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K-types and Invariants
for the Representations of $\mathrm{GSp}(4,\mathbb{R})$

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To my parents

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Contents

Abstract

Automorphic representations of the adelic group $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ are of importance in their relation to Siegel modular forms of degree 2. Given an automorphic representation π of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$, it decomposes into a product of admissible representations at each place. In the non-archimedean case, many useful results have been produced by Roberts and Schmidt. Here we find some invariants for the case of $\mathrm{GSp}(4, \mathbb{R})$, including the K-type structure, the L- and ϵ-factors, and the Gelfand-Kirillov dimension for all irreducible admissible representations.

Chapter 1

Preliminaries

1.1 Introduction

Automorphic representations of the adelic group $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ are of importance in their relation to Siegel modular forms of degree 2. Here we have $\mathbb{A}_{\mathbb{Q}}$ the adele ring of $\mathbb{Q}$ which is the restricted direct product of completions of $\mathbb{Q}$ at all places. Then for a field F, $\mathrm{GSp}(4, F)$ is defined as $g \in \mathrm{GL}(4, F)$ such that ${}^t gJg = \lambda(g)J$ where

$$J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \tag{1.1}$$

and $\lambda(g)$ is called the multiplier of g. There are then representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ with certain conditions which are automorphic representations.

Given an automorphic representation π on $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$, it decomposes into a product of representations on the groups $\mathrm{GSp}(4, \mathbb{Q}_\nu)$, as

$$\pi = \bigotimes_\nu \pi_\nu. \tag{1.2}$$

By studying these representations we can obtain information about automorphic representations. In particular, we are interested in irreducible admissible representations of $\mathrm{GSp}(4,\mathbb{Q}_\nu)$, so that for a maximal compact subgroup K_ν of $\mathrm{GSp}(4,\mathbb{Q}_\nu)$, $\pi_\nu|_{K_\nu}$ is unitary and when restricted in such a way, each irreducible representation appears with finite multiplicity. In the non-archimedean case, many useful results are presented in tables in [5]. Some similar results will be produced here for the archimedean case, so that we are dealing with $\mathrm{GSp}(4,\mathbb{R})$.

First, some helpful results about composition series of admissible representations of $\mathrm{Sp}(4,\mathbb{R})$ are collected from the work by Muić. Every irreducible admissible representation of $\mathrm{Sp}(4,\mathbb{R})$ is a contained in one of these composition series as a constituent of a representation produced by parabolic induction.

Then we move on to results about the L-factors and ϵ-factors of irreducible admissible $\mathrm{GSp}(4,\mathbb{R})$ representations. To do this, first the Langlands parameters are found. In this case, the Langlands parameters are admissible homomorphisms from the real Weil group, $W_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ to $\mathrm{GSp}(4,\mathbb{C})$. Given such a homomorphism, we may then decompose it into irreducible 1-dimensional and 2-dimensional representations of $W_\mathbb{R}$ with known L-factors and ϵ-factors. We find these factors both in the degree 4 spin case and the degree 5 standard case.

Next, as we are in the archimedean case, we have a maximal compact subgroup K of $\mathrm{Sp}(4,\mathbb{R})$ which we may use to examine the structure of $\mathrm{Sp}(4,\mathbb{R})$ representations by decomposing them as a sum of irreducible representations of K. Isomorphism classes of these representations of K are called K-types. By using the composition series produced earlier, the multiplicities of the K-types are determined for all irreducible representations of $\mathrm{Sp}(4,\mathbb{R})$ in this chapter.

Then in order to relate these results on $\mathrm{Sp}(4,\mathbb{R})$ to the case of $\mathrm{GSp}(4,\mathbb{R})$, there is an examination of the behavior of Langlands quotients of $\mathrm{GSp}(4,\mathbb{R})$ when restricted to $\mathrm{Sp}(4,\mathbb{R})$, so that the K-type structure and other properties may be determined for irreducible $\mathrm{GSp}(4,\mathbb{R})$ representations as well.

Finally, we consider the Gelfand-Kirillov dimension of these representations. For a Lie algebra representation of $\mathfrak{g}$, we view the representation as a finitely generated $U(\mathfrak{g})$ module V, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Then we take a generating subspace V_0 and make V a graded module with $V_n = U_n(\mathfrak{g})V_0$, where $U_n(\mathfrak{g}$ is generated by monomials in the enveloping algebra with exactly n elements. Then there exists a polynomial $d(n)$ with degree at most $\dim\mathfrak{g}$ with $d(n) = \dim V_n$ for large enough n. The degree of this polynomial is the Gelfand-Kirillov dimension. As the K-type structure of $\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{GSp}(4,\mathbb{R})$ representations is known, the Gelfand-Kirillov dimension of these representations can be calculated directly.

1.2 Basic definitions

Let us begin with some basic definitions. The general symplectic group $\mathrm{GSp}(4,\mathbb{R})$ is the set of $g \in \mathrm{GL}(4,\mathbb{R})$ such that ${}^t gJg = \lambda(g)J$ where

$$J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \tag{1.3}$$

and $\lambda(g)$ is the multiplier of g.

We will also make use of the subgroups $\mathrm{Sp}^{\pm}(4,\mathbb{R})$ consisting of all elements of $\mathrm{GSp}(4,\mathbb{R})$ with multiplier ±1 and $\mathrm{Sp}(4,\mathbb{R})$ consisting of all elements of $\mathrm{GSp}(4,\mathbb{R})$ with multiplier 1.

Also of importance is the Lie algebra of $\mathrm{Sp}(4,\mathbb{R})$, $\mathrm{sp}(4,\mathbb{R})$ consisting of the set of 4×4 matrices with entries in $\mathbb{R}$ with the condition that for $A \in \mathrm{sp}(4, \mathbb{R})$, $JA +^t AJ = 0$

Additionally for the matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we define

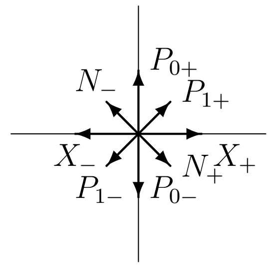
$$A' = \frac{1}{ad - bc} \left[\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right]. \tag{1.4}$$

1.3 Root structure

By [4] section 2.1 we there is a maximal compact subgroup K of $\mathrm{Sp}(4, \mathbb{R})$ consisting of all matrices of the form $AB - BA$. We may then take $\mathfrak{k}$ as the Lie algebra of K. Also from [4] we may use the following basis for the complexification $\mathrm{sp}(4, \mathbb{C})$.

$$\begin{aligned}
Z &= -i \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] & Z' &= -i \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{smallmatrix}\right] \\
N_+ &= \tfrac{1}{2} \left[\begin{smallmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{smallmatrix}\right] & N_- &= \tfrac{1}{2} \left[\begin{smallmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{smallmatrix}\right] \\
X_+ &= \tfrac{1}{2} \left[\begin{smallmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] & X_- &= \tfrac{1}{2} \left[\begin{smallmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] \\
P_{1+} &= \tfrac{1}{2} \left[\begin{smallmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{smallmatrix}\right] & P_{1-} &= \tfrac{1}{2} \left[\begin{smallmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -1 \\ -1 & 0 & -1 & 0 \end{smallmatrix}\right] \\
P_{0+} &= \tfrac{1}{2} \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{smallmatrix}\right] & P_{0-} &= \tfrac{1}{2} \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{smallmatrix}\right]
\end{aligned} \tag{1.5}$$

The root system of $\mathrm{sp}(4, \mathbb{C})$ is $\{(\pm 2, 0), (0, \pm 2), (\pm 1, \pm 1), (\pm 1, \mp 1)\}$ with $(-1, 1)$ and $(1, -1)$ the compact roots.



1.4 Parabolic induction

We will work extensively with parabolic induction from three subgroups of $\mathrm{GSp}(4,\mathbb{R})$. First is the standard Borel subgroup, consisting of elements of the form $\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$. We shall also need the Klingen subgroup, consisting of elements of the form $\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}$. Finally, we consider the Siegel subgroup, consisting of elements of the form $\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}$.

1.4.1 Parabolic induction on the Borel subgroup

To use parabolic induction the Borel subgroup, we take characters χ_1, χ_2, and σ on $\mathbb{R}$, and use them to define a representation

$$\begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} \mapsto \chi_1(a)\chi_2(b)\sigma(c) \tag{1.6}$$

on the Borel subgroup. We can then induce to the full group by taking the space of functions $f : \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ with the condition that

$$f(hg) = \left|a^2 b\right| |c|^{-\frac{3}{2}} \chi_1(a)\chi_2(b)\sigma(c)f(g) \tag{1.7}$$

for any $h \in \left[\begin{smallmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{smallmatrix}\right]$. Then we take the group action of right translation on this space, resulting in the induced representation we denote as $\chi_1 \times \chi_2 \rtimes \sigma$.

1.4.2 Parabolic induction on the Klingen

To use parabolic induction on the Klingen subgroup, we take a character χ on $\mathbb{R}$ and an irreducible representation of $\mathrm{GL}(2, \mathbb{R})$ which we call (π, V), and use them to define a representation

$$\begin{bmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \det A t^{-1} \end{bmatrix} \mapsto \chi(t)\pi(A) \tag{1.8}$$

on the Klingen subgroup, where $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We can then induce to the full group by taking the space of functions $f : \mathrm{GSp}(4, \mathbb{R}) \to V$ with the condition that

$$f(hg) = \left|t^2(\det A)^{-1}\right|^s \chi(t)\pi(A)f(g) \tag{1.9}$$

for any $h \in \left[\begin{smallmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \det A t^{-1} \end{smallmatrix}\right]$. Then we take the group action of right translation on this space, resulting in the induced representation we denote as $\chi \rtimes \pi$.

1.4.3 Parabolic induction on the Siegel subgroup

To use parabolic induction on the Siegel subgroup, we take a character σ on $\mathbb{R}$ and an irreducible representation of $\mathrm{GL}(2, \mathbb{R})$ which we call (π, V), and use them to define a representation

$$\begin{bmatrix} A & * \\ & cA' \end{bmatrix} \mapsto \sigma(c)\pi(A) \tag{1.10}$$

on the Siegel subgroup. We can then induce to the full group by taking the space of functions $f : \mathrm{GSp}(4,\mathbb{R}) \to V$ with the condition that

$$f(hg) = \left|\det A c^{-1}\right|^{\frac{3}{2}} \sigma(c)\pi(A)f(g) \tag{1.11}$$

for any $h \in \left[\begin{smallmatrix} A & * \\ & cA' \end{smallmatrix}\right]$. Then we take the group action of right translation on this space, resulting in the induced representation we denote as $\pi \rtimes \sigma$.

1.5 Discrete series representations

Frequently we will run across discrete series representations and limits of discrete series representations. Our notation shall be that for $p \in \mathbb{Z}_{>0}$, $X(p,+)$ is the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with minimal weight $p+1$ and $X(p,-)$ is the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with maximal weight $-p-1$. Similarly, $X(0,+)$ is the limit of discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with minimal weight 1 and $X(0,-)$ is the limit of discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with maximal weight -1.

In the case of $\mathrm{Sp}(4,\mathbb{R})$, when $(p,q) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, $p \neq q$, $X(p,q)$ shall denote the discrete series representation of $\mathrm{Sp}(4,\mathbb{R})$ with Harish-Chandra parameter (p,q). $X(p,-p)$ will denote the limit of discrete series with infinitesimal parameter $(p,-p)$. Further, $X^1(p,0)$ and $X^2(0,-p)$ will be the holomorphic and anti-holomorphic limits of discrete series with corresponding infinitesimal parameters and $X^2(p,0)$ and $X^1(0,-p)$ will be the large limits of discrete series with corresponding infinitesimal parameters.

For $\mathrm{GSp}(4,\mathbb{R})$, we will distinguish between holomorphic and large discrete series and limits of discrete series with the notation $X^{hol}(p,q)$ and $X^{large}(p,q)$.

Chapter 2

Composition series

We will gather together composition series for the principal series representations of $\mathrm{Sp}(4,\mathbb{R})$ for convenience. We use the results of [3] here. In the most general situation we have the induced representation $||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1$ with $s_1, s_2 \in \mathbb{C}$, $\epsilon_1, \epsilon_2 \in \{0, 1\}$. By way of Weyl transformations, we only need consider the case where $\mathrm{Re}(s_1) \geq \mathrm{Re}(s_2) \geq 0$. As described in Lemma 5.1 from [3] we see that the principal series is reducible in four different cases and irreducible otherwise. These four cases are

- s_2 is an integer such that $\epsilon_2 \equiv s_2 + 1 \pmod 2$
- s_1 is an integer such that $\epsilon_1 \equiv s_1 + 1 \pmod 2$
- $s_1 + s_2 \in \mathbb{Z}_{\neq 0}, \epsilon_1 + \epsilon_2 \equiv s_1 + s_2 + 1 \pmod 2$
- $s_1 - s_2 \in \mathbb{Z}_{\neq 0}, \epsilon_1 + \epsilon_2 \equiv s_1 - s_2 + 1 \pmod 2$.

2.1 Non-integral infinitesimal character

Let us first consider the cases where one or both of s_1, s_2 is non-integral.

- The first reducibility criterion is that s_2 is an integer such that $\epsilon_2 \equiv s_2 + 1 \pmod 2$. In this case, we have by Theorem 2.4 from [3] for $s_2 > 0$ that

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes (X(s_2,+) \oplus X(s_2,-)) \hookrightarrow ||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \twoheadrightarrow \tag{2.1}$$

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes V_{s_2}$$

  and for $s_2 = 0$ that

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \times \mathrm{sgn} \rtimes 1 \simeq ||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes (X(0,+) \oplus X(0,-)). \tag{2.2}$$

  In the event that $s_1 \notin \mathbb{Z}$, all constituents are irreducible by Theorem 12.1 in [3].

- The second case of reducibility is when $\epsilon_1 \equiv s_1 + 1 \pmod 2$, and we may use the intertwining operator defined as $B_1(t)$ in Lemma 7.3 from [3]. In this case, as long as $s_2 \notin \mathbb{Z}$, it gives an isomorphism

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \simeq ||^{s_2}\mathrm{sgn}^{\epsilon_2} \times ||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes 1, \tag{2.3}$$

  and we may proceed as in the first case.

- The third case of reducibility is when $s_1 - s_2 \in \mathbb{Z}_{\neq 0}, \epsilon_1 + \epsilon_2 \equiv s_1 - s_2 + 1 \pmod 2$. Then we have by Theorem 2.5 of [3] that

$$\delta(||^{\frac{s_1+s_2}{2}}\mathrm{sgn}^{\epsilon_2}, s_1 - s_2) \rtimes 1 \hookrightarrow ||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \twoheadrightarrow \tag{2.4}$$

$$\zeta(||^{\frac{s_1+s_2}{2}}\mathrm{sgn}^{\epsilon_2}, s_1 - s_2) \rtimes 1$$

Again by Theorem 12.1 in [3], all constituents are irreducible if $s_2 \notin \mathbb{Z}$.

- The final case of reducibility is when $s_1 + s_2 \in \mathbb{Z}_{\neq 0}, \epsilon_1 + \epsilon_2 \equiv s_1 + s_2 + 1 \pmod 2$, and we need to use the intertwining operator described in Lemma 7.2 from [3] as $A_1(t)$, which is an isomorphism

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \simeq ||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{-s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \tag{2.5}$$

  in this case as long as $s_2 \notin \mathbb{Z}$.

2.2 Integral infinitesimal character

Now we need to consider the cases where both $s_1, s_2 \in \mathbb{Z}$, where we first consider cases of the form $||^p\mathrm{sgn}^{\epsilon_1} \times ||^t\mathrm{sgn}^{\epsilon_2} \rtimes 1$ with $p > t > 0$, $p, t \in \mathbb{Z}$. We will use Theorems 2.4 and 2.5 from [3] frequently as they give the initial decomposition of these principal series representations.

- First, note that $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes 1$ is irreducible as it does not satisfy the criteria for reducibility.

- Next $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1$ has two composition series,

$$||^p\mathrm{sgn}^p \rtimes (X(t,+) \oplus X(t,-)) \hookrightarrow ||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1 \twoheadrightarrow ||^p\mathrm{sgn}^p \rtimes V_t \tag{2.6}$$

$$\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1 \hookrightarrow ||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1 \twoheadrightarrow \zeta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1. \tag{2.7}$$

  For the first composition series, Theorem 11.1 from [3] states that the constituents decompose as follows:

$$\mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1) \hookrightarrow ||^p\mathrm{sgn}^p \rtimes V_t \twoheadrightarrow \tag{2.8}$$

$$\mathrm{Lang}(||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1)$$

$$V_{1,+} \hookrightarrow ||^p\mathrm{sgn}^p \rtimes X(t,+) \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^p, X(t,+)) \tag{2.9}$$

$$V_{1,-} \hookrightarrow ||^p\mathrm{sgn}^p \rtimes X(t,-) \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^p, X(t,-)) \tag{2.10}$$

$$X(p,-t) \hookrightarrow V_{1,+} \twoheadrightarrow W_{1,+} \tag{2.11}$$

$$X(t,-p) \hookrightarrow V_{1,-} \twoheadrightarrow W_{1,-} \tag{2.12}$$

where $W_{1,+}$ has constituents $\mathrm{Lang}(\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t), 1)$ and $\mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,+))$ and $W_{1,-}$ has constituents $\mathrm{Lang}(\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t), 1)$ and $\mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,-))$.

For the second composition series we use that $\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1 \simeq \delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t}, p-t) \rtimes 1$, and by Theorem 10.3 from [3] the constituents decompose as follows:

$$W \hookrightarrow \delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1 \twoheadrightarrow \mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1) \tag{2.13}$$

where

$$\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1 \hookrightarrow W \twoheadrightarrow \tag{2.14}$$

$$\mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,-))$$

$$X(p,-t) \oplus X(t,-p) \hookrightarrow \delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1 \twoheadrightarrow \tag{2.15}$$

$$\mathrm{Lang}(\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1).$$

- In the third of these cases, $||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes 1$, we have the composition series

$$||^p\mathrm{sgn}^{p+1} \rtimes (X(t,+) \oplus X(t,-)) \hookrightarrow ||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes 1 \twoheadrightarrow \tag{2.16}$$

$$||^p\mathrm{sgn}^{p+1} \rtimes V_t.$$

We see by Lemma 9.4 of [3] that the constituents decompose as

$$||^t\mathrm{sgn}^{t+1} \rtimes X(p,+) \hookrightarrow ||^p\mathrm{sgn}^{p+1} \rtimes X(t,+) \twoheadrightarrow \tag{2.17}$$

$$\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(t,+)),$$

$$||^t\mathrm{sgn}^{t+1} \rtimes X(p,-) \hookrightarrow ||^p\mathrm{sgn}^{p+1} \rtimes X(t,-) \twoheadrightarrow \tag{2.18}$$

$$\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(t,-))$$

with $||^t\mathrm{sgn}^{t+1} \rtimes X(p,+)$, $||^t\mathrm{sgn}^{t+1} \rtimes X(p,-)$ and $||^p\mathrm{sgn}^{p+1} \rtimes V_t$ irreducible.

- In the final non-degenerate case, $||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1$, we have a composition series

$$\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1 \hookrightarrow ||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1 \twoheadrightarrow \zeta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1. \tag{2.19}$$

Again using Theorem 10.3 from [3], constituents further break down as

$$W \hookrightarrow \delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1 \twoheadrightarrow \mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1) \tag{2.20}$$

where

$$\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1 \hookrightarrow W \twoheadrightarrow \tag{2.21}$$

$$\mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,-))$$

$$X(p,-t) \oplus X(t,-p) \hookrightarrow \delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1 \twoheadrightarrow \tag{2.22}$$

$$\mathrm{Lang}(\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1).$$

Also by Theorem 10.6 from [3],

$$X(p,t) \oplus X(-t,-p) \hookrightarrow \zeta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1 \twoheadrightarrow \tag{2.23}$$

$$\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1).$$

There is also a second composition series we may use:

$$||^t\mathrm{sgn}^t \rtimes (X(p,+) \oplus X(p,-)) \hookrightarrow ||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1 \twoheadrightarrow ||^t\mathrm{sgn}^t \rtimes V_p. \tag{2.24}$$

By Theorem 10.1 and 10.6 (iv) in [3] the constituents decompose as

$$X(p,t) \oplus X(p,-t) \hookrightarrow ||^t\mathrm{sgn}^t \rtimes X(p,+) \twoheadrightarrow \mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,+)) \tag{2.25}$$

$$X(t,-p) \oplus X(-t,p) \hookrightarrow ||^t\mathrm{sgn}^t \rtimes X(p,-) \twoheadrightarrow \mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,-)) \tag{2.26}$$

$$\mathrm{Lang}(\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1) \oplus \mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1) \hookrightarrow \tag{2.27}$$

$$||^t\mathrm{sgn}^t \rtimes V_p \twoheadrightarrow \mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1).$$

2.3 Integral infinitesimal character, degenerate cases

Now we consider degenerate cases, starting with those where $p = t > 0$, giving us four possible cases.

- First, note $||^p\text{sgn}^p \times ||^p\text{sgn}^p \rtimes 1$ is irreducible.
- Next $||^p\text{sgn}^p \times ||^p\text{sgn}^{p+1} \rtimes 1$ has composition series:

$$||^p\text{sgn}^p \rtimes (X(p,+) \oplus X(p,-)) \hookrightarrow ||^p\text{sgn}^p \times ||^p\text{sgn}^{p+1} \rtimes 1 \twoheadrightarrow \tag{2.28}$$

$$||^p\text{sgn}^p \rtimes V_p.$$

From Theorem 10.4 in [3], the constituents decompose as follows:

$$X^1(p,-p) \hookrightarrow ||^p\text{sgn}^p \rtimes X(p,+) \twoheadrightarrow \text{Lang}(||^p\text{sgn}^p \rtimes X(p,+)), \tag{2.29}$$

$$X^2(p,-p) \hookrightarrow ||^p\text{sgn}^p \rtimes X(p,-) \twoheadrightarrow \text{Lang}(||^p\text{sgn}^p \rtimes X(p,-)), \tag{2.30}$$

and $||^p\text{sgn}^p \rtimes V_p$ is irreducible.

- Then $||^p\text{sgn}^{p+1} \times ||^p\text{sgn}^p \rtimes 1$ is isomorphic to $||^p\text{sgn}^p \times ||^p\text{sgn}^{p+1} \rtimes 1$ by the intertwining operator $B_1(t)$ as defined in Lemma 7.3 from [3].
- In the case $||^p\text{sgn}^{p+1} \times ||^p\text{sgn}^{p+1} \rtimes 1$ we have the composition series

$$||^p\text{sgn}^{p+1} \rtimes (X(p,+) \oplus X(p,-)) \hookrightarrow ||^p\text{sgn}^{p+1} \times ||^p\text{sgn}^{p+1} \rtimes 1 \twoheadrightarrow \tag{2.31}$$

$$||^p\text{sgn}^{p+1} \rtimes V_p,$$

All constituents are irreducible by Lemma 9.5 from [3].

Now we consider the cases where $p > t = 0$, giving us another collection of four cases.

- First, note that $||^p\text{sgn}^p \times 1 \rtimes 1$ is irreducible.
- Next for $||^p\text{sgn}^p \times \text{sgn} \rtimes 1$ we have that

$$\delta(||^{\frac{p}{2}}\text{sgn}, p) \rtimes 1 \hookrightarrow ||^p\text{sgn}^p \times \text{sgn} \rtimes 1 \twoheadrightarrow \zeta(||^{\frac{p}{2}}\text{sgn}, p) \rtimes 1. \tag{2.32}$$

By Theorems 10.7 and 11.2 from [3], the constituents decompose as follows:

$$X^2(p,0) \oplus X^1(0,-p) \hookrightarrow \delta(||^{\frac{p}{2}}\text{sgn}, p) \rtimes 1 \twoheadrightarrow \text{Lang}(\delta(||^{\frac{p}{2}}, p) \rtimes 1), \tag{2.33}$$

$$W' \hookrightarrow \zeta(||^{\frac{p}{2}}\text{sgn}, p) \rtimes 1 \twoheadrightarrow \text{Lang}(||^p\text{sgn}^p \times \text{sgn} \rtimes 1) \tag{2.34}$$

$$\text{Lang}(\delta(||^{\frac{p}{2}}, p) \rtimes 1) \hookrightarrow W' \twoheadrightarrow \tag{2.35}$$

$$\text{Lang}(||^p\text{sgn}^p \rtimes X(0,+) \oplus \text{Lang}(||^p\text{sgn}^p \rtimes X(0,-).$$

- In the third of these cases, $||^p\text{sgn}^{p+1} \times \text{sgn} \rtimes 1$, we have that

$$||^p\text{sgn}^{p+1} \times \text{sgn} \rtimes 1 = ||^p\text{sgn}^{p+1} \rtimes (X(0,+) \oplus X(0,-)). \tag{2.36}$$

The constituents decompose by Lemma 9.6 from [3] as

$$\text{sgn} \rtimes X(p,+) \hookrightarrow ||^p\text{sgn}^{p+1} \rtimes X(0,+) \twoheadrightarrow \text{Lang}(||^p\text{sgn}^{p+1} \rtimes X(0,+)), \tag{2.37}$$

$$\text{sgn} \rtimes X(p,-) \hookrightarrow ||^p\text{sgn}^{p+1} \rtimes X(0,-) \twoheadrightarrow \text{Lang}(||^p\text{sgn}^{p+1} \rtimes X(0,-)) \tag{2.38}$$

and $\mathrm{sgn} \rtimes X(p,+)$ and $\mathrm{sgn} \rtimes X(p,-)$ are irreducible.

- In the fourth case, $||^p \mathrm{sgn}^{p+1} \times 1 \rtimes 1$, we have a composition series

$$\delta(||^{\frac{p}{2}}, p) \rtimes 1 \hookrightarrow ||^p \mathrm{sgn}^{p+1} \times 1 \rtimes 1 \twoheadrightarrow \zeta(||^{\frac{p}{2}}, p) \rtimes 1. \tag{2.39}$$

  Constituents further break down by Theorem 10.7 of [3] as

$$X^2(p,0) \oplus X^1(0,-p) \hookrightarrow \delta(||^{\frac{p}{2}}, p) \rtimes 1 \twoheadrightarrow \mathrm{Lang}(\delta(||^{\frac{p}{2}}, p) \rtimes 1), \tag{2.40}$$

$$X^1(p,0) \oplus X^2(0,-p) \hookrightarrow \zeta(||^{\frac{p}{2}}, p) \rtimes 1 \twoheadrightarrow \mathrm{Lang}(||^p \mathrm{sgn}^{p+1} \times 1 \rtimes 1). \tag{2.41}$$

Finally, the last few cases to consider are when $p = t = 0$. These are fairly simple. Namely, $\mathrm{sgn} \times 1 \rtimes 1 \cong 1 \times \mathrm{sgn} \rtimes 1 \cong 1 \rtimes (X(0,+) \oplus X(0,-))$, $\mathrm{sgn} \times \mathrm{sgn} \rtimes 1 \cong \mathrm{sgn} \rtimes (X(0,+) \oplus X(0,-))$, and $1 \times 1 \rtimes 1$ is irreducible.

Chapter 3

Langlands parameters

Now we will determine the Langlands parameters for each irreducible representation of $\mathrm{GSp}(4,\mathbb{R})$.

