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GEOMETRIC QUANTIZATION OF A SEMI-GLOBAL MODEL
OF A FOCUS-FOCUS SINGULARITY

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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Dedicated

to

My parents

Narasimha Rao Sunkula

Visalakshmi Sunkula

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Abstract

Quantization of a classical mechanical system is an old problem in physics. In classical mechanics, the evolution of the system is given by a Hamiltonian vector field on a symplectic manifold ("phase space"). Geometric quantization is a procedure to construct a quantum system using the geometry of the classical phase space.

A completely integrable system is a symplectic manifold with a moment map. If the moment map has singularities, the geometric quantization of such system becomes difficult to construct. In such case one needs to use tools from algebraic geometry (sheaves, cohomologies, etc.) to quantize such a system.

The non-degenerate singularities of moment maps have been completely classified. In this dissertation we study a 4-dimensional symplectic manifold with a moment map that has a non-degenerate singularity of the so-called focus-focus type. A simple mechanical system with such a singularity is the spherical pendulum (a point mass moving without resistance on the surface of a sphere under the influence of the Earth's gravity field).

We compute the geometric quantization of a focus-focus singularity by constructing a fine resolution and computing the corresponding sheaf cohomology groups.

Chapter 1

Introduction

1.1 Motivation

The motion of a particle at a macroscopic level is governed by the laws of classical physics. In general a classical mechanical system can be modeled by a symplectic manifold and the space of functions on the manifold. The dynamics of such a system is described by deterministic equations of motion.

By means of the famous double split experiment, it has been observed that the laws of classical physics break down at a microscopic level. In particular, the experiment indicated that under certain circumstances particles can show interference patterns and that under certain conditions light showed behavior characteristic of a particle.

Heisenberg and Schrödinger provided two equivalent mathematical models which were able to reproduce the results from the experiments and make many other successfully tested predictions. These mathematical models, collectively known as quantum mechanics, describe the quantum behaviour of (point) particles under the influence of external forces. A quantum mechanical system can be modeled by a Hilbert space and the space of self-adjoint operators on it.

In an attempt to gain some insight into the features that were to be regarded as fundamental to any quantum version of a classical theory, Dirac emphasized the formal similarities between classical and quantum mechanics. According to him, one should expect that important concepts in classical mechanics correspond to important concepts in quantum mechanics. With an understanding of the general nature of the analogy between classical and quantum mechanics, one may hope to get laws and theorems in quantum mechanics appearing as generalizations of well-known results in classical mechanics.

Abstracting from the analogy found between classical mechanics and quantum mechanics, Dirac [8] formulated a general quantum condition, a guideline for passing from a given classical system to the corresponding quantum theory. This process in general is known as *quantization.* The original concept of quantization (which nowadays is referred to as *canonical quantization*), going back to Weyl [46], von Neumann [43], and Dirac [8], consists of assigning to each *classical observable* (i.e., a function $f(q,p)$, $(q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in T^*\mathbb{R}^n$), a self-adjoint operator $\mathcal{Q}(f)$ on a Hilbert space $\mathfrak{h}$. Summarizing, we have the following definition.

Definition 1.1.1. *A* full quantization *of* $\left(T^*\mathbb{R}^n, \omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i\right)$ *is a map taking each function* $f \in C^\infty(T^*\mathbb{R}^n, \mathbb{R})$ *to a self-adjoint operator* $\mathcal{Q}(f)$ *on a Hilbert space* $\mathfrak{h}$ *such that:*

(Q1) $\mathcal{Q}(f+g) = \mathcal{Q}(f) + \mathcal{Q}(g)$ *for each* $f, g \in C^\infty(T^*\mathbb{R}^n, \mathbb{R})$*;*

(Q2) $\mathcal{Q}(\lambda f) = \lambda \mathcal{Q}(f)$ *for each* $f \in C^\infty(T^*\mathbb{R}^n, \mathbb{R})$ *and* $\lambda \in \mathbb{R}$*;*

(Q3) $\mathcal{Q}(1_{T^*\mathbb{R}^n}) = \mathrm{Id}_{\mathfrak{h}}$*, where* $1_{T^*\mathbb{R}^n}$ *is the constant function 1 on* $T^*\mathbb{R}^n$*;*

(Q4) $[\mathcal{Q}(f), \mathcal{Q}(g)] = -\mathrm{i}\hbar\mathcal{Q}(\{f, g\})$ *for each* $f, g \in C^\infty(T^*\mathbb{R}^n, \mathbb{R})$*;*

(Q5) the operators $\mathcal{Q}(q_i)$ *and* $\mathcal{Q}(p_i)$ *are represented irreducibly in* $\mathfrak{h}$*.*

In 1946, Groenewold [14] proved that a full quantization of $(T^*\mathbb{R}^n, \omega)$ in the sense of Definition 1.1.1 is not possible (see also [6, 12, 19, 44]). Van Hove [42] suggested in 1951 that conditions (Q1)–(Q5) are too restrictive and gave the following definition.

Definition 1.1.2. *A* prequantization *of* $(T^*\mathbb{R}^n, \omega)$ *is a map taking smooth functions* $f \in C^\infty(T^*\mathbb{R}^n, \mathbb{R})$ *to self-adjoint operators on a Hilbert space* $\mathfrak{h}$ *satisfying conditions (Q1)–(Q4). The existence of such a prequantization is usually called the* Dirac problem*.*

In [42], Van Hove showed that there exists a prequantization of $(T^*\mathbb{R}^n, \omega)$ and that the Hilbert space $L^2(\mathbb{R}^n, \mathbb{C})$ and the operators

$$\mathcal{Q}(f) = -\mathrm{i}X_f - \langle \theta, X_f \rangle + f \ .$$

satisfy Definition 1.1.2, where $\theta = \sum_{i=1}^{n} p_i \,\mathrm{d}q_i$, X_f is the Hamiltonian vector field of f (see Definition 2.1.3), and $f \in C^\infty(T^*\mathbb{R}^n, \mathbb{R})$. Throughout this dissertation, we assume that the Planck's constant $\hbar$ is equal to 1.

Several attempts have been made to formalize the quantization procedure so as to apply it to general symplectic manifolds. In this dissertation we study a variant of quantization called *geometric quantization*. Proposed by Segal [33], Souriau [39] and Kostant [22, 23], geometric quantization sets as its goal the construction of quantum objects (Hilbert space and self-joint operators on them) by using the geometry of the corresponding classical system (symplectic manifold and functions on them).

The origins of geometric quantization lie not only in attempts by physicists to extend the known quantization procedures for simple mechanical systems to more general configurations and phase spaces, but also in the development of the theory of unitary representations by mathematicians.

In this dissertation, we study geometric quantization of manifolds foliated by certain integrable systems. In particular, we content ourselves to constructing a vector space $\mathcal{Q}(M)$ for geometric quantization of 4-dimensional symplectic manifold M, foliated by integrable systems which have a so-called *focus-focus singularity*. Even though the vector space $\mathcal{Q}(M)$ we construct does not have a Hilbert space structure, we call this geometric quantization, with an abuse of terminology. Despite the problems that may arise in order to define a Hilbert space structure on $\mathcal{Q}(M)$ and in defining operators, the first step is to compute the vector space $\mathcal{Q}(M)$.

1.2 Main result and literature review

In geometric quantization we start with the classical *phase space*: mathematically, a $2n$-dimensional manifold M endowed with a symplectic structure ω. We then choose a Hermitian line bundle $\mathbb{L}$ over M, equipped with a connection ∇, whose curvature is $\mathrm{curv}^{\nabla} = \omega$.

Start by considering the set of all square integrable sections of $\mathbb{L}$. The Hilbert space thus obtained, called a *prequantum Hilbert space*, is too big from a physical point of view – recall that the phase space is $2n$-dimensional (n coordinates and n momenta) while the wave function depends only on n variables. To obtain a quantum Hilbert space, one considers a subspace ("half") of the prequantum Hilbert space, i.e., one might consider the subspace of sections that depend only on the coordinates and are independent of the momenta. More generally, one can consider spaces of sections with covariant derivatives that are zero in some set of n directions. We refer to the set of directions in which the elements in the quantum space are covariantly constant as a *polarization*

(see Definition 2.2.18 for a precise definition of P). Then, in the simplest case, the geometric quantization is the vector space of sections which are covariantly flat in the P-directions.

Closely related to polarizations are integrable systems. An integrable system on a $2n$-dimensional symplectic manifold (M, ω) consists of n independent Poisson-commuting functions on M ("Hamiltonians"). Due to Arnold-Liouville theorem [1, Section 49A], an integrable system gives a Lagrangian fibration $M \to \mathbb{R}^n$ defined by the n- Hamiltonians. The leaves of such a fibration are the level sets of the map $M \to \mathbb{R}^n$.

In general, there are two main difficulties with these type of fibrations. The first is that there are usually no flat sections on the leaves – in fact, Rawnsley [31] showed that the existence of an S^1 action may be an obstruction for the existence of sections that are covariantly constant. Hence the simple approach to geometric quantization just gives the trivial vector space. In [23], Kostant proposed defining the geometric quantization to be the vector space of the total cohomology $\mathrm{H}^*(M, \mathcal{F})$ of the sheaf $\mathcal{F}$ of sections flat in the P-direction (see Definition 3.1.10),

$$\mathcal{Q}(M, \mathcal{F}) := \mathrm{H}^*(M, \mathcal{F}) = \bigoplus_{k \geq 0} \mathrm{H}^k(M, \mathcal{F}), \tag{1.1}$$

where $\mathrm{H}^k(M, \mathcal{F})$ are the k^{th} cohomology groups with values in the sheaf $\mathcal{F}$. This is an appealing and natural generalization: instead of using just the 0^{th} cohomology (global sections), the total cohomology is used.

The second difficulty with this approach is that the foliation determined by the fibration $M \to \mathbb{R}^n$ usually has singular fibers. To deal with this problem, we restrict to integrable systems with certain types of singularities, namely non-degenerate singularities. Non-degenerate singularities occur as combinations of three basic building blocks: elliptic, hyperbolic and focus-focus singularities. In Section 2.1.3 we will provide a detailed explanation of the singularity types.

In this dissertation, we are primarily interested in the case that the manifold M is 4-dimensional and the map $M \to \mathbb{R}^2$ is proper (i.e., the inverse image of a compact set is compact, which implies that the fibers are compact and generically are two-dimensional tori). The bulk of this dissertation deals with fibers that have a focus-focus singularity. In general, a fiber can have more than one singularity, but we further restrict to the case of a single singularity.

Theorem 1.2.1 (Main Theorem). *Let $\mu = (H, J) : M \to \mathbb{R}^2$ be a proper integrable system with non-degenerate singularities, let $\mathbb{L}$ be a trivial line bundle over M endowed with a Hermitian connection ∇ determined by the connection 1-form given by Lemma 4.3.2, and let $\mathcal{F}$ be the sheaf of P-flat sections. Assume that the fiber over $(0, 0) \in \mathbb{R}^2, \mu^{-1}(0, 0)$ is a focus-focus fiber. Then there exists an $\epsilon_0 > 0$ such that, when $\epsilon < \epsilon_0$, $M_\epsilon := \mu^{-1}(\{\,(H, J) : |(H, J)| < \epsilon\,\})$ satisfies*

$$\begin{aligned}
&\mathrm{H}^0(M_\epsilon, \mathcal{F}) = 0\ , \\
&\mathrm{H}^1(M_\epsilon, \mathcal{F}) = 0\ , \\
&\mathrm{H}^2(M_\epsilon, \mathcal{F}) = \{\, \textit{germs of functions at } 0 \in \mathbb{R}\,\},
\end{aligned}$$

that is,

$$\mathcal{Q}(M_\epsilon, \mathcal{F}) = \{\, \textit{germs of functions at } 0 \in \mathbb{R}\,\}.$$

Moreover, for $\epsilon' \leq \epsilon < \epsilon_0$, the inclusion $i : M_{\epsilon'} \to M_\epsilon$ induces an isomorphism $i^* : \mathrm{H}^*(M_{\epsilon'}, \mathcal{F}) \to \mathrm{H}^*(M_\epsilon, \mathcal{F})$ on sheaf cohomology.

In Lemma 4.3.3 we prove that in a small enough neighborhood of the focus-focus fiber, the focus-focus fiber itself is the only Bohr-Sommerfeld leaf (see Definition 2.2.20). Thus our Theorem 1.2.1 can be interpreted as saying that the only contribution to the cohomology comes from the Bohr-Sommerfeld leaf (the focus-focus fiber). The known results about the smooth and elliptic cases follow a similar pattern – only the Bohr-Sommerfeld leaves contribute to the cohomology.

In [36] Śniatycki studied the case when the polarization is given by a regular fibration $\pi : M \to B$. He proved that the cohomology groups appearing in (1.1) are all zero except in dimension n. Furthermore, he showed that $\mathrm{H}^n(M, \mathcal{F})$ can be computed by counting the Bohr-Sommerfeld leaves. More precisely, he proved the following result:

$$\mathrm{H}^n(M, \mathcal{F}) = \bigoplus_{b \in \mathrm{BS}} \mathbb{C} \ , \qquad \mathrm{H}^k(M, \mathcal{F}) = 0 \quad \text{for } k \neq n \ ,$$

where BS stands for the set of Bohr-Sommerfeld leaves. In [16] Hamilton studied the case where there exist elliptic singularities. He proves that if M is a compact locally toric $2n$-dimensional manifold then

$$\mathrm{H}^n(M, \mathcal{F}) = \bigoplus_{b \in \mathrm{BS}_0} \mathbb{C}, \qquad \mathrm{H}^k(M, \mathcal{F}) = 0 \quad \text{for } k \neq n. \tag{1.2}$$

Here BS_0 stands for the set of regular Bohr-Sommerfeld leaves (in particular, the singular leaves do not contribute). A locally toric manifold locally carries the structure of an integrable system with elliptic singularities only.

Hamilton and Miranda consider integrable systems with hyperbolic and elliptic singularities in [17]. They prove that for an integrable system on compact 2-dimensional symplectic manifold M with with non-degenerate singularities (which must be of hyperbolic or elliptic type)

$$\mathcal{Q}(M,\mathcal{F}) = \mathrm{H}^1(M,\mathcal{F}) = \bigoplus_{p\in\mathcal{H}}(\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{b\in \mathrm{BS}_0} \mathbb{C} \ , \tag{1.3}$$

where $\mathcal{H}$ stands for the set of hyperbolic singularities. A hyperbolic fiber can be thought of as a union of immersed circles and may or may not be a Bohr-Sommerfeld immersion. Hence hyperbolic singularities do not fit into the same framework as the previous theorems: they contribute to cohomology regardless of whether or not they are Bohr-Sommerfeld.

Following arguments in [27], the Čech cohomology spectral sequence can be effectively combined with our Theorem 1.2.1 and Hamilton's formula (1.2) to calculate the geometric quantization of more general 4-dimensional symplectic manifolds. As a simple example, we calculate the cohomology of the spherical pendulum:

Theorem 1.2.2. *Let $M = T^*S^2$ with the canonical symplectic form. Let $(q,p) = (q_1,q_2,q_3,p_1,p_2,p_3)$ be canonical coordinates (viewing M as a subspace of $\mathbb{R}^6$). Let $H = \frac{1}{2}p^2 + \epsilon q_3$ and $J = q \times p$. Then $\mu = (H,J) : M \to \mathbb{R}^2$ defines an integrable system and hence a polarization P and a sheaf $\mathcal{F}$ of P-flat sections. This system has elliptic singularities and a single focus-focus singularity. The cohomology groups associated to* (1.1) *are*

$$\mathrm{H}^2(M,\mathcal{F}) = \{\, \textit{germs of functions at } 0 \in \mathbb{R}\,\} \times \prod_{b\in \mathrm{BS}_0} \mathbb{C} \ ,$$

$$\mathrm{H}^k(M,\mathcal{F}) = 0 \ , \qquad k \neq 2 \ .$$

Hence, the geometric quantization of the spherical pendulum is

$$\mathcal{Q}(M,\mathcal{F}) = \{\, \textit{germs of functions at } 0 \in \mathbb{R} \,\} \times \prod_{b \in \mathrm{BS}_0} \mathbb{C}.$$

Proof. The image of the the map μ is diffeomorphic to a closed quadrant in $\mathbb{R}^2$. Call a point in the image an interior Bohr-Sommerfeld value if the fiber over that point is a Bohr-Sommerfeld leaf and has dimension 2. BS_0 is the set of interior Bohr-Sommerfeld values aside from the focus-focus value. We can cover the image of μ with a countable collection $\{\, U_i \,\}$ of open disks such that each disk contains at most one interior Bohr-Sommerfeld value, and the intersection of two or more disks contains no interior Bohr-Sommerfeld values. Let $V_i = \mu^{-1}(U_i)$ and let $\mathcal{U} = \{\, V_i \,\}$; $\mathcal{U}$ is an open cover of M. Then $\mathrm{H}^2(V_i,\mathcal{F})$ is isomorphic to $\mathbb{C}$ if U_i contains a point in BS_0, isomorphic to $\{\, \text{germs of functions at } 0 \in \mathbb{R} \,\}$ if U_i contains the focus-focus value, and 0 otherwise. The cohomology groups $\mathrm{H}^k(V_i,\mathcal{F})$ for $k \neq 2$ vanish for all V_i. The cohomology groups $\mathrm{H}^k(V_{i_0} \cap \cdots \cap V_{i_l},\mathcal{F})$ all vanish for $l \geq 1$ because there are no interior Bohr-Sommerfeld values in $U_{i_1} \cap \cdots \cap U_{i_l}$.

There exist a spectral sequence called Leray spectral sequence [13, page 463], whose E_2 term is given by

$$E_2^{p,q} = \mathrm{H}^p\big(\mu(M), \mathcal{R}^q_\mu \mathcal{F}\big),$$

which converges to $\mathrm{H}^{p+q}(M,\mathcal{F})$. Therefore, this fact can be used to compute the cohomology groups:

$$\mathrm{H}^m(M,\mathcal{F}) = \bigoplus_{p+q=m} E_2^{p,q}.$$

Here, $\mathcal{R}^q_\mu \mathcal{F}$ is the q^{th} direct image sheaf on $\mu(M)$ associated to the presheaf defined by

$$\mathcal{R}^q_\mu \mathcal{F}(U) := \mathrm{H}^q\big(\mu^{-1}(U), \mathcal{F}|\mu^{-1}(U)\big)\,, \qquad U \subset \mu(M).$$

The above discussion about $\mathrm{H}^k(V_i, \mathcal{F})$ implies that $\mathcal{R}^q_\mu \mathcal{F} = 0$, when $q \neq 2$. When $q = 2$, the sheaf is $\mathcal{R}^2_\mu \mathcal{F}$ is supported on the interior Bohr-Sommerfeld values; such a sheaf supported on a discrete set is called a skyscraper sheaf. A standard result about cohomology groups of a skyscraper sheaf is that the 0^{th} cohomology is the direct product of the towers over the discrete set where the sheaf is supported, and all other cohomology groups are zero. Using this, we obtain

$$E_2^{p,q} = \begin{cases} \{\,\text{germs of functions at } 0 \in \mathbb{R}\,\} \times \prod\limits_{b \in \mathrm{BS}_0} \mathbb{C} & \text{if } p = 0, q = 2\ , \\ \qquad\qquad\qquad 0 & \text{otherwise}\ . \end{cases}$$

Since $\mathrm{H}^m(M, \mathcal{F}) = \bigoplus\limits_{p+q=m} E_2^{p,q}$, this proves the result. $\square$

In [38], Solha studied the Kostant complex (see Section 3.3 of $\mathcal{F}$ in manifolds with focus-focus singularities. However, we believe there are unfixable errors in [38] that invalidate the proofs. In Chapter 7 we give an example that shows that the Kostant complex is not a resolution. In [28], Miranda, Presas and Solha use the results of [38] to calculate the geometric quantization of manifolds with focus-focus singularities. However, we believe that some results in [28] are wrong. In fact, we get a different answer than [28] – compare our Theorem 1.2.1 to their Theorem 5.1. Despite what we believe to be errors, we learned a great deal from the above papers and our work is indebted to them.

1.3 Overview of the dissertation

The plan of our exposition is the following.

In Chapter 2 we define some concepts of symplectic geometry and integrable systems, including the classification of non-degenerate singularities. We give definitions related to geometric quantization: Hermitian line bundles, connections, curvature, holonomy, polarization, (pre)quantization.

Chapter 3 is devoted to tools from sheaf cohomology that are needed in our subsequent computations, in particular, the sheaf $\mathcal{F}$ of sections that are covariantly constant along the chosen polarization. We construct fine resolutions for some singularities in low dimensions.

In Chapter 4 we develop a semi-global model for the focus-focus singularity, and in Chapter 5 we give some definitions and technical lemmas that we use later in our computations.

Chapter 6 contains the main results of the dissertation. There we compute the 0th, 1st, and 2nd cohomologies of the sheaf $\mathcal{F}$.

In the final Chapter 7, we construct a concrete example that contradicts some previously published results.

Chapter 2

Preliminaries

In this chapter we introduce some definitions, notations, and mathematical results symplectic geometry and geometric quantization that we will need.

2.1 Geometric formulation of classical mechanics

Symplectic geometry is an adequate mathematical framework for describing the Hamiltonian formalism of classical mechanics. It helps us to clearly and concisely formulate problems in classical physics and to understand their quantum counterparts. Moreover, symplectic geometry is a suitable starting point for geometric quantization.

2.1.1 Symplectic geometry

In this section we review some definitions and important theorems from symplectic geometry in the context of this dissertation; detailed exposition can be found, e.g., in [2, 4, 5, 24, 32].

Definition 2.1.1. *A* symplectic manifold *is a pair* (M, ω)*, where* M *is a finite dimensional manifold and* $\omega \in \Omega^2(M)$ *is a closed and non-degenerate* 2*-form. .*

In addition, if ω *is exact, i.e.,* $\omega = \mathrm{d}\theta$ *for some* $\theta \in \Omega^1(M)$*, then we say that* (M, ω) *is an* exact symplectic manifold.

Remark 2.1.2. As a consequence of the non-degeneracy condition on the 2-form ω in Definition 2.1.1, the following statements hold.

- A symplectic manifold M is even dimensional and orientable.
- If the symplectic manifold M has dimension $2n$, then $\omega^{\wedge n}$ is a volume form called the *Liouville volume form*, and the associated measure is called the *Liouville measure.*
- Associated to any $f \in C^\infty(M)$, there is a unique vector field $X_f \in \mathfrak{X}(M)$ defined by

$$\omega(X_f, \cdot) = -\mathrm{d}f. \tag{2.1}$$

▲

Definition 2.1.3. *Given any* $f \in C^\infty(M)$*, the vector field* X_f *in equation* (2.1) *is called the* Hamiltonian vector field *associated to* f*. The flow of the vector field* X_f *is called the* Hamiltonian flow *of* f*.*

Definition 2.1.4. *A diffeomorphism* $\phi : M \to N$ *between the symplectic manifolds* (M, ω) *and* (N, η) *is a* symplectomorphism *if* $\phi^*(\eta) = \omega$*.*

Theorem 2.1.5 (Darboux). *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold. Then at each point* $m \in M$ *there exists a symplectomorphism* ϕ *between a* *neighborhood* U_m *of* m *and a neighborhood* V *of* $0 \in \mathbb{R}^{2n}$ *such that*

$$\phi^*(\omega) = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i,$$

where $(q_1, \dots, q_n, p_1, \dots, p_n)$ *is a chart on* $\mathbb{R}^{2n}$.

The coordinates in which ω takes the form $\mathrm{d}q \wedge \mathrm{d}p := \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i$ are called *canonical coordinates.*

Definition 2.1.6. *Let* N *be a submanifold of a* $2n$*-dimensional symplectic manifold* (M, ω) *with the inclusion map* $\iota : N \hookrightarrow M$*. The submanifold* N *is called an* isotropic submanifold *if* $\iota^*\omega = 0$*. In addition, if the* $\dim N = n$*, then the submanifold* N *is called a* Lagrangian submanifold*.*

2.1.2 Integrable systems

Definition 2.1.7. *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold. The* Poisson bracket *induced by the* 2*-form* ω *on* $C^\infty(M)$ *is the* $\mathbb{R}$*-bilinear, skew-symmetric map*

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

defined by

$$\{f, g\} := \omega(X_f, X_g), \tag{2.2}$$

for any $f, g \in C^\infty(M)$.

The following lemma lists some fundamental properties of the Poisson bracket. For detailed proofs see, e.g., [25, Section 3].

Lemma 2.1.8. *The Poisson bracket* (2.2) *satisfies the following properties:*

- *it is antisymmetric and satisfies the Leibniz identity, i.e., for all* $f, g, h \in C^\infty(M)$,

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \; ; \tag{2.3}$$

- *if* $[\cdot, \cdot]$ *stands for the Lie bracket on vector fields and* X_f *is the Hamiltonian vector field of* $f \in C^\infty(M)$, *then the map*

$$\begin{array}{rcl} \big(C^\infty(M), \{\cdot, \cdot\}\big) & \to & \big(\mathfrak{X}(M), [\cdot, \cdot]\big) \\ f & \mapsto & X_f \end{array}$$

  is a Lie algebra homomorphism.

Definition 2.1.9. *An* integral *of a Hamiltonian function* $H \in C^\infty(M)$ *is a function that is invariant under the flow of the Hamiltonian vector field* X_H, *i.e., a function* $f \in C^\infty(M)$ *such that* $\{f, H\} = 0$.

Definition 2.1.10. *A* completely integrable Hamiltonian system (M, ω, F) *on a* $2n$*-dimensional symplectic manifold* (M, ω) *is given by a set of* n *smooth functions* $H_1, \ldots, H_n \in C^\infty(M)$, *that are functionally independent and Poisson-commuting, i.e.,*

$$\{H_i, H_j\} = \omega(X_{H_i}, X_{H_j}) = 0, \qquad i, j \in \{1, \ldots, n\}.$$

The map $F = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ *is called the* moment map.

The level sets of the moment map in a completely integrable system form a Lagrangian fibration $F : M \to \mathbb{R}^n$.

2.1.3 Classification of singularities

From a topological, dynamical and analytical viewpoint, the most interesting features of a completely integrable system on a symplectic manifold can be found in the singular fibers of the moment map $F = (H_1, \ldots, H_n)$ and in their surrounding neighborhoods. Singularities of a Hamiltonian system can be approached either through the dynamical systems viewpoint by studying the flow of vector fields or through the foliations of the phase space by the Hamiltonian functions.

In the case of completely integrable systems, both aspects are equivalent because the vector fields of the n functions $H_1, \ldots, H_n$ form a basis of the tangent spaces of the leaves of the foliation $H_i = \mathrm{const}_i$, at least for the regular points. In the following we will briefly describe the singularities in the dynamical systems viewpoint.

Definition 2.1.11. *Let $F = (H_1, \ldots, H_n)$ be the moment map of a completely integrable system on a $\mathbb{R}^{2n}$. A point $m \in \mathbb{R}^{2n}$ is said to be a* regular point *if*

$$\mathrm{rank}\{X_{H_1}(m), \ldots, X_{H_n}(m)\} = n \ .$$

If

$$\mathrm{rank}\{X_{H_1}(m), \ldots, X_{H_n}(m)\} = k, \qquad 0 \leq k < n \ ,$$

then a point $m \in \mathbb{R}^{2n}$ is said to be a singular point of rank k*. The value $F(m) \in \mathbb{R}^n$ is called a* regular value *if m is a regular point and a* singular value *if m is a singular point.*

Suppose that $m \in \mathbb{R}^{2n}$ is a singular point of rank k for a completely integrable system $F = (H_1, \ldots, H_n)$ on $\mathbb{R}^{2n}$. After replacing the H_i's with invertible linear combinations of H_j's if necessary, we may assume that

$$X_{H_1}(m) = \cdots = X_{H_{n-k}}(m) = 0,$$

and the X_{H_i}'s are linearly independent for $n - k < i \leq n$. The quadratic parts of $H_1, \ldots, H_{n-k}$ form an abelian subalgebra $\mathfrak{s}_m$ of the Lie algebra of quadratic forms, with the Poisson bracket as Lie bracket.

