)\theta$	$x \perp y$
$\circ$	V	$(x + y)(xy + \phi)$	$x \top y$
$\circ$	R'	$x + y$	$x + y$
$\times$	I	$x\bar{y} + \bar{x}y$	$x \times y$
$\times$	H	$(x + \bar{y})(\bar{x} + y)\theta + (x + y)(\bar{x} + \bar{y})\phi$	$x \underline{\times} y$
$\times$	V	$(x\bar{y} + \bar{x}y + \phi)(xy + \bar{x}\bar{y} + \theta)$	$x \overline{\times} y$
$\times$	R'	$(x + \bar{y})(\bar{x} + y)$	$x \bar{\times} y$
$\vec{o}$	I	$\bar{x}y$	$x \underset{o}{\rightarrow} y$
$\vec{o}$	H	$\bar{x}y + (\bar{x} + y)\theta$	$x \underset{\perp}{\rightarrow} y$
$\vec{o}$	V	$(\bar{x} + y)(\bar{x}y + \phi)$	$x \overset{\top}{\rightarrow} y$
$\vec{o}$	R'	$\bar{x} + y$	$x \underset{1}{\rightarrow} y$
$\overset{\leftarrow}{o}$	I	$x\bar{y}$	$x \underset{o}{\leftarrow} y$
$\overset{\leftarrow}{o}$	H	$x\bar{y} + (x + \bar{y})\theta$	$x \underset{\perp}{\leftarrow} y$
$\overset{\leftarrow}{o}$	V	$(x + \bar{y})(x\bar{y} + \phi)$	$x \overset{\top}{\leftarrow} y$
$\overset{\leftarrow}{o}$	R'	$x + \bar{y}$	$x \underset{1}{\leftarrow} y$
$\mid$	I	$\overline{(xy)}$	$x \mid y$
$\mid$	H	$(\bar{x} + \bar{y})(\bar{x}\bar{y} + \phi)$	$x \underline{\Vert} y$
$\mid$	V	$\bar{x}\bar{y} + (\bar{x} + \bar{y})\theta$	$x \overline{\Vert} y$
$\mid$	R'	$\overline{(x + y)}$	$x \Vert y$

defined for B: $\circ$, $+$, $\times$, $\bar{\bar{\times}}$, $\underset{0}{\rightarrow}$, $\underset{1}{\rightarrow}$, $\underset{0}{\leftarrow}$, $\underset{1}{\leftarrow}$, $|$, and $\|$. The $\perp$ and $\top$ operations for B^2 have already been indicated in Table XXII. Eight new expressions are indicated in Table XXV: $\underline{\times}$, $\overline{\times}$, $\underline{\rightarrow}$, $\overline{\rightarrow}$, $\underline{\leftarrow}$, $\overline{\leftarrow}$, $\underline{\|}$, and $\overline{\|}$. All of them involve either θ or ϕ or both and thus, along with $\perp$ and $\top$, apply to B^2 only. Dual and negative relationships for the twenty binary relations of B^2 are listed in Table XXVI.

Operational tables for each of the twenty binary relations of B^2 can be worked out from their definitions in terms of $\circ$ and $+$ by reference to Table XXII(a) and (b). Results are shown in Table XXVII.

Although all postulates of Table XIII except P1 apply to B^2 as well as to B, further postulates or definitions are needed for B^2. These are listed in Table XXVIII. With the aid of these and the postulates and theorems of Tables XIII and XIV, innumerable theorems for B^2 can be proved. Some of them are listed in Table XXIX. It will be noted that they are listed in dual pairs, where a dual expression is obtained in a manner similar to that used for B. Dual binary relations are interchanged, 0 and 1 are interchanges, and θ and ϕ are interchanged.

TABLE XXVI

DUAL AND NEGATIVE EXPRESSIONS OF B^2.

Expression	Dual	Negative
$x \circ y$	$x + y$	$x \mid y$
$x \perp y = xy + (x + y)\theta$	$x \top y$	$x \coprod y$
$x \top y = (x + y)(xy + \phi)$	$x \perp y$	$x \prod y$
$x + y$	$x \circ y$	$x \parallel y$
$x \times y = x\bar{y} + \bar{x}y$	$x \bar{\times} y$	$x \bar{\times} y$
$x \underline{\times} y = (x + \bar{y})(\bar{x} + y)\theta + (x + y)(\bar{x} + \bar{y})\phi$	$x \overline{\times} y$	$x \overline{\times} y$
$x \overline{\times} y = (x\bar{y} + \bar{x}y + \phi)(xy + \bar{x}\bar{y} + \theta)$	$x \underline{\times} y$	$x \underline{\times} y$
$x \bar{\times} y = (x + \bar{y})(\bar{x} + y)$	$x \times y$	$x \times y$
$x \underset{0}{\rightarrow} y = \bar{x}y$	$x \underset{1}{\rightarrow} y$	$x \underset{1}{\leftarrow} y$
$x \underline{\rightarrow} y = \bar{x}y + (\bar{x} + y)\theta$	$x \overline{\rightarrow} y$	$x \overline{\leftarrow} y$
$x \overline{\rightarrow} y = (\bar{x} + y)(\bar{x}y + \phi)$	$x \underline{\rightarrow} y$	$x \underline{\leftarrow} y$
$x \underset{1}{\rightarrow} y = \bar{x} + y$	$x \underset{0}{\rightarrow} y$	$x \underset{0}{\leftarrow} y$
$x \underset{0}{\leftarrow} y = x\bar{y}$	$x \underset{1}{\leftarrow} y$	$x \underset{1}{\rightarrow} y$
$x \underline{\leftarrow} y = x\bar{y} + (x + \bar{y})\theta$	$x \overline{\leftarrow} y$	$x \overline{\rightarrow} y$
$x \overline{\leftarrow} y = (x + \bar{y})(x\bar{y} + \phi)$	$x \underline{\leftarrow} y$	$x \underline{\rightarrow} y$
$x \underset{1}{\leftarrow} y = x + \bar{y}$	$x \underset{0}{\leftarrow} y$	$x \underset{0}{\rightarrow} y$
$x \mid y = \overline{(xy)} = \bar{x} + \bar{y}$	$x \parallel y$	$x \circ y$
$x \coprod y = (\bar{x} + \bar{y})(\bar{x}\bar{y} + \phi)$	$x \prod y$	$x \perp y$
$x \prod y = \bar{x}\bar{y} + (\bar{x} + \bar{y})\theta$	$x \coprod y$	$x \top y$
$x \parallel y = \overline{(x + y)} = \bar{x}\bar{y}$	$x \mid y$	$x + y$

TABLE XXVII

OPERATIONAL TABLES FOR B^2.

$\circ$	0	θ	φ	1
0	0	0	0	0
θ	0	θ	0	θ
φ	0	0	φ	φ
1	0	θ	φ	1

(a)

$\times$	0	θ	φ	1
0	0	θ	φ	1
θ	θ	0	1	φ
φ	φ	1	0	θ
1	1	φ	θ	0

(b)

$\vec{o}$	0	θ	φ	1
0	0	θ	φ	1
θ	0	0	φ	φ
φ	0	θ	0	θ
1	0	0	0	0

(c)

$\overset{\leftarrow}{o}$	0	θ	φ	1
0	0	0	0	0
θ	θ	0	θ	0
φ	φ	φ	0	0
1	1	φ	θ	0

(d)

$\vert$	0	θ	φ	1
0	1	1	1	1
θ	1	φ	1	φ
φ	1	1	θ	θ
1	1	φ	θ	0

(e)

$\perp$	0	θ	φ	1
0	0	θ	0	θ
θ	θ	θ	θ	θ
φ	0	θ	φ	1
1	θ	θ	1	1

(f)

$\underline{\times}$	0	θ	φ	1
0	θ	0	1	φ
θ	0	θ	φ	1
φ	1	φ	θ	0
1	φ	1	0	θ

(g)

$\underline{\vec{o}}$	0	θ	φ	1
0	θ	θ	1	1
θ	0	θ	φ	1
φ	θ	θ	θ	θ
1	0	θ	0	θ

(h)

$\underline{\overset{\leftarrow}{o}}$	0	θ	φ	1
0	θ	0	θ	0
θ	θ	θ	θ	θ
φ	1	φ	θ	0
1	1	1	θ	θ

(i)

$\underline{\Vert}$	0	θ	φ	1
0	1	φ	1	φ
θ	φ	φ	φ	φ
φ	1	φ	θ	0
1	φ	φ	0	0

(j)

$\top$	0	θ	φ	1
0	0	0	φ	φ
θ	0	θ	φ	1
φ	φ	φ	φ	φ
1	φ	1	φ	1

(k)

$\overline{\times}$	0	θ	φ	1
0	φ	1	0	θ
θ	1	φ	θ	0
φ	0	θ	φ	1
1	θ	0	1	φ

(ℓ)

$\overline{\vec{o}}$	0	θ	φ	1
0	φ	1	φ	1
θ	φ	φ	φ	φ
φ	0	θ	φ	1
1	0	0	φ	φ

(m)

$\overline{\overset{\leftarrow}{o}}$	0	θ	φ	1
0	φ	φ	0	0
θ	1	φ	θ	0
φ	φ	φ	φ	φ
1	1	φ	1	φ

(n)

$\overline{\Vert}$	0	θ	φ	1
0	1	1	θ	θ
θ	1	φ	θ	0
φ	θ	θ	θ	θ
1	θ	0	θ	0

(o)

$+$	0	θ	φ	1
0	0	θ	φ	1
θ	θ	θ	1	1
φ	φ	1	φ	1
1	1	1	1	1

(p)

$\bar{\bar{\times}}$	0	θ	φ	1
0	1	φ	θ	0
θ	φ	1	0	θ
φ	θ	0	1	φ
1	0	θ	φ	1

(q)

$\overset{\to}{1}$	0	θ	φ	1
0	1	1	1	1
θ	φ	1	φ	1
φ	θ	θ	1	1
1	0	θ	φ	1

(r)

$\overset{\leftarrow}{1}$	0	θ	φ	1
0	1	φ	θ	0
θ	1	1	θ	θ
φ	1	φ	1	φ
1	1	1	1	1

(s)

$\Vert$	0	θ	φ	1
0	1	φ	θ	0
θ	φ	φ	0	0
φ	θ	0	θ	0
1	0	0	0	0

(t)

TABLE XXVIII

POSTULATES AND DEFINITIONS FOR FOUR-VALUED BOOLEAN ALGEBRA

[P5a] $0 = (0,0)$	[P5b] $1 = (1,1)$
[P6a] $\theta = (0,1)$	[P6b] $\phi = (1,0)$

[P7] $x = 0$, or $x = \theta$, or $x = \phi$, or $x = 1$ for all $x \in B^2$.

[P8a] $\bar{x} = \phi$ if $x = \theta$	[P8b] $\bar{x} = \theta$ if $x = \phi$
[P9a] $x \perp y = xy + (x + y)\theta$	[P9b] $x \top y = (x + y)(xy + \phi)$

[P10a] $x \mathbin{\perp\!\!\!\!\times} y = (x + \bar{y})(\bar{x} + y)\theta + (x + y)(\bar{x} + \bar{y})\phi$

[P10b] $x \mathbin{\top\!\!\!\!\times} y = (x\bar{y} + \bar{x}y + \phi)(xy + \bar{x}\bar{y} + \theta)$

[P11a] $x \mathbin{\rightarrow\!\!\!\!\perp} y = \bar{x}y + (\bar{x} + y)\theta$	[P11b] $x \mathbin{\rightarrow\!\!\!\!\top} y = (\bar{x} + y)(\bar{x}y + \phi)$
[P12a] $x \mathbin{\leftarrow\!\!\!\!\perp} y = x\bar{y} + (x + \bar{y})\theta$	[P12b] $x \mathbin{\leftarrow\!\!\!\!\top} y = (x + \bar{y})(x\bar{y} + \phi)$
[P13a] $x \mathbin{\perp\!\!\perp} y = (\bar{x} + \bar{y})(\bar{x}\bar{y} + \phi)$	[P13b] $x \mathbin{\top\!\!\top} y = \bar{x}\bar{y} + (\bar{x} + \bar{y})\theta$

TABLE XXIX

THEOREMS FOR B^2

[T125a] $0 \cdot \theta = 0$	[T125b] $1 + \phi = 1$
[T126a] $0 \cdot \phi = 0$	[T126b] $1 + \theta = 1$
[T127a] $\theta \cdot 0 = 0$	[T127b] $\phi + 1 = 1$
[T128a] $\theta \cdot \theta = \theta$	[T128b] $\phi + \phi = \phi$
[T129a] $\theta \cdot \phi = 0$	[T129b] $\phi + \theta = 1$
[T130a] $\theta \cdot 1 = \theta$	[T130b] $\phi + 0 = \phi$
[T131a] $\phi \cdot 0 = 0$	[T131b] $\theta + 1 = 1$
[T132a] $\phi \cdot \theta = 0$	[T132b] $\theta + \phi = 1$
[T133a] $\phi \cdot \phi = \phi$	[T133b] $\theta + \theta = \theta$
[T134a] $\phi \cdot 1 = \phi$	[T134b] $\theta + 0 = \theta$
[T135a] $1 \cdot \theta = \theta$	[T135b] $0 + \phi = \phi$

TABLE XXIX (Continued)

[T137a]	$0 \times \theta = \theta$	[T137b]	$1 \bar{\times} \phi = \phi$
[T138a]	$0 \times \phi = \phi$	[T138b]	$1 \bar{\times} \theta = \theta$
[T139a]	$\theta \times 0 = \theta$	[T139b]	$\phi \bar{\times} 1 = \phi$
[T140a]	$\theta \times \theta = 0$	[T140b]	$\phi \bar{\times} \phi = 1$
[T141a]	$\theta \times \phi = 1$	[T141b]	$\phi \bar{\times} \theta = 0$
[T142a]	$\theta \times 1 = \phi$	[T142b]	$\phi \bar{\times} 0 = \theta$
[T143a]	$\phi \times 0 = \phi$	[T143b]	$\theta \bar{\times} 1 = \theta$
[T144a]	$\phi \times \theta = 1$	[T144b]	$\theta \bar{\times} \phi = 0$
[T145a]	$\phi \times \phi = 0$	[T145b]	$\theta \bar{\times} \theta = 1$
[T146a]	$\phi \times 1 = \theta$	[T146b]	$\theta \bar{\times} 0 = \phi$
[T147a]	$1 \times \theta = \phi$	[T147b]	$0 \bar{\times} \phi = \theta$
[T148a]	$1 \times \phi = \theta$	[T148b]	$0 \bar{\times} \theta = \phi$
[T149a]	$0 \underset{0}{\rightarrow} \theta = \theta$	[T149b]	$1 \underset{1}{\rightarrow} \phi = \phi$
[T150a]	$0 \underset{0}{\rightarrow} \phi = \phi$	[T150b]	$1 \underset{1}{\rightarrow} \theta = \theta$
[T151a]	$\theta \underset{0}{\rightarrow} 0 = 0$	[T151b]	$\phi \underset{1}{\rightarrow} 1 = 1$
[T152a]	$\theta \underset{0}{\rightarrow} \theta = 0$	[T152b]	$\phi \underset{1}{\rightarrow} \phi = 1$
[T153a]	$\theta \underset{0}{\rightarrow} \phi = \phi$	[T153b]	$\phi \underset{1}{\rightarrow} \theta = \theta$
[T154a]	$\theta \underset{0}{\rightarrow} 1 = \phi$	[T154b]	$\phi \underset{1}{\rightarrow} 0 = \theta$
[T155a]	$\phi \underset{0}{\rightarrow} 0 = 0$	[T155b]	$\theta \underset{1}{\rightarrow} 1 = 1$
[T156a]	$\phi \underset{0}{\rightarrow} \theta = \theta$	[T156b]	$\theta \underset{1}{\rightarrow} \phi = \phi$
[T157a]	$\phi \underset{0}{\rightarrow} \phi = 0$	[T157b]	$\theta \underset{1}{\rightarrow} \theta = 1$
[T158a]	$\phi \underset{0}{\rightarrow} 1 = \theta$	[T158b]	$\theta \underset{1}{\rightarrow} 0 = \phi$
[T159a]	$1 \underset{0}{\rightarrow} \theta = 0$	[T159b]	$0 \underset{1}{\rightarrow} \phi = 1$
[T160a]	$1 \underset{0}{\rightarrow} \phi = 0$	[T160b]	$0 \underset{1}{\rightarrow} \theta = 1$

TABLE XXIX (Continued)

[T161a]	$0 \underset{0}{\leftarrow} \theta = 0$	[T161b]	$1 \underset{1}{\leftarrow} \phi = 1$
[T162a]	$0 \underset{0}{\leftarrow} \phi = 0$	[T162b]	$1 \underset{1}{\leftarrow} \theta = 1$
[T163a]	$\theta \underset{0}{\leftarrow} 0 = \theta$	[T163b]	$\phi \underset{1}{\leftarrow} 1 = \phi$
[T164a]	$\theta \underset{0}{\leftarrow} \theta = 0$	[T164b]	$\phi \underset{1}{\leftarrow} \phi = 1$
[T165a]	$\theta \underset{0}{\leftarrow} \phi = \theta$	[T165b]	$\phi \underset{1}{\leftarrow} \theta = \phi$
[T166a]	$\theta \underset{0}{\leftarrow} 1 = 0$	[T166b]	$\phi \underset{1}{\leftarrow} 0 = 1$
[T167a]	$\phi \underset{0}{\leftarrow} 0 = \phi$	[T167b]	$\theta \underset{1}{\leftarrow} 1 = \theta$
[T168a]	$\phi \underset{0}{\leftarrow} \theta = \phi$	[T168b]	$\theta \underset{1}{\leftarrow} \phi = \theta$
[T169a]	$\phi \underset{0}{\leftarrow} \phi = 0$	[T169b]	$\theta \underset{1}{\leftarrow} \theta = 1$
[T170a]	$\phi \underset{0}{\leftarrow} 1 = 0$	[T170b]	$\theta \underset{1}{\leftarrow} 0 = 1$
[T171a]	$1 \underset{0}{\leftarrow} \theta = \phi$	[T171b]	$0 \underset{1}{\leftarrow} \phi = \theta$
[T172a]	$1 \underset{0}{\leftarrow} \phi = \theta$	[T172b]	$0 \underset{1}{\leftarrow} \theta = \phi$
[T173a]	$0 \mid \theta = 1$	[T173b]	$1 \parallel \phi = 0$
[T174a]	$0 \mid \phi = 1$	[T174b]	$1 \parallel \theta = 0$
[T175a]	$\theta \mid 0 = 1$	[T175b]	$\phi \parallel 1 = 0$
[T176a]	$\theta \mid \theta = \phi$	[T176b]	$\phi \parallel \phi = \theta$
[T177a]	$\theta \mid \phi = 1$	[T177b]	$\phi \parallel \theta = 0$
[T178a]	$\theta \mid 1 = \phi$	[T178b]	$\phi \parallel 0 = \theta$
[T179a]	$\phi \mid 0 = 1$	[T179b]	$\theta \parallel 1 = 0$
[T180a]	$\phi \mid \theta = 1$	[T180b]	$\theta \parallel \phi = 0$
[T181a]	$\phi \mid \phi = \theta$	[T181b]	$\theta \parallel \theta = \phi$
[T182a]	$\phi \mid 1 = \theta$	[T182b]	$\theta \parallel 0 = \phi$
[T183a]	$1 \mid \theta = \phi$	[T183b]	$0 \parallel \phi = \theta$
[T184a]	$1 \mid \phi = \theta$	[T184b]	$0 \parallel \theta = \phi$

TABLE XXIX (Continued)

[T185a]	$0 \perp 0 = 0$	[T185b]	$1 \top 1 = 1$
[T186a]	$0 \perp \theta = \theta$	[T186b]	$1 \top \phi = \phi$
[T187a]	$0 \perp \phi = 0$	[T187b]	$1 \top \theta = 1$
[T188a]	$0 \perp 1 = \theta$	[T188b]	$1 \top 0 = \phi$
[T189a]	$\theta \perp 0 = \theta$	[T189b]	$\phi \top 1 = \phi$
[T190a]	$\theta \perp \theta = \theta$	[T190b]	$\phi \top \phi = \phi$
[T191a]	$\theta \perp \phi = \theta$	[T191b]	$\phi \top \theta = \phi$
[T192a]	$\theta \perp 1 = \theta$	[T192b]	$\phi \top 1 = \phi$
[T193a]	$\phi \perp 0 = 0$	[T193b]	$\theta \top 1 = 1$
[T194a]	$\phi \perp \theta = \theta$	[T194b]	$\theta \top \phi = \phi$
[T195a]	$\phi \perp \phi = \phi$	[T195b]	$\theta \top \theta = \theta$
[T196a]	$\phi \perp 1 = 1$	[T196b]	$\theta \top 0 = 0$
[T197a]	$1 \perp 0 = \theta$	[T197b]	$0 \top 1 = \phi$
[T198a]	$1 \perp \theta = \theta$	[T198b]	$0 \top \phi = \phi$
[T199a]	$1 \perp \phi = 1$	[T199b]	$0 \top \theta = 0$
[T200a]	$1 \perp 1 = 1$	[T200b]	$0 \top 0 = 0$
[T201a]	$0 \pm 0 = \theta$	[T201b]	$1 \mp 1 = \phi$
[T202a]	$0 \pm \theta = \theta$	[T202b]	$1 \mp \phi = \phi$
[T203a]	$0 \pm \phi = 1$	[T203b]	$1 \mp \theta = 0$
[T204a]	$0 \pm 1 = 1$	[T204b]	$1 \mp 0 = 0$
[T205a]	$\theta \pm 0 = 0$	[T205b]	$\phi \mp 1 = 1$
[T206a]	$\theta \pm \theta = \theta$	[T206b]	$\phi \mp \phi = \phi$
[T207a]	$\theta \pm \phi = \phi$	[T207b]	$\phi \mp \theta = \theta$
[T208a]	$\theta \pm 1 = 1$	[T208b]	$\phi \mp 0 = 0$
[T209a]	$\phi \pm 0 = \theta$	[T209b]	$\theta \mp 1 = \phi$

TABLE XXIX (Continued)

