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DEDICATION

to

My Friends

For

their help and support during tough times

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CONTENTS

ABSTRACT

Let $\mathbf{F}$ be a characteristic zero differential field with an algebraically closed field of constants $\mathbf{C}$, $\mathbf{E} \supset \mathbf{K} \supset \mathbf{F}$ be no new constant extensions of $\mathbf{F}$ such that $\mathbf{K}$ is an extension by antiderivatives of $\mathbf{F}$, and let $\mathbf{E}$ contain antiderivatives $\mathfrak{y}_1, \cdots, \mathfrak{y}_n$ of $\mathbf{K}$. The antiderivatives $\mathfrak{y}_1, \cdots, \mathfrak{y}_n$ of $\mathbf{K}$ are called J-I-E antiderivatives if $\mathfrak{y}_i' \in \mathbf{K}$ satisfies certain conditions. We will provide a new proof for the Kolchin-Ostrowski theorem and generalize this theorem for a tower of extensions by J-I-E antiderivatives and use this generalized version of the theorem to classify the finitely differentially generated subfields of this tower. In the process, we will show that the J-I-E antiderivatives are algebraically independent over the ground differential field. An example of a J-I-E tower is the iterated antiderivative extensions of the field of rational functions $\mathbf{C}(x)$ generated by iterated logarithms, closed at each stage by all (translation) automorphisms. We analyze the algebraic and differential structure of these extensions. In particular, we show that the nth iterated logarithms and their translates are algebraically independent over the field generated by all lower level iterated logarithms. Our analysis provides an algorithm for determining the differential field generated by any rational expression in iterated logarithms.

Chapter 1

Introduction

All the fields considered in this thesis are of characteristic zero. If $\mathbf{F}$ is a field and $' : \mathbf{F} \to \mathbf{F}$ a linear map satisfying the condition $(uv)' = u'v + uv'$ for all $u, v \in \mathbf{F}$ then we will call the map $'$, a *derivation* of $\mathbf{F}$. A *differential field* is a field $\mathbf{F}$ with a derivation. If $\mathbf{F}$ is a differential field then one can easily see that $\mathbf{C} := \{c \in \mathbf{F} | c' = 0\}$ is also a differential field. We will call $\mathbf{C}$, the field of *constants* of $\mathbf{F}$. Let $\mathbf{E}$ and $\mathbf{F}$ be differential fields and let $\mathbf{E} \supseteq \mathbf{F}$. We say that $\mathbf{E}$ is a *differential field extension* of $\mathbf{F}$ if the derivation of $\mathbf{E}$ restricted to $\mathbf{F}$ is the derivation of $\mathbf{F}$. A differential field extension $\mathbf{E}$ of $\mathbf{F}$ will be called a *No New Constants* (NNC) extension of $\mathbf{F}$ if the field of constants of $\mathbf{E}$ and $\mathbf{F}$ are the same.

Let $\mathbf{E} \supset \mathbf{F}$ be a NNC extension. If $\mathfrak{x} \in \mathbf{E}$ and $\mathfrak{x}' \in \mathbf{F}$ then we call $\mathfrak{x}$ an *antiderivative* of an element (namely, $\mathfrak{x}'$) of $\mathbf{F}$, and if $\mathbf{E} = \mathbf{F}(\mathfrak{x}_1 \cdots, \mathfrak{x}_n)$ for some antiderivatives $\mathfrak{x}_1, \cdots, \mathfrak{x}_n \in \mathbf{E}$ of $\mathbf{F}$ then we will call $\mathbf{E}$ an *extension of* $\mathbf{F}$ *by antiderivatives*. If $\mathfrak{e} \in \mathbf{E}$ and $\frac{\mathfrak{e}'}{\mathfrak{e}} \in \mathbf{F}$ then we call $\mathfrak{e}$ an *exponential of an integral* of an element (namely, $\frac{\mathfrak{e}'}{\mathfrak{e}}$) of $\mathbf{F}$, and if $\mathbf{E} = \mathbf{F}(\mathfrak{e}_1 \cdots, \mathfrak{e}_m)$ for some exponentials of integrals $\mathfrak{e}_1 \cdots, \mathfrak{e}_m \in \mathbf{E}$ of $\mathbf{F}$ then we will call $\mathbf{E}$ *an extension of* $\mathbf{F}$ *by exponentials* *of integrals*.

In chapter 2 we will give a new proof for the following well known theorem: Let $\mathbf{F}$ be a differential field with an algebraically closed field of constants $\mathbf{C}$ and let $\mathbf{E} \supset \mathbf{F}$ be a NNC extension. Let $\mathfrak{x}_1, \cdots, \mathfrak{x}_n \in \mathbf{E}$, $\mathfrak{e}_1, \cdots, \mathfrak{e}_m \in \mathbf{E}$ where $\mathfrak{x}_i$'s are antiderivatives ($\mathfrak{x}_i' \in \mathbf{F}$) and $\mathfrak{e}_i$'s are exponentials of integrals ($\frac{\mathfrak{e}_i'}{\mathfrak{e}_i} \in \mathbf{F}$). Then $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$, $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ are algebraically dependent over $\mathbf{F}$ only if there are $c_i \in \mathbf{C}$, not all zero, such that $\sum_{i=1}^{n} c_i\mathfrak{x}_i \in \mathbf{F}$ or there are $n_i \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^{m} \mathfrak{e}_i^{n_i} \in \mathbf{F}$. Thus the algebraic dependence of $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$, $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ over $\mathbf{F}$ becomes a non trivial linear dependence of $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ over $\mathbf{F}$, or there is a non trivial power product relation among $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ over $\mathbf{F}$. This theorem is known as the Kolchin-Ostrowski theorem and it appears as theorem 2.3 in this thesis. A short note about the history of this theorem is also provided in the beginning of chapter 2.

