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BY

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Contents

Chapter 1

Introduction

One of the fundamental problems in the theory of automorphic forms or representations is the Ramanujan conjecture. Originally formulated by Ramanujan as estimation for the Fourier coefficients of the weight 12 holomorphic cusp form Δ over $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane $\mathfrak{h}$, the conjecture has been generalized to functions over a broader set of groups in terms of local representations of the associated automorphic forms. To review it, let $\mathcal{G}$ be a reductive algebraic group over a number field F, and let $\mathbb{A} := \otimes'_{\nu\leqslant\infty} F_\nu$ be the ring of adeles for F, where F_ν denotes the local field at a place ν. Then, one of the old versions of the Ramanujan conjecture can be stated as follows :

1.0.1. Conjecture: Let $\pi \simeq \otimes'_{\nu\leqslant\infty}\pi_\nu$ be an irreducible cuspidal representation of $\mathcal{G}(\mathbb{A})$, where π_ν denotes the local component of π at the place ν. Then π_ν is tempered for every $\nu \leqslant \infty$.

This naive version of the Ramanujan conjecture is known to be false with the first numerical counter examples being found by Saito and Kurokawa [12]. Adrianov [1], Maass [13] and Zagier [22] showed that the Saito-Kurokawa lift from elliptic cusp forms to holomorphic Siegel cusp forms of degree two always violates the conjecture.

Maass found explicit relations between the Fourier coefficients of the holomorphic Siegel cusp forms which characterize the image of the lift (cf. [13]). We shall refer to these as the Maass relations and to the image as the Maass space. In [9], Ikeda generalized the process of Saito-Kurokawa lifts for holomorphic Siegel cusp forms of higher degree. Kohnen and Kojima characterize the Maass space for Ikeda lifts again via a similar process as that of Maass (cf. [10], [11]). Both these proofs rely crucially on intermediate spaces of Jacobi forms.

While this naive version of the Ramanujan conjecture is strongly believed for the general linear groups, the generalized version is expected only for generic cuspidal representations of quasisplit reductive groups. Muto, Narita and Pitale in [15] provide a counterexample to the Ramanujan conjecture for $\mathrm{GL}_2(B)$ over the division quaternion algebra B with discriminant two. Note that $\mathrm{GL}_2(B)$ is an inner form of the split group GL_4. Unlike the cases of Saito-Kurokawa and Ikeda, the construction in [15] does not involve any intermediate spaces of Jacobi forms. Instead, given Fourier coefficients $c(N)$ of $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ which is an eigenfunction of the Atkin-Lehner involution, they directly define numbers $A(\beta)$ (cf. (3.3)). Then they show that these $A(\beta)$ are the Fourier coefficients of some $F_f \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ by using Maass Converse Theorem (cf. Theorem 3.1.1). Here $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ is the space of Maass forms on the 5-dimensional hyperbolic space with respect to $\mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is the Hurwitz order in B (see Section 2.2 for details). They further show that if f is a Hecke eigenform, then so is F_f and the representation $\Pi_F \simeq \otimes'_{p\leqslant\infty}\Pi_{F,p}$ of $\mathrm{GL}_2(B_{\mathbb{A}})$ corresponding to F_f is a counterexample to the Ramanujan conjecture. They also show that the image of Π_F under the global Jacquet-Langlands correspondence is the irreducible constituent of $\mathrm{Ind}_{P_{2,2}(\mathbb{A})}^{\mathrm{GL}_4(\mathbb{A})}(|\det|_{\mathbb{A}}^{-1/2}\sigma_f \times |\det|_{\mathbb{A}}^{1/2}\sigma_f)$, where σ_f is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to f.

The question we want to answer here is the same as the one Maass answered for the Saito-Kurokawa case in [13]. More precisely, we want to characterize the image of this lift, possibly in terms of recurrence relations between their Fourier coefficients. We tackle this problem by first noticing that $A(\beta)$ depends only on $K = |\beta|^2, u$ and n when $\beta = \varpi_2^u n\beta_0$ as in (3.2) (cf. 4.1.1).

Definition 1.0.1. Let $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ denote the subspace of $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ consisting of functions F whose Fourier coefficients $A(\beta)$ satisfy:

1. If $\beta = \varpi_2^u n\beta_0$ as in (3.2), then $A(\beta)$ depend only on $K = |\beta|^2$, u and n. We shall then write $A(\beta)$ as $A(K, u, n)$.

2. $A(K, u, n)$ satisfy the recurrence relations :

   (a) $A(K, u, n) = \frac{-3\epsilon}{\sqrt{2}} A(\frac{K}{2}, u-1, n) - A(\frac{K}{4}, u-2, n)$ for some $\epsilon \in \{\pm 1\}$,

   (b) $A(K, u, n) = \sum_{d|n} d \cdot A(\frac{K}{d^2}, u, 1)$.

Note that there are no intermediate spaces of Jacobi forms. As a result, we cannot just generalize any of the previous proofs of Maass, Kohnen or Kojima to this case. Instead we take a completely different approach.

We would like to mention that there is a recent work of Pitale-Saha-Schmidt (c.f. [17]) which provides a representation theoretic approach to Saito-Kurokawa lifts which does not use Fourier-Jacobi forms. We note however that the paper is only able to show a one way implication without Jacobi forms, where as we wish to prove both ways.

It is easy to see that the Fourier coefficients of F_f satisfy condition (1). To show that $A(K, u, n)$ also satisfy condition (2a) and (2b), we use Legendre's three-square theorem to isolate $c(N)$ as follows:

Proposition 1.0.1. *Let $N = 4^a b$, where a, b are a non-negative integers and $4 \nmid b$. With assumptions as in Theorem 4.3.1, we get*

$$c(-N) = \frac{A(2N, u, 1)}{\sqrt{2N}} + \epsilon \frac{A(N, u-1, 1)}{\sqrt{N}} \tag{1.1}$$

where

$$u = \begin{cases} 2a & \text{if } b \equiv 1, 3 \mod (4), \\ 2a+1 & \text{if } b \equiv 2 \mod (4). \end{cases}$$

.

Now, we manipulate the defining sum of $A(\beta)$ (c.f. (3.3)) using these $c(N)$ to show that $A(K, u, n)$ indeed satisfy the recurrence relation (2b). The relation (2a) follows from the fact that F_f is a Hecke eigenform at $p = 2$. Hence, we get

Theorem 1.0.2. *Let $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ be an Atkin-Lehner eigenform with eigenvalue $\epsilon \in \{\pm 1\}$ and which is a Hecke eigenform at $p = 2$. Then F_f obtained in Theorem 3.1.1 belongs to the Maass space $\mathcal{M}^*(GL_2(\mathcal{O}), r)$.*

This allows us to determine a "necessary" condition for any $F \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ to be a lift. We would like to show a theorem that this is also a "sufficient" condition. If $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$, we can still isolate $c(N)$ as before and now the question reduces to showing these are the Fourier coefficients of some $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$. As a first approach one can try to use the Maass converse theorem to show the automorphy of a function f with Fourier coefficients $\{c(N)\}$. The difficulty is that the analytic properties of the Dirichlet series associated with F do not translate into analytic properties of Dirichlet series obtained from $\{c(N)\}$. To approach this problem by representation theory, we first add the condition that F is a Hecke eigenform for all primes p and obtain the following theorem.

Theorem 1.0.3. *Let $F \in \mathcal{M}^*(GL_2(\mathcal{O}, r)$ such that F is a cuspidal Hecke eigenform. Then, there exists $f \in S(\Gamma_0(2), -(\frac{1}{4}+\frac{r^2}{4}))$, a Hecke eigenform, such that $F = F_f$.*

We denote by $\Pi_F \simeq \otimes_{p\leqslant\infty}\Pi_{F,p}$ the automorphic representation of $\mathrm{GL}_2(B_{\mathbb{A}})$ associated with F. Let the image of Π_F under the global Jacquet Langlands map be $\Pi \simeq \otimes_{p\leqslant\infty}\Pi_p$, a representation of $\mathrm{GL}_4(\mathbb{A})$. For a cuspidal representation σ of $\mathrm{GL}_2(\mathbb{A})$, we denote by $\mathrm{MW}(\sigma,2)$ the Langlands quotient of $\mathrm{Ind}_{P_{2,2}(\mathbb{A})}^{\mathrm{GL}_4(\mathbb{A})}(|\det|_{\mathbb{A}}^{1/2}\sigma \times |\det|_{\mathbb{A}}^{-1/2}\sigma)$, following the notation of Badulescu and Renard from [5]. The strategy of the proof now is to show that $\Pi = \mathrm{MW}(\sigma,2)$ for σ an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. We show that $\Pi_{F,p}$ is the unique irreducible constituent of some unramified principal series representation $\mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1\times\chi_2\times\chi_3\times\chi_4)$ where each χ_i is an unramified character of $\mathbb{Q}_p^\times$ described in the following proposition.

Proposition 1.0.2. *For every odd prime p, there is a $\lambda_p \in \mathbb{C}$ such that, up to the action of the Weyl group, χ_i are given by the formula*

$$\begin{aligned}\chi_1(p) &= p^{1/2}\frac{\lambda_p+\sqrt{\lambda_p^2-4}}{2}; \quad \chi_2(p) = p^{1/2}\frac{\lambda_p-\sqrt{\lambda_p^2-4}}{2};\\ \chi_3(p) &= p^{-1/2}\frac{\lambda_p+\sqrt{\lambda_p^2-4}}{2}; \quad \chi_4(p) = p^{-1/2}\frac{\lambda_p-\sqrt{\lambda_p^2-4}}{2}\end{aligned} \tag{1.2}$$

This is the most crucial result of the paper. The fact that χ_i are related in this special way and are not arbitrary is an important consequence of the action of the Hecke algebra and recurrence relations from Definition 1.1 (2b). For $p=2$, the structure of the local component $\Pi_{F,2}$ can be obtained from the action of the Hecke algebra and relation (2a). The component $\Pi_{F,\infty}$ follows from Section 6.1 of [15]. Conditions on the Satake parameters give us that Π is indeed of the form $\mathrm{MW}(\sigma,2)$ for some σ representation of $\mathrm{GL}_2(\mathbb{A})$. For an odd prime p, let χ_p be the unramified character of $\mathbb{Q}_p^\times$ such that $\chi_p(p) = \frac{\lambda_p+\sqrt{\lambda_p^2-4}}{2}$. At the prime $p=\infty$, let $\chi_\infty(a) = |a|^s$

where $s = \frac{\sqrt{-1}r}{2}$. For the prime $p = 2$, let χ be an unramified character of $\mathbb{Q}_2^\times$ with $\chi(2) = -\epsilon$ for ϵ as in condition (2a) of Definition 4.1.1.

Proposition 1.0.3. *Let $\sigma = \otimes_{p \leqslant \infty} \sigma_p$ be the irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ such that $\Pi = MW(\sigma, 2)$ as above. Then*

$$\sigma_p = \begin{cases} Ind_{\mathcal{B}_2(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\chi_p \times \chi_p^{-1}) & \text{for odd } p < \infty, \\ \chi St_{GL_2} & \text{for } p = 2, \\ Ind_{\mathcal{B}_2(\mathbb{R})}^{GL_2(\mathbb{R})}(\chi_\infty \times \chi_\infty^{-1}) & \text{for } p = \infty. \end{cases} \tag{1.3}$$

We then look at the distinguished vector in σ to find a function f associated to σ. As σ_2 is Steinberg and σ_∞ is principal series, we can show that $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ as required. We complete the proof by showing that $c(N)$ are indeed the Fourier coefficients of f implying $F = F_f$.

To generalize Theorem 1.0.3 to all $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ we first show that $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ is finite dimensional and has a Hecke eigenbasis of operators that commute with their adjoint. With this, proving that $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ is a Hecke invariant subspace suffices as this implies that $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ has a Hecke eigenbasis $F_i \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ which are lifts of $f_i \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$. By linearity of the defining condition (3.3), $F = \sum_i a_i F_i$ would be a lift of $\sum_i a_i f_i$.

We prove that $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ is Hecke invariant by showing that for all the Hecke operators T_i, the image under their action $T_i(F)$ satisfies the conditions of Definition 4.1.1. The condition that Fourier coefficients of $T_i(F)$ depend only on K, u and n is obtained by writing the coefficients of $T_i(F)$ in terms of $A(K, u, n)$ the Fourier coefficients of F. Since each of these coefficients depends only on K, u and n, so do the coefficients of $T_i(F)$. Condition (2a) is equivalent to F being a Hecke eigenform at prime $p = 2$ so it is valid for all $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$. Condition (2b) is shown by computing the recurrence sum for $(T_{i,p}F)(K,u,n)$ and showing that it is equal to $\sum_{d|n} d(T_{i,p}F)(K/d^2,u,1)$. Hence we get the result:

Theorem 1.0.4. *The following are equivalent.*

1. *F is a lift from an Atkin-Lehner eigenform $f \in S(\Gamma_0(2), -(\frac{1}{4}+\frac{r^2}{4}))$ with eigenvalue $\epsilon \in \{\pm 1\}$ and which is a Hecke eigenform at $p=2$.*

2. *F is an element of the space $\mathcal{M}^*(GL_2(\mathcal{O}), r)$*

Chapter 2

Automorphic forms on 5-dimensional hyperbolic space

2.1 Algebraic groups and the 5-dimensional hyperbolic space

Following the notation of Muto, Narita and Pitale in [15], let B be the definite quaternion algebra over $\mathbb{Q}$ with discriminant 2. In terms of the basis $\{1, i, j, k\}$, $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ with i, j, k satisfying

$$i^2 = j^2 = k^2 = -1, ij = -ji = k.$$

$\mathrm{GL}_2(B)$ will be the group of elements of $M_2(B)$ whose reduced norms are non-zero. Let $\mathbb{H} = B \otimes_{\mathbb{Q}} \mathbb{R}$ be the Hamilton quaternion algebra with $x \to \bar{x}$ the main involution of $\mathbb{H}$. Let $tr(x) = x + \bar{x}$ and $\nu(x) = x\bar{x}$ be the reduced trace and reduced norm of $x \in \mathbb{H}$ respectively, with $|x| = \sqrt{\nu(x)}$.

The general linear group $G := \mathrm{GL}_2(\mathbb{H})$ admits an Iwasawa decomposition

$$G = Z^+NAK,$$

where

$$Z^+ := \left\{ \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \middle| c \in \mathbb{R}_+^\times \right\}, N := \left\{ n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \middle| x \in \mathbb{H} \right\},$$
$$A := \left\{ a_y = \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{bmatrix} \middle| y \in \mathbb{R}^+ \right\}, K := \left\{ k \in G : {}^t\bar{k}k = 1_2 \right\}.$$

The quotient G/Z^+K can be realized as

$$\left\{ \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \middle| y \in \mathbb{R}_+^\times, x \in \mathbb{H} \right\}.$$

This gives a realization of the 5-dimensional real hyperbolic space.

2.2 Lie algebras

The Lie algebra $\mathfrak{g}$ of G is $M_2(\mathbb{H})$ and has an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$$

. Here

$$\mathfrak{z} := \left\{ \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \middle| c \in \mathbb{R} \right\}, \mathfrak{n} := \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \middle| x \in \mathbb{H} \right\},$$
$$\mathfrak{a} := \left\{ \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} \middle| t \in \mathbb{R} \right\}, \mathfrak{k} := \left\{ X \in M_2(\mathbb{H}) | \, {}^t\bar{X} + X = 0_2 \right\}.$$

where $\mathfrak{z}, \mathfrak{n}, \mathfrak{a}$ and $\mathfrak{k}$ are the Lie algebras of Z^+, N, A and K respectively.

To consider the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, let $H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and let α be the linear form of $\mathfrak{a}$ such that $\alpha(H) = 1$. Then $\{\pm 2\alpha\}$ is the set of roots for $(\mathfrak{g}, \mathfrak{a})$. For $z \in \mathbb{H}$ we put

$$E_{2\alpha}^{(z)} := \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}, E_{-2\alpha}^{(z)} := \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}.$$

The set $\{E_{2\alpha}^{(1)}, E_{2\alpha}^{(i)}, E_{2\alpha}^{(j)}, E_{2\alpha}^{(k)}\}$ (respectively $\{E_{-2\alpha}^{(1)}, E_{-2\alpha}^{(i)}, E_{-2\alpha}^{(j)}, E_{-2\alpha}^{(k)}\}$) form a basis of $\mathfrak{n}$ (respectively a basis of $\bar{\mathfrak{n}} := \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \middle| x \in \mathbb{H} \right\}$). Let $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{k}) := \{X \in \mathfrak{k} \mid [X, A] = 0 \ \ \forall A \in \mathfrak{a}\}$, then $\mathfrak{g}$ decomposes into

$$\mathfrak{g} = (\mathfrak{z} \oplus \mathfrak{z}_{\mathfrak{a}}(\mathfrak{k}) \oplus \mathfrak{a}) \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}.$$

Consider the Lie group $\mathrm{SL}_2(\mathbb{H})$ consisting of elements in $\mathrm{GL}_2(\mathbb{H})$ with their reduced norms 1. Its Lie algebra is $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{H})$ consisting of elements in $M_2(\mathbb{H})$ with their reduced traces zero.