[T210a]	$\phi \vec{\pm} \theta = \theta$	[T210b]	$\theta \vec{\mp} \phi = \phi$
[T211a]	$\phi \vec{\pm} \phi = \theta$	[T211b]	$\theta \vec{\mp} \theta = \phi$
[T212a]	$\phi \vec{\pm} 1 = \theta$	[T212b]	$\theta \vec{\mp} 0 = \phi$
[T213a]	$1 \vec{\pm} 0 = 0$	[T213b]	$0 \vec{\mp} 1 = 1$
[T214a]	$1 \vec{\pm} \theta = \theta$	[T214b]	$0 \vec{\mp} \phi = \phi$
[T215a]	$1 \vec{\pm} \phi = 0$	[T215b]	$0 \vec{\mp} \theta = 1$
[T216a]	$1 \vec{\pm} 1 = \theta$	[T216b]	$0 \vec{\mp} 0 = \phi$
[T217a]	$0 \overleftarrow{\pm} 0 = \theta$	[T217b]	$1 \overleftarrow{\mp} 1 = \phi$
[T218a]	$0 \overleftarrow{\pm} \theta = 0$	[T218b]	$1 \overleftarrow{\mp} \phi = 1$
[T219a]	$0 \overleftarrow{\pm} \phi = \theta$	[T219b]	$1 \overleftarrow{\mp} \theta = \phi$
[T220a]	$0 \overleftarrow{\pm} 1 = 0$	[T220b]	$1 \overleftarrow{\mp} 0 = 1$
[T221a]	$\theta \overleftarrow{\pm} 0 = \theta$	[T221b]	$\phi \overleftarrow{\mp} 1 = \phi$
[T222a]	$\theta \overleftarrow{\pm} \theta = \theta$	[T222b]	$\phi \overleftarrow{\mp} \phi = \phi$
[T223a]	$\theta \overleftarrow{\pm} \phi = \theta$	[T223b]	$\phi \overleftarrow{\mp} \theta = \phi$
[T224a]	$\theta \overleftarrow{\pm} 1 = \theta$	[T224b]	$\phi \overleftarrow{\mp} 0 = \phi$
[T225a]	$\phi \overleftarrow{\pm} 0 = 1$	[T225b]	$\theta \overleftarrow{\mp} 1 = 0$
[T226a]	$\phi \overleftarrow{\pm} \theta = \phi$	[T226b]	$\theta \overleftarrow{\mp} \phi = \theta$
[T227a]	$\phi \overleftarrow{\pm} \phi = \theta$	[T227b]	$\theta \overleftarrow{\mp} \theta = \phi$
[T228a]	$\phi \overleftarrow{\pm} 1 = 0$	[T228b]	$\theta \overleftarrow{\mp} 0 = 1$
[T229a]	$1 \overleftarrow{\pm} 0 = 1$	[T229b]	$0 \overleftarrow{\mp} 1 = 0$
[T230a]	$1 \overleftarrow{\pm} \theta = 1$	[T230b]	$0 \overleftarrow{\mp} \phi = 0$
[T231a]	$1 \overleftarrow{\pm} \phi = \theta$	[T231b]	$0 \overleftarrow{\mp} \theta = \phi$
[T232a]	$1 \overleftarrow{\pm} 1 = \theta$	[T232b]	$0 \overleftarrow{\mp} 0 = \phi$
[T233a]	$0 \underline{\times} 0 = \theta$	[T233b]	$1 \overline{\times} 1 = \phi$
[T234a]	$0 \underline{\times} \theta = 0$	[T234b]	$1 \overline{\times} \phi = 1$

TABLE XXIX (Continued)

[T235a]	$0 \underline{\times} \phi = 1$	[T235b]	$1 \overline{\times} \theta = 0$
[T236a]	$0 \underline{\times} 1 = \phi$	[T236b]	$1 \overline{\times} 0 = \theta$
[T237a]	$\theta \underline{\times} 0 = 0$	[T237b]	$\phi \overline{\times} 1 = 1$
[T238a]	$\theta \underline{\times} \theta = \theta$	[T238b]	$\phi \overline{\times} \phi = \phi$
[T239a]	$\theta \underline{\times} \phi = \phi$	[T239b]	$\phi \overline{\times} \theta = \theta$
[T240a]	$\theta \underline{\times} 1 = 1$	[T240b]	$\phi \overline{\times} 0 = 0$
[T241a]	$\phi \underline{\times} 0 = 1$	[T241b]	$\theta \overline{\times} 1 = 0$
[T242a]	$\phi \underline{\times} \theta = \phi$	[T242b]	$\theta \overline{\times} \phi = \theta$
[T243a]	$\phi \underline{\times} \phi = \theta$	[T243b]	$\theta \overline{\times} \theta = \phi$
[T244a]	$\phi \underline{\times} 1 = 0$	[T244b]	$\theta \overline{\times} 0 = 1$
[T245a]	$1 \underline{\times} 0 = \phi$	[T245b]	$0 \overline{\times} 1 = \theta$
[T246a]	$1 \underline{\times} \theta = 1$	[T246b]	$0 \overline{\times} \phi = 0$
[T247a]	$1 \underline{\times} \phi = 0$	[T247b]	$0 \overline{\times} \theta = 1$
[T248a]	$1 \underline{\times} 1 = \theta$	[T248b]	$0 \overline{\times} 0 = \phi$
[T249a]	$0 \underline{\|} 0 = 1$	[T249b]	$1 \overline{\|} 1 = 0$
[T250a]	$0 \underline{\|} \theta = \phi$	[T250b]	$1 \overline{\|} \phi = \theta$
[T251a]	$0 \underline{\|} \phi = 1$	[T251b]	$1 \overline{\|} \theta = 0$
[T252a]	$0 \underline{\|} 1 = \phi$	[T252b]	$1 \overline{\|} 0 = \theta$
[T253a]	$\theta \underline{\|} 0 = \phi$	[T253b]	$\phi \overline{\|} 1 = \theta$
[T254a]	$\theta \underline{\|} \theta = \phi$	[T254b]	$\phi \overline{\|} \phi = \theta$
[T255a]	$\theta \underline{\|} \phi = \phi$	[T255b]	$\phi \overline{\|} \theta = \theta$
[T256a]	$\theta \underline{\|} 1 = \phi$	[T256b]	$\phi \overline{\|} 0 = \theta$
[T257a]	$\phi \underline{\|} 0 = 1$	[T257b]	$\theta \overline{\|} 1 = 0$
[T258a]	$\phi \underline{\|} \theta = \phi$	[T258b]	$\theta \overline{\|} \phi = \theta$
[T259a]	$\phi \underline{\|} \phi = \theta$	[T259b]	$\theta \overline{\|} \theta = \phi$
[T260a]	$\phi \underline{\|} 1 = 0$	[T260b]	$\theta \overline{\|} 0 = 1$

TABLE XXIX (Continued)

[T261a] $1 \perp\!\!\!\perp 0 = \phi$	[T261b] $0 \top\!\!\!\top 1 = \theta$
[T262a] $1 \perp\!\!\!\perp \theta = \phi$	[T262b] $0 \top\!\!\!\top \phi = \theta$
[T263a] $1 \perp\!\!\!\perp \phi = 0$	[T263b] $0 \top\!\!\!\top \theta = 1$
[T264a] $1 \perp\!\!\!\perp 1 = 0$	[T264b] $0 \top\!\!\!\top 0 = 1$
[T265a] $x \perp y = y \perp x$	[T265b] $x \top y = y \top x$
[T266a] $x \mathbin{\perp\!\!\!\!\times} y = y \mathbin{\perp\!\!\!\!\times} x$	[T266b] $x \mathbin{\top\!\!\!\!\times} y = y \mathbin{\top\!\!\!\!\times} x$
[T267a] $x \perp\!\!\!\perp y = y \perp\!\!\!\perp x$	[T267b] $x \top\!\!\!\top y = y \top\!\!\!\top x$
[T268a] $\theta \perp x = \theta$	[T268b] $\phi \top x = \phi$
[T269a] $\phi \perp x = x$	[T269b] $\theta \top x = x$
[T270a] $\theta \mathbin{\perp\!\!\!\!\times} x = x$	[T270b] $\phi \mathbin{\top\!\!\!\!\times} x = x$
[T271a] $x \mathbin{\perp\!\!\!\!\times} \theta = x$	[T271b] $x \mathbin{\top\!\!\!\!\times} \phi = x$
[T272a] $\phi \mathbin{\perp\!\!\!\!\times} x = \bar{x}$	[T272b] $\theta \mathbin{\top\!\!\!\!\times} x = \bar{x}$
[T273a] $x \mathbin{\perp\!\!\!\!\times} \phi = \bar{x}$	[T273b] $x \mathbin{\top\!\!\!\!\times} \theta = \bar{x}$
[T274a] $\theta \mathbin{\perp\!\!\!\!\rightarrow} x = x$	[T274b] $\phi \mathbin{\top\!\!\!\!\rightarrow} x = x$
[T275a] $x \mathbin{\perp\!\!\!\!\rightarrow} \theta = \theta$	[T275b] $x \mathbin{\top\!\!\!\!\rightarrow} \phi = \phi$
[T276a] $\phi \mathbin{\perp\!\!\!\!\rightarrow} x = \theta$	[T276b] $\theta \mathbin{\top\!\!\!\!\rightarrow} x = \phi$
[T277a] $x \mathbin{\perp\!\!\!\!\rightarrow} \phi = \bar{x}$	[T277b] $x \mathbin{\top\!\!\!\!\rightarrow} \theta = \bar{x}$
[T278a] $\theta \mathbin{\leftarrow\!\!\!\!\perp} x = \theta$	[T278b] $\phi \mathbin{\leftarrow\!\!\!\!\top} x = \phi$
[T279a] $x \mathbin{\leftarrow\!\!\!\!\perp} \theta = x$	[T279b] $x \mathbin{\leftarrow\!\!\!\!\top} \phi = x$
[T280a] $\phi \mathbin{\leftarrow\!\!\!\!\perp} x = \bar{x}$	[T280b] $\theta \mathbin{\leftarrow\!\!\!\!\top} x = \bar{x}$
[T281a] $x \mathbin{\leftarrow\!\!\!\!\perp} \phi = \theta$	[T281b] $x \mathbin{\leftarrow\!\!\!\!\top} \theta = \phi$
[T282a] $\theta \perp\!\!\!\perp x = \phi$	[T282b] $\phi \top\!\!\!\top x = \theta$
[T283a] $x \perp\!\!\!\perp \theta = \phi$	[T283b] $x \top\!\!\!\top \phi = \theta$
[T284a] $\phi \perp\!\!\!\perp x = \bar{x}$	[T284b] $\theta \top\!\!\!\top x = \bar{x}$
[T285a] $x \perp\!\!\!\perp \phi = \bar{x}$	[T285b] $x \top\!\!\!\top \theta = \bar{x}$

TABLE XXIX (Continued)

[T286a] $x \perp \bar{x} = \theta$ [T286b] $x \top \bar{x} = \phi$

[T287a] $x \mathbin{\underline{\times}} x = \theta$ [T287b] $x \mathbin{\overline{\times}} x = \phi$

[T288a] $x \mathbin{\underline{\times}} \bar{x} = \phi$ [T288b] $x \mathbin{\overline{\times}} \bar{x} = \theta$

[T289a] $x \mathbin{\underline{>}} x = \theta$ [T289b] $x \mathbin{\overline{>}} x = \phi$

[T290a] $x \mathbin{\underline{>}} \bar{x} = \bar{x}$ [T290b] $x \mathbin{\overline{>}} \bar{x} = \bar{x}$

[T291a] $x \mathbin{\underline{<}} x = \theta$ [T291b] $x \mathbin{\overline{<}} x = \phi$

[T292a] $x \mathbin{\underline{<}} \bar{x} = x$ [T292b] $x \mathbin{\overline{<}} \bar{x} = x$

[T293a] $x \mathbin{\underline{\Vert}} x = \bar{x}$ [T293b] $x \mathbin{\overline{\Vert}} x = \bar{x}$

[T294a] $x \mathbin{\underline{\Vert}} \bar{x} = \phi$ [T294b] $x \mathbin{\overline{\Vert}} \bar{x} = \theta$

[T295a] $(x + y)(xy + \theta) = x \perp y$ [T295b] $xy + (x + y)\phi = x \top y$

[T296a] $xy\phi + (x + y)\theta = x \perp y$ [T296b] $(x + y + \theta)(xy + \phi) = x \top y$

[T297a] $(x + y + \phi)(xy + \theta) = x \perp y$
[T297b] $xy\theta + (x + y)\phi = x \top y$

[T298a] $(x\bar{y} + \bar{x}y)\phi + (xy + \bar{x}\bar{y})\theta = x \mathbin{\underline{\times}} y$
[T298b] $[(x + \bar{y})(\bar{x} + y) + \theta][(x + y)(\bar{x} + \bar{y}) + \phi] = x \mathbin{\overline{\times}} y$

[T299a] $[(x + \bar{y})(\bar{x} + y) + \phi][(x + y)(\bar{x} + \bar{y}) + \theta] = x \mathbin{\underline{\times}} y$
[T299b] $(x\bar{y} + \bar{x}y)\theta + (xy + \bar{x}\bar{y})\phi = x \mathbin{\overline{\times}} y$

[T300a] $(\bar{x} + y)(\bar{x}y + \theta) = x \mathbin{\underline{>}} y$ [T300b] $\bar{x}y + (\bar{x} + y)\phi = x \mathbin{\overline{>}} y$

[T301a] $\bar{x}y\phi + (\bar{x} + y)\theta = x \mathbin{\underline{>}} y$ [T301b] $(\bar{x} + y + \theta)(\bar{x}y + \phi) = x \mathbin{\overline{>}} y$

[T302a] $(\bar{x} + y + \phi)(\bar{x}y + \theta) = x \mathbin{\underline{>}} y$
[T302b] $\bar{x}y\theta + (\bar{x} + y)\phi = x \mathbin{\overline{>}} y$

[T303a] $(x + \bar{y})(x\bar{y} + \theta) = x \mathbin{\underline{<}} y$ [T303b] $x\bar{y} + (x + \bar{y})\phi = x \mathbin{\overline{<}} y$

[T304a] $x\bar{y}\phi + (x + \bar{y})\theta = x \mathbin{\underline{<}} y$ [T304b] $(x + \bar{y} + \theta)(x\bar{y} + \phi) = x \mathbin{\overline{<}} y$

[T305a] $(x + \bar{y} + \phi)(x\bar{y} + \theta) = x \mathbin{\underline{<}} y$
[T305b] $x\bar{y}\theta + (x + \bar{y})\phi = x \mathbin{\overline{<}} y$

[T306a] $\bar{x}\bar{y} + (\bar{x} + \bar{y})\phi = x \mathbin{\underline{\Vert}} y$ [T306b] $(\bar{x} + \bar{y})(\bar{x}\bar{y} + \theta) = x \mathbin{\overline{\Vert}} y$

[T307a] $\bar{x}\bar{y}\theta + (\bar{x} + \bar{y})\phi = x \mathbin{\underline{\Vert}} y$ [T307b] $(\bar{x} + \bar{y} + \phi)(\bar{x}\bar{y} + \theta) = x \mathbin{\overline{\Vert}} y$

TABLE XXIX (Continued)

[T308a] $(\bar{x} + \bar{y} + \theta)(\bar{x}\bar{y} + \phi) = x \mathbin{\underline{\|}} y$

[T308b] $\bar{x}\bar{y}\phi + (\bar{x} + \bar{y})\theta = x \mathbin{\overline{\|}} y$

[T309a] $(x \perp y)\theta = (x + y)\theta$ [T309b] $(x \top y) + \phi = xy + \phi$

[T310a] $(x \perp y)\phi = xy\phi$ [T310b] $(x \top y) + \theta = x + y + \theta$

[T311a] $(x \perp y) + \theta = xy + \theta$ [T311b] $(x \top y)\phi = (x + y)\phi$

[T312a] $(x \perp y) + \phi = x + y + \phi$ [T312b] $(x \top y)\theta = xy\theta$

[T313a] $(x \mathbin{\underline{\times}} y)\theta = (x + \bar{y})(\bar{x} + y)\theta = (x \mathbin{\bar{\times}} y)\theta$

[T313b] $(x \mathbin{\overline{\times}} y) + \phi = x\bar{y} + \bar{x}y + \phi = (x \times y) + \phi$

[T314a] $(x \mathbin{\underline{\times}} y)\phi = (x + y \ (\bar{x} + \bar{y})\phi = (x \times y)\phi$

[T314b] $(x \mathbin{\overline{\times}} y) + \theta = xy + \bar{x}\bar{y} + \theta = (x \mathbin{\bar{\times}} y) + \theta$

[T315a] $(x \mathbin{\underline{\times}} y) + \theta = (x + y)(\bar{x} + \bar{y}) + \theta = (x \times y) + \theta$

[T315b] $(x \mathbin{\overline{\times}} y)\phi = (xy + \bar{x}\bar{y})\phi = (x \mathbin{\bar{\times}} y)\phi$

[T316a] $(x \mathbin{\underline{\times}} y) + \phi = (x + \bar{y})(\bar{x} + y) + \phi = (x \mathbin{\bar{\times}} y) + \phi$

[T316b] $(x \mathbin{\overline{\times}} y)\theta = (x\bar{y} + \bar{x}y)\theta = (x \times y)\theta$

[T317a] $(x \mathbin{\underline{\rightarrow}} y)\theta = (\bar{x} + y)\theta = (x \underset{1}{\rightarrow} y)\theta$

[T317b] $(x \mathbin{\overline{\rightarrow}} y) + \phi = \bar{x}y + \phi = (x \underset{0}{\rightarrow} y) + \phi$

[T318a] $(x \mathbin{\underline{\rightarrow}} y)\phi = \bar{x}y\phi = (x \underset{0}{\rightarrow} y)\phi$

[T318b] $(x \mathbin{\overline{\rightarrow}} y) + \theta = \bar{x} + y + \theta = (x \underset{1}{\rightarrow} y) + \theta$

[T319a] $(x \mathbin{\underline{\rightarrow}} y) + \theta = \bar{x}y + \theta = (x \underset{0}{\rightarrow} y) + \theta$

[T319b] $(x \mathbin{\overline{\rightarrow}} y)\phi = (\bar{x} + y)\phi = (x \underset{1}{\rightarrow} y)\phi$

[T320a] $(x \mathbin{\underline{\rightarrow}} y) + \phi = \bar{x} + y + \phi = (x \underset{1}{\rightarrow} y) + \phi$

[T320b] $(x \mathbin{\overline{\rightarrow}} y)\theta = \bar{x}y\theta = (x \underset{0}{\rightarrow} y)\theta$

[T321a] $(x \mathbin{\underline{\leftarrow}} y)\theta = (x + \bar{y})\theta = (x \underset{1}{\leftarrow} y)\theta$

[T321b] $(x \mathbin{\overline{\leftarrow}} y) + \phi = x\bar{y} + \phi = (x \underset{0}{\leftarrow} y) + \phi$

[T322a] $(x \mathbin{\underline{\leftarrow}} y)\phi = x\bar{y}\phi = (x \underset{0}{\leftarrow} y)\phi$

[T322b] $(x \mathbin{\overline{\leftarrow}} y) + \theta = x + \bar{y} + \theta = (x \underset{1}{\leftarrow} y) + \theta$

[T323a] $(x \mathbin{\underline{\leftarrow}} y) + \theta = x\bar{y} + \theta = (x \underset{0}{\leftarrow} y) + \theta$

[T323b] $(x \mathbin{\overline{\leftarrow}} y)\phi = (x + \bar{y})\phi = (x \underset{1}{\leftarrow} y)\phi$

TABLE XXIX (Continued)

[T324a] $(x \overleftarrow{\perp} y) + \phi = x + y + \phi = (x \underset{1}{\leftarrow} y) + \phi$

[T324b] $(x \overleftarrow{\top} y)\theta = x\bar{y}\theta = (x \underset{0}{\leftarrow} y)\theta$

[T325a] $(x \perp y)x = (y + \theta)x$

[T325b] $(x \top y) + x = y\phi + x$

[T326a] $(x \perp y) + x = y\phi + x$

[T326b] $(x \top y)x = (y + \phi)x$

[T327a] $(x \underline{\times} y)x = (\bar{y}\phi + y\theta)x$

[T327b] $(x \overline{\times} y) + x = (\bar{y} + \theta)(y + \phi) + x$

[T328a] $(x \underline{\times} y) + x = (\bar{y} + \phi)(y + \theta) + x$

[T328b] $(x \overline{\times} y)x = (\bar{y}\theta + y\phi)x$

[T329a] $(x \underline{\times} y)y = (\bar{x}\phi + x\theta)y$

[T329b] $(x \overline{\times} y) + y = (\bar{x} + \theta)(x + \phi) + y$

[T330a] $(x \underline{\times} y) + y = (\bar{x} + \phi)(x + \theta) + y$

[T330b] $(x \overline{\times} y)y = (\bar{x}\theta + x\phi)y$

[T331a] $(x \overrightarrow{\perp} y)x = xy\theta$

[T331b] $(x \overrightarrow{\top} y) + x = x + y + \phi$

[T332a] $(x \overrightarrow{\perp} y) + x = x + y + \theta$

[T332b] $(x \overrightarrow{\top} y)x = xy\phi$

[T333a] $(x \overrightarrow{\perp} y)y = (\bar{x} + \theta)y$

[T333b] $(x \overrightarrow{\top} y) + y = \bar{x}\phi + y$

[T334a] $(x \overrightarrow{\perp} y) + y = \bar{x}\theta + y$

[T334b] $(x \overrightarrow{\top} y)y = (\bar{x} + \phi)y$

[T335a] $(x \overleftarrow{\perp} y)x = (\bar{y} + \theta)x$

[T335b] $(x \overleftarrow{\top} y) + x = \bar{y}\phi + x$

[T336a] $(x \overleftarrow{\perp} y) + x = \bar{y}\phi + x$

[T336b] $(x \overleftarrow{\top} y)x = (\bar{y} + \phi)x$

[T337a] $(x \overleftarrow{\perp} y)y = xy\theta$

[T337b] $(x \overleftarrow{\top} y) + y = x + y + \phi$

[T338a] $(x \overleftarrow{\perp} y) + y = x + y + \theta$

[T338b] $(x \overleftarrow{\top} y)y = xy\phi$

[T339a] $(x \underline{\|} y)x = x\bar{y}\phi$

[T339b] $(x \overline{\|} y) + x = x + \bar{y} + \theta$

[T340a] $(x \underline{\|} y) + x = y + \bar{y} + \phi$

[T340b] $(x \overline{\|} y)x = x\bar{y}\theta$

[T341a] $(x \underline{\|} y)y = \bar{x}y\phi$

[T341b] $(x \overline{\|} y) + y = \bar{x} + y + \theta$

[T342a] $(x \underline{\|} y) + y = \bar{x} + y + \phi$

[T342b] $(x \overline{\|} y)y = \bar{x}y\theta$

[T343a] $(x \perp y)(x \| y) = 0$

[T343b] $(x \top y) + (x \mid y) = 1$

[T344a] $(x \perp y)(x \mid y) = (x \times y)\theta$

[T344b] $(x \top y) + (x \| y) = (x \bar{\times} y) + \phi$

TABLE XXIX (Continued)