Definition 2.1.12. *A singular point m or rank k is said to be a* non-degenerate singular point of rank k *if the sub-algebra $\mathfrak{s}_m$ is a Cartan sub-algebra of the Lie algebra $\mathfrak{sp}(2n-2k, \mathbb{R})$ of the symplectic group* $\mathrm{Sp}(2n-2k, \mathbb{R})$.

Remark 2.1.13. In an obvious way, Definitions 2.1.11 and 2.1.12 can be carried over to a completely integrable system (M, ω, F) on a general $2n$-dimensional symplectic manifold.

▲

In 1936, Williamson [47] classified the Cartan subalgebras of the Lie algebra of the symplectic group.

Theorem 2.1.14 (Williamson). *Let $\mathfrak{s} \subset \mathfrak{sp}(2l; \mathbb{R})$ be a Cartan subalgebra. Then there exist canonical coordinates $(q_1, \ldots, q_l, p_1, \ldots, p_l)$ for $\mathbb{R}^{2l}$, a triple $(k_e, k_h, k_{ff}) \in \mathbb{Z}^3_{\geq 0}$ satisfying the condition $k_e + k_h + 2k_{ff} = l$, and a basis $h_1, \ldots, h_m$ of $\mathfrak{s}$ such that*

$$h_i = \begin{cases} \dfrac{q_i^2 + p_i^2}{2}, & i = 1, \ldots, k_e, \\ q_i p_i, & i = k_e + 1, \ldots, k_e + k_h, \\ \left\{ \begin{array}{l} q_i p_i + q_{i+1} p_{i+1}, \\ q_i p_{i+1} - q_{i+1} p_i \end{array} \right\}, & i = k_e + k_h + 1, k_e + k_h + 3, \ldots, k_e + k_h + 2k_{ff} - 1. \end{cases}$$

Additionally, if two Cartan subalgebras $\mathfrak{s}, \mathfrak{s}' \subset \mathfrak{sp}(2l;\mathbb{R})$ *are conjugate if and only if their corresponding triples are equal.*

The elements of the basis of $\mathfrak{s}$ *are called* elliptic blocks*,* hyperbolic blocks *or* focus-focus blocks *according to whether they are of the form* $\frac{(q_i^2+p_i^2)}{2}, q_ip_i$ *or a pair* $q_ip_i + q_{i+1}p_{i+1}, q_ip_{i+1} - q_{i+1}p_i$ *respectively.*

Zung [49] gave the following definition:

Definition 2.1.15. *Let* (M, ω, F) *be a* $2n$*-dimensional completely integrable system. The* Williamson type *of a non-degenerate singular point* m *of rank* k *is a quadruple* $(k, k_e, k_h, k_{ff}) \in \mathbb{Z}^4_{\geq 0}$ *satisfying the condition* $k + k_e + k_h + 2k_{ff} = n$*, where* (k_e, k_h, k_{ff}) *is the triple associated to the Cartan subalgebra* $\mathfrak{s}_m \subset \mathfrak{sp}(2n - 2k, \mathbb{R})$*.*

Given a completely integrable system $\big(M, \omega, F = (H_1, \ldots, H_n)\big)$, suppose $m \in M$ is a non-degenerate singularity of Williamson type (k, k_e, k_h, k_{ff}), then the following definition associates to such a quadruple a local model for the integrable system.

Definition 2.1.16. *Given a quadruple* $(k, k_e, k_h, k_{ff}) \in \mathbb{Z}^4_{\geq 0}$ *satisfying the condition* $k + k_e + k_h + 2k_{ff} = n$*, the* local model *of a singular point of Williamson type* (k, k_e, k_h, k_{ff}) *is a completely integrable system* $\big(\mathbb{R}^{2n}, \omega_0, F_k = (h_1, \ldots, h_n)\big)$*, where* $\omega_0 = \displaystyle\sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i$*, with*

$$h_i = \begin{cases} p_i & \text{if } i = 1, \ldots k \ , \\ \dfrac{(q_i^2 + p_i^2)}{2} & \text{if } i = k+1, \ldots, k + k_e \ , \\ q_ip_i & \text{if } i = k + k_e + 1, \ldots, k + k_e + k_h \ , \end{cases}$$

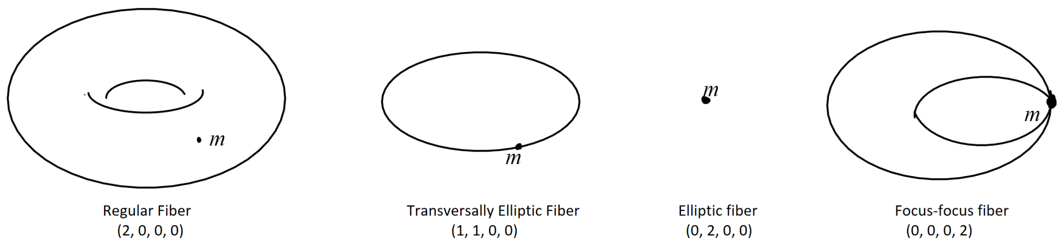

Figure 2.1: Some possible singularities of a 4-dimensional completely integrable system. Left to right: m regular point (Williamson type (2, 0, 0, 0)); m transversally elliptic singularity (Williamson type (1, 1, 0, 0)); m elliptic-elliptic singularity (Williamson type (0, 2, 0, 0)); m focus-focus singularity (Williamson type (0, 0, 0, 2)).

and the remaining h_i's (for $i = k+k_e+k_h+1, k+k_e+k_h+3, \ldots, k+k_e+k_h+2k_{ff}-1$) are focus-focus pairs $q_ip_i + q_{i+1}p_{i+1}$, $q_ip_{i+1} - q_{i+1}p_i$.

Eliasson established in [10, 11] (see also [26, 7]) that a small neighborhood of a non-degenerate singular point of Williamson type (k, k_e, k_h, k_{ff}) is equivalent to local model of Williamson type (k, k_e, k_h, k_{ff}).

Theorem 2.1.17. *Let $\big(M, \omega, F = (H_1, \ldots, H_n)\big)$ be a $2n$-dimensional integrable system, and let $m \in M$ be a non-degenerate singular point of Williamson type (k, k_e, k_h, k_{ff}). Then there exists open neighborhoods $U \subset M$ of m, $V \subset \mathbb{R}^{2n}$ of the origin, and a map $\phi : U \to V$ such that ϕ is a symplectomorphism and that $F = F_k \circ \phi$, where F_k is given in Definition 2.1.16.*

From Theorem 2.1.17, one can observe that the number of elliptic components k_e, hyperbolic components k_h, and focus-focus components k_{ff} completely determine the Cartan sub algebra formed by the completely integrable system (M, ω, F) with non degenerate singularities.

2.2 Geometric quantization

In this chapter we will briefly describe the geometric quantization scheme. Detailed exposition of different aspects of geometric quantization can be found, e.g., in [34, 35, 48, 3, 20, 9, 15].

2.2.1 Hermitian line bundles and connections

In this section we develop some necessary machinery for the geometric quantization procedure. In particular, we discuss the notion of a line bundle over a manifold, sections of the line bundle, covariant derivatives, curvature.

Definition 2.2.1. *A* (complex) line bundle *over a smooth manifold M is a smooth manifold $\mathbb{L}$ together with the following properties:*

- *the* projection $\pi : \mathbb{L} \to M$ *is a smooth surjective map;*
- *for all $m \in M$, the* fiber, $\mathbb{L}_m := \pi^{-1}(m)$, *over m is a 1-dimensional complex vector space.*

Definition 2.2.2. *A smooth map $\psi : M \to \mathbb{L}$ satisfying the condition that $\pi\big(\psi(m)\big) = m$ for all $m \in M$ is called a* section *of the line bundle. We denote the space of all sections by $\Gamma(\mathbb{L})$.*

Definition 2.2.3. *A* connection *on a line bundle $\mathbb{L}$ over M is a map*

$$\nabla : \mathfrak{X}(M) \times \Gamma(M) \to \Gamma(M)$$

that satisfies the following properties:

(a) $\nabla_X(\phi + \psi) = \nabla_X\phi + \nabla_X\psi$ *for all $X \in \mathfrak{X}(M)$, $\phi, \psi \in \Gamma(\mathbb{L})$,*

(b) $\nabla_X(f\psi) = X(f)\psi + f\nabla_X\psi$ *for all* $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, $\psi \in \Gamma(\mathbb{L})$.

Definition 2.2.4. *A* Hermitian structure *on a line bundle* $\mathbb{L}$ *over* M *is a choice of inner product* $(\cdot,\cdot)$ *on each fiber of* $\mathbb{L}$ *such that for each smooth section* ψ *of* $\mathbb{L}$, (ψ,ψ) *is a smooth function on* M. *A line bundle* $\mathbb{L}$ *together with a choice of Hermitian structure is called a* Hermitian line bundle*.*

If a connection ∇ *on a Hermitian line bundle* $\mathbb{L}$ *is compatible with the Hermitian structure on* $\mathbb{L}$*, i.e.,*

$$\big(\nabla_X\psi_1, \psi_2\big) + \big(\psi_1, \nabla_X\psi_2\big) = X(\psi_1,\psi_2), \qquad \forall \psi_1,\psi_2 \in \Gamma(M) \ , \tag{2.4}$$

then ∇ *is called a* Hermitian connection*.*

If $\mathbb{L}$ is a Hermitian line bundle over M endowed with a Hermitian connection ∇, then it is always possible to choose a locally defined smooth section ψ_0 near any point in M such that $(\psi_0,\psi_0) \equiv 1$; such a section ψ_0 is called a *unitary section*. Any section ψ of $\mathbb{L}$ can be written locally as $\psi = f\psi_0$, for a unique function $f \in C^\infty(M,\mathbb{C})$.

Remark 2.2.5. With respect to a unitary section ψ_0 the connection ∇ on $\mathbb{L}$ can be represented by a 1-form $\Theta \in \Omega^1(M)$ in the following way:

$$\nabla_X\psi_0 = -\mathrm{i}\langle\Theta, X\rangle\psi_0 \ , \tag{2.5}$$

and

$$\nabla_X(f\psi_0) = X(f)\psi_0 - \mathrm{i}f\langle\Theta, X\rangle\psi_0 \ . \tag{2.6}$$

The 1-form Θ satisfying (2.5) is called a *connection 1-form*. ▲

Definition 2.2.6. *For any Hermitian line bundle with Hermitian connection* ∇, *the* curvature 2-form curv^∇ *of the connection* ∇ *is defined by*

$$\mathrm{curv}^\nabla(X,Y)\,\psi = \mathrm{i}\big(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\big)\psi \tag{2.7}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$ *and section* $\psi \in \Gamma(\mathbb{L})$.

Proposition 2.2.7. *The curvature form is independent of the choice of the unitary section* ψ_0, *and* $\mathrm{curv}^\nabla = \mathrm{d}\Theta$.

Proof. Propositions 1 and 2 in [16]. □

2.2.2 Holonomy

Consider a Hermitian line bundle $\mathbb{L}$ over M with a Hermitian connection ∇.

Definition 2.2.8. *Let* γ *be a curve on* M, *with tangent vector* $\dot{\gamma}$, *and let* $\tilde{\gamma}$ *be the lifting of* γ *to* $\mathbb{L}$ *via a unitary section* ψ_0, *i.e., let* $\widetilde{\gamma}(t) = \psi_0(\gamma(t))$. *Then* $\tilde{\gamma}$ *is said to be* horizontal *if*

$$\nabla_{\dot{\gamma}}\psi_0 = 0 \tag{2.8}$$

for all points along the curve γ.

Definition 2.2.9. *Given a curve* $\gamma : [a,b] \to M$ *and a point* p *in the fiber over* $\gamma(a)$, *the lift* $\tilde{\gamma}$ *is uniquely determined by the condition that it is a horizontal lift of* γ *with* $\tilde{\gamma}(a) = p$. *The linear operator*

$$\Pi_s^{\dot{\gamma}} : \mathbb{L}_{\gamma(a)} \to \mathbb{L}_{\gamma(s)} : p = \widetilde{\gamma}(a) \mapsto \widetilde{\gamma}(s) \tag{2.9}$$

is called the parallel transport *from* $\gamma(a)$ *to* $\gamma(s)$ *along the curve* γ.

Definition 2.2.10. *If γ in Definition 2.2.9 is a loop, the map* (2.9) *is an automorphism of $\mathbb{L}_{\gamma(a)}$, called the* holonomy *around γ.*

With the help of (2.8) one can view holonomy as a map from and the map is given by

$$\mathrm{Hol} : \{\text{loops on } M\} \to S^1 : \gamma \mapsto \exp\left(\mathrm{i} \oint_\gamma \Theta\right) . \tag{2.10}$$

2.2.3 Geometric prequantization

Prequantization is the first step in the geometric quantization scheme (Definition 1.1.2), it is a simplification of the full quantization by ignoring the irreducibility condition (Q5) from Definition 1.1.2. For the case of symplectic manifold $M = T^*N, \omega = \mathrm{d}\theta$, a prequantization was constructed in 1960 by Segal [33]

who generalized the results of Koopman [21] and Van Hove [42]. This was done by considering the quantum operator

$$\mathcal{Q}(f) = f - \mathrm{i}X_f - \langle \theta, X_f \rangle \ , \tag{2.11}$$

and the (pre-)quantum Hilbert space is considered to be the space of smooth functions of compact support on M with the scalar product

$$\langle \phi_1, \phi_2 \rangle = \int_M \bar{\phi}_1 \phi_2 \,\mathrm{d}v \ , \tag{2.12}$$

where $v = \omega^n$ is the Liouville measure on M.

In an attempt to generalize the construction in (2.11) to a general symplectic manifold (M, ω), one needs to consider Hermitian line bundles over M, equipped with a Hermitian connection. However, to be able to define the prequantum operators as in (2.11), (M, ω) needs to be an exact symplectic manifold, i.e., there must exist $\theta \in \Omega^1(M)$ such that $\omega = \mathrm{d}\theta$. While this is not possible for every symplectic manifold, one can cover the manifold M by open sets U_α such that in each U_α the equality $\omega = \mathrm{d}\theta_\alpha$ holds for suitable 1-form θ_α on U_α. The problem with this approach is that the operators defined in (2.11) depend on θ_α which exists only locally. To be able to glue the operators together to one global operator, there needs to be an additional condition that the de Rham class $[\omega]$ is integral, i.e.,

$$\frac{1}{2\pi}\int_S \omega \in \mathbb{Z} \tag{2.13}$$

for every closed surface S in M.

Definition 2.2.11. *A symplectic manifold (M, ω) such that the de Rham class $[\omega]$ is integral is called* prequantizable. *A* prequantum line bundle *on (M, ω) is a Hermitian line bundle $\mathbb{L}$ over M endowed with a Hermitian connection ∇ that satisfies* $\mathrm{curv}^\nabla = \omega$. *We will denote the prequantum line bundle by $(M, \omega, \mathbb{L}, \nabla)$.*

Theorem 2.2.12. *A symplectic manifold (M, ω) is prequantizable if and only if there exists a prequantum line bundle $(M, \omega, \mathbb{L}, \nabla)$.*

Proof. See Proposition 8.3.1 from [48]. □

Remark 2.2.13. In this dissertation we will consider a globally trivial line bundle $\mathbb{L} = M \times \mathbb{C}$, endowed with Hermitian metric and a compatible Hermitian connection ∇, with curvature $\mathrm{curv}^\nabla = \omega$. If ψ_0 is the unit section and $\psi = f\psi_0$ (as in Remark 2.2.5), then we can identify ψ with f, and write

$$X(\psi) := X(f)\psi_0 \ , \qquad \psi = f\psi_0 \ ,$$

With this identification we can rewrite (2.6) as

$$\nabla_X \psi = X(\psi) - \mathrm{i}\langle \theta, X\rangle \psi, \quad \psi \in \Gamma(\mathbb{L}), \quad X \in \mathfrak{X}(M) \ , \tag{2.14}$$

where the 1-form θ is such that $\mathrm{d}\theta = \omega$. ▲

Given a prequantizable symplectic manifold (M, ω)

along with a prequantum line bundle $(M, \omega, \mathbb{L}, \nabla)$, the *prequantum Hilbert space* is defined to be a space of equivalence classes of square-integrable sections of $\mathbb{L}$ (two sections are equivalent if they are equal almost everywhere with respect to the Liouville measure). Suppose f is a smooth complex-valued function on M, the *prequantum operator* $\mathcal{Q}_{pre}(f)$ is the unbounded operator on the prequantum Hilbert space is given by

$$\mathcal{Q}_{pre}(f) := f - \mathrm{i}\nabla_{X_f}. \tag{2.15}$$

Note that (2.15) is same as (2.11).

2.2.4 Polarizations

As described in Section 2.2.3, the prequantization procedure only satisfies (Q1)–(Q4). To obtain a space from the prequantum Hilbert space such that the irreducibility condition (Q5) is satisfied, one needs to consider a subspace of the prequantum Hilbert space. To attain this, we introduce a new geometric structure called *polarization*. More details about polarizations can be found, e.g., in [48, 37, 40].

Definition 2.2.14. *Let (M, ω) be a $2n-$dimensional symplectic manifold. A* polarization *P of (M, ω) is a distribution in the complexified tangent bundle $TM^{\mathbb{C}}$ of M such that:*

1. *It is* Lagrangian*, i.e.,* $\omega(X,Y) = 0$ *for all* $X, Y \in P$*,,* $\dim P_m = n$ *for all* $m \in M$*.*

2. *It is* involutive*, that is* $[X,Y] \in P$*, for all* $X, Y \in P$*,*

3. *dim(*$P_m \cap \bar{P}_m \cap T_m M$*) is constant.*

Definition 2.2.15. *Let* (M,ω) *be a symplectic manifold. A polarization* P *on* M *is said to be a* real polarization *if* $P = \bar{P}$*.*

Remark 2.2.16. Suppose P is a real polarization, then $D := P \cap TM$ is a Lagrangian distribution in TM. Conversely, if D is a Lagrangian distribution of TM, then its complexification $D^{\mathbb{C}}$ is a real polarization. Hence, considering a real polarization in M is equivalent to taking a Lagrangian distribution in TM, hence M is foliated by Lagrangian submanifolds; the leaves of such a foliation are called *leaves of the polarization.* ▲

Theorem 2.2.17. *If* P *is a real polarization, then there exist a local basis of* $D = P \cap TM$ *made up of Hamiltonian vector fields.*

Proof. See [9, page 193]. □

Using Remark 2.2.16 and Theorem 2.2.17, we give the following definition for real polarization, which we use throughout this dissertation.

Definition 2.2.18. *A* non-degenerate integrable real polarization P *of a* $2n$*-dimensional symplectic manifold* M *is a (possibly singular) distribution*

$$P = \coprod_{m \in M} P_m\,, \quad P_m \subset T_m M\,, \tag{2.16}$$

such that for every $m \in M$ there exist n Poisson-commuting functions $H_1, \ldots, H_n$ on M with non-degenerate singularities (in the sense of 2.1.12) defined on a neighborhood U of m such that

$$\operatorname{span}\{ X_{H_1}(m'), \ldots, X_{H_n}(m') \} = P_{m'}$$

for every $m' \in U$.

Definition 2.2.19. *Let $(M, \omega, \mathbb{L}, \nabla)$ be a prequantizable symplectic manifold along with a prequantum line bundle, and P be a non-degenerate integrable real polarization on it. A smooth section $\psi \in \Gamma(\mathbb{L})$ is said to be* P-flat *if it is covariantly constant along P, i.e.,*

$$\nabla_X \psi = 0, \qquad \forall X \in P. \tag{2.17}$$

If P be a non-degenerate integrable real polarization, then there exists an integrable system $\big(M, \omega, \mu = (H_1, \ldots, H_n)\big)$ that gives this polarization. If $c \in \mathbb{R}^n$, then the leaves of the polarization are $\mu^{-1}(c)$.

Definition 2.2.20. *A leaf $\mu^{-1}(c)$ of the polarization P is called* Bohr-Sommerfeld leaf *if there exists a non-zero section $\psi : \mu^{-1}(c) \to \mathbb{L}$ such that $\nabla_X \psi = 0$, for all vector fields X tangent to the polarization P.*

2.2.5 Kostant's definition of geometric quantization

In general, the existence of P-flat sections along a polarization is not trivial. For example, consider the following

Example 2.2.21. Consider the manifold $\mathbb{R} \times S^1$ with coordinates (x, y) and symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Let $\mathbb{L}$ be the trivial line bundle with connection 1-form $\theta = x\,\mathrm{d}y$ with respect to the unitary section $\mathrm{e}^{\mathrm{i}x}$, and $P = \left\langle \frac{\partial}{\partial y} \right\rangle$. A section $s(x, y) = f(x, y)\mathrm{e}^{\mathrm{i}x}$ is flat section if it satisfies the following:

$$\left(\nabla_{\frac{\partial}{\partial y}} s\right)(x, y) = \left(\frac{\partial}{\partial y} - \mathrm{i}\left\langle \theta, \frac{\partial}{\partial y} \right\rangle\right) f(x, y)\mathrm{e}^{\mathrm{i}x} = \left(\frac{\partial f}{\partial y}(x, y) - \mathrm{i}xf(x, y)\right)\mathrm{e}^{\mathrm{i}x} = 0. \tag{2.18}$$

We then have $s(x, y) = g(x)\mathrm{e}^{\mathrm{i}xy}\mathrm{e}^{\mathrm{i}x}$, for some function g. Hence, $s(x, y)$ has a period of 2π in y if and only if $x \in \mathbb{Z}$. Thus, P-flat sections are only defined for the set of points with $x \in \mathbb{Z}$. ▲

As discussed earlier, the general idea of geometric quantization is to work with sections that are flat along the chosen polarization. But the P-flat sections in Example 2.2.21 are well-defined only on a subset of M, so one is forced to work with delta functions supported over these points in order to use flat sections as an analogue for quantum Hilbert space. Another methods is to deal with sheaves and higher order cohomology groups.

In this dissertation we use sheaf theory approach as suggested by Kostant in [23]. He suggested to associate quantum states to elements of higher cohomology groups, and to build the quantum phase space from these groups, by considering cohomology with coefficients in the sheaf $\mathcal{F}$ of P-flat sections (see Definition 3.1.10):

$$\mathcal{Q}(M, \mathcal{F}) := \bigoplus_{k \geq 0} \mathrm{H}^k(M, \mathcal{F}) \ , \tag{2.19}$$

where $\mathrm{H}^k(M; \mathcal{F})$ are the cohomology groups with values in sheaf $\mathcal{F}$.

Chapter 3

Sheaves appearing in geometric quantization

Kostant's definition of geometric quantization (2.19) requires us to compute the cohomology groups $\mathrm{H}^k(M, \mathcal{F})$ with coefficients in the sheaf $\mathcal{F}$ of P-flat sections (Definition 3.1.10). To this end, we will give definitions of the sheaves that are used in this dissertation, and will construct fine resolutions of $\mathcal{F}$ that will be used to compute $\mathrm{H}^k(M, \mathcal{F})$.

3.1 Sheaves

Definition 3.1.1. *Let X be a topological space. A* presheaf *$\mathcal{S}$ of modules on X assigns to every open set U of X a module $\mathcal{S}(U)$. It also assigns restriction maps: to any $V \subset U$, the presheaf assigns a map $\mathcal{S}(U) \to \mathcal{S}(V)$ such that if $W \subset V \subset U$ and $s \in \mathcal{S}(U)$, then*

$$s|_W = (s|_V)|_W, \tag{3.1}$$

and if $U = V$, then the restriction map is an identity.

Definition 3.1.2. *A presheaf $\mathcal{S}$ is a* sheaf *if the following properties hold:*

1. *If (U_i) is an open covering of an open set U, suppose that the sections $s_i \in \mathcal{S}(U_i)$ are such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, for each pair U_i, U_j of the open covering (U_i) of U, then there exists a section $s \in \mathcal{S}(U)$ such that $s|_{U_i} = s_i$, for each $U_i \subset U$.*

2. *If (U_i) is an open covering of an open set U, suppose that $s_1, s_2 \in \mathcal{S}(U)$ are such that $s_1|_{U_i} = s_2|_{U_i}$ for all $U_i \subset U$, then $s_1 = s_2$ on U.*

Definition 3.1.3. *Let $\mathcal{S}$ be a sheaf over the Topological space X. The* stalk *of $\mathcal{S}$ over $x \in X$ is the direct limit of $\mathcal{S}(U)$ with respect to the restriction maps:*

$$\mathcal{S}_x := \varinjlim_{x \in U} \mathcal{S}(U) \ .$$

Definition 3.1.4. *Let $\mathcal{C}^\infty_M$ denote the sheaf of smooth complex-valued functions on M; it is a sheaf of $\mathbb{C}$-algebras.*

Definition 3.1.5. *Let $\mathcal{P}$ stand for the sheaf of smooth vector fields tangent to the polarization P, i.e., for any open subset U of M,*

$$\mathcal{P}(U) = \{\, X \in \Gamma(TM|U) \mid X(m) \in P_m \text{ for all } m \in U \,\} \,. \tag{3.2}$$

We view $\mathcal{P}$ as a sheaf of $\mathcal{C}^\infty_M$-modules.

In the proof of the result that follows, we will need a parameterized version of Borel's Theorem (for a proof see, e.g., [30, Theorem I.1.3]).

Theorem 3.1.6 (Borel). *Let* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$*, and*

$$T : C^\infty(\mathbb{R}^{n+m}) \to C^\infty(\mathbb{R}^n)[[Y_1, \ldots, Y_m]] : f \mapsto (Tf)(x, y) = \sum_{\alpha \in \mathbb{Z}^m_+} D^\alpha_y f(x, 0) \cdot \frac{1}{\alpha!} Y^\alpha \tag{3.3}$$

be the Taylor expansion of a smooth function on $\mathbb{R}^{n+m}$ *by its partial derivatives with respect to* $y \in \mathbb{R}^m$*. Let* $m^\infty_{\mathbb{R}^n \times \{0\}}$ *be the kernel of* T*, i.e., the ideal of functions in* $C^\infty(\mathbb{R}^{n+m})$ *which are Taylor-flat along* $\mathbb{R}^m$ *on* $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^m$ *(in other words, whose partial derivatives with respect to* $y \in \mathbb{R}^m$ *vanish on* $\mathbb{R}^n \times \{0\}$*). Then the Taylor series mapping* (3.3) *gives an isomorphism*

$$C^\infty(\mathbb{R}^{n+m}) / m^\infty_{\mathbb{R}^n \times \{0\}} \overset{\cong}{\to} C^\infty(\mathbb{R}^n)[[Y_1, \ldots, Y_m]] \ . \tag{3.4}$$

Proposition 3.1.7. *Let* U *be an open subset of* M*, and* $H_1, \ldots, H_n$ *be as in Definition 2.2.18. Then* $X \in \mathcal{P}(U)$ *if and only if*

$$X = \sum_{j=1}^n f_j \, X_{H_j} \quad \textit{for some functions} \quad f_1, \ldots, f_n \in \mathcal{C}^\infty_M(U) \ . \tag{3.5}$$

Furthermore, the functions $f_1, \ldots, f_n$ *are uniquely determined by* $X, H_1, \ldots, H_n$*.*

Proof. The proposition is clear if U contains no singular points. Therefore, without loss of generality, we may assume that U is a small neighborhood of $0 \in \mathbb{R}^{2n}$ and that the Hamiltonian functions are in the form of the h_i, h_j, h_k from Definition 2.1.16. So we may assume H_i are elliptic blocks, H_j are hyperbolic blocks, and H_k, H_{k+1} are focus-focus blocks with i, j, k as in the theorem. Let I, J, K denote the sets for i, j, k from the theorem.