We introduce the differential operator Ω defined by the infinitesimal action of

$$\Omega := \frac{1}{32}H^2 - \frac{1}{4}H + \frac{1}{8}\sum_{z\in\{1,i,j,k\}} E_{2\alpha}^{(z)^2}. \tag{2.1}$$

This differential operator Ω coincides with the infinitesimal action of the Casimir element of $\mathfrak{g}_0$ on the space of right K-invariant smooth functions of G/Z^+. In what follows, we shall refer to it as the Casimir operator.

2.3 Automorphic forms

For $\lambda \in \mathbb{C}$ and a discrete subgroup $\Gamma \in \mathrm{SL}_2(\mathbb{R})$, we denote by $S(\Gamma, \lambda)$ the space of Maass cusp forms of weight 0 on the complex upper half plane $\mathfrak{h}$ whose eigenvalue with respect to the hyperbolic Laplacian is $-\lambda$.

For a discrete subgroup $\Gamma \subset \mathrm{GL}_2(\mathbb{H})$ and $r \in \mathbb{C}$, let $\mathcal{M}(\Gamma, r)$ denote the space of smooth functions F on $\mathrm{GL}_2(\mathbb{H})$ which satisfy the following conditions :

1. $\Omega \cdot F = -\frac{1}{2}\left(\frac{r^2}{4}+1\right)F$, where Ω is the Casimir operator defined in (2.1),
2. for any $(z, \gamma, g, k) \in Z^+ \times \Gamma \times G \times K$, we have $F(z\gamma gk) = F(g)$,
3. F is of moderate growth.

For automorphic forms of $\mathrm{SL}_2(\mathbb{R})$ we will concern ourselves only with the congruence subgroup $\Gamma_0(2) \in \mathrm{SL}_2(\mathbb{R})$ of level 2. For the choice of a discrete subgroup of $\mathrm{GL}_2(\mathbb{H})$, note that the definite quaternion algebra B has a unique maximal order $\mathcal{O}$ given by:

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+ij}{2},$$

called the Hurwitz order. The discrete subgroup we shall consider in this case will be $\mathrm{GL}_2(\mathcal{O})$.

We denote by

$$\mathcal{S} \coloneqq \mathbb{Z}(1-ij)+\mathbb{Z}(-i-ij)+\mathbb{Z}(-j-ij)+\mathbb{Z}(2ij) \tag{2.2}$$

the dual lattice of $\mathcal{O}$ with respect to the bilinear form on $\mathbb{H}\times\mathbb{H}$ defined by $\mathrm{Re} = \frac{1}{2}tr$. We denote by $\varpi_2 \coloneqq (1+i)$ which is the uniformizer of $B\otimes_{\mathbb{Q}}\mathbb{Q}_2$.

Lemma 2.3.1. *We have* $\mathcal{S} = \varpi_2\mathcal{O}$

In terms of $\mathcal{S}$, any $F \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ has a Fourier expansion of the form

$$F(n(x)a_y) = u(y) + \sum_{\beta\in S\smallsetminus\{0\}} A(\beta)y^2K_{\sqrt{-1}r}(2\pi|\beta|y)e^{2\pi\sqrt{-1}\mathrm{Re}(\beta x)}. \tag{2.3}$$

Here K_α, with $\alpha\in\mathbb{C}$, is the modified Bessel function, which satisfies the differential equation

$$y^2\frac{d^2K_\alpha}{dy^2}+y\frac{dK_\alpha}{dy}-(y^2+\alpha^2)K_\alpha = 0.$$

We call F cuspidal if $u(y)=0$.

Chapter 3

Lifting to GL(2) over a division quaternion algebra by Muto, Narita and Pitale

3.1 Construction of lift

We first define the set of primitive elements of $\mathcal{S}$, denoted $\mathcal{S}^{prim}$, by

$$\mathcal{S}^{prim} := \{\beta \in \mathcal{S} \setminus \{0\} \mid \varpi_2 | \beta, \varpi_2^2 \nmid \beta, n \nmid \beta \text{ for all odd integers n}\}. \tag{3.1}$$

Any $\beta \in \mathcal{S} \setminus \{0\}$ can then be uniquely written as

$$\beta = \varpi_2^u \cdot n \cdot \beta_0, \tag{3.2}$$

where u is a non-negative integer, n is an odd positive integer and $\beta_0 \in \mathcal{S}^{prim}$.

Let $c(N)$ be the Fourier coefficients of $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$. Assuming it is an Atkin-Lehner eigenfunction with eigenvalue $\epsilon \in \{\pm 1\}$, set

$$A(\beta) := |\beta| \sum_{t=0}^{u} \sum_{d|n} (-\epsilon)^t c\Big(- \frac{|\beta|^2}{2^{t+1} d^2} \Big). \tag{3.3}$$

With $A(\beta)$ as above, Muto, Narita and Pitale prove the following theorem.

Theorem 3.1.1 (Theorem 4.4 in [15])**.** *Let $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ with Fourier coefficients $c(N)$ and eigenvalue ϵ of the Atkin-Lehner involution. Define*

$$F_f(n(x)a_y) := \sum_{\beta \in \mathcal{S} \setminus \{0\}} A(\beta) y^2 K_{\sqrt{-1}r}(2\pi|\beta|y) e^{2\pi\sqrt{-1}Re(\beta x)} \tag{3.4}$$

with $\{A(\beta)\}_{\beta \in \mathcal{S} \setminus \{0\}}$ as in (3.3). Then we have $F_f \in \mathcal{M}(GL_2(\mathcal{O}), r)$ and F_f is a cusp form. Furthermore, $F_f \not\equiv 0$ if $f \not\equiv 0$.

The fundamental tool used in the proof is the following converse theorem of Maass [14].

Theorem 3.1.2 (Maass)**.** *Let $\{A(\beta)\}_{\beta \in S \setminus \{0\}}$ be a sequence of complex numbers such that*

$$A(\beta) = O(|\beta|^\kappa) \qquad (\exists \kappa > 0)$$

and put

$$F(n(x)a_y) := \sum_{\beta \in \mathcal{S} \setminus \{0\}} A(\beta) y^2 K_{\sqrt{-1}r}(2\pi|\beta|y) e^{2\pi\sqrt{-1}Re(\beta x)}.$$

For a Harmonic polynomial P on $\mathbb{H}$ of degree l we introduce

$$\xi(s, P) := \pi^{-2s} \Gamma\left(s + \frac{\sqrt{-1}r}{2}\right) \Gamma\left(s - \frac{\sqrt{-1}r}{2}\right) \sum_{\beta \in S \setminus \{0\}} A(\beta) \frac{P(\beta)}{|\beta|^{2s}},$$

which converges for $Re(s) > \frac{l+4+\kappa}{2}$. Let $\{P_{l,\nu}\}_\nu$ be a basis of Harmonic polynomials on $\mathbb{H}$ of degree l.

Then $F \in \mathcal{M}(\Gamma_T, r)$ is equivalent to the condition that, for any l, ν, the $\xi(s, P_{l,\nu})$ satisfies the following three conditions.

1. *It has analytic continuation to the whole complex plane.*
2. *It is bounded on any vertical strip of the complex plane.*
3. *The functional equation*

$$\xi(2 + l - s, P_{l,\nu}) = (-1)^l \xi(s, \hat{P}_{l,\nu})$$

   holds, where $\hat{P}(x) \coloneqq P(\bar{x})$ for $x \in \mathbb{H}$.

Here, Γ_T is the subgroup of $\mathrm{GL}_2(\mathcal{O})$ generated by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \quad (\beta \in \mathcal{O}).$$

3.2 Action of the Hecke operators

Let $\mathcal{G}(\mathbb{A}) = \mathrm{GL}_2(B_{\mathbb{A}})$, where $B_{\mathbb{A}}$ denotes the adelization of B, and let U be the compact subgroup of $\mathcal{G}(\mathbb{A})$ given by $\Pi_{p<\infty}\mathrm{GL}_2(\mathcal{O}_p)$, where $\mathcal{O}_p$ denotes the p-adic completion of $\mathcal{O}$ at a finite prime p. For a complex number $r \in \mathbb{C}$, the space of automorphic forms for $\mathcal{G}$, denoted $M(\mathcal{G}(\mathbb{A}), r)$, is defined as follows.

Definition 3.1. Let $M(\mathcal{G}(\mathbb{A}), r)$ be the space of smooth functions Φ on $\mathcal{G}(A)$ satisfying the following conditions:

1. $\Phi(z\gamma g u_f u_\infty) = \Phi(g)$ for any $(z, \gamma, g, u_f, u_\infty) \in Z_{\mathbb{A}} \times \mathcal{G}(\mathbb{Q}) \times \mathcal{G}(\mathbb{A}) \times U \times K$, where $Z_{\mathbb{A}}$ denotes the center of $\mathcal{G}(\mathbb{A})$,
2. $\Omega \cdot \Phi(g_\infty) = -\frac{1}{2}\left(\frac{r^2}{4} + 1\right)\Phi(g_\infty)$ for any $g_\infty \in \mathcal{G}(\mathbb{R}) = \mathrm{GL}_2(\mathbb{H})$,

3. Φ is of moderate growth.

The class number of $\mathcal{G}$ with respect to U is one, which means that $\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{Q})\mathcal{G}(\mathbb{R})U$. We can thus view $F \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ as a smooth function Φ_F on $\mathcal{G}(\mathbb{A})$ given by

$$\Phi_F(\gamma g_\infty u_f) = F(g_\infty) \quad \forall(\gamma, g_\infty, u_f) \in \mathcal{G}(\mathbb{Q}) \times \mathcal{G}(\mathbb{R}) \times U.$$

Hence, $M(\mathcal{G}(\mathbb{A}), r) \simeq \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$.

Section 5 of [15] proves that if f is a Hecke eigenform, then F_f is also a Hecke eigenform. For each place $p \leqslant \infty$ let $\mathcal{G}_p := \mathrm{GL}_2(B_p)$ for $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$. For any finite prime $p \neq 2$, we have $\mathrm{GL}_2(B_p) \simeq \mathrm{GL}_4(\mathbb{Q}_p)$. Let $\mathcal{O}_p$ be the p-adic completion of $\mathcal{O}$ as above. Then, for $p \neq 2$, $\mathcal{O}_p \simeq M_2(\mathbb{Z}_p)$ and $\mathrm{GL}_2(\mathcal{O}_p) \simeq \mathrm{GL}_4(\mathbb{Z}_p)$. Set $K_p = \mathrm{GL}_2(\mathcal{O}_p)$ for all $p < \infty$.

According to [20], the Hecke algebra of $\mathrm{GL}_2(B_p)$ with respect to $\mathrm{GL}_2(\mathcal{O}_p)$ is generated by:

$$\begin{cases} \{\varphi_1^{\pm 1}, \varphi_2\} & \text{if } p = 2 \\ \{\phi_1^{\pm 1}, \phi_2, \phi_3, \phi_4\} & \text{if } p \neq 2. \end{cases}$$

Here φ_1, φ_2 denote the characteristic functions of

$$K_2 \begin{bmatrix} \varpi_2 & 0 \\ 0 & \varpi_2 \end{bmatrix} K_2, K_2 \begin{bmatrix} \varpi_2 & 0 \\ 0 & 1 \end{bmatrix} K_2$$

respectively, whereas $\phi_1, \phi_2, \phi_3, \phi_4$ denote the characteristic functions of $K_p h_i K_p$

where $h_i, 1 \leqslant i \leqslant 4$ are

$$\begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}, \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

respectively for $p \neq 2$. We will define the set $C_p \coloneqq \{\alpha \in \mathcal{O} \mid \nu(\alpha) = p\}/\mathcal{O}^\times$. The following Proposition 5.8 from [15] allows us to explicitly compute the action of the Hecke operators on the Fourier coefficients $A(\beta)$ of $F \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$.

Proposition 3.2.1 (Proposition 5.8 from [15]). *1. Let $p = 2$ and $h = \begin{bmatrix} \varpi_2 & 0 \\ 0 & 1 \end{bmatrix}$. We obtain*

$$(K_2 h K_2 \cdot F)_\beta = 2(A(\beta\varpi_2^{-1}) + A(\beta\varpi_2)).$$

2. Let p be an odd prime and $\beta \in \mathcal{S} \smallsetminus \{0\}$.

(a) When $h = \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix}$,

$$(K_p h K_p \cdot F)_\beta = p(\sum_{\alpha \in C_p} A(\beta\bar{\alpha}^{-1}) + \sum_{\alpha \in C_p} A(\bar{\alpha}\beta)).$$

(b) When $h = \begin{bmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$,

$$(K_p h K_p \cdot F)_\beta = p(\sum_{\alpha \in C_p} A(\alpha^{-1}\beta) + \sum_{\alpha \in C_p} A(\beta\alpha)).$$

(c) When $h = \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$,

$$(K_p h K_p \cdot F)_\beta = (p^2 A(p^{-1}\beta) + p^2 A(p\beta) + p \sum_{(\alpha_1,\alpha_2) \in C_p \times C_p} A(\alpha_1^{-1}\beta\alpha_2)).$$

3.3 Automorphic representation corresponding to the lifting

The above action of the Hecke algebra allows us to find the Hecke eigenvalues for F_f in terms of the Hecke eigenvalues of f.

Proposition 3.3.1. *Let* $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ *be a Hecke eigenform with eigenvalue* λ_p *for every odd prime* p *and the Atkin-Lehner eigenvalue* ϵ*. Then* $F = F_f$ *as defined in Theorem 3.1.1 is a Hecke eigenform with eigenvalues* ${}_p\mu_1, {}_p\mu_2, {}_p\mu_3, {}_p\mu_4$ *for* $\phi_1, \phi_2, \phi_3, \phi_4$ *respectively at every odd prime* p *and* ${}_2\mu_1$, ${}_2\mu_2$ *for* φ_1, φ_2 *at* $p = 2$.

They are related as

$$ {}_p\mu_1 = 1, {}_p\mu_2 = {}_p\mu_4 = p(p+1)\lambda_p, {}_p\mu_3 = p^2\lambda_p^2 + p^3 + p \tag{3.5} $$

and

$$ {}_2\mu_1 = 1, {}_2\mu_2 = -3\sqrt{2}\epsilon. \tag{3.6} $$

This is proved in Proposition 5.9, 5.11 and 5.12 of [15]. Let the representation π_F denote the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(B_{\mathbb{A}})$ corresponding to F_f with $B_{\mathbb{A}} := \otimes'_{p\leqslant\infty} B_p$ generated by right translates of Φ_F (as in Definition 3.1). π_F is cuspidal as F_f is a cusp form and the irreducibility follows from the strong multiplicity-one result for $\mathrm{GL}_2(B_{\mathbb{A}})$ (c.f. [4],[5]). Let $\pi_F = \otimes'_p \pi_p$, where π_p an irreducible admissible representation of $\mathrm{GL}_2(B_p)$ for $p < \infty$ and π_∞ is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{H})$.

Let $\mathcal{B}_2$ and $\mathcal{B}_4$ denote the group of upper triangular matrices in GL_2 and GL_4 respectively. Then, for $p < \infty$ and odd, π_p is the unique spherical constituent of the unramified principal series representation $\mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1 \times \chi_2 \times \chi_3 \times \chi_4)$ where χ_i are unramified character of $\mathbb{Q}_p^\times$ given by

$$ \begin{aligned} \chi_1(p) = p^{1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}, \ \chi_2(p) = p^{1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2}, \\ \chi_3(p) = p^{-1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}, \ \chi_4(p) = p^{-1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2}. \end{aligned} \tag{3.7} $$

For $p = 2$, π_2 is the unique spherical constituent of the unramified principal series representation $\mathrm{Ind}_{\mathcal{B}_2(\mathbb{Q}_2)}^{\mathrm{GL}_2(B_2)}(\chi_1 \times \chi_2)$ with

$$ \chi_1(\varpi_2) = -\sqrt{2}\epsilon, \chi_2(\varpi_2) = -1/\sqrt{2}\epsilon. \tag{3.8} $$

At the prime $p = \infty$, the archimedian component π_∞ is isomorphic to the principal series $\mathrm{Ind}_{\mathcal{B}_2(\mathbb{H})}^{\mathrm{GL}_2(\mathbb{H})}(\chi_{\pm\sqrt{-1}r/2})$ with

$$\chi_s\left(\begin{bmatrix} a & * \\ 0 & d \end{bmatrix}\right) = \nu(ad^{-1})^s. \tag{3.9}$$

These local representations are explicitly constructed in Section 6 of [15].

Chapter 4

Maass space in $\mathcal{M}(\mathbf{GL}_2(\mathcal{O}), r)$

4.1 Definition of the Maass space

We will call the image of the lift constructed in Theorem 3.1.1 to $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ as the Maass space. To characterize the functions in the Maass space, we first define the following subspace.