3.1 Representations of GSp$(4,\mathbb{R})$

We consider how representations of $\mathrm{GSp}(4,\mathbb{R})$ relate to those of $\mathrm{Sp}(4,\mathbb{R})$. Given a representation of $\mathrm{Sp}(4,\mathbb{R})$, we may induce to $\mathrm{Sp}(4,\mathbb{R})^{\pm} = \mathrm{Sp}(4,\mathbb{R}) \sqcup \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \times \mathrm{Sp}(4,\mathbb{R})$. Then we may use the fact that $\mathrm{GSp}(4,\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathrm{Sp}(4,\mathbb{R})^{\pm}$ to obtain a representation of $\mathrm{GSp}(4,\mathbb{R})$ from a representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ and a character of $\mathbb{R}_{>0}$.

Given a representation of $\mathrm{GSp}(4,\mathbb{R})$, we may restrict it to $\mathrm{Sp}(4,\mathbb{R})$ and see how it decomposes. For this purpose, it is important that we observe that $\mathrm{Sp}(4,\mathbb{R})$ is a subgroup of index 2 of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$. This means that either an irreducible representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ is irreducible when restricted to $\mathrm{Sp}(4,\mathbb{R})$, and there is exactly one other representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ with the same restriction, or the $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ representation is not irreducible when restricted to $\mathrm{Sp}(4,\mathbb{R})$ and has submodules isomorphic to exactly two distinct irreducible representations of $\mathrm{Sp}(4,\mathbb{R})$.

Let us first consider the case of a $(\mathfrak{g}, K)$ module where $\mathfrak{g} = \mathrm{gsp}(4,\mathbb{R})$ and K is a maximal compact subgroup of $\mathrm{GSp}(4,\mathbb{R})$. We will consider such a (π, V_K) induced on the Borel parabolic, namely $\chi_1 \times \chi_2 \rtimes \sigma$ where χ_1, χ_2, and σ are characters of $\mathbb{R}^\times$. Such a representation has a standard model consisting of K-finite vectors in the space of smooth functions $f : \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ satisfying the property

$$f(hg) = |a^2 b||c|^{-\frac{3}{2}} \chi_1(a)\chi_2(b)\sigma(c)f(g) \tag{3.1}$$

for any $h \in \begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix}$. We take this representation and restrict to actions by elements of $\mathrm{Sp}(4,\mathbb{R})$. Then we map from this space of functions to another $\mathrm{Sp}(4,\mathbb{R})$-module consisting of functions $f : \mathrm{Sp}(4,\mathbb{R}) \to \mathbb{C}$ satisfying the property

$$f(hg) = |a^2 b| \chi_1(a)\chi_2(b)f(g) \tag{3.2}$$

for any $h \in \begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & b^{-1} & * \\ & & & a^{-1} \end{bmatrix}$. We shall show that the map given by restricting domain is bijective. Consider a function $f : \mathrm{GSp}(4,\mathbb{R}) \to \mathbb{C}$ such that its restriction to $\mathrm{Sp}(4,\mathbb{R})$ gives $f(g) = 0$ for $g \in \mathrm{Sp}(4,\mathbb{R})$. Then for any $g' \in \mathrm{GSp}(4,\mathbb{R})$, we may write $g' = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & c \end{bmatrix} g$ for some $g \in \mathrm{Sp}(4,\mathbb{R})$. This gives $f(g') = |c|^{-\frac{3}{2}}\sigma(c)f(g) = 0$ so the map is injective. To conclude that it is also surjective, note that we may extend a function f on $\mathrm{Sp}(4,\mathbb{R})$ with the above transformation property to a function on $\mathrm{GSp}(4,\mathbb{R})$ with suitable transformation properties by defining $f(g) = |c|^{-\frac{3}{2}}\sigma(c)f(h)$ where $g \in \mathrm{GSp}(4,\mathbb{R})$ and $h \in \mathrm{Sp}(4,\mathbb{R})$. This function restricted to $\mathrm{Sp}(4,\mathbb{R})$ will then give us the original function f. Then notice that by taking the K-finite functions obtained on $\mathrm{Sp}(4,\mathbb{R})$ by this bijection we have a model for the representation $\chi_1 \times \chi_2 \rtimes 1$ of $\mathrm{Sp}(4,\mathbb{R})$. That is, the restriction of $\chi_1 \times \chi_2 \rtimes \sigma$ to $\mathrm{Sp}(4,\mathbb{R})$ is isomorphic to $\chi_1 \times \chi_2 \rtimes 1$ as a $\mathrm{Sp}(4,\mathbb{R})$-module.

Next consider the case of a representation induced on the Siegel parabolic, namely $\pi \rtimes \sigma$ where σ is a character of $\mathbb{R}^\times$ and (π, V_π) is a representation on $\mathrm{GL}(2,\mathbb{R})$. Such a representation has a standard model consisting of the K-finite vectors in the space of smooth functions $f : \mathrm{GSp}(4,\mathbb{R}) \to V_\pi$ satisfying the property

$$f(hg) = |\det(A)c^{-1}|^{\frac{3}{2}}\sigma(c)\pi(A)f(g) \tag{3.3}$$

for any $h \in \left[\begin{smallmatrix} A & * \\ & cA' \end{smallmatrix}\right]$ where $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and we let $A' = \frac{1}{ad-bc}\left[\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right]$.

We take this representation and restrict to actions by elements of $\mathrm{Sp}(4,\mathbb{R})$. Then we map from this space of functions to another $\mathrm{Sp}(4,\mathbb{R})$-module consisting of functions $f : \mathrm{Sp}(4,\mathbb{R}) \to V_\pi$ satisfying the property

$$f(hg) = |\det(A)|^{\frac{3}{2}}\pi(A)f(g) \tag{3.4}$$

for any $h \in \left[\begin{smallmatrix} A & * \\ & A' \end{smallmatrix}\right]$ where $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We shall show that the map given by restricting domain is bijective. Consider a function $f : \mathrm{GSp}(4,\mathbb{R}) \to V_\pi$ such that its restriction to $\mathrm{Sp}(4,\mathbb{R})$ gives $f(h) = 0$ for $h \in \mathrm{Sp}(4,\mathbb{R})$. Then for any $g \in \mathrm{GSp}(4,\mathbb{R})$, we may write $g = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & c \end{bmatrix} h$ for some $h \in \mathrm{Sp}(4,\mathbb{R})$. This gives $f(g) = |c|^{-\frac{3}{2}}\sigma(c)f(h) = 0$ so the map is injective. To conclude that it is also surjective, note that we may extend a function f on $\mathrm{Sp}(4,\mathbb{R})$ with the above transformation property to a function on $\mathrm{GSp}(4,\mathbb{R})$ with suitable transformation properties by defining $f(g) = |c|^{-\frac{3}{2}}\sigma(c)f(h)$ where $g \in \mathrm{GSp}(4,\mathbb{R})$ and $h \in \mathrm{Sp}(4,\mathbb{R})$. Then notice that by taking the K-finite functions obtained on $\mathrm{Sp}(4,\mathbb{R})$ by this bijection we have a model for the representation $\pi \rtimes 1$ of $\mathrm{Sp}(4,\mathbb{R})$. That is, the restriction of $\pi \rtimes \sigma$ to $\mathrm{Sp}(4,\mathbb{R})$ is isomorphic to $\pi \rtimes 1$.

Finally we consider the case of a representation induced on the Klingen parabolic, namely $\chi \rtimes \pi$ where χ is a character of $\mathbb{R}^\times$ and (π, V_π) is a representation on $\mathrm{GL}(2,\mathbb{R})$. Such a representation has a standard model consisting of the K-finite vectors in the space of smooth functions $f : \mathrm{GSp}(4,\mathbb{R}) \to V_\pi$ satisfying the property

$$f(hg) = |t^2(ad-bc)^{-1}|\chi(t)\pi([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}])f(g) \tag{3.5}$$

for any $h \in \begin{bmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & (ad-bc)t^{-1} \end{bmatrix}$. We then take this representation and restrict to actions by elements of $\mathrm{Sp}(4,\mathbb{R})$. Then we map from this space of functions to another $\mathrm{Sp}(4,\mathbb{R})$-module consisting of functions $f : \mathrm{Sp}(4,\mathbb{R}) \to V_\pi$ satisfying the property

$$f(hg) = t^2\chi(t)\pi\left([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]\right)f(g) \tag{3.6}$$

for any $h \in \begin{bmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & t^{-1} \end{bmatrix}$ with $ad-bc=1$. We shall show that the map given by restricting domain is bijective. Consider a function $f : \mathrm{GSp}(4,\mathbb{R}) \to V_\pi$ such that its restriction to $\mathrm{Sp}(4,\mathbb{R})$ gives $f(h) = 0$ for $h \in \mathrm{Sp}(4,\mathbb{R})$. Then for any $g \in \mathrm{GSp}(4,\mathbb{R})$, we may write $g = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & e & \\ & & & e \end{bmatrix} h$ for some $h \in \mathrm{Sp}(4,\mathbb{R})$. This gives $f(g) = |e|^{-1}\pi\left([\begin{smallmatrix} 1 & \\ & e \end{smallmatrix}]\right)f(h) = 0$ so the map is injective. To conclude that it is also surjective, first observe that we may induce from a representation π on $\mathrm{SL}(2,\mathbb{R})$ to one on $\mathrm{GL}(2,\mathbb{R})$. Then we may extend a function f on $\mathrm{Sp}(4,\mathbb{R})$ with the above transformation property to a function on $\mathrm{GSp}(4,\mathbb{R})$ with suitable transformation properties by defining $f(g) = |e|^{-1}\pi([\begin{smallmatrix} 1 & \\ & e \end{smallmatrix}])f(h)$ where $g \in \mathrm{GSp}(4,\mathbb{R})$ and $h \in \mathrm{Sp}(4,\mathbb{R})$. Then notice that by taking the K-finite functions obtained on $\mathrm{Sp}(4,\mathbb{R})$ by this bijection we have a model for the representation $\chi \rtimes \pi|_{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{Sp}(4,\mathbb{R})$. We may then use information about restriction of $\mathrm{GL}(2,\mathbb{R})$ representations to

$\mathrm{SL}(2,\mathbb{R})$ for a given π to determine the behavior of the restriction of $\chi \rtimes \pi$.

3.2 Langlands classification

Let us consider the Langlands classification of $\mathrm{GSp}(4,\mathbb{R})$ representations. From Knapp, Theorem 14.92 [2], we know that we may take $S = MAN$ a standard cuspidal parabolic subgroup, σ a discrete series or nondegenerate limit of discrete series on M, and μ a character on the Lie algebra of A which we denote as $\mathfrak{a}$, with Re μ in the closed positive Weyl chamber. Inducing from the parabolic gives a representation with a unique irreducible quotient, and every irreducible admissible representation is obtained as a quotient of induction in this manner.

The first case is when we take the Borel parabolic. This gives induced representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$ with χ_1, χ_2 representations of $\mathrm{GL}(1,\mathbb{R})$ and σ a representation of $\mathrm{GSp}(0,\mathbb{R})$. We shall consider such representations in the form of $||^a \mathrm{sgn}^b, ||^c \mathrm{sgn}^d, ||^e \mathrm{sgn}^f$ where $(a,b,c,d,e,f) \in \mathbb{C} \times \{0,1\} \times \mathbb{C} \times \{0,1\} \times \mathbb{C} \times \{0,1\}$ with $\mathrm{Re}(a) \geq \mathrm{Re}(c) \geq 0$ and $\mathrm{Re}(a) + \mathrm{Re}(c) > 0$. Then we have a Langlands quotient $\mathrm{L}(\chi_1, \chi_2, \sigma)$. The case where $\mathrm{Re}(a) = \mathrm{Re}(c) = 0$ will be examined later.

Next, we consider when we take the Siegel parabolic subgroup. In this situation, we obtain the representation $\delta \rtimes \sigma$ with δ a discrete series on $\mathrm{GL}(2,\mathbb{R})$ and σ a representation of $\mathrm{GSp}(0,\mathbb{R})$. Then we have a Langlands quotient $\mathrm{L}(\delta, \sigma)$. Such a representation will be of the form $\mathrm{L}(\delta(||^s \mathrm{sgn}^\epsilon, \ell), ||^a \mathrm{sgn}^b)$ with $(s, \epsilon, \ell, a, b) \in \mathbb{C} \times \{0,1\} \times \mathbb{Z}_{>0} \times \mathbb{C} \times \{0,1\}$, and $\delta(||^s \mathrm{sgn}^\epsilon, \ell) = ||^s \mathrm{sgn}^\epsilon \otimes D_\ell$ where D_ℓ is a discrete series on $\mathrm{GL}(2)$ with weights $\ell + 1$ and above and $-\ell - 1$ and below and central character $\mathrm{sgn}^{\ell+1}$. We also restrict Re $s > 0$.

Third, we have the Langlands quotient of a Klingen induced representation on $\mathrm{GSp}(4,\mathbb{R})$, $\mathrm{Lang}(\chi \rtimes \pi)$. We have $\mathrm{L}(||^a \mathrm{sgn}^b \rtimes ||^c \mathrm{sgn}^d D_\ell)$ with $(a,b,c,d,\ell) \in$

$\mathbb{C} \times \{0,1\} \times \mathbb{C} \times \{0,1\} \times \mathbb{Z}_{\geq 0}$. We also restrict to Re $a > 0$.

Next are the discrete series and limits of discrete series on $\mathrm{GSp}(4,\mathbb{R})$. We denote these as $X^{hol}(p,q)$ for a holomorphic discrete series or limit of discrete series with Blattner parameter (p,q) and as $X^{large}(p,q)$ for a large discrete series or limit of discrete series with Blattner parameter (p,q). Finally, we have the case of irreducible tempered representations that are neither discrete series representations nor limits of discrete series representations. We obtain them by inducing as above, but with a unitary character, as in the following theorem.

Theorem 3.1. *Let η, η_1, η_2, and σ be unitary characters of $\mathbb{R}^\times$, $p \in \mathbb{Z}_{\geq 0}$, and $k \in \mathbb{Z}_{>0}$*

- *$\eta_1 \times \eta_2 \rtimes \sigma$ is irreducible.*
- *$\eta \rtimes \sigma D_p$ is reducible if and only if $\eta = 1$ and $p > 0$.*
- *When $p > 0$, $1 \rtimes \sigma D_p = X^{hol}(p,0) \oplus X^{large}(p,0)$.*
- *$\delta(\eta,k) \rtimes \sigma$ is irreducible.*

Proof.

- First consider $\eta_1 \times \eta_2 \rtimes \sigma$. When we restrict this representation to $\mathrm{Sp}(4,\mathbb{R})$, the representation we obtain is isomorphic to $\eta_1 \times \eta_2 \rtimes 1$ as a $\mathrm{Sp}(4,\mathbb{R})$- module. Then by Corollary 5.2 from [3], we have that $\eta_1 \times \eta_2 \rtimes 1$ is reducible if and only if $\eta_1 = \mathrm{sgn}$ or $\eta_2 = \mathrm{sgn}$. In this case, denoting the other character simply as η, then $\mathrm{sgn} \times \eta \rtimes 1 \simeq \eta \times \mathrm{sgn} \rtimes 1 \simeq \eta \rtimes X(0,+) \oplus \eta \rtimes X(0,-)$.

  In the event that $\eta_1 \times \eta_2 \rtimes 1$ is irreducible it follows that $\eta_1 \times \eta_2 \rtimes \sigma$ must also be irreducible. Then we consider the remaining case.

If either $\eta \times \text{sgn} \rtimes \sigma$ or $\text{sgn} \times \eta \rtimes \sigma$ is reducible, then it must decompose in the form $M \oplus N$ where M restricts to a representation isomorphic to $\eta \rtimes X(0,+)$ on $\text{Sp}(4,\mathbb{R})$ and N restricts to $\eta \rtimes X(0,-)$.

However, by the weight structure of $\eta \rtimes X(0,+)$ and $\eta \rtimes X(0,-)$ given in Muic [3], there can be no representation of $\text{GSp}(4,\mathbb{R})$ that restricts to $\eta \rtimes X(0,+)$ on $\text{Sp}(4,\mathbb{R})$, as the weights of any $\text{GSp}(4,\mathbb{R})$ representation must be closed under the map $(x,y) \mapsto (-y,-x)$. Therefore $\eta_1 \times \eta_2 \rtimes \sigma$ is irreducible.

- Next, we consider $\delta(\eta,k) \rtimes \sigma$. When restricted to $\text{Sp}(4,\mathbb{R})$ we obtain a representation isomorphic to $\delta(\eta,k) \rtimes 1$

  Then by Lemma 8.1 from [3], we have that $\delta(\eta,k) \rtimes 1$ is reducible if and only if k is even and $\eta \in \{1, \text{sgn}\}$ In this case, we have that $\delta(1,2p) \rtimes 1 \simeq \delta(\text{sgn},2p) \rtimes 1 \simeq X^1(p,-p) \oplus X^2(p,-p)$.

  Again, in the event that $\delta(\eta,k) \rtimes 1$ is irreducible it follows that $\delta(\eta,k) \rtimes \sigma$ must also be irreducible, so we consider the remaining case.

  If either $\delta(1,2k) \rtimes \sigma$ or $\delta(\text{sgn},2k) \rtimes \sigma$ is reducible, then it must decompose in the form $M \oplus N$ where M restricts to a representation isomorphic to $X^1(p,-p)$ on $\text{Sp}(4,\mathbb{R})$ and N restricts to $X^2(p,-p)$.

  However, by the weight structure of $X^1(p,-p)$ and $X^2(p,-p)$ given in Muic, there can be no representation of $\text{GSp}(4,\mathbb{R})$ that restricts to $X^1(p,-p)$ on $\text{Sp}(4,\mathbb{R})$, as the weights of any $\text{GSp}(4,\mathbb{R})$ representation must be closed under the map $(x,y) \mapsto (-y,-x)$. Therefore $\delta(\eta,k) \rtimes \sigma$ is irreducible.

- Finally, we consider representations of the form $\eta \rtimes \sigma D_p$. When we restrict $\eta \rtimes \sigma D_p$ to $\text{Sp}(4,\mathbb{R})$, we obtain that it decomposes as $\eta \rtimes X(p,+) \oplus \eta \rtimes X(p,-)$.

Similar to the above, by Lemma 8.1 from [3] we have that $\eta \rtimes X(p, \pm)$ is reducible in $\mathrm{Sp}(4, \mathbb{R})$ if and only if $\eta = 1$ and $p > 0$. In this case the representation decomposes as follows.

$$1 \rtimes X(p,+) \simeq X^1(p,0) \oplus X^2(p,0) \tag{3.7}$$

$$1 \rtimes X(p,-) \simeq X^1(0,-p) \oplus X^2(0,-p) \tag{3.8}$$

In the case that $\eta \rtimes X(p,+)$, $\eta \rtimes X(p,-)$ are irreducible we then have that if $\eta \rtimes \sigma D_p$ were to be reducible, it must have an irreducible component that restricts to $\eta \rtimes X(p,+)$. But this is again not possible by the weight structure of $\eta \rtimes X(p,+)$. In the case that we are considering $1 \rtimes \sigma D_p$, $p > 0$ it has a restriction to $\mathrm{Sp}(4, \mathbb{R})$ that decomposes as $X^1(p,0) \oplus X^2(p,0) \oplus X^1(0,-p) \oplus X^2(0,-p)$. At this point, we use a fact from Bump [1], Proposition 2.5.5 that each $\mathrm{GSp}(4, \mathbb{R})$ representation, when restricted to $\mathrm{Sp}(4, \mathbb{R})$, has irreducible components isomorphic to either one or two irreducible representations of $\mathrm{Sp}(4, \mathbb{R})$.

As the above restriction consists of four irreducible components, $1 \rtimes \sigma D_p$ cannot be irreducible and must have at least two irreducible components where each one restricts to a direct sum of two of the $\mathrm{Sp}(4, \mathbb{R})$ representations. From the weights of the irreducible $\mathrm{Sp}(4, \mathbb{R})$ representations, it must have at most two by similar arguments as before. A consideration of the weight structure of the $\mathrm{Sp}(4, \mathbb{R})$ representations gives that one irreducible component restricts to $X^1(p,0) \oplus X^2(0,-p)$ which are limits of holomorphic and antiholomorphic discrete series on $\mathrm{Sp}(4, \mathbb{R})$. The other component then restricts to $X^2(p,0) \oplus X^1(0,-p)$ which is a sum of limits of large discrete series of both types.

□

3.3 Langlands parameters

The local Langlands correspondence gives us a bijection between L-packets of irreducible admissible representations of $\mathrm{GSp}(4,\mathbb{R})$ and admissible homomorphisms $W_{\mathbb{R}} \to \mathrm{GSp}(4,\mathbb{C})$, where $W_{\mathbb{R}}$ is the real Weil group. Recall that the real Weil group is the group $W_{\mathbb{R}} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ with the usual multiplication on $\mathbb{C}^\times$ and j is an element with $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbb{C}$. From [6] section 3.1 we know that all representations of $W_{\mathbb{R}}$ are completely reducible and composed of one- and two-dimensional irreducible representations. Further, all possible one-dimensional representations are given by $\varphi_{+,t}$ and $\varphi_{-,t}$ as follows, where $t \in \mathbb{C}$, and $re^{i\theta} \in \mathbb{C}$:

$$\varphi_{+,t}(re^{i\theta}) = r^{2t},\ \varphi_{+,t}(j) = 1$$

$$\varphi_{-,t}(re^{i\theta}) = r^{2t},\ \varphi_{-,t}(j) = -1$$

The two-dimensional representations are all as follows, where $\ell \in \mathbb{Z}_{>0}$, $t \in \mathbb{C}$, and $re^{i\theta} \in \mathbb{C}$:

$$\varphi_{\ell,t}(re^{i\theta}) = \begin{bmatrix} r^{2t}e^{i\ell\theta} & \\ & r^{2t}e^{-i\ell\theta} \end{bmatrix}$$

$$\varphi_{\ell,t}(j) = \begin{bmatrix} & (-1)^\ell \\ 1 & \end{bmatrix}$$

We shall determine the Langlands parameters for each irreducible representation of $\mathrm{GSp}(4,\mathbb{R})$.

Langlands quotients supported on the minimal parabolic

First, we shall consider the case when we induce from diagonal representations of $\mathrm{GL}(1,\mathbb{R})$ and $\mathrm{GSp}(0,\mathbb{R}) \cong \mathbb{R}^\times$. In this situation, we obtain the representation $\chi_1 \times \chi_2 \rtimes \sigma$ with χ_1, χ_2 representations of $\mathrm{GL}(1,\mathbb{R})$ and σ a representation of $\mathrm{GSp}(0,\mathbb{R})$. We shall consider such representations in the form of $\mathrm{L}(||^a\mathrm{sgn}^b, ||^c\mathrm{sgn}^d, ||^e\mathrm{sgn}^f)$ where $(a,b,c,d,e,f) \in \mathbb{C}\times\{0,1\}\times\mathbb{C}\times\{0,1\}\times\mathbb{C}\times\{0,1\}$ with $\mathrm{Re}(a) \geq \mathrm{Re}(c) \geq 0$ and $\mathrm{Re}(a) + \mathrm{Re}(c) > 0$. Then we have a Langlands quotient $\mathrm{L}(\chi_1, \chi_2, \sigma)$.

By [5] equation (2.28) we have that the Langlands parameter of $\mathrm{L}(\chi_1, \chi_2, \sigma)$ is

$$W_\mathbb{R} \ni w \mapsto \begin{bmatrix} \chi_1\chi_2\sigma(w) & & & \\ & \chi_1\sigma(w) & & \\ & & \chi_2\sigma(w) & \\ & & & \sigma(w) \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C})$$

where we use χ_1, χ_2, and σ to mean their respective Langlands parameters.

In this case, we then have

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2(a+c+e)} & & & \\ & r^{2(a+e)} & & \\ & & r^{2(c+e)} & \\ & & & r^{2e} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^{b+d+f} & & & \\ & (-1)^{b+f} & & \\ & & (-1)^{d+f} & \\ & & & (-1)^{f} \end{bmatrix}.$$

In summary, using -1 to stand for $-$ and $+1$ to stand for $+$, we have that this representation decomposes as

$$\varphi_{(-1)^{b+d+f},a+c+e} \oplus \varphi_{(-1)^{b+f},a+e} \oplus \varphi_{(-1)^{d+f},c+e} \oplus \varphi_{(-1)^{f},e}. \tag{3.9}$$

We may also consider the degree 5 L-parameters given by composing with a homomorphism to $SO(5, \mathbb{C})$ given by [5] equations (A.2), (A.3) and (A.4). In this case of Langlands quotients supported on the minimal parabolic, this gives us

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2a} & & & & \\ & r^{2c} & & & \\ & & 1 & & \\ & & & r^{-2c} & \\ & & & & r^{-2a} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^{b} & & & & \\ & (-1)^{d} & & & \\ & & 1 & & \\ & & & (-1)^{d} & \\ & & & & (-1)^{b} \end{bmatrix}$$

We can see that the representation decomposes as

$$\varphi_{(-1)^b,a} \oplus \varphi_{(-1)^d,c} \oplus \varphi_{+,0} \oplus \varphi_{(-1)^d,-c} \oplus \varphi_{(-1)^b,-a}. \tag{3.10}$$

Langlands quotients supported on the Siegel parabolic

Next, we consider the case when we induce from representations of $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{GSp}(0,\mathbb{R})$ on the Siegel parabolic. In this situation, we obtain the representation $\delta \rtimes \sigma$ with δ an essentially square-integrable representation of $\mathrm{GL}(2,\mathbb{R})$ and σ a representation of $\mathrm{GSp}(0,\mathbb{R})$. Then we have a Langlands quotient $\mathrm{L}(\delta,\sigma)$. Such a representation will be of the form $\mathrm{L}(\delta(||^s\mathrm{sgn}^\epsilon,\ell), ||^a\mathrm{sgn}^b)$ with $(s,\epsilon,\ell,a,b) \in \mathbb{C}\times\{0,1\}\times\mathbb{Z}_{>0}\times\mathbb{C}\times\{0,1\}$, and $\delta(||^s\mathrm{sgn}^\epsilon,\ell) = ||^s\mathrm{sgn}^\epsilon \otimes D_\ell$ where D_ℓ is a discrete series on GL(2).