In section 2.3 and 2.4, we will give algorithms to compute the differential subfields of extensions by antiderivatives and exponentials of integrals of $\mathbf{F}$ when such an extension is purely transcendental over $\mathbf{F}$. Moreover, when $\mathbf{F}$ can be realized as the field of fractions of a polynomial ring over $\mathbf{C}$ that lives inside $\mathbf{F}$ then for any given intermediate differential subfield of this extension, our algorithm also computes the subgroup of differential automorphisms of our extension fixing that given differential subfield.

In chapter 3, section 3.1, we produce a method for generating algebraically independent iterated antiderivatives of $\mathbf{F}$ when $\mathbf{F}$ has a proper antiderivative extension. We call this special tower of extensions by iterated antiderivatives, the J-I-E tower. And, as an application, We will show that there is an infinite tower of extensions by iterated antiderivatives of $\mathbf{F}$ that is not imbeddable in any finite tower of Picard-Vessiot extensions of $\mathbf{F}$. In section 3.2 we classify the finitely differentially generated subfields of this tower. A J-I-E tower exist for any differential field $\mathbf{F}$ that has a proper antiderivative extension and it may contain non-elementary functions.

A tower of extensions by iterated logarithms is an example of J-I-E tower. For a vector $\vec{c} := (c_1, \cdots, c_n) \in \mathbf{C}^n$, where $\mathbf{C}$ is an algebraically closed-characteristic zero differential field with a trivial derivation, we call $\mathfrak{x}[\vec{c}, n] := \log(\log(\cdots \log(x + c_1) \cdots + c_{n-1}) + c_n)$ an iterated logarithm of level n. In chapter 4, we give meanings for these iterated logarithms and produce an algorithm to compute the differential subfields of differential field extensions by iterated logarithms. In the process, we will also show that the iterated logarithms are algebraically independent over $\mathbf{C}(x)$, where x is an element whose derivative equals 1. In Section 4.4 we will provide some examples of extensions by iterated logarithms and show how our algorithm works. These examples should also be viewed as examples for computing differential subfields of extensions by antiderivatives as well since the algorithms for both the settings works in a similar fashion.

1.1 Picard-Vessiot Theory

Here we will recall some definitions and state several results from differential Galois theory. One may find proofs for these results in [7]. Let $(\mathbf{F}, ')$ be a differential field with an algebraically closed field of constants $\mathbf{C}$ and let $\mathbf{E}$ be any differential field extension of $\mathbf{F}$. The differential Galois group $\mathbb{G}(\mathbf{E}|\mathbf{F})$ is the group of all differential automorphisms of $\mathbf{E}$ fixing every element of $\mathbf{F}$, that is, $\mathbb{G}(\mathbf{E}|\mathbf{F}) := \{\sigma \in Aut(\mathbf{E}|\mathbf{F}) | \sigma(u') = \sigma(u)' \; \forall u \in \mathbf{E}\}$. Sometimes we denote $\mathbb{G}(\mathbf{E}|\mathbf{F})$ by $\mathbb{G}$ without referring to ground differential field $\mathbf{F}$ and its extension $\mathbf{E}$. Let $L(y)$ be a monic homogeneous linear differential operator of order n over a differential field $\mathbf{F}$. A differential field extension $\mathbf{E} \supseteq \mathbf{F}$ is called a Picard-Vessiot(P-V) extension of $\mathbf{F}$ for $L(y)$ if the following conditions hold:

1. $\mathbf{E}$ is generated over $\mathbf{F}$ as a differential field by the set V of solutions of $L(y) = 0$ in $\mathbf{E}$ ($\mathbf{E} = \mathbf{F} < V >$)

2. $\mathbf{E}$ contains a full set of solutions of $L(y) = 0$ (there are $y_i \in V, 1 \leq i \leq n$, with the wronskian $w(y_1, \cdots, y_n) \neq 0$)

3. Every constant of $\mathbf{E}$ lies in $\mathbf{F}$, that is, $\mathbf{E}$ is a NNC extension of $\mathbf{F}$.

A Picard-Vessiot extension exists for a given monic homogeneous linear differential operator $L(y)$ in the case that the field of constants $\mathbf{C}$ of $\mathbf{F}$ is algebraically closed and it is unique up to differential automorphisms fixing $\mathbf{F}$. If $\mathbf{E}$ is a P-V extension of $\mathbf{F}$ then the set of all elements fixed by the differential Galois group $\mathbb{G}(\mathbf{E}|\mathbf{F})$ is $\mathbf{F}$, that is, $\mathbf{E}^{\mathbb{G}} = \{a \in \mathbf{E} \mid \sigma(a) = a \text{ for all } \sigma \in \mathbb{G}\} = \mathbf{F}$. The differential Galois group of a P-V extension is an algebraic matrix group over the field of constants.

If $\mathbf{E}_i$ is a Picard-Vessiot extension of $\mathbf{F}$ for $1 \leq i \leq n$ then there is a Picard-Vessiot extension $\mathbf{E}$ of $\mathbf{F}$ such that $\mathbf{E} \supseteq \mathbf{E}_i \supseteq \mathbf{F}$ and $\mathbf{E}$ is the compositum of its subfields $\mathbf{E}_i$.