Let $X \in \mathcal{P}(U)$ and write

$$\begin{aligned} X &= \sum_{i\in I}\left(A_i\frac{\partial}{\partial q_i}+B_i\frac{\partial}{\partial p_i}\right) \\ &\quad +\sum_{j\in J}\left(A_j\frac{\partial}{\partial q_j}+B_j\frac{\partial}{\partial p_j}\right) \\ &\quad +\sum_{k\in K}\left(A_k\frac{\partial}{\partial q_k}+A_{k+1}\frac{\partial}{\partial q_{k+1}}+B_k\frac{\partial}{\partial p_k}+B_{k+1}\frac{\partial}{\partial p_{k+1}}\right). \end{aligned}$$

Condition (3.5) then implies that, for every $m \in U$, for all $i \in I$, $j \in J$, $k \in K$,

$$\begin{aligned} \left(A_i\frac{\partial}{\partial q_i}+B_i\frac{\partial}{\partial p_i}\right)(m) &\in \mathrm{Span}\{\, X_{H_i}(m)\,\}, \\ \left(A_j\frac{\partial}{\partial q_j}+B_j\frac{\partial}{\partial p_j}\right)(m) &\in \mathrm{Span}\{\, X_{H_j}(m)\,\}, \\ \left(A_k\frac{\partial}{\partial q_k}+A_{k+1}\frac{\partial}{\partial q_{k+1}}+B_k\frac{\partial}{\partial p_k}+B_{k+1}\frac{\partial}{\partial p_{k+1}}\right)(m) &\in \mathrm{Span}\{\, X_{H_k}(m), X_{H_{k+1}}(m)\,\}. \end{aligned}$$

We will show the following: for each $i \in I$, $j \in J$, $k \in K$, there exists smooth functions f_i, f_j, f_k, f_{k+1} such that

$$\begin{aligned} A_i\frac{\partial}{\partial q_i}+B_i\frac{\partial}{\partial p_i} &= f_iX_{H_i}, \\ A_j\frac{\partial}{\partial q_j}+B_j\frac{\partial}{\partial p_j} &= f_jX_{H_j}, \\ A_k\frac{\partial}{\partial q_k}+A_{k+1}\frac{\partial}{\partial q_{k+1}}+B_k\frac{\partial}{\partial p_k}+B_{k+1}\frac{\partial}{\partial p_{k+1}} &= f_kX_{H_k}+f_{k+1}X_{H_{k+1}}. \end{aligned}$$

Consider the elliptic block i first. For each $m \in U$ we are given

$$A_i(m)\frac{\partial}{\partial q_i}+B_i(m)\frac{\partial}{\partial p_i} \in \mathrm{Span}\{\, X_{H_i}(m)\,\} = \mathrm{Span}\{\left(p_i\frac{\partial}{\partial q_i}-q_i\frac{\partial}{\partial p_i}\right)(m)\,\}\ .$$

The linear dependence of $A_i\frac{\partial}{\partial q_i}+B_i\frac{\partial}{\partial p_i}$ and $p_i\frac{\partial}{\partial q_i}-q_i\frac{\partial}{\partial p_i}$ implies that $A_iq_i+B_ip_i=0$

as functions on U. Then A_i vanishes on the hypersurface $\{p_i = 0\}$, hence $A_i = A'_i p_i$ for

$$A'_i(q_i, p_i) = \int_0^1 \frac{\partial A_i}{\partial p_i}(q_i, tp_i)\,\mathrm{d}t$$

(where all other coordinates are kept fixed). Similarly, B_i vanishes on the hypersurface $\{q_i = 0\}$, so $B_i = B'_i q_i$ for a smooth function B'_i constructed similarly to A'_i. Plugging back, we obtain $A'_i p_i q_i + B'_i q_i p_i = 0$ as functions on U, so dividing by $q_i p_i$ we obtain $B'_i = -A'_i$. Let $f_i = A'_i$, then

$$A_i \frac{\partial}{\partial q_i} + B_i \frac{\partial}{\partial p_i} = f_i p_i \frac{\partial}{\partial q_i} - f_i q_i \frac{\partial}{\partial p_i} = f_i X_{H_i}.$$

For the hyperbolic block H_j the reasoning is analogous, so we omit it.

Now consider a focus-focus block H_k, H_{k+1}. To make notation easier, assume $k = 1$ and let X_1 denote the part of X under consideration. Then we have

$$X_1(m) = \left(A_1 \frac{\partial}{\partial q_1} + A_2 \frac{\partial}{\partial q_2} + B_1 \frac{\partial}{\partial p_1} + B_2 \frac{\partial}{\partial p_2}\right)(m) \in \mathrm{Span}\{\, X_{H_1}(m), X_{H_2}(m)\,\}$$

for each $m \in U$. The Hamiltonian vectors on the right-hand side are

$$\begin{aligned} X_{H_1} &= q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} - p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2}, \\ X_{H_2} &= -q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} - p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}. \end{aligned}$$

The linear dependence of X_1, X_{H_1}, and X_{H_2} implies that the rank of the 3×4 matrix of their components is smaller than 3, which implies that A_1, A_2, B_1, and B_2 satisfy the equations

$$\begin{aligned} vA_1 &= -tB_1 - sB_2, \\ vA_2 &= -sB_1 + tB_2 \end{aligned} \tag{3.6}$$

with

$$u = q_1^2 + q_2^2, \quad v = p_1^2 + p_2^2, \quad s = q_1p_2 + q_2p_1, \quad t = q_1p_1 - q_2p_2. \tag{3.7}$$

(Note that $s^2 + t^2 = uv$.) We want to find smooth functions f_1 and f_2 such that $X_1 = f_1X_{H_1} + f_2X_{H_2}$. As remarked before, f_1 and f_2 exist as smooth functions at least away from the origin. Our goal is to show that they extend smoothly to the origin. Comparing the coefficients of $\frac{\partial}{\partial p_1}$ and $\frac{\partial}{\partial p_2}$, we obtain the equations (holding at least away from the origin)

$$\begin{aligned} vf_1 &= -p_1B_1 - p_2B_2, \\ vf_2 &= -p_2B_1 + p_1B_2. \end{aligned} \tag{3.8}$$

To express f_1 and f_2 from (3.8), we need to show that $(-p_1B_1 - p_2B_2)$ and $(-p_2B_1 + p_1B_2)$ are divisible by v. Using (3.6), we derive the relations

$$\begin{aligned} u(-p_1B_1 - p_2B_2) &= v(q_1A_1 + q_2A_2), \\ u(-p_2B_1 + p_1B_2) &= v(-q_2A_1 + q_1A_2). \end{aligned}$$

Thus it suffices to prove the following claim: if F and G are smooth functions on $\mathbb{R}^4$ with coordinates (q_1, q_2, p_1, p_2), the variables u and v are defined as in (3.7), and $uF = vG$, then G is divisible by u. (Clearly by symmetry we also have F is divisible by v.)

To this end, let $F, G : \mathbb{R}^4 \to \mathbb{C}$ be smooth functions of q_1, q_2, p_1, p_2. Let $z = q_1 + iq_2$ and $\bar{z} = q_1 - iq_2$. Let $R = \mathcal{C}^\infty(\mathbb{R}^2)[[z, \bar{z}]]$ be the ring of formal power series with coefficients in the ring of ($\mathbb{C}$-valued) smooth functions $\mathcal{C}^\infty(\mathbb{R}^2)$.

We think of $\mathcal{C}^\infty(\mathbb{R}^2)$ as functions of p_1, p_2. We write the natural map from $\mathcal{C}^\infty(\mathbb{R}^4) \to R$ as

$$W \mapsto \sum_{a \geq 0,\ b \geq 0} W_{a,b} z^a \bar{z}^b, \qquad W_{a,b} \in \mathcal{C}^\infty(\mathbb{R}^2) \ ;$$

by Borel's Theorem, this map is bijective. Since $z\bar{z} = u$, we have $(Fu)_{a+1,b+1} = F_{a,b}$ and $(Gv)_{a,b} = G_{a,b}v$ for all $a \geq 0$, $b \geq 0$. The equality $Fu = Gv$ then implies that $G_{0,0} = G_{0,1} = G_{1,0} = 0$,

and $F_{a-1,b-1} = G_{a,b}v$ as elements of $\mathcal{C}^\infty(\mathbb{R}^2)$ for all $a \geq 1$, $b \geq 1$. Therefore

$$\sum_{a \geq 0,\ b \geq 0} G_{a,b} z^a \bar{z}^b = z\bar{z} \sum_{a \geq 1,\ b \geq 1} G_{a,b} z^{a-1} \bar{z}^{b-1} = u \sum_{a \geq 1,\ b \geq 1} G_{a,b} z^{a-1} \bar{z}^{b-1} \ ,$$

that is, the image of G is divisible by $z\bar{z} = u$ inside R. By Borel's Theorem, there exists $G_1 \in \mathcal{C}^\infty(\mathbb{R}^4)$ such that

$$G_1 \mapsto \sum_{a \geq 1,\ b \geq 1} G_{a,b} z^{a-1} \bar{z}^{b-1} \in R.$$

It follows that

$$G - uG_1 \mapsto 0 \in R \ ,$$

i.e., $G - uG_1$ is Taylor-flat in the $(z, \bar{z})$-direction or, equivalently, in (q_1, q_2)-direction. It follows that there exists $G_2 \in \mathcal{C}^\infty(\mathbb{R}^4)$ such that $uG_2 = G - uG_1$. Thus $G = u(G_1 + G_2)$, proving that G is divisible by u. $\square$

Corollary 3.1.8. *The sheaf $\mathcal{P}$ is locally free of rank n, i.e., for every $m \in M$, there exists an open neighborhood U of m and an isomorphism $\mathcal{P}|U \cong \oplus^n \mathcal{C}^\infty_M|U$.*

Proof. Let U and $H_1, \dots, H_n$ be as in Proposition 3.1.7. Then, for any open subset V of U, define a map $\mathcal{P}(V) \to \mathcal{C}^\infty_M(V)$ that sends each $X = \sum_j f_j X_{H_j} \in \mathcal{P}(V)$ to its coefficient functions: $X \mapsto f_1 \oplus \cdots \oplus f_n$. This is an isomorphism by Proposition 3.1.7. $\square$

Example 3.1.9. Since the behavior of the vector fields at the singular points is at the heart of our study, in this simple example we illustrate the importance of nondegeneracy condition in Proposition 3.1.7. Let $(M, \omega) = (\mathbb{R}^2, \mathrm{d}p \wedge \mathrm{d}q)$ and consider two different Hamiltonian functions: $H = \frac{1}{2}(q^2 + p^2)$ and $K = H^2$. The origin of $\mathbb{R}^2$ is a non-degenerate singularity for H, and a degenerate singularity for K. The Hamiltonian vector fields of H and K are proportional to each other: $X_H = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$ and $X_K = 2HX_H$, so that they define the same polarization

$$P_m = \mathrm{span}\,\{X_H(m)\} = \mathrm{span}\,\{X_K(m)\}\ , \qquad m \in \mathbb{R}^2\ .$$

Clearly, $\dim P_{(0,0)} = 0$, and $\dim P_m = 1$ if $m \neq (0,0)$.

Let U be an open subset of $\mathbb{R}^2$ that contains $(0,0)$. It is easy to see that $X \in \mathcal{P}(U)$ if and only if there exists $f \in \mathcal{C}^\infty_M(U)$ such that $X = f\,X_H$. The "if" part of this claim is obvious. To prove the "only if" part, let $X = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial p}$ for some $a, b \in \mathcal{C}^\infty_M(U)$, and assume that $X \in \mathcal{P}(U)$. Then $\det \begin{bmatrix} a & b \\ p & -q \end{bmatrix} = 0$ or, equivalently, $-qa = pb$. Reasoning as in the "elliptic" part of the proof of Proposition 3.1.7, we conclude that $X = f\,X_H$. The uniqueness of f is easy to see: if $X = f\,X_H$ and $X = \tilde{f}\,X_H$, then $(f - \tilde{f})X_H = 0$. But since $X_H(m) \neq 0$ when $m \neq (0,0)$ and since $f(m) = \tilde{f}(m)$ for $m \neq (0,0)$ and, the smoothness of f

and $\tilde{f}$ implies that $f = \tilde{f}$ on U.

On the other hand, not every $X \in \mathcal{P}(U)$ can be written as a multiple of X_K. Take, for example, $X = X_H$ and assume that $X = g\, X_K$ for some $g \in \mathcal{C}^\infty_M(U)$. But since $X_K = 2H\, X_H$, this assumption implies that $2gH = 1$ on U which contradicts $H(0,0) = 0$. ▲

Definition 3.1.10. *Let $\mathcal{F}$ be the sheaf of P-*flat sections of $\mathbb{L}$*, i.e., sections of* $\mathbb{L}$ *that are covariantly constant in the direction of the polarization P: for any open set $U \subset M$,*

$$\mathcal{F}(U) = \{\, \psi \in \Gamma(\mathbb{L}|U) \mid \nabla_X \psi = 0 \textit{ for all } X \in \mathcal{P}(U)\,\}\,. \tag{3.9}$$

If U and $H_1, \ldots, H_n$ are as in Definition 2.2.18, then it is clear that $\psi \in \mathcal{F}(U)$ if and only if $\nabla_{X_{H_j}} \psi = 0$ for all $j = 1, \ldots, n$.

Definition 3.1.11. *If $\mathcal{S}$ is a sheaf of $\mathcal{C}^\infty_M$-modules, let $\Lambda^k \mathcal{S}$ stand for its kth exterior power ($k = 0, 1, 2, \ldots$). In other words, $\Lambda^k \mathcal{S}$ is a sheaf of $\mathcal{C}^\infty_M$-modules, defined for any open set $U \subset M$ by*

$$(\Lambda^k \mathcal{S})(U) = \Lambda^k(\mathcal{S}(U)) \ ,$$

where the right-hand side is the kth exterior power of the $\mathcal{C}^\infty_M(U)$-module $\mathcal{S}(U)$.

Remark 3.1.12. In the following two definitions, we remind the reader of the following: In general, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are sheaves, then the sheaf of homomorphisms from $\mathcal{F}_1$ to $\mathcal{F}_2$ is defined by the rule $U \mapsto \mathrm{Hom}(\mathcal{F}_1|U, \mathcal{F}_2|U)$. The more intuitive rule $U \mapsto \mathrm{Hom}(\mathcal{F}_1(U), \mathcal{F}_2(U))$ does not work because it is not possible in general to define restriction maps $\mathrm{Hom}(\mathcal{F}_1(U), \mathcal{F}_2(U)) \to \mathrm{Hom}(\mathcal{F}_1(V), \mathcal{F}_2(V))$ for $V \subset U$. However, in the case that the sheaves are sheaves of modules and $\mathcal{F}_1$ is free, the restriction maps can be defined in an obvious way, and both rules give the same sheaf. ▲

Definition 3.1.13. *Let* $\mathcal{L}^k_M$ *denote the sheaf of* $\mathbb{L}$-valued k-forms*, defined for any open set* $U \subset M$ *by*

$$\mathcal{L}^k_M(U) = \mathrm{Hom}_{\mathcal{C}^\infty_M(U)}\left(\Lambda^k \Gamma(TM|U), \Gamma(\mathbb{L}|U)\right) \ .$$

Using the isomorphism $\Gamma(\mathbb{L}|U) \cong \mathcal{C}^\infty_M(U)$*,* $\alpha \in \mathcal{L}^k_M(U)$ *is a skew-symmetric,* $\mathcal{C}^\infty_M(U)$*-multilinear map*

$$\alpha : \Gamma(TM|U) \times \cdots \times \Gamma(TM|U) \to \Gamma(\mathbb{L}|U) \cong \mathcal{C}^\infty_M(U) \ . \tag{3.10}$$

Thus, given k *vector fields* $X_1, \ldots, X_k \in \Gamma(TM|U)$*,* $\alpha(X_1, \ldots, X_k)$ *is a smooth section of* $\mathbb{L}|U$ *or, equivalently, a smooth complex-valued function on* U*.*

Definition 3.1.14. *Let* $\mathcal{L}^k_P$ *be the sheaf of* $\mathbb{L}$-valued polarized k-forms*, defined for an open set* $U \subset M$ *by*

$$\mathcal{L}^k_P(U) = \mathrm{Hom}_{\mathcal{C}^\infty_M(U)}\left(\Lambda^k \mathcal{P}(U), \Gamma(\mathbb{L}|U)\right) \ ;$$

"polarized" here means that the vector fields taken as arguments are tangent to the polarization P*. Similarly to* (3.10)*,* $\alpha \in \mathcal{L}^k_P(U)$ *can be thought of as a skew-symmetric,* $\mathcal{C}^\infty_M(U)$*-multilinear map*

$$\alpha : \mathcal{P}(U) \times \cdots \times \mathcal{P}(U) \to \Gamma(\mathbb{L}|U) \cong \mathcal{C}^\infty_M(U)$$

taking as arguments k *vector fields from* $\mathcal{P}(U)$*.*

The proof of the following important lemma is similar to the proof of Corollary 3.1.8, so we omit it.

Lemma 3.1.15. *Let $H_1, \ldots, H_n$ and U be as in Definition 2.2.18 and let $\alpha \in \mathcal{L}_P^k(U)$. For integers $1 \leq i_1 < \ldots < i_k \leq n$, let $\alpha_{i_1 \cdots i_k} := \alpha(X_{H_{i_1}}, \ldots, X_{H_{i_k}}) \in \mathcal{C}_M^\infty(U)$. Then α is uniquely determined by the set of smooth functions $\alpha_{i_1 \cdots i_k}$. Conversely, any set of smooth functions $\{\alpha_{i_1 \cdots i_k}\}_{1 \leq i_1 < \ldots < i_k \leq n}$ defines an $\alpha \in \mathcal{L}_P^k$. In other words,*

$$\mathcal{L}_P^k|U \cong \bigoplus^N \mathcal{C}_M^\infty|U, \tag{3.11}$$

where N is the number of k-tuples $(i_1, \ldots, i_k)$ satisfying $1 \leq i_1 < \ldots < i_k \leq n$.

Remark 3.1.16. From Definitions 3.1.13 and 3.1.14, it is obvious that

$$\mathcal{L}_M^0 = \mathcal{L}_P^0 \cong \mathcal{C}_M^\infty \ ,$$

and Lemma 3.1.15 makes it clear that $\mathcal{L}_P^k = 0$ (the 0 sheaf) for $k > n$. ▲

Remark 3.1.17. Throughout this dissertation, we will distinguish between the value of a function or a section at a point $m \in M$ and the germ of the function/section at a point. For example, if U is an open subset of M containing m, and $X \in \mathcal{P}(U)$ (recall Definition 3.1.5), then $X(m) \in P_m \subset T_m M$ is the value of X at m, while $X_m \in \mathcal{P}_m$ is the germ of X at m, and $\mathcal{P}_m$ is the stalk of $\mathcal{P}$ at m. ▲

Lemma 3.1.18. *There is a canonical morphism*

$$\Upsilon_P^k : \mathcal{L}_M^k \to \mathcal{L}_P^k \tag{3.12}$$

defined as follows: for $U \subset M$ open, Υ_P^k maps the $\mathbb{L}$-valued k-form $\alpha \in \mathcal{L}_M^k(U)$

to the $\mathbb{L}$-valued polarized k-form $\alpha|P \in \mathcal{L}_P^k(U)$, where $\alpha|P$ is α considered as a form that can only take as arguments vector fields tangent to P. In other words, if $\upsilon_P : P \to TM$ is the natural inclusion, then $\alpha|P$ is the pull-back $\upsilon_P^*\alpha$.

The stalk at $m \in M$ of the kernel of Υ_P^k is the set of all germs of k-forms α_m such that $(\alpha|P)_m = 0$. Moreover, for $1 \leq k \leq n$, $(\mathcal{L}_M^k)_m \to (\mathcal{L}_P^k)_m$ is surjective if and only if $\dim P_m = n$.

Proof. The description of the map Υ_P^k makes the statement about its kernel obvious.

Now consider surjectivity. Let U, $\alpha \in \mathcal{L}_P^k(U)$, $H_1, \ldots, H_n$ and $\alpha_{i_1 \cdots i_k}$ be as in Lemma 3.1.15, and let $m \in U$. If $\dim P_m = n$, then $X_{H_1}, \ldots, X_{H_n}$ are linearly independent near m, hence there exists a neighborhood $V \subset U$ of m and functions $q_1, \ldots, q_n \in \mathcal{C}_M^\infty(V)$ such that $(q_1, \ldots, q_n, H_1, \ldots, H_n)$ are canonical coordinates on V and, therefore, $X_{H_i} = \frac{\partial}{\partial q_i}$. Then the form

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \alpha_{i_1 \cdots i_k} \, \mathrm{d}q_{i_1} \wedge \cdots \wedge \mathrm{d}q_{i_k} \in \mathcal{L}_M^k(U)$$

obviously equals α when restricted to P.

Conversely, suppose $\dim P_m < n$, then (by changing to a new set of Hamiltonians if necessary) we may assume that $X_{H_1}(m) = 0$. Let $\alpha \in \mathcal{L}_P^k(U)$ be the section defined by $\alpha(X_{H_1}, \ldots, X_{H_k}) = 1$ and all other $\alpha(X_{H_{i_1}}, \ldots, X_{H_{i_k}}) = 0$. Since $X_{H_1}(m) = 0$, any form $\beta \in \mathcal{L}_M^k(U)$ satisfies $\beta(m)\,(X_{H_1}(m), \ldots, X_{H_k}(m)) = 0$. But since $\alpha(X_{H_1}, \ldots, X_{H_k}) = 1$, α_m is not in the image of $(\mathcal{L}_M^k)_m \to (\mathcal{L}_P^k)_m$. $\square$

Definition 3.1.19. *For $k \geq 0$, we let $\mathcal{L}_{M|P}^k$ denote the image of $\mathcal{L}_M^k$ inside $\mathcal{L}_P^k$ under the map Υ_P^k (3.12):*

$$\mathcal{L}_{M|P}^k := \Upsilon_P^k(\mathcal{L}_M^k) \subset \mathcal{L}_P^k \, . \tag{3.13}$$

Note that $\mathcal{L}^0_{M|P} = \mathcal{L}^0_P$.

3.2 Sheaf cohomology

In this section we collect several standard definitions and facts related to cohomology of sheaves, following [45, Chapter II]. We assume that the manifold M is a paracompact, Hausdorff topological space and $\mathcal{S}$ is a sheaf of abelian groups. We let $\mathcal{S}_m$ denote the stalk at m and, for a morphism $\phi : \mathcal{S} \to \mathcal{S}$, $\phi_m : \mathcal{S}_m \to \mathcal{S}_m$ denotes the morphism of stalks.

Let Γ stand for the contravariant functor of taking global sections of a sheaf, i.e., $\Gamma(\mathcal{S}) = \mathcal{S}(X)$ are the global sections of the sheaf $\mathcal{S}$.

Definition 3.2.1. *A sheaf* $\mathcal{S}$ *is fine if for any locally finite open cover* $\{U_\nu\}$ *of* M *there exists a family* $\{\eta_\nu\}$ *of sheaf morphisms* $\eta_\nu : \mathcal{S} \to \mathcal{S}$ *such that*

(i) $\sum_\nu \eta_\nu = \mathrm{Id}_{\mathcal{S}}$,

(ii) $\eta_\nu(\mathcal{S}_m) = 0$ *for all* m *in some neighborhood of the complement of* U_ν.

In the intended applications, our fine sheaves will be sheaves of modules. More precisely, we will make use of the following lemma:

Lemma 3.2.2. *Let* M *be a smooth manifold and let* $\mathcal{S}$ *be a sheaf of* $\mathcal{C}^\infty_M$*-modules. Then* $\mathcal{S}$ *is fine.*

Proof. Example 3.4 in [45]. □

Definition 3.2.3. *A fine resolution of a sheaf* $\mathcal{S}$ *of abelian groups is an exact sequence of sheaves*

$$0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots \tag{3.14}$$

such that $\mathcal{S}^i$ *is fine for each* i. *We denote this as* $0 \to \mathcal{S} \to \mathcal{S}^*$.

Remark 3.2.4. The exactness of (3.14) means that for each $m \in M$ the corresponding sequence of stalks is exact

$$0 \to \mathcal{S}_m \to \mathcal{S}_m^0 \to \mathcal{S}_m^1 \to \mathcal{S}_m^2 \to \cdots . \tag{3.15}$$

Explicitly, this means that given any open set U containing m and any section $s_i \in \mathcal{S}^i(U)$ such that $s_i \mapsto 0 \in \mathcal{S}^{i+1}(U)$ (where 0 is the zero section of $\mathcal{S}^{i+1}(U)$), there exists an open set $V \subset U$ containing m and a section $t_{i-1} \in \mathcal{S}^{i-1}(V)$ such that $t_{i-1} \mapsto s_i|V$; if $i = 0$ then $t_{i-1} \in \mathcal{S}(V)$. In other words, the exactness of (3.14) means that closed implies locally exact. ▲

Definition 3.2.5. *Let $0 \to \mathcal{S} \to \mathcal{S}^*$ be a fine resolution. For $i \geq 0$ let $\Psi_i : \mathcal{S}^i \to \mathcal{S}^{i+1}$ denote the morphism $\mathcal{S}^i \to \mathcal{S}^{i+1}$ (the $0 \to \mathcal{S} \to \mathcal{S}^0$ term is ignored), and $\Gamma(\Psi_i) : \Gamma(\mathcal{S}^i) \to \Gamma(\mathcal{S}^{i+1})$ be the induced morphism on global sections. Then the ith cohomology of the resolution $0 \to \mathcal{S} \to \mathcal{S}^*$ is defined as*

$$\mathrm{H}^i(X, \mathcal{S}; \{\mathcal{S}^*\}) = \mathrm{Ker}\,\Gamma(\Psi_i)/\,\mathrm{Im}\,\Gamma(\Psi_{i-1}) \ , \qquad i \geq 1 \ ,$$

and $\mathrm{H}^0(X, \mathcal{S}; \{\mathcal{S}^*\}) = \mathrm{Ker}\,\Gamma(\Psi_0)$.

If $\mathcal{S}$ is a fine sheaf, the cohomology groups defined in Definition 3.2.5 do not depend on the particular choice of a fine resolution of $\mathcal{S}$, which is the claim of the following

Lemma 3.2.6. *Let $0 \to \mathcal{S} \to \mathcal{S}_1^*$ and $0 \to \mathcal{S} \to \mathcal{S}_2^*$ be two fine resolutions. Then for each $i \geq 0$ there is a canonical isomorphism*

$$\mathrm{H}^i(X, \mathcal{S}; \{\mathcal{S}_1^*\}) \to \mathrm{H}^i(X, \mathcal{S}; \{\mathcal{S}_2^*\}).$$

Proof. This is Corollary 3.14 in [45] along with the fact that fine resolutions are acyclic (which follows from Theorem 3.11 (a),(2) and Proposition 3.5 in [45]). □

Lemma 3.2.6 justifies the following

Definition 3.2.7. *The sheaf cohomology of the fine sheaf $\mathcal{S}$ is defined as*

$$\mathrm{H}^i(X, \mathcal{S}) = \mathrm{H}^i(X, \mathcal{S}; \{\mathcal{S}^*\}).$$

where $0 \to \mathcal{S} \to \mathcal{S}^*$ is any fine resolution of $\mathcal{S}$.

3.3 de Rham resolution of $\mathcal{F}$

The goal of this section is to define a de Rham-like fine resolution of the sheaf $\mathcal{F}$. This presents no problems at the nonsingular points of the polarization P but special care needs to be taken at the singular points of P.

Definition 3.3.1. *For $0 \leq k \leq n-1$, we define the morphism $\mathrm{d}^\nabla : \mathcal{L}_P^k \to \mathcal{L}_P^{k+1}$ as follows: Let U be an open set, $\alpha \in \mathcal{L}_P^k(U)$, and $X_0, X_1, \ldots, X_k \in \mathcal{P}(U)$. Then $\mathrm{d}^\nabla \alpha \in \mathcal{L}_P^{k+1}(U)$ is defined by the formula*

$$(\mathrm{d}^\nabla \alpha)(X) = \nabla_X \alpha \quad \text{for } k = 0 \ ,$$

and

$$\begin{aligned}(\mathrm{d}^\nabla \alpha)(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i \, \nabla_{X_i} \big(\alpha(X_0, \ldots, \widehat{X}_i, \ldots, X_k) \big) \\ &+ \sum_{i<j} (-1)^{i+j} \, \alpha([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k) \ .\end{aligned} \tag{3.16}$$

for $1 \leq k \leq n-1$, where the hat means that the corresponding term is missing.