[T345a] $(x \perp y) + (x \mid y) = 1$ [T345b] $(x \top y)(x \parallel y) = 0$

[T346a] $(x \perp y) + (x \parallel y) = (x \bar{\times} y) + \theta$

[T346b] $(x \top y)(x \mid y) = (x \times y)\phi$

[T347a] $(x \times y)(x \mathrel{\times\!\!\!\!\!\perp} y) = (x \times y)\phi$

[T347b] $(x \bar{\times} y) + (x \mathrel{\times\!\!\!\!\!\top} y) = (x \bar{\times} y) + \theta$

[T348a] $(x \times y) + (x \mathrel{\times\!\!\!\!\!\top} y) = (x \times y) + \phi$

[T348b] $(x \bar{\times} y)(x \mathrel{\times\!\!\!\!\!\perp} y) = (x \bar{\times} y)\theta$

[T349a] $(x \mathrel{\perp\!\!\!\!\rightarrow} y)(x \underset{0}{\leftarrow} y) = 0$ [T349b] $(x \mathrel{\top\!\!\!\!\rightarrow} y) + (x \underset{1}{\leftarrow} y) = 1$

[T350a] $(x \mathrel{\perp\!\!\!\!\leftarrow} y)(x \underset{0}{\rightarrow} y) = 0$ [T350b] $(x \mathrel{\top\!\!\!\!\leftarrow} y) + (x \underset{1}{\rightarrow} y) = 1$

[T351a] $(x \mid y)(x \underline{\parallel} y) = x \underline{\parallel} y$ [T351b] $(x \parallel y) + (x \overline{\parallel} y) = x \overline{\parallel} y$

[T352a] $(x \mid y) + (x \overline{\parallel} y) = x \mid y$ [T352b] $(x \parallel y)(x \underline{\parallel} y) = x \parallel y$

CHAPTER V

THE APPLICATION OF BOOLEAN ALGEBRA OF ORDER TWO TO SWITCHING CIRCUITS

Basic Considerations

Two-valued Boolean algebra is a natural mathematical basis for switching devices. A one-to-one correspondence exists between on-off electrical devices and the true-false propositions of mathematical logic. Claude E. Shannon first applied Boolean algebra to electrical circuits in a thesis for the Master of Science degree from the Massachusetts Institute of Technology. An abstract of this thesis was presented as a paper at a meeting of the American Institute of Electrical Engineers, June 20-24, 1938, and later was published by that organization[1].

Two different viewpoints are possible with on-off devices. One can consider the circuit from the standpoint of transmission; either it conducts perfectly or it does not conduct at all. The alternate approach, used by Shannon, is to think in terms of "hindrance," where hindrance equals zero if a switch is closed and equals one if the switch is open. This notation has been followed by some colleagues[2] of Shannon at Bell Telephone Laboratories. The current trend, however,

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[1]Claude E. Shannon, "A Symbolic Analysis or Relay and Switching Circuits," AIEE Transactions, LVII (1938), 713-723.

[2]William Keister, A. E. Ritchie, and S. H. Washburn, The Design of Switching Circuits (New York, 1951).

seems to be to consider circuits in terms of transmission. Bennett,[3] Caldwell,[4] Huffman,[5] Richards,[6] and Serrell,[7] to name a few authors of recent publications, all use 0 for an open circuit and 1 for a closed circuit. This symbolism has also been used in a number of articles written by people at Bell Telephone Laboratories, including Hohn and Schissler,[8] Karnaugh,[9] and Washburn.[10] In this thesis, the idea of transmission will be used, with 0 representing an open circuit or a de-energized relay, and with 1 symbolizing a closed circuit or an energized relay.

As indicated on pages 33 and 34, the algebra of logic has two fundamental binary operations, the meet and the join. The ∘ symbol is used almost exclusively in switching circuits for the meet, and the + symbol is used for the join. In electrical circuits, two elements may be connected either in series or in parallel. The choice of 0 for an open circuit and 1 for a closed circuit dictates that the • symbol

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[3]W. S. Bennett, "Minimizing and Mapping Sequential Circuits," AIEE Transactions, LXXIV, pt. I (1955), 443-447.

[4]S. H. Caldwell, "The Recognition and Identification of Symmetric Switching Functions," AIEE Transactions, LXXIII, pt. I (1954), 142-147.

[5]D. A. Huffman, "The Synthesis of Sequential Switching Circuits, Part I," Journal of the Franklin Institute, CCLVII (1954), 161-190.

[6]R. K. Richards, Arithmetic Operations in Digital Computers (New York, 1955).

[7]Robert Serrell, "Elements of Boolean Algebra for the Study of Information-Handling Systems," Proceedings of the IRE, XLI (1953), 1366-1380.

[8]Franz E. Hohn and L. Robert Schissler, "Boolean Matrices and the Design of Combinational Relay Switching Circuits," The Bell System Technical Journal, XXIV (1955), 177-202.

[9]M. Karnaugh, "The Map Method for Systhesis of Combinational Logic Circuits," AIEE Transactions, LXXII, pt. I (1953), 593-599.

[10]S. H. Washburn, "An Application of Boolean Algebra to the Design of Electronic Switching Circuits," AIEE Transactions, LXII, pt. I (1953), 380-388.

must be used for a series circuit and the + symbol for a parallel circuit. The postulates of Table XIII can now be interpreted as indicated in Table XXX.

TABLE XXX

CIRCUIT INTERPRETATIONS OF TABLE XIII.

[P1] $x = 0$ or $x = 1$ for all $x \in B$

Any two-valued switching circuit must be either open or closed at any given instant.

[P2a] $\bar{x} = 1$ if $x = 0$

If a given circuit is open, the negative of the circuit is closed.

[P2b] $\bar{x} = 0$ if $x = 1$

If a given circuit is closed, the negative of the circuit is open.

[P3a] $x + 0 = 0 + x = x$

A circuit in parallel with an open circuit in either order, is the same as the original circuit alone.

[P3b] $x \cdot 1 = 1 \cdot x = x$

A circuit in series with a closed circuit, in either order, is the same as the original circuit alone.

[P4a] $x + 1 = 1 + x = 1$

A circuit in parallel with a closed circuit, in either order, is the same as a closed circuit, regardless of the state of transmission of the original circuit.

[P4b] $x \cdot 0 = 0 \cdot x = 0$

A circuit in series with an open circuit, in either order, is the same as an open circuit, regardless of the state of transmission of the original circuit.

Circuit Schematic Diagrams

Various ways have been used for representing switching circuits schematically. Shannon[11] uses the symbolism of Figure 7(a) to indicate the circuit X_{ab} from a to b, but does not regularly use the subscripts because it is assumed that $X_{ab} = X_{ba}$. Figure 7(b) is his schematic for X and Y in series, and Figure 7(c) represents X and Y in parallel. Keister, Ritchie, and Washburn[12] indicate a relay with "make" contacts that are closed when the relay is energized by a schematic like that of Figure 7(d). Figure 7(e) illustrates a relay with "break" contacts which open when the relay is energized. Karnaugh[13] uses the simplified schematic of Figure 7(f) for make contacts and that of Figure 7(g) for break contacts. Washburn[14] uses the symbolism of Figure 7(h), (i), and (j) for "and", "or", and "not" circuitry.

Following the scheme used by Shannon, but without the circles indicating terminals, Figure 8 shows some equivalent circuits based on selected theorems from Table XIV. In some instances, the right-hand circuit is obviously simpler; in others, the same numbers of contacts are required, but the circuits are arranged differently.

Circuit diagrams for the ten different binary relations for Boolean algebra of order two are shown in Figure 9. Because all operations can be expressed in terms of $\circ$, +, and negation, it seems there is little to be gained by using the other binary symbols. No known author has employed them in the study of switching circuits. Use of

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[11]Shannon, p. 714.
[12]Keister, Ritchie, and Washburn, pp. 12-14.
[13]Karnaugh, p. 596.
[14]Washburn, p. 381.

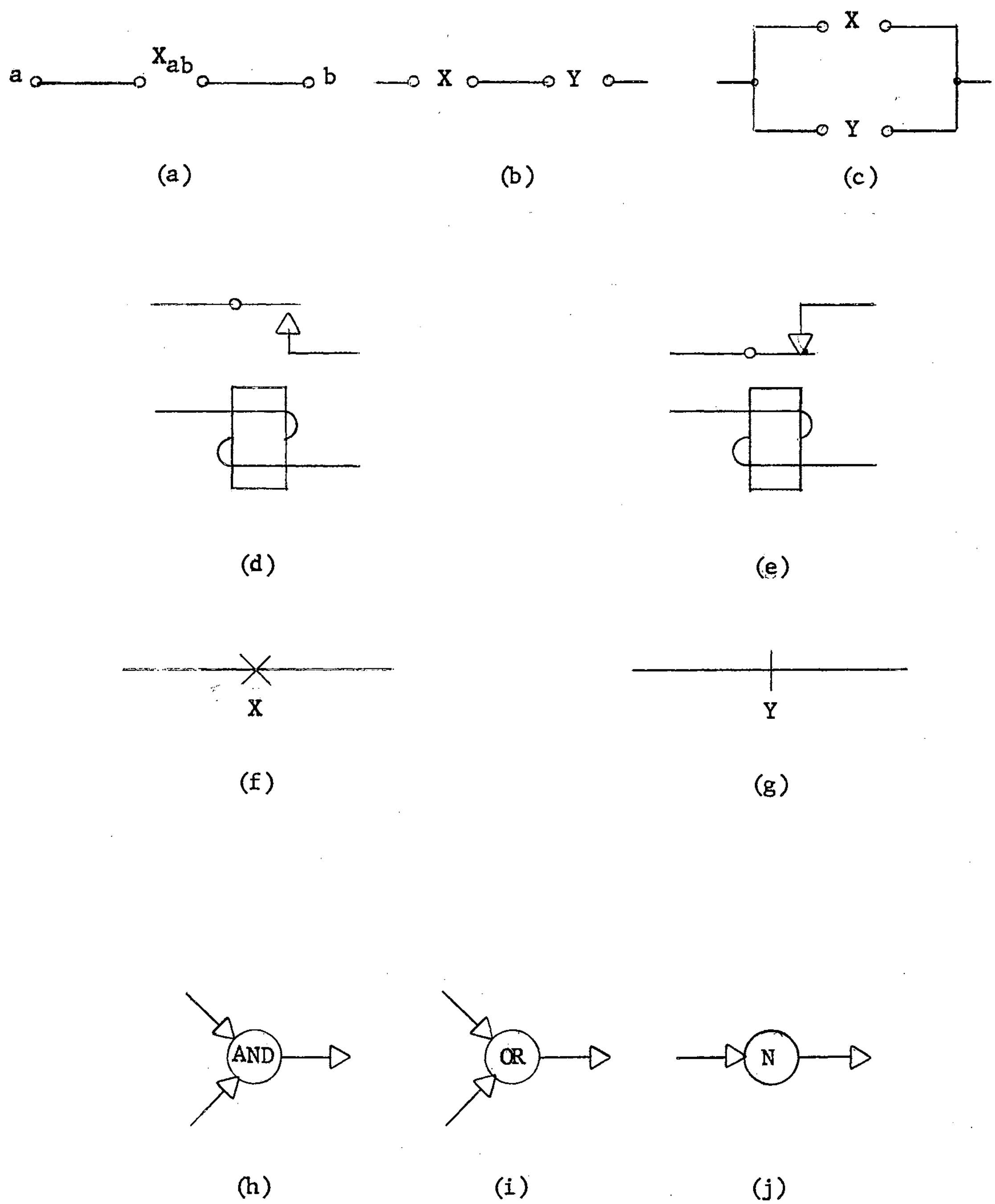
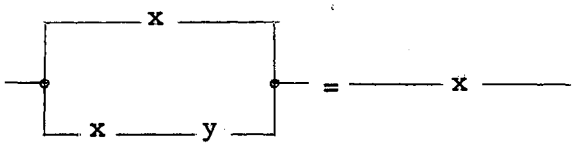


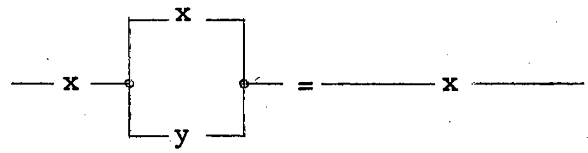
Fig. 7 Schematic Symbols for Switching Circuits.

[T13a] $x + (x \circ y) = x$



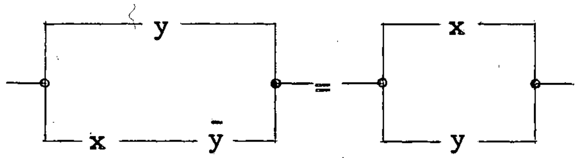
(a)

[T13b] $x \circ (x + y) = x$



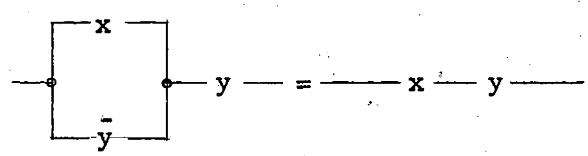
(b)

[T14a] $(x \circ \bar{y}) + y = x + y$



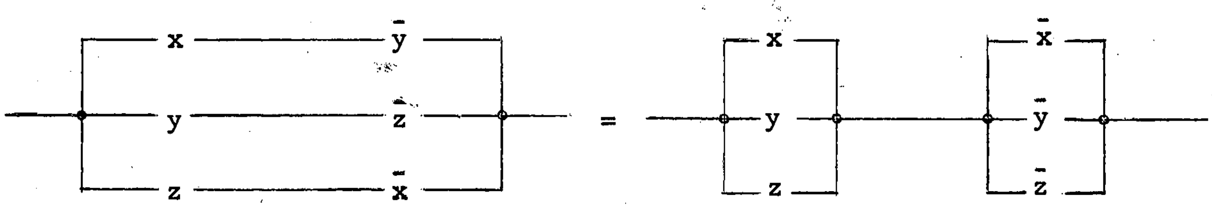
(c)

[T14b] $(x + \bar{y}) \circ y = x \circ y$



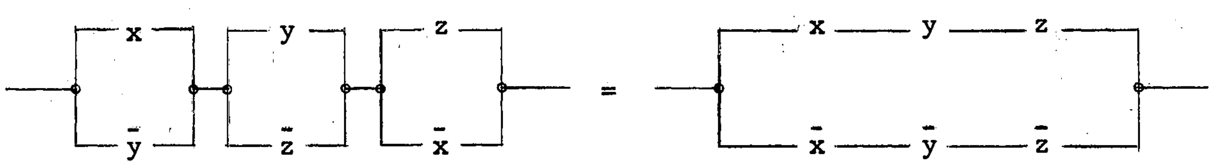
(d)

[T18a] $(x \circ \bar{y}) + (y \circ \bar{z}) + (z \cdot \bar{x}) = (x + y + z) \circ (\bar{x} + \bar{y} + \bar{z})$



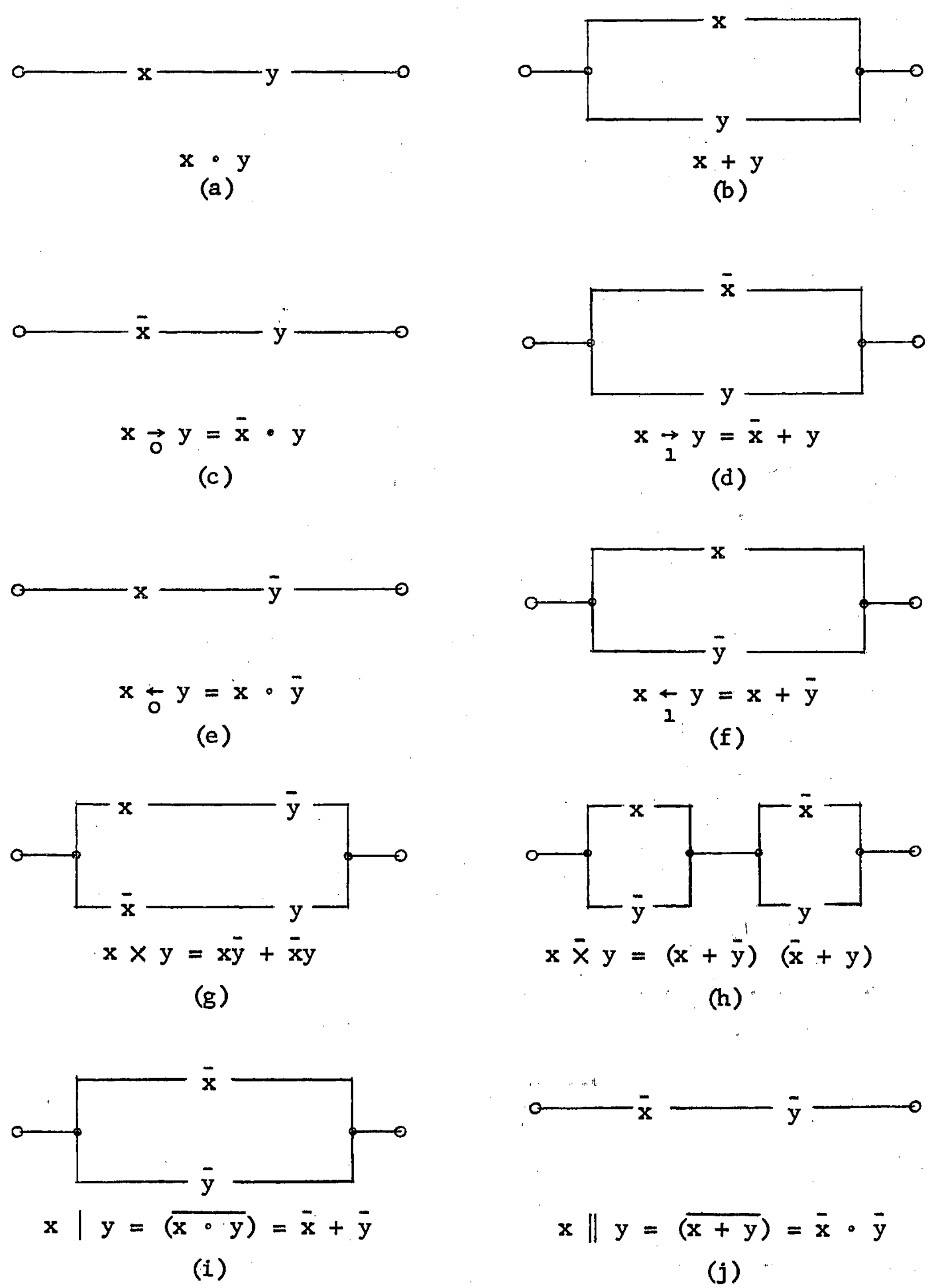
(e)

[T18b] $(x + \bar{y}) \circ (y + \bar{z}) \circ (z + \bar{x}) = (x \circ y \circ z) + (\bar{x} \cdot \bar{y} \circ \bar{z})$



(f)

Fig. 8. Circuit Diagrams for Some Theorems of Table XIV.



(a) $x \circ y$

(b) $x + y$

(c) $x \underset{0}{\rightarrow} y = \bar{x} \bullet y$

(d) $x \underset{1}{\rightarrow} y = \bar{x} + y$

(e) $x \underset{0}{\leftarrow} y = x \circ \bar{y}$

(f) $x \underset{1}{\leftarrow} y = x + \bar{y}$

(g) $x \times y = x\bar{y} + \bar{x}y$

(h) $x \,\bar{\times}\, y = (x + \bar{y})\ (\bar{x} + y)$

(i) $x \mid y = \overline{(x \circ y)} = \bar{x} + \bar{y}$

(j) $x \parallel y = \overline{(x + y)} = \bar{x} \circ \bar{y}$

Fig. 9 Circuit Diagrams for Ten Binary Relations.

the $\times$ operation and its negative $\bar{\times}$ does, however, offer considerable brevity. Moreover, these two circuits are frequently encountered in practice. $x \bar{\times} y$, for example, is a circuit which is 1 when x is 1 and y is 1 or when x is 0 and y is 0, but not when both are 1 or both are 0. This is the situation desired in control of a light from two different locations, where 1 may represent a toggle switch thrown in the "up" position and 0 indicates that it is in the "down" position. The same effect will be accomplished by using the $\times$ connection, where the light will be on if one switch is up and the other is down but not when both are up or both are down. The $\times$ relation is the "exclusive either -or"[15] of mathematical logic. This relation is encountered in binary addition for digital computers, where the summation output should be 1 if either of the two inputs is 1, but not if both inputs are 1. Huffman[16] uses the $\oplus$ symbol for this "cyclic addition", but this symbol has been used on page 10 of this thesis as a general binary operational sign, so its use for the "exclusive or" will be avoided.

No attempt has been made in this chapter to cover thoroughly the field of two-valued switching circuits, since books and periodicals have handled the subject adequately. This thesis has developed the mathematics of Boolean algebra of order two from an approach that is somewhat different from that used by the average electrical engineer. Such an approach is necessary for an understanding of the mathematical development of the Boolean algebra of order four. This chapter serves to illustrate the application of two-valued Boolean algebra to ordinary switching circuits, so that the reader will be

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[15] Paul Rosenbloom, The Elements of Mathematical Logic (New York, 1950).