By [5] (2.46), we have, using σ to also denote the parameter of σ and μ the parameter of δ, the Langlands parameter of $\mathrm{L}(\delta,\sigma)$ is

$$W_{\mathbb{R}} \ni w \mapsto \begin{bmatrix} \sigma(w)\ \det(\mu(w)) & & \\ & \sigma(w)\mu(w) & \\ & & \sigma(w) \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C})$$

In particular, we have

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2(a+2s)} & & & \\ & r^{2(a+s)}e^{i\ell\theta} & & \\ & & r^{2(a+s)}e^{-i\ell\theta} & \\ & & & r^{2a} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^{\ell+b+1} & & & \\ & & (-1)^{\ell+b} & \\ & (-1)^{b} & & \\ & & & (-1)^{b} \end{bmatrix}.$$

In summary, using -1 to stand for $-$ and $+1$ to stand for $+$, we have that this representation decomposes as

$$\varphi_{(-1)^{\ell+b+1},a+2s} \oplus \varphi_{\ell,a+s} \oplus \varphi_{(-1)^{b},a} \tag{3.11}$$

We may also consider the degree 5 L-parameters given by composing with a homomorphism to $SO(5,\mathbb{C})$. In this case of Langlands quotients supported on the Siegel parabolic, this gives us

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2s}e^{i\ell\theta} & & & & \\ & r^{2s}e^{-i\ell\theta} & & & \\ & & 1 & & \\ & & & r^{-2s}e^{i\ell\theta} & \\ & & & & r^{-2s}e^{-i\ell\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & 2(-1)^{\ell} & & & \\ \frac{1}{2} & & & & \\ & & 1 & & \\ & & & & 2 \\ & & & \frac{1}{2}(-1)^{\ell} & \end{bmatrix}$$

After suitable conjugation, we can see that the representation decomposes as

$$\varphi_{\ell,2s} \oplus \varphi_{+,0} \oplus \varphi_{\ell,-2s}. \tag{3.12}$$

Langlands quotients supported on the Klingen parabolic

Consider the Langlands quotient of a Klingen induced representation on GSp(4,$\mathbb{R}$), Lang($\chi \rtimes \pi$). Then by [5] equation (2.40), we have for χ being used to mean the parameter of χ and μ the parameter of π that

$$W_{\mathbb{R}} \ni w \mapsto \begin{bmatrix} \chi(w)\ \det(\mu(w))\mu(w)' & \\ & \mu(w) \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C})$$

In the most general case, we have L($||^a\mathrm{sgn}^b \rtimes ||^c\mathrm{sgn}^d D_\ell$) with $(a,b,c,\ell) \in \mathbb{C} \times \{0,1\} \times \mathbb{C} \times \mathbb{Z}_{\geq 0}$ so that we get

$$re^{it} \mapsto \begin{bmatrix} r^{2(a+c)}e^{i\ell\theta} & & & \\ & r^{2(a+c)}e^{-i\ell\theta} & & \\ & & r^{2c}e^{i\ell\theta} & \\ & & & r^{2c}e^{-i\ell\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & (-1)^{b+1} & & \\ (-1)^{b+\ell+1} & & & \\ & & & (-1)^{\ell} \\ & & 1 & \end{bmatrix}.$$

We can then decompose this representation as $\varphi_{\ell,a+c} \oplus \varphi_{\ell,c}$.

We may also consider the degree 5 L-parameters given by composing with a homomorphism to $SO(5,\mathbb{C})$. In this case of Langlands quotients supported on the

Klingen parabolic, this gives us

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2a} & & & & \\ & e^{2i\ell\theta} & & & \\ & & 1 & & \\ & & & e^{-2i\ell\theta} & \\ & & & & r^{-2a} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^{b+\ell} & & & & \\ & & & \frac{1}{2}(-1)^{\ell} & \\ & & -1 & & \\ & 2(-1)^{\ell} & & & \\ & & & & (-1)^{b+\ell} \end{bmatrix}$$

After suitable conjugation, we can see that the representation decomposes as

$$\varphi_{(-1)^{b+\ell},a} \oplus \varphi_{-,0} \oplus \varphi_{(-1)^{b+\ell},-a} \oplus \varphi_{2\ell,0}. \tag{3.13}$$

Irreducible essentially tempered representations

It remains to consider the irreducible essentially tempered representations of which there are several subtypes. First, we have the discrete series on $\mathrm{GSp}(4,\mathbb{R})$. From [6], for $\lambda_1 > \lambda_2 > 0$ both integers, the holomorphic discrete series X_{λ_1,λ_2} and the large discrete series $X_{\lambda_1,-\lambda_2}$ form a 2-element L-packet. Both then have Langlands parameter

$$re^{i\theta} \mapsto \begin{bmatrix} e^{i(\lambda_1-\lambda_2)\theta} & & & \\ & e^{i(\lambda_1+\lambda_2)\theta} & & \\ & & e^{-i(\lambda_1+\lambda_2)\theta} & \\ & & & e^{-i(\lambda_1-\lambda_2)\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & & & (-1)^{\lambda_1-\lambda_2} \\ & & (-1)^{\lambda_1+\lambda_2} & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

We can then decompose this representation as $\varphi_{\lambda_1+\lambda_2,0} \oplus \varphi_{\lambda_1-\lambda_2,0}$.

These representations have degree 5 L-parameters of the form:

$$re^{i\theta} \mapsto \begin{bmatrix} e^{2i\lambda_1\theta} & & & & \\ & e^{-2i\lambda_2\theta} & & & \\ & & 1 & & \\ & & & e^{2i\lambda_2\theta} & \\ & & & & e^{-2i\lambda_1\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & & & & (-2)^{\lambda_1+\lambda_2+1} \\ & & & \frac{1}{2} & \\ & & 1 & & \\ & 2 & & & \\ (-\frac{1}{2})^{\lambda_1+\lambda_2+1} & & & & \end{bmatrix}$$

After suitable conjugation, we can see that the representation decomposes as

$$\varphi_{2\lambda_2,0} \oplus \varphi_{+,0} \oplus \varphi_{2\lambda_1,0}. \tag{3.14}$$

Then there are the limits of discrete series. Of these, there are a holomorphic and a large limit of discrete series with Blattner parameter $\lambda = (p, 0)$, $p \in \mathbb{Z}_{>0}$. From [6] these form a 2-element L-packet with Langlands parameter

$$re^{i\theta} \mapsto \begin{bmatrix} e^{ip\theta} & & & \\ & e^{-ip\theta} & & \\ & & e^{ip\theta} & \\ & & & e^{-ip\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & (-1)^{p+1} & & \\ -1 & & & \\ & & & (-1)^p \\ & & 1 & \end{bmatrix}$$

We can then decompose this representation as $\varphi_{p,0} \oplus \varphi_{p,0}$

These representations have degree 5 L-parameters of the form:

$$re^{i\theta} \mapsto \begin{bmatrix} 1 & & & & \\ & e^{2ip\theta} & & & \\ & & 1 & & \\ & & & e^{-2ip\theta} & \\ & & & & 1 \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} 1 & & & & \\ & & & \frac{1}{2}(-1)^p & \\ & & -1 & & \\ & 2(-1)^p & & & \\ & & & & 1 \end{bmatrix}.$$

After suitable conjugation, we can see that the representation decomposes as

$$\varphi_{+,0} \oplus \varphi_{2p,0} \oplus \varphi_{-,0} \oplus \varphi_{+,0}. \tag{3.15}$$

There is also another type of limit of discrete series, namely the large limit of discrete series with Blattner parameter $\lambda = (p, -p)$, $p \in \mathbb{Z}_{>0}$. From [6] this has Langlands parameter

$$re^{it} \mapsto \begin{bmatrix} 1 & & & \\ & e^{2ip\theta} & & \\ & & e^{-2ip\theta} & \\ & & & 1 \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} -1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$$

And this representation decomposes as

$$\varphi_{-,0} \oplus \varphi_{2p,0} \oplus \varphi_{+,0} \tag{3.16}$$

These representations have degree 5 L-parameters of the form:

$$re^{i\theta} \mapsto \begin{bmatrix} e^{2ip\theta} & & & & \\ & e^{-2ip\theta} & & & \\ & & 1 & & \\ & & & e^{2ip\theta} & \\ & & & & e^{-2ip\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & 2 & & & \\ \frac{1}{2} & & & & \\ & & 1 & & \\ & & & & 2 \\ & & & \frac{1}{2} & \end{bmatrix}.$$

After suitable conjugation, we can see that the representation decomposes as

$$\varphi_{2p,0} \oplus \varphi_{+,0} \oplus \varphi_{2p,0}. \tag{3.17}$$

Finally, we have the case of irreducible tempered representations that are neither discrete series representations nor limits of discrete series representations. From Theorem 3.1 we may determine all such representations.

First we have the irreducible unitary principal series $||^a\mathrm{sgn}^b \times ||^c\mathrm{sgn}^d \rtimes ||^e\mathrm{sgn}^f$ where $(a, b, c, d, e, f) \in \mathbb{C} \times \{0, 1\} \times \mathbb{C} \times \{0, 1\} \times \mathbb{C} \times \{0, 1\}$ with $Re(a) = Re(c) = Re(e) = 0$. In this case, we obtain a similar Langlands parameter to the case of

Langlands quotients supported on the minimal parabolic, so that

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2(a+c+e)} & & & \\ & r^{2(a+e)} & & \\ & & r^{2(c+e)} & \\ & & & r^{2e} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^{b+d+f} & & & \\ & (-1)^{b+f} & & \\ & & (-1)^{d+f} & \\ & & & (-1)^{f} \end{bmatrix}.$$

In summary, using -1 to stand for $-$ and $+1$ to stand for $+$, we have that this representation decomposes as

$$\varphi_{(-1)^{b+d+f},a+c+e} \oplus \varphi_{(-1)^{b+f},a+e} \oplus \varphi_{(-1)^{d+f},c+e} \oplus \varphi_{(-1)^{f},e}. \tag{3.18}$$

These representations have degree 5 L-parameters of the form:

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2a} & & & & \\ & r^{2c} & & & \\ & & 1 & & \\ & & & r^{-2c} & \\ & & & & r^{-2a} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^b &&&& \\ & (-1)^d &&& \\ && 1 && \\ &&& (-1)^d & \\ &&&& (-1)^b \end{bmatrix}.$$

We can see that the representation decomposes as

$$\varphi_{(-1)^b,a} \oplus \varphi_{(-1)^d,c} \oplus \varphi_{+,0} \oplus \varphi_{(-1)^d,-c} \oplus \varphi_{(-1)^b,-c}. \tag{3.19}$$

Then there are the irreducible tempered representations of the form $\delta(||^s \text{sgn}^r, k) \rtimes ||^a \text{sgn}^b$ with $(s, k, a, b) \in \mathbb{C} \times \mathbb{Z}_{>0} \times \mathbb{C} \times \{0, 1\}$, with $Re(s) = Re(a) = 0$. Similar to the Langlands quotients supported on the Siegel parabolic, the Langlands parameter will be

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2(a+2s)} &&& \\ & r^{2(a+s)}e^{ik\theta} && \\ && r^{2(a+s)}e^{-ik\theta} & \\ &&& r^{2a} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^{k+b+1} &&& \\ && (-1)^{k+b} & \\ & (-1)^b && \\ &&& (-1)^b \end{bmatrix}.$$

In summary, using -1 to stand for $-$ and $+1$ to stand for $+$, we have that this representation decomposes as

$$\varphi_{(-1)^{k+b+1},a+2s} \oplus \varphi_{k,a+s} \oplus \varphi_{(-1)^b,a}. \tag{3.20}$$

These representations have degree 5 L-parameters of the form:

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2s}e^{ik\theta} & & & & \\ & r^{2s}e^{-ik\theta} & & & \\ & & 1 & & \\ & & & r^{-2s}e^{ik\theta} & \\ & & & & r^{-2s}e^{-ik\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & 2(-1)^k & & & \\ \frac{1}{2} & & & & \\ & & 1 & & \\ & & & & 2 \\ & & & \frac{1}{2}(-1)^k & \end{bmatrix}.$$

After suitable conjugation, we can see that the representation decomposes as

$$\varphi_{k,2s} \oplus \varphi_{+,0} \oplus \varphi_{k,-2s}. \tag{3.21}$$

Finally we have irreducible tempered representations of the form $||^a\text{sgn}^b \rtimes ||^c\text{sgn}^d D_\ell$ with $(a, b, c, \ell) \in \mathbb{C} \times \{0, 1\} \times \mathbb{C} \times \mathbb{Z}_{\geq 0}$ and $Re(a) = Re(c) = 0$, with either $||^a\text{sgn}^b \neq 1$ or $\ell = 0$. Then, similarly to the case of Langlands quotients induced on the Klingen, we obtain

$$re^{it} \mapsto \begin{bmatrix} r^{2(a+c)}e^{i\ell\theta} & & & \\ & r^{2(a+c)}e^{-i\ell\theta} & & \\ & & r^{2c}e^{i\ell\theta} & \\ & & & r^{2c}e^{-i\ell\theta} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} & (-1)^{b+1} & & \\ (-1)^{b+\ell+1} & & & \\ & & & (-1)^{\ell} \\ & & 1 & \end{bmatrix}.$$

We can then decompose this representation as

$$\varphi_{\ell,a+c} \oplus \varphi_{\ell,a}. \tag{3.22}$$

These representations have degree 5 L-parameters as follows:

$$re^{i\theta} \mapsto \begin{bmatrix} r^{2a} & & & & \\ & e^{2i\ell\theta} & & & \\ & & 1 & & \\ & & & 1e^{-2i\ell\theta} & \\ & & & & r^{-2a} \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} (-1)^{b+\ell} & & & & \\ & & & \frac{1}{2}(-1)^{\ell} & \\ & & -1 & & \\ & 2(-1)^{\ell} & & & \\ & & & & (-1)^{b+\ell} \end{bmatrix}.$$

After suitable conjugation, we can see that the representation decomposes as

$$\varphi_{(-1)^{b+\ell},a} \oplus \varphi_{-,0} \oplus \varphi_{(-1)^{b+\ell},-a} \oplus \varphi_{2\ell,0}. \tag{3.23}$$

The L- and ϵ- factors associated to these L-parameters are collected in the tables of appendix D. They are determined by using the L- and ϵ-factors associated to representations as given in Table 2 of [6] for example.

Chapter 4

K-types

We consider representations of $\mathrm{Sp}(4,\mathbb{R})$ by examining the weight lattices of representations of $\mathrm{Sp}(4,\mathbb{R})$, in particular their decomposition when restricted to the maximal compact subgroup, K. For later use, define

$$C(a,b,c) = \begin{cases} 2 & \text{if } a \equiv b \equiv c \pmod 2 \\ 0 & \text{if } a \equiv b \not\equiv c \pmod 2 \\ 1 & \text{if } a \not\equiv b \pmod 2 \end{cases}$$

4.1 Discrete series representations

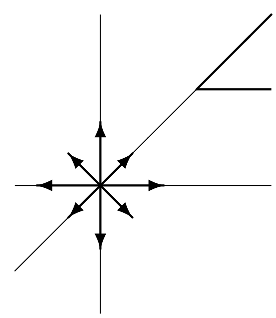
First, consider discrete series representations. The multiplicities of K-types of such representations may be determined by the Blattner formula.

$$M(\mu) = \sum_{w \in W_K} \epsilon(w) Q(w(\mu + \rho_c) - \lambda - \rho_n) \tag{4.1}$$

In this formula, W_K is the Weyl group of K, ϵ is the sign of w, and λ is the Harish-Chandra Parameter which is obtained from the Blattner Parameter by subtracting ρ_n and adding ρ_c, where ρ_n is the sum of non-compact positive roots and ρ_c is the sum of compact positive roots. Finally, $Q(r,s)$ is the number of ways (r,s) may be written as a sum of positive noncompact roots.

4.1.1 Holomorphic discrete series

Let $\lambda = (m,n)$ be the Harish-Chandra parameter of a discrete series representation π. Then in the holomorphic case, $m > n > 0$. We have $\rho_n = (\frac{3}{2}, \frac{3}{2})$ and $\rho_c = (\frac{1}{2}, -\frac{1}{2})$, so that if the Harish-Chandra parameter is $\lambda = (k-1, \ell-2)$ the Blattner parameter will be (k,ℓ) . Note that then $k \geq \ell > 2$ so that all Blattner parameters will be in the region shown.



In this case, for a K-type with lowest weight $\mu = (x,y)$, the Blattner formula gives a multiplicity of

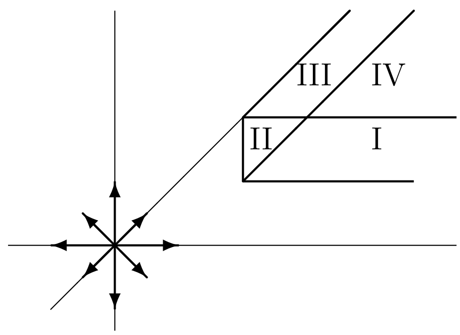
$$\begin{aligned} M(x,y) &= Q\left(\left(x+\frac{1}{2}, y-\frac{1}{2}\right) - \left(k+\frac{1}{2}, \ell-\frac{1}{2}\right)\right) \\ &\quad - Q\left(\left(y-\frac{1}{2}, x+\frac{1}{2}\right) - \left(k+\frac{1}{2}, \ell-\frac{1}{2}\right)\right) \\ &= Q(x-k, y-\ell) - Q(y-k-1, x-\ell+1) \end{aligned}$$

In this case, $Q(r,s) = \begin{cases} \left\lfloor \frac{\min(r,s)+2}{2} \right\rfloor & \text{if } r,s \geq 0,\ r \equiv s \pmod 2 \\ 0, & \text{otherwise} \end{cases}$

We can see this as in this case the positive noncompact roots are $(2,0)$, $(1,1)$, and $(0,2)$. Then we may see that, assuming $r \leq s$,

$$(r,s) = r(1,1) + \frac{s-r}{2}(0,2) = (r-2)(1,1) + (2,0) + \frac{s-r-2}{2}(0,2) = \dots$$

giving the value given above. So then the multiplicity of the K-type with $\mu = (x,y)$ reduces to four cases, as $x \geq y$ and $k \geq \ell$ so $y-k-1 < x-\ell+1$. In all cases, we assume $y - \ell \equiv x - k \pmod 2$ or else the multiplicity is 0. These cases are as shown in the diagram:



I: $\ell \leq y \leq k,\ y \leq x + \ell - k$ $\quad M(x,y) = \left\lfloor \frac{y-\ell+2}{2} \right\rfloor$

II: $x + \ell - k \leq y \leq k,\ x \geq k$ $\quad M(x,y) = \left\lfloor \frac{x-k+2}{2} \right\rfloor$

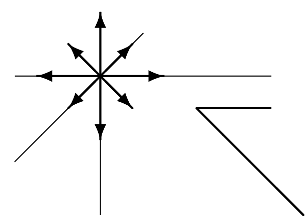
III: $y \geq k, x + \ell - k \leq y \leq x$ $\quad M(x,y) = \left\lfloor \frac{x-k+2}{2} \right\rfloor - \left\lfloor \frac{y-k+1}{2} \right\rfloor$

IV: $k \leq y \leq x + \ell - k$ $\quad M(x,y) = \left\lfloor \frac{y-\ell+2}{2} \right\rfloor - \left\lfloor \frac{y-k+1}{2} \right\rfloor$

Note that for $k = \ell$ only case IV occurs, giving a maximum multiplicity one. The antiholomorphic discrete series are symmetric to this case.

4.1.2 Large discrete series

We shall only consider one of the two cases of large discrete series - the other is symmetric. In the case considered here, $\rho_n = (\frac{3}{2}, -\frac{1}{2})$ and $\rho_c = (\frac{1}{2}, -\frac{1}{2})$. We shall call these large discrete series of the first type, and their symmetric counterparts we shall call large discrete series of the second type. Then a discrete series with a Blattner parameter of (k, ℓ), will have a Harish-Chandra parameter of $\lambda = (k-1, \ell)$. The large discrete series we are considering are those with Harish-Chandra parameters such that $k - 2 \geq -\ell > 0$, in the region below.



The multiplicity of a K-type with $\mu = (x, y)$ will be

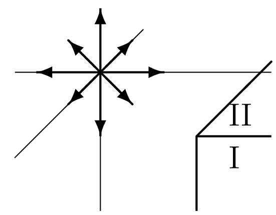
$$\begin{aligned} M(x,y) &= Q\left(x-k, y-\ell\right) - Q\left(\left(y-\frac{1}{2}, x+\frac{1}{2}\right) - \left(k+\frac{1}{2}, \ell-\frac{1}{2}\right)\right) \\ &= Q(x-k, y-\ell) - Q(y-k-1, x-\ell+1) \end{aligned}$$

In this case, $Q(r,s) = \begin{cases} \left\lfloor \frac{r+2}{2} \right\rfloor & \text{if } r \geq 0,\ s \leq 0, r \equiv s \pmod 2 \\ \left\lfloor \frac{r-s+2}{2} \right\rfloor & \text{if } r \geq s,\ s > 0, r \equiv s \pmod 2 \\ 0 & \text{otherwise} \end{cases}$

Here we see this, as the positive noncompact roots are $(1,1)$, $(2,0)$, and $(0,-2)$, so that we have

$$(r,s) = r(1,1) + \frac{r-s}{2}(0,-2) = (r-2)(1,1) + \frac{r-s-2}{2}(0,-2) + (2,0) = \dots$$

giving us the above values. Observe that $x - \ell + 1 \geq 0$ always , and also that $y \leq x$ and $k \geq -\ell > \ell$ so that $x - \ell + 1 > y - k - 1$ and the second term never contributes. Then we have two cases as depicted below



I: $y \leq \ell, x \geq k,$ $M(x,y) = \left\lfloor \frac{x-k+2}{2} \right\rfloor$

II: $\ell \leq y \leq x+\ell-k,$ $M(x,y) = \left\lfloor \frac{(x-k)-(y-\ell)+2}{2} \right\rfloor$

4.2 Induced representations

Now let us consider the multiplicities of K-types of induced representations of $\mathrm{Sp}(4,\mathbb{R})$. These are given by Muić, and we restate them here for our convenience.

4.2.1 Borel induced

Let $\eta_i = ||^s \operatorname{sgn}^{\epsilon_i}$ be characters of $\mathbb{R}^\times$. Then from [3] Lemma 6.1,

$$(\eta_1 \times \eta_2 \rtimes 1)|_{\mathbb{U}(2)}$$

$$\simeq \bigoplus_{x+\epsilon_1 \equiv y+\epsilon_2 \pmod 2} \#\left\{i; 0 \leq i \leq x-y, i \equiv x+\epsilon_1 \pmod 2\right\} V_{(x,y)}.$$

To rephrase this in a manner more suitable to our purposes, we note that one only obtains K-types $V_{(x,y)}$ with $x+y \equiv \epsilon_1+\epsilon_2 \pmod 2$ and the multiplicity is

$$M(x,y) = \frac{x-y+C(x,y,x+\epsilon_1)}{2}. \tag{4.2}$$

4.2.2 Siegel induced

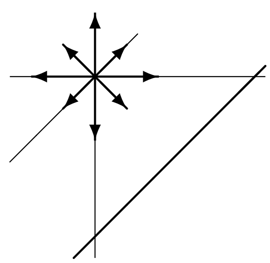
In this case we have $\delta(\eta, k)$ the unique irreducible subrepresentation of $\eta\,||^{\frac{k}{2}}\,\mathrm{sgn}^{k+1} \times \eta\,||^{-\frac{k}{2}}$ on $\mathrm{GL}_2(\mathbb{R})$. Then the induced representation decomposes as follows, by [3] Lemma 6.1:

$$(\delta(\eta, k) \rtimes 1)|_{U(2)} \simeq \bigoplus_{x-y-x\geq 0,\, x+y\equiv k+1 \pmod 2} \frac{x-y-k+1}{2} V_{(x,y)}.$$

We can rephrase this as stating that the K-type with highest weight (x, y) has multiplicity 0 when either $y > x - k - 1$ or $x + y \not\equiv k + 1 \pmod 2$ and otherwise has multiplicity

$$M(x, y) = \frac{x-y-k+1}{2}. \tag{4.3}$$

All K-types will be in the region under the line indicated below.



4.2.3 Klingen induced

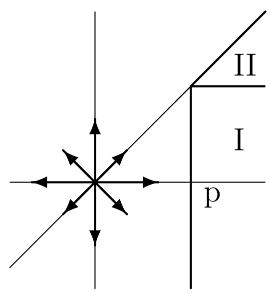
We consider three different types of representation on the Klingen subgroup from which we may induce. Recall that $X(s, +)$ is a discrete series or limit of discrete series with lowest weight $s + 1$, and $X(s, -)$ is a discrete series or limit of discrete series with highest weight $-s - 1$, and V_s is finite of dimension s. Also, $J(p)$ will be defined as the set of j such that $j \equiv p + 1 \pmod 2$, with $j \geq p + 1$ in the lowest weight $(+)$ case and $j \leq -p - 1$ in the highest weight $(-)$ case. We will also be inducing from a character $\eta = ||^a\,\mathrm{sgn}^\epsilon$. For all cases, the multiplicity will be nonzero only if $x + y \equiv p + \epsilon + 1$. From Muić [3] Lemma 6.1, we have the following:

Lowest Weight

When inducing from the lowest weight representation, we have

$$\eta \rtimes X(p,+)|_{U(2)} \simeq \bigoplus_{p \equiv x+y+\epsilon+1 \pmod 2} \# \{j \in J(p); y \leq j \leq x\} V_{(x,y)}.$$

This gives us that the multiplicity of the K-type (x, y) is nonzero in the following regions.



I: $y \leq p, x \geq p \quad M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor$

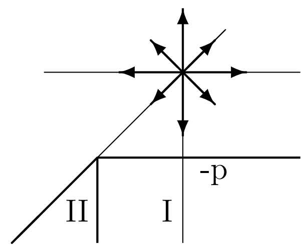
II: $y \geq p, x \geq p \quad M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor - \left\lfloor \frac{y-p}{2} \right\rfloor$

Highest Weight

When inducing from the highest weight representation, we have

$$\eta \rtimes X(p,-)|_{U(2)} \simeq \bigoplus_{p \equiv x+y+\epsilon+1 \pmod 2} \# \{j \in J(p); y \leq j \leq x\} V_{(x,y)}.$$

This gives us that the multiplicity of the K-type (x, y) is nonzero in the following regions.



I: $y \leq -p, x \geq -p \quad M(x,y) = \left\lfloor \frac{-p-y+1}{2} \right\rfloor$

II: $y \leq -p, x \leq -p \quad M(x,y) = \left\lfloor \frac{-p-y+1}{2} \right\rfloor - \left\lfloor \frac{-x-p}{2} \right\rfloor$

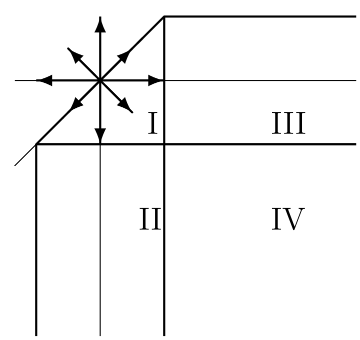
Finite

When inducing from the finite representation, we have

$$(\eta \rtimes V_p)|_{U(2)} \simeq$$

$$\bigoplus_{p \equiv k_1+k_2+\epsilon+1 \pmod 2} \#\{j; j \equiv p+1 \pmod 2, j \in [-p+1, p-1] \cap [y,x]\} V_{(x,y)}.$$

This gives us that the multiplicity of the K-type (x,y) is nonzero in the following regions.



I: $y \geq -p, x \leq p \quad M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor - \left\lfloor \frac{y-p}{2} \right\rfloor$

II: $y < -p, x \leq p \quad M(x,y) = \left\lfloor \frac{x+p+1}{2} \right\rfloor$

III: $y \geq -p, x > p \quad M(x,y) = \left\lfloor \frac{p-y+1}{2} \right\rfloor$

IV: $y < -p, x > p \quad M(x,y) = p$

4.3 Limits of discrete series

Recall our notation for limits of discrete series on $\mathrm{sl}(2,\mathbb{R})$. We denote by $X^1(p,0)$ the limit of holomorphic discrete series with Harish-Chandra parameter $(p,0)$ and by $X^2(0,-p)$ the corresponding limit of anti-holomorphic discrete series. Also, $X^2(p,0)$ and $X^1(p,-p)$ are limits of large discrete series of the first type, and $X^1(0,-p)$ and $X^2(p,-p)$ are of the second type.