Lemma 3.3.2. *The morphism* $\mathrm{d}^\nabla : \mathcal{L}_P^k \to \mathcal{L}_P^{k+1}$ *satisfies* $(\mathrm{d}^\nabla)^2 = 0$.

Proof. The proof that $(\mathrm{d}^\nabla)^2 = 0$ is similar to the proof that $\mathrm{d}^2 = 0$, so we omit it.

□

Lemma 3.3.3. *The inclusion* $0 \to \mathcal{F} \to \mathcal{L}_P^0$ *is the kernel of* $\mathrm{d}^\nabla : \mathcal{L}_P^0 \to \mathcal{L}_P^1$.

Proof. For $\psi \in \mathcal{L}_P^0$, $(\mathrm{d}^\nabla\psi)(X) = \nabla_X\psi$. Thus $\mathrm{d}^\nabla\psi = 0$ if and only if ψ is flat in the P direction, i.e., ψ is a section of $\mathcal{F}$ (3.9). □

Lemma 3.3.4. *Let* $m \in M$ *and suppose* $\dim P_m = n$. *Then the sequence of stalks*

$$0 \to \mathcal{F}_m \to (\mathcal{L}_P^0)_m \stackrel{\mathrm{d}^\nabla}{\to} (\mathcal{L}_P^1)_m \stackrel{\mathrm{d}^\nabla}{\to} \cdots \stackrel{\mathrm{d}^\nabla}{\to} (\mathcal{L}_P^n)_m \to 0 \tag{3.17}$$

is exact.

Proof. If $\dim P_m = n$, the polarization is non-singular in a neighborhood of m. In other words, there exists Hamiltonians $H_1, \ldots, H_n$ such that $X_{H_1}, \ldots, X_{H_n}$ are linearly independent vectors at each point in a neighborhood of m. Then the exactness of the sequence (3.17) is proved in [31, Theorem 3]. □

Remark 3.3.5. Lemma 3.3.4 can be stated briefly by saying that the Poincaré lemma ("closed implies locally exact") holds at the nonsingular points of P. ▲

By Lemma 3.2.2, the sheaves $\mathcal{L}_P^k$ of $\mathbb{L}$-valued polarized k-forms are fine. Thus $0 \to \mathcal{F} \to \mathcal{L}_P^*$ will be a fine resolution if P has no singular points. To obtain a fine resolution of $\mathcal{F}$ when P is singular, we will need to change $\mathcal{L}_P^*$ appropriately depending on the types of singularities that P has. This will be the goal in the next few subsections.

3.4 Examples of fine resolutions for $n = 1$

3.4.1 A fine resolution for an elliptic singularity, $n = 1$

We specialize to the case that the manifold M is a small open disk centered at the origin of $\mathbb{R}^2$ with canonical coordinates (q, p) and Liouville 1-form $\theta = \frac{1}{2}(p\,\mathrm{d}q - q\,\mathrm{d}p)$. Let $H = \frac{1}{2}(p^2 + q^2)$ and let P be the polarization generated by $X_H = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$. Assume that the only integral value H obtains in M is 0 (which occurs at the origin only). We start by considering the global $\mathbb{L}$-valued polarized 0- and 1-forms (note that in this case $\mathcal{L}_P^k = 0$ for $k \geq 2$ automatically).

Lemma 3.4.1. *Let $\alpha \in \mathcal{L}_P^1(M)$. Then there exists $\psi \in \mathcal{L}_P^0(M)$ satisfying $\mathrm{d}^\nabla \psi = \alpha$ if and only if $\langle \alpha, X_H \rangle$ vanishes at the origin.*

Proof. Let $\phi := \langle \alpha, X_H \rangle \in \mathcal{L}_P^0(M)$. Then $\mathrm{d}^\nabla \psi = \alpha$ holds if and only if $\nabla_{X_H} \psi = \phi$. Thus we need to find $\psi \in \mathcal{L}_P^0(M)$ satisfying $X_H(\psi) - \mathrm{i}H\psi = \phi$ (recall (2.14)). Let $t \in S^1$ be the angle coordinate such that (t, H) are canonical coordinates, so that $\omega = \mathrm{d}H \wedge \mathrm{d}t$ and $X_H = \frac{\partial}{\partial t}$. Writing $\psi = \psi(t, H)$, the equation for ψ becomes

$$\frac{\partial \psi}{\partial t} - \mathrm{i}H\psi = e^{\mathrm{i}Ht}\frac{\partial}{\partial t}(e^{-\mathrm{i}Ht}\psi) = \phi \ . \tag{3.18}$$

Introduce complex coordinates in M by setting $z = q - \mathrm{i}p$, then $z = \sqrt{2H}\mathrm{e}^{-\mathrm{i}t}$ and $\bar{z} = q + \mathrm{i}p = \sqrt{2H}\mathrm{e}^{\mathrm{i}t}$. Let $\psi_{a,b}$ and $\phi_{a,b}$ be the coefficients of $z^a \bar{z}^b$ in the Taylor expansion about the origin of ψ and ϕ, respectively. Since $H = \frac{1}{2}z\bar{z}$ and $\frac{\partial}{\partial t}(z^a \bar{z}^b) = \mathrm{i}(b - a)z^a \bar{z}^b$, a direct calculation shows that the Taylor coefficients of ψ must satisfy $0 = \phi_{0,0}$, $-\mathrm{i}\psi_{1,0} = \phi_{1,0}$, $\mathrm{i}\psi_{0,1} = \phi_{0,1}$, and

$$\mathrm{i}(b - a)\psi_{a,b} - \frac{\mathrm{i}}{2}\psi_{a-1,b-1} = \phi_{a,b} \ , \qquad a, b \geq 1 \ . \tag{3.19}$$

This system has a solution $\psi_{a,b}$ if and only if $\phi_{0,0} = 0$. Thus the vanishing of ϕ at the origin of $\mathbb{R}^2$ is a necessary condition for equation (3.18) to have a solution.

Now we assume that the value of ϕ at the origin of $\mathbb{R}^2$ is zero and show that this is also a sufficient condition. We may assume that ϕ has vanishing Taylor coefficients at the origin because we can replace ϕ with $(\phi - \nabla_{X_H}\widetilde{\psi})$, where $\widetilde{\psi}$ is a smooth section with Taylor coefficients $\widetilde{\psi}_{a,b}$ satisfying (3.19) (as guaranteed by Borel's Theorem, see, e.g., [18, Theorem 1.2.6]).

Integration of (3.18) yields

$$\psi(t,H) = e^{\mathrm{i}Ht}\left[\psi(0,H) + \int_0^t \mathrm{e}^{-\mathrm{i}Hs}\phi(s,H)\,\mathrm{d}s\right] .$$

The section ψ will be 2π-periodic in t if and only if $\psi(0,H) = \psi(2\pi,H)$, which is equivalent to

$$\psi(0,H) = \frac{1}{\mathrm{e}^{-2\pi\mathrm{i}H} - 1}\int_0^{2\pi} e^{-\mathrm{i}Hs}\phi(s,H)\,\mathrm{d}s .$$

Thus we get a unique solution away from the origin, i.e., for $H \neq 0$. A straightforward computation shows that we can rewrite ψ as

$$\begin{aligned}\psi(t,H) &= \frac{1}{\mathrm{e}^{-2\pi\mathrm{i}H} - 1}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}Hs}\phi(t+s,H)\,\mathrm{d}s \\ &= \frac{\mathrm{e}^{\mathrm{i}Ht}}{\mathrm{e}^{-2\pi\mathrm{i}H} - 1}\int_t^{t+2\pi} \mathrm{e}^{-\mathrm{i}Hs}\phi(s,H)\,\mathrm{d}s .\end{aligned} \tag{3.20}$$

Let Φ_t^H denote the time-t flow of X_H. We can rewrite (3.20) again more invariantly as

$$\psi(m) = \frac{1}{\mathrm{e}^{-2\pi\mathrm{i}H(m)} - 1}\int_0^{2\pi} e^{-\mathrm{i}H(m)s}\phi(\Phi_s^H(m))\,\mathrm{d}s , \qquad m \in M . \tag{3.21}$$

It is clear from this expression that ψ is smooth away from the origin of $\mathbb{R}^2$. It remains to check that ψ extends smoothly to the origin. Since ϕ has vanishing Taylor series at the origin, $\phi = H\widetilde{\phi}$ for some smooth section $\widetilde{\phi}$, and we obtain

$$\begin{aligned}\psi(m) &= \frac{1}{\mathrm{e}^{-2\pi\mathrm{i}H(m)} - 1} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}H(m)s} \phi(\Phi_s^H(m))\,\mathrm{d}s \\ &= \frac{H(m)}{\mathrm{e}^{-2\pi\mathrm{i}H(m)} - 1} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}H(m)s} \widetilde{\phi}(\Phi_s^H(m))\,\mathrm{d}s\ .\end{aligned}$$

With the H factor out front, it is clear that this expression extends smoothly to the origin. $\square$

Lemma 3.4.2. *The image of* $\mathrm{d}^\nabla : \mathcal{L}_P^0 \to \mathcal{L}_P^1$ *is the sheaf* $\mathcal{L}_{M|P}^1$ (3.13).

Proof. Let $(0,0)$ stand for the origin in $\mathbb{R}^2$ (written in (q,p) coordinates).

First consider the case $m \in M\backslash\{(0,0)\}$. In this case we can apply Lemma 3.1.18 with $k = n = 1$, according to which the map $(\Upsilon_P^1)_m : (\mathcal{L}_M^1)_m \to (\mathcal{L}_P^1)_m$ (recall (3.12)) between the stalks is surjective, so that $(\mathcal{L}_{M|P}^1)_m = (\Upsilon_P^1)_m\big((\mathcal{L}_M^1)_m\big) = (\mathcal{L}_P^1)_m$. On the other hand, Lemma 3.3.4 guarantees the exactness of the sequence $0 \to \mathcal{F}_m \to (\mathcal{L}_P^0)_m \xrightarrow{\mathrm{d}^\nabla} (\mathcal{L}_P^1)_m \to 0$, so that $\mathrm{d}^\nabla\big((\mathcal{L}_P^0)_m\big) = (\mathcal{L}_P^1)_m$. Putting these facts together, we obtain $\mathrm{d}^\nabla\big((\mathcal{L}_P^0)_m\big) = (\mathcal{L}_{M|P}^1)_m$.

To prove that $\mathrm{d}^\nabla\big((\mathcal{L}_P^0)_{(0,0)}\big) = (\mathcal{L}_{M|P}^1)_{(0,0)}$, we first show that $\mathrm{d}^\nabla\big((\mathcal{L}_P^0)_{(0,0)}\big) \subset (\mathcal{L}_{M|P}^1)_{(0,0)}$. Let U be an open neighborhood of $(0,0)$ and $\psi_{(0,0)} \in (\mathcal{L}_P^0)_{(0,0)}$ be the germ of a section $\psi \in \mathcal{L}_P^0(U)$ at $(0,0)$. Then $\mathrm{d}^\nabla\psi \in \mathcal{L}_P^1(U)$ is the $\mathbb{L}$-valued polarized 1-form defined by $X_H \mapsto \phi := \nabla_{X_H}\psi = X_H(\psi) - \mathrm{i}H\psi$, so that

$$(\mathrm{d}^\nabla\psi)_{(0,0)} = \big\{(X_H)_{(0,0)} \mapsto \phi_{(0,0)}\big\} \in (\mathcal{L}_P^1)_{(0,0)}\ .$$

But the values of X_H and H at the origin are $X_H(0,0) = 0$ and $H(0,0) = 0$, so the value of $\phi(0,0)$ is also 0. Hence there exist sections $\phi_1, \phi_2 \in \mathcal{L}_P^0(U)$ such that

$\phi = p\phi_1 + q\phi_2$. Let $\alpha_{(0,0)} \in (\mathcal{L}^1_M)_{(0,0)}$ be the germ at $(0,0)$ of the $\mathbb{L}$-valued 1-form $\alpha = \phi_1 \,\mathrm{d}q - \phi_2 \,\mathrm{d}p \in \mathcal{L}^1_M(U)$. Then $\langle \alpha, X_H \rangle = \phi$, hence $\mathrm{d}^\nabla \psi = \alpha | P = \Upsilon^1_P(\alpha) \in \mathcal{L}^1_{M|P}(U)$ (the notation $\alpha|P$ was introduced in Lemma 3.1.18).

It remains to show that d^∇ maps $(\mathcal{L}^0_P)_{(0,0)}$ onto $(\mathcal{L}^1_{M|P})_{(0,0)}$. Let $\alpha = \phi_1 \,\mathrm{d}q - \phi_2 \,\mathrm{d}p \in \mathcal{L}^1_M(U)$ and $\phi = \langle \alpha, X_H \rangle = p\phi_1 + q\phi_2 \in \mathcal{L}^0_P(U)$, as above. Then the image $\Upsilon^1_P(\alpha) \in \mathcal{L}^1_{M|P}$ of α in $\mathcal{L}^1_P(U)$ has the form $X_H \mapsto \phi$. Since $\phi(0,0) = 0$, Lemma 3.4.1 guarantees the existence of $\psi \in \mathcal{L}^0_P(U)$ with $\nabla_{X_H}\psi = \phi$. Thus, d^∇ maps $(\mathcal{L}^0_P)_{(0,0)}$ onto $(\mathcal{L}^1_{M|P})_{(0,0)}$. □

Since $\mathcal{L}^0_P$ is trivially equal to $\mathcal{L}^0_{M|P}$, we can restate our results as follows:

Proposition 3.4.3. *If M is a two dimensional symplectic manifold and P is a polarization with only elliptic singularities, then the sequence*

$$0 \to \mathcal{F} \to \mathcal{L}^0_{M|P} \xrightarrow{\mathrm{d}^\nabla} \mathcal{L}^1_{M|P} \to 0$$

is a fine resolution of $\mathcal{F}$.

3.4.2 A fine resolution for a hyperbolic singularity, $n = 1$

Again, we consider the case that M is a small open disk centered at the origin of $\mathbb{R}^2$ with canonical coordinates (q,p) and Liouville 1-form $\theta = \frac{1}{2}(p\,\mathrm{d}q - q\,\mathrm{d}p)$. Let $H = pq$ and let P be the polarization generated by the Hamiltonian vector field $X_H = q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}$. We assume that the only integral value H obtains in M is 0, which occurs on the union of the coordinate axes.

The following two lemmas are taken from [29, Section 6], where one can find detailed proofs, so we only indicate the ideas.

Lemma 3.4.4. *Suppose that $\rho \in \mathcal{L}^0_P(M)$ is Taylor flat at the origin. Then there exists a solution $\psi \in \mathcal{L}^0_P(M)$ to the equation*

$$\nabla_{X_H}\psi = \rho \ .$$

Proof. This is [29, Lemma 6.2]. For $p, q > 0$ the solution is given by the integral

$$\psi(q,p) = \int_{-t}^{0} \mathrm{e}^{-\mathrm{i}Hs}\rho(\mathrm{e}^{s}q, \mathrm{e}^{-s}p)\,\mathrm{d}s \ , \qquad t = \frac{1}{2}\ln\frac{q}{p} \ .$$

Similar expressions hold in other quadrants. It is shown in [29] that ψ extends smoothly to M. □

Lemma 3.4.5. *There exists a solution $\psi \in \mathcal{L}^0_P(M)$ to the equation*

$$\nabla_{X_H}\psi = \phi \ , \qquad \phi \in \mathcal{L}^0_P(M)$$

if and only if $\phi(0,0) = 0$.

Proof. Let $\psi_{a,b}$ and $\phi_{a,b}$ be the $q^a p^b$ Taylor coefficient at the origin of ψ and ϕ, respectively.

They must satisfy the relations

$$\begin{aligned}
0 &= \phi_{0,0} \ , \\
\psi_{a,a} &= \mathrm{i}\phi_{a+1,a+1} \ , && a \geq 1 \ , \\
\psi_{0,b} &= \frac{\phi_{0,b} + \mathrm{i}\psi_{0,b-1}}{b} \ , && b \geq 1 \ , \\
\psi_{a,0} &= \frac{-\phi_{a,0} - \mathrm{i}\psi_{a-1,0}}{a} \ , && a \geq 1 \ , \\
\psi_{a,b} &= \frac{\phi_{a,b} + \mathrm{i}\psi_{a-1,b-1}}{b-a} \ , && a \geq 1 \ , \ b \geq 1 \ , \ a \neq b \ .
\end{aligned}$$

The first relation imposes the condition $\phi_{0,0} = 0$, while the rest of the relations can be solved recursively to yield a unique solution for $\psi_{a,b}$ (for details see the proof of [29, Lemma 6.1]). Thus, a necessary condition for existence of a solution is $\phi(0,0) = 0$.

On the other hand, if $\phi(0,0) = 0$, then we have shown that there exists ψ such that $(\nabla_{X_H}\psi - \phi)$ is Taylor flat at the origin. Hence by Lemma 3.4.4 there exists ψ such that $\nabla_{X_H}\psi = \phi$. □

Lemma 3.4.6. *The image of* $\mathrm{d}^\nabla : \mathcal{L}^0_P \to \mathcal{L}^1_P$ *is the sheaf* $\mathcal{L}^1_{M|P}$.

Proof. If $m \in M \setminus \{(0,0)\}$, then $\dim P_m = 1$, so the morphism Υ^1_P from (3.12) is surjective, therefore $\left(\mathcal{L}^1_{M|P}\right)_m = (\mathcal{L}^1_P)_m$ by Lemma 3.1.18. On the other hand, Lemma 3.3.4 guarantees that $\mathrm{d}^\nabla\left((\mathcal{L}^0_P)_m\right) = (\mathcal{L}^1_P)_m$.

Now let $m = (0,0)$. First we show that $\mathrm{d}^\nabla\left((\mathcal{L}^0_P)_{(0,0)}\right) \subset (\mathcal{L}^1_{M|P})_{(0,0)}$. Let $\psi_{(0,0)} \in (\mathcal{L}^0_P)_{(0,0)}$, U be an open neighborhood of $(0,0)$, and $\psi \in \mathcal{L}^0_P(U)$ be a section with germ $\psi_{(0,0)}$ at $(0,0)$. Then $\mathrm{d}^\nabla\psi \in \mathcal{L}^1_P(U)$ is the $\mathbb{L}$-valued polarized 1-form defined by $X_H \mapsto \phi := \nabla_{X_H}(\psi) = X_H(\psi) - \mathrm{i}H\psi$, so that

$$(\mathrm{d}^\nabla\psi)_{(0,0)} = \left\{(X_H)_{(0,0)} \mapsto \phi_{(0,0)}\right\} \in (\mathcal{L}^1_P)_{(0,0)} \ .$$

Since $X_H(0,0) = 0$ and $H(0,0) = 0$, we have $\phi(0,0) = 0$, so there exist sections $\phi_1, \phi_2 \in \mathcal{L}^0_P(U)$ such that $\phi = p\phi_1 + q\phi_2$. Let $\alpha_{(0,0)} \in (\mathcal{L}^1_M)_{(0,0)}$ be the germ of the 1-from $\alpha = \phi_2\,\mathrm{d}q - \phi_1\,\mathrm{d}p \in \mathcal{L}^1_M(U)$. Then $\langle\alpha, X_H\rangle = \phi$, hence $\mathrm{d}^\nabla\psi = \alpha|P = \Upsilon^1_P(\alpha) \in \mathcal{L}^1_{M|P}(U)$.

Finally, we show that d^∇ maps $(\mathcal{L}^0_P)_{(0,0)}$ onto $(\mathcal{L}^1_{M|P})_{(0,0)}$. Let $\alpha = \phi_2\,\mathrm{d}q - \phi_1\,\mathrm{d}p \in \mathcal{L}^1_M(U)$ and let $\phi = \langle\alpha, X_H\rangle = p\phi_1 + q\phi_2$. Then the image $\Upsilon^1_P(\alpha) \in \mathcal{L}^1_{M|P}$ of α in $\mathcal{L}^1_P(U)$ has the form $X_H \mapsto \phi$. Since $\phi(0,0) = 0$, there exists ψ with $\nabla_{X_H}\psi = \phi$ by Lemma 3.4.5, therefore d^∇ maps $(\mathcal{L}^0_P)_{(0,0)}$ onto $(\mathcal{L}^1_{M|P})_{(0,0)}$. □

Thus, we have proved the following:

Proposition 3.4.7. *If M is a two dimensional symplectic manifold and P is a polarization with only hyperbolic singularities, then the sequence*

$$0 \to \mathcal{F} \to \mathcal{L}^0_{M|P} \xrightarrow{\mathrm{d}^\nabla} \mathcal{L}^1_{M|P} \to 0$$

is a fine resolution of $\mathcal{F}$.

3.4.3 Summary for $n = 1$

Combining Propositions 3.4.3 and 3.4.7, we get the following theorem:

Theorem 3.4.8. *Let M be a two dimensional symplectic manifold and P a polarization with non-degenerate singularities. Then the sequence*

$$0 \to \mathcal{F} \to \mathcal{L}^0_{M|P} \xrightarrow{\mathrm{d}^\nabla} \mathcal{L}^1_{M|P} \to 0$$

is a fine resolution of $\mathcal{F}$.

Definition 3.4.9. *For M a two dimensional symplectic manifold and P a polarization with non-degenerate singularities, we call the sequence $0 \to \mathcal{F} \to \mathcal{L}^*_{M|P}$ the de Rham resolution of $\mathcal{F}$. It is a fine resolution.*

3.5 A fine resolution for a focus-focus singularity, $n = 2$

The purpose of this subsection is twofold – we discuss the resolution for a focus-focus singularity in a 4-dimensional symplectic manifold, and introduce some notations that will be used in the rest of the dissertation.

Let M be a small open disk centered at the origin of $\mathbb{R}^4$ with canonical coordinates $(q,p) = (q_1, q_2, p_1, p_2)$ and canonical 1-form

$$\theta_0 = \frac{1}{2}(p\,\mathrm{d}q - q\,\mathrm{d}p) := \frac{1}{2}\sum_{i=1}^{2}(p_i\,\mathrm{d}q_i - q_i\,\mathrm{d}p_i)\ . \tag{3.22}$$

Let

$$\mu = (H, J) : M \to \mathbb{R}^2$$

be the moment map given by

$$\mu(q,p) = \big(H(q,p), J(q,p)\big) = (q_1p_1 + q_2p_2, q_1p_2 - q_2p_1)\ . \tag{3.23}$$

The Hamiltonian vector fields generated by the functions H and J are

$$\begin{aligned} X_H &= q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2} - p_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial p_2}\ , \\ X_J &= -q_2\frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} - p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}\ . \end{aligned} \tag{3.24}$$

Since $X_H(q,p)$ and $X_J(q,p)$ are linearly independent when $(q,p) \neq (0,0)$, the polarization P generated by X_H and X_J is nonsingular when $(q,p) \neq (0,0)$. At $(0,0)$, however, both X_H and X_J vanish, and the system has a singularity of focus-focus type. We will assume that the disk M is so small that the only integral value that the functions H and J obtain in M is 0, and this happens at the focus-focus point $(0,0)$.

The covariant derivatives of a section ψ are (recall (2.14))

$$\nabla_{X_H}\psi = X_H(\psi) - \mathrm{i}H\psi\ , \qquad \nabla_{X_J}\psi = X_J(\psi) - \mathrm{i}J\psi\ .$$

For convenience, we introduce complex coordinates

$$z_1 := q_1 + \mathrm{i}q_2, \qquad z_2 := p_1 + \mathrm{i}p_2 \ . \tag{3.25}$$

In these coordinates, the functions H and J can be written as

$$H(z_1, z_2) = \Re(\bar{z}_1 z_2) \ , \qquad J = \Im(\bar{z}_1 z_2) \ , \tag{3.26}$$

so that the moment map $\mu = H + \mathrm{i}J$ can be considered as a function from a small disk in $\mathbb{C}^2$ centered at 0 to $\mathbb{C}$. In these coordinates, the Hamiltonian vector fields X_H and X_J have the form

$$\begin{aligned} X_H(z_1, z_2) &= z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \ , \\ X_J(z_1, z_2) &= \mathrm{i}z_1 \frac{\partial}{\partial z_1} - \mathrm{i}\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \mathrm{i}z_2 \frac{\partial}{\partial z_2} - \mathrm{i}\bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \ . \end{aligned} \tag{3.27}$$

A flow for time t in the X_H direction and time s in the X_J direction is given by the formula

$$\Phi_t^H \circ \Phi_s^J(z_1, z_2) = (\mathrm{e}^{t+\mathrm{i}s} z_1, \mathrm{e}^{-t+\mathrm{i}s} z_2) \ ; \tag{3.28}$$

since X_H and X_J commute, their flows commute as well. The flow of X_J is 2π-periodic. If t and s parameterize the X_H and X_J flows, respectively, we can view (t, s, H, J) as symplectic coordinates; they are related to (q, p) by (3.23) and (3.26) and

$$t = \frac{1}{2} \ln \frac{|z_1|}{|z_2|} \ , \qquad s = \arg z_1 \in S^1 \ .$$

When $z_1 \neq 0$ and $z_2 \neq 0$, this gives well-defined, smooth coordinates (modulo the 2π jump in the ϕ coordinate) such that

$$X_H = \frac{\partial}{\partial t} \ , \qquad X_J = \frac{\partial}{\partial s} \ .$$

Lemma 3.5.1. *Let*

$$U = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \epsilon, |z_2| < \epsilon\} \ ,$$

with $\epsilon > 0$ small enough so that $U \subset M$. Then $\mathcal{F}(U) = \{0\}$. As a consequence, the stalk of $\mathcal{F}$ at the focus-focus point $(0,0) \in M$ is trivial: $\mathcal{F}_{(0,0)} = \{0\}$.

Proof. Let $\psi \in \mathcal{F}(U)$. Then $\nabla_{X_J}\psi = 0$ is equivalent to $\frac{\partial}{\partial s}\psi - \mathrm{i}J\psi = 0$, which implies that

$$\psi(t, s, H, J) = \mathrm{e}^{\mathrm{i}Js}\, \psi(t, 0, H, J) \ ,$$

where $\psi(t, 0, H, J)$ represents the "initial condition" of ψ. The function ψ must be 2π-periodic on the S^1-orbits of X_J, which forces $\mathrm{e}^{2\pi \mathrm{i}J} = 1$ or $\psi(t, 0, H, J) = 0$. Since $\mathrm{e}^{2\pi \mathrm{i}J} = 1$ only when $J = 0$, which is a hypersurface inside U, we must have $\psi(t, 0, H, J) = 0$ for all $(t, 0, H, J)$ in U. This implies that $\psi = 0$, i.e., $\mathcal{F}(U) = \{0\}$ and, hence, $\mathcal{F}_{(0,0)} = \{0\}$. □

Definition 3.5.2. *Let $\mathcal{L}^k_{0,P}$ stand for the sheaf of $\mathcal{C}^\infty_M$-modules of $\mathbb{L}$-valued polarized k-forms satisfying*

$$\mathcal{L}^k_{0,P}(U) = \{\alpha \in \mathcal{L}^k_P(U) : \alpha|V = 0 \text{ on some neighborhood } V \text{ of } (0,0) \in M\} \tag{3.29}$$

on any open set $U \subset M$.

The next lemma is obvious but important.

Lemma 3.5.3. *The stalk of $\mathcal{L}^k_{0,P}$ at any point $m \neq (0,0)$ is $(\mathcal{L}^k_{0,P})_m = (\mathcal{L}^k_P)_m$, and at $m = (0,0)$ it is $(\mathcal{L}^k_{0,P})_{(0,0)} = \{0\}$.*

As an easy corollary we obtain the main result of this section:

Proposition 3.5.4. *The inclusion map $0 \to \mathcal{F} \to \mathcal{L}^0_P$ maps $\mathcal{F}$ into $\mathcal{L}^0_{0,P}$. The morphism d^∇ maps $\mathcal{L}^k_{0,P}$ into $\mathcal{L}^{k+1}_{0,P}$. The sequence $0 \to \mathcal{F} \to \mathcal{L}^*_{0,P}$ is a fine resolution of $\mathcal{F}$.*

Proof. For the first statement, let $U \subset M$ be an open set and $\psi \in \mathcal{F}(U)$. If $(0,0) \in U$, then ψ vanishes on some neighborhood of $(0,0)$ by Lemma 3.5.1, hence $\psi \in \mathcal{L}^0_{0,P}(U)$.