[16] Huffman, p. 163.

able to understand more readily the application of four-valued Boolean algebra in the following chapter.

CHAPTER VI

THE APPLICATION OF BOOLEAN ALGEBRA OF ORDER FOUR TO SWITCHING CIRCUITS

Unilateral Devices

As indicated in the preceding chapter, it is possible to have between two points, p and q, a circuit that can be considered to be a perfect conductor. To such a circuit, the number 1 can be assigned. There may be between p and q a circuit that can be considered a perfect insulator. It would be given a value of 0. Of course, it is possible to have between p and q a resistance of almost any ohmic value, but such resistances are not considered to enter into switching circuits. There is, however, a type of circuit that may be included in the switching-circuit field, that is, a circuit which is, to all practical purposes, a perfect conductor in one direction and a perfect insulator in the other direction. Current may, for example, flow readily from q to p but not from p to q. Thermionic diodes are conductors of this type. If the anode of such a diode is negative with respect to its cathode, no appreciable conduction takes place. With the opposite polarity, the diode conducts relatively well. Actually some leakage current may flow when the anode is negative and the diode will show a few ohms of resistance when the anode is positive, but in relation to other circuit components it may usually be thought of as a perfect diode. The recently developed silicon and germanium diodes have similar characteristics.

The ordinary mathematics of switching circuits will not handle such devices. Conductance from p to q would be 0, but conductance from q to p would be 1. The circuit may be considered as an ordered couple (0,1), where the first element represents conductance from p to q and the second, the conductance from q to p. Figure 10(c) shows the conventional symbol for a diode, connected so that it will readily conduct positive current from q to p but not from p to q. This might be visualized as shown in Figure 10(d) where two directions are indicated, the top lead for current from p to q, the bottom for current from q to p. The diode so connected could be assigned a value of θ, the symbol assigned to the couple (0,1) on page 51. If connections to the diode were reversed, as shown in Figure 10(e), $\phi = (1,0)$ would represent the conductance. An open circuit would be indicated by $0 = (0,0)$ and a closed circuit by $1 = (1,1)$.

<u>Reference Direction</u>

It is obvious that to express the circuit element shown in Figure 10(c), for example, a reference direction needs to be established. Since it is customary in electronic circuitry to have input terminals at the left and output terminals at the right, a simple x can be used for the conductance of Figure 11(a), implying a left-to-right reference direction. Thus, in Figure 11(b) the conductance is labeled as $\theta = (0,1)$ implying that left-to-right conductance is 0 and right-to-left is 1. If, then, in going from left to right through a network, one first encounters the perpendicular line of a diode, it is labeled θ. If the direction from left to right is with the arrow, conductance is labeled ϕ. In cases where a left-to-right direction is not clear,

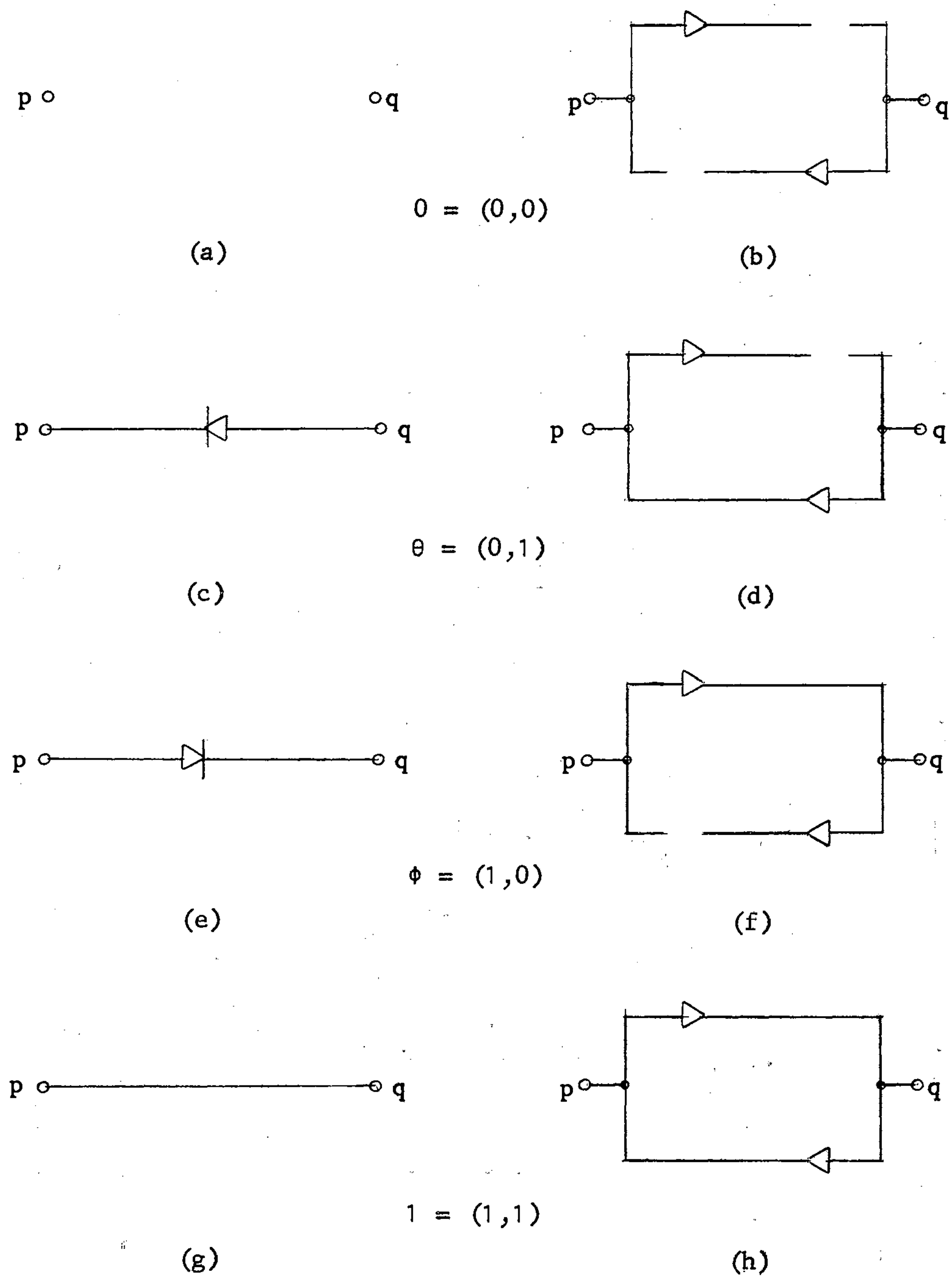

Fig. 10. Possible Circuits between p and q.

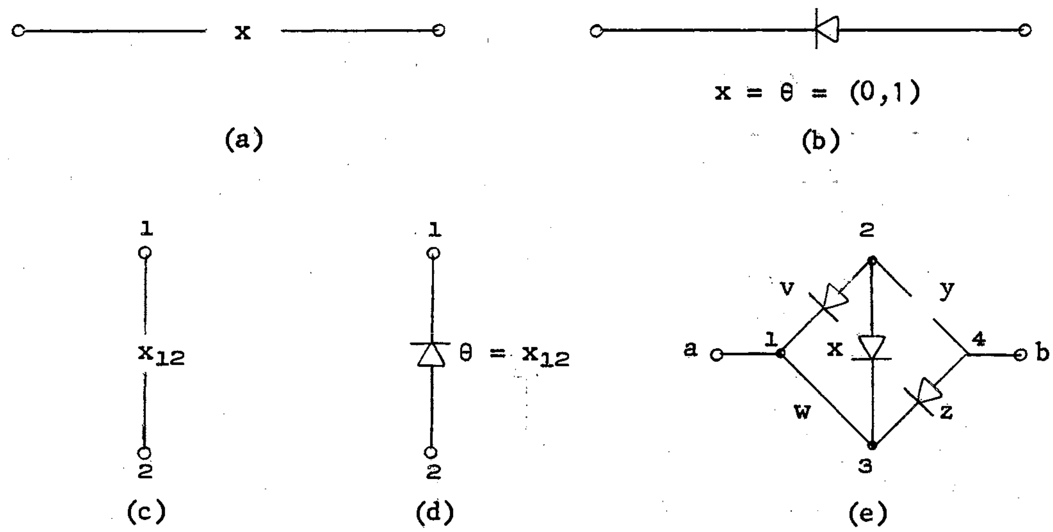

Fig. 11. Illustrations of Reference Directions.

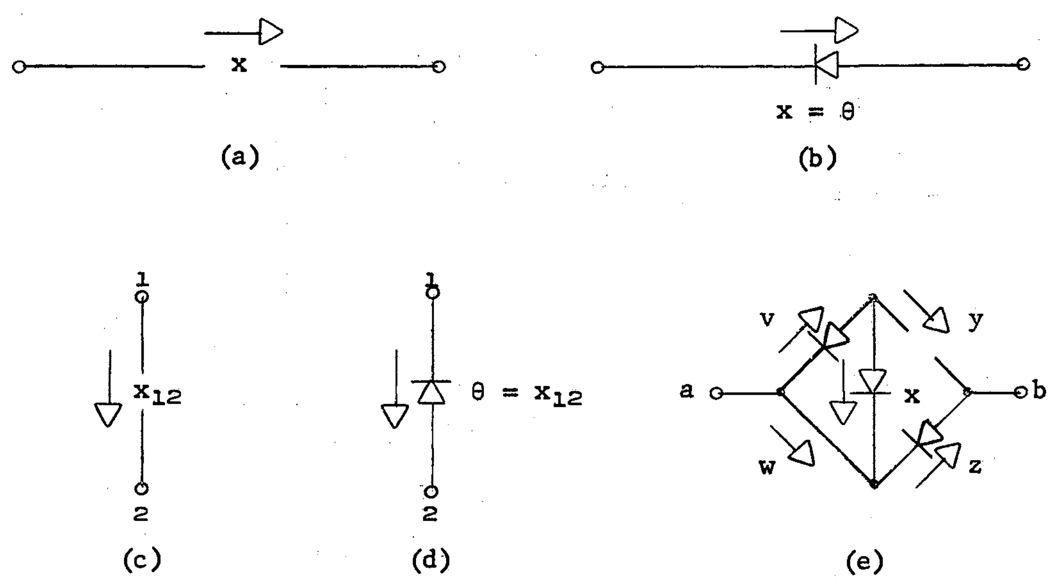

Fig. 12. Alternate Scheme for Indicating Reference Direction.

points at the two ends of each diode should be numbered so that the direction from the lower number to the higher number can be used as a reference direction. This is indicated in Figure 11(c) and (d). For the bridge circuit of Figure 11(e), by this lower-to-higher scheme, $v = \theta$, $w = 1$, $x = \phi$, $y = 0$, and $z = \theta$. An alternate scheme for indicating reference direction is to draw an arrow beside each circuit element, as shown in Figure 12.

In tracing through possible paths from input to output, it is sometimes necessary to go in a direction opposite to the reference direction. Under those circumstances, an underscore will be placed below the letter for the circuit element. Thus, for the bridge of Figure 11(e) or Figure 12(e), the conductance from a to b $= C_{ab} = vy + wz + vxz + w\underline{x}y$. In terms of circuit values, $C_{ab} = \theta \cdot 0 + 1 \cdot \theta + \theta \cdot \phi \cdot \theta + 1 \cdot \theta \cdot 0$. It will be observed that the underscore operation does not change the values of 0 and 1 but does interchange θ and ϕ. $\underline{\ \ } = \begin{pmatrix} 0 & \theta & \phi & 1 \\ 0 & \phi & \theta & 1 \end{pmatrix}$.

By reference to Table XXII (a) and (b) or to Table XXVII (a) and (p), the value of the conductance C_{ab} can be simplified. $C_{ab} = \theta \cdot 0 + 1 \cdot \theta + \theta \cdot \phi \cdot \theta + 1 \cdot \theta \cdot 0 = 0 + \theta + 0 + 0 = \theta$. The bridge, then, is equivalent to a single diode between a and b, with a connected to the cathode, indicated by the perpendicular line.

<u>Schematic Diagrams of Binary Relations for B^2</u>

Circuit diagrams for the twenty binary relations of Table XXVII are shown in Figures 13 and 14. Those shown on the top and bottom lines of both figures are also applicable to B, since neither θ nor ϕ is included as a definite circuit element. These ten are identical to those for B shown in Figure 9. The remaining ten are reasonably complex circuits. Therefore, if such connections are frequently encoun-

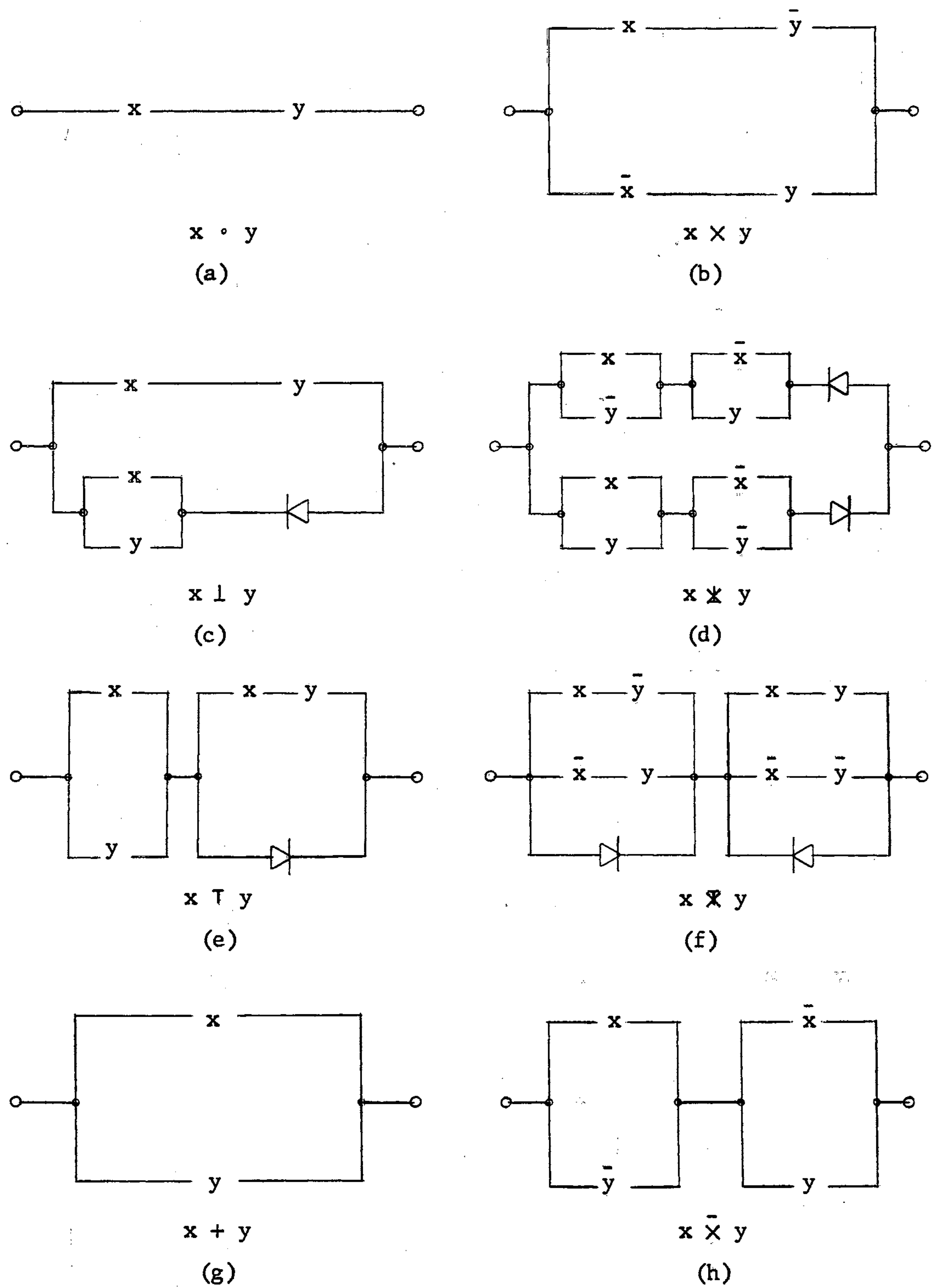

Fig. 13. Circuit Diagrams for Eight Binary Relations for B^2.

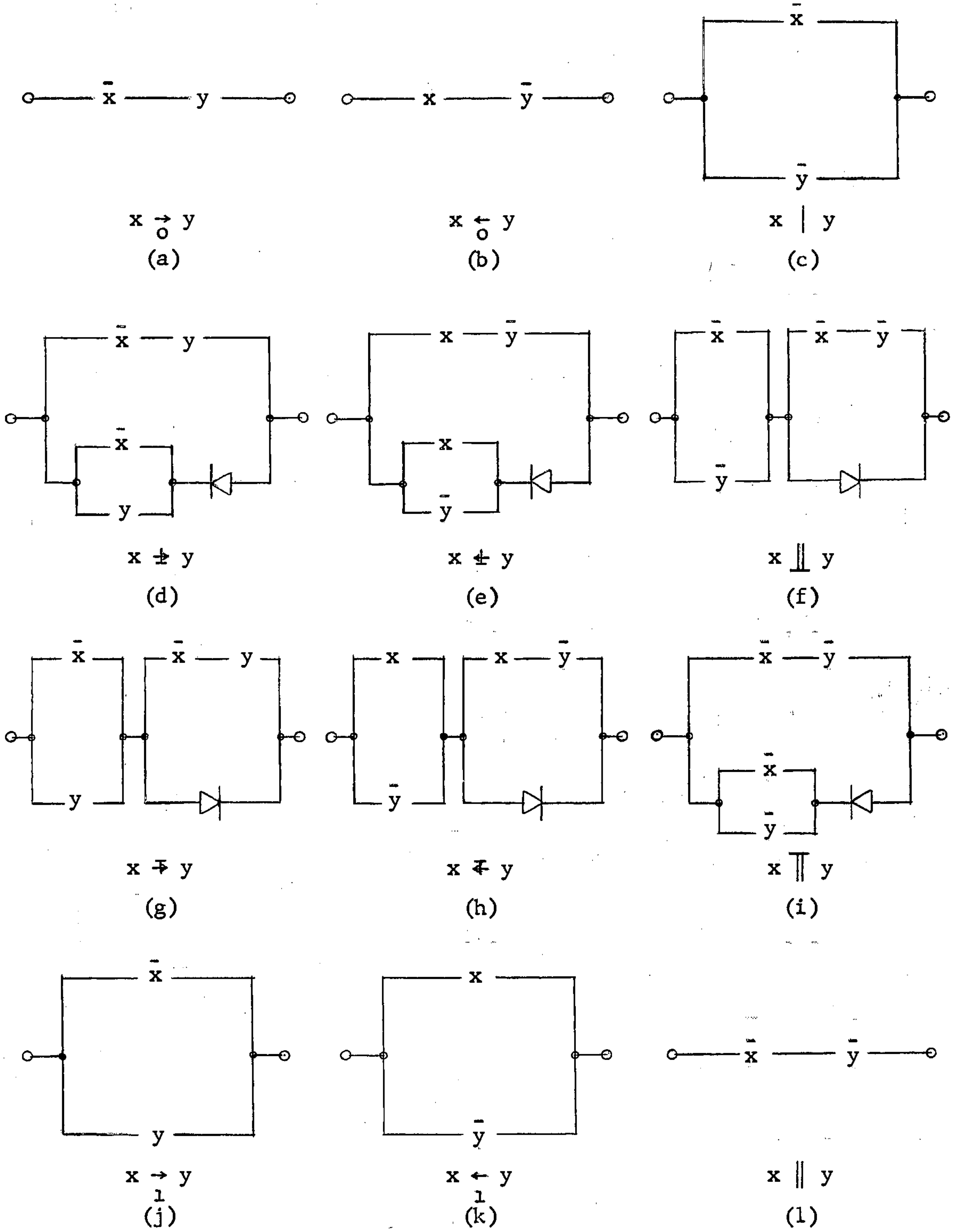

Fig. 14. Circuit Diagrams for Twelve Other Binary Relations for B^2

tered, the shorthand symbolism would be well worth adopting. If such circuits are rarely encountered, little would be gained by the use of symbols other than $\cdot$ and $+$. The use of unidirectional devices in switching circuits has not as yet been thoroughly exploited. It is hoped that the development of this four-valued logic will facilitate their use. Possibly then such relations as $x \curlywedge y$ will prove to be more important than is presently apparent.

Representation of Voltage

The four elements of B^2 can be used to represent different kinds of voltages. As with conductance, various magnitudes are not considered. If a voltage is to be used to operate a relay or a lamp, for example, either there is enough voltage or not enough for satisfactory operation. Boolean algebra can, however, distinguish a positive voltage from a negative one.

As with conductance, it is necessary first to establish a reference direction, most conveniently done by means of an arrow. The voltage then can be considered as a couple, where the first element of the couple is 1 if the voltage will tend to send current in the direction of the arrow and the second element is 1 if the voltage will tend to send current in a direction opposite to that of the arrow. In each case, the value of the element is 0 if it is not 1. Figure 15 illustrates the notation. As is customary, the battery symbol designates a d-c source, with a short line for the negative terminal and a long line for the positive one. Since $V = (1,1) = 1$ must represent a voltage that will tend to send current in either direction, and because equal currents simultaneously flowing in both directions would produce a net effect of zero current, the only physically realizable voltage that is appropriate must be alternating.