4.3.1 $X^1(p,0)$

First, we consider $X^1(p,0)$ with Harish-Chandra parameter $(p,0)$ and lowest weight $(p+1,2)$. By Proposition 2.5 in [4] we have that the lowest weight module $N(k,\ell)$ is irreducible for $\ell \geq 2$, so the lowest weight module $N(p+1,2)$ is irreducible. Since $X^1(p,0)$ is a lowest weight module with the same lowest weight, we may obtain the multiplicities of K-types from Lemma 2.7 of [5] which are identical with those given by the Blattner formula for a holomorphic discrete series with such a Harish-Chandra parameter. The multiplicities of $X^2(0,-p)$ will be symmetric to these and are therefore also given by the Blattner formula as if it were an anti-holomorphic discrete series.

4.3.2 $X^2(p,0)$

Next, from Lemma 8.1 in Muić [3],

$$1 \rtimes X(p,+) \simeq X^1(p,0) \oplus X^2(p,0).$$

We may then restrict to K to use this to determine that the multiplicities of K-types $X^1(p,0)$ and $X^2(p,0)$ sum together to give the multiplicities of K-types of $1 \rtimes X(p,+)$.

Then we may determine the multiplicities of K-types of $X^2(p,0)$. Recall that the multiplicities of K-types of $1 \rtimes X(p,+)$ are as follows, nonzero only when $x+y \equiv p+1 \pmod 2$:

Ia: $y < p, x \geq p \quad M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor$

IIa: $y \geq p, x \geq p \quad M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor - \lfloor \frac{y-p}{2} \rfloor$

Also recall that the multiplicities of K-types of $X^1(p,0)$ are as follows, from above, nonzero only when $x+y \equiv p+1 \pmod 2$:

Ib: $2 \leq y \leq p+1,\ y \leq x-p+1 \quad M(x,y) = \lfloor \frac{y}{2} \rfloor$

IIb: $x-p+1 \leq y \leq p+1,\ x \geq p+1 \quad M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor$

IIIb: $y \geq p+1, x-p+1 \leq y \leq x \quad M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor - \lfloor \frac{y-p}{2} \rfloor$

IVb: $p+1 \leq y \leq x-p+1 \quad M(x,y) = \lfloor \frac{y}{2} \rfloor - \lfloor \frac{y-p}{2} \rfloor$

Notice that II and III, and I and IV agree in the case $y = p$ so we may use them interchangeably in such a case.

Then by subtraction, we can see that the multiplicities of K-types of $X^2(p,0)$ are:

Ia-Ib: $2 \leq y \leq p,\ y \leq x-p+1 \quad M(x,y) = \frac{x-p-y+1}{2}$

Ia-IIb: $x-p+1 \leq y \leq p,\ x \geq p+1 \quad M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor - \lfloor \frac{x-p+1}{2} \rfloor = 0$

Ia: $y \leq 2,\ x \geq p+1 \quad M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor$

IIa-IIIb: $y \geq p+1, x-p+1 \leq y \leq x \quad M(x,y) = 0$

IIa-IVb: $p+1 \leq y \leq x-p+1 \quad M(x,y) = \frac{x-p-y+1}{2}$

Recall that if the Blattner formula applied here, it would give multiplicities of

I: $y \leq 0, x \geq p+1, \quad M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor$

II: $0 \leq y \leq x-p-1, \quad M(x,y) = \frac{x-p-y+1}{2}$

Then we conclude that the multiplicities of K-types of $X^2(p,0)$ are in fact given by the Blattner formula. The multiplicities of $X^1(0,-p)$ are symmetric and are therefore also given by the Blattner formula.

4.3.3 $X^1(p,-p)$

The remaining limits of discrete series to consider are $X^1(p,-p)$ and $X^2(p,-p)$, both limits of large discrete series. We may use Lemma 8.1 in Muić [3], which states

$$\delta(1,2p) \rtimes 1 \simeq X^1(p,-p) \oplus X^2(p,-p).$$

Because we know that $X^1(p,-p)$ only has K-types of type (x,y) for $x \geq p+1$ and $X^2(p,-p)$ only has K-types of type (x,y) for $y \leq -p-1$ we can entirely determine the multiplicities of K-types in $X^1(p,-p)$ with $y \geq -p$. They agree with those given by the Blattner formula for a large discrete series with Blattner parameter $(p+1,-p)$ as expected. That is, for $-p \leq y \leq x-2p-1$, we have $M(x,y) = \lfloor \frac{x-y-2p+1}{2} \rfloor$. However, we must use another method to determine the multiplicities in $X^1(p,-p)$ for $y < -p$

From Theorem 10.4 in Muić [3],

$$||^p \operatorname{sgn}^p \rtimes X(p,+) = X^1(p,-p) + \operatorname{Lang}\left(||^p \operatorname{sgn}^p \rtimes X(p,+)\right)$$

We may use this to determine the multiplicities of K-types (x,y) in $\operatorname{Lang}\left(|| \operatorname{sgn}^p \rtimes X(p,+)\right)$ for $y \geq -p$. In particular we may note that both $||^p \operatorname{sgn}^p \rtimes X(p,+)$ and $X^1(p,-p)$ have multiplicities of $M(x,-p) = \frac{x-p+1}{2}$ for $y=-p$, so that $\operatorname{Lang}\left(||^p \operatorname{sgn}^p \rtimes X(p,+)\right)$ has no K-types with weight $y=-p$.
$\operatorname{Lang}\left(||^p \operatorname{sgn}^p \rtimes X(p,+)\right)$ does have a K-type at $(p+1,-p+2)$. By examining commutators, we can determine this K-type is then a lowest weight. Specifically, we know that if we take a vector v in this K-type with weight $(p+1,-p+2)$, and apply elements from our basis for $\operatorname{sp}(4,\mathbb{C})$, the following occurs. We have

$N_+P_{0-}v = P_{0-}N_+v = 0$, but we know that $P_{0-}v$ is not a highest weight vector in a K-type, as there is no K-type in Lang $(||^p \operatorname{sgn}^p \rtimes X(p,+))$ with highest weight $(p+1,-p)$, so $P_{0-}v = 0$. Once this is known, also note by commutation relations, $N_+P_{1-}v = P_{1-}N_+ = 0$. But we know that there are no K-types with maximal weight $(p,-p+1)$ in $||^p \operatorname{sgn}^p \rtimes X(p,+)$, so that $P_{1-}v = 0$. Then also note $N_+N_+X_-v = X_-N_+N_+v = 0$. But there are no K-types with maximal weight $(p,-p+1)$ or $(p-1,-p+2)$ in $||^p \operatorname{sgn}^p \rtimes X(p,+)$, so that $X_-v = 0$. Then the K-type at $(p+1,-p+2)$ is a lowest weight K-type for Lang $(||^p \operatorname{sgn}^p \rtimes X(p,+))$, so that the Langlands quotient is a lowest weight representation with lowest weight $\lambda = (p+1,-p+2)$.

Using Proposition 2.5 from [4], we know that this lowest weight representation is irreducible and can determine its multiplicities. In particular, it has no K-types with $y < -p$, so that we can determine the remaining multiplicities of $X^1(p,-p)$ are $M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor$ for $y < -p, x \geq p+1$, which are the same as those which would be given by the Blattner formula for a large representation with Blattner parameter of $(p+1,-p)$. Similarly, the multiplicities of $X^2(p,-p)$ are symmetric, and thus those given by a large representation with Blattner parameter $(p,-p-1)$.

4.4 Langlands quotients

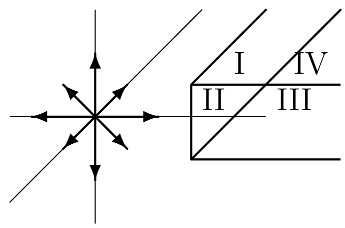
We may use the preceding facts about composition series and K-types of induced representations and discrete series representations to determine the K-types of all Langlands quotients.

4.4.1 Quotients of the Klingen

First, consider those with $p > t > 0$, $p,t \in \mathbb{Z}$.

Lang$(||^t\mathbf{sgn}^t \rtimes X(p,+))$

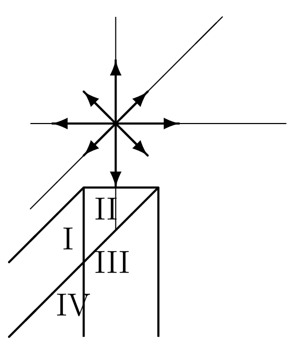
We know from (2.25) that $\text{Lang}(||^t\text{sgn}^t \rtimes X(p,+))$ is a quotient of $||^t\text{sgn}^t \rtimes X(p,+)$ by $X(p,t) \oplus X(p,-t)$. By using previously determined multiplicities for these representations, we find that the K-types for this representation are given by two regions as follows, nonzero only for $x+y \equiv p+t+1 \pmod 2$.



$$\begin{array}{lll} \text{I:} & y \geq t, x-t-p \leq y \leq x+t-p & M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor - \left\lfloor \frac{y-t}{2} \right\rfloor \\ \text{II:} & x-t-p \leq y \leq t, x \geq p & M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor \\ \text{III:} & y \leq x-t-p, -t \leq y \leq t & M(x,y) = \left\lfloor \frac{y+t}{2} \right\rfloor \\ \text{IV:} & t \leq y \leq x-t-p & M(x,y) = t \end{array} \tag{4.4}$$

Lang$(||^t\mathbf{sgn}^t \rtimes X(p,-))$

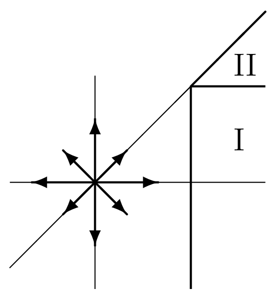
We know from (2.26) that $\text{Lang}(||^t\text{sgn}^t \rtimes X(p,-))$ is a quotient of $||^t\text{sgn}^t \rtimes X(p,-)$ by $X(-t,-p) \oplus X(t,-p)$. By the same process as above, we find that the K-types for this representation are given as follows, nonzero only for $x+y \equiv p+t+1$ (mod 2).



$$\begin{array}{llll} \text{I:} & x \leq -t, x-t-p \leq y \leq x+t-p & M(x,y) = \left\lfloor \frac{y+p+1}{2} \right\rfloor - \left\lfloor \frac{-t-x}{2} \right\rfloor \\ \text{II:} & x-t-p \leq y \leq -p, x \leq t & M(x,y) = \left\lfloor \frac{y+p+1}{2} \right\rfloor \\ \text{III:} & y \leq x-t-p, -t \leq x \leq t & M(x,y) = \left\lfloor \frac{t-x}{2} \right\rfloor \\ \text{IV:} & y \leq x-t-p, x \leq -t & M(x,y) = t \end{array} \tag{4.5}$$

Lang$(||^t\mathbf{sgn}^{t+1} \rtimes X(p,+))$

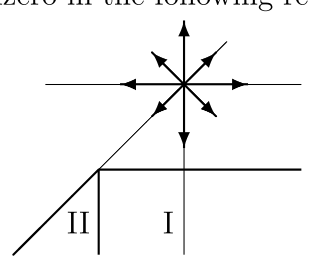
We have that $\text{Lang}(||^t\text{sgn}^{t+1} \rtimes X(p,+)) \cong ||^t\text{sgn}^{t+1} \rtimes X(p,+)$ as it is irreducible. Therefore the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+t \pmod 2$.



$$\begin{array}{lll} \text{I:} & y \leq p, x \geq p & M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor \\ \text{II:} & y \geq p, x \geq p & M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor - \left\lfloor \frac{y-p}{2} \right\rfloor \end{array}$$

Lang$(||^t\mathbf{sgn}^{t+1} \rtimes X(p,-))$

Similarly, $\text{Lang}(||^t\text{sgn}^{t+1} \rtimes X(p,-)) \cong ||^t\text{sgn}^{t+1} \rtimes X(p,-)$, so then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+t \pmod 2$.

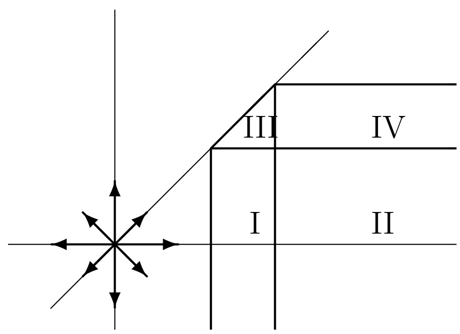


I:	$y \leq -p, x \geq -p$	$M(x,y) = \lfloor \frac{-p-y+1}{2} \rfloor$
II:	$y \leq -p, x \leq -p$	$M(x,y) = \lfloor \frac{-p-y+1}{2} \rfloor - \lfloor \frac{-x-p}{2} \rfloor$

Lang$(||^p\mathbf{sgn}^{p+1} \rtimes X(t,+))$

Next, note by (2.17) that Lang$(||^p\text{sgn}^{p+1} \rtimes X(t,+))$ is a quotient of $||^p\text{sgn}^{p+1} \rtimes X(t,+)$ by $||^t\text{sgn}^{t+1} \rtimes X(p,+)$.

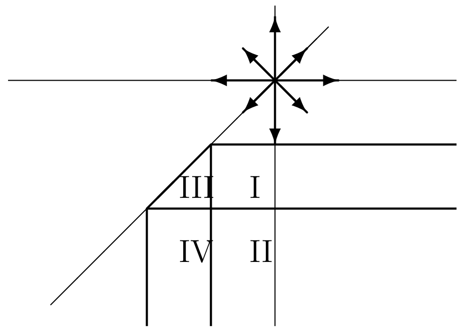
Therefore by our results above on the multiplicity of K-types of the multiplicity of $||^t\text{sgn}^{t+1} \rtimes X(p,+)$, it follows that the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+t \pmod 2$.



I:	$t \leq x \leq p, y \leq t$	$M(x,y) = \lfloor \frac{x-t+1}{2} \rfloor$
II:	$x \geq p, y \leq t$	$M(x,y) = \lfloor \frac{x-t+1}{2} \rfloor - \lfloor \frac{x-p+1}{2} \rfloor$
III:	$x \leq p, t \leq y \leq x$	$M(x,y) = \lfloor \frac{x-t+1}{2} \rfloor - \lfloor \frac{y-t}{2} \rfloor$
IV:	$x \geq p, t \leq y \leq p$	$M(x,y) = \lfloor \frac{x-t+1}{2} \rfloor - \lfloor \frac{y-t}{2} \rfloor - \lfloor \frac{x-p+1}{2} \rfloor$

Lang$(||^p\mathbf{sgn}^{p+1} \rtimes X(t,-))$

Similarly, by (2.18), Lang$(||^p\text{sgn}^{p+1} \rtimes X(t,-))$ is a quotient of $||^p\text{sgn}^{p+1} \rtimes X(t,-)$ by $||^t\text{sgn}^{t+1} \rtimes X(p,-)$. Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+t \pmod 2$.



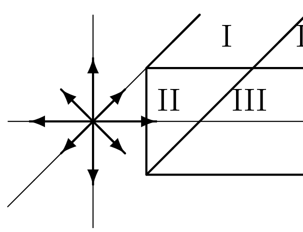
I:	$x \geq -t, -p \leq y \leq -t$	$M(x,y) = \left\lfloor \frac{-y-t+1}{2} \right\rfloor$
II:	$x \geq -t, y \leq -p$	$M(x,y) = \left\lfloor \frac{-y-t+1}{2} \right\rfloor - \left\lfloor \frac{-y-p+1}{2} \right\rfloor$
III:	$x \leq p, t \leq y \leq x$	$M(x,y) = \left\lfloor \frac{-y-t+1}{2} \right\rfloor - \left\lfloor \frac{x+p}{2} \right\rfloor$
IV:	$-p \leq x \leq -t, y \leq -p$	$M(x,y) = \left\lfloor \frac{-y-t+1}{2} \right\rfloor - \left\lfloor \frac{-y-p+1}{2} \right\rfloor - \left\lfloor \frac{x+p}{2} \right\rfloor$

Now consider those cases with $p > t = 0$

Lang$(||^p$**sgn**$^p \rtimes X(p,+))$

First, by (2.29) Lang$(||^p \mathrm{sgn}^p \rtimes X(p,+))$ is a quotient of $||^p \mathrm{sgn}^p \rtimes X(p,+)$ by $X^1(p,-p)$.

Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x + y \equiv 1 \pmod 2$.

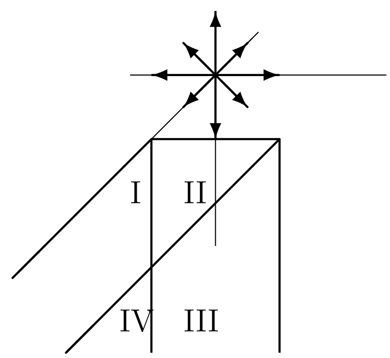


$$
\begin{array}{lll}
\text{I:} & p \leq y \leq x, y \geq x - 2p & M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor - \left\lfloor \frac{y-p}{2} \right\rfloor \\
\text{II:} & x \geq p, x - 2p \leq y \leq x & M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor \\
\text{III:} & y \leq x - 2p, -p \leq y \leq p & M(x,y) = \left\lfloor \frac{y+p}{2} \right\rfloor \\
\text{IV:} & p \leq y \leq x - 2p & M(x,y) = p
\end{array}
\tag{4.6}
$$

Lang$(||^p\mathbf{sgn}^p \rtimes X(p,-))$

Now by (2.30), $\mathrm{Lang}(||^p\mathrm{sgn}^p \rtimes X(p,-))$ is a quotient of $||^p\mathrm{sgn}^p \rtimes X(p,-)$ by $X^2(p,-p)$.

Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv 1 \pmod 2$.

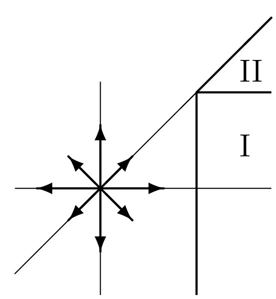


$$\begin{array}{lll} \text{I:} & x \le -p, x-2p \le y \le x & M(x,y) = \left\lfloor \frac{y+p+1}{2} \right\rfloor - \left\lfloor \frac{-p-x}{2} \right\rfloor \\ \text{II:} & x-2p \le y \le -p, x \le p & M(x,y) = \left\lfloor \frac{y+p+1}{2} \right\rfloor \\ \text{III:} & y \le x-2p, -p \le x \le p & M(x,y) = \left\lfloor \frac{p-x}{2} \right\rfloor \\ \text{IV:} & y \le x-2p, x \le -p & M(x,y) = p \end{array} \tag{4.7}$$

Lang$(||^p\mathbf{sgn}^{p+1} \rtimes X(p,+))$

We have $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(p,+)) \cong ||^p\mathrm{sgn}^{p+1} \rtimes X(p,+)$ by Lemma 9.5 in [3].

Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv 1 \pmod 2$.



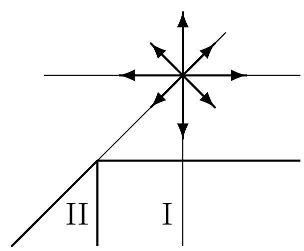
I: $y \leq p, x \geq p \quad M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor$

II: $y \geq p, x \geq p \quad M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor - \left\lfloor \frac{y-p}{2} \right\rfloor$

Lang$(||^p\mathbf{sgn}^{p+1} \rtimes X(p,-))$

$\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(p,-)) \cong ||^p\mathrm{sgn}^{p+1} \rtimes X(p,-)$ as it is irreducible.

Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x + y \equiv 1 \pmod 2$.



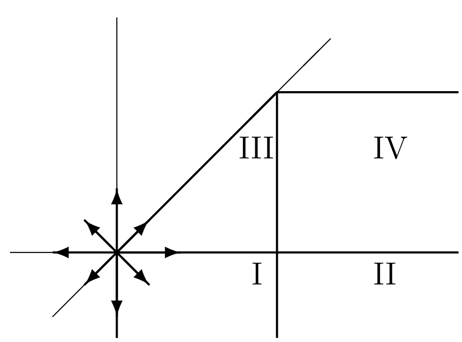
I: $y \leq -p, x \geq -p \quad M(x,y) = \left\lfloor \frac{-p-y+1}{2} \right\rfloor$

II: $y \leq -p, x \leq -p \quad M(x,y) = \left\lfloor \frac{-p-y+1}{2} \right\rfloor - \left\lfloor \frac{-x-p}{2} \right\rfloor = \frac{x-y+C(x,y,p+1)}{2}$

Lang$(||^p\mathbf{sgn}^{p+1} \rtimes X(0,+))$

Note that $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(0,+))$ is a quotient of $||^p\mathrm{sgn}^{p+1} \rtimes X(0,+)$ by $\mathrm{sgn} \rtimes X(p,+)$ from (2.37)

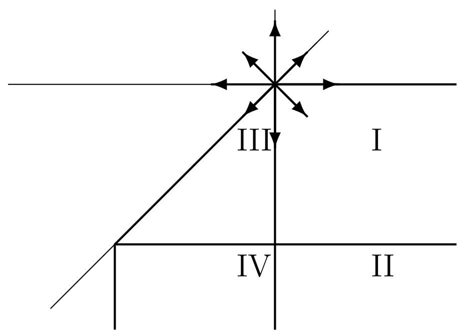
Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x + y \equiv p + 1 \pmod 2$.



I:	$0 \le x \le p, y \le 0$	$M(x,y) = \lfloor \frac{x+1}{2} \rfloor$
II:	$x \ge p, y \le 0$	$M(x,y) = \lfloor \frac{x+1}{2} \rfloor - \lfloor \frac{x-p+1}{2} \rfloor$
III:	$x \le p, 0 \le y \le x$	$M(x,y) = \lfloor \frac{x+1}{2} \rfloor - \lfloor \frac{y}{2} \rfloor$
IV:	$x \ge p, 0 \le y \le p$	$M(x,y) = \lfloor \frac{x+1}{2} \rfloor - \lfloor \frac{y}{2} \rfloor - \lfloor \frac{x-p+1}{2} \rfloor$

Lang$(||^p\mathbf{sgn}^{p+1} \rtimes X(0,-))$

We see by (2.38) that $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(0,-))$ is a quotient of $||^p\mathrm{sgn}^{p+1} \rtimes X(0,-)$ by $\mathrm{sgn} \rtimes X(p,-)$, so then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+1 \pmod 2$.



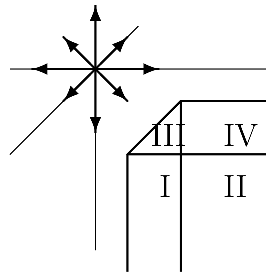
I:	$-p \le y \le 0, x \ge 0$	$M(x,y) = \lfloor \frac{-y+1}{2} \rfloor$
II:	$y \le -p, x \ge 0$	$M(x,y) = \lfloor \frac{-y+1}{2} \rfloor - \lfloor \frac{-y-p+1}{2} \rfloor$
III:	$x \le p, -p \le y \le x$	$M(x,y) = \lfloor \frac{-y+1}{2} \rfloor - \lfloor \frac{-x}{2} \rfloor$
IV:	$y \ge -p, -p \le x \le 0$	$M(x,y) = \lfloor \frac{-y+1}{2} \rfloor - \lfloor \frac{-x}{2} \rfloor - \lfloor \frac{-y-p+1}{2} \rfloor$

4.4.2 Langlands quotients of the Siegel

Lang$(\delta(||^{\frac{p-t}{2}}\mathbf{sgn}^t, p+t) \rtimes 1)$

Observe that $\mathrm{Lang}(\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1)$ is a quotient of $\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1$ by $X(p,-t) \oplus X(t,-p)$ by (2.22).

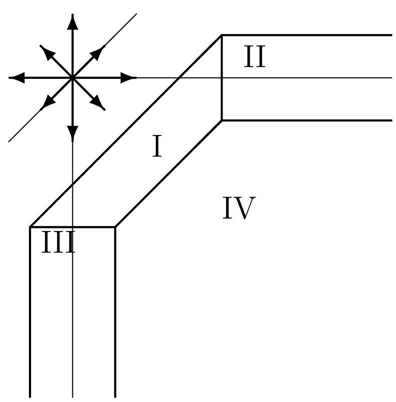
Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+t+1 \pmod 2$.



I:	$y \le -p, t \le x \le p$	$M(x,y) = \left\lfloor \frac{x-t+1}{2} \right\rfloor$
II:	$y \le -p, x \ge p$	$M(x,y) = \frac{p-(t+1)+C(p,t+1,x)}{2}$
III:	$x \le p, -p \le y \le x-t-p$	$M(x,y) = \frac{x-y-p-t+1}{2}$
IV:	$-p \le y \le -t, x \ge p$	$M(x,y) = \left\lfloor \frac{-y-t+1}{2} \right\rfloor$

Lang$(\delta(||^{\frac{p+t}{2}}\mathbf{sgn}^{t+1}, p-t) \rtimes 1)$

Now note from (2.20) and (2.21) that Lang$(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1)$ appears in a composition series such that we may determine its K-types by taking the multiplicities of those in $\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1$ and subtracting multiplicities given in $\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^{t}, p+t) \rtimes 1$, Lang$(||^t\mathrm{sgn}^t \rtimes X(p,+))$, and Lang$(||^t\mathrm{sgn}^t \rtimes X(p,-))$. This gives us nonzero multiplicities as follows, for $x+y \equiv p+t+1 \pmod 2$:



I: $y \geq -p, x \leq p, x-p-t+1 \leq y \leq x-p+t+1$ $M(x,y) = \frac{x-y-p+t+1}{2}$

II: $-t \leq y \leq t, x \geq p$ $M(x,y) = \left\lfloor \frac{t-y+1}{2} \right\rfloor$

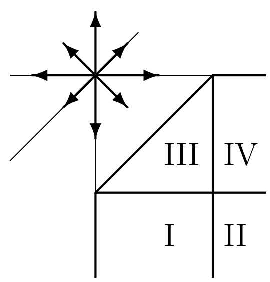
III: $y \leq -p, -t \leq x \leq t$ $M(x,y) = \left\lfloor \frac{x+t+1}{2} \right\rfloor$

IV: $y \leq -t, x \geq p, y \leq x-p-t+1$ $M(x,y) = t$

Lang$(\delta(||^{\frac{p}{2}}\,\mathbf{sgn}^p, p) \rtimes 1)$

Next, by (2.40), Lang$(\delta(||^{\frac{p}{2}}\,\mathrm{sgn}^p, p) \rtimes 1)$ is a quotient of $\delta(||^{\frac{p}{2}}\,\mathrm{sgn}^p, p) \rtimes 1$ by $X^1(0,-p) \oplus X^2(p,0)$.

Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x + y \equiv p+1 \pmod 2$.



I: $y \leq -p, 0 \leq x \geq p$ $M(x,y) = \left\lfloor \frac{x+1}{2} \right\rfloor$

II: $y \leq -p, x \geq p$ $M(x,y) = \frac{p-1+C(p,1,x)}{2}$

III: $x \leq p, -p \leq y \leq x-p$ $M(x,y) = \frac{x-y-p+1}{2}$

IV: $-p \leq y \leq 0, x \geq p$ $M(x,y) = \left\lfloor \frac{-y+1}{2} \right\rfloor$

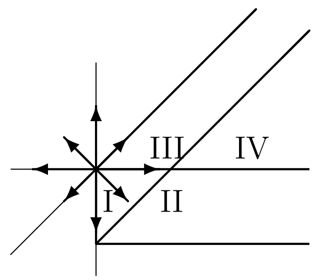
4.4.3 Remaining cases

The following representations induced from the Klingen require knowledge of K-types of Langlands quotients of the Siegel, so they appear here.