The second statement follows from the simple fact that each term in the right hand side of (3.16) vanishes on some neighborhood of the $(0,0)$ (by the definition of $\mathcal{L}^k_{0,P}$).

For the last statement we need to show that $0 \to \mathcal{F} \to \mathcal{L}^*_{0,P}$ is exact. In other words, for each $m \in M$, the following sequence of stalks is exact:

$$0 \to \mathcal{F}_m \to (\mathcal{L}^0_{0,P})_m \to (\mathcal{L}^1_{0,P})_m \to (\mathcal{L}^2_{0,P})_m \to 0.$$

If $m = (0,0)$, then this sequence is exact because all stalks are $\{0\}$ by Lemmas 3.5.1 and 3.5.3. If $m \neq (0,0)$, then the polarization P is nonsingular at m, and the sequence is exact by Lemmas 3.5.1 and 3.3.4. $\square$

Chapter 4

Focus-focus singularity and the semi-global model

In this section we will describe the model manifold on which we calculate the cohomology groups associated with geometric quantization (see (2.19)). Vũ Ngọc in [41] introduced a topological invariant called *Taylor series invariant* which completely characterizes the neighborhood of a focus-focus fiber. In Section 4.1 we will briefly describe the Taylor series invariant, in Section 4.2 we will follow Section 6 of [41] to construct a neighborhood of the focus-focus fiber with a given Taylor series invariant. In Section 4.3 we will construct a Liouville 1-form used to define a connection ∇ and then show that the focus-focus fiber is the only Bohr-Sommerfeld fiber in a neighborhood of the focus-focus fiber.

4.1 The Taylor series invariant

Let (N, ω_N) be a 4-dimensional symplectic manifold and let $\mu = (H, J) : N \to \mathbb{R}^2$ be a proper moment map, i.e., the pre-image of a compact set is compact and

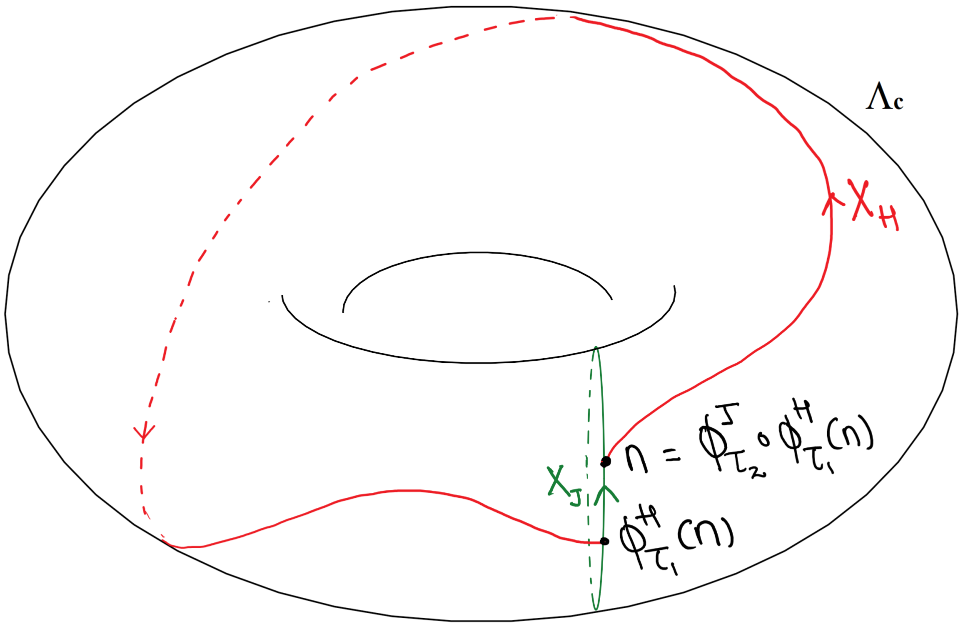

Figure 4.1: On the definition of the times $\tau_1(c)$ and $\tau_2(c)$ on the regular fiber Λ_c.

in particular each fiber $\mu^{-1}(c_1, c_2)$ is compact for all $c = (c_1, c_2) \in \mathbb{R}^2$. Assume that this integrable system has a unique singular point at $n_0 \in N$ such that $\mu(n_0) = (0, 0)$ and the singularity at n_0 is of focus-focus type. In addition let us consider the following identification that we use throughout the dissertation,

$$\mathbb{R}^2 \to \mathbb{C},$$

$$c = (c_1, c_2) \mapsto c_1 + \mathrm{i}c_2.$$

Let us now consider a point n on a regular fiber $\Lambda_c := \mu^{-1}(c)$, $c \neq 0$. Let $S^1 \cdot n$ be the X_J-orbit of the point n. Denote by $\tau_1(c)$ the first time n returns to $S^1 \cdot n$ under the flow of X_H:

$$\tau_1(c) := \min\left\{t > 0 : \Phi^H_t(n) \in S^1 \cdot n\right\}. \tag{4.1}$$

Denote by $\tau_2(c)$ the time needed for the point $\Phi^H_{\tau_1(c)}(n)$ to reach n under the flow of X_J:

$$\tau_2(c) := \min\left\{s \geq 0 : \Phi^J_s \circ \Phi^H_{\tau_1(c)}(n) = n\right\}. \tag{4.2}$$

The quantities $\tau_1(c)$ and $\tau_2(c)$ are represented pictorially in Figure 4.1. In [41] Vũ Ngọc proved that τ_1 and τ_2 are independent of the choice of $n \in \Lambda_c$. Clearly, c approaches $0 \in \mathbb{C}$, $\tau_1(c)$ tends to ∞. For some determination of the complex logarithm, define

$$\sigma_1(c) := \tau_1(c) + \ln|c|\ , \qquad \sigma_2(c) := \tau_2(c) - \arg c. \tag{4.3}$$

In [41] Vũ Ngọc showed that $\sigma_1(c)$ and $\sigma_2(c)$ extend to smooth single-valued functions around the origin and that,

$$\sigma := \sigma_1 \mathrm{d}c_1 + \sigma_2 \mathrm{d}c_2 \tag{4.4}$$

is a closed 1-form.

Definition 4.1.1. *Let $S \in C^\infty(\mathbb{R}^2)$ be the unique function such that*

$$\mathrm{d}S = \sigma, \qquad S(0,0) = 0,$$

where σ is the one form given by (4.4)*. The Taylor series of S at $(0,0)$ denoted by $(S)^\infty$ is the* Taylor series invariant *of the completely integrable system (N, ω_N, μ) defined above.*

Let S be the unique smooth function as in Definition 4.1.1, denote by S_1, S_2 the partial derivatives of $S \in C^\infty(\mathbb{R}^2)$ with respect to the first and second variables respectively,

$$S_1(c) := \frac{\partial S(c)}{\partial c_1}; \qquad S_2(c) := \frac{\partial S(c)}{\partial c_2}, \quad c = (c_1, c_2).$$

Then, since $\mathrm{d}S = \sigma$, we have $\sigma_1(c) = S_1(c)$ and $\sigma_2(c) = S_2(c)$. Hence, the times $\tau_1(c)$ and $\tau_2(c)$ can be expressed as

$$\tau_1(c) = S_1(c) - \ln|c| \ , \qquad \tau_2(c) = S_2(c) + \arg c. \tag{4.5}$$

4.2 Semi-global model

Vũ Ngọc in his seminal paper [41] proved the following classification result.

Theorem 4.2.1. *The set of equivalence classes of the germs of singular Lagrangian fibrations of focus-focus type at the focus-focus leaf is in natural bijection with $\mathbb{R}[[X,Y]]_0$. Here $\mathbb{R}[[X,Y]]$ is the algebra of real formal power series in two variables, and $\mathbb{R}[[X,Y]]_0$ is the subspace of such series with vanishing constant term.*

The Taylor series invariant $(S)^\infty$ from Definition 4.1.1 is considered to be an element of $\mathbb{R}[[X,Y]]_0$, and it classifies the foliations in a neighborhood of the focus-focus fiber. In more detail, this means that another system has the same Taylor series invariant near a focus-focus singularity if and only if there is a symplectomorphism which takes a foliated neighborhood of the singular fiber to a foliated neighborhood of the singular fiber preserving the leaves of the foliation and sending the singular fiber to the singular fiber. Moreover, given a Taylor series expansion of some function, Vũ Ngọc constructed a neighborhood of focus-focus fiber with the given Taylor series expansion as its Taylor series invariant.

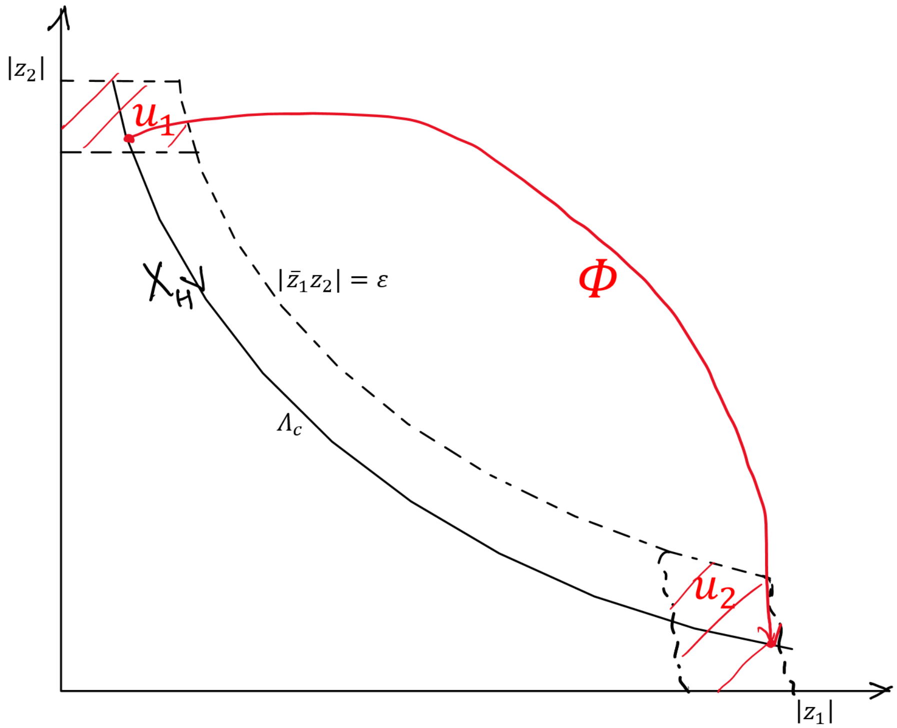

Figure 4.2: On the construction of Vũ Ngọc's semi-global model.

In the following we will give a brief of this construction. In particular we will construct a symplectic manifold (M_ϵ, ω) along with a moment map $\mu : M_\epsilon \to D_\epsilon = \{ c \in \mathbb{C} : |c| < \epsilon \}$ such that:

- the fibers of the moment map, $\Lambda_c = \mu^{-1}(c)$ are compact Lagrangian tori, when $c \in \mathbb{C} \setminus \{0\}$,
- the fiber $\Lambda_0 = \mu^{-1}(0)$ of the moment map, is a focus-focus fiber (a pinched torus) which has a single singular point of focus-focus type,
- the Taylor series invariant $(S)^\infty$ (recall Definition 4.1.1) is the Taylor series expansion at $0 \in D_\epsilon$ of a given smooth function $S : D_\epsilon \to \mathbb{R}$ with $S(0) = 0$.

To begin the construction let us start with the local model described in Section 3.5. Choose a number $\epsilon > 0$, small enough such that the pre-image of the moment map (3.23), $\mu^{-1}(D_\epsilon)$ lies inside the open disk M, i.e, $\mu^{-1}(D_\epsilon) \subset M$. Clearly, the leaves $\mu^{-1}(c)$ of the foliation of M are not compact Lagrangian tori and so our first goal will be to make the leaves $\mu^{-1}(c)$ into tori Λ_c.

Let $S : D_\epsilon \to \mathbb{R}$ be a smooth function with $S(0) = 0$, $S_1(c), S_2(c)$ be the partial derivatives of S with respect the first and second variables, and let $\tau_1(c), \tau_2(c)$ be defined by the formula (4.5). With these notations, let us define the following:

$$
\begin{aligned}
\Phi := \Phi^J_{\tau_2(c)} \circ \Phi^H_{\tau_1(c)} : \mu^{-1}(D_\epsilon) &\to \mu^{-1}(D_\epsilon) \\
(z_1, z_2) &\mapsto \Phi^J_{\tau_2(c)} \circ \Phi^H_{\tau_1(c)}(z_1, z_2), \qquad (z_1, z_2) \in \mu^{-1}(c).
\end{aligned}
\tag{4.6}
$$

The map Φ is the combined X_H and X_J flow by the times $\tau_1(c)$ and $\tau_2(c)$ acting on the points $(z_1, z_2) \in \mu^{-1}(c) \subset \mu^{-1}(D_\epsilon)$. Equations (3.28), (4.1), (4.2), (3.26), and (4.5) yield

$$
\begin{aligned}
\Phi(z_1, z_2) &= \Phi^J_{\tau_2(c)} \circ \Phi^H_{\tau_1(c)}(z_1, z_2) \\
&= \left(\mathrm{e}^{\tau_1(c)+\mathrm{i}\tau_2(c)}\, z_1, \mathrm{e}^{-\tau_1(c)+\mathrm{i}\tau_2(c)}\, z_2\right) \\
&= \left(\mathrm{e}^{S_1(c)+\mathrm{i}S_2(c)}\, \bar{c}^{-1} z_1, \mathrm{e}^{-S_1(c)+\mathrm{i}S_2(c)}\, c z_2\right) \\
&= \left(\mathrm{e}^{S_1(c)+\mathrm{i}S_2(c)}\, \bar{z}_2^{-1}, \mathrm{e}^{-S_1(c)+\mathrm{i}S_2(c)}\, \bar{z}_1 z_2^2\right) .
\end{aligned}
\tag{4.7}
$$

Let $\delta > 0$ be a very small positive number, and define

$$
\begin{aligned}
U_1 &:= \{(z_1, z_2) \in \mathbb{C}^2 : 1 - \delta < |z_2| < 1 + \delta\} \cap \mu^{-1}(D_\epsilon) \, ; \\
U_2 &:= \Phi(U_1).
\end{aligned}
$$

The sets U_1, U_2 are represented pictorially in the Figure 4.2. In this figure $\mu^{-1}(D_\epsilon)$ is represented by the domain between the dashed line with equation

$|\bar{z}_1 z_2| = \epsilon$ and the $|z_1|, |z_2|$ axes.

Remark 4.2.2. Figure 4.2 should be interpreted with care because two dimensions are missing from it. If, say, we think of the figure as representing the 3-dimensional manifold $\{J = 0\}$ (which will play an important role later), then each point in the solid curve represents a circle (obtained by the flow of X_J), and each segment of this curve represents a cylinder, the direction along the curve being the direction of the flow of X_H. The origin of the coordinate system represents the focus-focus point, and the focus-focus torus will be constructed from the coordinate axes in the figure. ▲

If we choose δ small enough such that $U_1 \cap U_2 = \emptyset$, then Φ maps U_1 symplectomorphically onto its image, U_2 [41, Lemma 6.1].

With the help of the symplectomorphism Φ, we finish the construction of the manifold M_ϵ (4.9). By identifying a point $m_1 \in U_1$ with a point $m_2 \in U_2$ as follows:

$$m_1 \sim m_2 \quad \Longleftrightarrow \quad m_2 = \Phi(m_1)\ . \tag{4.8}$$

Let M_ϵ be the set consisting of U_1, U_2, and all the points "between them", with U_1 and U_2 identified by (4.8);

$$M_\epsilon := \overline{\left\{\Phi_s^J \circ \Phi_t^H(m) : s \in [0, 2\pi),\ t \in [0, \tau_1\big(\mu(m)\big)], m \in U_1\right\} / \sim}\ . \tag{4.9}$$

The purpose of taking the closure is to include the points from the focus-focus torus in M_ϵ. Here we reintroduce the notation $\Lambda_c := \mu^{-1}(c)$ for the (pinched) tori foliating M_ϵ.

The map Φ defined in (4.7)) is a symplectomorphism and hence, the new manifold M_ϵ obtained by the identification of U_1 and U_2 inherits the symplectic form from $\mu^{-1}(D_\epsilon)$.

4.3 Bohr-Sommerfeld fibers for the semi-global model

In this section, we will show that the focus-focus fiber is the only Bohr-Sommerfeld fiber in a neighborhood of the focus-focus fiber. To this end, we first construct a 1-form on the semi-global model. In section 4.2 we constructed a semi-global model for the focus-focus torus with the help of the symplectomorphism Φ. However, we want the map Φ to be an exact symplectomorphism, i.e., Φ also preserves the 1-form θ, so that the manifold M_ϵ obtained after identifying U_1 and U_2 would have a globally defined 1-form θ such that $\omega_0 = \mathrm{d}\theta$. To this end, we choose (arbitrarily) that on U_2 we require that $\theta|U_2 = \theta_0|U_2$, and in the Lemma below compute the pull-back $\Phi^*(\theta|U_2) \in \Omega^1(U_1)$, which will be used to construct a globally defined 1-form θ.

Lemma 4.3.1. *The pull-back of the diffeomorphism $\Phi : U_1 \to U_2$ is given by*

$$\Phi^*(\theta_0|U_2) = \Big\{\theta_0 + \mathrm{d}\left[-H + H\cdot(S_1\circ\mu) + J\cdot(S_2\circ\mu) - S\circ\mu\right]\Big\}\Big|\, U_1\ .$$

Proof. For brevity, in the calculations below we temporarily write $S(c)$ instead of $S(\mu(z_1, z_2))$, and similarly for $S_1(c)$ and $S_2(c)$. From (4.7), we have $z_1 \circ \Phi(z_1, z_2) =$

$e^{S_1(c)+iS_2(c)}\,\bar{z}_2^{-1}$, $z_2 \circ \Phi(z_1, z_2) = e^{-S_1(c)+iS_2(c)}\,\bar{z}_1 z_2^2$, so

$$
\begin{aligned}
\Phi^*(z_2\,d\bar{z}_1) &= (z_2 \circ \Phi)\,d(\bar{z}_1 \circ \Phi) \\
&= e^{-S_1(c)+iS_2(c)}\,\bar{z}_1 z_2^2\,d\big[e^{S_1(c)-iS_2(c)}\,z_2^{-1}\big] \\
&= \bar{z}_1 z_2\,d\big[S_1(c) - iS_2(c)\big] - \bar{z}_1\,dz_2
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\Phi^*(z_1\,d\bar{z}_2) &= (z_1 \circ \Phi)\,d(\bar{z}_2 \circ \Phi) \\
&= e^{S_1(c)+iS_2(c)}\,\bar{z}_2^{-1}\,d\big[e^{-S_1(c)-iS_2(c)}\,z_1\bar{z}_2^2\big] \\
&= -z_1\bar{z}_2\,d\big[S_1(c) + iS_2(c)\big] + \bar{z}_2\,dz_1 + 2z_1\,d\bar{z}_2\ .
\end{aligned}
$$

Using the expressions(3.22), 3.25, $\theta_0 = \frac{1}{2}\Re\,(z_2\,d\bar{z}_1 - z_1\,d\bar{z}_2)$. We obtain

$$
\begin{aligned}
\Phi^*\left(\theta_0|U_2\right) &= \frac{1}{2}\Re\left[\Phi^*(z_2\,d\bar{z}_1 - z_1\,d\bar{z}_2)\right] \\
&= \frac{1}{2}\Re\,(\bar{z}_2\,dz_1 - \bar{z}_1\,dz_2) - \Re\,(\bar{z}_2\,dz_1 + z_1\,d\bar{z}_2) \\
&\quad + \frac{1}{2}\Re\left\{c\,d\big[S_1(c) - iS_2(c)\big] + \bar{c}\,d\big[S_1(c) + iS_2(c)\big]\right\} \\
&= \theta_0 - 2\Re\,d\overline{(H + iJ)} + \Re\left\{(H + iJ)\,d\big[S_1(c) - iS_2(c)\big]\right\} \\
&= \theta_0 - dH + H\,dS_1 + J\,dS_2 \\
&= \theta_0 + d\left(-H + H\cdot S_1 + J\cdot S_2 - S\right).
\end{aligned}
$$

$\square$

Let $\chi_0 : \mu^{-1}(D_\epsilon) \to \mathbb{R}$ be a function satisfying the conditions

(a) χ_0 does not depend on J, i.e., it is constant on the flow lines of X_J;

(b) $\chi_0|U_1 \equiv 1$, $\chi_0|U_2 \equiv 0$.

Define the function

$$\chi := \chi_0 \cdot [-H + H \cdot (S_1 \circ \mu) + J \cdot (S_2 \circ \mu) - S \circ \mu] : \mu^{-1}(D_\epsilon) \to \mathbb{R} \ , \qquad (4.10)$$

and let θ be the 1-form on $\mu^{-1}(D_\epsilon)$ defined by

$$\theta := \theta_0 + \mathrm{d}\chi \in \Omega^1\big(\mu^{-1}(D_\epsilon)\big) \ . \qquad (4.11)$$

Lemma 4.3.1 implies that θ satisfies $\Phi^*(\theta|U_2) = \theta|U_1$, so that, if we endow $\mu^{-1}(D_\epsilon)$ with the 1-form θ given by (4.11), then the map $\Phi : U_1 \to U_2$ (4.7) preserves θ (and, hence, is an exact symplectomorphism).

Lemma 4.3.2. *The 1-form θ defined on $\mu^{-1}(D_\epsilon)$ in equation* (4.11) *induces a well-defined 1-form on M_ϵ. Abusing notation, we will denote θ to be the induced 1-form.*

Proof. This follows from the fact the $\Phi^*(\theta|U_2) = \theta|U_1$. $\square$

From now on we consider the manifold M_ϵ constructed in (4.9) endowed with the 1-form $\theta \in \Omega^1(M_\epsilon)$ (4.11) and symplectic form $\omega = \mathrm{d}\theta = \omega_0 \in \Omega^2(M_\epsilon)$. We call this *semi-global model* [41] for an open neighborhood of the focus-focus torus $\Lambda_{(0,0)}$ (note that we slightly modified Vũ Ngọc's model to make M_ϵ an exact symplectic manifold).

Since the functions H, J, $S_1 \circ \mu$, and $S_2 \circ \mu$ are all constants of motion, and χ_0 is independent of J, we have $X_J(\chi) = 0$ (recall (4.10)), so $\langle \theta, X_J \rangle = \langle \theta_0, X_J \rangle = J$, which implies that

$$\langle \theta, X_J \rangle = J. \qquad (4.12)$$

In the following, we will show that the focus-focus fiber $\Lambda_{(0,0)}$ is the only

Bohr-Sommerfeld fiber (recall Definition 2.2.20) for the semi-global model for the focus-focus singularity. This property plays an important role in the calculation of the cohomologies in Chapter 6.

Lemma 4.3.3. *If $\epsilon > 0$ is small enough, then the only Bohr-Sommerfeld fiber in M_ϵ is the focus-focus fiber $\Lambda_{(0,0)}$.*

Proof. First, we will show that when $c = c_1 + \mathrm{i}c_2 \in D_\epsilon \setminus \{0\}$, the fibers Λ_c are not Bohr-Sommerfeld fiber.

Suppose that ψ is a P-flat section on Λ_c, then $\nabla_{X_H}\psi = 0 = \nabla_{X_J}\psi$. Let $m \in \Lambda_c$, then the condition for P-flat implies that $\Pi^H_{-t}\Pi^J_{-s}\psi\big(\Phi^H_t \circ \Phi^J_s(m)\big) = \psi(m)$ for all $t \in \mathbb{R}, s \in [0, 2\pi]$. Recalling the defnitions $\tau_1(c), \tau_2(c)$ from (4.1) and (4.2), we require $\Pi^H_{-\tau_1(c)}\Pi^J_{-\tau_2(c)}$ and $\Pi^J_{2\pi}$ to be trivial. In order to compute these, we calculate the action integrals along the paths γ_H and γ_J defined below.

Let γ_H be the path starting at m, going along the flow of X_H until it hits $S^1 \cdot m$ at time $\tau_1(c)$, and then going along the X_J-flow until it returns to m and let γ_J be the path starting at m and going along the flow of X_J until it returns to m at time 2π. The curves γ_H, γ_J are illustrated in Figure 4.3

Using (4.10), (4.11), the fact that the change of χ_0 along the path γ_H is $\Delta\chi_0 = -1$, and that $c = c_1 + \mathrm{i}c_2 = \mu(m) = H(m) + \mathrm{i}J(m)$, we obtain the action

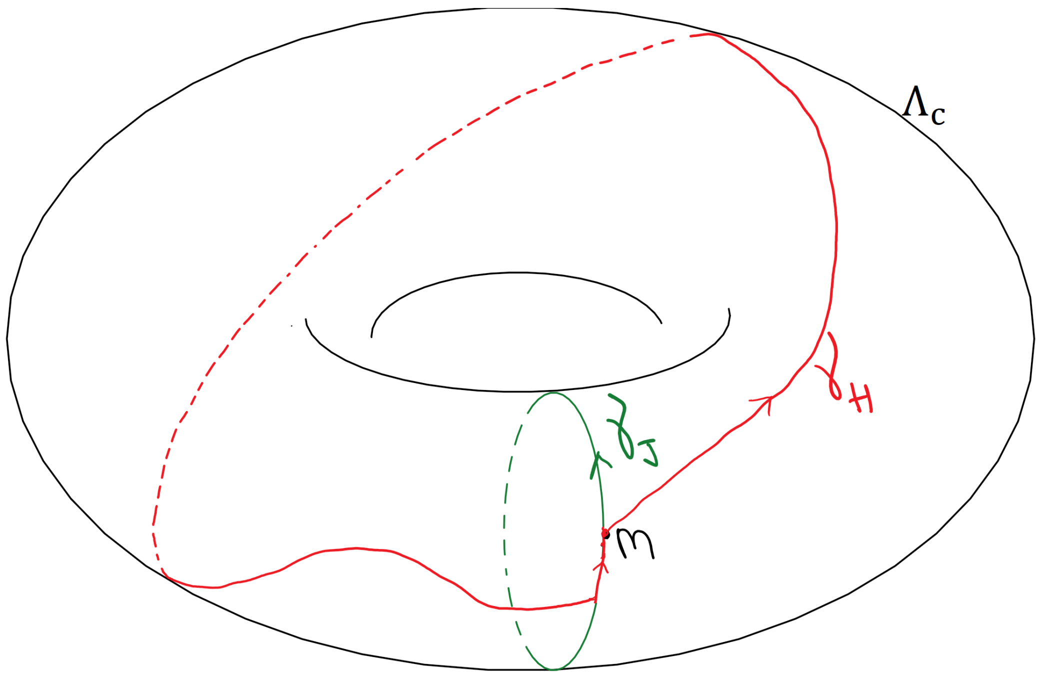

Figure 4.3: Curves γ_H, γ_J starting at $m \in \Lambda_c$.

integral around the loop γ_H:

$$\begin{aligned}
\mathcal{A}(\gamma_H) &= \int_0^{\tau_1(c)} \langle \theta, X_H \rangle \circ \Phi_t^H(m)\,\mathrm{d}t + \int_0^{\tau_2(c)} \langle \theta, X_J \rangle \circ \Phi_s^J \circ \Phi_{\tau_1(c)}^H(m)\,\mathrm{d}s \\
&= \int_0^{\tau_1(c)} \Big\{ H + \big[-H + H \cdot (S_1 \circ \mu) + J \cdot (S_2 \circ \mu) \\
&\qquad\qquad - S \circ \mu \big]\, X_H(\chi_0) \Big\} \circ \Phi_t^H(m)\,\mathrm{d}t + c_2 \tau_2(c) \\
&= c_1 \tau_1(c) + \big[-c_1 + c_1 S_1(c) + c_2 S_2(c) - S(c) \big] \cdot \Delta\chi_0 + c_2 \tau_2(c) \\
&= c_1 \tau_1(c) + c_1 - c_1 S_1(c) - c_2 S_2(c) + S(c) + c_2 \tau_2(c) \\
&= c_1 [S_1(c) - \ln|c|] + c_1 - c_1 S_1(c) - c_2 S_2(c) + S(c) + c_2 [S_2(c) + \arg c] \\
&= -c_1 \ln|c| + c_1 + S(c) + c_2 \arg c \,.
\end{aligned}$$

Using (4.12), we obtain the action integral along the loop γ_J:

$$\mathcal{A}(\gamma_J) = \int_0^{2\pi} \langle \theta, X_J \rangle \circ \Phi_s^J(m)\, \mathrm{d}s = \int_0^{2\pi} J \circ \Phi_s^J(m)\, \mathrm{d}s = 2\pi c_2 \ .$$

If $\epsilon > 0$ is small, then $|c_2| \leq \epsilon$ is also small, and it is clear that $\mathcal{A}(\gamma_2)$ is an integer multiple of 2π exactly when $c_2 = 0$. Hence, the holomomy $\Pi_{2\pi}^J$ along γ_J is trivial only when $c_2 = 0$. Thus, the only fibers which could be Bohr-Sommerfeld are the fibers Λ_c such that $c = c_1$.