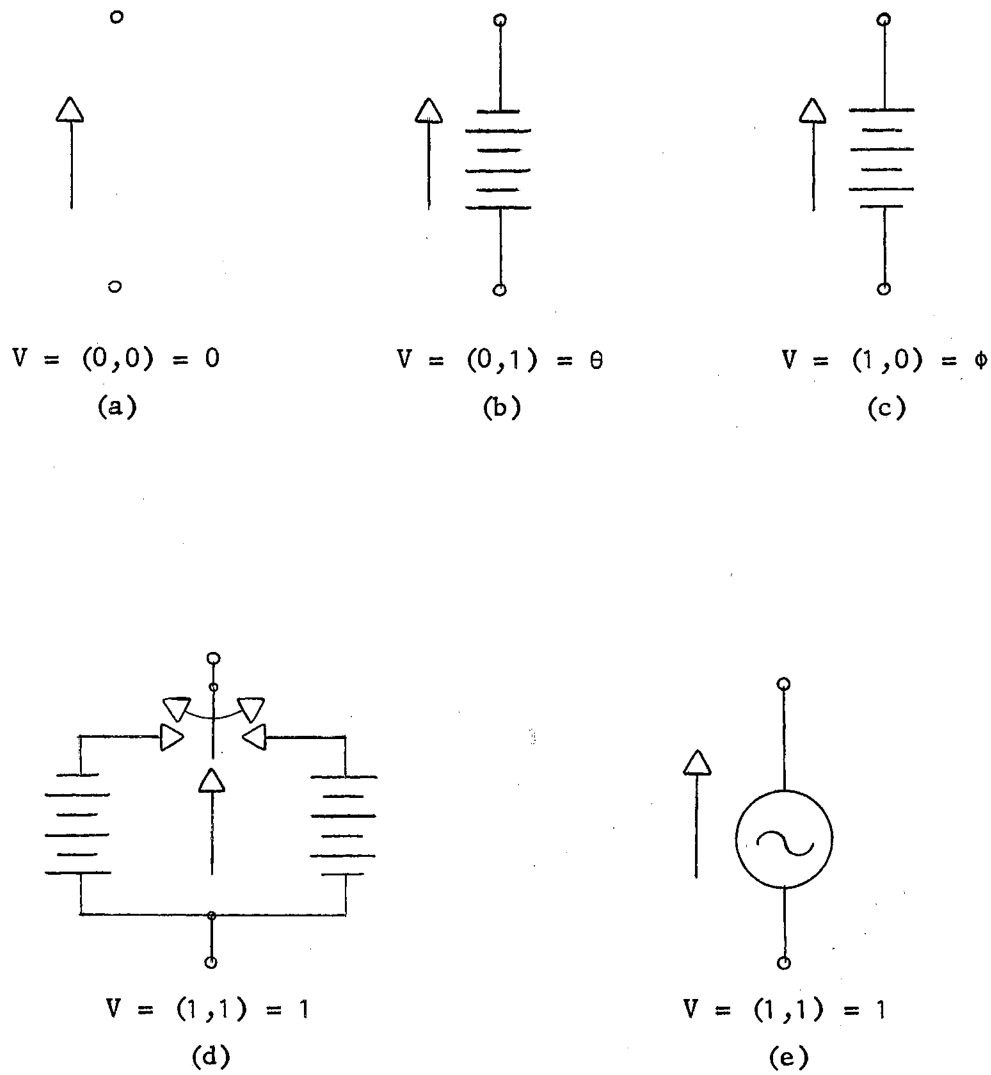

Fig. 15. Symbols for Different Kinds of Voltage.

One representation would be that of Figure 15(d), where the switch at the top swings back and forth, producing alternately positive and negative voltage at the top terminal with respect to the bottom one. The other possibility is shown in Figure 15(e), that is, a simple a-c generator. For either of these two situations, it must be realized that at any one instant the current will tend to flow in only one direction. It is assumed that the resultant average current or the rms current will be the important aspect, not the instantaneous value.

Voltage with Series Conductance

As far as output terminals are concerned, any voltage V in series with conductance C can be treated as a simple source S equal to the product $V \cdot C$, as illustrated in Figure 16. The reference directions for V and C must be the same. Obviously, if either V or C is zero, S will be zero. If $V = 1$, $S = C$, and if $C = 1$, $S = V$.

If a complex two-terminal network can be simplified to an equivalent two-terminal network, the latter can be used in conjunction with V to evaluate a source. For example, Figure 17(a) shows voltage equal to 1 in series with the bridge circuit of Figure 12(e). Since the bridge circuit has a conductance of θ, $V \cdot C = S = \theta$, so the source can be considered as a battery with the polarity as indicated. A different bridge network is illustrated in Figure 17(b). The possible paths through the bridge yield the expression $\theta\phi + \phi\theta + \theta x\theta + \phi\underline{x} = 0 + 0 + x\theta + \underline{x}\phi = x\theta + \underline{x}\phi$. If x is 0 or ϕ, $S = 0$. If x is θ or 1, $S = 1$.

Polarized Relays

If ample current is sent through the coil of an ordinary relay, its contacts will be operated, regardless of the direction of the current. In either Figure 18(a) or Figure 18(b) the relay will, therefore, be operated. If provision is made to avoid chatter, alternating

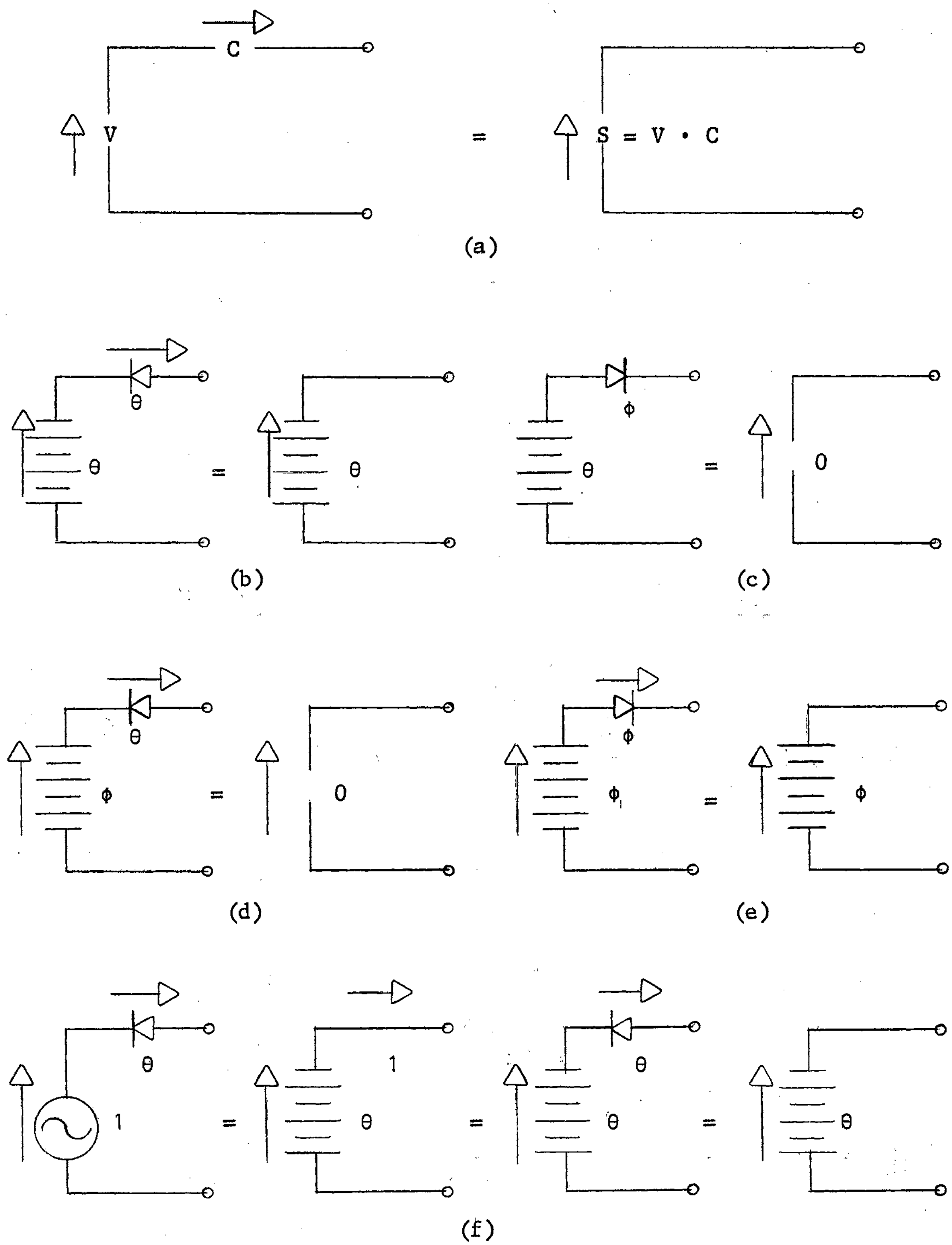

Fig. 16. Equivalent Voltage Sources.

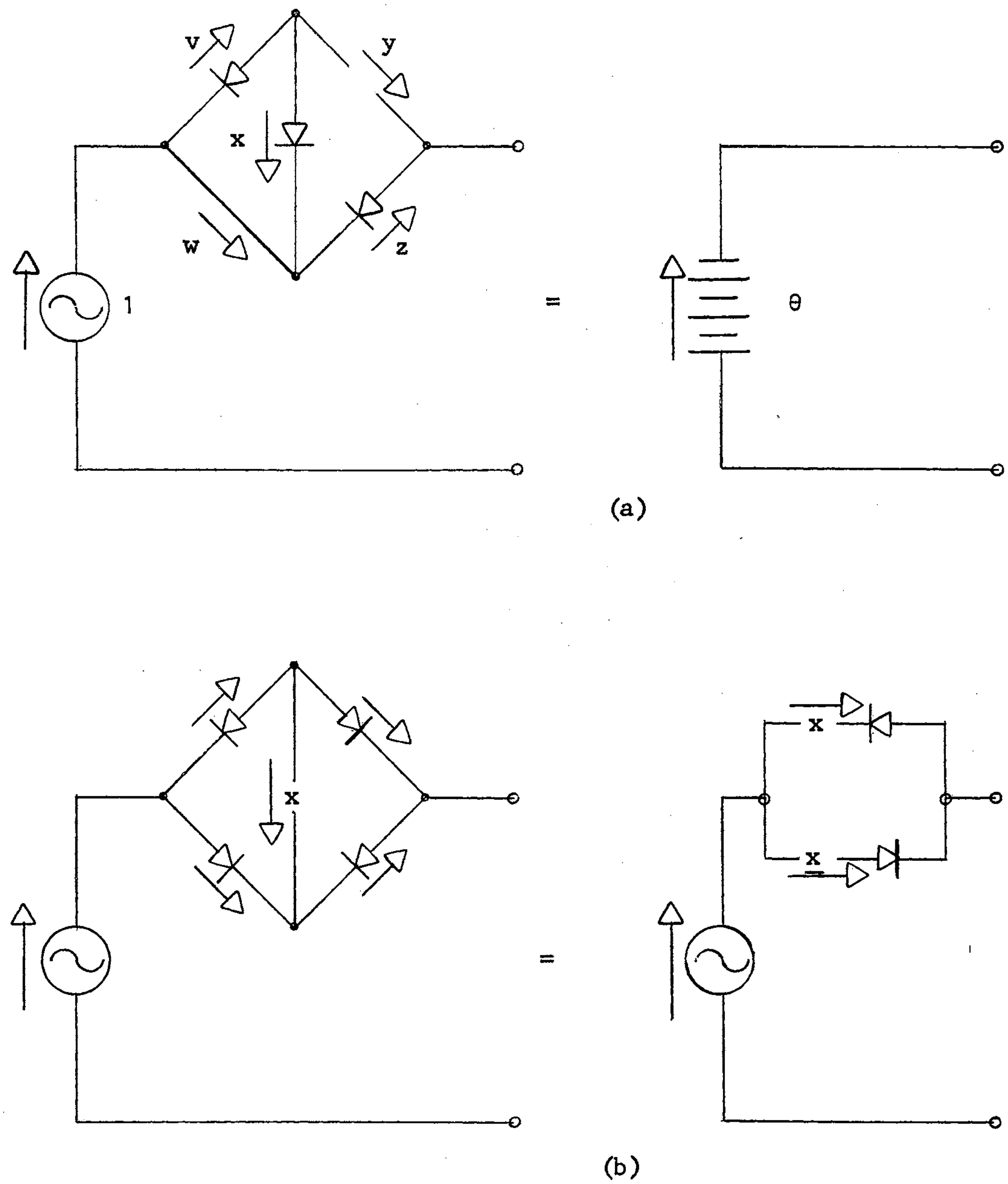

Fig. 17. Equivalents of More Complex Circuits.

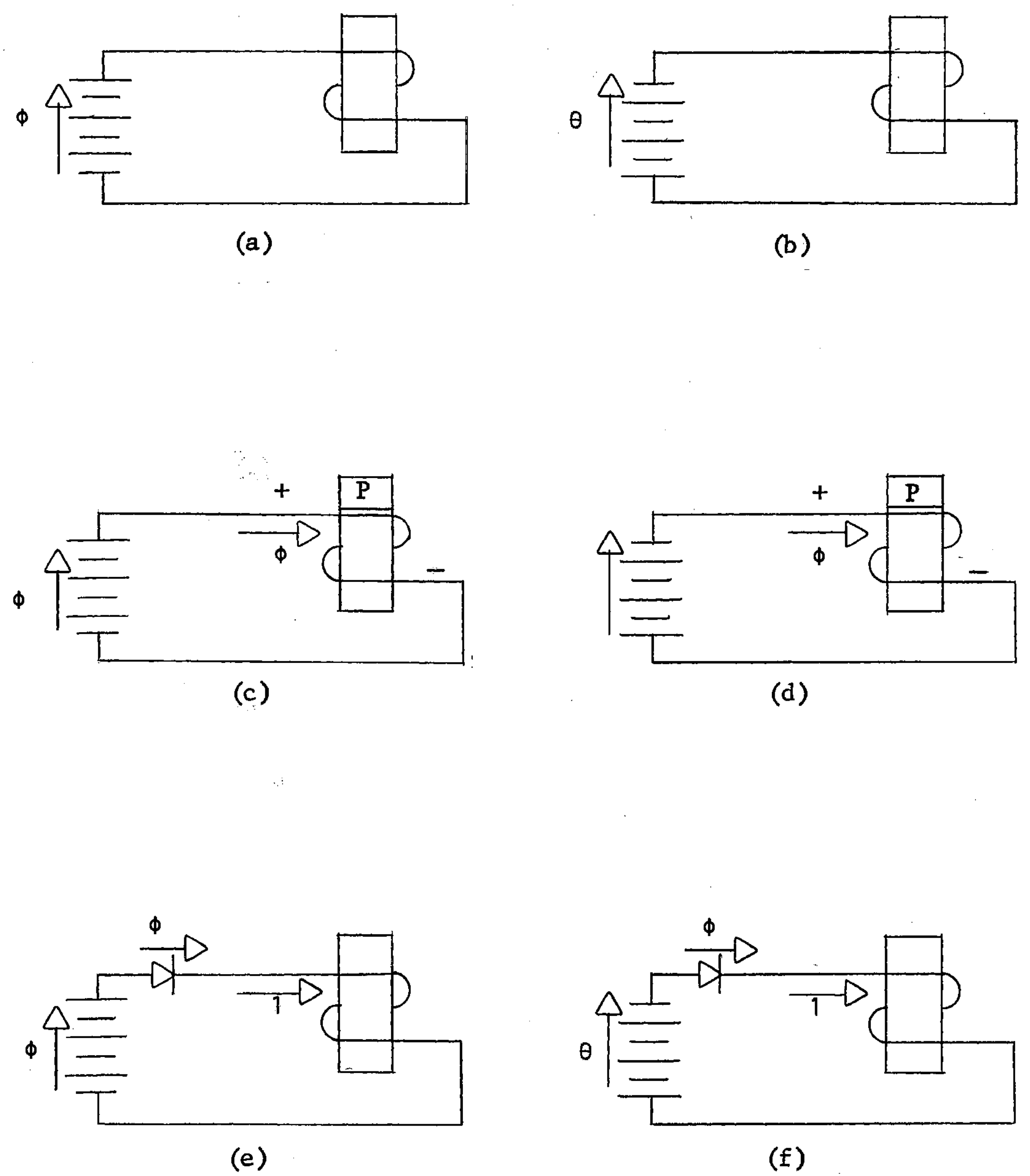

Fig. 18. Various Relay Coil Connections.

current will be satisfactory. In the usual a-c relay, part of the pole face is encircled by a copper ring which introduces a phase shift so that the net pull on the armature does not decrease enough to cause chatter. This type of relay will operate satisfactorily on direct current if the magnitude of the current is the same as the rms value for which it is designed. A source designated θ, ϕ, or 1 would then actuate the relay.

Certain relays are "biased" by a permanent magnet so that they respond to one polarity of voltage but not to the opposite polarity. A typical representation[1] for such a polarized relay coil is shown in Figure 18(c), where the polarity marks indicate to which terminals of a battery the two leads should be connected to actuate the relay. The relay shown in Figure 18(c) would be operated, that of Figure 18(d) would not. Although current will flow in both instances, the current in Figure 18(c) will be effective in actuating the relay, that in (d) will not. One might, then, label a relay with an arrow alongside one lead and mark the arrow with θ, ϕ, or 1 to indicate the direction of conductance which will effect actuation of the relay. If this is done, then the product VC_e will indicate whether the relay will operate. If $VC_e = 0$, the relay will not operate; for any other value of the product, the relay will operate. It will be noted that even though $VC_e = 0$, current might be flowing in one direction, but it would not actuate the relay. Such is the case in Figure 18(d).

The equivalent of a polarized relay can be constructed by placing

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[1]William Keister, A. E. Ritchie, and S. H. Washburn, The Design of Switching Circuits (New York, 1951), p. 19.

a diode in series with an ordinary relay which would operate with current in either direction. That is to say, the effective conductance of the relay alone is 1. For Figure 18(e) and (f), the series combination of the diode and the relay give a conductance of $\phi \cdot 1 = \phi$. The combination of Figure 18(e) is thus the equivalent of the relay shown in (c). Similarly (d) and (f) are equivalents. In this case, however, it should be remembered that current flows in (d) but not in (f). In neither of them is there any current to effect the operation of the relay.

Diodes in Circuit Simplification

In analysis of ordinary two-valued switching circuits, the following type of function is frequently encountered: $F = ab + \bar{a}\bar{b} + a\bar{b}c$. The circuit for this expression in series with a battery, is shown in Figure 19(a). An obvious simplification is indicated in Figure 19(b), which corresponds to $F = a(b + \bar{b}c) + \bar{a}\bar{b}$. At first glance, the circuit of Figure 19(c) might seem to be an equivalent, since paths ab, $\bar{a}\bar{b}$, and $a\bar{b}c$ exist through it. The circuit is incorrect, however, because of the "sneak path" $\bar{a}bc$. The expression for the circuit of Figure 19(d) is $ab + \bar{a}\bar{b} + a\phi c\bar{b} + \bar{a}c\theta b$. If this circuit is connected in series with a battery as shown in (d), the equivalent source is $\phi(ab + \bar{a}\bar{b} + a\phi c\bar{b} + \bar{a}c\theta b) = \phi ab + \phi\bar{a}\bar{b} + \phi ac\bar{b} + \phi\theta\bar{a}cb = \phi ab + \phi\bar{a}\bar{b} + \phi ac\bar{b} + 0 = \phi(ab + \bar{a}\bar{b} + a\bar{b}c)$, which is the same as that of Figure 19(a).

Diodes in Translator Circuits

Diodes are frequently used in translator circuits for digital computers, for example, to convert from one code to another. Figure 20 shows a circuit for translating from a decimal code to a binary code, where the combination of binary lamps that will be lighted depends on which digital switches D are closed. Since the decimal number

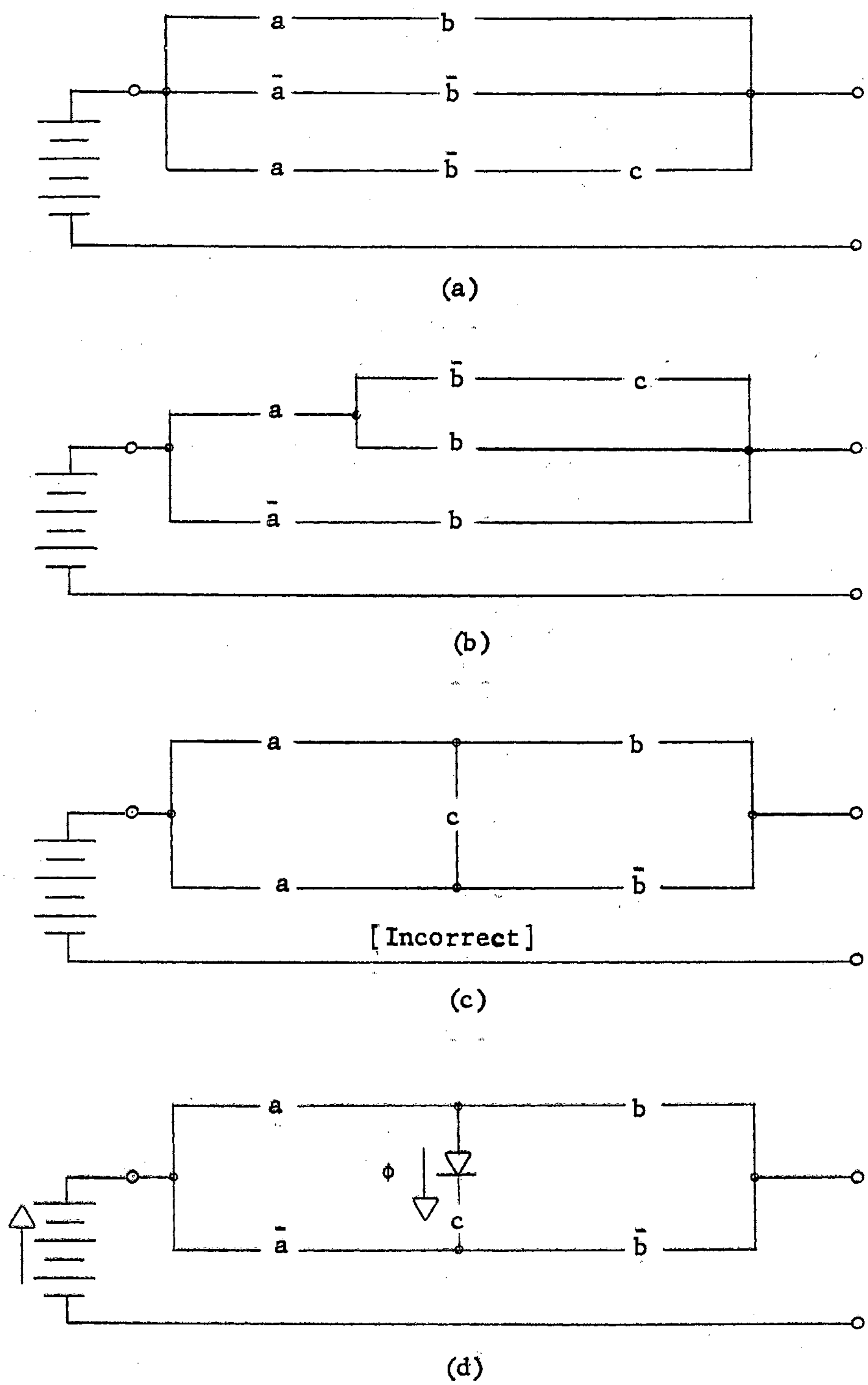


Fig. 19. Steps in the Development of a Circuit for ab + $\bar{a}\bar{b}$ + $a\bar{b}c$.