There is a Fundamental theorem in this context. Let $\mathbf{F}$ be a differential field with algebraically closed field of constants $\mathbf{C}$, and let $\mathbf{E} \supseteq \mathbf{F}$ be a P-V extension. Then the differential Galois group of $\mathbf{E}$ over $\mathbf{F}$ is naturally an algebraic group over $\mathbf{C}$ and there is a lattice inverting bijective correspondence between

$$\{\mathbf{E} \supseteq \mathbf{K} \supseteq \mathbf{F} \mid \mathbf{K} \text{ is an intermediate differential field}\}$$

and

$$\{\mathbb{H} \leq \mathbb{G}(\mathbf{E}|\mathbf{F}) \mid \mathbb{H} \text{ is a Zariski closed subgroup of } \mathbb{G}(\mathbf{E}|\mathbf{F})\}$$

given by

$$\mathbf{K} \mapsto \mathbb{G}(\mathbf{E}|\mathbf{K}) \text{ and } \mathbb{H} \mapsto \mathbf{E}^{\mathbb{H}}.$$

The intermediate field $\mathbf{K}$ is a P-V extension of $\mathbf{F}$ if and only if the subgroup $\mathbb{H} = \mathbb{G}(\mathbf{E}|\mathbf{K})$ is normal in $\mathbb{G}$; if it is, then

$$\mathbb{G}(\mathbf{E}^{\mathbb{H}}|\mathbf{F}) = \frac{\mathbb{G}(\mathbf{E}|\mathbf{F})}{\mathbb{H}}.$$

Let $\mathbb{G}^0(\mathbf{E}|\mathbf{F})$ be the connected component of the identity in $\mathbb{G}(\mathbf{E}|\mathbf{F})$, and let $\mathbf{E}^0$ be the corresponding intermediate field. Then $\mathbf{E}^0$ is the algebraic closure of $\mathbf{F}$ in $\mathbf{E}$, $\mathbf{E}^0$ is a finite Galois extension of $\mathbf{F}$ with Galois group $\frac{\mathbb{G}(\mathbf{E}|\mathbf{F})}{\mathbb{G}^0(\mathbf{E}|\mathbf{F})}$, and the transcendence degree of $\mathbf{E}$ over $\mathbf{E}^0$ is $\dim(\mathbb{G}^0(\mathbf{E}|\mathbf{F}))$.

Analogous to the algebraic closure of a given field, we may define a Picard-Vessiot closure of a given differential field $\mathbf{F}$. The Picard-Vessiot closure $\mathbf{F}_1$ of $\mathbf{F}_0 := \mathbf{F}$ is a differential field extension of $\mathbf{F}_0$ such that

- $\mathbf{F}_1$ is a union of Picard-Vessiot extensions of $\mathbf{F}_0$
- Every Picard-Vessiot extension of $\mathbf{F}_0$ has an isomorphic copy in $\mathbf{F}_1$.

The Picard-Vessiot closure $\mathbf{F}_1$ of $\mathbf{F}_0$ need not be "closed". That is, there are linear homogeneous differential equations over $\mathbf{F}_1$ whose solutions may not be in $\mathbf{F}_1$ (see theorem 3.7). This leads us to consider a chain of Picard-Vessiot closures of $\mathbf{F}_0$. A finite tower of Picard-Vessiot closures of $\mathbf{F}_0$ is a chain

$$\mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \mathbf{F}_2 \subseteq \cdots \subseteq \mathbf{F}_n,$$

where $\mathbf{F}_0 := \mathbf{F}, n \in \mathbb{N}$ and $\mathbf{F}_i$ is the Picard-Vessiot closure of $\mathbf{F}_{i-1}$, for all $1 \leq i \leq n$. Finally we define the complete Picard-Vessiot closure $\mathbf{F}_\infty$ of $\mathbf{F}$ as the union $\cup_{i=0}^{\infty} \mathbf{F}_i$. The differential field $\mathbf{F}_\infty$ is "closed". If $\mathbf{E}$ is a normal differential subfield of $\mathbf{F}_\infty$ then every automorphism of $\phi \in \mathbb{G}(\mathbf{E}|\mathbf{F})$ extends to an automorphism $\Phi \in \mathbb{G}(\mathbf{F}_\infty|\mathbf{F})$ and every automorphism $\Phi \in \mathbb{G}(\mathbf{F}_\infty|\mathbf{F})$ also restricts to a $\phi \in \mathbb{G}(\mathbf{E}|\mathbf{F})$. We also note that the fixed field of $\mathbb{G}(\mathbf{F}_\infty|\mathbf{F})$ is $\mathbf{F}$. For details see [9].

Chapter 2

The Kolchin-Ostrowski Theorem

Let $\mathbf{F}$ be a differential field with an algebraically closed field of constants $\mathbf{C}$. Sometimes we will denote the field of constants $\mathbf{C}$ of $\mathbf{F}$ by $\mathbf{C}_{\mathbf{F}}$. Let us recall some definitions from chapter 1.

Definition 2.1. Let $\mathbf{E} \supset \mathbf{F}$ be a differential field extension of $\mathbf{F}$. An element $\mathfrak{x} \in \mathbf{E}$ is called an *antiderivative* of an element of $\mathbf{F}$ if $\mathfrak{x}' \in \mathbf{F}$. A No New Constant(NNC) extension $\mathbf{E} \supset \mathbf{F}$ is called an *extension by antiderivatives* (or an *antiderivative extension*) of $\mathbf{F}$ if for $i = 1, 2, \cdots, n$ there exists $\mathfrak{x}_i \in \mathbf{E}$ such that $\mathfrak{x}_i' \in \mathbf{F}$ and $\mathbf{E} = \mathbf{F}(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n)$.

Definition 2.2. Let $\mathbf{E} \subset \mathbf{F}$ be a differential field extension of $\mathbf{F}$. An element $\mathfrak{e} \in \mathbf{E}$ is called an *exponential of an integral* of an element of $\mathbf{F}$ if $\frac{\mathfrak{e}'}{\mathfrak{e}} \in \mathbf{F}$. A NNC extension $\mathbf{E} \supset \mathbf{F}$ is an extension by *exponential of integrals* of $\mathbf{F}$ if for $i = 1, 2, \cdots, n$ there exists $\mathfrak{e}_i \in \mathbf{E}$ such that $\frac{\mathfrak{e}_i'}{\mathfrak{e}_i} \in \mathbf{F}$ and $\mathbf{E} = \mathbf{F}(\mathfrak{e}_1, \mathfrak{e}_2, \cdots, \mathfrak{e}_n)$.