Since $S(0,0) = 0$, by approximating $S(c_1)$ by its first order Taylor series expansion $c_1 S_1(c_1)$, we obtain $\mathcal{A}(\gamma_H) = -c_1 \ln|c_1| + c_1 + c_1 S_1(c)$, using (4.5), we get that $\mathcal{A}(\gamma_H) = c_1[1 + \tau_1(c_1)]$. Since $\tau_1(c) > 0$, this expression is not zero when $c_1 \neq 0$.

Now, let $c = (0,0)$, then note that equation (4.10), (4.11) implies that $\langle \theta, X_H \rangle = 0$, and $\langle \theta, X_J \rangle = J = 0$ on $\Lambda_{(0,0)}$. Together this implies that the action integral around any loop in $\Lambda_{(0,0)} = 0$ and hence, the focus-focus fiber $\Lambda_{(0,0)}$ is a Bohr-Sommerfeld fiber.

$\square$

Chapter 5

Geometric tools

In this section we breifly describe some geometric tools for cirlce actions. Most of the definitions and some proofs were given by Rawnsley and Solha [31, 38].

Definition 5.0.1. *Let $X \in \Gamma(\mathcal{P})$, and $\Phi_t^X : M \to M$ be the flow of X. Denote by Π_t^X the operator of* parallel transport *in $\mathbb{L}$ along the integral curves of X, i.e., for $\psi \in \Gamma(\mathcal{L}_P^0)$,*

$$(\nabla_X \psi)(m) = \lim_{t \to 0} \frac{\Pi_{-t}^X \, \psi(\Phi_t^X(m)) - \psi(m)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} \, \Pi_{-t}^X \, \psi(\Phi_t^X(m)) \bigg|_{t=0} . \tag{5.1}$$

When $X = X_H$, we will denote its flow by Φ_t^H and the parallel transport along it by Π_t^H; similarly for $X = X_J$.

Definition 5.0.2. *Define the action $\widetilde{\Phi}_t^{X*}$ of the flow Φ_t^X on the $\mathbb{L}$-valued polarized k-forms by*

$$\begin{aligned} \Big[\big(\widetilde{\Phi}_t^{X*} \alpha \big) \, (X_1, \ldots, X_k) \Big] \, (m) &= \Pi_{-t}^X \left[\alpha \big(\Phi_{t*}^X X_1, \ldots, \Phi_{t*}^X X_1 \big) \circ \Phi_t^X(m) \right] \\ &= \Pi_{-t}^X \big[\alpha(\Phi_t^X(m)) \big(T_m \Phi_t^X \cdot X_1(m), \ldots, T_m \Phi_t^X \cdot X_k(m) \big) \big] \ , \end{aligned} \tag{5.2}$$

where $\alpha \in \Gamma(\mathcal{L}_P^k)$, $X, X_1, \ldots, X_k \in \Gamma(\mathcal{P})$, *and* $T_m\Phi_t^X$ *is the derivative of* Φ_t^X *at* $m \in M$. *When* $X = X_J$ *or* $X = X_H$, *we write* $\widetilde{\Phi}_t^{H*}$ *and* $\tilde{\Phi}_t^{J*}$ *instead of* $\widetilde{\Phi}_t^{X_H*}$ *and* $\tilde{\Phi}_t^{X_J*}$.

Definition 5.0.3. *The* covariant Lie derivative *along the vector field* $X \in \Gamma(\mathcal{P})$ *acting on for* $\mathbb{L}$*-valued polarized* k*-forms is defined by*

$$\pounds_X^{\nabla}\alpha := \lim_{t\to 0} \frac{\widetilde{\Phi}_t^{X*}\alpha - \alpha}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{\Phi}_t^{X*}\alpha\bigg|_{t=0} \;, \qquad \alpha \in \Gamma(\mathcal{L}_P^k)\;. \tag{5.3}$$

In the following lemma we collect several facts about the concepts introduced above.

Lemma 5.0.4. *The statements in the first several parts of this lemma are general, while the last parts are about the particular case of the vector fields* X_H *and* X_J.

(a) *If* $\Phi_t^{X*} := (\Phi_{-t}^X)_*$ *stands for the pull-back of vector fields, then for* $\alpha \in \Gamma(\mathcal{L}_P^k)$ *and* $X, X_1, \ldots, X_k \in \Gamma(\mathcal{P})$,

$$\left(\widetilde{\Phi}_t^{X*}\alpha\right)\left(\Phi_t^{X*}X_1, \ldots, \Phi_t^{X*}X_k\right) = \widetilde{\Phi}_t^{X*}\big(\alpha\left(X_1, \ldots, X_k\right)\big)\;. \tag{5.4}$$

(b) *The operator* $\pounds_X^{\nabla}$ *acting on* $\mathbb{L}$*-valued 0-forms is the covariant derivative:*

$$\pounds_X^{\nabla}\psi = \nabla_X\psi\;, \qquad \psi \in \Gamma(\mathcal{L}_P^0)\;. \tag{5.5}$$

(c) For any vector fields $X, Y \in \Gamma(\mathcal{P})$, the following relations hold on $\Gamma(\mathcal{L}^k_P)$:

$$\iota_{\Phi^{X*}_t Y} \circ \widetilde{\Phi}^{X*}_t = \widetilde{\Phi}^{X*}_t \circ \iota_Y \ , \tag{5.6}$$

$$\iota_{\Phi^{X*}_t Y} \circ \pounds^\nabla_X = \pounds^\nabla_X \circ \iota_Y \ , \tag{5.7}$$

$$\pounds^\nabla_{\Phi^{X*}_t Y} \circ \widetilde{\Phi}^{X*}_t = \widetilde{\Phi}^{X*}_t \circ \pounds^\nabla_Y \ . \tag{5.8}$$

In particular, for any vector field X, the contraction ι_X, the action $\widetilde{\Phi}^{X*}_t$ on $\Gamma(\mathcal{L}^k_P)$, and the covariant Lie derivative $\pounds^\nabla_X$ commute with one another.

(d) The evolution of $\widetilde{\Phi}^{X*}_t$ is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{\Phi}^{X*}_t = \widetilde{\Phi}^{X*}_t \circ \pounds^\nabla_X = \pounds^\nabla_X \circ \widetilde{\Phi}^{X*}_t \ . \tag{5.9}$$

(e) For any vector field $X \in \Gamma(\mathcal{P})$, the covariant exterior derivative commutes with the action $\widetilde{\Phi}^{X*}_t$ on $\Gamma(\mathcal{L}^k_P)$ and with the covariant Lie derivative $\pounds^\nabla_X$:

$$\mathrm{d}^\nabla \circ \widetilde{\Phi}^{X*}_t = \widetilde{\Phi}^{X*}_t \circ \mathrm{d}^\nabla \ , \tag{5.10}$$

$$\mathrm{d}^\nabla \circ \pounds^\nabla_X = \pounds^\nabla_X \circ \mathrm{d}^\nabla \ . \tag{5.11}$$

(f) The covariant Lie derivative satisfies the Leibniz rule*: for $X, X_1, \ldots, X_k \in \Gamma(\mathcal{P})$ and $\alpha \in \Gamma(\mathcal{L}^k_P)$,*

$$\begin{aligned} \pounds^\nabla_X \big[\alpha\big(X_1, \ldots, X_k\big)\big] &= \big(\pounds^\nabla_X \alpha\big)\big(X_1, \ldots, X_k\big) \\ &\quad + \sum_{i=1}^{k} \alpha\big(X_1, \ldots, \pounds^\nabla_X X_i, \ldots X_k\big) \ . \end{aligned} \tag{5.12}$$

(g) The covariant Lie derivative satisfies a relation analogous to the Cartan

magic formula*: for any vector field $X \in \Gamma(\mathcal{P})$ and $\alpha \in \Gamma(\mathcal{L}_P^k)$ with $k \geq 1$,*

$$\pounds_X^\nabla = \mathrm{d}^\nabla \circ \iota_X + \iota_X \circ \mathrm{d}^\nabla \ . \tag{5.13}$$

(h) The parallel transport operators along the flows of X_H and X_J commute:

$$\Pi_t^H \circ \Pi_s^J = \Pi_s^J \circ \Pi_t^H \ . \tag{5.14}$$

(i) The actions of the flows of X_H and X_J on $\mathbb{L}$-valued polarized k-forms commute:

$$\widetilde{\Phi}_t^{H*} \circ \widetilde{\Phi}_s^{J*} = \widetilde{\Phi}_s^{J*} \circ \widetilde{\Phi}_t^{H*} \ . \tag{5.15}$$

(ii) The operator $\pounds_{X_H}^\nabla$ commutes with Φ_t^{J*}; $\pounds_{X_J}^\nabla$ commutes with Φ_t^{H*}:

$$\pounds_{X_H}^\nabla \circ \widetilde{\Phi}_t^{J*} = \widetilde{\Phi}_t^{J*} \circ \pounds_{X_H}^\nabla \ , \qquad \pounds_{X_J}^\nabla \circ \widetilde{\Phi}_t^{H*} = \widetilde{\Phi}_t^{H*} \circ \pounds_{X_J}^\nabla \ ; \tag{5.16}$$

the operators $\pounds_{X_H}^\nabla$ and $\pounds_{X_J}^\nabla$ commute:

$$\pounds_{X_H}^\nabla \circ \pounds_{X_J}^\nabla = \pounds_{X_J}^\nabla \circ \pounds_{X_H}^\nabla \ . \tag{5.17}$$

Proof. To simplify the notations while still revealing the ideas, in the proofs below we will use $\mathbb{L}$-valued polarized 0-forms and 1-forms instead of $\mathbb{L}$-valued polarized k-forms.

Properties (5.4) and (5.5) can be observed directly from the definitions (5.2) and (5.3).

The identity (5.6) follows directly from (5.4): for $\alpha \in \Gamma(\mathcal{L}^1_P)$,

$$\iota_{\widetilde{\Phi}_t^{X*}Y} \circ \widetilde{\Phi}_t^{X*}\alpha = \langle \widetilde{\Phi}_t^{X*}\alpha, \widetilde{\Phi}_t^{X*}Y\rangle = \widetilde{\Phi}_t^{X*}\langle \alpha, Y\rangle = \widetilde{\Phi}_t^{X*} \circ \iota_Y \alpha \ .$$

Differentiating both sides of (5.6) with respect to t and setting $t = 0$, we obtain (5.7).

To derive (5.8), use that the flow of the pull-back $\Phi_t^{X*}Y$ of the vector field Y is

$$s \mapsto \Phi_s^{\Phi_t^{X*}Y} = \Phi_{-t}^X \circ \Phi_s^Y \circ \Phi_t^X \ ,$$

which implies that

$$\Pi_s^{\Phi_t^{X*}Y} = \Pi_{-t}^X \circ \Pi_s^Y \circ \Pi_t^X \ .$$

Using these facts, we have for $\psi \in \Gamma(\mathcal{L}^0_P)$

$$\begin{aligned}
\left(\pounds^{\nabla}_{\Phi_t^{X*}Y} \circ \widetilde{\Phi}_t^{X*}\psi\right)(m) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[\widetilde{\Phi}_s^{(\Phi_t^{X*}Y)*}\left(\widetilde{\Phi}_t^{X*}\psi\right)\right](m) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Pi_{-s}^{\Phi_t^{X*}Y} \circ \Pi_{-t}^X \left[\psi \circ \Phi_t^X \circ \Phi_s^{\Phi_t^{X*}Y}(m)\right] \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Pi_{-t}^X \circ \Pi_{-s}^Y \left[\psi \circ \Phi_s^Y \circ \Phi_t^X(m)\right] \\
&= \Pi_{-t}^X \left[\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(\Pi_{-s}^Y \circ \psi \circ \Phi_s^Y\right)\left(\Phi_t^X(m)\right)\right] \\
&= \Pi_{-t}^X \left[(\pounds^{\nabla}_Y\psi)(\Phi_t^X(m))\right] \\
&= \left(\widetilde{\Phi}_t^{X*} \circ \pounds^{\nabla}_Y\psi\right)(m) \ .
\end{aligned}$$

The mutual commutativity of ι_X, $\widetilde{\Phi}_t^{X*}$, and $\pounds^{\nabla}_X$ follows from the fact that a vector field is invariant with respect to its flow, i.e., $\Phi_{t*}^X X = X$.

The proof of (5.9) for a $\mathbb{L}$-valued polarized 0-form ψ goes as follows:

$$
\begin{aligned}
\big[\widetilde{\Phi}_t^{X*} \circ \pounds_X^{\nabla}\psi\big](m) &= \Pi_{-t}^X\big[\big(\pounds_X^{\nabla}\psi\big)\big(\Phi_t^X(m)\big)\big] \\
&= \Pi_{-t}^X \left[\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Pi_{-s}^X \left(\psi \circ \Phi_s^X\right) \circ \Phi_t^X(m) \right] \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Pi_{-(t+s)}^X \left[\psi \circ \Phi_{t+s}^X(m)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \Pi_{-t}^X \left[\psi \circ \Phi_t^X(m)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{\Phi}_t^{X*}\psi(m) \ ;
\end{aligned}
$$

the rest of the statement comes from (5.8) with $Y = X$, and $\Phi_{t*}^X X = X$.

Property (5.10) follows from Definition 3.3.1 and the identities (5.5), (5.8), and (5.4): for $\alpha \in \Gamma(\mathcal{L}_P^1)$,

$$
\begin{aligned}
&\big(\mathrm{d}^{\nabla} \circ \widetilde{\Phi}_t^{X*}\alpha\big)\big(\Phi_t^{X*}X_0, \Phi_t^{X*}X_1\big) \\
&\qquad = \nabla_{\Phi_t^{X*}X_0}\big\langle \widetilde{\Phi}_t^{X*}\alpha, \Phi_t^{X*}X_1\big\rangle - \nabla_{\Phi_t^{X*}X_1}\big\langle \widetilde{\Phi}_t^{X*}\alpha, \Phi_t^{X*}X_0\big\rangle \\
&\qquad\qquad\qquad\qquad - \big\langle \widetilde{\Phi}_t^{X*}\alpha, \pounds_{\Phi_t^{X*}X_0}^{\nabla} \circ \Phi_t^{X*}X_1\big\rangle \\
&\qquad = \mathcal{L}_{\Phi_t^{X*}X_0}^{\nabla}\big\langle \widetilde{\Phi}_t^{X*}\alpha, \Phi_t^{X*}X_1\big\rangle - \mathcal{L}_{\Phi_t^{X*}X_1}^{\nabla}\big\langle \widetilde{\Phi}_t^{X*}\alpha, \Phi_t^{X*}X_0\big\rangle \\
&\qquad\qquad\qquad\qquad - \big\langle \widetilde{\Phi}_t^{X*}\alpha, \Phi_t^{X*} \circ \pounds_{X_0}^{\nabla}X_1\big\rangle \\
&\qquad = \mathcal{L}_{\Phi_t^{X*}X_0}^{\nabla} \circ \widetilde{\Phi}_t^{X*}\big\langle \alpha, X_1\big\rangle - \mathcal{L}_{\Phi_t^{X*}X_1}^{\nabla} \circ \widetilde{\Phi}_t^{X*}\big\langle \alpha, X_0\big\rangle - \widetilde{\Phi}_t^{X*}\big\langle \alpha, \pounds_{X_0}^{\nabla}X_1\big\rangle \\
&\qquad = \widetilde{\Phi}_t^{X*} \circ \mathcal{L}_{X_0}^{\nabla}\big\langle \alpha, X_1\big\rangle - \widetilde{\Phi}_t^{X*} \circ \mathcal{L}_{X_1}^{\nabla}\big\langle \alpha, X_0\big\rangle - \widetilde{\Phi}_t^{X*}\big\langle \alpha, \pounds_{X_0}^{\nabla}X_1\big\rangle \\
&\qquad = \widetilde{\Phi}_t^{X*}\left[\nabla_{X_0}\big\langle \alpha, X_1\big\rangle - \nabla_{X_1}\big\langle \alpha, X_0\big\rangle - \big\langle \alpha, \pounds_{X_0}^{\nabla}X_1\big\rangle\right] \\
&\qquad = \widetilde{\Phi}_t^{X*}\left[\big(\mathrm{d}^{\nabla}\alpha\big)(X_0, X_1)\right] \\
&\qquad = \big(\widetilde{\Phi}_t^{X*} \circ \mathrm{d}^{\nabla}\alpha\big)\big(\Phi_t^{X*}X_0, \Phi_t^{X*}X_1\big) \ .
\end{aligned}
$$

To obtain (5.11), differentiate (5.10) with respect to t and set $t = 0$.

The Leibniz rule (5.12) is proved as usual, and, together with Definition 3.3.1 and (5.5), it implies the Cartan magic formula (5.13).

The commutativity (5.14) of the parallel transport along the integral lines of X_H and X_J follows from the vanishing of the curvature (recall Definition 2.2.6) along the leaves of the foliation P (or, equivalently, from the fact that the P is Lagrangian), and the commutativity of X_H and X_J.

The commutativity of X_H and X_J implies the commutativity of their flows Φ^H_t and Φ^J_s which, together with (5.14), yields (5.15). Finally, (5.16) and (5.17) are infinitesimal versions of (5.15). $\square$

The operator introduced in the definition below was introduced by Rawnsley [31] and plays an important role in the rest of the dissertation.

Definition 5.0.5. *Let* $\mathcal{J} : \Gamma(\mathcal{L}^k_P) \to \Gamma(\mathcal{L}^{k-1}_P)$ *be the operator*

$$\mathcal{J}\alpha = \iota_{X_J} \int_0^{2\pi} \widetilde{\Phi}^{J*}_s \alpha \; \mathrm{d}s \; . \tag{5.18}$$

Since we will be using this formula extensively, below we write it in detail in the particular case of an $\mathbb{L}$-valued polarized 1-form $\alpha \in \Gamma(\mathcal{L}^1_P)$: using (5.2), (5.4), and the fact that each vector field X is invariant with respect to its own flow ($\Phi^{X*}_t X = X$), we obtain

$$\begin{aligned} \mathcal{J}\alpha &= \int_0^{2\pi} \big\langle \widetilde{\Phi}^{J*}_s \alpha, X_J \big\rangle \; \mathrm{d}s = \int_0^{2\pi} \big\langle \widetilde{\Phi}^{J*}_s \alpha, \Phi^{J*}_s X_J \big\rangle \; \mathrm{d}s \\ &= \int_0^{2\pi} \widetilde{\Phi}^{J*}_s \big\langle \alpha, X_J \big\rangle \; \mathrm{d}s = \int_0^{2\pi} \widetilde{\Pi}^J_{-s} \circ \big\langle \alpha, X_J \big\rangle \circ \Phi^J_s \; \mathrm{d}s \; . \end{aligned} \tag{5.19}$$

The flow of X_J is 2π-periodic for each $m \in M$ (except at the focus-focus point). The parallel transport over a closed loop starting at $m \in M$ is $\Pi^J_{2\pi}(m) \in \mathrm{Hom}(\mathbb{L}_m, \mathbb{L}_m) \cong \mathbb{C}$ since $\mathbb{L}$ is a line bundle. Moreover, since the fiber metric is compatible with the connection, $\Pi^J_{2\pi}$ has modulus 1. This motivates the following

Definition 5.0.6. *Let* $\mathrm{Hol}(m)$ *be the holonomy around a closed loop of the flow of* X_J*, starting at* $m \in M$*:*

$$\mathrm{Hol} := \Pi^J_{2\pi} : M \to S^1 \subset \mathbb{C} : m \mapsto \mathrm{Hol}(m) := \Pi^J_{2\pi}(m) \ . \tag{5.20}$$

Proposition 5.0.7. *The following identities hold:*

$$\begin{aligned} \mathcal{J} \circ \nabla\psi &= \big(\mathrm{Hol}^{-1} - 1\big)\,\psi \ , \quad \psi \in \Gamma(\mathcal{L}^0_P) \ , \\ \big(\mathrm{d}^\nabla \circ \mathcal{J} + \mathcal{J} \circ \mathrm{d}^\nabla\big)\,\alpha &= \big(\mathrm{Hol}^{-1} - 1\big)\,\alpha \ , \quad \alpha \in \Gamma(\mathcal{L}^k_P) \ , \ k \geq 1 \ . \end{aligned}$$

Proof. Using consecutively (5.19), (5.5), (5.8), (5.9), and (5.20), we obtain

$$\begin{aligned} \mathcal{J} \circ \nabla\psi &= \int_0^{2\pi} \widetilde{\Phi}^{J*}_s \big\langle \nabla\psi, X_J \big\rangle \,\mathrm{d}s = \int_0^{2\pi} \widetilde{\Phi}^{J*}_s \circ \nabla_{X_J}\psi \,\mathrm{d}s \\ &= \int_0^{2\pi} \widetilde{\Phi}^{J*}_s \circ \pounds^\nabla_{X_J}\psi \,\mathrm{d}s = \int_0^{2\pi} \mathcal{L}^\nabla_{X_J} \circ \widetilde{\Phi}^{J*}_s\psi \,\mathrm{d}s \\ &= \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}s}\,\widetilde{\Phi}^{J*}_s\psi \,\mathrm{d}s = \big(\widetilde{\Phi}^{J*}_s\psi\big)\Big|_{s=0}^{2\pi} = \big(\mathrm{Hol}^{-1} - 1\big)\,\psi \ . \end{aligned}$$

Similarly, (5.18), (5.13), and (5.10) give us

$$\begin{aligned}
\left(\mathrm{d}^\nabla \circ \mathcal{J} + \mathcal{J} \circ \mathrm{d}^\nabla\right)\alpha &= \mathrm{d}^\nabla \circ \iota_{X_J} \int_0^{2\pi} \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s + \iota_{X_J} \int_0^{2\pi} \widetilde{\Phi}_s^{J*}(\mathrm{d}^\nabla \alpha)\,\mathrm{d}s \\
&= \int_0^{2\pi} \mathrm{d}^\nabla \circ \iota_{X_J} \circ \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s + \int_0^{2\pi} \iota_{X_J} \circ \mathrm{d}^\nabla \circ \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s \\
&= \int_0^{2\pi} \left(\pounds^\nabla_{X_J} - \iota_{X_J} \circ \mathrm{d}^\nabla\right) \circ \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s + \int_0^{2\pi} \iota_{X_J} \circ \mathrm{d}^\nabla \circ \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s \\
&= \int_0^{2\pi} \pounds^\nabla_{X_J} \circ \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s = \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}s} \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s \\
&= \left(\widetilde{\Phi}_s^{J*}\alpha\right)\Big|_{s=0}^{2\pi} = \left(\mathrm{Hol}^{-1} - 1\right)\alpha \ .
\end{aligned}$$

□

Proposition 5.0.8. *For* $\alpha \in \Gamma(\mathcal{L}^1_P)$,

$$\nabla_{X_J}(\mathcal{J}\alpha) = 0 \ .$$

Proof. Using (5.18) and (5.6) with $Y = X$, we obtain

$$\begin{aligned}
\nabla_{X_J}(\mathcal{J}\alpha) &= \nabla_{X_J} \circ \iota_{X_J} \int_0^{2\pi} \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \widetilde{\Phi}_t^{J*} \circ \iota_{X_J} \int_0^{2\pi} \widetilde{\Phi}_s^{J*}\alpha \,\mathrm{d}s \\
&= \iota_{X_J} \circ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_0^{2\pi} \widetilde{\Phi}_{s+t}^{J*}\,\alpha \,\mathrm{d}s = \iota_{X_J} \circ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_0^{2\pi} \widetilde{\Phi}_{s_1}^{J*}\alpha \,\mathrm{d}s_1 = 0 \ ,
\end{aligned}$$

where we have set $s_1 = s + t$ and used the 2π-periodicity of the integrand. □

Proposition 5.0.9. *For* $\alpha \in \Gamma(\mathcal{L}^1_P)$ *that satisfies* $\mathrm{d}^\nabla \alpha = 0$*, we have*

$$\nabla_{X_H}(\mathcal{J}\alpha) = \left(\mathrm{Hol}^{-1} - 1\right)\langle \alpha, X_H \rangle \ .$$

Proof. We use (5.19), (5.5), (5.8), (5.20), and the fact that $\mathrm{d}^\nabla \alpha = 0$ is equivalent to $\nabla_{X_H}\big\langle\alpha, X_J\big\rangle = \nabla_{X_J}\big\langle\alpha, X_H\big\rangle$ to obtain

$$
\begin{aligned}
\nabla_{X_H}(\mathcal{J}\alpha) &= \nabla_{X_H}\int_0^{2\pi}\widetilde{\Phi}_s^{J*}\langle\alpha, X_J\rangle\,\mathrm{d}s = \pounds_{X_H}^{\nabla}\int_0^{2\pi}\widetilde{\Phi}_s^{J*}\langle\alpha, X_J\rangle\,\mathrm{d}s \\
&= \int_0^{2\pi}\widetilde{\Phi}_s^{J*}\circ\pounds_{X_H}^{\nabla}\langle\alpha, X_J\rangle\,\mathrm{d}s = \int_0^{2\pi}\widetilde{\Phi}_s^{J*}\circ\nabla_{X_H}\langle\alpha, X_J\rangle\,\mathrm{d}s \\
&= \int_0^{2\pi}\widetilde{\Phi}_s^{J*}\circ\nabla_{X_J}\langle\alpha, X_H\rangle\,\mathrm{d}s = \int_0^{2\pi}\nabla_{X_J}\circ\widetilde{\Phi}_s^{J*}\langle\alpha, X_H\rangle\,\mathrm{d}s \\
&= \int_0^{2\pi}\pounds_{X_J}^{\nabla}\circ\widetilde{\Phi}_s^{J*}\langle\alpha, X_H\rangle\,\mathrm{d}s = \int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}s}\,\widetilde{\Phi}_s^{J*}\langle\alpha, X_H\rangle\,\mathrm{d}s \\
&= \Big[\widetilde{\Phi}_s^{J*}\langle\alpha, X_H\rangle\Big]\Big|_{s=0}^{2\pi} = \big(\mathrm{Hol}^{-1} - 1\big)\,\langle\alpha, X_H\rangle\ .
\end{aligned}
$$

$\square$

Chapter 6

Computations of the cohomology groups

With the help of the lemmas in Chapter 5, we will compute the sheaf cohomology of the sheaf $\mathcal{F}$ of P-flat sections of $\mathbb{L}$. Recall that in Section 3.5 we obtained the fine resolution

$$0 \to \mathcal{F} \to \mathcal{L}^0_{0,P} \xrightarrow{\nabla} \mathcal{L}^1_{0,P} \xrightarrow{\mathrm{d}^\nabla} \mathcal{L}^2_{0,P} \to 0 \ . \tag{6.1}$$

In Section 6.1 we will find $\mathrm{H}^0(M_\epsilon, \mathcal{F})$ and $\mathrm{H}^1(M_\epsilon, \mathcal{F})$, and in Section 6.2 we will perform the complicated calculation of $\mathrm{H}^2(M_\epsilon, \mathcal{F})$.