Definition 4.1.1. Let $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ denote the subspace of cusp forms F in $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ Fourier coefficients $A(\beta)$ satisfy:

1. If $\beta = \varpi_2^u n \beta_0$ as in (3.2), then $A(\beta)$ depend only on $K = |\beta|^2$, u and n. We shall then write $A(\beta)$ as $A(K, u, n)$.

2. $A(K, u, n)$ satisfy the recurrence relations :

   (a) $A(K, u, n) = \frac{-3\epsilon}{\sqrt{2}} A(\frac{K}{2}, u-1, n) - A(\frac{K}{4}, u-2, n)$ for some $\epsilon \in \{\pm 1\}$,

   (b) $A(K, u, n) = \sum_{d|n} d \cdot A\left(\frac{K}{d^2}, u, 1\right)$.

We will define $A(K, u, n) = 0$ if u is negative. These recurrence relations are similar to those of Maass in the case of Saito-Kurokawa lifts in [13].

4.2 Isolating c(-N)

All our information about the Fourier coefficients $\{A(\beta)\}_{\beta\in S\smallsetminus\{0\}}$ of F_f is obtained from the Fourier coefficients $c(-N)$ of f from equation (3.3). To do any successful manipulation of $A(\beta)$, we would ideally like to have a formula for $c(-N)$ in terms of the Fourier coefficients $A(\beta)$.

Proposition 4.2.1. *Let $f \in S(\Gamma_0(2), -(\frac{1}{4}+\frac{r^2}{4}))$ be an Atkin-Lehner eigenform with eigenvalue $\epsilon \in \{\pm 1\}$ and which is a Hecke eigenform at $p = 2$. We will denote by the Fourier coefficients of f by $\{c(N)\}$. Let F_f be as in Theorem 3.1.1. For $N = 4^a b$ with a, b non-negative integers and $4 \nmid b$, we get*

$$c(-N) = \frac{A(2N, u, 1)}{\sqrt{2N}} + \epsilon \frac{A(N, u-1, 1)}{\sqrt{N}} \tag{4.1}$$

where

$$u = \begin{cases} 2a & \text{if } b \equiv 1, 3 \mod (4) \\ 2a+1 & \text{if } b \equiv 2 \mod (4) \end{cases}$$

Note that there is more than one β with the same K, u and n. However, by construction in equation (3.3) all such β give the same $A(\beta)$. As such, $c(-N)$ is well defined in terms of β representatives of $A(K, u, n)$. We will need the following lemmas for the proof of Proposition 4.2.1.

Lemma 4.2.1. $\beta = (x + yi + zj + wij) \in \mathcal{S}^{prim}$ *iff* $|\beta|^2 \equiv 2 \mod 4$ *and* $\gcd(\beta) := \gcd(x, y, z, w) = 1$.

Proof. First, we prove the following claim.

Claim 1. Let $\beta \in \mathcal{O}$. Then $\beta \in \mathcal{S}$ iff $|\beta|^2 \equiv 0, 2 \mod 4$.

Proof of claim. Simplifying the condition from (2.2), we see that $x + y + z + w \equiv 0$

mod 2 and therefore $x^2+y^2+z^2+w^2 \equiv 0 \mod 2$ or equivalently $x^2+y^2+z^2+w^2 \equiv 0,2 \mod 4$. If $\beta \in \mathcal{O}$ such that $|\beta|^2 \equiv 0 \mod 2$ then by parity conditions $x+y+z+w \equiv 0 \mod 2$ implying $\beta \in \mathcal{S}$. Hence, $\beta \in S$ iff $|\beta|^2 \equiv 0,2 \mod 4$. $\square$

Now, consider an element $\beta_1 \in \mathcal{S}$ with $\gcd(\beta_1) = 1$ such that $\beta_1 \notin \mathcal{S}^{prim}$. Then by definition of $\mathcal{S}^{prim}$ in (3.1) this means $\beta_1 = \varpi_2^2\beta$ for some $\beta \in \mathcal{O}$. Now, $|\varpi_2^2|^2 = 4$ hence, $4||\beta_1|^2$ implying $|\beta_1|^2 \equiv 0 \mod 4$.

Conversely, for $\beta \in \mathcal{S}$ if $|\beta|^2 \equiv 0 \mod 4$ then $|\varpi_2^{-1}\beta|^2 \equiv 0,2 \mod 4$. Since $\varpi_2^{-1} = \frac{1-i}{2}$ it is an easy verification that $\varpi_2^{-1}\beta \in \mathcal{O}$. Then by claim, $\varpi_2^{-1}\beta \in \mathcal{S}$. Therefore, $\beta \in \varpi_2\mathcal{S}$ which is to say $\beta \notin \mathcal{S}^{prim}$.

Therefore, $\beta \in \mathcal{S}$ satisfies $|\beta|^2 \equiv 2 \mod 4$ with $\gcd(\beta) = 1$ iff $\beta \notin \varpi_2 S$ and equivalently $\beta \in \mathcal{S}^{prim}$ as required. $\square$

For any N, the easiest way for there to exist a β with $\gcd(\beta) = 1$ and $|\beta|^2 = 2N$ is if $w = 1$ with $x^2+y^2+z^2 = 2N-1$. By Legendre's three square Theorem, an odd number $2N-1$ cannot be written as a sum of three squares iff $2N-1 \equiv 7 \mod 8 \Leftrightarrow 2N \equiv 0 \mod 8 \Leftrightarrow N \equiv 0 \mod 4$. However, if $x^2+y^2+z^2 = 2N-1$ and x, y, z are all odd then $4|(x^2+y^2+z^2+1)$ and hence $\beta \notin \mathcal{S}^{prim}$.

Lemma 4.2.2. *If $N \equiv 1,3 \mod 4$, then there is a $\beta \in \mathcal{S}^{prim}$ such that*

$$c(-N) = \frac{A(\beta)}{\sqrt{2N}}. \tag{4.2}$$

Proof of Lemma 4.2.2. If $N \equiv 1,3 \mod 4$ then $2N-1 \equiv 1,5 \mod 8$ respectively. Therefore, by Legendre's theorem, there exist $x, y, z \in \mathbb{Z}$, not all odd, such that $2N-1 = x^2+y^2+z^2$. Hence, by Lemma 4.2.1, $\beta = (x+yi+zj+ij) \in \mathcal{S}^{prim}$ and

$\beta = \varpi_2^0 \cdot 1 \cdot \beta$. Then (3.3) becomes

$$A(\beta) = |\beta| c\Big(\frac{-|\beta|^2}{2}\Big) = \sqrt{2N} c(-N).$$

Solving for $c(-N)$ gives the required result. $\square$

If $N \equiv 2 \mod 4$, we can still find $x, y, z \in \mathbb{Z}$ such that $x^2+y^2+z^2+1^2 = 2N$. As $|\beta|^2 = 2N \equiv 0 \mod 4$, $\beta \in \mathcal{S}$ but $\beta \notin \mathcal{S}^{prim}$ by Lemma 4.2.1. However, $\varpi_2^{-2}\beta \notin \mathcal{S}$ as $|\varpi_2^{-2}\beta|^2 \equiv 1 \mod 4$. Therefore $\beta = \varpi_2^1 \cdot 1 \cdot \beta_0$ for some $\beta_0 \in \mathcal{S}^{prim}$.

If $N \equiv 0 \mod 4$ then one cannot find $x, y, z \in \mathbb{Z}$ such that $x^2 + y^2 + z^2 = 2N$. In that case, write $N = 4^a b$ with $4 \nmid b$. Then, we can find x, y and z such that $x^2 + y^2 + z^2 + 1 = 2b$. This allows us to find $\beta \in \mathcal{S}$ such that $|\beta|^2 = 2N$ and $\beta = \varpi_2^{2a} \cdot 1 \cdot \beta_0$ if $b \equiv 1, 3 \mod 4$ and $\beta = \varpi_2^{2a+1} \cdot 1 \cdot \beta_0$ if $b \equiv 2 \mod 4$ with $\beta_0 \in \mathcal{S}^{prim}$.

Lemma 4.2.3. *If $N \equiv 0, 2 \mod 4$, then $\exists \beta \in \mathcal{S}$ such that*

$$c(-N) = \frac{A(\beta)}{\sqrt{2N}} + \epsilon \frac{A(\varpi_2^{-1}\beta)}{\sqrt{N}}. \tag{4.3}$$

Proof of Lemma 4.2.3. Let $\beta = \varpi_2^u \cdot 1 \cdot \beta_0 \in \mathcal{S}$ with $|\beta|^2 = 2N$ as in the remark before the lemma. Unlike Lemma 4.2.2, in this case $u \neq 0$. Following (3.3), we get

$$\begin{aligned} A(\beta) =& |\beta| \sum_{t=0}^{u} (-\epsilon)^t c\left(-\frac{|\beta|^2}{2^{t+1}}\right) \\ =& \sqrt{2N} c(-N) + \sqrt{2N} \sum_{t=1}^{u} (-\epsilon)^t c\left(-\frac{|\beta|^2}{2^{t+1}}\right) \end{aligned}$$

But

$$\begin{aligned}\sum_{t=1}^{u}(-\epsilon)^t c\left(-\frac{|\beta|^2}{2^{t+1}}\right) &= \frac{-\epsilon}{\sqrt{N}}\sqrt{N}\sum_{t=0}^{u-1}(-\epsilon)^t c\left(-\frac{|\beta|^2/2}{2^{t+1}}\right)\\ &= \frac{-\epsilon}{\sqrt{N}}\sqrt{N}\sum_{t=0}^{u-1}(-\epsilon)^t c\left(-\frac{|\varpi_2^{-1}\beta|^2}{2^{t+1}}\right)\\ &= \frac{-\epsilon}{\sqrt{N}}A(\varpi_2^{-1}\beta).\end{aligned}$$

Rearranging the terms gives us the required result. □

Proof of Proposition 4.2.1. Let $N = 4^a b$ with a and b as in the statement of the proposition. If $b \equiv 1, 3 \mod 4$ then by Lemma 4.2.2, we can find $\beta' \in \mathcal{S}^{prim}$ such that $|\beta'|^2 = 2b$. Then $\beta = \varpi_2^{2a}\beta'$ has $|\beta|^2 = 2N$, $u = 2a$ and $n = 1$. Hence, in this case $A(\beta) = A(2N, 2a, 1)$. Then $A(\varpi_2^{-1}\beta) = A(N, 2a-1, 1)$ and the first condition of the proposition follows from Lemma 4.2.3. If $a = 0$ then we are in case of Lemma 4.2.2 with $A(N, -1, 1) = 0$.

If $b \equiv 2 \mod 4$ then we can find $\beta' = \varpi_2 \cdot 1 \cdot \beta_0 \in \mathcal{S}$ such that $\beta = \varpi_2^{2a}\beta'$ satisfies $A(\beta) = A(2N, 2a+1, 1)$ and $A(\varpi_2^{-1}\beta) = A(N, 2a, 1)$. The second condition of the proposition now follows from Lemma 4.2.3. □

4.3 First result

Now that we have an explicit formula for the Fourier coefficients $c(-N)$ in terms of the Fourier coefficients $A(K, u, n)$ of F_f, we can prove our first result towards identifying the image of the lift.

Theorem 4.3.1. *Let $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ be an Atkin-Lehner eigenform with eigenvalue $\epsilon \in \{\pm 1\}$ and which is a Hecke eigenform at $p = 2$. Then F_f obtained in Theorem 3.1.1 belongs to the subspace $\mathcal{M}^*(GL_2(\mathcal{O}), r)$.*

Proof. The Fourier coefficients of F_f are given in terms of the Fourier coefficients of f as in equation (3.3) by:

$$A(\beta) = |\beta| \sum_{t=0}^{u} \sum_{d|n} (-\epsilon)^t c\Big(- \frac{|\beta|^2}{2^{t+1} d^2} \Big).$$

The only properties of β used in here are $|\beta|, u$ and n. Replacing $|\beta|$ with $|\beta|^2$ we can say that the Fourier coefficients $A(\beta)$ of F_f are depend only on $|\beta|^2, u$ and n, satisfying the first condition of Definition 4.1.1.

To prove equation (2b), we use the value of $c(-N)$ from (4.1) and substitute $A(K, u, n)$ for $A(\beta)$. Doing so, we get:

$$\begin{aligned}
A(K, u,n) &= \sqrt{K} \sum_{t=0}^{u} \sum_{d|n} (-\epsilon)^t c\left(- \frac{K}{2^{t+1} d^2} \right) \\
&= \sqrt{K} \sum_{t=0}^{u} \sum_{d|n} (-\epsilon)^t \left(\frac{A(K/(2^t d^2), u-t, 1)}{\sqrt{K/(2^t d^2)}} \right. \\
&\quad \left. + \epsilon \frac{A(K/(2^{t+1} d^2), u-t-1, 1)}{\sqrt{K/(2^{t+1} d^2)}} \right) \\
&= \sqrt{K} \sum_{d|n} \left(\sum_{t=0}^{u} \left((-\epsilon)^t \frac{A((K/d^2)/2^t, u-t, 1)}{\sqrt{(K/d^2)/2^t}} \right. \right. \\
&\quad \left. \left. - (-\epsilon)^{t+1} \frac{A((K/d^2)/2^{t+1}, u-(t+1), 1)}{\sqrt{(K/d^2)/2^{t+1}}} \right) \right) \\
&= \sqrt{K} \sum_{d|n} \frac{A(K/d^2, u, 1)}{\sqrt{K/d^2}} \\
&= \sum_{d|n} d A(K/d^2, u, 1)
\end{aligned}$$

For equation (2a), note that F_f is Hecke eigenform by Proposition 5.9 of [15] for $p = 2$ since we have assumed the same for f. Then, by Proposition 5.10 of [15], the

Fourier coefficients of the lift F_f satisfy

$$2(A(\beta\varpi_2) + A(\beta\varpi_2^{-1})) = -3\sqrt{2}\epsilon A(\beta)$$

with $A(\beta\varpi_2^{-1}) = 0$ if $u = 0$. Writing it in terms of K, u and n, we get

$$2(A(2K, u+1, n) + A(K/2, u-1, n)) = -3\sqrt{2}\epsilon A(K, u, n)$$

or equivalently

$$A(K, u, n) = \frac{-3\epsilon}{\sqrt{2}} A(\frac{K}{2}, u-1, n) - A(\frac{K}{4}, u-2, n)$$

for $u \geqslant 1$ with $A(\frac{K}{4}, u-2, n) = 0$ for $u = 1$. □

We now wish to prove the converse of Theorem 4.3.1 which is to show that our 'necessary' condition is also 'sufficient'. We do so first in Theorem 6.1.1 for the case of Hecke eigenforms and then in Theorem 6.3.1 for the general case.

Chapter 5

The Jacquet-Langlands correspondence for $\mathbf{GL}(2, B) \leftrightarrow \mathbf{GL}(4)$

5.1 Description of the automorphic representation

If F is a cuspidal Hecke eigenform, let the automorphic representation associated with it be denoted by $\Pi_F \simeq \otimes'_{p \leqslant \infty} \Pi_{F,p}$. At every prime p, the local component $\Pi_{F,p}$ is a spherical representation of $\mathrm{GL}_2(B_p)$ with $B_p = B \otimes \mathbb{Q}_p$. The representation is cuspidal since the Hecke eigenform F is cuspidal.

For every odd $p < \infty$ we have $\mathrm{GL}_2(B_p) \cong \mathrm{GL}_4(\mathbb{Q}_p)$. From Section 5.2 and 6.1 of [15], we have $\Pi_{F,p}$ is the unique irreducible constituent of some unramified principal series representation $\mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1 \times \chi_2 \times \chi_3 \times \chi_4)$ where each χ_i is an unramified character of $\mathbb{Q}_p$.

For $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$, the characters $\chi_1, \chi_2, \chi_3, \chi_4$ have a special form. This is proved in the next proposition.

Proposition 5.1.1. *For every odd prime p, there is a $\lambda_p \in \mathbb{C}$ such that, up to the action of the Weyl group, χ_i are given by the formula*

$$\chi_1(p) = p^{1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}; \quad \chi_2(p) = p^{1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2};$$
$$\chi_3(p) = p^{-1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}; \quad \chi_4(p) = p^{-1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2} \tag{5.1}$$

The proof of the proposition will use [15, Lemma 5.10].