In this section we will prove the Kolchin-Ostrowski theorem, which states

Theorem 2.3. *(Kolchin-Ostrowski) Let $\mathbf{E} \supset \mathbf{F}$ be a NNC differential field extension and let $\mathfrak{x}_1, \cdots, \mathfrak{x}_n \in \mathbf{E}$, $\mathfrak{e}_1, \cdots, \mathfrak{e}_m \in \mathbf{E} \backslash \{0\}$ be such that $\mathfrak{x}_i$ is an antiderivative* *of an element* $\mathbf{F}$ *for each* i ($\mathfrak{x}_i' \in \mathbf{F}$) *and* $\mathfrak{e}_i$ *is an exponential of an integral of an element of* $\mathbf{F}$ *for each* i ($\frac{\mathfrak{e}_j'}{\mathfrak{e}_j} \in \mathbf{F}$). *Then either* $\mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{e}_1, \cdots, \mathfrak{e}_m$ *are algebraically independent over* $\mathbf{F}$ *or there exist* $(c_1, \cdots, c_n) \in \mathbf{C}^n \setminus \{0\}$ *such that* $\sum_{i=1}^n c_i \mathfrak{x}_i \in \mathbf{F}$ *or there exist* $(r_1, \cdots, r_m) \in \mathbb{Z}^n \setminus \{0\}$ *such that* $\prod_{j=1}^m \mathfrak{e}_j^{r_j} \in \mathbf{F}$.

In his paper [10], A. Ostrowski proves that a set of antiderivatives $\{\mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ of $\mathbf{F}$ is either algebraically independent over $\mathbf{F}$ or there are constants $c_i \in \mathbf{C}$ not all zero such that $\sum_{i=1}^n c_i \mathfrak{x}_i \in \mathbf{F}$. In his setting, $\mathbf{F}$ is a differential field of meromorphic functions and $\mathbf{C} = \mathbb{C}$, the field of complex numbers. Later, Ostrowski's result was generalized by Kolchin [6] to theorem 2.3. In their papers [4] and [11], J. Ax and M. Rosenlicht also presented proofs of theorem 2.3. The proof we are going to present is elementary and differ from the proofs listed above.

2.1 Algebraic Dependence of Antiderivatives

Theorem 2.4. *Let* $\mathbf{E} \supset \mathbf{F}$ *be a differential field extension and let* $\mathfrak{x} \in \mathbf{E}$ *be an antiderivative. Then either* $\mathfrak{x}$ *is transcendental over* $\mathbf{F}$ *or* $\mathfrak{x} \in \mathbf{F}$.

Proof. Let $\mathbf{C_F}$ denote the field of constants of $\mathbf{F}$ and suppose that $\mathfrak{x}$ is algebraic over $\mathbf{F}$. Then there is a monic irreducible polynomial $P(x) = \sum_{i=0}^n a_i x^i \in \mathbf{F}[x]$ such that $P(\mathfrak{x}) = 0$. Note that $(P(\mathfrak{x}))' = 0$, that is $\mathfrak{x}$ is a solution of the polynomial

$$\sum_{i=1}^n (i a_i \mathfrak{x}' - a_{i-1}') x^{i-1} \in \mathbf{F}[x].$$

Since the degree of the above polynomial $< n$, it has to be the zero polynomial. In particular $n\mathfrak{x}' = a_{n-1}'$, that is $(\mathfrak{x} - b)' = 0$, where $b := \frac{a_{n-1}}{n} \in \mathbf{F}$. Observe that $\mathfrak{x} - b$ is algebraic over $\mathbf{F}$ (since $\mathfrak{x}$ and b are algebraic) and therefore there is a monic irreducible polynomial $Q(x) = \sum_{i=0}^m b_i x^i \in \mathbf{F}[x]$ such that $Q(\mathfrak{x} - b) = 0$.

Again taking the derivative of the equation $Q(\mathfrak{x}-b)=0$, we note that $\mathfrak{x}-b$ is a solution of the polynomial

$$\sum_{i=1}^{m} b'_{i-1}x^{i-1} \in \mathbf{F}[x].$$

Since the degree of the above polynomial is $< m$, it has to be the zero polynomial. Thus $b_i \in \mathbf{C_F}$ and therefore the polynomial $Q(x)$ has coefficients in $\mathbf{C}$. Since $\mathbf{C_F}$ is algebraically closed and $\mathfrak{x}-b$ is a zero of $Q(x)$ we obtain $\mathfrak{x}-b \in \mathbf{C_F}$. Thus $\mathfrak{x}-b=c$ for some $c \in \mathbf{C_F}$ and since $b \in \mathbf{F}$ we then obtain $\mathfrak{x}=b+c \in \mathbf{F}$.
Note that we do not require the constants of $\mathbf{F}$ and $\mathbf{E}$ to be the same to prove this theorem. The above theorem is also proved in [5], page 23 and [7], page 7. □