6.1 Calculation of $\mathrm{H}^0(M_\epsilon, \mathcal{F})$ and $\mathrm{H}^1(M_\epsilon, \mathcal{F})$

The 0th cohomology of $\mathcal{F}$ is easily computed in the following

Lemma 6.1.1. *The 0th cohomology group of $\mathcal{F}$ is trivial:* $\mathrm{H}^0(M_\epsilon, \mathcal{F}) = \{ 0 \}$.

Proof. By definition, $\mathrm{H}^0(M_\epsilon, \mathcal{F})$ is the set $\mathcal{F}(M_\epsilon) = \Gamma(\mathcal{F})$ of global P-flat sections of $\mathbb{L}$. Let $\psi \in \mathcal{F}(M_\epsilon)$. By Lemma 4.3.3, the restriction of ψ to the non-singular torus fibers is 0. Since the non-singular fibers form are dense in M_ϵ, by continuity we obtain that $\psi \equiv 0$ on M_ϵ. $\square$

We now calculate $\mathrm{H}^1(M_\epsilon, \mathcal{F})$, recall that

$$\mathrm{H}^1(M_\epsilon, \mathcal{F}) = \mathrm{Ker}\big\{\Gamma(\mathcal{L}^1_{0,P}) \overset{\mathrm{d}^\nabla}{\to} \Gamma(\mathcal{L}^2_{0,P})\big\} / \, \mathrm{Im}\big\{\Gamma(\mathcal{L}^0_{0,P}) \overset{\nabla}{\to} \Gamma(\mathcal{L}^1_{0,P})\big\}. \tag{6.2}$$

In the lemma below we will find conditions on $\alpha \in \Gamma(\mathcal{L}^1_{0,P})$ that guarantee that, when α is exact, i.e, we will find conditions for the existence of a $\psi \in \Gamma(\mathcal{L}^0_{0,P})$ such that $\alpha = \mathrm{d}^\nabla \psi$.

Lemma 6.1.2. *The first cohomology group of $\mathcal{F}$ is trivial:* $\mathrm{H}^1(M_\epsilon, \mathcal{F}) = \{\, 0 \,\}$.

Proof. The proof consists of two steps. In the first step we will show that, if $\alpha \in \Gamma(\mathcal{L}^1_{0,P})$ is closed, that is, if $\mathrm{d}^\nabla \alpha = 0$ then $\mathcal{J}\alpha \equiv 0$ when $J = 0$, and in the second step we will use this fact to find $\mathrm{H}^1(M, \mathcal{F})$.

Let $\alpha \in \Gamma(\mathcal{L}^1_{0,P})$ satisfy $\mathrm{d}^\nabla \alpha = 0$.

According to Propositions 5.0.8 and 5.0.9, when $J = 0$, both $\nabla_{X_J}(\mathcal{J}\alpha)$ and $\nabla_{X_H}(\mathcal{J}\alpha)$ are identically 0, so that $\mathcal{J}\alpha \in \Gamma(\mathcal{L}^0_{0,P})$ is a P-flat section on the tori with $J = 0$. But, according to Lemma 4.3.3, the focus-focus fiber is the only Bohr-Sommerfeld torus in M_ϵ, i.e., the only torus which admits a non-zero P-flat section. By the same reasoning as in the proof of Lemma 6.1.1, we conclude that $\mathcal{J}\alpha \equiv 0$ when $J = 0$.

Now we will show that the vanishing of $\mathrm{d}^\nabla \alpha$ implies the existence of $\psi \in \Gamma(\mathcal{L}^0_{0,P})$ such that $\alpha = \mathrm{d}^\nabla \psi$. Proposition 5.0.7 and the closedness of α imply that

$$\left(\mathrm{Hol}^{-1} - 1\right) \alpha = \mathrm{d}^\nabla \circ \mathcal{J}\alpha + \mathcal{J} \circ \mathrm{d}^\nabla \alpha = \mathrm{d}^\nabla \circ \mathcal{J}\alpha \; .$$

Define $\psi := \frac{1}{\mathrm{Hol}^{-1} - 1} \mathcal{J}\alpha$ on $M_\epsilon \setminus \{J = 0\}$, then $\mathrm{d}^\nabla \psi = \alpha$ in this domain. The only thing that remains to be proven is that ψ can be extended smoothly to all of M_ϵ.

From the first step of the proof we know that $\mathcal{J}\alpha$ vanishes when $J = 0$. Also, $\mathcal{J}\alpha$ vanishes identically in a neighborhood of the singular point because $\alpha \in \Gamma(\mathcal{L}^1_{0,P})$. Thus, since the hypersurface $J = 0$ is smooth away from the singular point, and the gradient of J is nonzero away from the singular point, $\mathcal{J}\alpha$ is divisible by J. Furthermore, by the construction of M_ϵ in Section 4.2, $J = 0$ is the only place where $\mathrm{Hol} = 1$. Thus ψ extends smoothly to all of M_ϵ.

□

6.2 Calculation of $\mathrm{H}^2(M_\epsilon, \mathcal{F})$

To compute

$$\mathrm{H}^2(M_\epsilon, \mathcal{F}) = \Gamma(\mathcal{L}^2_{0,P}) / \mathrm{Im}\big\{\Gamma(\mathcal{L}^1_{0,P}) \overset{\mathrm{d}^\nabla}{\to} \Gamma(\mathcal{L}^2_{0,P})\big\} ,$$

we need to find conditions on a $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ that guarantee its exactness. We start by proving a lemma that restates the exactness of β as a condition that holds on $\{J = 0\} \subset M_\epsilon$.

Lemma 6.2.1. *The form $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ is exact if and only if there exists $\psi \in \Gamma(\mathcal{L}^0_{0,P})$ such that $\nabla\psi = \mathcal{J}\beta$ when $J = 0$.*

Proof. First let us assume that $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ is exact, i.e., that there exists $\alpha \in \Gamma(\mathcal{L}^1_{0,P})$ such that $\mathrm{d}^\nabla\alpha = \beta$. Proposition 5.0.7 then implies that for $J = 0$ (i.e., $\mathrm{Hol} = 1$),

$$\mathcal{J}\beta = \mathcal{J} \circ \mathrm{d}^\nabla\alpha = \big(\mathrm{Hol}^{-1} - 1\big)\,\alpha - \mathrm{d}^\nabla \circ \mathcal{J}\alpha = \mathrm{d}^\nabla(-\mathcal{J}\alpha) = \nabla(-\mathcal{J}\alpha) .$$

If we define $\psi = -\mathcal{J}\alpha \in \Gamma(\mathcal{L}^0_{0,P})$, then on the set $\{J=0\}$ we have $\nabla\psi = \mathcal{J}\beta$.

Conversely, assume that there exists a $\psi \in \Gamma(\mathcal{L}^0_{0,P})$ such that $\nabla\psi = \mathcal{J}\beta$ when $\{J=0\}$. Define $\alpha_1 := \mathcal{J}\beta - \nabla\psi$, then by hypothesis α_1 vanishes when $J=0$. Define $\alpha := \dfrac{1}{\mathrm{Hol}^{-1}-1}\alpha_1$ on $M_\epsilon \setminus \{J=0\}$.

Then the commutativity of d^∇ and the parallel transport (which implies that $\mathrm{d}^\nabla(\mathrm{Hol}^{-1}-1) = (\mathrm{Hol}^{-1}-1)\mathrm{d}^\nabla$), Lemma 3.3.2 ($\mathrm{d}^\nabla \circ \nabla\psi \equiv 0$), Proposition 5.0.7, and the trivial fact that $\mathrm{d}^\nabla\beta \equiv 0$ imply that, for $J \neq 0$,

$$\begin{aligned}\mathrm{d}^\nabla\alpha = \mathrm{d}^\nabla \frac{\mathcal{J}\beta - \nabla\psi}{\mathrm{Hol}^{-1}-1} &= (\mathrm{Hol}^{-1}-1)^{-1}\,\mathrm{d}^\nabla \circ \mathcal{J}\beta \\ &= (\mathrm{Hol}^{-1}-1)^{-1}\,\left[(\mathrm{Hol}^{-1}-1)\beta - \mathcal{J}\circ \mathrm{d}^\nabla\beta\right] = \beta\ .\end{aligned}$$

The only thing that remains to be proven is that α can be extended smoothly to all of M_ϵ.

We have shown that $\alpha_1 = \mathcal{J}\beta - \nabla\psi$ vanishes when $J=0$; it also vanishes identically in a neighborhood of the focus-focus point because $\alpha_1 \in \Gamma(\mathcal{L}^1_{0,P})$. Thus, since the hypersurface $\{J=0\}$ is smooth away from the focus-focus point, and the gradient of J is nonzero away from the focus-focus point, α_1 is divisible by J. Furthermore, by the construction of M_ϵ in Section 4.2, $\{J=0\}$ is the only place in M_ϵ where $\mathrm{Hol}=1$. Thus, α extends smoothly to all of M_ϵ. □

6.2.1 Solving $\nabla\psi = \mathcal{J}\beta$ on $\{J=0\}$

Lemma 6.2.1 reduces the problem of proving the exactness of $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ to finding $\psi \in \Gamma(\mathcal{L}^0_{0,P})$ that satisfies $\nabla\psi = \mathcal{J}\beta$ on $\{J=0\}$, which we analyze in this section. The set $\{J=0\}$ is foliated by the tori $\Lambda_{(\xi,0)}$ with ξ in some open interval of $\mathbb{R}$ containing 0.

First we derive two solutions, ψ_1 and ψ_2, of $\nabla\psi = \mathcal{J}\beta$ on $\{J=0\}$ that hold in two different subsets of $\{J=0\}$.

The equation $\nabla\psi = \mathcal{J}\beta$ is equivalent to the simultaneous validity of the equations $\nabla_{X_H}\psi = \langle\mathcal{J}\beta, X_H\rangle$ and $\nabla_{X_J}\psi = \langle\mathcal{J}\beta, X_J\rangle = 0$. To solve $\nabla_{X_H}\psi = \langle\mathcal{J}\beta, X_H\rangle$, apply $\widetilde{\Phi}_t^{H*}$ to both sides and use that $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\Phi}_t^{H*} = \widetilde{\Phi}_t^{H*}\circ\nabla_{X_H}$ (cf. (5.5) and (5.9)):

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\Phi}_t^{H*}\psi = \widetilde{\Phi}_t^{H*}\langle\mathcal{J}\beta, X_H\rangle\ ,$$

which integrates to

$$\widetilde{\Phi}_t^{H*}\psi(m) - \psi(m) = \int_0^t \widetilde{\Phi}_{t_1}^{H*}\langle\mathcal{J}\beta, X_H\rangle(m)\,\mathrm{d}t_1\ . \tag{6.3}$$

On the other hand, $\nabla_{X_J}\psi = 0$ is equivalent to $\widetilde{\Phi}_s^{J*}\psi = \psi$ for any s, which implies

$$\widetilde{\Phi}_t^{H*}\circ\widetilde{\Phi}_s^{J*}\psi(m) - \psi(m) = \int_0^t \widetilde{\Phi}_{t_1}^{H*}\langle\mathcal{J}\beta, X_H\rangle(m)\,\mathrm{d}t_1\ ,$$

or, equivalently,

$$\psi(\Phi_t^H\circ\Phi_s^J(m)) = \Pi_t^H\circ\Pi_s^J\left[\psi(m) + \int_0^t \widetilde{\Phi}_{t_1}^{H*}\langle\mathcal{J}\beta, X_H\rangle(m)\,\mathrm{d}t_1\right]\ . \tag{6.4}$$

From (6.4) we first derive a solution ψ_1 on $\{J=0\}\setminus\Lambda_{(0,0)}$. Recall from (4.1) and (4.2) that, if $m\in\Lambda_{(\xi,0)}$ with $\xi\neq 0$, then $\Phi^H_{\tau_1(\xi)}\circ\Phi^J_{\tau_2(\xi)}(m) = m$. Using this in (6.4), we obtain the solution ψ_1 of $\nabla\psi = \mathcal{J}\beta$ on $\{J=0\}\setminus\Lambda_{(0,0)}$:

$$\begin{aligned}\psi_1(m) = \left[\Pi^H_{-\tau_1(\xi)}\circ\Pi^J_{-\tau_2(\xi)} - 1\right]^{-1}\int_0^{\tau_1(\xi)} \widetilde{\Phi}_{t_1}^{H*}\langle\mathcal{J}\beta, X_H\rangle(m)\,\mathrm{d}t_1\ ,&\\ m\in\{J=0\}\setminus\Lambda_{(0,0)}\ .&\end{aligned} \tag{6.5}$$

Since we seek a solution $\psi\in\Gamma(\mathcal{L}^0_{0,P})$ of $\nabla\psi = \mathcal{J}\beta$ on $\{J=0\}$, there must exist an open neighborhood V of the focus-focus point in which ψ is identically 0. If ψ_1 from (6.5) is a solution, then the integral in the right-hand side of (6.5) must vanish when $m \in V$.

On the other hand, if m is in the same neighborhood V, then, imposing the initial condition $\psi(m) = 0$ in (6.4), we obtain another solution $\psi_2 \in \Gamma(\mathcal{L}^0_{0,P})$ on the fiber $\Lambda_{\mu(m)}$:

$$\begin{aligned} \psi_2(\Phi^H_t \circ \Phi^J_s(m)) = \Pi^H_t \circ \Pi^J_s \int_0^t \widetilde{\Phi}^{H*}_{t_1} \langle \mathcal{J}\beta, X_H \rangle (m)\, \mathrm{d}t_1 \ , \\ m \in \Lambda_{\mu(m)} \ , \quad m \in V \cap \{J = 0\} \ . \end{aligned} \tag{6.6}$$

The expression (6.6) defines a solution on a neighborhood of the focus-focus fiber $\Lambda_{(0,0)}$ inside $\{J = 0\}$ provided that the integral in the right-hand side of (6.6) vanishes for values of t such that $\Phi^H_t \circ \Phi^J_s(m) \in V$.

To formulate simple conditions for existence of ψ_1 (6.5) and ψ_2 (6.6), we introduce some notations. Let V_β be an open neighborhood of the focus-focus point such that $\beta|V_\beta \equiv 0$; without loss of generality, we assume that it is a ball of radius $R < 1$ centered at the focus-focus point. Choose a number $\xi_1 \in \mathbb{R}$ such that

$$0 < \xi_1 < \frac{R^2}{2} \ . \tag{6.7}$$

Choose a number $\eta \in \mathbb{R}$ such that

$$\sqrt{\xi_1} < \eta < \frac{R}{\sqrt{2}} \ . \tag{6.8}$$

Then it is easy to show that for any $\xi \in (-\xi_1, \xi_1)$, the conditions (6.7) and (6.8) imply that the points $\left(\eta, \frac{\xi}{\eta}\right)$ and $\left(\frac{\xi}{\eta}, \eta\right)$ belong to $V_\beta \cap \Lambda_{(\xi,0)}$. Using (3.28), if $\xi \neq 0$, that the shortest time that the X_H flow needs to take the point $\left(\frac{\xi}{\eta}, \eta\right)$ to the X_J-orbit of $\left(\eta, \frac{\xi}{\eta}\right)$ is $2\ln\eta - \ln|\xi| > 0$. The quantities V_β, R, η are illustrated in Figure 6.1.

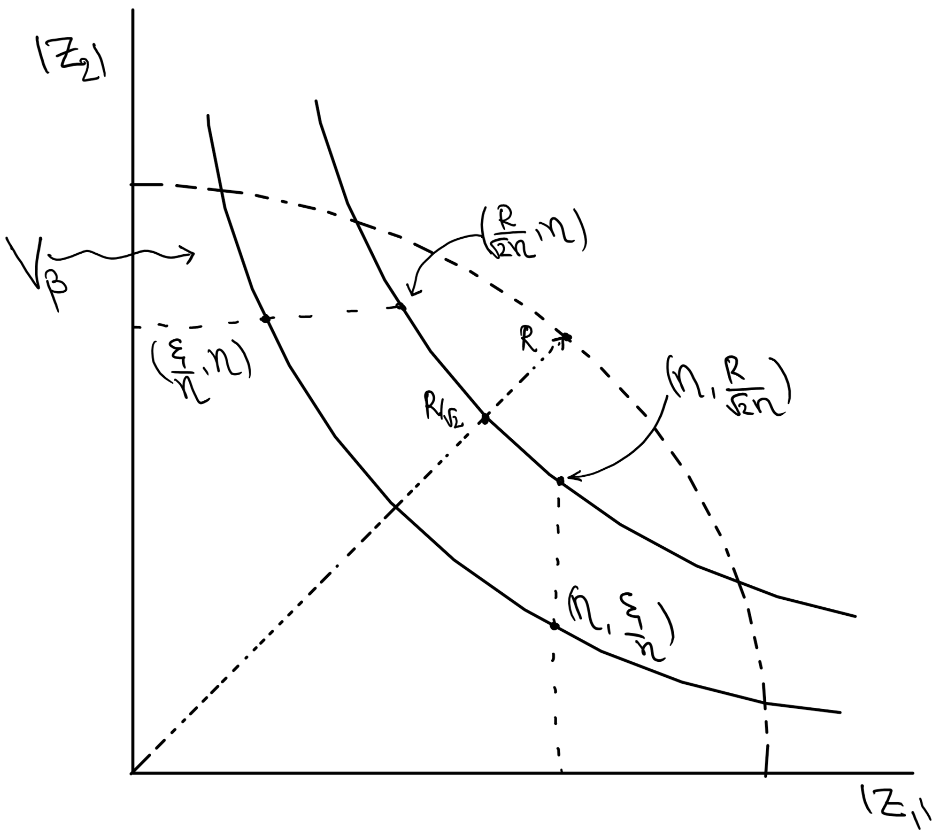

Figure 6.1: On the definition of η, R, V_β.

Using (4.1) and (4.5), we can write the shortest time for the X_H flow to take the point $\left(\eta, \frac{\xi}{\eta}\right)$ to the X_J-orbit of $\left(\frac{\xi}{\eta}, \eta\right)$ as

$$\tau_1(\xi) - \big(2\ln\eta - \ln|\xi|\big) = S_1(\xi) - 2\ln\eta > 0\ . \tag{6.9}$$

Using that $S_1(\xi)$ is a smooth function of ξ in a neighborhood of $\xi = 0$ in $\mathbb{R}$ (recall Sec. 4.1), we use the rightmost expression in (6.9) as a definition for the shortest

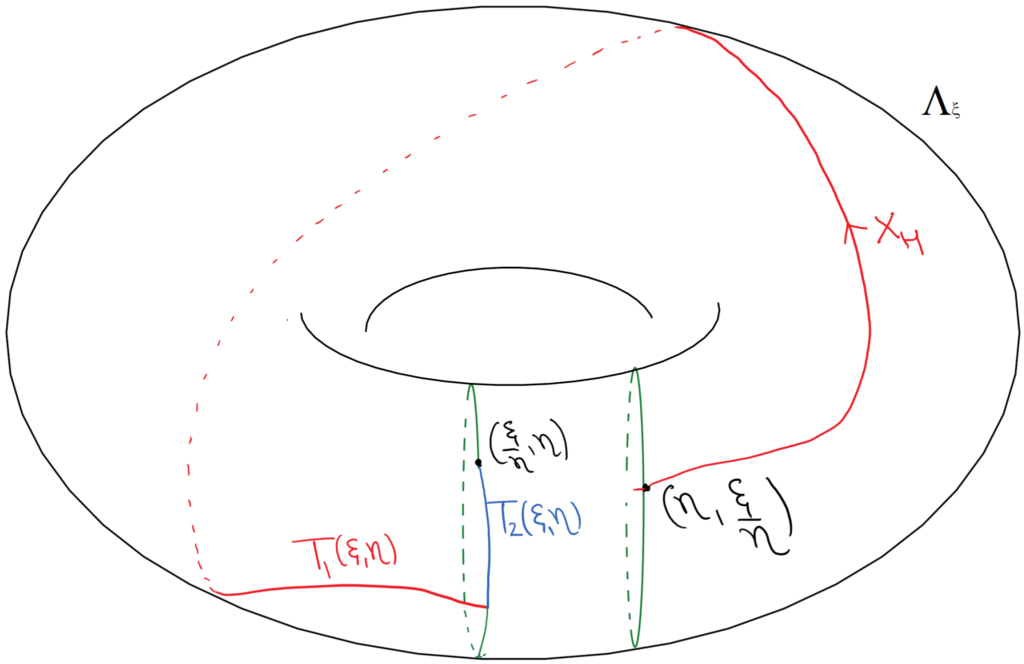

Figure 6.2: On the definition of the times $T_1(\xi,\eta), T_2(\xi,\eta)$.

time for the X_H flow to take $\left(\eta, \frac{\xi}{\eta}\right)$ to the X_J-orbit of $\left(\frac{\xi}{\eta}, \eta\right)$, for any $\xi \in (-\xi_1, \xi_1)$:

$$T_1(\xi,\eta) = S_1(\xi) - 2\ln\eta > 0 \ . \tag{6.10}$$

We also define

$$T_2(\xi,\eta) = \begin{cases} 0 & \text{if } \xi \geq 0 \ , \\ \pi & \text{if } \xi < 0 \ , \end{cases} \tag{6.11}$$

then

$$\Phi^H_{T_1(\xi,\eta)} \circ \Phi^J_{T_2(\xi,\eta)} \left(\eta, \frac{\xi}{\eta}\right) = \left(\frac{\xi}{\eta}, \eta\right) \ .$$

The meaning of $T_1(\xi,\eta)$ is represented pictorially in Figure 6.2.

We are ready to give the following

Definition 6.2.2. *Given $\beta \in \Gamma(\mathcal{L}^2_{0,P})$, let V_β, R, ξ_1, η, and $T_1(\xi,\eta)$ be chosen as* *above. Define the function*

$$\mathcal{A}_\beta : (-\xi_1, \xi_1) \to \mathbb{L} : \xi \mapsto \mathcal{A}_\beta(\xi) \in \mathbb{L}_{\left(\eta, \frac{\xi}{\eta}\right)}$$

be the mapping defined by one of the following equivalent expressions:

$$\begin{aligned} \mathcal{A}_\beta(\xi) &:= \int_0^{T_1(\xi,\eta)} \left(\widetilde{\Phi}_t^{H*}\langle \mathcal{J}\beta, X_H\rangle\right)\left(\eta, \tfrac{\xi}{\eta}\right) \mathrm{d}t \\ &= \int_0^{T_1(\xi,\eta)} \int_0^{2\pi} \left[\widetilde{\Phi}_t^{H*} \circ \widetilde{\Phi}_s^{J*}\left(\beta(X_J, X_H)\right)\right]\left(\eta, \tfrac{\xi}{\eta}\right) \mathrm{d}s \, \mathrm{d}t \ . \end{aligned} \tag{6.12}$$

Remark 6.2.3. Since the function $T_1(\xi, \eta)$ (6.10) is smooth in ξ for any fixed $\eta > 0$, $\mathcal{A}_\beta(\xi)$ depends smoothly on ξ. ▲

Remark 6.2.4. Although the definition (6.14) of $\mathcal{A}_\beta$ depends on the choice of the number η (satisfying (6.8)), the dependence is immaterial in our consideration. Despite that, below we will give an alternative expression $\widetilde{\mathcal{A}}_\beta$ that does not depend on the choice of η (see Definition 6.2.6 and Lemma 6.2.7 below). This expression for $\widetilde{\mathcal{A}}_\beta$ can be considered as a complex-valued function of $\xi \in (-\xi_1, \xi_1)$ if a local trivialization of $\mathbb{L}$ is chosen in a small neighborhood of the focus-focus point, which without loss of generality can be identified with V_β defined above. ▲

Lemma 6.2.5. *Let $\beta \in \Gamma(\mathcal{L}^2_{0,P})$, and V_β, R, ξ_1, η, and $T_1(\xi, \eta)$ be chosen as above. The following relation connects the expressions for ψ_1 (defined by* (6.5) *on $\{J = 0\} \setminus \Lambda_{(0,0)}$), ψ_2 (defined by* (6.6) *on a small neighborhood of the focus-focus torus inside $\{J = 0\}$), and $\mathcal{A}_\beta$* (6.12) *where the domains of ψ_1 and ψ_2 overlap:*

$$\begin{aligned} &\psi_1(\Phi_t^H \circ \Phi_s^J(\eta, \tfrac{\xi}{\eta})) - \psi_2(\Phi_t^H \circ \Phi_s^J(\eta, \tfrac{\xi}{\eta})) \\ &\qquad\qquad = \left[\Pi^H_{-\tau_1(\xi)} \circ \Pi^J_{-\tau_2(\xi)} - 1\right]^{-1} \Pi_t^H \Pi_s^J \mathcal{A}_\beta(\xi) \ , \end{aligned} \tag{6.13}$$

for any values of t *and* s.

Proof. We have

$$\begin{aligned}&\psi_1(\Phi^H_t\circ\Phi^J_s(\eta,\tfrac{\xi}{\eta}))\\&\qquad=\left[\Pi^H_{-\tau_1(\xi)}\circ\Pi^J_{-\tau_2(\xi)}-1\right]^{-1}\int_0^{\tau_1(\xi)}\widetilde{\Phi}^{H*}_{t_1}\langle\mathcal{J}\beta,X_H\rangle(\Phi^H_t\circ\Phi^J_s(\eta,\tfrac{\xi}{\eta}))\,\mathrm{d}t_1\\&\qquad=\left[\Pi^H_{-\tau_1(\xi)}\circ\Pi^J_{-\tau_2(\xi)}-1\right]^{-1}\Pi^H_t\circ\Pi^J_s\int_t^{\tau_1(\xi)+t}\widetilde{\Phi}^{H*}_{u}\langle\mathcal{J}\beta,X_H\rangle(\eta,\tfrac{\xi}{\eta})\,\mathrm{d}u\ .\end{aligned}$$

We write the integral as $\displaystyle\int_t^{\tau_1(\xi)}=\int_0^{\tau_1(\xi)}+\int_{\tau_1(\xi)}^{\tau_1(\xi)+t}-\int_0^t$, and notice that

$$\int_{\tau_1(\xi)}^{\tau_1(\xi)+t}\widetilde{\Phi}^{H*}_{u}\langle\mathcal{J}\beta,X_H\rangle(\eta,\tfrac{\xi}{\eta})\,\mathrm{d}u=\Pi^H_{-\tau_1(\xi)}\circ\Pi^J_{-\tau_2(\xi)}\int_0^t\widetilde{\Phi}^{H*}_{u}\langle\mathcal{J}\beta,X_H\rangle(\eta,\tfrac{\xi}{\eta})\,\mathrm{d}u\ .$$

Plug back to obtain

$$\begin{aligned}&\psi_1(\Phi^H_t\circ\Phi^J_s(\eta,\tfrac{\xi}{\eta}))\\&\qquad=\left[\Pi^H_{-\tau_1(\xi)}\circ\Pi^J_{-\tau_2(\xi)}-1\right]^{-1}\Pi^H_t\circ\Pi^J_s\int_0^{\tau_1(\xi)}\widetilde{\Phi}^{H*}_{u}\langle\mathcal{J}\beta,X_H\rangle(\eta,\tfrac{\xi}{\eta})\,\mathrm{d}u\\&\qquad\quad+\Pi^H_t\circ\Pi^J_s\int_0^t\widetilde{\Phi}^{H*}_{u}\langle\mathcal{J}\beta,X_H\rangle(\eta,\tfrac{\xi}{\eta})\,\mathrm{d}u\ .\end{aligned}$$

Using (6.6) and (6.12), we can rewrite this relation as (6.13). □

As mentioned in Remark 6.13, we now define a smooth complex-valued function $\widetilde{\mathcal{A}}_\beta$ that we will show is independent on the choice of η.

Definition 6.2.6. *In the notations of Definition 6.2.2, choose a local trivialization of* $\mathbb{L}$ *in an open set that contains* V_β*, and define the function*

$$\widetilde{\mathcal{A}}_\beta:(-\xi_1,\xi_1)\to\mathbb{C}:$$

$$\widetilde{\mathcal{A}}_\beta(\xi) = \mathrm{e}^{-\mathrm{i}[\xi \ln \eta + \chi(\eta, \frac{\xi}{\eta})]}\, \mathcal{A}_\beta(\xi)\ . \tag{6.14}$$

Here, the function ξ is defined as in (4.10)

Clearly, Remark 6.2.3 and (4.10) implies that $\widetilde{\mathcal{A}}_\beta$ is a smooth function.