Lang$(||^p\mathbf{sgn}^p \rtimes X(0,+))$

Now we see from Theorem 11.2 in [3] that $\mathrm{Lang}(||^p\mathrm{sgn}^p \rtimes X(0,+))$ has K-types that can be determined by taking those of $||^p\mathrm{sgn}^p \rtimes X(0,+)$ and removing those from $X^2(p,0)$ and $\mathrm{Lang}(\delta(||^{\frac{p}{2}}\,\mathrm{sgn}^p, p) \rtimes 1)$

Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+1 \pmod 2$.

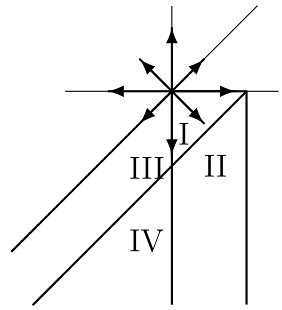


$$\begin{array}{lll} \text{I:} & x-p \le y \le 0, x \ge 0 & M(x,y) = \left\lfloor \frac{x+1}{2} \right\rfloor \\ \text{II:} & -p \le y \le 0, y \le x-p & M(x,y) = \left\lfloor \frac{y+p}{2} \right\rfloor \\ \text{III:} & y \ge 0, x-p \le y \le x & M(x,y) = \left\lfloor \frac{x+1}{2} \right\rfloor - \left\lfloor \frac{y}{2} \right\rfloor \\ \text{IV:} & 0 \le y \le x-p & M(x,y) = \left\lfloor \frac{p}{2} \right\rfloor \end{array} \tag{4.8}$$

Lang$(||^p\mathbf{sgn}^p \rtimes X(0,-))$

Next we see from Theorem 11.2 in [3] that $\mathrm{Lang}(||^p\mathrm{sgn}^p \rtimes X(0,-))$ has K-types that can be determined by taking those of $||^p\mathrm{sgn}^p \rtimes X(0,-)$ and removing those from $X^1(0,-p)$ and $\mathrm{Lang}(\delta(||^{\frac{p}{2}}\,\mathrm{sgn}^p, p) \rtimes 1)$

Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+1 \pmod 2$.

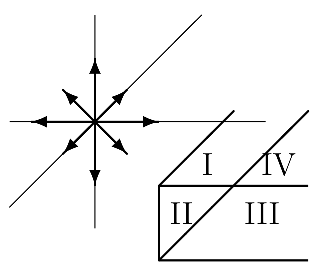


$$
\begin{array}{lll}
\text{I:} & 0 \le x \le y+p, y \le 0 & M(x,y) = \left\lfloor \frac{-y+1}{2} \right\rfloor \\
\text{II:} & 0 \le x \le p, y \ge x+p & M(x,y) = \left\lfloor \frac{-x+p}{2} \right\rfloor \\
\text{III:} & x \le 0, y \le x \le y+p & M(x,y) = \left\lfloor \frac{-y+1}{2} \right\rfloor - \left\lfloor \frac{-x}{2} \right\rfloor \\
\text{IV:} & y+p \le x \le 0 & M(x,y) = \left\lfloor \frac{p}{2} \right\rfloor
\end{array}
\tag{4.9}
$$

Lang$(||^p\mathbf{sgn}^p \rtimes X(t,+))$

We can see from (2.9) that $\text{Lang}(||^p\text{sgn}^p \rtimes X(t,+))$ is a quotient of $||^p\text{sgn}^p \rtimes X(t,+)$ by what was called $V_{1,+}$ with constituents $X(p,-t)$, $\text{Lang}(\delta(||^{\frac{p-t}{2}}\text{sgn}^t, p-t), 1)$, and $\text{Lang}(||^t\text{sgn}^t \rtimes X(p,+))$.

Therefore by our results on the multiplicity of K-types, we can calculate the following multiplicities for K-types (x,y), nonzero only when $x+y \equiv p+t \pmod 2$.

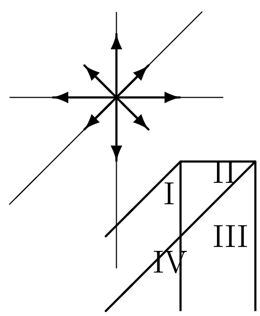


$$
\begin{array}{lll}
\text{I:} & y \geq -t, x-t-p \leq y \leq x-2t & M(x,y) = \left\lfloor \frac{x-t+1}{2} \right\rfloor - \left\lfloor \frac{y+t}{2} \right\rfloor \\
\text{II:} & x-t-p \leq y \leq -t, x \geq t & M(x,y) = \left\lfloor \frac{x-t+1}{2} \right\rfloor \\
\text{III:} & y \leq x-t-p, -p \leq y \leq -t & M(x,y) = \left\lfloor \frac{y+p}{2} \right\rfloor \\
\text{IV:} & -t \leq y \leq x-t-p & M(x,y) = \left\lfloor \frac{p-t}{2} \right\rfloor
\end{array}
\tag{4.10}
$$

Lang$(||^p\mathbf{sgn}^p \rtimes X(t,-))$

We can see from (2.10) that $\text{Lang}(||^p\text{sgn}^p \rtimes X(t,-))$ is a quotient of $||^p\text{sgn}^p \rtimes X(t,-)$ by what was called $V_{1,-}$ with constituents $X(t,-p)$, $\text{Lang}(\delta(||^{\frac{p-t}{2}}\text{sgn}^t, p-t), 1)$, and $\text{Lang}(||^t\text{sgn}^t \rtimes X(p,-))$.

Therefore by our results on the multiplicity of K-types, we can calculate the following multiplicities for K-types (x,y), nonzero only when $x+y \equiv p+t \pmod 2$.



$$
\begin{array}{lll}
\text{I:} & x \leq t, -y-2t \leq x \leq -y-t-p & M(x,y) = \left\lfloor \frac{-y-t+1}{2} \right\rfloor - \left\lfloor \frac{-y+t}{2} \right\rfloor \\
\text{II:} & -t \leq x \leq -y-t-p, x \geq t & M(x,y) = \left\lfloor \frac{-y-t+1}{2} \right\rfloor \\
\text{III:} & -x \leq -y-t-p, t \leq x \leq p & M(x,y) = \left\lfloor \frac{-x+p}{2} \right\rfloor \\
\text{IV:} & -y-t-p \leq x \leq t & M(x,y) = \left\lfloor \frac{p-t}{2} \right\rfloor
\end{array}
\tag{4.11}
$$

4.4.4 Langlands induced from the Borel

First, consider those with $p > t > 0$

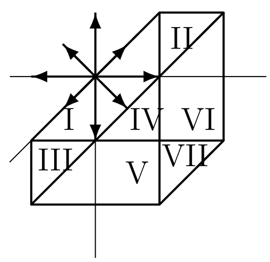
Lang$(||^p\mathbf{sgn}^p \times ||^t\mathbf{sgn}^t \rtimes 1)$

First, $\mathrm{Lang}(||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes 1) \cong ||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes 1$, so we obtain multiplicities of $M(x,y) = \frac{x-y+C(x,y,\epsilon_1+1)}{2}$, for $x+y \equiv p+t \pmod 2$.

Lang$(||^p\mathbf{sgn}^p \times ||^t\mathbf{sgn}^{t+1} \rtimes 1)$

Next, from (2.8), $\mathrm{Lang}(||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1)$ is a quotient of $||^p\mathrm{sgn}^p \rtimes V_t$ by $\mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1)$.

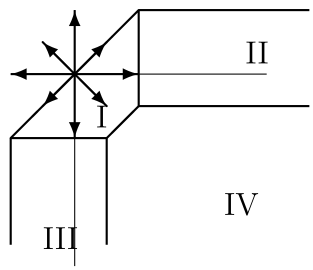
Then the multiplicity of the K-type (x,y) is nonzero in the following regions, for $x+y \equiv p+t+1 \pmod 2$.



I:	$y \geq x+t-p, y \geq -t, x \leq t$	$M(x,y) = \left\lfloor \frac{x-t+1}{2} \right\rfloor - \left\lfloor \frac{y-t}{2} \right\rfloor$
II:	$y \geq x+t-p, y \leq t, t \leq x \leq p$	$M(x,y) = \left\lfloor \frac{t-y+1}{2} \right\rfloor$
III:	$y \geq x+t-p, -p \leq y \leq -t, x \geq -t$	$M(x,y) = \left\lfloor \frac{x+t+1}{2} \right\rfloor$
IV:	$y \leq x+t-p, y \geq -t, x \leq t$	$M(x,y) = \frac{p-(t+1)+C(x,y,t+1)}{2}$
V:	$y \leq x+t-p, -p \leq y \leq -t, -t \leq x \leq t$	$M(x,y) = \left\lfloor \frac{y+p}{2} \right\rfloor$
VI:	$y \leq x+t-p, -t \leq y \leq t, t \leq x \leq p$	$M(x,y) = \left\lfloor \frac{p-x}{2} \right\rfloor$
VII:	$y \geq x-t-p, y \leq -t, x \geq t$	$M(x,y) = \frac{t-x+p+y-1}{2}$

Lang$(||^p\mathbf{sgn}^{p+1} \times ||^t\mathbf{sgn}^t \rtimes 1)$

Now note by (2.27) that $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1)$ has K-types that can be determined by taking those from $||^t\mathrm{sgn}^t \rtimes V_p$ and subtracting those from $\mathrm{Lang}(\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^{t+1}, p+t) \rtimes 1)$ and $\mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1)$.

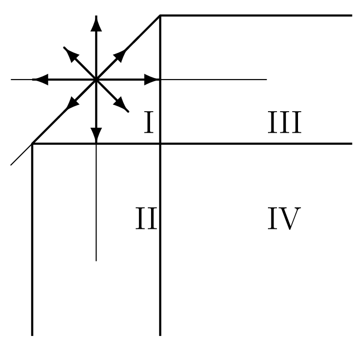


I:	$y \geq x+t-p, y \geq -p, x \leq p$	$M(x,y) = \left\lfloor \frac{x-p+1}{2} \right\rfloor - \left\lfloor \frac{y-p}{2} \right\rfloor$
II:	$t \leq y \leq p, x \geq p$	$M(x,y) = \left\lfloor \frac{p-y+1}{2} \right\rfloor$
III:	$y \leq -p, t \leq x \leq p$	$M(x,y) = \left\lfloor \frac{x+p+1}{2} \right\rfloor$
IV:	$y \leq t, y \leq x+t-p, x \geq -t$	$M(x,y) = \left\lfloor \frac{p+t}{2} \right\rfloor$

Lang$(||^p\mathbf{sgn}^{p+1} \times ||^t\mathbf{sgn}^{t+1} \rtimes 1)$

Next we have $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes 1) \cong ||^p\mathrm{sgn}^{p+1} \rtimes V_t$

Then the multiplicity of the K-type (x, y) is nonzero in the following regions, for $x + y \equiv p + t \pmod 2$.

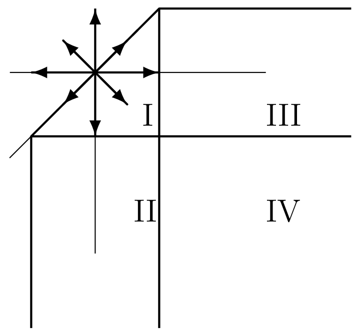


I:	$y \geq -t, x \leq t$	$M(x,y) = \left\lfloor \frac{x-t+1}{2} \right\rfloor - \left\lfloor \frac{y-t}{2} \right\rfloor$
II:	$y < -t, x \leq t$	$M(x,y) = \left\lfloor \frac{x+t+1}{2} \right\rfloor$
III:	$y \geq -t, x > t$	$M(x,y) = \left\lfloor \frac{t-y+1}{2} \right\rfloor$
IV:	$y < -t, x > t$	$M(x,y) = t$

Then consider those with $p = t > 0$

Lang$(||^p\mathbf{sgn}^p \times ||^p\mathbf{sgn}^{p+1} \rtimes 1)$

We see that $\mathrm{Lang}(||^p\mathrm{sgn}^p \times ||^p\mathrm{sgn}^{p+1} \rtimes 1) \cong ||^p\mathrm{sgn}^p \rtimes V_p$. Then the multiplicity of the K-type (x, y) is nonzero for $x + y \equiv p \pmod 2$.

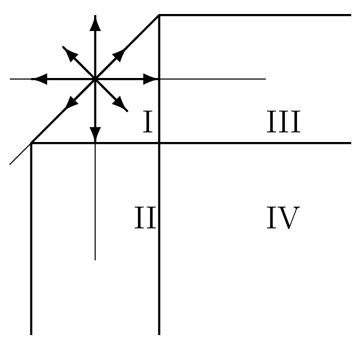


I:	$y \geq -p, x \leq p$	$M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor - \lfloor \frac{y-p}{2} \rfloor$
II:	$y < -p, x \leq p$	$M(x,y) = \lfloor \frac{x+p+1}{2} \rfloor$
III:	$y \geq -p, x > p$	$M(x,y) = \lfloor \frac{p-y+1}{2} \rfloor$
IV:	$y < -p, x > p$	$M(x,y) = p$

Lang$(||^p\mathbf{sgn}^{p+1} \times ||^p\mathbf{sgn}^{p+1} \rtimes 1)$

Similarly, $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^p\mathrm{sgn}^{p+1} \rtimes 1) \cong ||^p\mathrm{sgn}^{p+1} \rtimes V_p$

Then the multiplicity of the K-type (x, y) is nonzero for $x + y \equiv p + 1 \pmod 2$.



I:	$y \geq -p, x \leq p$	$M(x,y) = \lfloor \frac{x-p+1}{2} \rfloor - \lfloor \frac{y-p}{2} \rfloor$
II:	$y < -p, x \leq p$	$M(x,y) = \lfloor \frac{x+p+1}{2} \rfloor$
III:	$y \geq -p, x > p$	$M(x,y) = \lfloor \frac{p-y+1}{2} \rfloor$
IV:	$y < -p, x > p$	$M(x,y) = p$

4.5 Non-integer coefficients

Now we consider $||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1$ where at least one of s_1, s_2 is not an integer. These representations are reducible if one of four conditions are met, and are otherwise irreducible with K-types as given above. The first reducibility criterion is that $\epsilon_2 \equiv s_2 + 1 \pmod 2$. In this case, we have for $s_2 > 0$ that

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes (X(s_2,+) \oplus X(s_2,-)) \hookrightarrow ||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \twoheadrightarrow ||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes V_{s_2}$$

and for $s_2 = 0$ that

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \times \mathrm{sgn} \rtimes 1 \simeq ||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes (X(0,+) \oplus X(0,-))$$

Here we only consider the case that $s_1 \notin \mathbb{Z}$, so all constituents are irreducible.

For the case of $\epsilon_1 \equiv s_1 + 1 \pmod 2$, we note that as long as $s_2 \notin \mathbb{Z}$, it gives an isomorphism

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \simeq ||^{s_2}\mathrm{sgn}^{\epsilon_2} \times ||^{s_1}\mathrm{sgn}^{\epsilon_1} \rtimes 1,$$

at which point the representation reduces as above.

Now for the case where $s_1 - s_2 \in \mathbb{Z}_{\neq 0}, \epsilon_1 + \epsilon_2 \equiv s_1 + s_2 + 1 \pmod 2$, we have that

$$\delta(||^{\frac{s_1+s_2}{2}}\mathrm{sgn}^{\epsilon_2}, s_1 - s_2) \rtimes 1 \hookrightarrow ||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \twoheadrightarrow \zeta(||^{\frac{s_1+s_2}{2}}\mathrm{sgn}^{\epsilon_2}, s_1 - s_2) \rtimes 1$$

We consider the case where at least one and therefore both of $s_1, s_2 \notin \mathbb{Z}$, in which case the constituents are irreducible.

For the case $s_1+s_2 \in \mathbb{Z}_{\neq 0}, \epsilon_1+\epsilon_2 \equiv s_1+s_2+1 \pmod 2$, we have an isomorphism

$$||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1 \simeq ||^{s_1}\mathrm{sgn}^{\epsilon_1} \times ||^{-s_2}\mathrm{sgn}^{\epsilon_2} \rtimes 1$$

in this case as long as $s_2 \notin \mathbb{Z}$. Then the representation reduces as above.

Chapter 5

Restriction of Langlands quotients

5.1 Relation between Langlands quotients of GSp$(4,\mathbb{R})$ and Sp$(4,\mathbb{R})$

Earlier we considered restriction of induced representations from $\mathrm{GSp}(4,\mathbb{R})$ to $\mathrm{Sp}(4,\mathbb{R})$, now we will consider the restriction of Langlands quotients of $\mathrm{GSp}(4,\mathbb{R})$ to $\mathrm{Sp}(4,\mathbb{R})$. There are several cases to be examined. First, we will need a lemma.

Lemma 5.1. *Let (π, V) be a unitary representation of $GSp(4,\mathbb{R})$. Then the following are equivalent:*

- *π is a discrete series representation.*
- *$\pi|_{Sp(4,\mathbb{R})} = \tau_1 \oplus \ldots \oplus \tau_n$ where τ_i are discrete series representations on $Sp(4,\mathbb{R})$.*

Proof. First, suppose (π, V) is a unitary discrete series representation of $\mathrm{GSp}(4,\mathbb{R})$.

Then $\int_{\mathrm{Sp}(4,\mathbb{R})^{\pm}} |\langle\pi(g)v_1, v_2\rangle|^2\, dg < \infty$, so that $\int_{\mathrm{Sp}(4,\mathbb{R})} |\langle\pi(g)v_1, v_2\rangle|^2\, dg < \infty$. But then $\pi|_{\mathrm{Sp}(4,\mathbb{R})} = \tau_1 \oplus \ldots \oplus \tau_n$ for some collection of irreducible representations of $\mathrm{Sp}(4, \mathbb{R})$, and it follows that for each such representation τ_i we will have $\int_{\mathrm{Sp}(4,\mathbb{R})} |\langle\tau_i(g)v_1, v_2\rangle|^2\, dg < \infty$. Then each τ_i is a discrete series representation on $\mathrm{Sp}(4, \mathbb{R})$.

Next, let us consider the other direction. Let (π, V) be a unitary representation of $\mathrm{GSp}(4, \mathbb{R})$ with $\pi|_{\mathrm{Sp}(4,\mathbb{R})} = \tau_1 \oplus \ldots \oplus \tau_n$ where τ_i are discrete series representations on $\mathrm{Sp}(4, \mathbb{R})$. Then each $\int_{\mathrm{Sp}(4,\mathbb{R})} |\langle\tau_i(g)v_1, v_2\rangle|^2\, dg < \infty$, so that $\int_{\mathrm{Sp}(4,\mathbb{R})} |\langle\pi(g)v_1, v_2\rangle|^2\, dg < \infty$. But

$$\int_{\mathrm{Sp}(4,\mathbb{R})^{\pm}} |\langle\pi(g)v_1, v_2\rangle|^2\, dg = \tag{5.1}$$

$$\int_{\mathrm{Sp}(4,\mathbb{R})} |\langle\pi(g)v_1, v_2\rangle|^2\, dg + \int_{\mathrm{Sp}(4,\mathbb{R})} \left|\left\langle \pi(g)\pi\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}\right) v_1, v_2 \right\rangle\right|^2 dg < \infty,$$

so that (π, V) is a discrete series representation. □

5.2 Representations induced from the Siegel parabolic subgroup

Unless stated otherwise, we assume that $p > t > 0$, $p, t \in \mathbb{Z}$ for the following.

Case of $\delta(||^{\frac{(p-t)}{2}}\mathbf{sgn}^t, p+t) \rtimes \sigma$

We have from (2.15) that as representations on $\mathrm{Sp}(4, \mathbb{R})$,

$$X(p, -t) \oplus X(t, -p) \hookrightarrow \delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t) \rtimes 1 \twoheadrightarrow \mathrm{Lang}(\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t), 1).$$

Now, we may extend $\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t)\rtimes 1$ to a representation on $\mathrm{GSp}(4,\mathbb{R})$, namely $\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t)\rtimes\sigma$. Then this gives us an action of $\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix}$ on $X(p,-t)\oplus X(t,-p)$. Under this action, $X(p,-t)$ must be mapped to another irreducible subrepresentation of $\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t)\rtimes 1$. However, as $\mathrm{Lang}(\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t), 1)$ is a Langlands quotient, we know that the only irreducible subrepresentations are $X(p,-t)$ and $X(t,-p)$. By examining the weight structure, we can determine that it must be mapped to $X(t,-p)$, and therefore $X(t,-p)$ must be mapped to $X(p,-t)$. Then we may extend $X(p,-t)\oplus X(t,-p)$ to a $\mathrm{GSp}(4,\mathbb{R})$ representation.

As we may extend the induced representation and the kernel, therefore we may extend the quotient, $\mathrm{Lang}(\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t), 1)$, to a $\mathrm{GSp}(4,\mathbb{R})$ representation. We then note that extension of $\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t)\rtimes 1$ in this manner gives us a representation $\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t)\rtimes\sigma$, and that extension of the irreducible representation $\mathrm{Lang}(\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t), 1)$ gives us an irreducible quotient of $\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t)\rtimes\sigma$. This must be the Langlands quotient, as it is a unique irreducible quotient. From this we can see that the restriction of $\mathrm{Lang}(\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t), \sigma)$ to $\mathrm{Sp}(4,\mathbb{R})$ is $\mathrm{Lang}(\delta(||^{\frac{(p-t)}{2}}\mathrm{sgn}^t, p+t), 1)$.

Case of $\delta(||^{\frac{p}{2}}, p)\rtimes\sigma$, $p>0$

From (2.29), we know that $X^2(p,0)\oplus X^1(0,-p)\hookrightarrow\delta(||^{\frac{p}{2}}, p)\rtimes 1\twoheadrightarrow\mathrm{Lang}(\delta(||^{\frac{p}{2}}, p), 1)$. Now we can extend $\delta(||^{\frac{p}{2}}, p)\rtimes 1$ to $\mathrm{GSp}(4,\mathbb{R})$ and we may extend $X^2(p,0)\oplus X^1(0,-p)$ also, as similar to previous cases we may examine weights to determine $\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix}X^2(p,0)=X^1(0,p)$. Then $\mathrm{Lang}(\delta(||^{\frac{p}{2}}, p), 1)$ extends to an irreducible quotient. We can see this quotient must be $\mathrm{Lang}(\delta(||^{\frac{p}{2}}, p), \sigma)$ as it is the unique irreducible quotient. Therefore the restriction of $\mathrm{Lang}(\delta(||^{\frac{p}{2}}, p), \sigma)$ to $\mathrm{Sp}(4,\mathbb{R})$ is $\mathrm{Lang}(\delta(||^{\frac{p}{2}}, p), 1)$.

5.3 Representations induced from the Klingen parabolic subgroup

We continue to assume that $p > t > 0$, $p, t \in \mathbb{Z}$ for the following.

Case of $||^t\mathbf{sgn}^t \rtimes ||^c\mathbf{sgn}^d D_p$

From composition series, we see that, as $\mathrm{Sp}(4,\mathbb{R})$ representations,

$$X(p,t) \oplus X(p,-t) \hookrightarrow ||^t\mathrm{sgn}^t \rtimes X(p,+) \twoheadrightarrow \mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,+))$$

and

$$X(t,-p) \oplus X(-t,-p) \hookrightarrow ||^t\mathrm{sgn}^t \rtimes X(p,-) \twoheadrightarrow \mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,-)).$$

Then we know that $X(p,t) \oplus X(p,-t) \oplus X(t,-p) \oplus X(-t,-p) \hookrightarrow ||^t\mathrm{sgn}^t \rtimes (X(p,+) \oplus X(p,-)) \twoheadrightarrow \mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,-))$.

Now we can extend $X(p,+) \oplus X(p,-)$ from a representation of $\mathrm{SL}(2,\mathbb{R})$ to one of $\mathrm{GL}(2,\mathbb{R})$. Then we may extend $||^t\mathrm{sgn}^t \rtimes (X(p,+) \oplus X(p,-))$ to $\mathrm{GSp}(4,\mathbb{R})$, giving $||^t\mathrm{sgn}^t \rtimes \sigma D_p$. As a result, we may also extend $X(p,t) \oplus X(p,-t) \oplus X(t,-p) \oplus X(-t,-p)$. To do so, consider $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} X(p,t)$. We know this must be mapped to another irreducible subrepresentation, and by considering weights, we know that it must be $X(-t,-p)$. Similarly we may determine that $X(p,-t)$ is mapped to $X(t,-p)$.

Then we may extend the quotient, $\mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,-))$. Suppose the extension is not irreducible. Then it must have a subrepresentation restricting to $\mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,+))$. However, from the weight structure of $\mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,+))$, there can be no $\mathrm{GSp}(4,\mathbb{R})$ representation that restricts to only $\mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,+))$. Then the extension of the direct sum must be an irreducible quotient.

Since this is an irreducible quotient of $||^t\mathrm{sgn}^t \rtimes D_p$, we can see this quotient must be $\mathrm{Lang}(||^t\mathrm{sgn}^t, ||^c\mathrm{sgn}^d D_p)$. Then we can see that the restriction of $\mathrm{Lang}(||^t\mathrm{sgn}^t, ||^c\mathrm{sgn}^d D_p)$ to $\mathrm{Sp}(4,\mathbb{R})$ is
$\mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^t, X(p,-))$.

Case of $||^t\mathbf{sgn}^{t+1} \rtimes ||^c\mathbf{sgn}^d D_p$

From known composition series, we see that $||^t\mathrm{sgn}^{t+1} \rtimes X(p,+)$ and $||^t\mathrm{sgn}^{t+1} \rtimes X(p,-)$ are irreducible. We may then extend $X(p,+) \oplus X(p,-)$ to $\mathrm{GL}(2,\mathbb{R})$ so that $||^t\mathrm{sgn}^{t+1} \rtimes (X(p,+) \oplus X(p,-))$ extends to $||^t\mathrm{sgn}^{t+1} \rtimes \sigma D_p$. Then as above, this representation must be irreducible as a $\mathrm{GSp}(4,\mathbb{R})$ representation. Therefore the restriction of $\mathrm{Lang}(||^t\mathrm{sgn}^{t+1} \rtimes ||^c\mathrm{sgn}^d D_p)$ from $\mathrm{GSp}(4,\mathbb{R})$ to $\mathrm{Sp}(4,\mathbb{R})$ is $\mathrm{Lang}(||^t\mathrm{sgn}^{t+1}, X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^{t+1}, X(p,-))$.