2.1.1 Galois Group

Let $\mathbf{E} \supseteq \mathbf{F}$ be an extension by antiderivatives $\mathfrak{x}_1, \cdots, \mathfrak{x}_n \in \mathbf{E} \setminus \mathbf{C}$ of $\mathbf{F}$. That is, $\mathbf{E}=\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_n)$, $\mathfrak{x}'_i \in \mathbf{F}$ and $\mathfrak{x}_i \notin \mathbf{C}$ for all $1 \leq i \leq n$. Since $\mathbf{E}$ is a NNC extension of $\mathbf{F}$, the differential subfield $\mathbf{E}_i=\mathbf{F}(\mathfrak{x}_i)$ of $\mathbf{E}$ is also a NNC extension of $\mathbf{F}$. Let $f_i := \mathfrak{x}'_i \in \mathbf{F}$ and observe that

$$\mathfrak{x}''_i = \frac{f'_i}{f_i}\mathfrak{x}'_i.$$

Thus $\mathfrak{x}_i$ is a solution of a second order linear homogeneous differential equation over $\mathbf{F}$. Moreover, if V_i is the vector space spanned by the unity $1 \in \mathbf{C}$ and $\mathfrak{x}_i$ over $\mathbf{C}$ then $\mathbf{E}_i = \mathbf{F}\langle V_i \rangle$–the differential field generated by $\mathbf{F}$ and V_i. The full set of solutions of the differential equation $Y'' = \frac{f'}{f}Y'$ is the vector space V_i. Thus we see that $\mathbf{E}_i$ is a Picard-Vessiot extension of $\mathbf{F}$. Since a compositum of Picard-Vessiot extensions is again a Picard-Vessiot extension(see [7], page 28-29),

$\mathbf{E} := \mathbf{E}_1 \cdot \mathbf{E}_2 \cdots \mathbf{E}_n$ is also a Picard-Vessiot extension of $\mathbf{F}$.

Assume that $\mathfrak{x}_i \notin \mathbf{F}$ for each i. If $\sigma \in \mathbb{G}(\mathbf{E}_i|\mathbf{F})$ then

$$\sigma(\mathfrak{x}_i)' = \sigma(\mathfrak{x}_i') = \sigma(f_i) = f_i = \mathfrak{x}_i'. \tag{2.1}$$

Thus $\sigma(\mathfrak{x}_i)' = \mathfrak{x}_i'$, which implies $\big(\sigma(\mathfrak{x}_i) - \mathfrak{x}_i\big)' = 0$. Since $\mathbf{E}$ is a NNC extension of $\mathbf{F}$, there is a $c_{i\sigma} \in \mathbf{C}$ such that $\sigma(\mathfrak{x}_i) - \mathfrak{x}_i = c_{i\sigma}$, that is, $\sigma(\mathfrak{x}_i) = \mathfrak{x}_i + c_{i\sigma}$. On the other hand, for any $c \in \mathbf{C}$, the automorphism $\sigma_{ic} : \mathbf{E}_i \to \mathbf{E}_i$ defined as $\sigma_{ic}(\mathfrak{x}_i) = \mathfrak{x}_i + c$ and $\sigma(f) = f$ for all $f \in \mathbf{F}$ can be readily seen as a differential automorphism. Thus $\mathbb{G}(\mathbf{E}_i|\mathbf{F})$ injects into $(\mathbf{C}, +)$ as an algebraic subgroup for each i. Since $(\mathbf{C}, +)$ has no non trivial algebraic subgroups and since $\mathfrak{x}_i \notin \mathbf{F}$, from the fundamental theorem, we see that $\mathbb{G}(\mathbf{E}_i|\mathbf{F}) \simeq (\mathbf{C}, +)$ and that the extension $\mathbf{E}_i$ of $\mathbf{F}$ has no intermediate differential subfields. Any automorphism of $\mathbf{E}$ fixing $\mathbf{F}$ is completely determined by its action on $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ and thus we have a map $\sigma \mapsto (c_{1\sigma}, \cdots, c_{n\sigma})$, an algebraic group homomorphism from $\mathbb{G}$ to $(\mathbf{C}, +)^n$. This map is clearly injective. From this observation, we see that the differential Galois group $\mathbb{G}(\mathbf{E}|\mathbf{F})$ is isomorphic to an algebraic subgroup of $(\mathbf{C}, +)^n$. Note that $\mathbb{G}(\mathbf{E}|\mathbf{F})$ could be a proper algebraic subgroup of $(\mathbf{C}, +)^n$; depending on whether all the antiderivatives are algebraically independent over $\mathbf{F}$ or not. We will discuss about the nature of the algebraic dependence of antiderivatives in the next theorem.

We will do a similar analysis for the extensions by exponentials of integrals of $\mathbf{F}$ in section 2.2.

Theorem 2.5. *Let $\mathbf{E} \supset \mathbf{F}$ be a NNC differential field extension and for $1 = 1, 2, \cdots, n$ let $\mathfrak{x}_i \in \mathbf{E}$ be antiderivatives of $\mathbf{F}$. Then either $\mathfrak{x}_i$'s are algebraically independent over $\mathbf{F}$ or there is a tuple $(c_1, \cdots, c_n) \in \mathbf{C}^n \backslash \{0\}$ such that $\sum_{i=1}^{n} c_i \mathfrak{x}_i \in$*

$\mathbf{F}$.

Proof 1. First we will present Kolchin's proof. Observe that $\mathbf{E} = \mathbf{F}(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n)$ is a Picard-Vessiot extension of $\mathbf{F}$ and for every $\sigma \in \mathbb{G}(\mathbf{E}|\mathbf{F})$ we see that $\sigma(\mathfrak{x}_i) = \mathfrak{x}_i + c_{i\sigma}$. Thus, as noted earlier, $\mathbb{G}(\mathbf{E}|\mathbf{F})$ imbeds into $(\mathbf{C}^n, +)$ as an algebraic subgroup. Suppose that the $\mathfrak{x}_i's$ are algebraically dependent and say $\mathfrak{x}_1$ is algebraic over $\mathbf{F}(\mathfrak{x}_2, \mathfrak{x}_3, \cdots, \mathfrak{x}_n)$. We may also assume that $\mathfrak{x}_i$'s$\notin \mathbf{F}$ for any i (otherwise there is nothing to prove).