Lemma 5.1.1 ([15, Lemma 5.10])**.** *Let $\beta \in \mathcal{S}^{prim}$. Then*

$$\#\{\alpha \in C_p : p|\beta\alpha\} = \#\{\alpha \in C_p : p|\alpha\beta\} = \begin{cases} 1 & \text{if } p \mid |\beta|^2, \\ 0 & \text{if } p \nmid |\beta|^2 \end{cases}$$

In addition, p^2 does not divide $\alpha\beta$ or $\beta\alpha$ for any $\alpha \in C_p$

Here $C_p \coloneqq \{\alpha \in \mathcal{O} | \nu(\alpha) = p\}/\mathcal{O}^\times$ with $\#(C_p) = (p+1)$. In terms of $A(K, u, n)$, if $A(\beta) = A(K, 0, 1)$ with $p \nmid K$ then $A(\alpha\beta) = A(\beta\alpha) = A(pK, 0, 1)$ for every $\alpha \in C_p$. If $p \mid K$ then there are unique $\alpha_1, \alpha_2 \in C_p$ (not necessarily different) such that $A(\alpha_1\beta) = A(\beta\alpha_2) = A(pK, 0, p)$ and $A(\alpha\beta) = A(\beta\alpha) = A(pK, 0, 1)$ in every other case.

Proof of Proposition 5.1.1. It is enough to show that the Hecke eigenvalues ${}_p\mu_1$, ${}_p\mu_2$, ${}_p\mu_3$, ${}_p\mu_4$ for F satisfy the equation (3.5). The fact that this is enough follows from the proof of Proposition 6.2 from [15]. We will follow notation of Proposition 3.2.1 for the Hecke algebra and refer to diagonal matrices given before it by h_2, h_3 and h_4 respectively.

Since the Maass form F is non-zero, at least one of the Fourier coefficients $A(K, u, n)$ is non-zero. This implies, from the recurrence conditions of Definition 4.1.1, that there exists at least one K such that $A(K, 0, 1) \neq 0$. Let $K = p^n K_0$ where K_0 is co-prime to p. Then we claim that

$$\lambda_p = \frac{A(p^{n+1}K_0, 0, 1) + A(p^{n-1}K_0, 0, 1)}{A(p^n K_0, 0, 1)} \tag{5.2}$$

with $A(p^{n-1}K_0, 0, 1) = 0$ if $n = 0$.

Case 1: $K = p^0 K_0$ i.e. to say $p \nmid K$. Let β such that $A(\beta) = A(K, 0, 1)$. By Lemma 5.1.1, for every $\alpha \in C_p$, we have $\beta\alpha^{-1} \notin \mathcal{O}$ and $p \nmid \overline{\alpha}\beta$. Therefore, $A(\beta\bar{\alpha}^{-1}) = 0$ and $A(\bar{\alpha}\beta) = A(pK, 0, 1)$ for every $\alpha \in C_p$. Then, condition 2(a) of Proposition 3.2.1 implies

$$\begin{aligned}
(K_p h_2 K_p \cdot F)_\beta =& p\big(\sum_{\alpha \in C_p} A(\beta\bar{\alpha}^{-1}) + \sum_{\alpha \in C_p} A(\bar{\alpha}\beta)\big) \\
=& p\big(\sum_{\alpha \in C_p} 0 + \sum_{\alpha \in C_p} A(pK, 0, 1)\big) \\
=& p(p+1)A(pK, 0, 1).
\end{aligned}$$

Hence, we get ${}_p\mu_2 = p(p+1)\lambda_p$ with λ_p given in (5.2) as required. The same exact argument also proves that ${}_p\mu_4 = p(p+1)\lambda_p$.

To show ${}_p\mu_3 = p^2\lambda_p^2 + p^3 + p$, we use condition 2(c) of Propositon 3.2.1 and get

$$\begin{aligned}(K_p h_3 K_p \cdot F)_\beta =& \Big(p^2 A(p^{-1}\beta) + p^2 A(p\beta) + p \sum_{(\alpha_1,\alpha_2)\in C_p\times C_p} A(\alpha_1^{-1}\beta\alpha_2)\Big) \\ =& p^2\cdot 0 + p^2 A(p^2K,0,p) + p((p+1)A(K,0,1)) \\ =& p^2 A(p^2K,0,p) + (p^2+p)A(K,0,1) \\ =& p^2(pA(K,0,1) + A(p^2K,0,1)) \\ &+ p(p+1)A(K,0,1) \qquad (5.3) \\ =& p^3 A(K,0,1) + pA(K,0,1) \\ &+ p^2(A(p^2K,0,1) + A(K,0,1)).\end{aligned}$$

The $A(p^{-1}\beta) = 0$ since $p \nmid \beta$. By Lemma 5.1.1, for each α_2 there is a unique α_1 such that $\alpha_1^{-1}\beta\alpha_2 \in \mathcal{O}$. Hence, there are total $(p+1)$ copies of $A(K,0,1)$, one for each α_2. For (5.3), we are using our recurrence relation (2b) of Definition 4.1.1 to expand $A(p^2K,0,p) = pA(K,0,1) + A(p^2K,0,1)$.

It now suffices to prove $A(p^2K,0,1) + A(K,0,1) = \lambda_p^2 A(K,0,1)$ to show ${}_p\mu_3 = p^2\lambda_p^2 + p^3 + p$ as in (3.5). However, we know $\lambda_p^2 A(K,0,1) = \lambda_p(\lambda_p A(K,0,1)) = \lambda_p(A(pK,0,1))$ from the argument for ${}_p\mu_2$. Let β' be such that $A(\beta') = A(pK,0,1)$.

Then, it follows that

$$
\begin{aligned}
p(p+1)\lambda_p A(pK,0,1) =& (K_p h_2 K_p \cdot F)_{\beta'} \\
=& p\big(\sum_{\alpha\in C_p} A(\beta'\bar{\alpha}^{-1}) + \sum_{\alpha\in C_p} A(\bar{\alpha}\beta')\big) \\
=& p(A(K,0,1) + A(p^2K,0,p) \\
& + pA(p^2K,0,1)) \qquad (5.4) \\
=& p(A(K,0,1) + pA(K,0,1) + A(p^2K,0,1) \\
& + pA(p^2K,0,1)) \qquad (5.5) \\
=& p(p+1)(A(K,0,1) + A(p^2K,0,1)).
\end{aligned}
$$

We use Lemma 5.1.1 to expand out the sums to obtain (5.4). Since $p|pK$, there exists a unique $\alpha \in C_p$ such that $\alpha^{-1}\beta' \in \mathcal{O}$ and $A(\beta'\bar{\alpha}^{-1})$ $= A(K,0,1)$ for that α. $A(\beta'\bar{\alpha}^{-1}) = 0$ in the other p cases. In the second sum, there exists a unique $\alpha \in C_p$ such that $A(\bar{\alpha}\beta') = A(p^2K,0,p)$. In the other p cases, $A(\bar{\alpha}\beta') = A(p^2K,0,1)$. We use the recurrence relation (2b) of Definition 4.1.1 again to obtain (5.5). Hence, ${}_p\mu_3 = p^2\lambda_p + p^3 + p$ as required, completing the first case.

Case 2: $K = p^n K_0$ with $n > 0$. Let β be such that $A(\beta) = A(p^n K_0, 0, 1)$ where

K_0 is an even number co-prime to p.

$$\begin{aligned}(K_p h_2 K_p \cdot F)_\beta =& p\big(\sum_{\alpha \in C_p} A(\beta\bar{\alpha}^{-1}) + \sum_{\alpha \in C_p} A(\bar{\alpha}\beta)\big) \\ =& p(A(p^{n-1}K_0, 0, 1) + A(p^{n+1}K_0, 0, p) \\ &+ pA(p^{n+1}K_0, 0, 1)) \\ =& p(A(p^{n-1}K_0, 0, 1) + A(p^{n+1}K_0, 0, 1) \\ &+ pA(p^{n-1}K_0, 0, 1) + pA(p^{n+1}K_0, 0, 1)) \\ =& p(p+1)(A(p^{n-1}K_0, 0, 1) + A(p^{n+1}K_0, 0, 1)).\end{aligned} \tag{5.6}$$

We use Lemma 5.1.1 again to write the sums in terms of $A(K, u, n)$. As $p|p^n K_0$ but $p \nmid \beta$, there exists a unique $\alpha \in C_p$ such that $\beta\alpha^{-1} \in \mathcal{O}$, for which $A(\beta\bar{\alpha}^{-1}) = A(p^{n-1}K_0, 0, 1)$. As before, $A(\beta\bar{\alpha}^{-1}) = 0$ in all the other p cases. In the second sum, there exists a unique $\alpha \in C_p$ such that $A(\bar{\alpha}\beta) = A(p^{n+1}K_0, 0, p)$. In the other p cases, $A(\bar{\alpha}\beta) = A(p^{n+1}K_0, 0, 1)$. We obtain equation (5.6) then by using the recurrence relation (2b) of Definition 4.1.1 to expand $A(p^{n+1}K_0, 0, p)$. Hence, we get ${}_p\mu_2 = p(p+1)\lambda_p$ with λ_p given in (5.2) as required. Once again, the same exact argument also proves that ${}_p\mu_4 = p(p+1)\lambda_p$.

To show that ${}_p\mu_3 = \lambda_p^2 p^2 + p^3 + p$, we have to consider two subcases: $n = 1$ and $n \geqslant 2$. We will set both cases up and prove them together. **Subcase 1:** Letting

$n = 1$, we get

$$
\begin{aligned}
(K_p h_3 K_p \cdot F)_\beta =& \Big(p^2 A(p^{-1}\beta) + p^2 A(p\beta) + p \sum_{(\alpha_1,\alpha_2)\in C_p\times C_p} A(\alpha_1^{-1}\beta\alpha_2) \Big) \\
=& p^2 \cdot 0 + p^2 A(p^3K_0, 0, p) + p((p+1)A(pK_0, 0, 1) \\
&+ pA(pK_0, 0, 1)) \\
=& p^2(pA(pK_0, 0, 1) + A(p^3K_0, 0, 1)) \\
&+ (2p^2 + p)A(pK_0, 0, 1) \qquad (5.7) \\
=& p^3A(pK_0, 0, 1) + pA(pK_0, 0, 1) + p^2(A(p^3K_0, 0, 1) \\
&+ 2A(pK_0, 0, 1)).
\end{aligned}
$$

We will again use Lemma 5.1.1 to simplify the terms in the summation. The first term, $A(p^{-1}\beta) = 0$ as $p \nmid \beta$. Since $p|pk$, there exist unique $\alpha_2' \in C_p$ such that $p|\beta\alpha_2'$. For that α_2', $A(\alpha_1^{-1}\beta\alpha_2') = A(pK_0, 0, 1)$ for every $\alpha_1 \in C_p$. In the other p cases of α_2's, there exists a unique $\alpha_1 \in C_p$ such that $\alpha_1^{-1}\beta\alpha_2 \in \mathcal{O}$ and $A(\alpha_1^{-1}\beta\alpha_2') = A(pK_0, 0, 1)$. We use the recurrence relation (2b) of Definition 4.1.1 to expand $A(p^3K_0, 0, p)$ to obtain (5.7).

Subcase 2: $n \geqslant 2$

$$\begin{aligned}
&(K_p h_3 K_p \cdot F)_\beta \\
&\qquad =\Big(p^2 A(p^{-1}\beta) + p^2 A(p\beta) + p \sum_{(\alpha_1,\alpha_2)\in C_p\times C_p} A(\alpha_1^{-1}\beta\alpha_2)\Big) \\
&\qquad =p^2 \cdot 0 + p^2 A(p^{n+2}K_0, 0, p) + p(pA(p^n K_0, 0, 1) \\
&\qquad +A(p^n K_0, 0, p) + pA(p^n K_0, 0, 1)) \\
&\qquad =p^2(pA(p^n K_0, 0, 1) + A(p^{n+2}K_0, 0, 1)) \\
&\qquad +p((2p+1)A(p^n K_0, 0, 1) + pA(p^{n-2}K_0, 0, 1)) \qquad (5.8) \\
&\qquad =p^2(A(p^{n+2}K_0, 0, 1) + A(p^{n-2}K_0, 0, 1) + 2A(p^n K_0, 0, 1)) \\
&\qquad +(p^3 + p)A(p^n K_0, 0, 1).
\end{aligned}$$

Once again, using Lemma 5.1.1 to simplify the summation, we get $A(p^{-1}\beta) = 0$ as $p \nmid \beta$. Since $p | p^n K_0$, there exists a unique $\alpha_2' \in C_p$ such that $p|\beta\alpha_2'$. Since now $p \mid \beta\alpha_2'$, there exists a unique $\alpha_1 \in C_p$ such that $(\alpha_1^{-1}\beta\alpha_2') = A(p^n K_0, 0, p)$. In other p cases of α_1, $A(\alpha_1^{-1}\beta\alpha_2') = A(p^n K_0, 0, 1)$. In the other p cases of α_2's there exist unique α_1's such that $\alpha_1^{-1}\beta\alpha_2 \in \mathcal{O}$ and $A(\alpha_1^{-1}\beta\alpha_2') = A(p^n K_0, 0, 1)$. We use recurrence relation (2b) twice, more precisely, once for the $p^2 A(p^{n+2}K_0, 0, p)$ and once for $A(p^n K_0, 0, p)$ to obtain (5.8).

To prove ${}_p\mu_3 = p^2\lambda_p + p^3 + p$ in both subcase 1 and subcase 2, it suffices to show that $A(p^{n+2}K_0, 0, 1) + A(p^{n-2}K_0, 0, 1) + 2A(p^n K_0, 0, 1) = \lambda_p^2 A(p^n K_0, 0, 1)$ where $A(p^{n-2}K_0, 0, 1) = 0$ for the first subcase. For this, we use the identity $\lambda_p\big(A(p^{n+1}K_0, 0, 1) + A(p^{n-1}K_0, 0, 1)\big) = \lambda_p^2 A(p^n K_0, 0, 1)$ from before. All that is left to show now is that $A(p^{n+2}K_0, 0, 1) + A(p^n K_0, 0, 1) = \lambda_p A(p^{n+1}K_0, 0, 1)$ and $A(p^n K_0, 0, 1) + A(p^{n-2}K_0, 0, 1) = \lambda_p A(p^{n-1}K_0, 0, 1)$. Both of these are easy to prove and follow from the computation of $(K_p h_2 K_p \cdot F)_\beta$ done at the start of **Case 2**.

Thus, the Hecke eigenvalues ${}_p\mu_1, {}_p\mu_2, {}_p\mu_3, {}_p\mu_4$ for F satisfy the equation (3.5) as required. Rest of the proof follows from the proof of Proposition 6.2 in [15]. □

Proposition 5.1.1 gives us the exact structure of $\Pi_{F,p}$ for all odd primes p. Next we give a description of $\Pi_{F,2}$ and $\Pi_{F,\infty}$.

Proposition 5.1.2. *a) The local component* $\Pi_{F,2}$ *is the unique irreducible constituent of the unramified principal series representation* $Ind_{\mathcal{B}_2(B_2)}^{GL_2(B_2)}(\chi_1 \times \chi_2)$ *with* χ_1, χ_2 *unramified characters of* $B_2^\times$ *such that*

$$\chi_1(\varpi_2) = -\sqrt{2}\epsilon, \chi_2(\varpi_2) = -1/\sqrt{2}\epsilon.$$

b) At the prime $p = \infty$*, the archimedean component* $\Pi_{F,\infty}$ *is isomorphic to the principal series representation* $Ind_{\mathcal{B}_2(\mathbb{H})}^{GL_2(\mathbb{H})}(\chi_{\pm\frac{\sqrt{-1}r}{2}})$ *where*

$$\chi_s\left(\begin{bmatrix} a & * \\ 0 & d \end{bmatrix}\right) = \nu(ad^{-1})^s$$

Proof of Proposition 5.1.2. a) For the structure of $\Pi_{F,2}$, it is again enough to show that the Hecke eigenvalues ${}_2\mu_1$, ${}_2\mu_2$ satisfy the equation (3.6). The proof of this is simpler than the odd prime case. From the Maass space condition (2a) in Definition 4.1.1, we have

$$A(2K, u+1, n) = \frac{-3\epsilon}{\sqrt{2}}A(K, u, n) - A(\frac{K}{2}, u-1, n)$$

which gives us

$$2(A(2K, u+1, n) + A(K/2, u-1, n)) = -3\sqrt{2}\epsilon A(K, u, n).$$

Let $\beta \in \mathcal{S}$ such that $A(\beta) = A(K, u, n)$. Then, in terms of β, the above condition can be written as

$$2(A(\beta\varpi_2) + A(\beta\varpi_2^{-1})) = -3\sqrt{2}\epsilon A(\beta).$$

Comparing with condition 1 of Proposition 3.1, we get that the Hecke eigenvalue ${}_2\mu_2 = -3\sqrt{2}\epsilon$. Rest of the argument follows from Sections 5.2 and 6.1 of [15].

b) The proof for the structure of $\Pi_{F,\infty}$ is the same as in Section 6.1 of [15] since we still have a Maass form with Casimir eigenvalue $-\frac{1}{2}\left(\frac{r^2}{4}+1\right)$. □

5.2 Jacquet Langlands correspondence

Let $B_{\mathbb{A}}$ denote the adelization of B with $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ as before. Badulescu and Renard in Theorem 1.4 of [5] give a map $\mathbf{G}$ from the automorphic representations on $\mathrm{GL}_2(B_{\mathbb{A}})$ to those on $\mathrm{GL}_4(\mathbb{A})$. Let $D\mathrm{GL}_2(B_{\mathbb{A}})$ and $D\mathrm{GL}_4(\mathbb{A})$ denote the discrete series representations of $\mathrm{GL}_2(B_{\mathbb{A}})$ and $\mathrm{GL}_4(\mathbb{A})$ respectively. For $G' = \mathrm{GL}_2(B_2)$ or $\mathrm{GL}_2(\mathbb{H})$ and $G = \mathrm{GL}_4(\mathbb{Q}_2)$ or $\mathrm{GL}_4(\mathbb{R})$ respectively, we denote by $\mathcal{C}(G')$ the category of smooth representations of G' (in the non-archimedean case) or the category of Harish-Chandra modules (in the archimedean case) with a fixed maximal compact subgroup K of G'. Let $\mathcal{R}(G')$ denote the Grothendieck group of the category of finite length representations in $\mathcal{C}(G')$. If $g \in \mathrm{GL}_n(B_p)$ in the non-archimedean case for some n or $g \in \mathrm{GL}_n(\mathbb{H})$ for the archimedean case, we say that g is regular semisimple if the characteristic polynomial of g has distinct roots in an algebraic closure of $\mathbb{Q}_p$ or $\mathbb{C}$ respectively. If $\pi \in \mathcal{R}(G)$, then we denote by Θ_π the function character of π as a locally constant map, stable under conjugation, defined on the set of regular semisimple elements of G. We say that $g' \in G$ corresponds to $g \in \mathrm{GL}_4(\mathbb{Q}_2)$ or $\mathrm{GL}_4(\mathbb{R})$ respectively if g and g' are regular semisimple and have the same characteristic polynomial and we write $g \leftrightarrow g'$. For a unitary irreducible smooth representation u of G, we say that u is compatible if there is a unique unitary smooth irreducible representation u' of G' such that $\Theta_u(g) = \varepsilon(u)\Theta_{u'}(g')$ for any $g \leftrightarrow g'$, where $\varepsilon(u) \in \{-1, 1\}$. We denote the map $u \to u'$ by $|\mathbf{LJ}_v|$ where $v = 2$ or ∞. We say a discrete series π of $\mathrm{GL}_4(\mathbb{A})$ is $B-$*compatible* if π_v is compatible at both $v = 2$ and $v = \infty$.