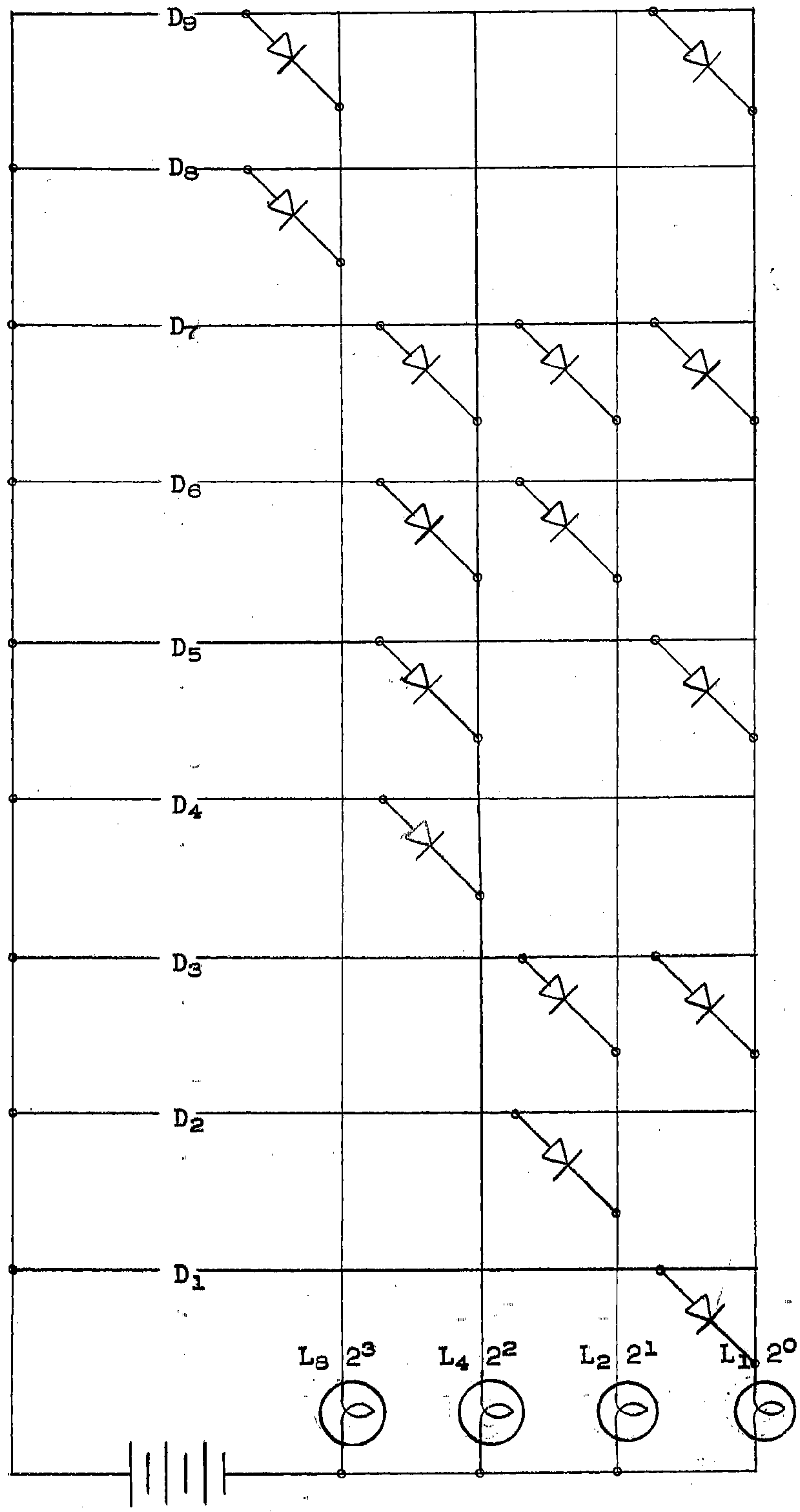


Fig. 20. A Decimal-to-Binary Translator.

$3 = 2^1 + 2^0$, lamps L_2 and L_1, should be lighted when D_3 is closed. Similarly, $5 = 2^2 + 2^0$, so lamps L_4 and L_1, should be lighted when D_5 is closed. At first, it might be thought that direct connections should be made from L_2 and L_1, to D_3 and from L_4 and L_1 to D_5. If this were done, however, there would be a conductive path to light L_2 when D_5 is closed. To avoid these spurious paths, it is common practice to employ diodes, as shown in Figure 20, so that there will be conduction in the proper direction, but not in the wrong direction. Four-valued Boolean algebra is a discipline that will enable one to handle with mathematical precision such circuits.

<u>Diodes in Control Circuits</u>

Figure 21 shows a circuit for controlling three relays through two wires. Table XXXI indicates effective currents through the three relays. As indicated in a previous paragraph, for any value other than zero the relay will be operated. It will be noticed that with this circuit arrangement all four possible combinations of operation of K_1 and K_2 are achieved. K_3 is operated for all but the first position. If make contacts on K_1 and K_2 were paralleled, the result would be the equivalent of a set of make contacts on K_3, so that K_3 could be eliminated. If, however, lamps were substituted for the relays, there might be an occasion when it would be advantageous to have a third lamp lighted whenever either of the other two were lighted. The fact that this circuit needs only two wires between the switch location and that of the relays or lamps could be a decided advantage if the locations were widely separated.

<u>Use of Theorems for B^2</u>

The theorems for B^2 have potentialities for circuit development. As an example, T309a states that $(x \perp y)\ \theta = (x + y)\ \theta$. By T310a,

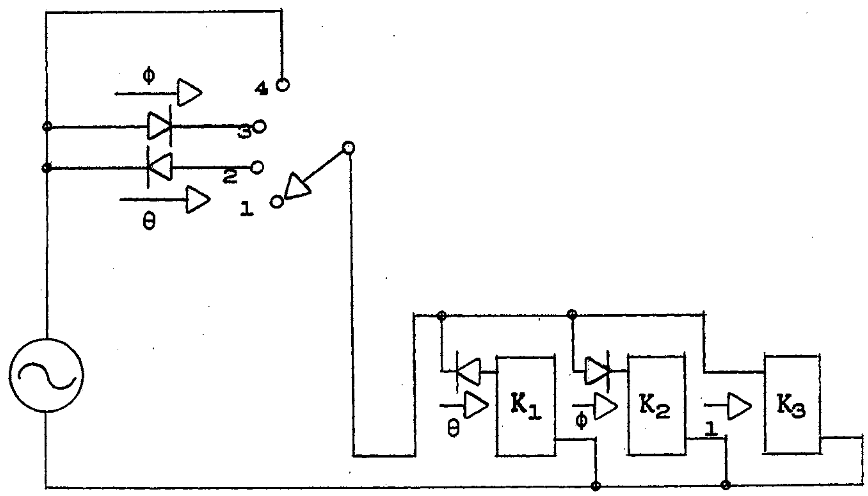


Fig. 21. Two-Wire Control of Three Relays.

TABLE XXXI

VALUES FOR THE CIRCUIT OF FIGURE 20.

Switch Position	Equivalent Source	Effective Currents K_1	K_2	K_3
1	0	0	0	0
2	θ	θ	0	θ
3	ϕ	0	ϕ	ϕ
4	1	θ	ϕ	1

$(x \perp y)\ \phi = xy\phi$. The first expression is for "x or y in series with θ." The second expression means "x and y in series with ϕ." Figure 22(a) shows $x \perp y$ in series with a voltage V. If $V = \theta$, the equivalent circuit is shown in Figure 22(b). If $V = \phi$, the circuit of Figure 22(c) results. This suggests the use of the circuit of Figure 22 (d), in which the lamp L_1 will light if either switch x or switch y is closed, and L_2 will light if both switches are closed. However, because diodes are required in series with the lamps to distinguish directions of conduction, it might appear better to have used the circuit of Figure 22(e), which uses only two diodes, whereas the $(x \perp y)$ circuit of (d) uses three. If, on the other hand, the lamps are to be operated at some distance from the switches, circuit (e) requires three wires between locations, but (d) needs only two. Figure 22(e) could be modified, as shown in (f), to use only two wires between locations, but then it would require four diodes instead of the three for the $(x \perp y)$ circuit.

Experimental Apparatus

For experimentally constructing ordinary switching circuits and four-valued switching circuits, the apparatus of Plate I has been developed. Connections to the various elements are made by plugging into banana jacks. Leads have stackable banana plugs on each end. Sources of supply available on the panel are 115 volts ac and 48 volts dc. Ten single-pole, double-throw switches are included, as well as three four-pole, double-throw, spring-return ones. Both neon and incandescent indicator lamps are available. Two relay banks have two-winding relays, the coils of which can be connected in an additive or a subtractive manner. All relay coils are equipped with series resistors so that shunt control can be utilized. A "matrix"

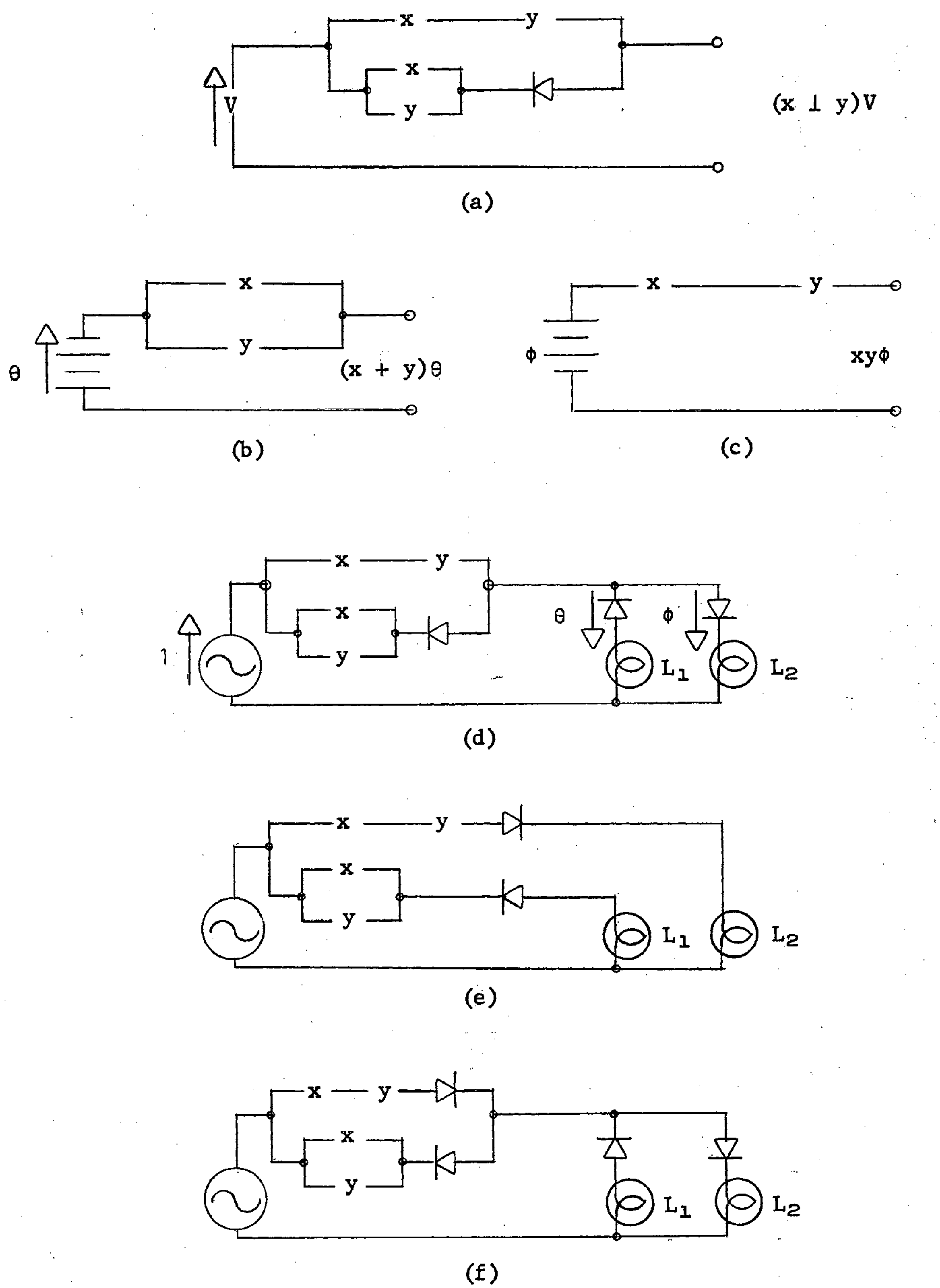
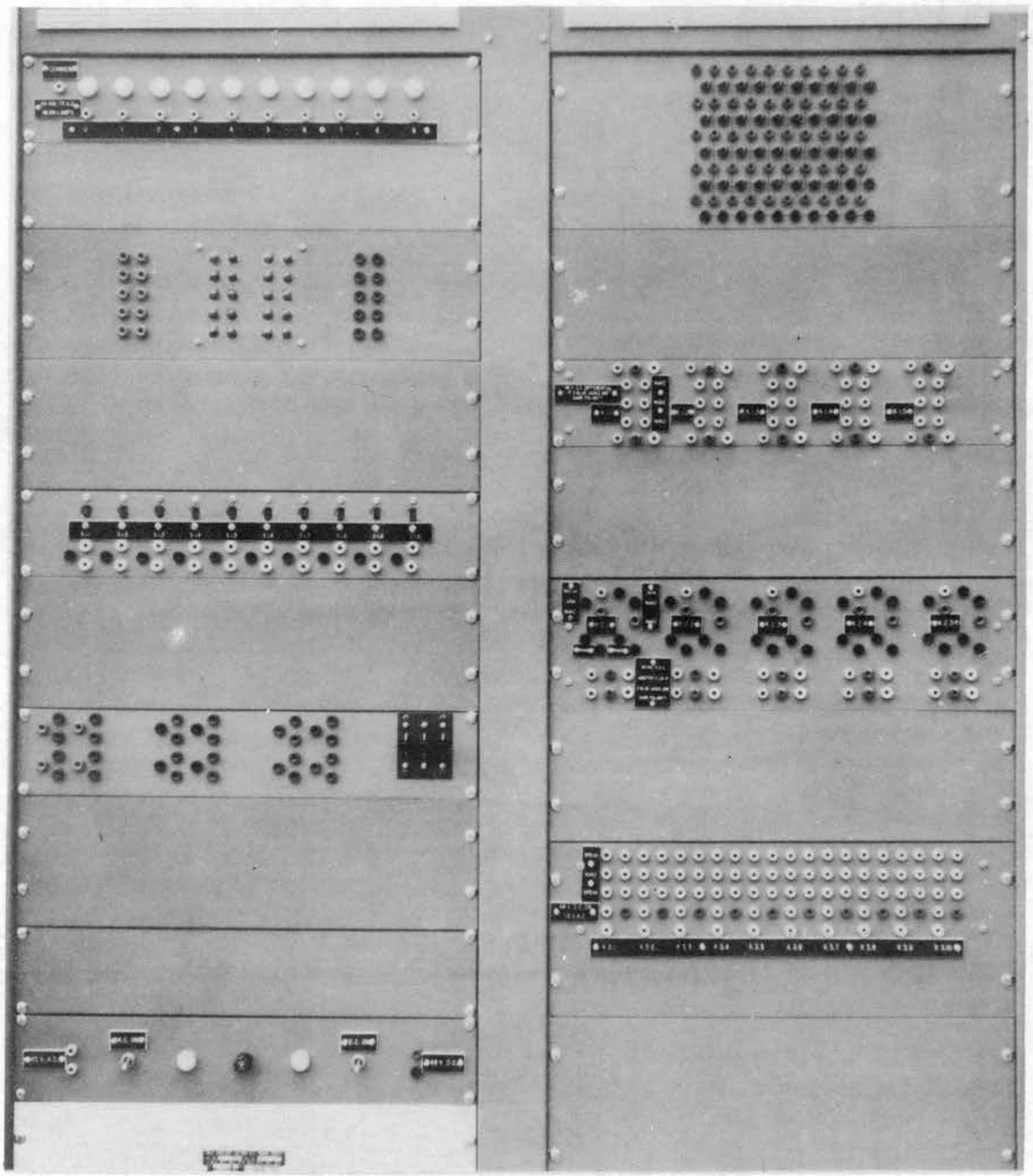


Fig. 22. Certain "And" and "Or" Circuits.

Plate I

Experimental Apparatus for Constructing Switching Circuits



board in the upper right corner of the picture facilitates connection of translator circuits like that shown in Figure 20. A number of diodes are available, mounted on General Radio double plugs, for constructing four-valued switching circuits.

The apparatus has been used to check experimentally a number of two-valued and four-valued switching circuits. It is believed that the apparatus is particularly valuable as a means of testing the mathematically developed circuits and of revealing incorrect solutions like that of Figure 19(c).

CHAPTER VII

MATRICES FOR FOUR-VALUED BOOLEAN ALGEBRA

Adaptation of the Hohn-Schissler Theory

Hohn and Schissler[1] of Bell Telephone Laboratories have effectively developed the theory of Boolean matrices for a two-valued algebra. This chapter will show how their system can be used with the four-valued Boolean algebra developed in this thesis.

Since four-valued logic allows the use of diodes that conduct well in one direction but not in the other, reference directions must be carefully observed. For example, their Figure 2[2] should have direction arrows establishing reference directions for the variables, as shown in Figure 23. Their Figure 3[3] is similarly adapted in Figure 24; their Figure 4[4] becomes Figure 25. It should be noted that, since no directivity was indicated by Hohn and Schissler, the directions assigned are arbitrary. A definite type of assymmetry will, however, result for certain matrices if all nodes are numbered and the reference direction is established from the lower number to the higher number for each element.

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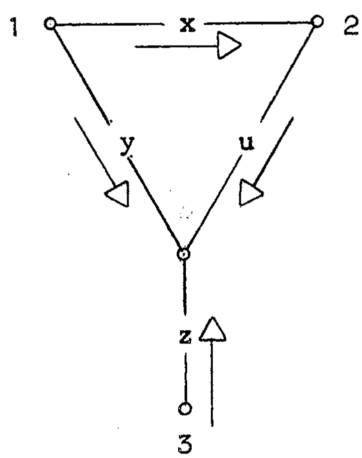
[1]Franz E. Hohn and L. Robert Schissler, "Boolean Matrices and the Design of Combinational Relay Switching Circuits," The Bell System Technical Journal, XXXIV (1955), 177-202.

[2]Ibid., p. 179.

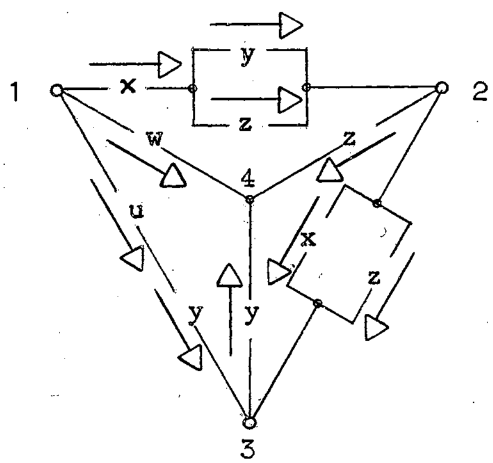
[3]Ibid., p. 181.

[4]Ibid., p. 184.



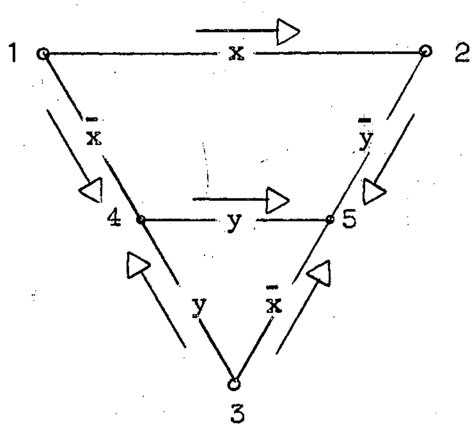
$$\begin{bmatrix} 1 & x + y\underline{u} & xu\underline{z} + y\underline{z} \\ \underline{x} + \underline{y}u & 1 & \underline{x}y\underline{z} + u\underline{z} \\ \underline{xu}z + \underline{y}z & x\underline{y}z + \underline{u}z & 1 \end{bmatrix}$$

Fig. 23. Adaptation of Figure 2 of Hohn and Schissler.



$$C = \begin{bmatrix} 1 & x(y + z) & uy & w \\ \underline{x}(\underline{y} + \underline{z}) & 1 & x + z & z \\ \underline{uy} & \underline{x} + \underline{z} & 1 & y \\ \underline{w} & \underline{z} & \underline{y} & 1 \end{bmatrix}$$

Fig. 24. Adaptation of Figure 3 of Hohn and Schissler.



$$P = \begin{bmatrix} 1 & x & 0 & \bar{x} & 0 \\ \underline{x} & 1 & 0 & 0 & \bar{y} \\ 0 & 0 & 1 & y & \bar{x} \\ \underline{\bar{x}} & 0 & \underline{y} & 1 & y \\ 0 & \underline{\bar{y}} & \underline{\bar{x}} & \underline{y} & 1 \end{bmatrix}$$

Fig. 25. Adaptation of Figure 4 of Hohn and Schissler.