Since $\mathfrak{x}_1$ is an antiderivative of an element of $\mathbf{F}$ and $\mathfrak{x}_1$ is algebraic over $\mathbf{F}(\mathfrak{x}_2, \mathfrak{x}_3, \cdots, \mathfrak{x}_n)$ from theorem 2.4 we obtain $\mathfrak{x}_1 \in \mathbf{F}(\mathfrak{x}_2, \mathfrak{x}_3, \cdots, \mathfrak{x}_n)$ and thus $\mathbb{G}(\mathbf{E}|\mathbf{F}) \hookrightarrow (\mathbf{C}^n, +)$ is not a surjection. In particular, if $\sigma \in \mathbb{G}(\mathbf{E}|\mathbf{F})$ fixes $\mathfrak{x}_2, \cdots, \mathfrak{x}_n$ then σ fixes $\mathfrak{x}_1$ too. Therefore

$$\mathbb{G}(\mathbf{E}|\mathbf{F}) = \{(d_1, d_2, \cdots, d_n) \in \mathbf{C}^n | L_i(d_1, d_2, \cdots, d_n) = 0, 1 \leq i \leq t\},$$

where L_i is a linear homogeneous polynomial, which we sometimes call as linear forms, over $\mathbf{C}$ for each i. Now for any $\sigma \in \mathbb{G}(\mathbf{E}|\mathbf{F})$ and $L \in \{L_i | 1 \leq i \leq t\}$,

$$\begin{aligned}
\sigma(L(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n)) &= L(\sigma(\mathfrak{x}_1), \sigma(\mathfrak{x}_2), \cdots, \sigma(\mathfrak{x}_n)) \\
&= L(\mathfrak{x}_1 + d_1, \mathfrak{x}_2 + d_2, \cdots, \mathfrak{x}_n + d_n) \\
&= L(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) + L(d_1, d_2, \cdots, d_n) \\
&= L(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) \quad \text{since } L(d_1, \cdots, d_n) = 0
\end{aligned}$$

and thus $L(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) \in \mathbf{E}^{\mathbb{G}(\mathbf{E}|\mathbf{F})}$. From Galois theory we know that $\mathbf{E}^{\mathbb{G}(\mathbf{E}|\mathbf{F})} = \mathbf{F}$. Hence $L(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n) = \sum_{i=1}^{n} c_i \mathfrak{x}_i \in \mathbf{F}$ for some $c_i \in \mathbf{C}$. □

Proof 2. This proof does not require Galois theory. For every tuple $(c_1, \cdots, c_n)$

$\in \mathbf{C}^n \setminus \{0\}$ let us assume that $\sum_{i=1}^n c_i\mathfrak{x}_i \notin \mathbf{F}$. Theorem 2.4 and our assumption that $\sum_{i=1}^n c_i\mathfrak{x}_i \notin \mathbf{F}$ guarantees us a nonempty algebraically independent subset S of $\{\mathfrak{x}_i | 1 \leq i \leq n\}$ over $\mathbf{F}$. We may assume that $S = \{\mathfrak{x}_2, \mathfrak{x}_3, \cdots, \mathfrak{x}_n\}$. Again from theorem 2.4, we see that $\mathfrak{x}_1$ is transcendental over $\mathbf{F}(S)$ or $\mathfrak{x}_1 \in \mathbf{F}(S)$. We will show that the latter case is not possible and this will prove the theorem.

Suppose that $\mathfrak{x}_1 \in \mathbf{F}(S)$ and let t be the largest positive integer such that

$$\sum_{i=1}^{t} c_i\mathfrak{x}_i \in \mathbf{F}(S_t),$$

where $c_i \in \mathbf{C}, c_1 = 1$ and $S_t := S \setminus \{\mathfrak{x}_i \big| 2 \leq i \leq t\}$.

Since $|S| < \infty$ and $t \geq 1$, such a t exist and since $\sum_{i=1}^n c_i\mathfrak{x}_i \notin \mathbf{F}$, $S_t \neq \emptyset$. In particular, $t < n$ and thus $\mathfrak{x}_{t+1} \in S_t$. For notational convenience let $\mathfrak{x} := \mathfrak{x}_{t+1}$. We write

$$\sum_{i=1}^{t} c_i\mathfrak{x}_i = \frac{P}{Q}$$

where $P := \sum_{i=0}^r a_i\mathfrak{x}^i$, $Q := \sum_{i=0}^s b_i\mathfrak{x}^i$, $b_s = 1$, $a_r \neq 0, a_i, b_i \in \mathbf{K} := \mathbf{F}(S_t \setminus \{\mathfrak{x}\})$ and $(P, Q) = 1$. Differentiating the above equation, we get $\sum_{i=1}^t c_i\mathfrak{x}_i' = \frac{P'Q - PQ'}{Q^2}$ and thus

$$fQ^2 = P'Q - PQ', \tag{2.2}$$

where $f := \sum_{i=1}^t c_i\mathfrak{x}_i'$. If $f = 0$ then $(\sum_{i=1}^t c_i\mathfrak{x}_i)' = 0$ and since $\mathbf{E}$ is a NNC extension of $\mathbf{F}$, $\sum_{i=1}^t c_i\mathfrak{x}_i \in \mathbf{C} \subset \mathbf{F}$, a contradiction to our assumption that $\sum_{i=1}^t c_i\mathfrak{x}_i \notin \mathbf{F}$. Thus $f \neq 0$. Now suppose that $\deg Q \geq 1$. From the above equation we see that Q divides $P'Q - PQ'$, which implies Q divides PQ' and since $(P, Q) = 1$, Q divides Q'. Thus $s = \deg Q \leq \deg Q'$. But then $\deg Q' = \deg((s\mathfrak{x}' + b_{s-1}')\mathfrak{x}^{s-1} + \cdots + b_1\mathfrak{x}' + b_0') \leq s - 1$, a contradiction. Thus $\deg Q = 0$, that is $Q \in \mathbf{K}$.