Case of $||^p\mathbf{sgn}^p \rtimes ||^c\mathbf{sgn}^d D_t$, $p > t > 0$

From (2.9) and (2.10) we see that as $\mathrm{Sp}(4,\mathbb{R})$ representations,

$$V_{1,+} \hookrightarrow ||^p\mathrm{sgn}^p \rtimes X(t,+) \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^p, X(t,+))$$

$$V_{1,-} \hookrightarrow ||^p\mathrm{sgn}^{p+} \rtimes X(t,-) \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^p, X(t,-)).$$

As we can extend the constituents of $V_{1,+} \oplus V_{1,-}$ to a $\mathrm{GSp}(4,\mathbb{R})$ representation we can then extend $V_{1,+} \oplus V_{1,-}$ itself. Also, $||^p\mathrm{sgn}^p \rtimes (X(t,+) \oplus X(t,-))$ extends to $||^p\mathrm{sgn}^p \rtimes ||^c\mathrm{sgn}^d D_t$.

We may then extend the quotient of $\mathrm{Lang}(||^p\mathrm{sgn}^p, X(t,+)) \oplus$

$\text{Lang}(||^p\text{sgn}^p, X(t,-))$, which must then be irreducible in $\text{GSp}(4,\mathbb{R})$ by its weight structure, similar to the previous cases.

Since the extension of the quotient is an irreducible quotient of $||^p\text{sgn}^p \rtimes ||^c\text{sgn}^d D_t$ it must be $\text{Lang}(||^p\text{sgn}^p \rtimes ||^c\text{sgn}^d D_t)$. From this we conclude that the restriction of $\text{Lang}(||^p\text{sgn}^p \rtimes ||^c\text{sgn}^d D_t)$ from $\text{GSp}(4,\mathbb{R})$ to $\text{Sp}(4,\mathbb{R})$ is $\text{Lang}(||^p\text{sgn}^p, X(t,+)) \oplus \text{Lang}(||^p\text{sgn}^p, X(t,-))$.

Case of $||^p\textbf{sgn}^{p+1} \rtimes ||^c\textbf{sgn}^d D_t$, $p > t > 0$

From (2.17) and (2.18) we know that, as $\text{Sp}(4,\mathbb{R})$ representations,

$$||^t\text{sgn}^{t+1} \rtimes X(p,+) \hookrightarrow ||^p\text{sgn}^{p+1} \rtimes X(t,+) \twoheadrightarrow \text{Lang}(||^p\text{sgn}^{p+1}, X(t,+))$$

$$||^t\text{sgn}^{t+1} \rtimes X(p,-) \hookrightarrow ||^p\text{sgn}^{p+1} \rtimes X(t,-) \twoheadrightarrow \text{Lang}(||^p\text{sgn}^{p+1}, X(t,-)).$$

We may then extend $||^p\text{sgn}^{p+1} \rtimes (X(t,+) \oplus X(t,-))$ as above to $||^p\text{sgn}^{p+1} \rtimes ||^c\text{sgn}^d D_t$. Similarly, $||^t\text{sgn}^{t+1} \rtimes X(p,+)$ and $||^t\text{sgn}^{t+1} \rtimes X(p,-)$ are irreducible, and we have determined that they extend to $||^t\text{sgn}^{t+1} \rtimes ||^c\text{sgn}^d(X(t,+) \oplus X(t,-))$ which must be itself irreducible.

We may then extend the quotient of $\text{Lang}(||^p\text{sgn}^{p+1}, X(t,+)) \oplus \text{Lang}(||^p\text{sgn}^{p+1}, X(t,-))$, which must then be irreducible in $\text{GSp}(4,\mathbb{R})$ by its weight structure, similar to before.

Since the extension of the quotient is an irreducible quotient of $||^p\text{sgn}^{p+1} \rtimes ||^c\text{sgn}^d D_t$ it must be $\text{Lang}(||^p\text{sgn}^{p+1} \rtimes ||^c\text{sgn}^d D_t)$. From this we conclude that the restriction of $\text{Lang}(||^p\text{sgn}^{p+1} \rtimes ||^c\text{sgn}^d D_t)$ from $\text{GSp}(4,\mathbb{R})$ to $\text{Sp}(4,\mathbb{R})$ is $\text{Lang}(||^p\text{sgn}^{p+1}, X(t,+)) \oplus \text{Lang}(||^p\text{sgn}^{p+1}, X(t,-))$.

Case of $||^p\mathbf{sgn}^p \rtimes ||^c\mathbf{sgn}^d D_p$, $p > 0, \quad p \in \mathbb{Z}$

From (2.25) and (2.26) , we see that, as $\mathrm{Sp}(4,\mathbb{R})$ representations,

$$X^1(p,-p) \hookrightarrow ||^p\mathrm{sgn}^p \rtimes X(p,+) \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^p, X(p,+))$$

$$X^2(p,-p) \hookrightarrow ||^p\mathrm{sgn}^p \rtimes X(p,-) \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^p, X(p,-)).$$

But then we have that

$$X^1(p,-p) \oplus X^2(p,-p) \hookrightarrow ||^p\mathrm{sgn}^p \rtimes (X(p,+) \oplus X(p,-)) \twoheadrightarrow$$

$$\mathrm{Lang}(||^p\mathrm{sgn}^p, X(p,+)) \oplus \mathrm{Lang}(||^p\mathrm{sgn}^p, X(t,-)).$$

Similar to above, we may extend $||^p\mathrm{sgn}^p \rtimes (X(p,+) \oplus X(p,-))$ to a $\mathrm{GSp}(4,\mathbb{R})$ representation. We may then also extend $X^1(p,-p) \oplus X^2(p,-p)$ to an irreducible $\mathrm{GSp}(4,\mathbb{R})$ representation in a consistent manner. This allows us to extend the quotient, which must also be irreducible by the weights of the $\mathrm{Sp}(4,\mathbb{R})$ summands, so it is the Langlands quotient $\mathrm{Lang}(||^p\mathrm{sgn}^p, \sigma D_p)$. Then we see that the restriction of $\mathrm{Lang}(||^p\mathrm{sgn}^p, ||^c\mathrm{sgn}^d D_p)$ to $\mathrm{Sp}(4,\mathbb{R})$ is precisely $\mathrm{Lang}(||^p\mathrm{sgn}^p, X(p,+)) \oplus \mathrm{Lang}(||^p\mathrm{sgn}^p, X(p,-))$.

Case of $||^p\mathbf{sgn}^{p+1} \rtimes ||^c\mathbf{sgn}^d D_p$, $p > 0$

From (2.27), we see that $||^p\mathrm{sgn}^{p+1} \rtimes X(p,+)$ and $||^p\mathrm{sgn}^{p+1} \rtimes X(p,-)$ are irreducible, so that as above and considering their weights, their extension to $\mathrm{GSp}(4,\mathbb{R})$ is the irreducible representation $||^p\mathrm{sgn}^{p+1} \rtimes ||^c\mathrm{sgn}^d D_p$. Then the restriction of $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1}, ||^c\mathrm{sgn}^d D_p)$ from $\mathrm{GSp}(4,\mathbb{R})$ to $\mathrm{Sp}(4,\mathbb{R})$ is
$\mathrm{Lang}(||^p\mathrm{sgn}^{p+1}, X(p,+)) \oplus \mathrm{Lang}(||^p\mathrm{sgn}^{p+1}, X(p,-))$.

Case of sgn $\rtimes ||^c$sgn$^d D_p$

From (2.33), we see that sgn $\rtimes X(p,+)$ and sgn $\rtimes X(p,-)$ are irreducible, so that as above sgn $\rtimes D_p$ is the irreducible extension of their direct sum. Then the restriction of Lang(sgn, $||^c\text{sgn}^d D_p$) from GSp$(4,\mathbb{R})$ to Sp$(4,\mathbb{R})$ is Lang(sgn, $X(p,+)$) $\oplus$ Lang(sgn, $X(p,-)$).

Case of $||^p$sgn$^p \rtimes ||^c$sgn$^d D_0$

We may use (2.35), stating that

$$X^2(p,0) \oplus X^1(0,-p) \hookrightarrow \delta(||^{\frac{p}{2}}\text{sgn}, p) \rtimes 1 \twoheadrightarrow \text{Lang}(\delta(||^{\frac{p}{2}}, p) \rtimes 1)$$

to conclude that the restriction of Lang($||^p\text{sgn}^p, ||^c\text{sgn}^d D_0$) from GSp$(4,\mathbb{R})$ to Sp$(4,\mathbb{R})$ is Lang($||^p\text{sgn}^p, X(0,+)$) $\oplus$ Lang($||^p\text{sgn}^p, X(0,-)$).

Case of $||^p$sgn$^{p+1} \rtimes ||^c$sgn$^d D_0$

From (2.32) and (2.33), we see that sgn $\rtimes X(p,\pm) \hookrightarrow ||^p\text{sgn}^{p+1} \rtimes X(0,\pm) \twoheadrightarrow$ Lang($||^p\text{sgn}^{p+1}, X(0,\pm)$). Similar to previous cases we may extend the direct sum of the middle terms in the composition series to GSp$(4,\mathbb{R})$, and then extend the direct sum of the first terms in a consistent manner. This then allows us to extend the quotient which must be irreducible by considering the weights of its summands as Sp$(4,\mathbb{R})$ representations. It then follows that the restriction of Lang($||^p\text{sgn}^{p+1}, ||^c\text{sgn}^d D_0$) to Sp$(4,\mathbb{R})$ is Lang($||^p\text{sgn}^{p+1}, X(0,+)$) $\oplus$ Lang($||^p\text{sgn}^{p+1}, X(0,-)$).

Case of $\eta \rtimes ||^c\mathrm{sgn}^d D_0$, η unitary

By [3] Lemma 6.1, representations of the form $\eta \rtimes X(0, \pm)$ are irreducible when η is unitary. We may extend the direct sum of both of these representations to $\mathrm{GSp}(4, \mathbb{R})$, giving $\eta \rtimes ||^c\mathrm{sgn}^d D_0$. By the weights of its summands, it must be an irreducible $\mathrm{GSp}(4, \mathbb{R})$ representation. Therefore is is its own Langlands quotient, so the restriction of $\mathrm{Lang}(\eta, ||^c\mathrm{sgn}^d D_0)$ to $\mathrm{Sp}(4, \mathbb{R})$ is $\mathrm{Lang}(\eta, X(0, +)) \oplus \mathrm{Lang}(\eta, X(0, -))$.

5.4 Representations induced from the Borel parabolic subgroup

First let us consider representations on $\mathrm{GSp}(4, \mathbb{R})$ of the form $||^p\mathrm{sgn}^b \times ||^t\mathrm{sgn}^d \rtimes ||^e\mathrm{sgn}^f$ with $a, b, c, d \in \mathbb{Z}$ and both $a \geq c \geq 0$ and $a + c > 0$. This then breaks down further into cases which will be addressed.

Case of $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes ||^e\mathrm{sgn}^f$ with $c > 0$

We may use Lemma 7.1 from [3] which states that $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes 1$ is irreducible with these conditions in combination with the fact that $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes ||^e\mathrm{sgn}^f$ restricts to $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes 1$ when we restrict action to $\mathrm{Sp}(4, \mathbb{R})$ to conclude $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes ||^e\mathrm{sgn}^f$ is irreducible. Then $\mathrm{Lang}(||^p\mathrm{sgn}^p, ||^t\mathrm{sgn}^t, ||^e\mathrm{sgn}^f)$ must restrict to $\mathrm{Lang}(||^p\mathrm{sgn}^p, ||^t\mathrm{sgn}^t, 1)$.

Case of $||^p\mathbf{sgn}^{p+1} \times ||^t\mathbf{sgn}^t \rtimes ||^e\mathbf{sgn}^f$ with $p > t > 0$

We may use (2.23), stating that

$$X(p,t) \oplus X(-t,-p) \hookrightarrow \zeta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1 \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1)$$

to see that $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1}, ||^t\mathrm{sgn}^t, ||^e\mathrm{sgn}^f)$ must restrict to $\mathrm{Lang}(||^{p+1}\mathrm{sgn}^p, ||^t\mathrm{sgn}^t, 1)$.

From (2.8), we see that

$$\mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1 \hookrightarrow ||^p\mathrm{sgn}^p \rtimes V_t \twoheadrightarrow \mathrm{Lang}(||^p\mathrm{sgn}^p, ||^t\mathrm{sgn}^{t+1}, 1)$$

. Similar to previous cases we may extend the direct sum of the middle term in the composition series to $\mathrm{GSp}(4,\mathbb{R})$, and then extend the first term in a consistent manner. This then allows us to extend the quotient which must be irreducible by considering the weights of its summands as $\mathrm{Sp}(4,\mathbb{R})$ representations. It then follows that the restriction of $\mathrm{Lang}(||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes ||^e\mathrm{sgn}^f$ to $\mathrm{Sp}(4,\mathbb{R})$ is $\mathrm{Lang}(||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1$.

Case of $||^p\mathbf{sgn}^{p+1} \times ||^t\mathbf{sgn}^{t+1} \rtimes ||^e\mathbf{sgn}^f$ with $p > t > 0$

From (2.16), we have that $||^p\mathrm{sgn}^{p+1} \rtimes V_t = \mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes 1)$. We can also see from (2.16) that $||^p\mathrm{sgn}^p \rtimes X(t,+) \oplus ||^p\mathrm{sgn}^p \rtimes X(t,-) \hookrightarrow ||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes 1 \twoheadrightarrow ||^p\mathrm{sgn}^{p+1} \rtimes V_t$. Then we may see that the restriction of $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes ||^e\mathrm{sgn}^f)$ to $\mathrm{Sp}(4,\mathbb{R})$ is $\mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes 1)$.

Case of $||^p\mathbf{sgn}^p \times ||^p\mathbf{sgn}^p \rtimes ||^e\mathbf{sgn}^f$

We may use Lemma 7.1 from [3] which states that $||^p\text{sgn}^p \times ||^p\text{sgn}^p \rtimes 1$ is irreducible with these conditions in combination with the fact that $||^p\text{sgn}^p \times ||^p\text{sgn}^p \rtimes ||^e\text{sgn}^f$ restricts to $||^p\text{sgn}^p \times ||^p\text{sgn}^p \rtimes 1$ when we restrict action to $\text{Sp}(4, \mathbb{R})$ to conclude $||^p\text{sgn}^p \times ||^p\text{sgn}^p \rtimes ||^e\text{sgn}^f$ is irreducible. Then $\text{Lang}(||^p\text{sgn}^p, ||^p\text{sgn}^p, ||^e\text{sgn}^f)$ must restrict to $\text{Lang}(||^p\text{sgn}^p, ||^p\text{sgn}^p, 1)$.

Case of $||^p\mathbf{sgn}^p \times ||^p\mathbf{sgn}^{p+1} \rtimes ||^e\mathbf{sgn}^f$**,** $p > 0$

From (2.24), we have that $||^p\text{sgn}^p \rtimes V_p = \text{Lang}(||^p\text{sgn}^p \times ||^p\text{sgn}^{p+1} \rtimes 1)$. Then we can also see from (2.24) that $||^p\text{sgn}^p \rtimes X(p, +) + ||^p\text{sgn}^p \rtimes X(p, -) \hookrightarrow ||^p\text{sgn}^p \times ||^p\text{sgn}^{p+1} \rtimes 1 \twoheadrightarrow ||^p\text{sgn}^p \rtimes V_p$. Then we may see that the restriction of $\text{Lang}(||^p\text{sgn}^p \times ||^p\text{sgn}^{p+1} \rtimes ||^e\text{sgn}^f)$ to $\text{Sp}(4, \mathbb{R})$ is $\text{Lang}(||^p\text{sgn}^p \times ||^p\text{sgn}^{p+1} \rtimes 1)$.

Case of $||^p\mathbf{sgn}^{p+1} \times ||^p\mathbf{sgn}^{p+1} \rtimes ||^e\mathbf{sgn}^f$**,** $p > 0$

From [3] Lemma 7.5, we have that $||^p\text{sgn}^{p+1} \rtimes V_p = \text{Lang}(||^p\text{sgn}^{p+1} \times ||^p\text{sgn}^{p+1} \rtimes 1)$. Then by (2.27), we can see that $||^p\text{sgn}^{p+1} \rtimes X(p, +) \oplus ||^p\text{sgn}^{p+1} \rtimes X(p, -) \hookrightarrow ||^p\text{sgn}^{p+1} \times ||^p\text{sgn}^{p+1} \rtimes 1 \twoheadrightarrow ||^p\text{sgn}^{p+1} \rtimes V_p$. Then we may see that the restriction of $\text{Lang}(||^p\text{sgn}^{p+1} \times ||^p\text{sgn}^{p+1} \rtimes ||^e\text{sgn}^f)$ to $\text{Sp}(4, \mathbb{R})$ is $\text{Lang}(||^p\text{sgn}^{p+1} \times ||^p\text{sgn}^{p+1} \rtimes 1)$.

Case of $||^p\mathbf{sgn}^p \times 1 \rtimes ||^e\mathbf{sgn}^f$**,** $p > 0$

We may use Lemma 7.1 from [3] which states that $||^p\text{sgn}^p \times 1 \rtimes 1$ is irreducible with these conditions in combination with the fact that $||^p\text{sgn}^p \times 1 \rtimes ||^e\text{sgn}^f$ restricts to $||^p\text{sgn}^p \times 1 \rtimes 1$ when we restrict action to $\text{Sp}(4, \mathbb{R})$ to conclude $||^p\text{sgn}^p \times 1 \rtimes ||^e\text{sgn}^f$ is irreducible. Then $\text{Lang}(||^p\text{sgn}^p, 1, ||^e\text{sgn}^f)$ must restrict to $\text{Lang}(||^p\text{sgn}^p, 1, 1)$.

Finally, there is a remaining case that is resolved here.

Case of $\delta(||^{\frac{p+t}{2}}\mathbf{sgn}^t, p-t) \rtimes \sigma$, $p > t > 0$

From (2.13) and (2.14), we have two exact sequences:

$$W \hookrightarrow \delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1 \twoheadrightarrow \mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t), 1)$$

and

$$\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1 \hookrightarrow W \twoheadrightarrow \mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,-).$$

Now that we know both $\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1$ and

$$\mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,+)) \oplus \mathrm{Lang}(||^t\mathrm{sgn}^t \rtimes X(p,-)$$

are restrictions of $\mathrm{GSp}(4,\mathbb{R})$ representations, W is as well. Then we may conclude $\mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t), 1)$ is the restriction to $\mathrm{Sp}(4,\mathbb{R})$ of a quotient of $\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes \sigma$ on $\mathrm{GSp}(4,\mathbb{R})$. Such a quotient must then be irreducible as a $\mathrm{GSp}(4,\mathbb{R})$, and so it must be that $\mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t), \sigma)$, when restricted to $\mathrm{Sp}(4,\mathbb{R})$, restricts to $\mathrm{Lang}(\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t), 1)$.

Chapter 6

Gelfand-Kirillov dimension

We now consider the Gelfand-Kirillov dimension of irreducible admissible representations of $\mathrm{Sp}(4, \mathbb{R})$. These results will be collected in the tables in Appendix A.

6.1 Definitions

Here we deal with representations as $(\mathfrak{g}, K)$ modules, with $\mathfrak{g} = \mathrm{sp}(4, \mathbb{C})$, as we wish to work with the complexification of the Lie algebra $\mathrm{sp}(4, \mathbb{R})$. For a Lie algebra representation of $\mathfrak{g}$, we view the representation as a finitely generated $U(\mathfrak{g})$ module V, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. We then use the basis for $\mathfrak{g}$ of

$$\{Z, Z', P_{0+}, P_{0-}, P_{1+}, P_{1-}, X_+, X_-, N_+, N_-\} \tag{6.1}$$

as defined in (1.5). By the Poincaré–Birkhoff–Witt theorem, we may then take a basis for $U(\mathfrak{g})$ consisting of elements of the form

$$Z^{\alpha_1} Z'^{\alpha_2} P_{0+}^{\alpha_3} P_{0-}^{\alpha_4} P_{1+}^{\alpha_5} P_{1-}^{\alpha_6} X_+^{\alpha_7} X_-^{\alpha_8} N_+^{\alpha_9} N_-^{\alpha_{10}} \tag{6.2}$$

where each $\alpha_i \in \mathbb{Z}_{\geq 0}$. Define $U_n(\mathfrak{g})$ as the subspace generated by those basis elements with degree at most n. Then we take a generating subspace V_0 of V and make V a graded module with $V_n = U_n(\mathfrak{g})V_0$. Then there exists a polynomial $d(n)$ with degree at most $\dim\mathfrak{g}$ with $d(n) = \dim V_n$ for large enough n by [7] Lemma 2.1. The degree d of this polynomial is the Gelfand-Kirillov dimension of the representation V. For the representation (π, V), we shall define $\mathrm{Dim}(\pi) = d$.

Lemma 6.1. *Let (π, V) be an irreducible admissible representation of* $\mathrm{Sp}(4, \mathbb{R})$*. If $v \in V$ is in a K-type $V_{(k,\ell)}$, then $U_1(\mathfrak{g})v$ contains only elements from K-types $V_{(k',\ell')}$ where $k+\ell-2 \leq k'+\ell' \leq k+\ell+2$ and $k-\ell-2 \leq k'-\ell' \leq k-\ell+2$. Further, for $n \in \mathbb{Z}_{>0}$, $U_n(\mathfrak{g})v$ contains only elements of K-types of the form $V_{(k',\ell')}$ where $k+\ell-2n \leq k'+\ell' \leq k+\ell+2n$ and $k-\ell-2n \leq k'-\ell' \leq k-\ell+2n$.*

Proof. First we assume that v is a highest weight vector in its K-type, with weight (k, ℓ). Note that Zv, $Z'v$,N_+v,N_-v are all either 0 or in the same K-type.

We then consider that X_+v has weight $(k+2, \ell)$, and $N_+X_+v = X_+N_+v = 0$, so that X_+v has highest weight and must belong to $V_{(k+2,\ell)}$. Then note $P_{1+}v$ has weight $(k+1, \ell+1)$, and $N_+P_{1+}v = P_{1+}N_+v + 2X_+v = 2X_+v$, so that $P_{1+}v$ is a sum of highest weight vectors from $V_{(k+1,\ell+1)}$ and vectors from $V_{(k+2,\ell)}$. Now we examine $P_{0+}v$, which has weight $(k, \ell+2)$, and $N_+P_{0+}v = P_{0+}N_+v + P_{1+}v = 2P_{1+}v$, so that $P_{0+}v$ is a sum of highest weight vectors from $V_{(k,\ell+2)}$ and vectors from $V_{(k+1,\ell+1)}$ and $V_{(k+2,\ell)}$.

Continuing, note that $P_{0-}v$ has weight $(k, \ell-2)$, and $N_+P_{0-}v = P_{0-}N_+v = 0$, so that $P_{0-}v$ has highest weight and must belong to $V_{(k,\ell-2)}$. Then note $P_{1-}v$ has weight $(k-1, \ell-1)$, and $N_+P_{1-}v = P_{1-}N_+v - 2P_{0-}v = -2P_{0-}v$, so that $P_{1-}v$ is a sum of highest weight vectors from $V_{(k-1,\ell-1)}$ and vectors from $V_{(k,\ell-2)}$. Now we examine X_-v, which has weight $(k-2, \ell)$, and $N_+X_-v = X_-N_+v - P_{1-}v = -P_{1-}v$, so that $X_- v$ is a sum of highest weight vectors from $V_{(k-2,\ell)}$ and vectors from $V_{(k-1,\ell-1)}$ and $V_{(k,\ell-2)}$.

Now we shall consider the case where v is not a highest weight vector in its K-type, using induction. We have completed the base case, so suppose that for v such that $v = N_-^n v'$ for v' a highest weight vector, we have that $U_1(\mathfrak{g})v$ contains elements from K-types $V_{(k',\ell')}$ where $k+\ell-2 \leq k'+\ell' \leq k+\ell+2$ and $k-\ell-2 \leq k'-\ell' \leq k'-\ell+2$. If v is such that $v = N_-^{n+1}v'$ for v' a highest weight vector, then $v = N_- N_-^n v'$. For any $X \in U_1(\mathfrak{g})$,$[X, N_-] = Y \in U_1(\mathfrak{g})$. But then $Xv = XN_-N_-^n v' = N_- X N_-^n v' + Y N_-^n v'$. Then by hypothesis, $XN_-^n v'$ and $YN_-^n v'$ have K-types in the desired region, and so too will $N_- X N_-^n v'$, so that it follows Xv will also.

This proves that $U_1(\mathfrak{g})v$ contains elements from K-types $V_{(k',\ell')}$ where $k+\ell-2 \leq k'+\ell' \leq k+\ell+2$ and $k-\ell-2 \leq k'-\ell' \leq k-\ell+2$.

By induction, as $U_{n+1}(\mathfrak{g})v = U_1(\mathfrak{g})U_n(\mathfrak{g})v$, we conclude that $U_n(\mathfrak{g})v$ is in a K-type $V_{(k',\ell')}$ where $k+\ell-2n \leq k'+\ell' \leq k+\ell+2n$ and $k-\ell-2n \leq k'-\ell' \leq k-\ell+2n$. □

Proposition 6.2. *Let* (π, V) *be an irreducible admissible representation of* $\mathrm{Sp}(4,\mathbb{R})$. *If the multiplicity of* K*-types contained in* π *is bounded, then Dim*$(\pi) \leq 3$.

Proof. Choose some $v \in V$. Then $\mathbb{C}v = V_0$ is a generating subspace for V. By hypothesis, the multiplicity of any K-type is bounded by some integer N. Also, v is contained in a K-type $V_{(k,\ell)}$, and we shall let M be the maximum of $|k|$ and $|\ell|$. Then using Lemma 6.1, we conclude that $U_n(\mathfrak{g})v$ can only contain elements of K-types $V_{(k',\ell')}$ where $-M-2n \leq k' \leq M+2n$ and $-M-2n \leq \ell' \leq M+2n$. Each such K-type has a multiplicity of at most N, and contains elements with at most $2M + 4n + 1$ distinct weights. Using these facts to obtain an upper bound, we find $\dim U_n(\mathfrak{g})v \leq N(2M + 4n + 1)(2M + 4n + 1)^2$, which is of degree three in n, so that $\mathrm{Dim}(\pi) \leq 3$.

□

6.2 Gelfand-Kirillov dimensions for lowest weight modules

Now let us consider the irreducible representations that are realizable as lowest weight modules. These cases include the holomorphic discrete series, limits of holomorphic discrete series. Also, by examining results on K-types from Chapter 4, $\mathrm{Lang}(||^t \mathrm{sgn}^t \rtimes X(p, \pm))$ for $p > t > 0$ is lowest weight by (4.4) and (4.5), $\mathrm{Lang}(||^p \mathrm{sgn}^p \rtimes X(p, \pm))$ for $p > 0$ is lowest weight by (4.6) and (4.7), $\mathrm{Lang}(||^p \mathrm{sgn}^p \rtimes X(0, \pm))$ for $p > 0$ is lowest weight by (4.8) and (4.9), and $\mathrm{Lang}(||^p \mathrm{sgn}^p \rtimes X(t, \pm))$ for $p > t > 0$ is lowest weight by (4.10) and (4.11). In all of these cases we can determine the Gelfand-Kirillov dimension by using Lemma 2.3 from [7]. Each of these representations is can be realized in the form $X = \mathbb{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{b}} V$ with V the lowest weight K-type, and $\mathfrak{b}$ in our case is the subspace with basis

$$\{Z, Z', P_{0-}, P_{1-}, X_-, N_+, N_-\} . \tag{6.3}$$

Then $\mathrm{Dim}X = \mathrm{Dim}V + \dim \mathfrak{g}/\mathfrak{b} = 0 + 3$. Each of these representations therefore has Gelfand-Kirillov dimension $d = 3$.