If each node is identified, a primitive connection matrix[5] is the array of all elements p_{ij} between nodes i and j, with all nodes taken into account. For Figure 23, the primitive connection matrix is as

$$P = \begin{bmatrix} 1 & x & 0 & y \\ \underline{x} & 1 & 0 & u \\ 0 & 0 & 1 & z \\ \underline{y} & \underline{u} & \underline{z} & 1 \end{bmatrix}$$

would be expected, $p_{ii} = 1$, since that is a connection of a node with itself. The main diagonal must, therefore, consist entirely of 1's. Because all directions were established from a lower numbered node to a higher numbered one, the elements above the main diagonal do not have underscore marks, but all variables below the main diagonal are underscored. 0, of course, indicates an open circuit in either direction. As indicated on page 86, the underscore operation does not affect either 0 or 1.

The output matrix shown beside the circuit of Figure 23 traces all paths between terminals 1, 2, and 3, the output terminals of this three-terminal matrix. A third type of matrix discussed by Hohn and Schissler[6] is the connection matrix, which includes non-terminal and terminal nodes, but does not demand as many non-terminal nodes as the primitive matrix does. This is illustrated in Figure 24, where no node was identified between terminals 1 and 2 or between terminals 1 and 3.

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[5]Hohn and Schissler, p. 180.
[6]Ibid.

For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, Hohn and Schissler state[7] the following definitions and properties.

(1) Equality: $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j.

(2) Sum: $A + B = [a_{ij} + b_{ij}]$, that is, the sum is formed by adding corresponding elements. The sum of two switching matrices is again a switching matrix. It corresponds to connecting the elements a_{ij} and b_{ij} in parallel between nodes i and j throughout the circuit.

(3) Logical Product: $A * B = [a_{ij} \cdot b_{ij}]$, that is, the logical product is found by multiplying corresponding elements throughout. The logical product of two switching matrices is again a switching matrix. It corresponds to connecting the elements a_{ij} and b_{ij} in series between i and j.

(4) Complement: $A' = [\alpha_{ij}]$ where $\alpha_{ij} = a_{ij}'$ if $i \neq j$, but $\alpha_{ii} = 1$ for all i. This operation corresponds to replacing all the two-terminal circuits corresponding to the a_{ij} $(i \neq j)$ by their complements, recognizing the fact that the connection of a terminal to itself is invariable.

(5) Inclusion: $A \leqq B$ ("A is included in B" or "A is contained in B") if and only if $a_{ij} \leqq b_{ij}$ for all i and j. Also, $B \geqq A$ is equivalent to $A \leqq B$. If $A \leqq B$, then any combination of values of the input variables which results in a path from i to j in the circuit corresponding to A, also results in such a path in the circuit corresponding to B.

(6) Zero Matrix: The zero matrix Z has $a_{ij} = 0$ for $i \neq j$ but $a_{ii} = 1$ for all i. This corresponds to open circuits between all pairs of terminals.

(7) Universal Matrix: The universal matrix U has $a_{ij} = 1$ for all i and j. It corresponds to short circuits between all pairs of terminals.

(8) Matrix Product:

$$AB = \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right) .$$

The rule here is the same as for ordinary matrices. A^p means AA ... A to p factors. The matrix product of two switching matrices is again a switching matrix, but since the product of symmetric matrices is not necessarily symmetric, this product does not always have meaning in the case of relay switching circuits.

(9) Multiplication by a Scalar: $\alpha A = A\alpha = [\beta_{ij}]$ where α belongs to S and $\beta_{ij} = \alpha a_{ij}$ if $i \neq j$, but $\beta_{ii} = 1$ for all i. Thus αA is again a switching matrix.

(10) Transpose: $A^T = [\alpha_{ij}]$ where $\alpha_{ij} = a_{ji}$.

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[7]Hohn and Schissler, pp. 181-182.

The preceding definitions and properties fit four-valued Boolean algebra as well as the two-valued kind, provided reference directions are strictly adhered to for each element. The complement A' has been represented in this thesis by $\tilde{A}$. The statement on inclusion needs some clarification. In B^2, $0 \leqq \theta$, $0 \leqq \phi$, $0 \leqq 1$, $\theta \leqq 1$, and $\phi \leqq 1$. Consistent with these coverage relations, the statement would need to read as follows: "If $A \leqq B$, then any combination of values of the input variables which results in a [bilateral] path [1] ..." Obviously, since $\theta \leqq 1$, $a_{ij} = \theta$ would mean that b_{ij} could be either θ or 1. The "S" referred to under "Multiplication by a Scalar" is the set of Boolean switching functions, for which this thesis has used B. It can be extended to include B^2. If a primitive matrix P is based on directions from a lower numbered node to a higher numbered one, all variables below the main diagonal will be underscored. Its <u>transpose</u> P^T will therefore have all variables above the main diagonal underscored. For a matrix A based on two-valued elements, the transpose A^T would be identical to A.

In the appendix[8] to their article, Hohn and Schissler include the following basic properties for switching matrices A, B, and C. All apply equally well to four-valued switching circuits.

$A + A = A$	$A * A' = Z$
$A * A = A$	$U + A = U$
$A + B = B + A$	$U * A = A$
$A * B = B * A$	$A + A' = U$
$A + (B + C) = (A + B) + C$	$(A * B)' = A' + B'$
$A * (B * C) = (A * B) * C$	$(A + B)' = A' * B'$

[8]Hohn and Schissler, pp. 201-202.

$A + (B * C) = (A + B) * (A + C)$ $\qquad$ $(A')' = A$

$A * (B + C) = (A * B) + (A * C)$ $\qquad$ $A + (A * B) = A$

$Z + A = A$ $\qquad$ $A + (A' * B) = A + B$

$Z * A = Z$ $\qquad$ $A \leqq A$

$A \leqq B$ and $B \leqq A$ if and only if $A = B$

$A \leqq B$ and $B \leqq C$ imply $A \leqq C$

$A \leqq B$ if and only if $A * B = A$

$A \leqq B$ if and only if $A + B = B$

$Z \leqq A \leqq U$ for all A

$AB \neq BA$ ordinarily $\qquad$ $U^p = U$

$A(B + C) = AB + AC$ $\qquad$ $Z^p = Z$

$(A + B)C = AC + BC$ $\qquad$ $(A^p)^q = A^{pq}$

$AZ = ZA = A$ $\qquad$ $A^pA^q = A^{p+q}$

$(A^T)^T = A$ $\qquad$ $AU = UA = U$

$(A^T)^1 = (A^1)^T$ $\qquad$ $(AB)C = A(BC)$

$(A + B)^T = A^T + B^T$ $\qquad$ $A(B * C) \leqq AB * AC$

$(A * B)^T = A^T * B^T$ $\qquad$ $(A * B)C \leqq AC * BC$

$(AB)^T = B^TA^T$

$A \leqq B$ implies $AC \leqq BC$ and $CA \leqq CB$, but not conversely.

Reduced Connection Matrices

Because reference directions must be carefully observed for four-valued switching circuits, their matrices will differ from those for two-valued circuits by virtue of the underscores required for the former. Figure 25 illustrates the primitive matrix, with directions considered, for the circuit of Figure 4[9] in the article of Hohn and Schissler. Removal of a non-terminal node r, as they indicate, is

[9]Hohn and Schissler, p. 184.

accomplished by adding to each c_{ij} the product of the entry c_{ir} in row i and column r by the entry c_{rj} in row r and column j, then deleting row r and column r. Deletion of terminal 5, then terminal 4 results in the following matrices.

$$C_{(5} = \begin{bmatrix} 1 & x & 0 & \bar{x} \\ \underline{x} & 1 & \underline{\bar{x}}\bar{y} & \underline{y}\bar{y} \\ 0 & \bar{x}\underline{y} & 1 & y + \bar{x}\underline{y} \\ \underline{\bar{x}} & y\underline{\bar{y}} & \underline{y} + \underline{\bar{x}}y & \end{bmatrix}$$

$$C_{(4(5} = \begin{bmatrix} 1 & x + \bar{x}(y\underline{\bar{y}}) & \bar{x}(\underline{y} + \underline{\bar{x}}y) \\ \underline{x} + \underline{\bar{x}}(\underline{y}\bar{y}) & 1 & \underline{\bar{x}}\bar{y} + \underline{y}\bar{y}(\underline{y} + \underline{\bar{x}}y) \\ \underline{\bar{x}}(y + \bar{x}\underline{y}) & \bar{x}\underline{\bar{y}} + y\underline{\bar{y}}(y + \bar{x}\underline{y}) & 1 \end{bmatrix}$$

By T14a, $x + \bar{x}(y\underline{\bar{y}}) = x + y\underline{\bar{y}})$, and $\underline{x} + \underline{\bar{x}}(\underline{y}\bar{y}) = \underline{x} + \underline{y}\bar{y}$. By T13b, $\underline{y}(\underline{y} + \underline{\bar{x}}y) = \underline{y}$, and $y(y + \bar{x}\underline{y}) = y$. The matrix $C_{(4(5}$ can therefore be simplified.

$$C_{(4(5} = \begin{bmatrix} 1 & x + y\underline{\bar{y}} & \bar{x}(\underline{y} + \underline{\bar{x}}y) \\ \underline{x} + \underline{y}\bar{y} & 1 & (\underline{\bar{x}} + \underline{y})\bar{y} \\ \underline{\bar{x}}(y + \bar{x}\underline{y} & (\bar{x} + y)\bar{y} & 1 \end{bmatrix}$$

In contrast with the matrix given[10] by Hohn and Schissler,

$$C_{(4(5} \quad \begin{bmatrix} 1 & x & \bar{x}y \\ x & 1 & \bar{x}\bar{y} \\ \bar{x}y & \bar{x}\bar{y} & 1 \end{bmatrix},$$

the matrix for four-valued switching circuits is, as would be expected,

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[10]Hohn and Schissler, p. 185.

more elaborate. It can be readily reduced to that of Hohn and Schissler if it is recognized that, when the only allowable values are 0 and 1, $y\underline{\bar{y}} = 0$, $\bar{x}(\underline{y} + \underline{\bar{x}}y) = \bar{x}(y + \bar{x}y) = \bar{x}y(1 + \bar{x}) = \bar{x}y$, and $(\bar{x} + \underline{y})\bar{y} = (\bar{x} + y)\bar{y} = \bar{x}\bar{y}$.

<u>Two-Terminal Network Connections</u>

Figure 26 shows the allowable values for a two-terminal network, including for each case the matrix expression for the conductances. Figure 27 indicates the two possible ways in which two two-terminal networks can be connected. The series connection shown in Figure 27(a) would be indicated in matrix notation as follows:

$$C = A * B = \begin{bmatrix} 1 & x_{12} \\ x_{21} & 1 \end{bmatrix} * \begin{bmatrix} 1 & x_{34} \\ x_{43} & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_{12}x_{34} \\ x_{21}x_{43} & 1 \end{bmatrix}.$$

The parallel connection shown in Figure 27(b) yields

$$C = A + B = \begin{bmatrix} 1 & x_{12} \\ x_{21} & 1 \end{bmatrix} + \begin{bmatrix} 1 & x_{34} \\ x_{43} & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_{12} + x_{43} \\ x_{21} + x_{43} & 1 \end{bmatrix}.$$

A somewhat different approach would be to consider that each terminal of the original elements is a discrete node but that joining them in series gives rise to the following conductances: $x_{13} = x_{31} = 0$, $x_{14} = x_{41} = 0$, $x_{23} = x_{32} = 1$, $x_{24} = x_{42} = 0$. The resultant primitive matrix will then be

$$P = \left[\begin{array}{cc|cc} 1 & x_{12} & 0 & 0 \\ x_{21} & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & x_{34} \\ 0 & 0 & x_{43} & 1 \end{array}\right].$$

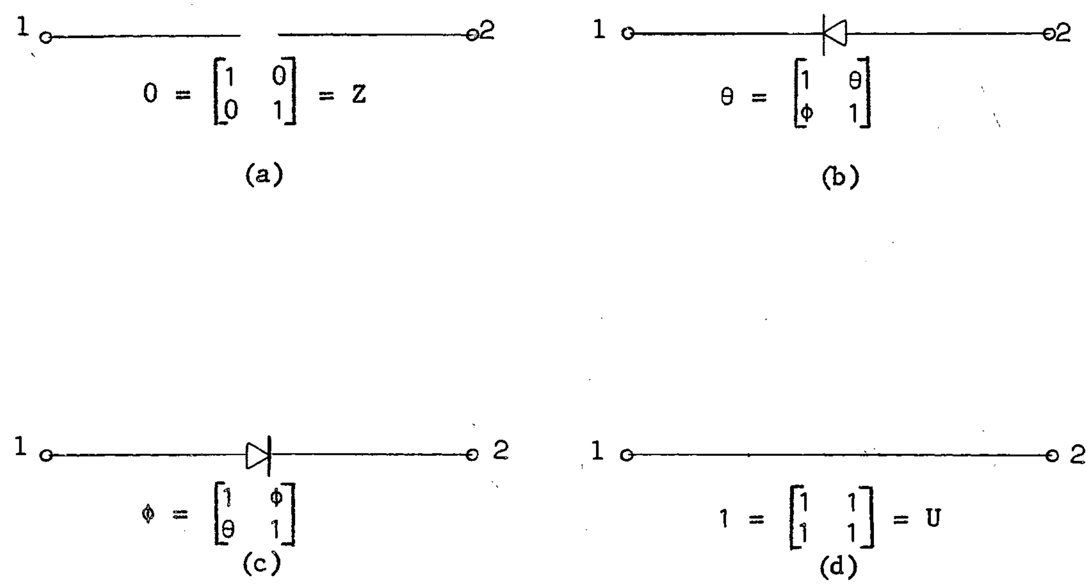

Fig. 26. Basic Two-Terminal Networks.

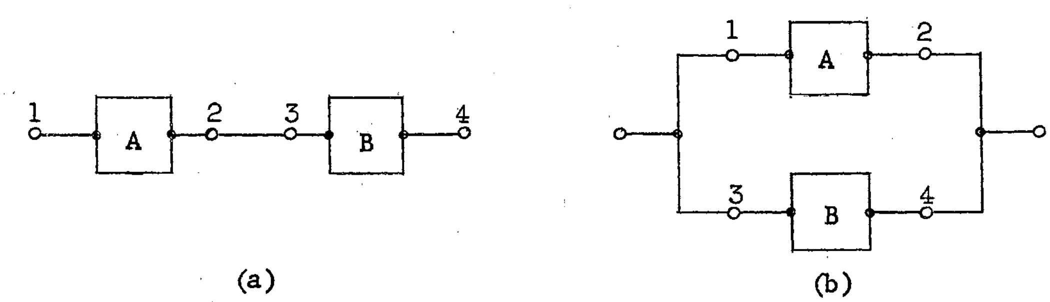

Fig. 27. Series and Parallel Connections for Two Two-Terminal Networks.

Now the Hohn-Schissler method can be used to eliminate first node three, then node two, resulting in the following connection matrices:

$$C_{(3} = \begin{bmatrix} 1 & x_{12} & 0 \\ x_{21} & 1 & x_{34} \\ 0 & x_{43} & 1 \end{bmatrix}, \text{ and } C_{(2(3} = \begin{bmatrix} 1 & x_{12}x_{34} \\ x_{21}x_{43} & 1 \end{bmatrix}.$$

It will be observed that $C_{(2(3}$ is identical to $C = A * B$.

Partitioning[11] of the primitive matrix P is indicated by dashed lines. It could have been written

$$P = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \text{ where } \alpha_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \alpha_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$\alpha_{11} = A$, and $\alpha_{22} = B$. It will be noted that $\alpha_{12} = \alpha_{21}{}^{T}$.

For the parallel connection shown in Figure 27(b), the following conductances are introduced: $x_{13} = x_{31} = 1$, $x_{14} = x_{41} = 0$, $x_{23} = x_{32} = 0$, and $x_{24} = x_{42} = 1$. The resultant primitive matrix is

$$P = \left[\begin{array}{cc|cc} 1 & x_{12} & 1 & 0 \\ x_{21} & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & x_{34} \\ 0 & 1 & x_{43} & 1 \end{array}\right].$$

Elimination of nodes produces the following connection matrices:

$$C_{(4} = \begin{bmatrix} 1 & x_{12} & 1 \\ x_{21} & 1 & x_{43} \\ 1 & x_{34} & 1 \end{bmatrix} \qquad C_{(3(4} = \begin{bmatrix} 1 & x_{12} + x_{34} \\ x_{21} + x_{43} & 1 \end{bmatrix}$$

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[11] E. A. Guillemin, The Mathematics of Circuit Analysis (New York, 1950), pp. 48-53.

$$C_{(3} = \begin{bmatrix} 1 & x_{12} & x_{34} \\ x_{21} & 1 & 1 \\ x_{43} & 1 & 1 \end{bmatrix} \qquad C_{(2(3} = \begin{bmatrix} 1 & x_{12} + x_{34} \\ & \\ x_{21} + x_{43} & 1 \end{bmatrix}$$

As would be expected, the results are the same whether nodes three and four are eliminated or nodes two and three. As indicated by the expressions $x_{12} + x_{34}$ and $x_{21} + x_{43}$, x_{12} and x_{34} were placed in parallel. C = A + B.

Partitioning of the primitive matrix for the parallel connection would yield

$$P = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ & \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \text{ where } \alpha_{12} = \alpha_{21} = \begin{bmatrix} 1 & 0 \\ & \\ 0 & 1 \end{bmatrix} = Z,\ \alpha_{11} = A,$$

and $\alpha_{22} = B$.

<u>Three-Terminal Network Connections</u>

Two three-terminal networks are shown in Figure 28. If these are connected in cascade, as indicated in Figure 29, $x_{14} = x_{41} = 0$, $x_{15} = x_{51} = 0$, $x_{16} = x_{61} = 0$, $x_{24} = x_{42} = 1$, $x_{25} = x_{62} = 0$, $x_{26} = x_{62}, = 0$, $x_{34} = x_{43} = 0$, $x_{35} = x_{53} = 0$, $x_{36} = x_{63} = 1$. The partitioned primitive matrix $P = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$, where $\alpha_{11} = [A]$, $\alpha_{22} = [B]$, $\alpha_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $\alpha_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In expanded form,

$$P = \left[\begin{array}{ccc|ccc} 1 & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & 1 & x_{23} & 1 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & x_{45} & x_{46} \\ 0 & 0 & 0 & x_{54} & 1 & x_{56} \\ 0 & 0 & 1 & x_{64} & x_{65} & 1 \end{array}\right]$$

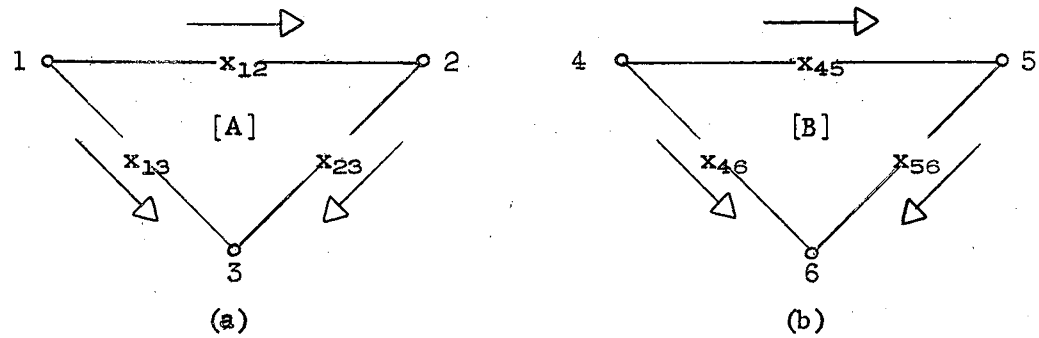

Fig. 28. Basic Three-Terminal Networks.

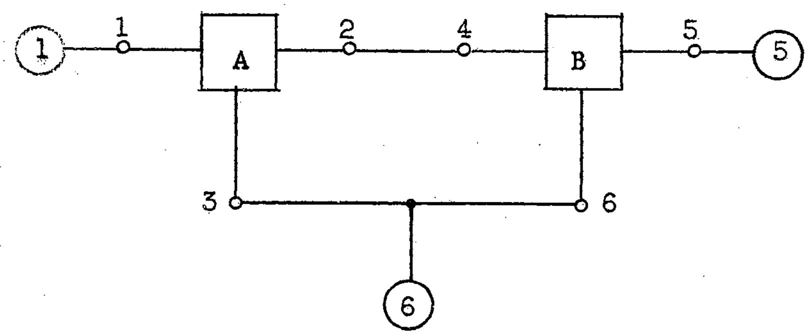

Fig. 29. Cascade Connection of Two Three-Terminal Networks.

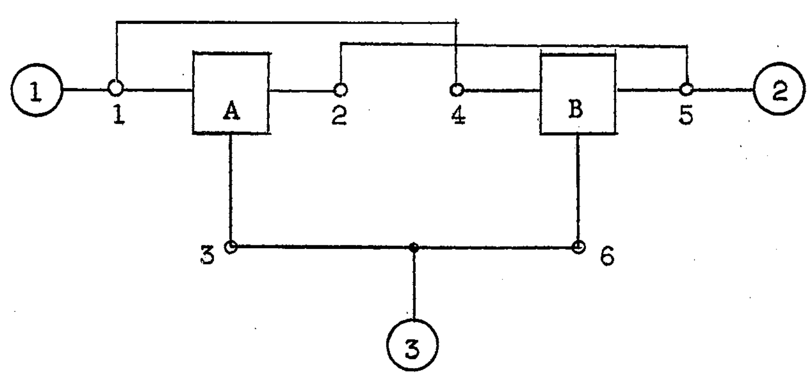

Fig. 30. Parallel Connection of Two Three-Terminal Networks.