Hence we may assume that $\sum_{i=1}^{t} c_i \mathfrak{x}_i = P$ and note that

$$f = P'. \tag{2.3}$$

Case 1: $\deg(P) = 0$, that is $P \in \mathbf{K} = \mathbf{F}(S_t \setminus \{\mathfrak{x}\})$.

Then $\sum_{i=1}^{t} c_i \mathfrak{x}_i = P \in \mathbf{F}(S_t \setminus \{\mathfrak{x}\})$. Since $\mathfrak{x} = \mathfrak{x}_{t+1}$, we obtain $\sum_{i=1}^{t+1} c_i \mathfrak{x}_i \in \mathbf{F}(S_t \setminus \{\mathfrak{x}_{t+1}\})$, where $c_{t+1} := 0$. This contradicts the maximality of t.

Case 2: $\deg(P) > 1$

From equation 2.3 we see that

$$f = a_r' \mathfrak{x}^r + (r a_r \mathfrak{x}' + a_{r-1}')\mathfrak{x}^{r-1} + \cdots + a_1 \mathfrak{x}' + a_0'. \tag{2.4}$$

Thus comparing the coefficients of $\mathfrak{x}^r$ we get $a_r' = 0$, that is $a_r \in \mathbf{C}$. Since $r-1 \geq 1$ comparing the coefficients of $\mathfrak{x}^{r-1}$, we get

$$\begin{aligned} r a_r \mathfrak{x}' + a_{r-1}' &= 0 \\ \Longrightarrow \mathfrak{x}' &= \left(\frac{-a_{r-1}}{r a_r}\right)' \\ \Longrightarrow \mathfrak{x} &= \frac{-a_{r-1}}{r a_r} + c_1 \end{aligned}$$

for some $c_1 \in \mathbf{C}$ and thus $\mathfrak{x} = \frac{-a_{r-1}}{r a_r} + c_1 \in \mathbf{K}$, a contradiction to the assumption that $\mathfrak{x}$ is transcendental over $\mathbf{K}$.

Case 3: $\deg P = 1$

Finally if $\deg P = 1$ then $P = a_1 \mathfrak{x} + a_0 = \sum_{i=1}^{t} c_i \mathfrak{x}_i$ and therefore taking the derivative we have

$$a_1' \mathfrak{x} + a_1 \mathfrak{x}' + a_0' = f.$$

Thus comparing the coefficients, we obtain $a_1' = 0$ that is $a_1 \in \mathbf{C}$ and $a_1 \mathfrak{x}' + a_0' = f$.

Now letting $c_{t+1} := -a_1$ and substituting $\mathfrak{x}_{t+1}$ for $\mathfrak{x}$, we get $\sum_{i=1}^{t+1} c_i\mathfrak{x}_i = a_0 \in \mathbf{K} = \mathbf{F}(S_t \setminus \{\mathfrak{x}_{t+1}\})$ and this again contradicts the maximality of t. Hence the theorem. □

2.2 Exponentials of Integrals

Here we will prove theorems analogous to theorems 2.4 and 2.5 for the exponential of an integral setting.

Theorem 2.6. *Let $\mathbf{E} \supset \mathbf{F}$ be a differential field extension. If there is a $\mathfrak{e} \in \mathbf{E}$ such that $\frac{\mathfrak{e}'}{\mathfrak{e}} \in \mathbf{F}$ then either $\mathfrak{e}$ is transcendental over $\mathbf{F}$ or there is an $n \in \mathbb{N}$ such that $\mathfrak{e}^n \in \mathbf{F}$.*

Proof. Suppose that $\mathfrak{e}$ is algebraic over $\mathbf{F}$, $\frac{\mathfrak{e}'}{\mathfrak{e}} = f \in \mathbf{F}$ and let $P(x) = \sum_{i=0}^{n} a_i x^i \in \mathbf{F}[x]$ be the monic irreducible polynomial of $\mathfrak{e}$. Then $P(\mathfrak{e}) = 0$ and therefore $(P(\mathfrak{e}))' = 0$, which implies $\mathfrak{e}$ is a solution of the polynomial

$$P_1 := nfx^n + \sum_{i=0}^{n-1}(a_i' - ia_i f)x^i \in \mathbf{F}[x].$$

Since P is the monic irreducible polynomial of $\mathfrak{e}$, we have $nfP = P_1$. Thus comparing the coefficients of nfP and P_1 we obtain $nfa_0 = a_0'$ and since $nf\mathfrak{e}^n = (\mathfrak{e}^n)'$, we obtain $\left(\frac{\mathfrak{e}^n}{a_0}\right)' = 0$ (P is irreducible so $a_0 \neq 0$). Note that $\mathfrak{e}$ and a_0 are algebraic over $\mathbf{F}$ so $\frac{\mathfrak{e}^n}{a_0}$ is also algebraic over $\mathbf{F}$. Since $(\frac{\mathfrak{e}^n}{a_0})' = 0$, as in the proof of theorem 2.4, we obtain $\frac{\mathfrak{e}^n}{a_0} = c \in \mathbf{C_F}$ and thus $\mathfrak{e}^n = ca_0 \in \mathbf{F}$.
This theorem is also proved in [5], page 24 and [7], page 8. □

Theorem 2.7. *Let $\mathbf{E} \supset \mathbf{F}$ be a NNC differential field extension and for $i = 1, 2, \cdots, n$ let $\mathfrak{e}_i \in \mathbf{E} \setminus \{0\}$ be such that $\frac{\mathfrak{e}'}{\mathfrak{e}} \in \mathbf{F}$. Then either $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ are* *algebraically independent or there exist* $(k_1, \cdots, k_n) \in \mathbb{Z}^n \setminus \{0\}$ *such that the power product* $\prod_{i=1}^n \mathfrak{e}_i^{k_i} \in \mathbf{F}$.