6.3 Gelfand-Kirillov dimensions for large representations

Next, we will consider the large representations, as by Vogan [7] Theorem 6.2, the large irreducible representations are precisely those irreducible representations with Gelfand-Kirillov dimension 4 and they are, up to infinitesimal equivalence, those representations that are a subrepresentation of the Borel induced representation $||^{s_1}\text{sgn}^{\epsilon_1} \times ||^{s_2}\text{sgn}^{\epsilon_2} \rtimes 1$ with $Re(s_1) \geq Re(s_2) \geq 0$.

Breaking down into subcases, we first consider the case when at least one of s_1, s_2 is non-integer. Then either $||^{s_1}\text{sgn}^{\epsilon_1} \times ||^{s_2}\text{sgn}^{\epsilon_2} \rtimes 1$ is irreducible, in which case it is a large irreducible representation, or it reduces. If it reduces, there are four cases to address.

- In the case $s_2 \in \mathbb{Z}$ and $\epsilon_2 \equiv s_2 + 1 \pmod 2$, then we see by (2.1) that $||^{s_1}\text{sgn}^{\epsilon_1} \rtimes X(s_2, +)$ and $||^{s_1}\text{sgn}^{\epsilon_1} \rtimes X(s_2, -)$ are irreducible subrepresentations.

- If $s_2 \notin \mathbb{Z}$, $s_1 \in \mathbb{Z}$ and $\epsilon_1 \equiv s_1 + 1 \pmod 2$, then we see by (2.1) and (2.3) that $||^{s_1}\text{sgn}^{\epsilon_1} \rtimes X(s_2, +)$ and $||^{s_1}\text{sgn}^{\epsilon_1} \rtimes X(s_2, -)$ are irreducible subrepresentations.

- If $s_1 - s_2 = k \in \mathbb{Z}_{\neq 0}$ and $\epsilon_1 - \epsilon_2 \equiv k + 1 \pmod 2$, then we see by (2.4) that $\delta(||^{\frac{s_1+s_2}{2}}\text{sgn}^{\epsilon_2}, s_1 - s_2) \rtimes 1$ is a irreducible subrepresentation of $||^{s_1}\text{sgn}^{\epsilon_1} \times ||^{s_2}\text{sgn}^{\epsilon_2} \rtimes 1$.

- If $s_1 + s_2 = k \in \mathbb{Z}_{\neq 0}$ and $\epsilon_1 + \epsilon_2 \equiv k + 1 \pmod 2$, then we see by (2.4) and (2.5) that $\delta(||^{\frac{s_1+s_2}{2}}\text{sgn}^{\epsilon_2}, s_1 - s_2) \rtimes 1$ is a irreducible subrepresentation.

In the situation where $p, t \in \mathbb{Z}$, there are several subcases to examine, for which the composition series will be helpful. We will handle the case of $p = t = 0$ later. Until then, we assume $p \geq t \geq 0$ and $p + t > 0$.

- First, note that $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^t \rtimes 1$ is irreducible by section 2.2.

- Next, consider $||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^{t+1} \rtimes 1$. This then has subrepresentations $||^p\mathrm{sgn}^{p+1} \rtimes X(t,+)$ and $||^p\mathrm{sgn}^{p+1} \rtimes X(t,-)$ by (2.16). Then, if $p > t > 0$, we see that that $||^p\mathrm{sgn}^{p+1} \rtimes X(t,+)$ has an irreducible subrepresentation $||^t\mathrm{sgn}^{t+1} \rtimes X(p,+) = \mathrm{Lang}(||^t\mathrm{sgn}^{t+1} \rtimes X(p,+))$ by (2.17). Similarly, $||^p\mathrm{sgn}^{p+1} \rtimes X(t,-)$ has an irreducible subrepresentation $||^t\mathrm{sgn}^{t+1} \rtimes X(p,-) = \mathrm{Lang}(||^t\mathrm{sgn}^{t+1} \rtimes X(p,-))$ by (2.18). Now consider the degenerate cases, beginning with $p = t > 0$. In this case, we have that $||^p\mathrm{sgn}^{p+1} \rtimes X(p,+) = \mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(p,+))$ and $||^p\mathrm{sgn}^{p+1} \rtimes X(p,-) = \mathrm{Lang}(||^p\mathrm{sgn}^{p+1} \rtimes X(p,-))$ are irreducible subrepresentations from (2.31). Finally, when $p > t = 0$, we have by (2.36), (2.37) and (2.38) that $\mathrm{sgn} \rtimes X(p,+)$ and $\mathrm{sgn} \rtimes X(p,-)$ are irreducible subrepresentations of $||^p\mathrm{sgn}^{p+1} \times \mathrm{sgn}^1 \rtimes 1$ and are therefore large.

- Then, consider $||^p\mathrm{sgn}^{p+1} \times ||^t\mathrm{sgn}^t \rtimes 1$. When $p > t$ this has the subrepresentation $\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1$ by (2.19). Then for the case $p > t > 0$, we have that $\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1$ is a subrepresentation of $\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1$ by (2.20) and (2.21). Further, the large discrete series $X(p,-t)$ and $X(t,-p)$ are subrepresentations of $\delta(||^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1$ by (2.22). Next, we consider the degenerate case $p > t = 0$. Then the limits of large discrete series $X^2(p,0)$ and $X^1(0,-p)$ are subrepresentations of $\delta(||^{\frac{p}{2}}, p) \rtimes 1$ by (2.39) and (2.40). Then there is the degenerate case $p = t > 0$, which by (2.29) and (2.30) gives us that $X^1(p,-p)$ and $X^2(p,-p)$ are large irreducible representations.

- Finally, consider $||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1$. This then has subrepresentations $||^p\mathrm{sgn}^p \rtimes X(t,+)$, $||^p\mathrm{sgn}^p \rtimes X(t,-)$ and $\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1$. As $\delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t+1}, p-t) \rtimes 1 \simeq \delta(||^{\frac{p+t}{2}}\mathrm{sgn}^{t}, p-t) \rtimes 1$, we need not examine it further. For the case $p > t > 0$ we have by (2.9),(2.10),(2.11), and (2.12) that $||^p\mathrm{sgn}^p \rtimes X(t,\pm)$ contain the large discrete series $X(p,-t)$ and $X(t,-p)$ as subrepresentations. The first degenerate case is where $p > t = 0$ in which case by (2.32) and (2.33), $\delta(||^{\frac{p}{2}}\mathrm{sgn}, p) \rtimes 1$ contains $X^2(p,0)$ and $X^1(0,-p)$ as subrepresentations, so they are large. In the second degenerate case, with $p = t > 0$, we see that $X^1(p,-p)$ is an irreducible subrepresentation of $||^p\mathrm{sgn}^p \rtimes X(p,+)$, and $X^2(p,-p)$ is an irreducible subrepresentation of $||^p\mathrm{sgn}^p \rtimes X(p,-)$ by (2.28), (2.29), and (2.30) so they are large representations.

6.4 Gelfand-Kirillov dimensions in the remaining cases

Now we move on to the remaining non-large representations. First, we have the finite representation, $\mathrm{Lang}(||^p\mathrm{sgn}^p \times ||^t\mathrm{sgn}^{t+1} \rtimes 1)$. As this representation has finite dimension, $U_n(\mathfrak{g})v$ will have constant dimension for sufficiently large n, so that it has a Gelfand-Kirillov dimension of 0.

Finally, there are those representations that contain a wedge of K-types in the sense that they contain an element v and a subspace consisting of all elements of the form $N_-^\alpha X_+^\beta P_{0-}^\gamma v$ with $\alpha, \beta, \gamma \in \mathbb{Z}_{>0}$. Then we may find a lower bound for the Gelfand-Kirillov dimension by taking a subspace of $U_n(\mathfrak{g})v$ generated as a vector space by elements of the form $N_-^\alpha X_+^\beta P_{0-}^\gamma v$ with $\alpha + \beta + \gamma \leq n$.

For the purposes of a lower bound, we may assume v be an element of the K-type $V_{(0,0)}$. Then each $X_+^{\beta} P_{0-}^{\gamma} v$ is a highest weight vector of $V_{(2\beta,-2\gamma)}$, and then each $N_-^{\alpha} X_+^{\beta} P_{0-}^{\gamma} v$ is a distinct nonzero vector as long as $\alpha \leq 2\beta + 2\gamma$. Then when $\alpha + \beta + \gamma = n$, letting $m = \left\lfloor \frac{n}{3} \right\rfloor$ we have at least $\sum_{i=m+1}^{n} (i+1) = \frac{(n+1)(n+2)}{2} - \frac{(m+2)(m+3)}{2} \geq \frac{(n+1)(n+2)}{2} - \frac{(\frac{n}{2}+2)(\frac{n}{2}+3)}{2} = \frac{3n^2+2n-16}{8}$ elements. This gives a lower bound on the dimension of $U_n(\mathfrak{g})v$ of $\sum_{i=1}^{n} \frac{3i^2+2i-16}{8} = \frac{3n(n+1)(2n+1)+3n(n+1)-96n}{144}$ so that the Gelfand-Kirillov dimension is at least 3. This holds true for any representation with a similar wedge of K-types. For such representations that are not large representations, we also know that the Gelfand-Kirillov dimension must be strictly less than 4 by Vogan [7] Theorem 6.2 as that is an equivalent condition to being a large representation. Then all such representations must have Gelfand-Kirillov dimension $d = 3$.

Appendix A

Constituents of induced representations

The following tables give all irreducible constituents of each induced representation and their Gelfand-Kirillov dimension. This is derived from the composition series presented in chapter 2 and the work in chapter 6 on Gelfand-Kirillov dimension.

Table A.1: Klingen induced from finite

Representation	Irreducible constituents	Dim
	$s \notin \mathbb{Z}$	
$\lvert\rvert^s \operatorname{sgn}^\epsilon \rtimes V_p$	Irreducible	3
	$p, t \in \mathbb{Z}, p > t > 0$	
$\lvert\rvert^p \operatorname{sgn}^p \rtimes V_t$	$\mathrm{Lang}(\delta(\lvert\rvert^{\frac{p+t}{2}} \operatorname{sgn}^{t+1}, p-t) \rtimes 1)$ $\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^p \times \lvert\rvert^t \operatorname{sgn}^{t+1} \rtimes 1)$	3 0
$\lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes V_t$	$\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \times \lvert\rvert^t \operatorname{sgn}^{t+1} \rtimes 1)$	3
$\lvert\rvert^t \operatorname{sgn}^t \rtimes V_p$	$\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \times \lvert\rvert^t \operatorname{sgn}^t \rtimes 1)$ $\mathrm{Lang}(\delta(\lvert\rvert^{\frac{p-t}{2}} \operatorname{sgn}^t, p+t) \rtimes 1)$ $\mathrm{Lang}(\delta(\lvert\rvert^{\frac{p+t}{2}} \operatorname{sgn}^t, p-t) \rtimes 1)$	3 3 3
$\lvert\rvert^t \operatorname{sgn}^{t+1} \rtimes V_p$	$\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes X(t,+))$ $\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes X(t,-))$ $\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \times \lvert\rvert^t \operatorname{sgn}^{t+1} \rtimes 1)$	3 3 3
	$p, t \in \mathbb{Z}, p = t > 0$	
$\lvert\rvert^p \operatorname{sgn}^p \rtimes V_p$	$\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^p \times \lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes 1)$	3
$\lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes V_p$	$\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \times \lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes 1)$	3
	$p, t \in \mathbb{Z}, p > t = 0$	
$\operatorname{sgn} \rtimes V_p$	$\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes X(0,+))$ $\mathrm{Lang}(\lvert\rvert^p \operatorname{sgn}^{p+1} \rtimes X(0,-))$	3 3
$1 \rtimes V_p$	$\mathrm{Lang}(\delta(\lvert\rvert^{\frac{p}{2}} \operatorname{sgn}^p, p) \rtimes 1)$	3

Table A.2: Klingen induced from discrete series and limits of discrete series

Representation	Irreducible constituents	Dim
$s \notin \mathbb{Z}$		
$\|\|^s \operatorname{sgn}^\epsilon \rtimes X(p, \pm)$	Irreducible	4
$p, t \in \mathbb{Z}, p > t > 0$		
$\|\|^t \operatorname{sgn}^t \rtimes X(p, +)$	$X(p, -t)$	4
	$X(p, t)$	3
	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^t \rtimes X(p, +))$	3
$\|\|^t \operatorname{sgn}^t \rtimes X(p, -)$	$X(t, -p)$	4
	$X(t, p)$	3
	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^t \rtimes X(p, -))$	3
$\|\|^t \operatorname{sgn}^{t+1} \rtimes X(p, \pm)$	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^{t+1} \rtimes X(p, \pm))$	4
$\|\|^p \operatorname{sgn}^p \rtimes X(t, +)$	$X(p, -t)$	4
	$\operatorname{Lang}(\delta(\|\|^{\frac{p-t}{2}} \operatorname{sgn}^t, p+t) \rtimes 1)$	3
	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^t \rtimes X(p, +)$	3
	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^t \rtimes X(t, +)$	3
$\|\|^p \operatorname{sgn}^p \rtimes X(t, -)$	$X(t, -p)$	4
	$\operatorname{Lang}(\delta(\|\|^{\frac{p-t}{2}} \operatorname{sgn}^t, p+t) \rtimes 1)$	3
	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^t \rtimes X(p, -)$	3
	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^t \rtimes X(t, -)$	3
$\|\|^p \operatorname{sgn}^{p+1} \rtimes X(t, \pm)$	$\operatorname{Lang}(\|\|^t \operatorname{sgn}^{t+1} \rtimes X(p, \pm))$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \rtimes X(t, \pm))$	3
$p, t \in \mathbb{Z}, p > t = 0$		
$\|\|^p \operatorname{sgn}^p \rtimes X(p, +)$	$X^1(p, -p)$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(p, +))$	3
$\|\|^p \operatorname{sgn}^p \rtimes X(p, -)$	$X^2(p, -p)$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(p, -))$	3
$\|\|^p \operatorname{sgn}^{p+1} \rtimes X(p, \pm)$	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \rtimes X(p, \pm))$	4

Table A.3: Klingen induced from discrete series and limits of discrete series, continued

Representation	Irreducible constituents	Dim
$p, t \in \mathbb{Z}, p = t > 0$		
$\|\|^p \operatorname{sgn}^p \rtimes X(0,+)$	$X^2(p,0)$	4
	$\operatorname{Lang}(\delta(\|\|^{\frac{p}{2}}, p) \rtimes 1)$	3
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(0,+))$	3
$\|\|^p \operatorname{sgn}^p \rtimes X(0,-)$	$X^1(0,-p)$	4
	$\operatorname{Lang}(\delta(\|\|^{\frac{p}{2}}, p) \rtimes 1)$	3
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(0,-))$	3
$\|\|^p \operatorname{sgn}^{p+1} \rtimes X(0,\pm)$	$\operatorname{sgn} \rtimes X(p,\pm)$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \rtimes X(0,\pm))$	3
$\operatorname{sgn} \rtimes X(p,\pm)$	Irreducible	4
$1 \rtimes X(p,+)$	$X^2(p,0)$	4
	$X^1(p,0)$	3
$1 \rtimes X(p,-)$	$X^1(0,-p)$	4
	$X^2(0,-p)$	3
$1 \rtimes X(0,\pm)$	Irreducible	4

Table A.4: Siegel induced

Representation	Irreducible constituents	Dim
	$s \notin \mathbb{Z}$	
$\delta(\|\|^{s}, k) \rtimes 1$	Irreducible	4
	$p, t \in \mathbb{Z}, p > t > 0$	
$\delta(\|\|^{\frac{p-t}{2}}, p+t) \rtimes 1$	$X(p, -t)$	4
	$X(t, -p)$	4
	$\mathrm{Lang}(\delta(\|\|^{\frac{p-t}{2}}, p+t) \rtimes 1)$	3
$\delta(\|\|^{\frac{p+t}{2}}, p-t) \rtimes 1$	$X(p, -t)$	4
	$X(t, -p)$	4
	$\mathrm{Lang}(\delta(\|\|^{\frac{p-t}{2}}, p+t) \rtimes 1)$	3
	$\mathrm{Lang}(\|\|^{t}\,\mathrm{sgn}^{t} \rtimes X(p, +))$	3
	$\mathrm{Lang}(\|\|^{t}\,\mathrm{sgn}^{t} \rtimes X(p, -))$	3
	$\mathrm{Lang}(\delta(\|\|^{\frac{p+t}{2}}, p-t) \rtimes 1)$	3
	$p, t \in \mathbb{Z}, p = t > 0$	
$\delta(1, 2p) \rtimes 1$	$X^{1}(p, -p)$	4
	$X^{2}(p, -p)$	4
	$p, t \in \mathbb{Z}, p > t = 0$	
$\delta(\|\|^{\frac{p}{2}}, p) \rtimes 1$	$X^{2}(p, 0)$	4
	$X^{1}(0, -p)$	4
	$\mathrm{Lang}(\delta(\|\|^{\frac{p}{2}}, p) \rtimes 1)$	3

Table A.5: Borel induced

Representation	Irreducible constituents	Dim
	$p, t \in \mathbb{Z}, p > t > 0$	
$\vert\vert^p \operatorname{sgn}^p \times \vert\vert^t \operatorname{sgn}^t \rtimes 1$	Irreducible	4
$\vert\vert^t \operatorname{sgn}^{t+1} \times \vert\vert^p \operatorname{sgn}^p \rtimes 1$	$X(p, -t)$	4
	$\mathrm{Lang}(\delta(\vert\vert^{\frac{p-t}{2}} \operatorname{sgn}^t, p+t) \rtimes 1)$	3
	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^t \rtimes X(p, +)$	3
	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^t \rtimes X(t, +)$	3
	$X(t, -p)$	4
	$\mathrm{Lang}(\delta(\vert\vert^{\frac{p-t}{2}} \operatorname{sgn}^t, p+t) \rtimes 1)$	3
	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^t \rtimes X(p, -)$	3
	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^t \rtimes X(t, -)$	3
	$\mathrm{Lang}(\delta(\vert\vert^{\frac{p+t}{2}} \operatorname{sgn}^{t+1}, p-t) \rtimes 1)$	3
	$\mathrm{Lang}(\vert\vert^p \operatorname{sgn}^p \times \vert\vert^t \operatorname{sgn}^{t+1} \rtimes 1)$	0
$\vert\vert^p \operatorname{sgn}^{p+1} \times \vert\vert^t \operatorname{sgn}^{t+1} \rtimes 1$	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^{t+1} \rtimes X(p, +))$	4
	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^{t+1} \rtimes X(p, -))$	4
	$\mathrm{Lang}(\vert\vert^p \operatorname{sgn}^{p+1} \rtimes X(t, +))$	3
	$\mathrm{Lang}(\vert\vert^p \operatorname{sgn}^{p+1} \rtimes X(t, -))$	3
	$\mathrm{Lang}(\vert\vert^p \operatorname{sgn}^{p+1} \times \vert\vert^t \operatorname{sgn}^{t+1} \rtimes 1)$	3
$\vert\vert^p \operatorname{sgn}^{p+1} \times \vert\vert^t \operatorname{sgn}^t \rtimes 1$	$X(p, -t)$	4
	$X(p, t)$	3
	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^t \rtimes X(p, +))$	3
	$X(t, -p)$	4
	$X(t, p)$	3
	$\mathrm{Lang}(\vert\vert^t \operatorname{sgn}^t \rtimes X(p, -))$	3
	$\mathrm{Lang}(\vert\vert^p \operatorname{sgn}^{p+1} \times \vert\vert^t \operatorname{sgn}^t \rtimes 1)$	3
	$\mathrm{Lang}(\delta(\vert\vert^{\frac{p-t}{2}} \operatorname{sgn}^t, p+t) \rtimes 1)$	3
	$\mathrm{Lang}(\delta(\vert\vert^{\frac{p+t}{2}} \operatorname{sgn}^t, p-t) \rtimes 1)$	3

Table A.6: Borel induced, continued

Representation	Irreducible constituents	Dim
$p, t \in \mathbb{Z}, p = t > 0$		
$\|\|^p \operatorname{sgn}^{p+1} \times \|\|^p \operatorname{sgn}^p \rtimes 1$	$X^1(p, -p)$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(p, +))$	3
	$X^2(p, -p)$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(p, -))$	3
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \times \|\|^p \operatorname{sgn}^{p+1} \rtimes 1)$	3
$\|\|^p \operatorname{sgn}^{p+1} \times \|\|^p \operatorname{sgn}^{p+1} \rtimes 1$	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \rtimes X(p, +))$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \rtimes X(p, -))$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \times \|\|^p \operatorname{sgn}^{p+1} \rtimes 1)$	3
$p, t \in \mathbb{Z}, p > t = 0$		
$\|\|^p \operatorname{sgn}^{p+1} \times \operatorname{sgn} \rtimes 1$	$\operatorname{sgn} \rtimes X(p, +)$	4
	$\operatorname{sgn} \rtimes X(p, -)$	4
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \rtimes X(0, +))$	3
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \rtimes X(0, -))$	3
$\|\|^p \operatorname{sgn}^{p+1} \times 1 \rtimes 1$	$X^2(p, 0)$	4
	$X^1(p, 0)$	3
	$X^1(0, -p)$	4
	$X^2(0, -p)$	3
	$\operatorname{Lang}(\delta(\|\|^{\frac{p}{2}} \operatorname{sgn}^p, p) \rtimes 1)$	3
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^{p+1} \times 1 \rtimes 1)$	3
$\operatorname{sgn} \times \|\|^p \operatorname{sgn}^p \rtimes 1$	$X^2(p, 0)$	4
	$\operatorname{Lang}(\delta(\|\|^{\frac{p}{2}}, p) \rtimes 1)$	3
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(0, +))$	3
	$X^1(0, -p)$	4
	$\operatorname{Lang}(\delta(\|\|^{\frac{p}{2}}, p) \rtimes 1)$	3
	$\operatorname{Lang}(\|\|^p \operatorname{sgn}^p \rtimes X(0, -))$	3

Appendix B

Restriction of representations

These tables contain the results of restricting Langlands quotients of $\mathrm{GSp}(4, \mathbb{R})$ to $\mathrm{Sp}(4, \mathbb{R})$, as determined by chapter 5.

Table B.1: Langlands quotients supported on the Klingen parabolic

$p, t \in \mathbb{Z}, p > t > 0$	
$\mathrm{Lang}(\|\|^t \mathrm{sgn}^t, \|\|^c \mathrm{sgn}^d D_p)$	$\mathrm{Lang}(\|\|^t \mathrm{sgn}^t, X(p,+)) \oplus \mathrm{Lang}(\|\|^t \mathrm{sgn}^t, X(p,-))$
$\mathrm{Lang}(\|\|^t \mathrm{sgn}^{t+1}, \|\|^c \mathrm{sgn}^d D_p)$	$\mathrm{Lang}(\|\|^t \mathrm{sgn}^{t+1}, X(p,+)) \oplus \mathrm{Lang}(\|\|^t \mathrm{sgn}^{t+1}, X(p,-))$
$\mathrm{Lang}(\|\|^p \mathrm{sgn}^p, \|\|^c \mathrm{sgn}^d D_t)$	$\mathrm{Lang}(\|\|^p \mathrm{sgn}^p, X(t,+)) \oplus \mathrm{Lang}(\|\|^p \mathrm{sgn}^p, X(t,-))$
$\mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, \|\|^c \mathrm{sgn}^d D_t)$	$\mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, X(t,+)) \oplus \mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, X(t,-))$
$p \in \mathbb{Z}, p > 0$	
$\mathrm{Lang}(\|\|^p \mathrm{sgn}^p, \|\|^c \mathrm{sgn}^d D_p)$	$\mathrm{Lang}(\|\|^p \mathrm{sgn}^p, X(p,+)) \oplus \mathrm{Lang}(\|\|^p \mathrm{sgn}^p, X(p,-))$
$\mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, \|\|^c \mathrm{sgn}^d D_p)$	$\mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, X(p,+)) \oplus \mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, X(p,-))$
$\mathrm{Lang}(\|\|^p \mathrm{sgn}^p, \|\|^c \mathrm{sgn}^d D_0)$	$\mathrm{Lang}(\|\|^p \mathrm{sgn}^p, X(0,+)) \oplus \mathrm{Lang}(\|\|^p \mathrm{sgn}^p, X(0,-))$
$\mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, \|\|^c \mathrm{sgn}^d D_0)$	$\mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, X(0,+)) \oplus \mathrm{Lang}(\|\|^p \mathrm{sgn}^{p+1}, X(0,-))$
$s \notin \mathbb{Z}, \mathbb{Z} \ni p \geq 0$	
$\mathrm{Lang}(\|\|^s \mathrm{sgn}^\epsilon, \|\|^c \mathrm{sgn}^d D_p)$	$\mathrm{Lang}(\|\|^s \mathrm{sgn}^\epsilon, X(p,+)) \oplus \mathrm{Lang}(\|\|^s \mathrm{sgn}^\epsilon, X(p,-))$

Table B.2: Langlands quotients supported on the Siegel parabolic

$p, t \in \mathbb{Z}, p > t > 0$	
$\mathrm{Lang}(\delta(\|\|^{\frac{p-t}{2}} \mathrm{sgn}^{\epsilon}, p+t), \sigma)$	$\mathrm{Lang}(\delta(\|\|^{\frac{p-t}{2}} \mathrm{sgn}^{\epsilon}, p+t), 1)$
$\mathrm{Lang}(\delta(\|\|^{\frac{p+t}{2}} \mathrm{sgn}^{\epsilon}, p-t), \sigma)$	$\mathrm{Lang}(\delta(\|\|^{\frac{p+t}{2}} \mathrm{sgn}^{\epsilon}, p-t), 1)$
$p \in \mathbb{Z}, p > 0$	
$\mathrm{Lang}(\delta(\|\|^{\frac{p}{2}} \mathrm{sgn}^{\epsilon}, p), \sigma)$	$\mathrm{Lang}(\delta(\|\|^{\frac{p}{2}} \mathrm{sgn}^{\epsilon}, p), 1)$
$s - \frac{k}{2} \notin \mathbb{Z}$	
$\mathrm{Lang}(\delta(\|\|^{s} \mathrm{sgn}^{\epsilon}, k), \sigma)$	$\mathrm{Lang}(\delta(\|\|^{s} \mathrm{sgn}^{\epsilon}, k), 1)$

Table B.3: Langlands quotients supported on the minimal parabolic

$p, t \in \mathbb{Z}, p > t > 0$	
$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p}, \|\|^{t} \mathrm{sgn}^{t}, \sigma)$	$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p}, \|\|^{t} \mathrm{sgn}^{t}, 1)$
$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p+1}, \|\|^{t} \mathrm{sgn}^{t}, \sigma)$	$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p+1}, \|\|^{t} \mathrm{sgn}^{t}, 1)$
$p \in \mathbb{Z}, p > 0$	
$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p}, \|\|^{p} \mathrm{sgn}^{p}, \sigma)$	$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p}, \|\|^{p} \mathrm{sgn}^{p}, 1)$
$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p}, \|\|^{p} \mathrm{sgn}^{p+1}, \sigma)$	$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p}, \|\|^{p} \mathrm{sgn}^{p+1}, 1)$
$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p+1}, \|\|^{p} \mathrm{sgn}^{p+1}, \sigma)$	$\mathrm{Lang}(\|\|^{p} \mathrm{sgn}^{p+1}, \|\|^{p} \mathrm{sgn}^{p+1}, 1)$
$a, b \in \mathbb{C} \setminus \mathbb{Z},\ Re(a) \geq Re(b) \geq 0$	
$\mathrm{Lang}(\|\|^{a} \mathrm{sgn}^{\epsilon_1}, \|\|^{b} \mathrm{sgn}^{\epsilon_2}, \sigma)$	$\mathrm{Lang}(\|\|^{a} \mathrm{sgn}^{\epsilon_1}, \|\|^{b} \mathrm{sgn}^{\epsilon_2}, 1)$

Appendix C

Decomposition tables

Here we collect tables with multiple decompositions of the Borel induced representations of $\mathrm{GSp}(4,\mathbb{R})$. The full induced is in the upper left with two different decompositions in its row and column. The remaining boxes contain the irreducible constituents of each larger constituent. This information derives from the composition series in chapter 2. Note that for these tables, sgn is abbreviated as simply s due to space constraints.