This will reduce to

$$C = \begin{bmatrix} 1 & x_{12}x_{45} & x_{13} + x_{12}(x_{23} + x_{46}) \\ x_{21}x_{54} & 1 & x_{56} + x_{54}(x_{23} + x_{46}) \\ x_{31} + x_{21}(x_{32} + x_{64}) & x_{65} + x_{45}(x_{32} + x_{64} & 1 \end{bmatrix}$$

If the two three-terminal networks of Figure 28 are placed in parallel, as indicated in Figure 30, $x_{14} = x_{41} = 1$, $x_{15} = x_{51} = 0$, $x_{16} = x_{61} = 0$, $x_{24} = x_{42} = 0$, $x_{25} = x_{52} = 1$, $x_{26} = x_{62} = 0$, $x_{34} = x_{43} = 0$, $x_{35} = x_{53} = 0$, $x_{36} = x_{63} = 1$. Now, for the partitioned primitive matrix, $\alpha_{11} = [A]$, $\alpha_{22} = [B]$, $\alpha_{12} = \alpha_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Z$.

Other Network Connections

The procedures indicated above can, of course, be extended to cover various connections for different networks. When networks are placed in parallel, $\alpha_{12} = \alpha_{21} = Z$, the zero matrix of appropriate order. Perhaps a more direct approach for parallel connections is to make sure that, for two matrices A and B to be joined in parallel, elements to be paralleled occupy corresponding positions in the two matrices, then simply to form the matrix sum A + B.

An example of the use of matrices for four-valued switching circuits is the solution of the network shown in Figure 31, where two four-terminal networks are to be placed in cascade by connecting terminals two and five together and terminals four and seven together. The identities of only terminals one and eight are to be retained. That is, it is desired to find an equivalent two-terminal circuit for terminals one and eight. A solution follows.

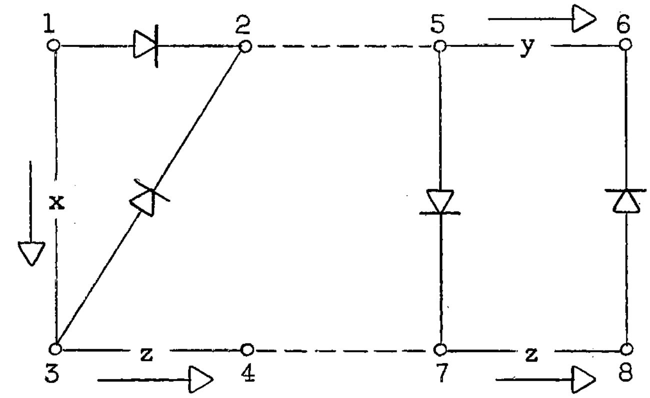

Fig. 31. A Four-Valued Switching Circuit to be Solved by Matrices.

$$\alpha_{11} = \begin{bmatrix} 1 & \phi & x & 0 \\ \theta & 1 & \theta & 0 \\ \underline{x} & \phi & 1 & z \\ 0 & 0 & \underline{z} & 1 \end{bmatrix} \qquad \alpha_{22} = \begin{bmatrix} 1 & y & \phi & 0 \\ \underline{y} & 1 & 0 & \theta \\ \theta & 0 & 1 & z \\ 0 & \phi & \underline{z} & 1 \end{bmatrix}$$

$$\alpha_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \alpha_{21} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & \phi & x & 0 & 0 & 0 & 0 & 0 \\ \theta & 1 & \theta & 0 & 1 & 0 & 0 & 0 \\ \underline{x} & \phi & 1 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{z} & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & y & \phi & 0 \\ 0 & 0 & 0 & 0 & \underline{y} & 1 & 0 & \theta \\ 0 & 0 & 0 & 1 & \theta & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & \phi & \underline{z} & 1 \end{bmatrix}$$

$$C_{(7} = \begin{bmatrix} 1 & \phi & x & 0 & 0 & 0 & 0 \\ \theta & 1 & \theta & 0 & 1 & 0 & 0 \\ \underline{x} & \phi & 1 & z & 0 & 0 & 0 \\ 0 & 0 & \underline{z} & 1 & \theta & 0 & z \\ 0 & 1 & 0 & \phi & 1 & y & \phi z \\ 0 & 0 & 0 & 0 & \underline{y} & 1 & \theta \\ 0 & 0 & 0 & \underline{z} & \theta\underline{z} & \phi & 1 \end{bmatrix}$$

$$C_{(6(7} = \begin{bmatrix} 1 & \phi & x & 0 & 0 & 0 \\ \theta & 1 & \theta & 0 & 1 & 0 \\ \underline{x} & \phi & 1 & z & 0 & 0 \\ 0 & 0 & \underline{z} & 1 & \theta & z \\ 0 & 1 & 0 & \phi & 1 & (\theta y + \phi z) \\ 0 & 0 & 0 & \underline{z} & (\phi\underline{y} + \theta\underline{z}) & 1 \end{bmatrix}$$

$$C_{(5(6(7} = \begin{bmatrix} 1 & \phi & x & 0 & 0 \\ \theta & 1 & \theta & \phi & (\theta y + \phi z) \\ \underline{x} & \phi & 1 & z & 0 \\ 0 & \theta & \underline{z} & 1 & (z + \theta y) \\ 0 & (\phi \underline{y} + \theta \underline{z}) & 0 & (\underline{z} + \phi \underline{y}) & 1 \end{bmatrix}$$

$$C_{(4(5(6(7} = \begin{bmatrix} 1 & \phi & x & 0 \\ \theta & 1 & (\theta + \phi \underline{z}) & (\theta y + \phi z) \\ \underline{x} & (\phi + \theta z) & 1 & z \\ 0 & (\phi \underline{y} + \theta \underline{z}) & \underline{z} & 1 \end{bmatrix}$$

$$C_{(3(4(5(6(7} = \begin{bmatrix} 1 & (\phi + xz) & xz \\ (\theta + \underline{xz}) & 1 & (\theta y + z) \\ \underline{xz} & (\phi \underline{y} + \underline{z}) & 1 \end{bmatrix}$$

$$C_{(2(3(4(5(6(7} = \begin{bmatrix} 1 & z(x + \phi) \\ \underline{z}(\underline{x} + \theta) & 1 \end{bmatrix}$$

The circuit of Figure 31 is, therefore, equivalent to that shown in Figure 32.

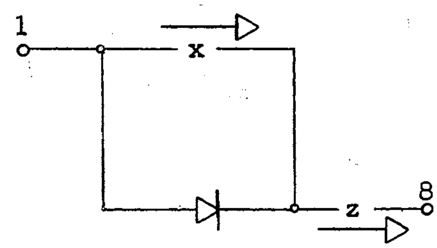

Fig. 32. A Simplification of Figure 31.

Various theorems, notably T13a and T14a, were involved in obtaining the reduced connection matrices. Obviously, the solution is not easy, but orderly steps do lead to a correct solution. The solution can be checked by perfect induction or by use of experimental apparatus like that shown in Plate I.

CHAPTER VIII

SUMMARY AND CONCLUSIONS

Résumé

In the twenty years since two-valued Boolean algebra was first applied to switching circuits, it has become a widely adopted method for analysis and simplification of digital control circuits. In recent years, non-thermionic diodes have been greatly improved, and they are frequently used in control circuits. A need has, therefore, arisen for a mathematical technique for handling diodes as a part of switching circuits. A logical approach would be to use a four-valued Boolean algebra which would allow four possible states of conduction: an open circuit, conduction in one direction only, conduction in the opposite direction only, and conduction equally well in both directions.

Since technical literature contains very little information on four-valued Boolean algebra, it was decided that the mathematical theory should be developed from modern algebra in such a way as to provide a good foundation for the electrical engineer who wished to use and understand the four-valued logic in the study of switching circuits. This meant that the mathematical basis of two-valued Boolean algebra needed to be developed from sound principles so that it could be logically expanded into a four-valued algebra. The mathematical basis of Boolean algebras is covered in the second chapter, including group theory, lattice theory, link-preserving transformations, and the direct union of two lattices. The next chapter develops from that basis the Boolean algebra of order two, including a number of binary operations and theorems. The Boolean algebra of order four is then presented as the direct union of two-valued Boolean algebras.

The application of the mathematics to switching circuits is then demonstrated. A brief coverage of conventional two-valued switching circuits is given, followed by an analysis of the use of four-valued logic in switching circuits that include diodes, different kinds of voltage sources, and polarized relays.

In the seventh chapter the use of matrices in the analysis of four-valued switching circuits is demonstrated. It is shown that essentially the only modification of the existing theory of Boolean matrices for two-valued circuits is rigorous adherence to reference directions. Partitioned matrices are used in demonstrating how two network matrices can be combined when the two networks are interconnected.

Appraisal

The use of four-valued Boolean algebra for analysis of simple switching circuits with one or two diodes may be difficult to justify. One can often tell by looking at a circuit what it will do, without any need for a knowledge of lattice theory. A good electrician can, for that matter, wire up a "three-way" switch without thinking in terms of $x \bar{\times} y$. The use of matrix theory for analyzing two switches in parallel would be unnecessarily complicated.

On the other hand, the reason for the adoption of two-valued switching algebra is the fact that it will enable one to analyze rigorously and to simplify complex circuits. By the same token, the

Boolean algebra of order four is a strict mathematical discipline for precise handling of switching circuits which include diodes. It is a particularly useful concept in that it is fully compatible with regular two-valued logic, reducing automatically to it when the allowable values are confined to zero and one.

Potentialities

The use of four-valued Boolean algebra for analysis of switching circuits with diodes seems to have definite advantages. Its true test will, of course, be its application to practical problems. It is hoped that engineers in industry can soon be informed of the theory developed in this thesis so that they can test its merit in the solution of problems encountered in the fields of telephone switching, digital computers, and automatic control. Perhaps, in commercial application, even the unusual binary operations for two-valued logic shown on page 39 will prove useful. In addition to sending copies of this thesis to friends in industry who are working with switching circuits, the author hopes to be able to present a paper on the subject at a future meeting of one of the technical societies, so as to evoke comment from practicing engineers and perhaps to lead to adoption of the technique where applicable.

BIBLIOGRAPHY

Auerbach, I. L., and S. B. Disson. "Magnetic Elements in Arithmetic and Control Circuits." Electrical Engineering, LXXIV (September, 1955), 766-770.

Anderson, A. E. "Transistors in Switching Circuits." Proceedings of the IRE, XL (November, 1952), 1541-1558.

Anderson, J. R. "Ferroelectric Materials As Storage Elements for Digital Computers and Switching Systems." AIEE Transactions, LXXI, pt. I (January, 1953), 301-395.

Beckenbach, Edwin F., ed. Modern Mathematics for the Engineer. New York: McGraw-Hill, 1956.

Bennett, W. S. "Minimizing and Mapping Sequential Circuits." AIEE Transactions, LXXIV, pt. I (September, 1955), 443-447.

Berkeley, E. C. A Summary of Symbolic Logic and Its Practical Applications. Newtonville, Mass.: E. C. Berkeley, 1951.

Beter, R. H., W. E. Bradley, R. B. Brown, and M. Rubinoff. "Surface Barrier Transistor Switching Circuits." IRE Convention Record, III, pt. IV (1955), 139-145.

Birkhoff, Garrett. Lattice Theory. 2d ed. New York: Colloquium Publications, 1948.

_______ and Saunders Mac Lane. A Survey of Modern Algebra. Rev. ed. New York: Macmillan, 1953.

Boole, George. An Investigation of the Laws of Thought. New York: Dover, 1951 (Reprint).

Brown, D. R., and N. Rochester. "Rectifier Networks for Multiposition Switching." Proceedings of the IRE, XXXVII (February, 1949), 139-147.

Burks, A. W., and J. B. Wright. "Theory of Logical Nets." Proceedings of the IRE, XLI (October, 1953), 1357-1365.

Bush, G. L. "The Diode Matrix as a Component in Relay Switching Circuits." AIEE Transactions, LXXII, pt. I (January, 1954), 833-838.

Caldwell, S. H. "The Recognition and Identification of Symmetric Switching Functions." AIEE Transactions, LXXIII, pt. I, (May, 1954), 142-147.

Davis, R. L. "The Number of Structures of Finite Relations." Proceedings of the American Mathematical Society, IV (June, 1953), 486-495.

Evans, W. G., W. G. Hall, and R. I. Van Nice. "Magnetic Logic Circuits for Industrial Control Systems." AIEE Transactions, LXXV, pt. II (July, 1956), 166-171.

Felker, J. H. "Typical Block Diagrams for Transistor Digital Computor." AIEE Transactions, LXXI, pt. I (July, 1952), 175-182.

_______. "Regenerative Amplifier for Digital Computer Applications." Proceedings of the IRE, XL (November, 1952), 1584-1596.

Finzi, L. A., and G. C. Feth. "Magnetic Amplifiers in Bistable Operation." AIEE Transactions, LXXIV, pt. I (November, 1955), 592-593.

Gilbert, E. N. "N-Terminal Switching Circuits." Bell System Technical Journal, XXX (1951), 668-688.

Guillemin, E. A. The Mathematics of Circuit Analysis. New York: John Wiley and Sons, 1950.

Guterman, S., R. D. Kodis, and S. Ruhman. "Logical and Control Functions Performed with Magnetic Cores." Proceedings of the IRE, XLIII (March, 1955), 291-298.

Hamming, R. W. "Error Detecting and Error Correcting Codes." Bell System Technical Journal, XXVIII (1949), 59-98.

Hartree, D. R. Calculating Instruments and Machines. Urbana: University of Illinois Press, 1949.

Harvard Computation Laboratory Staff. Synthesis of Electronic Computing and Control Circuits. Cambridge: Harvard University Press, 1951.

Henle, R. A. "A Multistable Transistor Circuit." AIEE Transactions, LXXIV, pt. I (November, 1955), 568-571.

Hilbert, D., and W. Ackermann. Grundzüge der theoretischen Logic. 2d. ed. Berlin: Springer, 1938.

Hohn, Franz E. "Some Mathematical Aspects of Switching." American Mathematical Monthly, LXII (1955), 75-90.

_______ and L. Robert Schissler. "Boolean Matrices and the Design of Combinational Relay Switching Circuits." The Bell System Technical Journal, XXXIV (January, 1955), 177-202.

Huffman, D. A. "The Synthesis of Sequential Switching Circuits, Part I." Journal of the Franklin Institute, CCLVII (March, 1954), 161-190

_______. "The Synthesis of Sequential Switching Circuits, Part II." Journal of the Franklin Institute, CCLVII (April, 1954), 275-303.

Hussey, Luther W. "Semiconductor Diode Gates." The Bell System Technical Journal, XXXII (September, 1953), 1137-1153.

Joel, A. E. "Electronics in Telephone Switching Systems." AIEE Transactions, LXX, pt. I (January, 1957), 701-709.

Karnough, M. "The Map Method for Synthesis of Combinational Logic Circuits." AIEE Transactions, LXXII, pt. I (November, 1953), 593-599.

_______. "Pulse Switching Circuits Using Magnetic Cores." Proceedings of the IRE, XLIII (May, 1955), 576-584.

Katz, A. "A Magnetic Amplifier Switching Matrix." AIEE Transactions, LXXV, pt. I (May, 1956), 236-241.

Keister, William, A. E. Ritchie, and S. H. Washburn. The Design of Switching Circuits. New York: D. Van Nostrand and Company, 1951.

King, G. W., G. W. Brown, and L. N. Ridenour. Proceedings of the IRE, XLI (October, 1953), 1421-1428.

Kiss, Stephen A. Transformations on Lattices and Structures of Logic. New York: Stephen A. Kiss, 1947.

Langer, Susanne K. An Introduction to Symbolic Logic. 2d. ed. New York: Dover Publications, 1953.

Lee, C. Y. "Switching Functions on an N-Dimensional Cube." AIEE Transactions, LXXIII, pt. I (September, 1954), 289-291.

_______ and W. H. Chen. "Several-Valued Combinational Switching Circuits." AIEE Transactions, LXXV, pt. I (July, 1956), 278-283.

Leiner, A. L., W. A. Notz, J. L. Smith, and A. Weinberger. "System Design of the SEAC and DYSEAC." IRE Transactions, EC-III (June, 1954), 8-22.

Lewis, I. A. D. "A Symbolic Method for the Solution of Some Switching and Relay Circuit Problems." Proceedings of the IRE, XCVIII (May, 1951), 181-191.

Lewis, C. I., and C. H. Langford. Symbolic Logic. New York: Appleton-Century, 1932.

Luce, R. D. "Note on Boolean Matrix Theory." Proceedings of the American Mathematical Society, (1952), 382-388.

MacDuffee, Cyrus Colton. An Introduction to Abstract Algebra. New York: John Wiley and Sons, 1940.

Martin, N. M. "The Sheffer Functions of 3-Valued Logic." Journal of Symbolic Logic, XIX (1954), 45-51.

McCluskey, E. J., Jr. "Detection of Group Invariance or Total Symmetry of a Boolean Function." The Bell System Technical Journal, XXXV (November, 1956), 1445-1453.

_______. "Minimization of Boolean Functions." The Bell System Technical Journal, XXXV (November, 1956), 1417-1444.

Mealy, George H. "A Method for Synthesizing Sequential Circuits." The Bell System Technical Journal, XXXIV (September, 1955), 1045-1079.

Minnick, R. C. "Magnetic Switching Circuits." Journal of Applied Physics, XXV (April, 1954), 479-485.

Montgomerie, G. A. "Sketch of an Algebra for Relay and Contactor Circuits." Journal of the Institution of Electrical Engineers, XCV, pt. II (1948), 355-364.

Moore, E. F., and C. E. Shannon. "Machine Aid for Switching Circuit Design." Proceedings of the IRE, XLI (October, 1953), 1348-1351.

Murray, F. J. "Mechanisms and Robots." Journal of the Association for Computing Machinery. II (April, 1955), 16-82.

Nelson, E. C. "An Algebraic Theory for Use in Digital Computer Design." IRE Transactions, EC-III (September, 1954), 12-21.

Plotkin, M. Binary Codes with Specified Minimum Distance. Philadelphia: University of Pennsylvania Press, 1952.

Post, E. L. "Introduction to a General Theory of Elementary Propositions." American Journal of Mathematics, XLIII (1921), 163-185.

Prihar, Z. "Topological Properties of Telecommunication Networks." Proceedings of the IRE, XLIV (July, 1956), 927-933.

Quine, W. V. "The Problem of Simplifying Truth-Functions." The American Mathematical Monthly, LIX (October, 1952), 521-531.

_______. "A Way to Simplify Truth Functions." The American Mathematical Monthly, LXII (November, 1955), 627-631.

Rajchman, Jan A. "Static Magnetic Matrix Memory and Switching Circuits." RCA Review, (June, 1951), 183-201.

Ramey, R. A. "The Single-Core Magnetic Amplifier as a Computer Element." AIEE Transactions, LXXI, pt. I (January, 1953), 442-446.

Raspanti, Matthew. "A Complex Algebra for Relay Circuits." Electrical Engineering, LXXII (November, 1953), 992-993.

Richards, R. K. Arithmetic Operations in Digital Computers. New York: D. Van Nostrand and Company, 1955.

Rosenbloom, P. C. The Elements of Mathematical Logic. New York: Dover Publications, 1950.

Rosser, J. B., and A. R. Turquette. "Axiom Schemes for M-Valued Propositional Calculi." Journal of Symbolic Logic, X (1945), 61-82.

Royer, G. H. "A Switching Transistor D-C to A-C Converter Having an Output Frequency Proportional to the D-C Input Voltage." AIEE Transactions, LXXIV, pt. I (July, 1955), 322-326.

Schaefer, David H. "A Rectifier Algebra." AIEE Transactions, LXXIII, pt. I (January, 1955), 679-682.

Serrell, Robert. "Elements of Boolean Algebra for the Study of Information-Handling Systems." Proceedings of the IRE, XLI (October, 1953), 1366-1380.

Shannon, Claude E. "A Symbolic Analysis of Relay and Switching Circuits." AIEE Transactions, LVII (1938), 713-723.

_______. "The Synthesis of Two-Terminal Switching Circuits." The Bell System Technical Journal, XXVIII (January, 1949), 59-98.

_______. "Computers and Automata." Proceedings of the IRE, XLI (October, 1953), 1234-1241.

Sheffer, H. M. "A Set of Five Independent Postulates for Boolean Algebras." Transactions of the American Mathematical Society, XIV (1913), 481-488.

Smith, Lloyd P. Mathematical Methods for Scientists and Engineers. New York: Prentise-Hall, 1953.

Trent, R. L. "Two Transistor Binary Counter." Electronics, XXV (July, 1952), 100-101.

Trousdale, Robert B. "The Symmetrical Transistor as a Bilateral Switching Element." AIEE Transactions, LXXV, pt. I (September, 1956), 400-403.

Turnbull, H. W. The Theory of Determinants. London: Blackie and Son Ltd., 1929.

Van Allen, R. L. "A Variable Frequency Magnetic-Coupled Multivibrator." AIEE Transactions, LXXIV, pt. I (July, 1955), 356-361.

Van Nice, R. I. "Magnetic Logic Circuit Control System Design Considerations." AIEE Transactions, LXXV, pt. I (November, 1956), 595-600.

Veitch, E. W. "A Chart Method for Simplifying Truth-Functions." Proceedings of the Association for Computing Machinery, (May, 1952), 128-133.

Washburn, S. H. "Relay 'Trees' and Symmetric Circuits." AIEE Transactions. LXVIII, pt. I (1949), 582-586.

_______. "An Application of Boolean Algebra to the Design of Electronic Switching Circuits." AIEE Transactions, LXXII, pt. I (September, 1953), 380-388.

Wedderburn, J. H. M. Lectures on Matrices. New York: American Mathematical Society, 1934.

Whitehead, A. N., and B. Russell. Principia Mathematica. 2d. ed. I, II, III. Cambridge, England: The University Press, 1925-1927.

Wilder, Raymond L. Introduction to the Foundations of Mathematics. New York: John Wiley and Sons, 1952.

Yokelson, B. J., and W. Ulrich. "Engineering Multistage Diode Logic Circuits." AIEE Transactions, LXXIV, pt. I (September, 1955), 466-475.

Zeheb, D., and W. P. Cayword. "A Symbolic Method for Synthesis of 2-Terminal Switching Circuits." AIEE Transactions, LXXIII, pt. I (January, 1955), 690-693.

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