Theorem 5.2.1 (Theorem 1.4 of [5]). *a) There is a unique map $\boldsymbol{G} : DGL_2(B_{\mathbb{A}}) \to DGL_4(\mathbb{A})$ such that for every $\pi' \in DGL_2(B_{\mathbb{A}})$, if $\pi = \boldsymbol{G}(\pi')$, then one has:*

- *π is B-compatible*
- *if $v \neq 2, \infty$, then $\pi_v = \pi'_v$*
- *if $v = 2$ or $v = \infty$, then $|\boldsymbol{LJ}_v|(\pi_v) = \pi'_v$*

The map $\boldsymbol{G}$ is injective. The image of $\boldsymbol{G}$ is the set of all B-compatible discrete series of $GL_4(\mathbb{A})$.

b) If $\pi' \in DGL_2(B_{\mathbb{A}})$, then the multiplicity of π' in the discrete spectrum is one (multiplicity one theorem).

c) If $\pi', \pi'' \in DGL_2(B_{\mathbb{A}})$ and $\pi'_v \simeq \pi''_v$ for almost all v, then $\pi' = \pi''$ (strong multiplicity one theorem)

For a representation π of GL_n, we will say $\pi = \mathrm{MW}(\sigma, k)$ if a discrete series representation π is the unique irreducible quotient of the induced representation $\nu^{(k-1)/2}\sigma \times \nu^{(k-3)/2}\sigma \times \ldots \times \nu^{-(k-1)/2}\sigma$. Here σ is cuspidal and ν is the global character given by product of local characters i.e. absolute value of reduced norm. For GL_4, the possible values of k will be 1,2 or 4. In each of these cases, σ will be a cuspidal representation of GL_4, GL_2 and GL_1 respectively over the appropriate group.

Let Π denote the image of Π_F under $\mathbf{G}$. We will use following results from Proposition 18.2 of [5] to find conditions on Π.

Proposition 5.2.1 (Proposition 18.2 of [5]). *Let $\pi = MW(\rho, k)$ be a representation of $GL_4(\mathbb{A})$.*

a) There exists $k_\rho \in \{1, 2\}$ such that π is B-compatible if and only if $k_\rho \mid k$.

b) Let π' be a discrete series of $GL_2(B_\mathbb{A})$ and let $\pi = \boldsymbol{G}(\pi')$. Then π' is cuspidal if and only if π is of the form $MW(\rho, k_\rho)$.

By Proposition 18.2 part b) of [5], since Π_F is cuspidal, its image Π is of the form $\mathrm{MW}(\sigma, k_\sigma)$. By Proposition 18.2 part a) of [5], $k_\sigma \mid d$ when the dimension of the division algebra is d^2. In our case, the division algebra is a quaternion algebra, so $d = 2$. Hence, $k_\sigma \mid 2$ implying $k_\sigma = 2$ or $k_\sigma = 1$. The latter condition is same as σ being cuspidal.

Proposition 5.2.2. *Let $F \in \mathcal{M}^*(GL_2(\mathcal{O}), r)$ be a cuspidal Hecke eigenform with Π_F the associated representation of $GL_2(B_\mathbb{A})$. Then $\boldsymbol{G}(\Pi_F) = MW(\sigma, 2)$ for some cuspidal representation σ of $GL_2(\mathbb{A})$.*

Proof. We will show that $\mathbf{G}(\Pi_F) = \Pi$ is not cuspidal, which is equivalent to showing $k_\sigma \neq 1$. Since $k_\sigma = 1$ or $k_\sigma = 2$, this proves the proposition.

For the sake of contradiction, assume $k_\sigma = 1$. Therefore, Π is a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A})$. Then, by equation (14) of Sarnak [19], for every odd prime p we have

$$\left|\log_p \left(\left|\alpha_i \left(\Pi_p\right)\right|_p \right)\right| \leqslant \frac{1}{2} - \frac{1}{4^2+1}.$$

Here $\alpha_i(\Pi_p)$ denotes the i-th Satake parameter of Π at prime p, $|\ |_p$ denotes the p-adic valuation and the outer $|\ |$ denotes the standard absolute value. We have $\alpha_i(\Pi_p) = \chi_i(p)$ with $\chi_i(p)$ as given in (5.1). This, in particular, tells us that

$$\left|\log_p\left(\left|p^{1/2}\frac{\lambda_p \pm \sqrt{\lambda_p^2-4}}{2}\right|_p\right)\right| \leqslant \frac{1}{2} - \frac{1}{4^2+1}$$

and

$$\left|\log_p\left(\left|p^{-1/2}\frac{\lambda_p \pm \sqrt{\lambda_p^2-4}}{2}\right|_p\right)\right| \leqslant \frac{1}{2} - \frac{1}{4^2+1}.$$

Therefore, we can write

$$\left|\log_p\left(\left|p^{\pm 1/2}\right|_p\right) + \log_p\left(\left|\frac{\lambda_p \pm \sqrt{\lambda_p^2-4}}{2}\right|_p\right)\right| \leqslant \frac{1}{2} - \frac{1}{17}. \tag{5.9}$$

Let $\log_p\left(\left|\frac{\lambda_p+\sqrt{\lambda_p^2-4}}{2}\right|_p\right) = \alpha^+$ and $\log_p\left(\left|\frac{\lambda_p-\sqrt{\lambda_p^2-4}}{2}\right|_p\right) = \alpha^-$ for convenience of notation. Note that $\alpha^+ + \alpha^- = 0$. Then (5.9) implies that

$$\left|\frac{1}{2} + \alpha^+\right| \leqslant \frac{1}{2} - \frac{1}{17} \qquad \left|\frac{1}{2} + \alpha^-\right| \leqslant \frac{1}{2} - \frac{1}{17}$$
$$\left|\frac{-1}{2} + \alpha^+\right| \leqslant \frac{1}{2} - \frac{1}{17} \qquad \left|\frac{-1}{2} + \alpha^-\right| \leqslant \frac{1}{2} - \frac{1}{17}$$

In particular, we get that

$$-\frac{1}{2} + \frac{1}{17} \leqslant \frac{1}{2} + \alpha^+ \leqslant \frac{1}{2} - \frac{1}{17} \quad \text{and} \quad -\frac{1}{2} + \frac{1}{17} \leqslant \frac{-1}{2} + \alpha^+ \leqslant \frac{1}{2} - \frac{1}{17}.$$

Simplifying, we get

$$-1 + \frac{1}{17} \leqslant \alpha^+ \leqslant -\frac{1}{17} \quad \text{and} \quad \frac{1}{17} \leqslant \alpha^+ \leqslant 1 - \frac{1}{17}$$

both of which cannot be simultaneously true. This gives us a contradiction to our starting assumption that $k_\sigma = 1$. Hence, $k_\sigma \neq 1$ which implies $k_\sigma = 2$ as required. $\square$

From Proposition 5.2.2 we obtain an irreducible cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$. We will next describe the local components of σ and use that to construct $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ which can lift to F.

5.3 Description of σ

Let σ be as from Proposition 5.2.2, with $\sigma \simeq \otimes_{p \leqslant \infty} \sigma_p$. For an odd prime p, let χ_p be the unramified character of $\mathbb{Q}_p^\times$ such that $\chi_p(p) = \frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}$ for λ_p as in Proposition 5.1.1. At the prime $p = \infty$, let $\chi_\infty(a) = |a|^s$ where $s = \frac{\sqrt{-1}r}{2}$. For the prime $p = 2$, let χ be an unramified character of $\mathbb{Q}_2^\times$ with $\chi(2) = -\epsilon$ for ϵ as in condition (2a) of Definition 4.1.1.

Proposition 5.3.1. *Let $\sigma \simeq \otimes_{p \leqslant \infty} \sigma_p$ be the irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ from Proposition 5.2.2. Then*

$$\sigma_p = \begin{cases} Ind_{\mathcal{B}_2(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\chi_p \times \chi_p^{-1}) & \text{for odd } p < \infty, \\ \chi St_{GL_2} & \text{for } p = 2, \\ Ind_{\mathcal{B}_2(\mathbb{R})}^{GL_2(\mathbb{R})}(\chi_\infty \times \chi_\infty^{-1}) & \text{for } p = \infty. \end{cases} \tag{5.10}$$

Proof. We will use the local Jacquet-Langlands map

$$\mathbf{C} : \mathrm{GL}_2(B_p) \to \mathrm{GL}_4(\mathbb{Q}_p)$$

to explicitly write down σ_p at each prime p. We will use the same notation for the map from $\mathrm{GL}_2(\mathbb{H})$ to $\mathrm{GL}_4(\mathbb{R})$. The groups $\mathrm{GL}_2(B_p)$ and $\mathrm{GL}_4(\mathbb{Q}_p)$ are isomorphic for every odd prime p, hence following section 1.5 of [5] the maps is identity for odd prime p. For $p = 2$, the map is described in Theorem 3.2 of [4] and in Section 1.3 of [5] for $p = \infty$.

Since the **C** map is identity at every odd prime p, we have $\Pi_{F,p} = \Pi_p$ where Π_p is the local component of Π at prime p. Let $P_{2,2}$ denote the 2,2-parabolic subgroup of GL_4. From Proposition 5.2.2 we know that $\Pi_{F,p} = \mathrm{MW}(\sigma_p, 2)$. We also know that $\Pi_{F,p}$ is the spherical constituent of $\mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1 \times \chi_2 \times \chi_3 \times \chi_4)$. We will denote $\mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1 \times \chi_2 \times \chi_3 \times \chi_4)$ by $\mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi')$ for ease of notation. The reduced norm here is just $\nu = |\det|$.

If $\Pi_{F,p} = \mathrm{MW}(\sigma_p, 2)$, then such a σ_p is unique (see [18] Section 8). Hence, to show the structure of σ_p, it is enough to prove the following claim:

Claim 2. For every odd prime p,

$$\mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1 \times \chi_2 \times \chi_3 \times \chi_4) \simeq \mathrm{Ind}_{P_{2,2}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\nu^{1/2}\sigma_p \times \nu^{-1/2}\sigma_p)$$

for $\sigma_p = \mathrm{Ind}_{\mathcal{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_p \times \chi_p^{-1})$.

Using method similar to 6.5 in [16], define the map

$$L : \mathrm{Ind}_{P_{2,2}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\nu^{1/2}\sigma_p \times \nu^{-1/2}\sigma_p) \to \mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi')$$

by

$$(Lh)(g) := (h(g))(I_2, I_2).$$

Here h is a function in $\mathrm{Ind}_{P_{2,2}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\nu^{1/2}\sigma_p \times \nu^{-1/2}\sigma_p)$ and I_n is the identity matrix in $\mathrm{GL}_n(\mathbb{Q}_p)$. We have to show this map is well defined and is an isomorphism.

To show that L is well-defined we have to prove that for any $A \in \mathcal{B}_4$, Lh satisfies

$(Lh)(Ag) = \delta_{\mathcal{B}_4}^{1/2}(A)\chi'(A)(Lh)(g)$. For $A = \begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & d \end{bmatrix}$

we have

$$\begin{aligned}
&\delta_{\mathcal{B}_4}^{1/2}(A)\chi'(A)(Lh(g)) = \\
&\quad = |a^3bc^{-1}d^{-3}|^{1/2}\chi_1(a)\chi_2(b)\chi_3(c)\chi_4(d)(Lh(g)) \\
&\quad = |a|^{3/2}|b|^{1/2}|c|^{-1/2}|d|^{-3/2}|a|^{1/2}\chi_p(a)|b|^{1/2}\chi_p^{-1}(b) \\
&\quad\quad |c|^{-1/2}\chi_p(c)|d|^{-1/2}\chi_p^{-1}(d)(Lh(g)) \\
&\quad = |a|^2|b||c|^{-1}|d|^{-2}\chi_p(a)\chi_p^{-1}(b)\chi_p(c)\chi_p^{-1}(d)(Lh(g))
\end{aligned}$$

On the other hand

$$
\begin{aligned}
&(Lh)(Ag) = \\
&= \left|\det \begin{bmatrix} a & * \\ 0 & b \end{bmatrix}\right| \left|\det \begin{bmatrix} c & * \\ 0 & d \end{bmatrix}\right|^{-1} \nu^{1/2}\left(\begin{bmatrix} a & * \\ 0 & b \end{bmatrix}\right) \nu^{-1/2}\left(\begin{bmatrix} c & * \\ 0 & d \end{bmatrix}\right) \\
&\left(\left(\sigma_p\left(\begin{bmatrix} a & * \\ 0 & b \end{bmatrix}\right), \sigma_p\left(\begin{bmatrix} c & * \\ 0 & d \end{bmatrix}\right)\right) h(g)\right)(I_2, I_2) \\
&= |a||b||c|^{-1}|d|^{-1}|a|^{1/2}|b|^{1/2}|c|^{-1/2}|d|^{-1/2} h(g)\left(\begin{bmatrix} a & * \\ 0 & b \end{bmatrix} \times \begin{bmatrix} c & * \\ 0 & d \end{bmatrix}\right) \\
&= |a||b||c|^{-1}|d|^{-1}|a|^{1/2}|b|^{1/2}|c|^{-1/2}|d|^{-1/2} \\
&\quad |a|^{1/2}|b|^{-1/2}\chi_p(a)\chi_p^{-1}(b)|c|^{1/2}|d|^{-1/2}\chi_p(c)\chi_p^{-1}(d)h(g)(I_2, I_2) \\
&= |a|^2|b||c|^{-1}|d|^{-2}\chi_p(a)\chi_p^{-1}(b)\chi_p(c)\chi_p^{-1}(d)h(g)(I_2, I_2) \\
&= |a|^2|b||c|^{-1}|d|^{-2}\chi_p(a)\chi_p^{-1}(b)\chi_p(c)\chi_p^{-1}(d)(Lh(g)) \\
&= \delta_{\mathcal{B}_4}^{1/2}(A)\chi'(A)(Lh(g))
\end{aligned}
$$

To prove injectivity, we look at two functions h_1 and h_2 in $\mathrm{Ind}_{P_{2,2}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\nu^{1/2}\sigma_p \times \nu^{-1/2}\sigma_p)$. By definition, then $Lh_1 = Lh_2$ implies $h_1(g)(I_2, I_2) = h_2(g)(I_2, I_2)$ for every g. Applying $((\nu^{1/2}\sigma_p)(s_1) \times (\nu^{-1/2}\sigma_p)(s_2))$ for $s_1, s_2 \in \mathrm{GL}_2(\mathbb{Q}_p)$ to both sides, we get $h_1(g)(s_1, s_2) = h_2(g)(s_1, s_2)$. Therefore, $h_1 = h_2$.