Table C.1: Irreducible decomposition, $p, t \in \mathbb{Z}_{>0}$

$\vert\vert^t \mathrm{s}^{t+1} \times \vert\vert^p \mathrm{s}^p \rtimes \sigma$	$\zeta(\vert\vert^{\frac{p+t}{2}} \mathrm{s}^{t+1}, p-t) \rtimes \sigma$	$\delta(\vert\vert^{\frac{p-t}{2}} \mathrm{s}^{t+1}, p+t) \rtimes \sigma$	W
$\vert\vert^t \mathrm{s}^t \rtimes D_p$	$X(p,t)$	$X(p,-t)$	$L(\vert\vert^t \mathrm{s}^t \rtimes D_p)$
	$L(\vert\vert^t \mathrm{s}^t \rtimes D_t)$	$L(\delta(\vert\vert^{\frac{p-t}{2}} \mathrm{s}^{t+1}, p+t) \rtimes \sigma)$	
$\vert\vert^p \mathrm{s}^p \rtimes V_t$	$L(\vert\vert^t \mathrm{s}^{t+1} \times \vert\vert^p \mathrm{s}^p \rtimes \sigma)$		$L(\delta(\vert\vert^{\frac{p+t}{2}} \mathrm{s}^{t+1}, p-t) \rtimes \sigma)$

Table C.2: Irreducible decomposition, $p \in \mathbb{Z}_{>0}$

$\|\|^p \mathrm{s}^{p+1} \times \|\|^p \mathrm{s}^p \rtimes \sigma$	$\zeta(\|\|^p \mathrm{s}^{p+1}, 0) \rtimes \sigma$	$\sigma(\mathrm{s}^{p+1}, 0) \rtimes \sigma$
$\|\|^p \mathrm{s}^p \rtimes D_p$	$L(\|\|^p \mathrm{s}^p \rtimes D_p)$	$X(p, -p)$
$\|\|^p \mathrm{s}^p \rtimes V_p$	$L(\|\|^p \mathrm{s}^{p+1} \times \|\|^p \mathrm{s}^p \rtimes \sigma)$	

Table C.3: Irreducible decomposition, $p \in \mathbb{Z}_{>0}$

$\mathrm{s} \times \|\|^p \mathrm{s}^p \rtimes \sigma$	$\zeta(\|\|^{\frac{p}{2}} \mathrm{s}, p) \rtimes \sigma$	$\delta(\|\|^{\frac{p}{2}} \mathrm{s}, p) \rtimes \sigma$
$\|\|^p \mathrm{s}^p \rtimes D_0$	$L(\|\|^p \mathrm{s}^p \rtimes D_0)$ $L(\delta(\|\|^{\frac{p}{2}} \mathrm{s}, p) \rtimes \sigma)$	$X^{large}(p, 0)$

Table C.4: Irreducible decomposition, $p, t \in \mathbb{Z}_{>0}$

$\|\|^p \mathrm{s}^{p+1} \times \|\|^t \mathrm{s}^t \rtimes \sigma$	$\zeta(\|\|^{\frac{p+t}{2}} \mathrm{s}^t, p-t) \rtimes \sigma$	$\delta(\|\|^{\frac{p-t}{2}} \mathrm{s}^t, p+t) \rtimes \sigma$	W
$\|\|^t \mathrm{s}^t \rtimes D_p$	$X(p, t)$	$X(p, -t)$	$L(\|\|^t \mathrm{s}^t \rtimes D_p)$
$\|\|^t \mathrm{s}^t \rtimes V_p$	$L(\|\|^p \mathrm{s}^{p+1} \times \|\|^t \mathrm{s}^t \rtimes \sigma)$	$L(\delta(\|\|^{\frac{p-t}{2}} \mathrm{s}^t, p+t) \rtimes \sigma)$	$L(\delta(\|\|^{\frac{p+t}{2}} \mathrm{s}^t, p-t) \rtimes \sigma)$

Table C.5: Irreducible decomposition, $p \in \mathbb{Z}_{>0}$

$\|\|^p \mathrm{s}^{p+1} \times \|\|^p \mathrm{s}^p \rtimes \sigma$	$\zeta(\|\|^p \mathrm{s}^p, 0) \rtimes \sigma$	$\sigma(\mathrm{s}^p, 0) \rtimes \sigma$
$\|\|^p \mathrm{s}^p \rtimes D_p$	$L(\|\|^p \mathrm{s}^p \rtimes D_p)$	$X(p, -p)$
$\|\|^p \mathrm{s}^p \rtimes V_p$	$L(\|\|^p \mathrm{s}^{p+1} \times \|\|^p \mathrm{s}^p \rtimes \sigma)$	

Table C.6: Irreducible decomposition, $p \in \mathbb{Z}_{>0}$

$\|\|^p \mathrm{s}^{p+1} \times 1 \rtimes \sigma$	$\zeta(\|\|^{\frac{p}{2}}, p) \rtimes \sigma$	$\delta(\|\|^{\frac{p}{2}}, p) \rtimes \sigma$
$1 \rtimes D_p$	$X^{hol}(p, 0)$	$X^{large}(p, 0)$
$1 \rtimes V_p$	$L(\|\|^p \mathrm{s}^{p+1} \times 1 \rtimes \sigma)$	$L(\delta(\|\|^{\frac{p}{2}} \mathrm{s}, p) \rtimes \sigma)$

Table C.7: Irreducible decomposition, $p, t \in \mathbb{Z}_{>0}$

$\|\|^p \mathrm{s}^{p+1} \times \|\|^t \mathrm{s}^{t+1} \rtimes \sigma$	$\|\|^p \mathrm{s}^{p+1} \rtimes D_t$	$\|\|^p \mathrm{s}^{p+1} \rtimes V_t$
$\|\|^t \mathrm{s}^{t+1} \rtimes V_p$	$L(\|\|^p \mathrm{s}^{p+1} \rtimes D_t)$	$L(\|\|^p \mathrm{s}^{p+1} \times \|\|^t \mathrm{s}^{t+1} \rtimes \sigma)$
$\|\|^t \mathrm{s}^{t+1} \rtimes D_p$	$L(\|\|^t \mathrm{s}^{t+1} \rtimes D_p)$	

Table C.8: Irreducible decomposition, $p \in \mathbb{Z}_{>0}$

$\|\|^p \mathrm{s}^{p+1} \times \|\|^p \mathrm{s}^{p+1} \rtimes \sigma$	$\|\|^p \mathrm{s}^{p+1} \rtimes D_p$	$\|\|^p \mathrm{s}^{p+1} \rtimes V_p$

Table C.9: Irreducible decomposition, $p \in \mathbb{Z}_{>0}$

$\|\|^p \mathrm{s}^{p+1} \times \mathrm{s} \rtimes \sigma$	$\|\|^p \mathrm{s}^{p+1} \rtimes D_0$
$\mathrm{s} \rtimes V_p$	$L(\|\|^p \mathrm{s}^{p+1} \rtimes D_0)$
$\mathrm{s} \rtimes D_p$	$L(\mathrm{s} \rtimes D_p)$

Appendix D

L- and ϵ-factors

The following tables give the L- and ϵ-factors of all irreducible $\mathrm{GSp}(4,\mathbb{R})$ representations. These are calculated in chapter 3.

Table D.1: Degree 4 L-factors

Representation	Irreducibles	$\mathrm{L}(s,\varphi)$	$\epsilon(s,\varphi,\psi)$
$\mathrm{L}(\|\|^a\mathrm{sgn}, \|\|^c\mathrm{sgn}, \|\|^e\mathrm{sgn})$	$\varphi_{-,a+c+e} \oplus \varphi_{+,a+e} \oplus \varphi_{+,c+e} \oplus \varphi_{-,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e+1)\Gamma_{\mathbb{R}}(s+a+e)$ $\Gamma_{\mathbb{R}}(s+c+e)\Gamma_{\mathbb{R}}(s+e+1)$	-1
$\mathrm{L}(\|\|^a\mathrm{sgn}, \|\|^c\mathrm{sgn}, \|\|^e)$	$\varphi_{+,a+c+e} \oplus \varphi_{-,a+e} \oplus \varphi_{-,c+e} \oplus \varphi_{+,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e)\Gamma_{\mathbb{R}}(s+a+e+1)$ $\Gamma_{\mathbb{R}}(s+c+e+1)\Gamma_{\mathbb{R}}(s+e)$	-1
$\mathrm{L}(\|\|^a\mathrm{sgn}, \|\|^c, \|\|^e\mathrm{sgn})$	$\varphi_{+,a+c+e} \oplus \varphi_{+,a+e} \oplus \varphi_{-,c+e} \oplus \varphi_{-,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e)\Gamma_{\mathbb{R}}(s+a+e)$ $\Gamma_{\mathbb{R}}(s+c+e+1)\Gamma_{\mathbb{R}}(s+e+1)$	-1
$\mathrm{L}(\|\|^a, \|\|^c\mathrm{sgn}, \|\|^e\mathrm{sgn})$	$\varphi_{+,a+c+e} \oplus \varphi_{-,a+e} \oplus \varphi_{+,c+e} \oplus \varphi_{-,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e)\Gamma_{\mathbb{R}}(s+a+e+1)$ $\Gamma_{\mathbb{R}}(s+c+e)\Gamma_{\mathbb{R}}(s+e+1)$	-1
$\mathrm{L}(\|\|^a\mathrm{sgn}, \|\|^c, \|\|^e)$	$\varphi_{-,a+c+e} \oplus \varphi_{-,a+e} \oplus \varphi_{+,c+e} \oplus \varphi_{+,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e+1)\Gamma_{\mathbb{R}}(s+e)$ $\Gamma_{\mathbb{R}}(s+a+e+1)\Gamma_{\mathbb{R}}(s+c+e)$	-1
$\mathrm{L}(\|\|^a, \|\|^c\mathrm{sgn}, \|\|^e)$	$\varphi_{-,a+c+e} \oplus \varphi_{+,a+e} \oplus \varphi_{-,c+e} \oplus \varphi_{+,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e+1)\Gamma_{\mathbb{R}}(s+a+e)$ $\Gamma_{\mathbb{R}}(s+c+e+1)\Gamma_{\mathbb{R}}(s+e)$	-1
$\mathrm{L}(\|\|^a, \|\|^c, \|\|^e\mathrm{sgn})$	$\varphi_{-,a+c+e} \oplus \varphi_{-,a+e} \oplus \varphi_{-,c+e} \oplus \varphi_{-,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e+1)$ $\Gamma_{\mathbb{R}}(s+a+e+1)$ $\Gamma_{\mathbb{R}}(s+c+e+1)\Gamma_{\mathbb{R}}(s+e+1)$	1
$\mathrm{L}(\|\|^a, \|\|^c, \|\|^e)$	$\varphi_{+,a+c+e} \oplus \varphi_{+,a+e} \oplus \varphi_{+,c+e} \oplus \varphi_{+,e}$	$\Gamma_{\mathbb{R}}(s+a+c+e)\Gamma_{\mathbb{R}}(s+a+e)$ $\Gamma_{\mathbb{R}}(s+c+e)\Gamma_{\mathbb{R}}(s+e)$	1

Table D.2: Degree 4 L-factors

Representation	Irreducibles	$\mathrm{L}(s,\varphi)$	$\epsilon(s,\varphi,\psi)$
$\mathrm{L}(\delta(\|\|^r, 2k+1), \|\|^a\mathrm{sgn})$	$\varphi_{-,a+2r} \oplus \varphi_{2k+1,a+r} \oplus \varphi_{-,a}$	$\Gamma_{\mathbb{R}}(s+a+2r+1)\Gamma_{\mathbb{R}}(s+a+1)$ $\Gamma_{\mathbb{C}}(s+a+r+k+\frac{1}{2})$	$(-1)^k$
$\mathrm{L}(\delta(\|\|^r, 2k), \|\|^a\mathrm{sgn})$	$\varphi_{+,a+2r} \oplus \varphi_{2k,a+r} \oplus \varphi_{-,a}$	$\Gamma_{\mathbb{R}}(s+a+2r)\Gamma_{\mathbb{R}}(s+a+1)$ $\Gamma_{\mathbb{C}}(s+a+r+k)$	$(-1)^{k+1}$
$\mathrm{L}(\delta(\|\|^r, 2k+1), \|\|^a)$	$\varphi_{+,a+2r} \oplus \varphi_{2k+1,a+r} \oplus \varphi_{+,a}$	$\Gamma_{\mathbb{R}}(s+a+2r)\Gamma_{\mathbb{R}}(s+a)$ $\Gamma_{\mathbb{C}}(s+a+r+k+\frac{1}{2})$	$(-1)^k$
$\mathrm{L}(\delta(\|\|^r, 2k), \|\|^a)$	$\varphi_{-,a+2r} \oplus \varphi_{2k,a+r} \oplus \varphi_{+,a}$	$\Gamma_{\mathbb{R}}(s+a+2r+1)\Gamma_{\mathbb{R}}(s+a)$ $\Gamma_{\mathbb{C}}(s+a+r+k)$	$(-1)^{k+1}$
$\mathrm{L}(\|\|^a\mathrm{sgn}^b \rtimes \|\|^c\mathrm{sgn}^d D_\ell)$	$\varphi_{\ell,a+c} \oplus \varphi_{\ell,a}$	$\Gamma_{\mathbb{C}}(s+a+c+\frac{\ell}{2})\Gamma_{\mathbb{C}}(s+a+\frac{\ell}{2})$	$(-1)^\ell$

Table D.3: Degree 4 L-factors

Representation	Irreducibles	$\mathrm{L}(s,\varphi)$	$\epsilon(s,\varphi,\psi)$
$X_{\lambda_1,\lambda_2}, \lambda_1 > \lambda_2 > 0$	$\varphi_{\lambda_1+\lambda_2,0} \oplus \varphi_{\lambda_1-\lambda_2,0}$	$\Gamma_{\mathbb{C}}(s+\frac{\lambda_1+\lambda_2}{2})\Gamma_{\mathbb{C}}(s+\frac{\lambda_1-\lambda_2}{2})$	$(-1)^{\lambda_1+1}$
$X_{\lambda_1,-\lambda_2}, \lambda_1 > \lambda_2 > 0$	$\varphi_{\lambda_1+\lambda_2,0} \oplus \varphi_{\lambda_1-\lambda_2,0}$	$\Gamma_{\mathbb{C}}(s+\frac{\lambda_1+\lambda_2}{2})\Gamma_{\mathbb{C}}(s+\frac{\lambda_1-\lambda_2}{2})$	$(-1)^{\lambda_1+1}$
$X_{p,0}^{large}, p > 0$	$\varphi_{p,0} \oplus \varphi_{p,0}$	$\Gamma_{\mathbb{C}}(s+\frac{p}{2})\Gamma_{\mathbb{C}}(s+\frac{p}{2})$	$(-1)^{p+1}$
$X_{p,0}^{hol}, p > 0$	$\varphi_{p,0} \oplus \varphi_{p,0}$	$\Gamma_{\mathbb{C}}(s+\frac{p}{2})\Gamma_{\mathbb{C}}(s+\frac{p}{2})$	$(-1)^{p+1}$
$X_{p,-p}^1, p > 0$	$\varphi_{-,0} \oplus \varphi_{2p,0} \oplus \varphi_{+,0}$	$\Gamma_{\mathbb{C}}(s+p)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$	$(-1)^{p+1}$
$X_{p,-p}^2, p > 0$	$\varphi_{-,0} \oplus \varphi_{2p,0} \oplus \varphi_{+,0}$	$\Gamma_{\mathbb{C}}(s+p)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$	$(-1)^{p+1}$

Table D.4: Degree 5 L-factors

Representation	Irreducibles	$L(s,\varphi)$	$\epsilon(s,\varphi,\psi)$
$L(\vert\vert^a\mathrm{sgn}, \vert\vert^c\mathrm{sgn}, \vert\vert^e\mathrm{sgn})$	$\varphi_{-,a}\oplus\varphi_{-,c}\oplus\varphi_{+,0}\oplus\varphi_{-,-c}\oplus\varphi_{-,-a}$	$\Gamma_{\mathbb{R}}(s-\frac{a}{2}+1)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-\frac{c}{2}+1)\Gamma_{\mathbb{R}}(s+a+1)\Gamma_{\mathbb{R}}(s+c+1)$	1
$L(\vert\vert^a\mathrm{sgn}, \vert\vert^c\mathrm{sgn}, \vert\vert^e)$	$\varphi_{-,a}\oplus\varphi_{-,c}\oplus\varphi_{+,0}\oplus\varphi_{-,-c}\oplus\varphi_{-,-a}$	$\Gamma_{\mathbb{R}}(s+a+1)\Gamma_{\mathbb{R}}(s+c+1)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-\frac{a}{2}+1)\Gamma_{\mathbb{R}}(s-\frac{c}{2}+1)$	1
$L(\vert\vert^a\mathrm{sgn}, \vert\vert^c, \vert\vert^e\mathrm{sgn})$	$\varphi_{-,a}\oplus\varphi_{+,c}\oplus\varphi_{+,0}\oplus\varphi_{+,-c}\oplus\varphi_{-,-a}$	$\Gamma_{\mathbb{R}}(s+a+1)\Gamma_{\mathbb{R}}(s+c)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-\frac{a}{2}+1)\Gamma_{\mathbb{R}}(s-\frac{c}{2})$	-1
$L(\vert\vert^a, \vert\vert^c\mathrm{sgn}, \vert\vert^e\mathrm{sgn})$	$\varphi_{+,a}\oplus\varphi_{-,c}\oplus\varphi_{+,0}\oplus\varphi_{-,-c}\oplus\varphi_{+,-a}$	$\Gamma_{\mathbb{R}}(s+a)\Gamma_{\mathbb{R}}(s+c+1)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-\frac{a}{2})\Gamma_{\mathbb{R}}(s-\frac{c}{2}+1)$	-1
$L(\vert\vert^a\mathrm{sgn}, \vert\vert^c, \vert\vert^e)$	$\varphi_{-,a}\oplus\varphi_{+,c}\oplus\varphi_{+,0}\oplus\varphi_{+,-c}\oplus\varphi_{-,-a}$	$\Gamma_{\mathbb{R}}(s+a+1)\Gamma_{\mathbb{R}}(s+c)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-\frac{a}{2}+1)\Gamma_{\mathbb{R}}(s-\frac{c}{2})$	-1
$L(\vert\vert^a, \vert\vert^c\mathrm{sgn}, \vert\vert^e)$	$\varphi_{+,a}\oplus\varphi_{-,c}\oplus\varphi_{+,0}\oplus\varphi_{-,-c}\oplus\varphi_{+,-a}$	$\Gamma_{\mathbb{R}}(s+a)\Gamma_{\mathbb{R}}(s+c+1)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-\frac{a}{2})\Gamma_{\mathbb{R}}(s-\frac{c}{2}+1)$	-1
$L(\vert\vert^a, \vert\vert^c, \vert\vert^e\mathrm{sgn})$	$\varphi_{+,a}\oplus\varphi_{+,c}\oplus\varphi_{+,0}\oplus\varphi_{+,-c}\oplus\varphi_{+,-a}$	$\Gamma_{\mathbb{R}}(s+a)\Gamma_{\mathbb{R}}(s+c)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-a)\Gamma_{\mathbb{R}}(s-c)$	1
$L(\vert\vert^a, \vert\vert^c, \vert\vert^e)$	$\varphi_{+,a}\oplus\varphi_{+,c}\oplus\varphi_{+,0}\oplus\varphi_{+,-c}\oplus\varphi_{+,-a}$	$\Gamma_{\mathbb{R}}(s+a)\Gamma_{\mathbb{R}}(s+c)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-a)\Gamma_{\mathbb{R}}(s-c)$	1

Table D.5: Degree 5 L-factors

Representation	Irreducibles	$L(s,\varphi)$	$\epsilon(s,\varphi,\psi)$
$L(\delta(\vert\vert^r, 2k+1), \vert\vert^a\mathrm{sgn})$	$\varphi_{2k+1,2r}\oplus\varphi_{+,0}\oplus\varphi_{2k+1,-2r}$	$\Gamma_{\mathbb{C}}(s+2r+k+\frac{1}{2})\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s-2r+k+\frac{1}{2})$	1
$L(\delta(\vert\vert^r, 2k), \vert\vert^a\mathrm{sgn})$	$\varphi_{2k,2r}\oplus\varphi_{+,0}\oplus\varphi_{2k,-2r}$	$\Gamma_{\mathbb{C}}(s+2r+k)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s-2r+k)$	-1
$L(\delta(\vert\vert^r, 2k+1), \vert\vert^a)$	$\varphi_{2k+1,2r}\oplus\varphi_{+,0}\oplus\varphi_{2k+1,-2r}$	$\Gamma_{\mathbb{C}}(s+2r+k)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s-2r+k)$	1
$L(\delta(\vert\vert^r, 2k), \vert\vert^a)$	$\varphi_{2k,2r}\oplus\varphi_{+,0}\oplus\varphi_{2k,-2r}$	$\Gamma_{\mathbb{C}}(s+2r+k+\frac{1}{2})\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s-2r+k+\frac{1}{2})$	-1
$L(\vert\vert^a\mathrm{sgn}\rtimes\vert\vert^c\mathrm{sgn}^d D_{2k})$	$\varphi_{-,a}\oplus\varphi_{-,0}\oplus\varphi_{-,-a}\oplus\varphi_{4k,0}$	$\Gamma_{\mathbb{R}}(s+a+1)\Gamma_{\mathbb{R}}(s-a+1)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+2k)$	$-i$
$L(\vert\vert^a\mathrm{sgn}\rtimes\vert\vert^c\mathrm{sgn}^d D_{2k+1})$	$\varphi_{+,a}\oplus\varphi_{-,0}\oplus\varphi_{+,-a}\oplus\varphi_{4k+2,0}$	$\Gamma_{\mathbb{R}}(s+a)\Gamma_{\mathbb{R}}(s-a)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+2k+1)$	$-i$
$L(\vert\vert^a\rtimes\vert\vert^c\mathrm{sgn}^d D_{2k})$	$\varphi_{+,a}\oplus\varphi_{-,0}\oplus\varphi_{+,-a}\oplus\varphi_{4k,0}$	$\Gamma_{\mathbb{R}}(s+a)\Gamma_{\mathbb{R}}(s-a)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+2k)$	i
$L(\vert\vert^a\rtimes\vert\vert^c\mathrm{sgn}^d D_{2k+1})$	$\varphi_{-,a}\oplus\varphi_{-,0}\oplus\varphi_{-,-a}\oplus\varphi_{4k+2,0}$	$\Gamma_{\mathbb{R}}(s+a+1)\Gamma_{\mathbb{R}}(s-a+1)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+2k+1)$	i

Table D.6: Degree 5 L-factors

Representation	Irreducibles	$L(s,\varphi)$	$\epsilon(s,\varphi,\psi)$
$X_{\lambda_1,\lambda_2}, \lambda_1>\lambda_2>0$	$\varphi_{2\lambda_2,0}\oplus\varphi_{+,0}\oplus\varphi_{2\lambda_1,0}$	$\Gamma_{\mathbb{C}}(s+\lambda_2)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+\lambda_1)$	$(-1)^{\lambda_1+\lambda_1+1}$
$X_{\lambda_1,-\lambda_2}, \lambda_1>\lambda_2>0$	$\varphi_{2\lambda_2,0}\oplus\varphi_{+,0}\oplus\varphi_{2\lambda_1,0}$	$\Gamma_{\mathbb{C}}(s+\lambda_2)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+\lambda_1)$	$(-1)^{\lambda_1+\lambda_1+1}$
$X^{large}_{2p,0}, p>0$	$\varphi_{+,0}\oplus\varphi_{2p,0}\oplus\varphi_{-,0}\oplus\varphi_{+,0}$	$(\Gamma_{\mathbb{R}}(s))^2\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+p)$	$(-1)^{p+1}$
$X^{large}_{2p-1,0}, p>0$	$\varphi_{+,0}\oplus\varphi_{2p-1,0}\oplus\varphi_{-,0}\oplus\varphi_{+,0}$	$(\Gamma_{\mathbb{R}}(s))^2\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+p-\frac{1}{2})$	$(-1)^{p}$
$X^{hol}_{p,0}, p>0$	$\varphi_{+,0}\oplus\varphi_{2p,0}\oplus\varphi_{-,0}\oplus\varphi_{+,0}$	$(\Gamma_{\mathbb{R}}(s))^2\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+p)$	$(-1)^{p+1}$
$X^{hol}_{2p-1,0}, p>0$	$\varphi_{+,0}\oplus\varphi_{2p-1,0}\oplus\varphi_{-,0}\oplus\varphi_{+,0}$	$(\Gamma_{\mathbb{R}}(s))^2\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+p-\frac{1}{2})$	$(-1)^{p}$
$X^1_{p,-p}, p>0$	$\varphi_{2p,0}\oplus\varphi_{+,0}\oplus\varphi_{2p,0}$	$\Gamma_{\mathbb{C}}(s+p)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+p)$	-1
$X^2_{p,-p}, p>0$	$\varphi_{2p,0}\oplus\varphi_{+,0}\oplus\varphi_{2p,0}$	$\Gamma_{\mathbb{C}}(s+p)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+p)$	-1

Bibliography

[1] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR 1431508

[2] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001, An overview based on examples, Reprint of the 1986 original. MR 1880691

[3] Goran Muić, *Intertwining operators and composition series of generalized and degenerate principal series for* $\mathrm{Sp}(4,\mathbb{R})$, Glas. Mat. Ser. III **44(64)** (2009), no. 2, 349–399. MR 2587308

[4] Ameya Pitale, Abhishek Saha, and Ralf Schmidt, *A note on the growth of nearly holomorphic vector-valued Siegel modular forms*, L-functions and automorphic forms, Contrib. Math. Comput. Sci., vol. 10, Springer, Cham, 2017, pp. 185–193. MR 3931455

[5] Brooks Roberts and Ralf Schmidt, *Local newforms for GSp(4)*, Lecture Notes in Mathematics, vol. 1918, Springer, Berlin, 2007. MR 2344630

[6] Ralf Schmidt, *Archimedean aspects of Siegel modular forms of degree 2*, Rocky Mountain J. Math. **47** (2017), no. 7, 2381–2422. MR 3748235

[7] David A. Vogan, Jr., *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), no. 1, 75–98. MR 0506503