Lemma 6.2.7. *The function $\widetilde{\mathcal{A}}_\beta$ defined by* (6.14) *is independent of the choice of η satisfying* (6.7) *and* (6.8).

Proof. Let η and η' satisfy (6.7) and (6.8), and $\xi \in (-\xi_1, \xi_1)$; without loss of generality, assume that $\eta < \eta'$. Using (3.28), we obtain that in time interval of length

$$T' = \ln \eta' - \ln \eta\ , \tag{6.15}$$

the X_H flow takes the point $(\frac{\xi}{\eta'}, \eta')$ to $(\frac{\xi}{\eta}, \eta)$, and the point $(\eta, \frac{\xi}{\eta})$ to $(\eta', \frac{\xi}{\eta'})$:

$$\Phi^H_{T'}(\tfrac{\xi}{\eta'}, \eta') = (\tfrac{\xi}{\eta}, \eta)\ , \qquad \Phi^H_{T'}(\eta, \tfrac{\xi}{\eta}) = (\eta', \tfrac{\xi}{\eta'})\ . \tag{6.16}$$

Figure 6.3 illustrates the meaning of the quantities in (6.15) and (6.16).

We will now compute $\widetilde{\mathcal{A}}_\beta(\xi)$ (6.14) for the choices $\eta = \eta$ and $\eta = \eta'$; we will denote the corresponding functions by $\widetilde{\mathcal{A}}^{(\eta)}_\beta(\xi)$ and $\widetilde{\mathcal{A}}^{(\eta')}_\beta(\xi)$. We will need an expression for the parallel transport operator $\Pi^H_{-T'}$ which can be derived from (4.10) and (4.11) similarly to the calculations in the proof of Lemma 4.3.3:

$$\begin{aligned}
\Pi^H_{T'} &= \exp\left\{ \mathrm{i} \int_0^{T'} \langle \theta, X_H \rangle \circ \Phi^H_t\big(\eta, \tfrac{\xi}{\eta}\big)\, \mathrm{d}t \right\}\Bigg|_{(H,J)=(\xi,0),\, T'=\ln\eta'-\ln\eta} \\
&= \exp\left\{ \mathrm{i}\left[\xi T' + \chi\big(\Phi^H_{T'}\big(\eta, \tfrac{\xi}{\eta}\big)\big) - \chi\big(\eta, \tfrac{\xi}{\eta}\big) \right] \right\} \\
&= \mathrm{e}^{\mathrm{i}[\xi \ln \eta' + \chi(\eta', \frac{\xi}{\eta'})]}\, \mathrm{e}^{-\mathrm{i}[\xi \ln \eta + \chi(\eta, \frac{\xi}{\eta})]}\ ,
\end{aligned} \tag{6.17}$$

where we have also used (6.15).

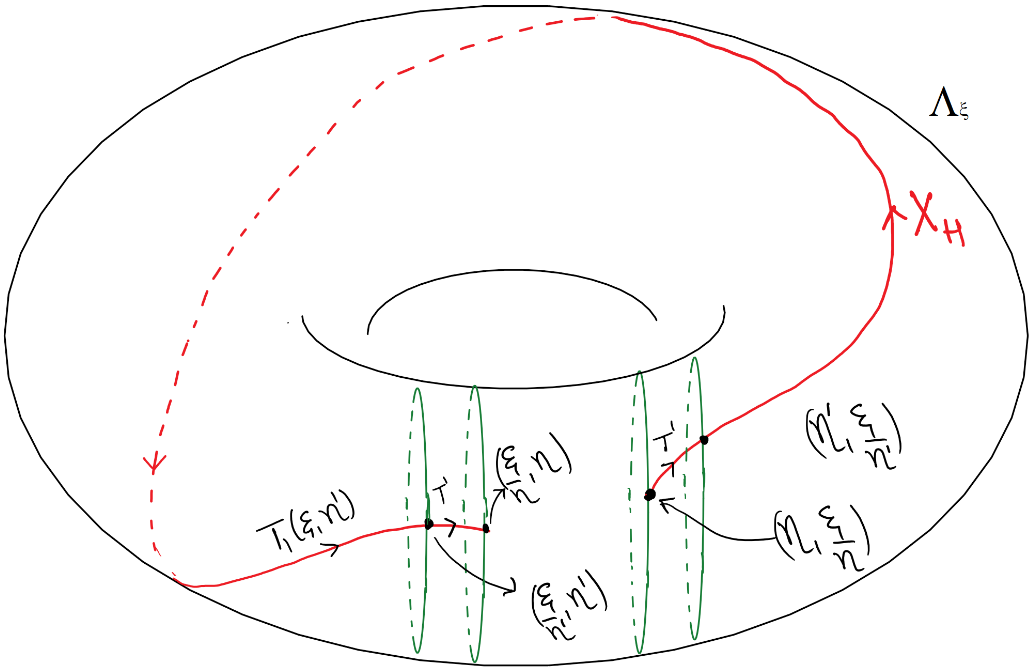

Figure 6.3: On the definition of the times T' and $T_1(\xi,\eta')$.

To compare $\widetilde{\mathcal{A}}^{(\eta)}_\beta(\xi)$ and $\widetilde{\mathcal{A}}^{(\eta')}_\beta(\xi)$, we first note that β is zero on the X_H-trajectory from $(\eta,\frac{\xi}{\eta})$ to $(\eta',\frac{\xi}{\eta'})$, and also from $(\frac{\xi}{\eta'},\eta')$ to $(\frac{\xi}{\eta},\eta)$. Using this observation, performing an elementary change of variables, and using (6.10), (6.15), (6.17), we obtain

$$\begin{aligned}
\widetilde{\mathcal{A}}^{(\eta)}_\beta(\xi) &= \mathrm{e}^{-\mathrm{i}[\xi\ln\eta+\chi(\eta,\frac{\xi}{\eta})]}\int_0^{T_1(\xi,\eta)}\big(\widetilde{\Phi}^{H*}_t\langle\mathcal{J}\beta,X_H\rangle\big)\big(\eta,\tfrac{\xi}{\eta}\big)\,\mathrm{d}t\\
&= \mathrm{e}^{-\mathrm{i}[\xi\ln\eta+\chi(\eta,\frac{\xi}{\eta})]}\int_{T'}^{T_1(\xi,\eta)-T'}\big(\widetilde{\Phi}^{H*}_t\langle\mathcal{J}\beta,X_H\rangle\big)\big(\eta,\tfrac{\xi}{\eta}\big)\,\mathrm{d}t\\
&= \mathrm{e}^{-\mathrm{i}[\xi\ln\eta+\chi(\eta,\frac{\xi}{\eta})]}\int_0^{T_1(\xi,\eta)-2T'}\big(\widetilde{\Phi}^{H*}_{T'+t_1}\langle\mathcal{J}\beta,X_H\rangle\big)\big(\eta,\tfrac{\xi}{\eta}\big)\,\mathrm{d}t_1\\
&= \mathrm{e}^{-\mathrm{i}[\xi\ln\eta+\chi(\eta,\frac{\xi}{\eta})]}\,\Pi^H_{-T'}\int_0^{T_1(\xi,\eta)-2T'}\big(\widetilde{\Phi}^{H*}_{t_1}\langle\mathcal{J}\beta,X_H\rangle\big)\big(\Phi^H_{T'}\big(\eta,\tfrac{\xi}{\eta}\big)\big)\,\mathrm{d}t_1\\
&= \mathrm{e}^{-\mathrm{i}[\xi\ln\eta'+\chi(\eta',\frac{\xi}{\eta'})]}\int_0^{T_1(\xi,\eta')}\big(\widetilde{\Phi}^{H*}_{t_1}\langle\mathcal{J}\beta,X_H\rangle\big)\big(\eta',\tfrac{\xi}{\eta'}\big)\,\mathrm{d}t_1\\
&= \widetilde{\mathcal{A}}^{(\eta')}_\beta(\xi)\ .
\end{aligned}$$

$\square$

Lemma 6.2.8. *Let $\beta \in \Gamma(\mathcal{L}^2_{0,P})$. There exists a $\psi \in \Gamma(\mathcal{L}^0_{0,P})$ such that $\nabla\psi = \langle \mathcal{J}\beta, X_H\rangle$ on the subset $\{\, J = 0\,\}$ if and only if the germ at 0 of the smooth function $\widetilde{\mathcal{A}}_\beta$ vanishes.*

Proof. First assume that the germ of $\widetilde{\mathcal{A}}_\beta$ (or equivalently, of $\mathcal{A}_\beta$) vanishes at 0. Define $\psi \in \mathcal{L}^0_{0,P}(\{J = 0\})$ by

$$\psi(m) := \begin{cases} \psi_1(m) & \text{for } m \in \{J = 0\} \setminus \Lambda_{(0,0)}\ , \\ \psi_2(m) & \text{for } m \in \text{a small neighborhood of } \Lambda_{(0,0)} \text{ inside } \{J = 0\}\ , \end{cases}$$

where ψ_1 and ψ_2 are defined by (6.5) and (6.6), respectively. Since the germ of $\mathcal{A}_\beta$ vanishes at 0, Lemma 6.2.5 guarantees that ψ is well-defined on $\{J = 0\}$ and that it vanishes on an open neighborhood of the focus-focus point. The section ψ on $\{J = 0\}$ can be extended in a smooth fashion to M_ϵ, which gives us a $\psi \in \Gamma(\mathcal{L}^0_{0,P})$.

Conversely, assume that there exists $\psi \in \Gamma(\mathcal{L}^0_{0,P})$ such that $\nabla\psi = \mathcal{J}\beta$. Assume that ψ vanishes in the open neighborhood V_β defined above (such that $\beta|V_\beta \equiv 0$). Choose ξ_1 and η satisfying (6.7) and (6.8). Then $\psi(\eta, \frac{\xi}{\eta}) = 0$ and $\nabla_{X_H}\psi = \langle \mathcal{J}\beta, X_H\rangle$ on $\{J = 0\}$ imply that

$$\psi\big(\Phi^H_t(\eta, \tfrac{\xi}{\eta})\big) = \Pi^H_t \int_0^t \big(\widetilde{\Phi}^{H*}_t \langle \mathcal{J}\beta, X_H\rangle\big)(\eta, \tfrac{\xi}{\eta})\ \mathrm{d}t\ , \quad t \in \mathbb{R}$$

(recall (6.3)). The definition (6.10) of $T_1(\xi, \eta)$ implies that $\psi\big(\Phi^H_{T_1(\xi,\eta)}(\eta, \frac{\xi}{\eta})\big) = 0$, so for every $\xi \in (-\xi_1, \xi_1)$, we have

$$\begin{aligned}
0 &= \psi\big(\Phi^H_{T_1(\xi,\eta)}(\eta, \tfrac{\xi}{\eta})\big) \\
&= \Pi^H_{T_1(\xi,\eta)} \int_0^{T_1(\xi,\eta)} \big(\widetilde{\Phi}^{H*}_t \langle \mathcal{J}\beta, X_H\rangle\big)(\eta, \tfrac{\xi}{\eta})\ \mathrm{d}t \\
&= \Pi^H_{T_1(\xi,\eta)}\, \mathcal{A}_\beta(\xi)\ .
\end{aligned}$$

Therefore $\mathcal{A}_\beta$ vanishes identically on $(-\xi_1, \xi_1)$, hence the germ at 0 of the function $\widetilde{\mathcal{A}}_\beta$ vanishes. □

We summarize the above results in the following

Theorem 6.2.9. *The polarized $\mathbb{L}$-valued 2-form $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ is exact if and only if the germ of the smooth function $\widetilde{\mathcal{A}}_\beta(\xi)$ defined in* (6.14) *vanishes at $\xi = 0$.*

6.2.2 Completion of the calculation of $\mathrm{H}^2(M, \mathcal{F})$

Here we apply Theorem 6.2.9 to finish the calculation of $\mathrm{H}^2(M, \mathcal{F})$.

Definition 6.2.10. *Let the map*

$$\kappa : \Gamma(\mathcal{L}^2_{0,P}) \to \{\, \textit{germs of smooth } \mathbb{C}\textit{-valued functions at } 0 \in \mathbb{R}\,\}$$

be defined by

$$\kappa(\beta) = (\textit{the germ of the function } \widetilde{\mathcal{A}}_\beta \textit{ at } 0 \in \mathbb{R})\ , \tag{6.18}$$

where $\widetilde{\mathcal{A}}_\beta$ is defined in (6.14).

Clearly, κ is a $\mathbb{C}$-linear map.

Lemma 6.2.11. *Let*

$$\begin{aligned}
&\mathscr{V} = \{\textit{the smooth functions from } \mathbb{R} \textit{ to } \mathbb{C} \textit{ whose germs vanish at 0}\}\ , \\
&\mathscr{V}_0 = \{\textit{the germs at 0 of functions from } \mathscr{V}\ \}\ .
\end{aligned} \tag{6.19}$$

The map κ defined in (6.18) *is surjective; moreover, it maps $\mathrm{d}^\nabla(\Gamma(\mathcal{L}^1_{0,P}))$ onto the set $\mathscr{V}_0$.*

Proof. To prove the surjectivity of κ, for any f_0 be the germ at 0 of a smooth $\mathbb{C}$-valued function we will construct a $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ such that $\kappa(\beta) = f_0$. Recall that $\widetilde{\mathcal{A}}_\beta$ is defined by (6.14); we use the notations introduced in Section 6.2.1. For ξ_1 and η satisfying (6.7) and (6.8), and $\xi \in (-\xi_1, \xi_1)$, the continuous function $T_1(\xi, \eta)$ defined in (6.10) takes values in some interval $[T_{1,\min}, T_{1,\max}]$; let $g : \mathbb{R} \to \mathbb{R}$ be a function whose support is in the interval $(0, T_{1,\min})$ which satisfies $\displaystyle\int_{\mathbb{R}} g(t)\,\mathrm{d}t = \frac{1}{2\pi}$. Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function whose germ at 0 is f_0, and $\mathbf{e}$ be a unit vector in $\mathbb{L}_{(\eta,\, \frac{\xi}{\eta})}$. Define $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ on $\{J = 0\}$ by

$$\beta(X_J, X_H)(\Phi^H_t \Phi^J_s(\eta, \tfrac{\xi}{\eta})) = \mathrm{e}^{i[\xi \ln \eta + \chi(\eta, \frac{\xi}{\eta})]}\, g(t) f(\xi)\, \Pi^H_t \Pi^J_s \mathbf{e} \in \mathbb{L}_{\Phi^H_t \Phi^J_s(\eta, \frac{\xi}{\eta})}\ . \tag{6.20}$$

From (6.14) and (6.20), we obtain

$$\begin{aligned}
\widetilde{\mathcal{A}}_\beta(\xi) &= \mathrm{e}^{-\mathrm{i}[\xi \ln \eta + \chi(\eta, \frac{\xi}{\eta})]} \int_0^{T_1(\xi,\eta)} \big(\widetilde{\Phi}^{H*}_t \langle \mathcal{J}\beta, X_H\rangle\big)\big(\eta, \tfrac{\xi}{\eta}\big)\,\mathrm{d}t \\
&= \mathrm{e}^{-\mathrm{i}[\xi \ln \eta + \chi(\eta, \frac{\xi}{\eta})]} \int_0^{T_1(\xi,\eta)} \int_0^{2\pi} \Pi^H_{-t} \Pi^J_{-s} \beta(X_J, X_H)\big(\Phi^H_t \Phi^J_s\big(\eta, \tfrac{\xi}{\eta}\big)\big)\,\mathrm{d}s\,\mathrm{d}t \\
&= f(\xi) \int_0^{T_1(\xi,\eta)} \int_0^{2\pi} g(t)\,\mathbf{e}\,\mathrm{d}s\,\mathrm{d}t = f(\xi)\ .
\end{aligned}$$

Now we will prove that $\kappa(\mathrm{d}^\nabla(\Gamma(\mathcal{L}^1_{0,P}))) = \mathscr{V}_0$. First we show that if $f \in C^\infty(\mathbb{R})$ such that its germ at 0, f_0, is in $\kappa\left(\mathrm{d}^\nabla(\Gamma(\mathcal{L}^1_{0,P}))\right)$, then $f_0 \in \mathscr{V}_0$. Since

$f_0 \in \kappa(\mathrm{d}^\nabla(\Gamma(\mathcal{L}^1_{0,P})))$, there exists $\alpha \in \Gamma(\mathcal{L}^1_{0,P})$ such that f_0 equals the germ of $\widetilde{\mathcal{A}}_{\mathrm{d}^\nabla\alpha}$ at 0. Since $\mathrm{d}^\nabla\alpha$ is exact, Theorem 6.2.9 guarantees that the germ of $\widetilde{\mathcal{A}}_{\mathrm{d}^\nabla\alpha}$ vanishes at 0, hence $f_0 \in \mathscr{V}_0$.

On the other hand, suppose $f \in C^\infty(\mathbb{R})$ is such that $f_0 \in \mathscr{V}_0$. By the surjectivity of κ, there exists $\beta \in \Gamma(\mathcal{L}^2_{0,P})$ such that f_0 equals the germ of $\widetilde{\mathcal{A}}_\beta$ at 0. Since $f_0 \in \mathscr{V}_0$, the germ of $\widetilde{\mathcal{A}}_\beta$ vanishes at 0, and Theorem 6.2.9 implies that β is exact, i.e., there exists $\alpha \in \Gamma(\mathcal{L}^1_{0,P})$ such that $\beta = \mathrm{d}^\nabla\alpha$. This means that $f_0 = \kappa(\mathrm{d}^\nabla\alpha)$, i.e., $\mathscr{V}_0 \subseteq \kappa(\mathrm{d}^\nabla(\Gamma(\mathcal{L}^1_{0,P})))$. $\square$

Lemma 6.2.11 implies immediately that

$$\begin{aligned} \mathrm{H}^2(M,\mathcal{F}) &= \Gamma(\mathcal{L}^2_{0,P})/\mathrm{d}^\nabla(\Gamma(\mathcal{L}^1_{0,P})) \\ &= \{\text{Germs of smooth functions } \mathbb{R}\to\mathbb{C} \text{ at } 0\ \}/\mathscr{V}_0 \\ &= \{\text{Germs of smooth functions } \mathbb{R}\to\mathbb{C} \text{ at } 0\ \}\ , \end{aligned}$$

where $\mathscr{V}_0$ is defined in (6.19). Hence, we prove the following:

Theorem 6.2.12. *Let $\mu : M \to \mathbb{R}^2$ be an integrable system with non-degenerate singularities. Equip M with a trivial line bundle with connection determined by the 1-form of Lemma 4.3.2, and $\mathcal{F}$ is the the sheaf of P-flat sections of* $\mathbb{L}$ (3.9). *Assume that the $\mu^{-1}(0,0)$ is a focus-focus torus. Then there exists an $\epsilon_0 > 0$ such that $M_\epsilon := \mu^{-1}(\{\,(H,J) : |(H,J)| < \epsilon\,\})$ satisfies*

$$\mathrm{H}^0(M_\epsilon,\mathcal{F}) = 0, \quad \mathrm{H}^1(M_\epsilon,\mathcal{F}) = 0, \quad \mathrm{H}^2(M_\epsilon,\mathcal{F}) = \{\,\textit{germs of functions at } 0 \in \mathbb{R}\,\}$$

whenever $\epsilon < \epsilon_0$.

Chapter 7

Example

Consider the local model $M = \mathbb{R}^4$ for a focus-focus singularity given in Section 3.5. In the following we will construct a polarized $\mathbb{L}$-valued 1-form α which is closed but not exact.

Let $\zeta = \zeta(z_1, z_2) : M \to \mathbb{C}$ be the function defined as follows:

$$\zeta(z_1, z_2) = \begin{cases} 0 & \text{if } H(z_1, z_2) = 0 \ , \\ \dfrac{1}{2} H(z_1, z_2) \operatorname{Log} \dfrac{z_1}{z_2} & \text{if } H(z_1, z_2) \neq 0 \ , \end{cases} \tag{7.1}$$

with $H(z_1, z_2)$ defined in (3.26); note that $H(z_1, z_2) \neq 0$ implies that $z_1 \neq 0$ and $z_2 \neq 0$. Choose the branch cut of $\operatorname{Log} z$ in (7.1) to be the negative imaginary axis (i.e., at $z = -\mathrm{i}a$ with $a \in \mathbb{R}$, $a > 0$). Using that $H(z_1, z_2) = |z_1||z_2| \cos \operatorname{Arg} \frac{z_1}{z_2}$, one can easily see that $\zeta(z_1, z_2)$ vanishes when $\frac{z_1}{z_2}$ belongs to the branch cut of Log, and that ζ is bounded on any compact subset of M.

Lemma 7.0.1. *Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a smooth function which is Taylor flat at 0. Then $\mathrm{e}^{\mathrm{i}\zeta} \lambda(H)$ is a smooth function on M. Moreover, $\nabla_{X_H} \big(\mathrm{e}^{\mathrm{i}\zeta} \lambda(H) \big) = 0$.*

Proof. From the definitions, it is easy to show that $\mathrm{e}^{\mathrm{i}\zeta} \lambda(H)$ is a smooth function on M. From (3.27) and $X_H(H)=0$ we obtain

$$\begin{aligned}
X_H\big(\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\big) &= \mathrm{i}\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)X_H(\zeta)+\mathrm{e}^{\mathrm{i}\zeta}\lambda'(H)X_H(H)\\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)X_H(\zeta)\\
&= \frac{\mathrm{i}}{2}H\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\left(z_1\frac{\partial}{\partial z_1}+\bar{z}_1\frac{\partial}{\partial \bar{z}_1}-z_2\frac{\partial}{\partial z_2}-\bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right)\mathrm{Log}\frac{z_1}{z_2}\\
&= \mathrm{i}H\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\ ,
\end{aligned}$$

therefore $\nabla_{X_H}\big(\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\big)=X_H\big(\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\big)-\mathrm{i}H\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)=0.$ □

Lemma 7.0.2. *Let $\alpha\in\mathcal{L}^1_P(M)$ be the 1-form with $\langle\alpha,X_H\rangle=0$, $\langle\alpha,X_J\rangle=\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)$. Then α is closed. Moreover, as long as the germ of λ at 0 is not 0, the equation $\nabla\psi=\alpha$ does not have a solution $\psi\in\mathcal{L}^0_P(M)$ in any open ball centered at the origin, i.e., α is not locally exact.*

Proof. The closedness of α follows from (3.16), the hypothesis, and Lemma 7.0.1:

$$(\mathrm{d}^\nabla\alpha)(X_H,X_J)=\nabla_{X_H}\langle\alpha,X_J\rangle-\nabla_{X_J}\langle\alpha,X_H\rangle=\nabla_{X_H}\big(\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\big)=0\ .$$

To prove that α is not locally exact, assume for contradiction that there exists an open neighborhood of the origin of M (which without loss of generality can be assumed to be an open ball centered at the origin) in which the equation $\nabla\psi=\alpha$ has a solution ψ. This is equivalent to the system of equations

$$\nabla_{X_H}\psi=0,\quad \nabla_{X_J}\psi=\mathrm{e}^{\mathrm{i}\zeta}\lambda(H).$$

Solving $\nabla_{X_J}\psi=\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)$, obtain

$$\psi(\Phi^J_s(z_1,z_2))=\Pi^J_s\left[\psi(z_1,z_2)+\int_0^s\Pi^J_{-s_1}\left[\big(\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\big)\circ\Phi^J_{s_1}(z_1,z_2)\right]\mathrm{d}s_1\right]\ .$$

Using that ψ is 2π-periodic in s, we obtain that on $\{J=0\}$ we must have

$$\int_0^{2\pi} \left(\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)\right) \circ \Phi^J_{s_1}(z_1,z_2)\ \mathrm{d}s_1 = 0.$$

Using (3.27), it is easy to show that the function $\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)$ is constant on X_J-orbits in $\{J=0\}$, hence the above integral is equal to $2\pi\mathrm{e}^{\mathrm{i}\zeta(z_1,z_2)}\lambda(H(z_1,z_2))$. The $\mathrm{e}^{\mathrm{i}\zeta}$ term is never 0, so the integral vanishes if and only if $\lambda(H(z_1,z_2))=0$. Therefore, it does not vanish for all (z_1,z_2) with $J(z_1,z_2)=0$ unless λ is identically 0, hence the solution ψ does not exist unless the germ of λ at 0 is 0. $\square$

Lemma 7.0.3. *The polarized $\mathbb{L}$-valued 1-form α defined in Lemma 7.0.2 is in $\mathcal{L}^1_{M|P}(M)$. In other words, there exists a (non-polarized) $\mathbb{L}$-valued 1-form $\alpha_1 \in \mathcal{L}^1_M(M)$ such that $\alpha_1|P=\alpha$, i.e., $\langle\alpha_1,X_H\rangle = \langle\alpha,X_H\rangle$ and $\langle\alpha_1,X_J\rangle=\langle\alpha,X_J\rangle$.*

Proof. On $M\cap\{z_1\neq 0, z_2\neq 0\}$, define the non-polarized 1-form

$$\alpha_1 = \frac{\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)}{2\mathrm{i}}\left(\frac{\mathrm{d}z_1}{z_1}+\frac{\mathrm{d}z_2}{z_2}\right)\ .$$

Using (3.27), it is easy to show that

$$\langle\alpha_1,X_H\rangle = 0 = \langle\alpha,X_H\rangle\ ,\qquad \langle\alpha_1,X_J\rangle = \mathrm{e}^{\mathrm{i}\zeta}\lambda(H) = \langle\alpha,X_J\rangle\ .$$

However, $\lambda(H)/z_1$ is defined only when $z_1\neq 0$, and similarly for $\lambda(H)/z_2$. In the following we will complete the proof by showing that $\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)/z_1$ and $\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)/z_2$ extend to smooth functions on M. We first note that

$$|H(z_1,z_2)| = \left|\frac{\bar{z}_1z_2+z_1\bar{z}_2}{2}\right| \leq |z_1||z_2| \quad \text{for } z_1\neq 0\ .$$

Since λ is Taylor flat at 0, for each non-negative integer n there exists a constant c_n such that $|\lambda(H)| \leq c_n|H|^n$ whenever $|H|$ is small. Thus for any bounded set and any non-negative integer n there exists a constant c'_n such that $|\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)| \leq c'_n|z_1z_2|^n$ for all (z_1, z_2) in this set. It follows that $\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)/z_1$ extends smoothly to $z_1 = 0$ by defining it to be 0 when $z_1 = 0$. Similarly, $\mathrm{e}^{\mathrm{i}\zeta}\lambda(H)/z_2$ extends smoothly to $z_2 = 0$. □

The preceding lemmas provide a counterexample to Theorem 6.1 in [38]: in the notation of [38], the above results show that $H^1(S^\bullet_P(L)) \neq 0$ with $n = 2$ and $k_f = 1$ (k_f is our k_{ff} from Theorem 2.1.14). The proofs of the lemmas above also show that neither $0 \to \mathcal{F} \to \mathcal{L}^*_P$ nor $0 \to \mathcal{F} \to \mathcal{L}^*_{M|P}$ is a resolution of $\mathcal{F}$ (the exactness of the sequence of stalks fails at the focus-focus point). The mistake in the proof of Theorem 6.4 in [38] seems to be that the estimate in equation (34) in that paper is not correct.

Theorem 6.4 in [38] also appears as Theorem 4.2 in [28]. Theorem 5.1 in [28] is not correct because its proof relies upon Theorem 4.2 in that paper.

In [38], the de Rham resolution for the sheaf $\mathcal{F}$ is used as a resolution. According to Definition 3.1 in [38], the sheaves in the de Rham resolution are the ones defined in our Definition 3.1.14. However, in Section 6.6 of [38], the sheaves in the de Rham resolution are taken to be those in our Definition 3.1.19. The same discrepancy appears in [29]: see the definitions in Section 4 versus the proof of Proposition 6.2 in that paper, for example. At nonsingular points, the two definitions agree, but they do not agree at singular points.

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