To show that it is an isomorphism, we construct an inverse map

$$
\tilde{L} : \mathrm{Ind}_{\mathcal{B}_4(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi') \to \mathrm{Ind}_{P_{2,2}(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\nu^{1/2}\sigma_p \times \nu^{-1/2}\sigma_p)
$$

given by

$$(\tilde{L}h)(g)(b_1,b_2) = h\left(\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} g\right)$$

for $b_1, b_2 \in \mathcal{B}_2(\mathbb{Q}_p)$. We can verify that it is well defined by similar computation as above and it is easy to see that $L \circ \tilde{L}$ is identity. Hence L is an isomorphism of representation.

Claim 3. At prime $p = \infty$,

$$|\mathbf{C}|(\Pi_{F,\infty}) \simeq \mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\nu^{1/2}\sigma_\infty \times \nu^{-1/2}\sigma_\infty)$$

for $\sigma_\infty = \mathrm{Ind}_{\mathcal{B}_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi_\infty \times \chi_\infty^{-1})$ with $\chi_\infty(a) = |a|^s$ and $s = \frac{\sqrt{-1}r}{2}$.

For $p = \infty$, note that calculations in Section 6 of [15] for the description of Π_∞ are for a general element $F \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ and are independent of any lifting properties. Hence, $\Pi_{F,\infty}$ is the irreducible component of $\mathrm{Ind}_{\mathcal{B}_2(\mathbb{H})}^{\mathrm{GL}_2(\mathbb{H})}(\chi'_s \times \chi_s'^{-1})$ with $\chi'_s = \nu'^s(x)$ and $s = \frac{\sqrt{-1}r}{2}$. Here ν' denotes the reduced norm of $\mathbb{H}$ at infinity and is equal to the square root of the absolute value.

Following Section 1.3 from [4], the image of the Jacquet-Langlands correspondence is $\mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\xi_s \times \xi_s^{-1})$ where $\xi_s = |\mathbf{C}|(\chi'_s)$ and $\xi_s^{-1} = |\mathbf{C}|(\chi_s'^{-1})$. Here ξ_s, ξ_s^{-1} are characters of $\mathrm{GL}_2(\mathbb{R})$ with $\xi_s = \chi_s \circ \det$ and $\chi_s(a) = |a|^s$ for $s = \frac{\sqrt{-1}r}{2}$. Therefore, $\Pi_\infty = \mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\xi_s \times \xi_s^{-1})$. Now, $\mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\xi_s \times \xi_s^{-1})$ is the irreducible quotient of $\mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\tau_s \times \tau_{-s})$ where $\tau_s = \mathrm{Ind}_{\mathcal{B}_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{1/2}\ \chi_s \times |\ |^{-1/2}\ \chi_s)$. Hence, we obtain the following isomorphism

$$\begin{aligned}
&\mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\tau_s \times \tau_{-s}) \\
&\qquad \simeq \mathrm{Ind}_{\mathcal{B}_4(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(|\ |^{1/2}\,\chi_s \times |\ |^{-1/2}\,\chi_s \times |\ |^{1/2}\,\chi_{-s} \times |\ |^{-1/2}\,\chi_{-s}) \\
&\qquad \simeq \mathrm{Ind}_{\mathcal{B}_4(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(|\ |^{1/2}\,\chi_s \times |\ |^{1/2}\,\chi_{-s} \times |\ |^{-1/2}\,\chi_s \times |\ |^{-1/2}\,\chi_{-s}) \\
&\qquad \simeq \mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\nu^{1/2}\sigma_s \times \nu^{-1/2}\sigma_s)
\end{aligned}$$

where $\sigma_s = \mathrm{Ind}_{\mathcal{B}_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi_s \times \chi_{-s})$ and $\nu = |\det|$.

However, we know from global Jaquet Langlands of Section 5.2 that Π_∞ is also of the form $MW(\sigma_\infty, 2)$ which is the irreducible quotient of $\mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\nu^{1/2}\sigma_\infty \times \nu^{-1/2}\sigma_\infty)$. Hence, by uniqueness, we get $\sigma_\infty = \sigma_s$ as in the claim.

Structure of σ_p for p odd and $p = \infty$ cases is proved by the claims above. We will now show case $p = 2$.

Let ρ and ρ' be unitary representations of $\mathrm{GL}_2(\mathbb{Q}_2)$ and $B_2^\times$ respectively such that $\mathbf{C}(\rho') = \rho$. At prime $p = 2$, Theorem 3.2 from [5] tells us $\mathbf{C}(\overline{u}(\rho', 2)) = u(\rho, 2)$ where

$$u(\rho, k) = Lg(\Pi_{i=0}^{k-1}\nu^{(k-1)/2-i}\rho), \quad \overline{u}(\rho', k) = Lg(\Pi_{i=0}^{k-1}\nu'^{(k-1)/2-i}\rho')$$

with Lg denoting the unique irreducible quotient. Here, $\nu = |\det|$ and ν' is the reduced norm. In our case, we have $u(\rho, 2) = \Pi_2$ and $\overline{u}(\rho', 2) = \Pi_{F,2}$. We know $\Pi_{F,2}$, unlike at odd primes p, is a representation of $\mathrm{GL}_2(B_2)$. On the other hand have $\Pi_2 = \mathrm{MW}(\sigma_2, 2)$, hence $k = 2$. Therefore, $\Pi_{F,2} = \mathrm{Ind}_{\mathcal{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(B_2)}(\chi_1 \times \chi_2) = \mathrm{MW}(\rho', 2)$. Hence, ρ' is a one dimensional representation of $B^\times$ given by a character $\chi' = \chi \circ \nu'$ for an unramified character χ of $\mathbb{Q}_2^\times$. Comparing with Proposition 5.2, we get $\chi'(\varpi_2) = -\epsilon$. According to Section 56 of [7] such a character corresponds to twisted

Steinberg representation χSt of $\mathrm{GL}_2(\mathbb{Q}_2)$. Hence, $\sigma_2 = \chi St$ with $\chi(2) = -\epsilon$.

□

Chapter 6

Main Theorem

6.1 Distinguished vector in σ

Theorem 4.3.1 showed the 'necessary' condition that if a given $F \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ is a lift then $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$. We wish to prove a converse of Theorem 4.3.1, i.e. the 'sufficient' condition. We will first show this under the extra hypothesis that F is a Hecke eigenform and in the last section prove it in all generality for all $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$.

Theorem 6.1.1. *Let $F \in \mathcal{M}^*(GL_2(\mathcal{O}), r)$ such that F is a cuspidal Hecke eigenform. Then, there is a $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$, a Hecke eigenform, such that $F = F_f$.*

From Section 5.3, we know that σ_p is unramified principal series at every prime $p \neq 2$. Hence, the new vector at every prime $p \neq \{2, \infty\}$ is the unique spherical vector ψ_p stable under $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$. At $p = \infty$, we have the unique weight zero fixed vector ψ_∞ which is stable under $K_\infty = O_2(\mathbb{R})$.

At $p = 2$, the representation is an unramified twist of Steinberg and hence the conductor is p. Therefore, the new vector ψ_2 is invariant under

$$K_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big| a, d \in \mathbb{Z}_2^\times, b \in \mathbb{Z}_2, c \in 2\mathbb{Z}_2 \right\}.$$

Let $\psi = \otimes_{p \leqslant \infty} \psi_p \in V_\sigma$. It satisfies

$$\psi(z\gamma g k) = \psi(g) \quad \text{for } \gamma \in \mathrm{GL}_2(\mathbb{Q}), z \in Z(\mathrm{GL}_2(\mathbb{A})), k \in \Pi_{p \leqslant \infty} K_p.$$

For $g_\infty = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$, let $g_\infty(i) = \frac{ai+b}{ci+d} = \tau \in \mathfrak{h}$. Consider the function $f_\psi : \mathfrak{h} \to \mathbb{C}$ associated to ψ defined as $f_\psi(\tau) = f_\psi(g_\infty(i)) = \psi(g_\infty \otimes_{p<\infty} 1_p)$ where 1_p is the identity of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then, for $\gamma \in \Gamma_0(2)$ we have

$$\begin{aligned} f_\psi(\gamma(\tau)) &= \psi((\gamma g_\infty) \otimes_{p<\infty} 1_p) \\ &= \psi((\otimes_{p\leqslant\infty} \gamma^{-1})((\gamma g_\infty) \otimes_{p<\infty} 1_p)) && \because \otimes_{p\leqslant\infty}\gamma^{-1} \in \mathrm{GL}_2(\mathbb{Q}) \\ &= \psi(g_\infty \otimes_{p<\infty} \gamma^{-1}) \\ &= \psi((g_\infty \otimes_{p<\infty} 1_p)k) && k = (1 \otimes_{p<\infty} \gamma^{-1}) \\ &= \psi(g_\infty \otimes_{p<\infty} 1_p) && \because k \in \Pi_{p\leqslant\infty} K_p \\ &= f_\psi(\tau) \end{aligned}$$

Hence f_ψ is invariant under the action of $\Gamma_0(2)$. Since the local representation σ_∞ at $p = \infty$ associated with the vector ψ_∞ is principal series, the function f_ψ is a Maass form.

Following Lemma 9 from [3] for $n = 1$, the map $\psi \to f_\psi$ is Hecke equivariant. The structure of σ_p from Section 5.3 allows us to find the Hecke eigenvalues for f_ψ at all odd prime $p < \infty$. Following Proposition 3.1.2 of [21], the function f_ψ

is an eigenfunction of the Atkin-Lehner involution with eigenvalue $-\chi(2) = \epsilon$ from (5.10) and Hecke eigenvalue $\lambda_2 = \chi(2) = -\epsilon$. By Proposition 4.6.6 of [6], the Hecke eigenvalue for odd primes p with $\sigma_p = \mathrm{Ind}_{\mathcal{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_p \times \chi_p^{-1})$ would be $(\chi_p(p) + \chi_p^{-1}(p)) = \lambda_p$. Note that we are using the action of the Hecke algebra as in (30) of [3] here, hence the lack of $p^{1/2}$.

The eigenvalue for the hyperbolic Laplacian is obtained from the Hecke eigenvalue at infinity as by Proposition 2.5.4 from [6]. Following the notation from Bump [6], in this case, $s_1 = \frac{\sqrt{-1}r}{2}$ and $s_2 = -\frac{\sqrt{-1}r}{2}$. Hence, $s = \frac{1}{2}(\sqrt{-1}r + 1) = \frac{1+\sqrt{-1}r}{2}$. Then, the eigenvalue for the Laplacian is given by $s(1-s) = \left(\frac{1}{4} + \frac{r^2}{4}\right)$. Hence, f_ψ belongs to $S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ as required. We will do an additional verification that this f_ψ does indeed lift to F. This will complete the proof of Theorem 6.1.1.

6.2 Fourier coefficients of f_ψ

Let $N = 4^a b$, where a, b are non-negative integers and $4 \nmid b$. For $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ we can define a sequence of numbers

$$c(-N) := \frac{A(2N, u, 1)}{\sqrt{2N}} + \epsilon \frac{A(N, u-1, 1)}{\sqrt{N}} \tag{6.1}$$

where

$$u = \begin{cases} 2a & \text{if } b \equiv 1, 3 \mod (4) \\ 2a+1 & \text{if } b \equiv 2 \mod (4) \end{cases}$$

It can be proved that this sequence of numbers $\{c(-N)\}$ satisfy (3.3) in terms of K, u and n just by reversing the argument in proof of Theorem 4.3.1.

$$\begin{aligned}
A(K, u,n) =& \sum_{d|n} dA(K/d^2, u, 1) \\
=&\sqrt{K}\sum_{d|n} \frac{A(K/d^2, u, 1)}{\sqrt{K/d^2}} \\
=&\sqrt{K}\sum_{d|n}\left(\sum_{t=0}^{u}\left((-\epsilon)^t\frac{A((K/d^2)/2^t, u-t, 1)}{\sqrt{(K/d^2)/2^t}}\right.\right. \\
&\left.\left. - (-\epsilon)^{t+1}\frac{A((K/d^2)/2^{t+1}, u-(t+1), 1)}{\sqrt{(K/d^2)/2^{t+1}}}\right)\right) \\
=&\sqrt{K}\sum_{t=0}^{u}\sum_{d|n}(-\epsilon)^t\left(\frac{A(K/(2^t d^2), u-t, 1)}{\sqrt{K/(2^t d^2)}}\right. \\
&\left. + \epsilon\frac{A(K/(2^{t+1}d^2), u-t-1, 1)}{\sqrt{K/(2^{t+1}d^2)}}\right) \\
=&\sqrt{K}\sum_{t=0}^{u}\sum_{d|n}(-\epsilon)^t c\left(-\frac{K}{2^{t+1}d^2}\right)
\end{aligned}$$

We will show that $f_\psi \in S(\Gamma_0(2), -(\frac{1}{4}+\frac{r^2}{4}))$ lifts to F by showing that these $\{c(N)\}$ are the coefficients of f for all $N < 0$. This is to say, if $c_\psi(N)$ are the Fourier coefficients of f_ψ, then $c(-N) = c_\psi(-N)$ for all $N > 0$.

For every odd prime p, since f_ψ is a Hecke eigenform, $c_\psi(N)$ satisfy

$$p^{\frac{1}{2}}c_\psi(pN) + p^{\frac{-1}{2}}c_\psi(N/p) = \lambda_p c_\psi(N). \tag{6.2}$$

by equation (5.7) of [15], with $c_\psi(N/p) = 0$ if $p \nmid N$. Rewriting, we get

$$c_\psi(pN) = p^{-1/2}\lambda_p c_\psi(N) - p^{-1}c_\psi(N/p). \tag{6.3}$$

Since f is a Hecke eigenform at prime $p = 2$, from equation (5.6) of [15], its Fourier coefficients also satisfy

$$c_\psi(2N) = -\frac{\epsilon}{2} c_\psi(N). \tag{6.4}$$

Equations (6.3) and (6.4) together allows us to write $c_\psi(-N)$ in terms of $c_\psi(-1)$, λ_p and ϵ for all N. This also shows that $c_\psi(-1)$ is in fact non-zero.

Lemma 6.2.1. *The sequence of numbers* $\{c(N)\}$ *as defined in* (6.1) *satisfy equations* (6.2) *and* (6.4)

Proof. To prove $\{c(N)\}$ satisfy (6.2), we will use that

$$\lambda_p = \frac{A(p^{n+1}K_0, u, 1) + A(p^{n-1}K_0, u, 1)}{A(p^n K_0, u, 1)}.$$

Note that this statement is slightly different than our claim in (5.2) since we no longer assume $u = 0$. This statement is still true however, since the Hecke computation from Proposition 3.1 (b) holds true for any general $A(K, u, n)$ and not just with $A(K, 0, 1)$ as we used for Proposition 5.1.1. It can also be calculated explicitly via the same argument as in proof of Proposition 5.1.1. In the computation below, we assume $c(-N/p) = 0$ and $A(N/p, u-1, 1) = 0$ are 0 if $p \nmid N$.

$$
\begin{aligned}
p^{\frac{1}{2}} c(-pN) + p^{\frac{-1}{2}} c(-N/p) \\
&= \left(p^{1/2} \frac{A(2pN, u, 1)}{\sqrt{2pN}} + \epsilon p^{1/2} \frac{A(pN, u-1, 1)}{\sqrt{pN}} \right) \\
&+ \left(p^{-1/2} \frac{A(2N/p, u, 1)}{\sqrt{2N/p}} + \epsilon p^{-1/2} \frac{A(N/p, u-1, 1)}{\sqrt{N/p}} \right) \\
&= \frac{A(2pN, u, 1)}{\sqrt{2N}} + \epsilon \frac{A(pN, u-1, 1)}{\sqrt{N}} \\
&+ \frac{A(2N/p, u, 1)}{\sqrt{2N}} + \epsilon \frac{A(N/p, u-1, 1)}{\sqrt{N}} \\
&= \left(\frac{A(2pN, u, 1) + A(2N/p, u, 1)}{\sqrt{2N}} \right) \\
&+ \left(\epsilon \frac{A(pN, u-1, 1) + A(N/p, u-1, 1)}{\sqrt{N}} \right) \\
&= \frac{\lambda_p A(2N, u, 1)}{\sqrt{2N}} + \epsilon \frac{\lambda_p A(N, u-1, 1)}{\sqrt{N}} \\
&= \lambda_p \left(\frac{A(2N, u, 1)}{\sqrt{2N}} + \epsilon \frac{A(N, u-1, 1)}{\sqrt{N}} \right) \\
&= \lambda_p c(-N)
\end{aligned}
$$

Thus, $\{c(-N)\}$ satisfy (6.2). To show equation (6.4), we use the condition (2a) from Definition 4.1.1. From $c(-N)$ as in (6.1), we get

$$
\begin{aligned}
c(-2N) &= \frac{A(4N, u+1, 1)}{\sqrt{4N}} + \epsilon \frac{A(2N, u, 1)}{\sqrt{2N}} \\
&= \left(\frac{-3\epsilon}{\sqrt{2}} \frac{A(2N, u, 1)}{\sqrt{4N}} - \frac{A(4N, u-1, 1)}{\sqrt{4N}} \right) + \epsilon \frac{A(2N, u, 1)}{\sqrt{2N}} \\
&= \frac{-3\epsilon}{2} \frac{A(2N, u, 1)}{\sqrt{2N}} + \epsilon \frac{A(2N, u, 1)}{\sqrt{2N}} - \frac{A(N, u-1, 1)}{2\sqrt{N}} \\
&= \frac{-\epsilon}{2} \frac{A(2N, u, 1)}{\sqrt{2N}} - \frac{1}{2} \frac{A(N, u-1, 1)}{\sqrt{N}} \\
&= \frac{-\epsilon}{2} c(-N)
\end{aligned}
$$

□

Since $c(N)$ satisfy (6.2) and (6.4), $c(-1)$ is also not 0. Then, we can normalize the Fourier coefficients $c_\psi(-N)$ so that $c_\psi(-1) = c(-1)$. Since both $\{c(-N)\}$ and $\{c_\psi(-N)\}$ satisfy (6.2) and (6.4), this implies $c(-N) = c_\psi(-N)$ for all N. Therefore, the Fourier coefficients of f_ψ satisfy (3.3) and hence, it is a Hecke eigenform in $S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ whose lift $F_f = F$, completing the proof of Theorem 6.1.1.

6.3 Main result for non-Hecke eigenforms

We would like to generalize the result of Theorem 6.1.1 to all $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$. We will do so by proving that $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ has a Hecke eigenbasis and showing that the Maass space $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ is stable under the action of all the Hecke operators given in Proposition 3.2.1. If $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ is stable, then it has a Hecke eigenbasis $\{F_i\}$ which are lifts of some $\{f_i\}$ for $f_i \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ by Theorem 6.1.1. Then by linearity of the defining condition (3.3), $F = \sum_i a_i F_i$ would be a lift of $\sum_i a_i f_i \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$. Let $\Gamma \subset \mathrm{GL}_2(\mathcal{O})$ be a subgroup of finite index. For Maass forms F, G over $\mathcal{M}(\Gamma, r)$ with one of them cuspidal, we can define their Petersson inner product by

$$\langle F, G \rangle = \frac{1}{Vol(\Gamma \backslash \mathrm{GL}_2(\mathcal{O}))} \int_{\Gamma \backslash \mathrm{GL}_2(\mathbb{H})/Z^+K} F(g)\overline{G(g)} dg \tag{6.5}$$

where the Haar measure dg is given by $\frac{dxdy}{y^2}$ when $g = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$ as in Section 2.1.

Proposition 6.3.1. *$\mathcal{M}(GL_2(\mathcal{O}), r)$ has a basis of forms that are simultaneous eigenvectors of the Hecke algebra $\otimes \mathcal{H}(G_p, K_p)$ and the subspace of cusp forms has an* *orthogonal basis of Hecke eigenfunctions with respect to the Petersson inner product on the 5-dimensional hyperbolic as in* (6.5)

Proof. The Hecke algebra acting on $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ is $\otimes\mathcal{H}(G_p, K_p)$ as in Section 5.2 of [15]. By Theorem 6 from Section 8 of [20], the algebra $\mathcal{H}(G_p, K_p)$ is commutative for every prime p.

By Theorem 1 on page 8 of [8], $\dim(\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)) < \infty$. This means the space $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ is a finite dimensional vector space with a commutative algebra $\otimes\mathcal{H}(G_p, K_p)$ of operators acting on it. The final step in the proof is to show that the operators commute with their adjoint with respect to the Petersson inner product.

Let g be such that $K_p g K_p$ is one of the generators of the Hecke algebra $\mathcal{H}(G_p, K_p)$ which according to Proposition 3.2.1 are h_1, h_2, h_3 and h_4 for odd p and $\begin{bmatrix} \varpi_2 & 0 \\ 0 & \varpi_2 \end{bmatrix}$ and $\begin{bmatrix} \varpi_2 & 0 \\ 0 & 1 \end{bmatrix}$ for $p = 2$. Let $K_p g K_p = \sqcup_i K_p g_i = \sqcup_i g_i K_p$. Then $K_p g^{-1} K_p = \sqcup_i K_p g_i^{-1}$ and $K_p g^{-1} z K_p = \sqcup_i K_p g_i^{-1} z$ for $z \in Z$ an element of the center. Let F, G be cusp forms in $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$. To find the adjoint operator under the Petersson inner product of (6.5), we can look at

$$
\begin{aligned}
\langle T(g)F, G\rangle = \sum_i \langle F|_{g_i}, G\rangle &= \sum_i \langle F|_{g_i}, G|_z\rangle \\
&= \sum_i \langle F|_{g_i}, G|_z|_{g_i^{-1}}|_{g_i}\rangle \\
&= \sum_i \langle F, G|_{zg_i^{-1}}\rangle \\
&= \sum_i \langle F, T(zg^{-1})G\rangle.
\end{aligned}
$$

Hence, to show that the Hecke operators commute with their adjoint, it suffices to show that $T(zg^{-1})$ is a generator up to a element of the center. Note that while F, G are cusp forms with respect to $\mathrm{GL}_2(\mathcal{O})$, the forms $G|_{zg^{-1}}$ might be modular only with respect to some smaller subgroup Γ hence hence we define Petersson inner product for general subgroup Γ rather than just $\mathrm{GL}_2(\mathcal{O})$.

Now, taking $z = \begin{bmatrix} \varpi_2 & 0 \\ 0 & \varpi_2 \end{bmatrix}$ and the Weyl group element $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in K_2$, we can see that $K_2 g K_2 = K_2 w z g^{-1} w K_2$. Similarly, taking $z = \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}$ and $w = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \in K_p$, we can show that $K_p h_2 K_p = K_p w z h_4^{-1} w K_p$, $K_p h_3 K_p = K_p w z h_3^{-1} w K_p$ and $K_p h_4 K_p = K_p w z h_2^{-1} w K_p$. Letting $T_{p,i}$ denote the Hecke operator of $K_p h_i K_p$, this shows

$$T_{2,1}^* = T_{2,1}, \quad T_{2,2}^* = T_{2,2},$$

and

$$T_{p,1}^* = T_{p,1}, \quad T_{p,2}^* = T_{p,4}, \quad T_{p,3}^* = T_{p,3}, \quad T_{p,4}^* = T_{p,2}$$

for every prime odd $p > 2$. □

Theorem 6.3.1. *The following are equivalent.*

1. *F is a lift from an Atkin-Lehner eigenform $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ with eigenvalue $\epsilon \in \{\pm 1\}$ and which is a Hecke eigenform at $p = 2$.*

2. *F is an element of the space $\mathcal{M}^*(GL_2(\mathcal{O}), r)$*

Proof. As mentioned before, it is enough to show that $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ is stable under the action of the Hecke Algebra. We will prove that for any Hecke operator $T_{p,i}$ and any $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$, the image of the action $T_{p,i}F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ by showing $T_{p,i}F$ satisfies all the conditions of Definition 4.1.1.

Condition 1 of Definition 4.1.1 follows from the fact that we can write the coefficients $A'(\beta)$ of $T_{p,i}F$ in terms of $A(K,u,n)$ using Proposition 3.2.1 by case by case decomposition similar to proof of Proposition 5.1.1. We showed in proof of Proposition 5.1.2 that condition (2a) is actually equivalent to F being a Hecke eigenform at prime $p=2$. Hence, $T_{2,2}F = -(3\sqrt{2}\epsilon)F$ for all $F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$. Checking the recurrence relations for $T_{p,i}F$ where p is an odd prime requires computation. We will show this case by case with $A(\beta) = A(p^m K, u, p^l n)$ where $p \nmid Kn$. Our cases will be for $T_{p,2}F$ and $T_{p,3}F$ with $m = 2l$ and $m > 2l$. The computation for $T_{p,4}$ is identical to $T_{p,2}$ and hence will not be shown separately.

To simplify computation, we will use a simpler version of our recurrence relation.

$$\begin{aligned} A(p^m K, u, p^l n) &= \sum_{d \mid p^l n} dA\left(\frac{p^m K}{d^2}, u, 1\right) \\ &= \sum_{i=0}^{l} \sum_{d' \mid n} p^i d' A\left(\frac{p^m K}{p^{2i} d'^2}, u, 1\right) \qquad (6.6) \\ &= \sum_{i=0}^{l} p^i \sum_{d' \mid n} d' A\left(\frac{p^{m-2i} K}{d'^2}, u, 1\right) \\ &= \sum_{i=0}^{l} p^i A(p^{m-2i} K, u, n). \qquad (6.7) \end{aligned}$$

To obtain (6.6), we wrote $d = p^i d'$ and split the sum over $d \mid p^l n$ into sum over $0 \leqslant i \leqslant l$ and $d' \mid d$.

For ease of notation, we will refer to the Fourier coefficients of the $T_{p,i}F$ in terms of K, u and n as $T_{p,i}F(K,u,n)$. In terms of recurrence relation (6.7), the result we want to show will be

$$T_{p,i}F(p^m K, u, p^l n) = \sum_{i=0}^{l} p^i T_{p,i}F(p^{m-2i} K, u, n).$$

Case 1: We will start with computation for $T_{p,2}F$ and $m = 2l$. By convention any term $A(p^m K, u, p^l n) = 0$ if either l or m is negative. In this case, p^l exactly divides $p^m K$ so $\beta = p^l \beta_0$ with $p \nmid |\beta_0|^2$. Therefore, by Lemma 5.1.1, we have

$$
\begin{aligned}
&T_{p,2}F(p^m K, u, p^l n) = \\
&\qquad =(p+1)A(p^{m+1}K, u, p^l n) + (p+1)A(p^{m-1}K, u, p^{l-1}n) \\
&\qquad =(p+1)(A(p^{m+1}K, u, p^l n) + A(p^{m-1}K, u, p^{l-1}n))
\end{aligned}
$$

and

$$
\begin{aligned}
T_{p,2}F(p^m K, u, n) =& pA(p^{m+1}K, u, n) + A(p^{m+1}K, u, pn) \\
&+ A(p^{m-1}K, u, n) \\
=& pA(p^{m+1}K, u, n) + pA(p^{m-1}K, u, n) \\
&+ A(p^{m+1}K, u, n) + A(p^{m-1}K, u, n) \\
=& (p+1)(A(p^{m+1}K, u, n) + A(p^{m-1}K, u, n)).
\end{aligned}
$$

Note, we are using our recurrence relation from step 2 to step 3 to expand the terms with pn. Hence,

$$
\begin{aligned}
&\sum_{i=0}^{l} p^i T_{p,2}F(p^{m-2i}K, u, n) = \\
&\qquad = \sum_{i=0}^{l-1} p^i T_{p,2}F(p^{m-2i}K, u, n) + p^l T_{p,2}F(K, u, n) \\
&\qquad = \sum_{i=0}^{l-1} p^i (p+1)(A(p^{m+1-2i}K, u, n) + A(p^{m-1-2i}K, u, n)) \\
&\qquad\quad + p^l (p+1)(A(pK, u, n) + 0)
\end{aligned}
$$

We compare that with

$$
\begin{aligned}
T_{p,2}F(p^mK,u,p^ln) &= (p+1)(A(p^{m+1}K,u,p^ln) + A(p^{m-1}K,u,p^{l-1}n)) \\
=&(p+1)\left(\sum_{i=0}^{l} p^iA(p^{m+1-2i}K,u,n) + \sum_{i=0}^{l-1} p^iA(p^{m-1-2i}K,u,n)\right) \\
=&(p+1)\left(\sum_{i=0}^{l-1} p^iA(p^{m+1-2i}K,u,n) + \sum_{i=0}^{l-1} p^iA(p^{m-1-2i}K,u,n)\right) \\
&+ p^l(p+1)A(pK,u,n)
\end{aligned}
$$

to obtain the desired result.

Case 2: Now, consider the computation for $T_{p,2}F$ with $m > 2l$. The expansion for $T_{p,2}F(p^mK,u,n)$ from **Case 1** is still valid here. Hence,

$$
\begin{aligned}
&\sum_{i=0}^{l} p^iT_{p,2}F(p^{m-2i}K,u,n) = \\
&\qquad = \sum_{i=0}^{l} p^i(p+1)(A(p^{m+1-2i}K,u,n) + A(p^{m-1-2i}K,u,n))
\end{aligned}
$$

In this case, $\beta = p^l \beta_0$ but $p \mid |\beta_0|^2$. Therefore, by Lemma 5.1.1, we have

$$
\begin{aligned}
&T_{p,2}F(p^m K, u, p^l n) = \\
&\quad = pA(p^{m+1}K, u, p^l n) + A(p^{m+1}K, u, p^{l+1}n) \\
&\quad + pA(p^{m-1}K, u, p^{l-1}n) + A(p^{m-1}K, u, p^l n) \\
&\quad = p\sum_{i=0}^{l} p^i A(p^{m+1-2i}K, u, n) + \sum_{i=0}^{l+1} p^i A(p^{m+1-2i}K, u, n) \\
&\quad + p\sum_{i=0}^{l-1} p^i A(p^{m-1-2i}K, u, n) + \sum_{i=0}^{l} p^i A(p^{m-1-2i}K, u, n) \\
&\quad = (p+1)\sum_{i=0}^{l} p^i A(p^{m+1-2i}K, u, n) + p^{l+1}A(p^{m-1-2l}K, u, n) \\
&\quad + (p+1)\sum_{i=0}^{l-1} p^i A(p^{m-1-2i}K, u, n) + p^l A(p^{m-1-2l}K, u, n) \\
&\quad = (p+1)\sum_{i=0}^{l} p^i A(p^{m+1-2i}K, u, n) + (p+1)p^l A(p^{m-1-2l}K, u, n) \\
&\quad + (p+1)\sum_{i=0}^{l-1} p^i A(p^{m-1-2i}K, u, n) \\
&\quad = (p+1)\left(\sum_{i=0}^{l} p^i A(p^{m+1-2i}K, u, n) + \sum_{i=0}^{l-1} p^i A(p^{m-1-2i}K, u, n)\right).
\end{aligned}
$$

Comparing the two sums we have the equality.

Case 3: Now we move on to the computation for $T_{p,3}F$, starting with $m = 2l$. Once again we have p^l exactly dividing $p^m K$ so $\beta = p^l \beta_0$ with $p \nmid |\beta_0|^2$. Therefore, by Lemma 5.1.1, we have

$$T_{p,3}F(p^mK,u,p^ln) =$$
$$=p^2A(p^{m-2}K,u,p^{l-1}n) + p^2A(p^{m+2}K,u,p^{l+1}n)$$
$$+p(p(p+1)A(p^mK,u,p^{l-1}n) + (p+1)A(p^mK,u,p^ln))$$
$$=p^2A(p^{m-2}K,u,p^{l-1}n) + p^2A(p^{m+2}K,u,p^{l+1}n)$$
$$+(p^2+p)(pA(p^mK,u,p^{l-1}n) + A(p^mK,u,p^ln))$$

and

$$T_{p,3}F(p^mK,u,n) =$$
$$=p^2(0) + p^2A(p^{m+2}K,u,pn)$$
$$+p(pA(p^mK,u,n) + pA(p^mK,u,n) + A(p^mK,u,pn))$$
$$=p^2A(p^{m+2}K,u,pn) + 2p^2A(p^mK,u,n) + pA(p^mK,u,pn)$$
$$=p^2(pA(p^mK,u,n) + A(p^{m+2}K,u,n)) + 2p^2A(p^mK,u,n)$$
$$+p(pA(p^{m-2}K,u,n) + A(p^mK,u,n))$$
$$=p^2A(p^{m+2}K,u,n) + p^2A(p^{m-2}K,u,n)$$
$$+(p^3+2p^2+p)A(p^mK,u,n).$$

Hence,

$$\sum_{i=0}^{l} p^i T_{p,3} F(p^{m-2i}K, u, n) =$$

$$= \sum_{i=0}^{l} p^i (p^2 A(p^{m-2-2i}K, u, n) + p^2 A(p^{m+2-2i}K, u, n)$$

$$+(p^3 + 2p^2 + p) A(p^{m-2i}K, u, n))$$

$$= \sum_{i=0}^{l-1} p^i (p^2 A(p^{m-2-2i}K, u, n) + p^2 A(p^{m+2-2i}K, u, n)$$

$$+(p^3 + 2p^2 + p) A(p^{m-2i}K, u, n))$$

$$+p^l (p^2 A(p^2 K, u, n) + (p^3 + 2p^2 + p) A(K, u, n))$$

We separated the term of $i = l$ since it expands differently for the case of p^{m-2-2i}.

Comparing it with

$$
\begin{aligned}
T_{p,3}F(p^mK,u,p^ln) = & \\
&= p^2A(p^{m-2}K,u,p^{l-1}n) + p^2A(p^{m+2}K,u,p^{l+1}n) \\
&+ (p^2+p)(pA(p^mK,u,p^{l-1}n) + A(p^mK,u,p^ln)) \\
&= p^2\sum_{i=0}^{l-1} p^iA(p^{m-2-2i}K,u,n) + p^2\sum_{i=0}^{l+1} p^iA(p^{m+2-2i}K,u,n) \\
&+ (p^2+p)\left(p\sum_{i=0}^{l-1} p^iA(p^{m-2i}K,u,n) + \sum_{i=0}^{l} p^iA(p^{m-2i}K,u,n)\right) \\
&= p^2\sum_{i=0}^{l-1} p^iA(p^{m-2-2i}K,u,n) + p^2\sum_{i=0}^{l-1} p^iA(p^{m+2-2i}K,u,n) \\
&+ p^2p^lA(p^2K,u,n) + p^2p^{l+1}A(K,u,n) \\
&+ (p^2+p)\left(p\sum_{i=0}^{l-1} p^iA(p^{m-2i}K,u,n) + \sum_{i=0}^{l} p^iA(p^{m-2i}K,u,n)\right) \\
&+ (p^2+p)p^lA(K,u,n) \\
&= \sum_{i=0}^{l-1} p^i(p^2A(p^{m-2-2i}K,u,n) + p^2A(p^{m+2-2i}K,u,n) \\
&+ (p^3+2p^2+p)A(p^{m-2i}K,u,n)) \\
&+ p^l(p^2A(p^2K,u,n) + (p^3+2p^2+p)A(K,u,n))
\end{aligned}
$$

we get that the two sums are equal as required.

Case 4: Finally, let $m > 2l$ with $A(p^{m-2-2l}K,u,n) = 0$ if $m = 2l+1$. Then, we have

$$\sum_{i=0}^{l} p^i T_{p,3} F(p^{m-2i}K, u, n) =$$

$$= \sum_{i=0}^{l} p^i (p^2 A(p^{m+2-2i}K, u, pn) + 2p^2 A(p^{m-2i}K, u, n)$$

$$+ pA(p^{m-2i}K, u, pn))$$

$$= \sum_{i=0}^{l} p^i (p^2 (A(p^{m+2-2i}K, u, n) + pA(p^{m-2i}K, u, n))$$

$$+ 2p^2 A(p^{m-2i}K, u, n) + p(A(p^{m-2i}K, u, n) + pA(p^{m-2-2i}K, u, n)))$$

$$= \sum_{i=0}^{l} p^i (p^2 A(p^{m-2-2i}K, u, n) + p^2 A(p^{m+2-2i}K, u, n)$$

$$+ (p^3 + 2p^2 + p) A(p^{m-2i}K, u, n)$$

Note, we are using our recurrence relation from step 1 to step 2 to expand the terms with pn.

We compare that with

$$
\begin{aligned}
T_{p,3}F(p^mK, u,p^ln) = {} & p^2A(p^{m-2}K, u, p^{l-1}n) \\
& +p^2A(p^{m+2}K, u, p^{l+1}n) + p(p^2A(p^mK, u, p^{l-1}n) \\
& +2pA(p^mK, u, p^ln) + A(p^mK, u, p^{l+1}n)) \\
& =p^2\sum_{i=0}^{l-1} p^iA(p^{m-2-2i}K, u, n) + p^2\sum_{i=0}^{l+1} p^iA(p^{m+2-2i}K, u, n) \\
& +p^3\sum_{i=0}^{l-1} p^iA(p^{m-2i}K, u, n) + 2p^2\sum_{i=0}^{l} p^iA(p^mK, u, n) \\
& +p\sum_{i=0}^{l+1} p^iA(p^mK, u, n) \\
& =p^2\sum_{i=0}^{l-1} p^iA(p^{m-2-2i}K, u, n) + p^2\sum_{i=0}^{l} p^iA(p^{m+2-2i}K, u, n) \\
& +p^2p^{l+1}A(p^{m-2l}K, u, n) + p^3\sum_{i=0}^{l-1} p^iA(p^{m-2i}K, u, n) \\
& +2p^2\sum_{i=0}^{l} p^iA(p^mK, u, n) + p\sum_{i=0}^{l} p^iA(p^mK, u, n) \\
& +pp^{l+1}A(p^{m-2-2l}K, u, n).
\end{aligned}
$$

Rearranging the sum, we get

$$
\begin{aligned}
T_{p,3}F(p^mK, u, p^ln) = {} & \sum_{i=0}^{l} p^i(p^2A(p^{m-2-2i}K, u, n) \\
& + \sum_{i=0}^{l} p^ip^2A(p^{m+2-2i}K, u, n) \\
& + \sum_{i=0}^{l} p^i(p^3 + 2p^2 + p)A(p^{m-2i}K, u, n) \\
& = \sum_{i=0}^{l} p^iT_{p,3}F(p^{m-2i}K, u, n)
\end{aligned}
$$

as required. □

Appendix A

Converse theorem for $\Gamma_0(4)$

A.1 Converse Theorem

Following the method of construction of Muto, Narita and Pitale in [15], a possible approach to the problem is via a proper converse theorem. If we could use the two recurrence relations from Definition 4.1.1 and Proposition 4.2.1, we can perhaps infer about the analytic properties of $C(-N)$ from $A(\beta)$. The following proposition fulfills the role of the required converse theorem for the case $N = 4$.

Proposition A.1. *If a Maass form $f \in S(\Gamma_0(2), -(\frac{1}{4}+\frac{r^2}{4}))$ has a Fourier expansion $\sum_{-\infty}^{\infty} a_n \sqrt{y} K_\nu(2\pi|n|y)e^{2\pi inx}$ with $a_0 = 0$ and if*

$$\Lambda(s, f) := N^{(\frac{s-1/2}{2})}\pi^{(-s+\epsilon)}\Gamma\Big(\frac{s+\epsilon+\nu}{2}\Big)\Gamma\Big(\frac{s+\epsilon-\nu}{2}\Big)\sum \frac{a_n}{n^s}$$

satisfies the functional equation $\Lambda(s, f) = (-1)^\epsilon \Lambda(1-s, f)$ with $\epsilon = 0$ if $a_n = a_{-n}$ and $\epsilon = 1$ if $a_n = -a_{-n}$ for $N \leqslant 4$, then f is a Maass form over $\Gamma_0(N)$.

We will use Lemma 1.9.2 from [6] to prove this proposition.

Lemma A.2 (1.9.2). *If f is an eigenvector for the Laplace operator, then $f(iy) = \frac{\partial f(iy)}{\partial x} = 0$ for all $y > 0 \Rightarrow f(z) = 0$ for all z.*

Proof. Action of γ doesn't change Laplace invariance and leaves the eigenvalue unchanged. Hence, $h(z)$ is also a Laplace eigenfunction and we can use the above lemma to it. From the existence of Fourier expansion, we know that $f(z)$ is invariant under $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Since these two matrices generate $\Gamma_0(N)$ for $N \leqslant 4$, we would have shown $f(z)$ is $\Gamma_0(N)$ invariant.Now $\left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right]$. So showing that $f(iy) = \pm f(\gamma'(iy))$ for $\gamma' = \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]$ is enough.

If f is an odd Maass form with expansion $\sum_{-\infty}^{\infty} a_n \sqrt{y} K_\nu(2\pi|n|y) e^{2\pi i n x}$ with $a_0 = 0$ then $f(x+iy) = \sum_1^{\infty} a_n \sqrt{y} K_\nu(2\pi|n|y) i \sin(2\pi n x)$. The terms of $e^{2\pi i n z}$ involving $\cos(2\pi n x)$ cancel for a_n and a_{-n} and hence are omitted. Then $f(iy) = \sum_1^{\infty} a_n i \sqrt{y} K_\nu(2\pi|n|y) \mathrm{sin}(0) = 0$ for all $y > 0$.

If f is an even form then $\frac{\partial f}{\partial x}$ is odd and vice versa. Hence, showing that $f(iy) = f(\gamma'(iy))$ for an even form and $\frac{\partial f}{\partial x}(iy) = \frac{\partial f}{\partial x}(\gamma'(iy))$ for an odd form is sufficient.

Even Case: We know from Equation 1.9.10 from [6] that if $f(x+iy)$ is an even form with Fourier expansion $\sum_{-\infty}^{\infty} a_n \sqrt{y} K_\nu(2\pi|n|y) e^{2\pi i n x}$ with $a_0 = 0$, then

$$\int_0^{\infty} f(iy) y^{s-\frac{1}{2}} \frac{dy}{y} = \frac{1}{2}\pi^{-s}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)L(s,f) \tag{A.1}$$

where $L(s,f) = \sum \frac{a_n}{n^s}$. Defining

$$\Lambda(s,f) := N^{\left(\frac{s-1/2}{2}\right)}\pi^{-s}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)L(s,f) \tag{A.2}$$

we get

$$\int_0^{\infty} f(iy) y^{s-\frac{1}{2}} \frac{dy}{y} = \frac{1}{2} N^{-\left(\frac{s-1/2}{2}\right)}\Lambda(s,f).$$

Therefore,

$$f(iy) = 2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{s-1/2}{2})}\Lambda(s,f)y^{-(s-1/2)}ds$$

by the Mellin inversion formula 1.5.5 from [6]. If $\Lambda(s,f) = \Lambda(1-s,f)$, then

$$f(iy) = 2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{s-1/2}{2})}\Lambda(1-s,f)y^{-(s-1/2)}ds.$$

Let $s' = 1-s$, then $ds' = -ds$, $s-1/2 = 1/2-s'$ and $N^{-(\frac{s-1/2}{2})} = N^{-(\frac{s'-1/2}{2})}N^{(s'-1/2)}$.

$$\begin{aligned}
\therefore f(iy) =& 2\frac{-1}{2\pi i}\int_{(c)} N^{-(\frac{s'-1/2}{2})}\Lambda(s',f)N^{(s'-1/2)}y^{(s'-1/2)}ds' \\
=& 2\frac{-1}{2\pi i}\int_{(c)} N^{-(\frac{s'-1/2}{2})}\Lambda(s',f)(Ny)^{(s'-1/2)}ds' \\
=& -2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{s-1/2}{2})}\Lambda(s,f)\left(\frac{1}{Ny}\right)^{-(s-1/2)} ds \\
=& -f\left(\frac{i}{Ny}\right) \\
=& -f\left(\frac{-1}{iNy}\right)
\end{aligned}$$

Therefore, $f(iy) = \pm f(\gamma'(iy))$ where $\gamma' = \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$.

Odd Case: If $f(z)$ is an odd form, then consider $g(z) = \frac{1}{4\pi i}\frac{\partial f}{\partial x}(z)$. In this case, we prove the relation $g(iy) = \pm\frac{1}{Ny^2}g(\gamma'(iy))$ which is essentially the equality of first partials. The Fourier expansion is $g(z) = \sum_{-\infty}^{\infty} na_n\sqrt{y}K_\nu(2\pi|n|y)e^{2\pi inx}$. Therefore,

$$\begin{aligned}
&\int_0^\infty g(iy)y^{(s+1)-\frac{1}{2}}\frac{dy}{y} \\
&\qquad =\frac{1}{2}\pi^{-(s+1)}\Gamma\left(\frac{s+1+\nu}{2}\right)\Gamma\left(\frac{s+1-\nu}{2}\right)\sum_1^\infty \frac{na_n}{n^{s+1}} \\
&\qquad =\frac{1}{2}\pi^{-(s+1)}\Gamma\left(\frac{s+1+\nu}{2}\right)\Gamma\left(\frac{s+1-\nu}{2}\right)L(s,f)
\end{aligned}$$

where $L(s,f) = \sum_1^\infty \frac{a_n}{n^s}$ as before.

So define

$$\Lambda(s,f) = N^{(\frac{s+1-1/2}{2})}\pi^{-(s+1)}\Gamma\left(\frac{s+1+\nu}{2}\right)\Gamma\left(\frac{s+1-\nu}{2}\right)L(s,f).$$

Then

$$\int_0^\infty f(iy)y^{s+1-\frac{1}{2}}\frac{dy}{y} = \frac{1}{2}N^{-(\frac{s+1-1/2}{2})}\Lambda(s,f).$$

Therefore,

$$g(iy) = 2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{s+1-1/2}{2})}\Lambda(s,f)y^{-(s+1-1/2)}ds$$

by the Mellin inversion formula from 1.5.5 from [6]. Now if $\Lambda(s,f) = -\Lambda(1-s,f)$, then

$$g(iy) = 2\frac{-1}{2\pi i}\int_{(c)} N^{-(\frac{s+1/2}{2})}\Lambda(1-s,f)y^{-(s+1/2)}ds.$$

Let $s' = 1-s$, then $ds' = -ds$, $s+1/2 = 3/2-s'$. Also, let $y' = \frac{1}{Ny}$. Therefore,

$y = 1/Ny'$ and $Ny' = 1/y$

$$\begin{aligned}
\therefore g(iy) =& 2\frac{-1}{2\pi i}\int_{(c)} N^{-(\frac{3/2-s'}{2})}\Lambda(s',f)y^{(s'-3/2)}ds' \\
=& 2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{3/2-s'}{2})}\Lambda(s',f)(Ny')^{(s'-3/2)}ds' \\
=& 2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{3/2-s'}{2})}\Lambda(s',f)(Ny')^{-(s+1/2)}(Ny')^2ds' \\
=& 2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{3/2-s'}{2})}\Lambda(s',f)N^{-(s+1/2)}y'^{-(s+1/2)}(Ny')^2ds' \\
=& 2\frac{1}{2\pi i}\int_{(c)} N^{-1}N^{-(\frac{s'+1/2}{2})}\Lambda(s',f)y'^{-(s+1/2)}(Ny')^2ds \\
=& 2\frac{1}{2\pi i}\int_{(c)} N^{-(\frac{s'+1/2}{2})}\Lambda(s',f)\left(\frac{1}{Ny}\right)^{-(s+1/2)} N^{-1}\left(\frac{1}{y}\right)^2 ds \\
=& 2\frac{1}{2\pi i}\left(\frac{1}{Ny^2}\right)\int_{(c)} N^{-(\frac{s'+1/2}{2})}\Lambda(s',f)\left(\frac{1}{Ny}\right)^{-(s+1/2)} ds \\
=& \left(\frac{1}{Ny^2}\right) g\left(\frac{i}{Ny}\right) \\
=& \frac{1}{Ny^2} g\left(\frac{-1}{iNy}\right)
\end{aligned}$$

as required. □

A.2 Discussion of possible application

Let

$$\zeta(s,P) := \pi^{-2s}\Gamma\left(s+\frac{\sqrt{-1}r}{2}\right)\Gamma\left(s-\frac{\sqrt{-1}r}{2}\right)\sum_{\beta\in S\setminus\{0\}} A(\beta)\frac{P(\beta)}{|\beta|^{2s}}$$

where P is a harmonic polynomial of degree l. Then $\zeta(s,P)$ converges for $Re(s) > \frac{l+4+k}{2}$. Let $\{P_{l,\nu}\}_\nu$ be the basis of Harmonic polynomials of degree l on $\mathbb{H}$. Then, Maass converse theorem implies F with coefficients $A(B)$ belongs to $\mathcal{M}(\Gamma_T; r)$ if for all $l \in \mathbb{Z}$ and for all ν, the following 3 conditions are satisfied:

1. $\zeta(s, P_{l,\nu})$ has analytic continuation to whole complex plane,
2. $\zeta(s, P_{l,\nu})$ is bounded on any vertical strip of the complex plane,
3. the functional equation $\zeta(2+l-s, P_{l,\nu}) = (-1)^l \zeta(s, \hat{P}_{l,\nu})$ holds, where $\hat{P}_{l,\nu}(x) := P(\overline{x})$ for all $x \in \mathbb{H}$.

We would like to use this fact with a suitable P to garner information about $C(-N)$. However, for each N there are a lot of β with $|\beta|^2 = 2N$, so we need to make a smart choice for P.

First observation: l cannot be odd. For odd l, $P(\beta) = -P(-\beta)$ where as $A(\beta) = A(-\beta)$. Hence, $\zeta(s, P) = 0$ for every odd degree harmonic P.

Second observation: By Lemma 4.2.1, β can be primitive only when $d = 1$ and $u = 0$ if and only if $|\beta|^2 \equiv 2 \pmod 4$. Hence, u for β is basically all the extra powers of 2 in β. Now d for β and $\overline{\beta}$ are equal. At the same time, $|\beta| = |\overline{\beta}|$, hence both β and $\overline{\beta}$ have the same u. Therefore, they have the same absolute value, common odd divisor as well as power of ϖ_2. In other words, $A(\beta) = A(\overline{\beta})$.

With both of these observations, we can look for polynomials which might be suitable. However, even after testing various polynomials none of them appear to be helpful. Either too little terms are eliminated or no good terms remain and computation complexity increases too fast with every increase in degree. A heuristic reason for the lack of suitable polynomials is that the information we are trying to gain is too specific in the whole space. So this approach seems not to be working.

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DEDICATION

to

My parents

Vivek Moreshwar Wagh, and

Shubhada Vivek Wagh

For

Encouraging me to follow my dreams