Proof. The proof of this theorem very much mimics the proof of theorem 2.5. Let us assume that $\prod_{i=1}^n \mathfrak{e}_i^{k_i} \notin \mathbf{F}$ for any $(k_1, \cdots, k_n) \in \mathbb{Z}^n \setminus \{0\}$. Then from theorem 2.6 we see that there is a nonempty algebraically independent set $S \subset \{\mathfrak{e}_i | 1 \leq i \leq n\}$ and we may assume that $S = \{\mathfrak{e}_2, \cdots, \mathfrak{e}_n\}$. From theorem 2.6 we see that either $\mathfrak{e}_1$ is transcendental over $\mathbf{F}(S)$ or there is a $k_1 \in \mathbb{N}$ such that $\mathfrak{e}_1^{k_1} \in \mathbf{F}(S)$. We will show that the latter is not possible and this will prove the theorem.

Suppose that there is a $k_1 \in \mathbb{N}$ such that $\mathfrak{e}_1^{k_1} \in \mathbf{F}(S)$. Let t be the largest positive integer such that the power product

$$\prod_{i=1}^{t} \mathfrak{e}_i^{k_i} \in \mathbf{F}(S_t),$$

where $k_i \in \mathbb{Z}$ for $2 \leq i \leq t$ and $S_t = S \setminus \{\mathfrak{e}_i | 2 \leq i \leq t\}$. Since $\prod_{i=1}^n \mathfrak{e}_i^{k_i} \notin \mathbf{F}$ we obtain $S_t \neq \emptyset$. Indeed $\mathfrak{e}_{t+1} \in S_t$. Let $\mathfrak{e} := \mathfrak{e}_{t+1}$ and write

$$\prod_{i=1}^{t} \mathfrak{e}_i^{k_i} = \frac{P}{Q}$$

where $P := \sum_{i=0}^l a_i \mathfrak{e}^i$, $Q := \sum_{i=0}^m b_i \mathfrak{e}^i$, $(P, Q) = 1$ $b_m = 1$, $a_l \neq 0, a_i, b_i \in \mathbf{F}(S_t \setminus \{\mathfrak{e}\})$. Differentiating the above equation, we get

$$(\prod_{i=1}^{t} \mathfrak{e}_i^{k_i})' = \frac{P'Q - PQ'}{Q^2},$$

Let $f_i := \frac{\mathfrak{e}_i'}{\mathfrak{e}_i}$, $g := \frac{\mathfrak{e}'}{\mathfrak{e}}$, $P = \sum_{i=0}^{l} a_i \mathfrak{e}^i$ and $Q = \sum_{i=0}^{m} b_i \mathfrak{e}^i$. Note that $g, f_i \in \mathbf{F}$ and

$$\begin{aligned}(\prod_{i=1}^{t} \mathfrak{e}_i^{k_i})' &= \sum_{j=1}^{t} (\mathfrak{e}_j^{k_j})' \prod_{i=1, i\neq j}^{t} \mathfrak{e}_i^{k_i} \\ &= \sum_{j=1}^{t} k_j \mathfrak{e}_j' \mathfrak{e}_j^{k_j-1} \prod_{i=1, i\neq j}^{t} \mathfrak{e}_i^{k_i},\end{aligned}$$

which implies

$$(\prod_{i=1}^{t} \mathfrak{e}_i^{k_i})' = \big(\sum_{i=1}^{t} k_j f_j\big) \prod_{i=1}^{t} \mathfrak{e}_i^{k_i} \tag{2.5}$$

and thus $\left(\frac{P}{Q}\right)' = \big(\sum_{i=1}^{t} k_j f_j\big) \frac{P}{Q}$. Hence

$$QP' - PQ' = \big(\sum_{i=1}^{t} k_j f_j\big) PQ. \tag{2.6}$$

Since

$$\begin{aligned}QP' - PQ' &= ((a_l' + l a_l g)\mathfrak{e}^{l+m} + \cdots + a_0' b_0) \\ &\quad - (m a_l g \mathfrak{e}^{l+m} + \cdots + a_0 b_0') \\ &= (a_l' + (l-m) a_l g)\mathfrak{e}^{l+m} + \cdots + a_0' b_0 - a_0 b_0',\end{aligned}$$

and

$$PQ = a_l \mathfrak{e}^{l+m} + (a_l b_{m-1} + a_{l-1})\mathfrak{e}^{l+m-1} + \cdots + a_0 b_0,$$

substituting in equation 2.6 we get

$$\begin{aligned}&(a_l' + (l-m) a_l g)\mathfrak{e}^{l+m} + \cdots + a_0' b_0 - a_0 b_0' = \big(\sum_{i=1}^{t} k_j f_j\big)(a_l \mathfrak{e}^{l+m} \\ &+ (a_l b_{m-1} + a_{l-1})\mathfrak{e}^{l+m-1} + \cdots + a_0 b_0).\end{aligned}$$

The LHS and RHS are polynomial in $\mathfrak{e}$ with coefficients in $\mathbf{F}(S_t \setminus \{\mathfrak{e}\})$. Since $\mathbf{E} \supset \mathbf{F}$ is a NNC extension and $\prod_{i=1}^{t} \mathfrak{e}_i^{k_i} \notin \mathbf{F}$ we have $\sum_{i=1}^{t} k_j f_j \neq 0$ and therefore both the LHS and RHS are of degree $l+m$. Thus comparing the coefficients of $\mathfrak{e}^{l+m}$ we get

$$a_l' + (l-m)a_l g = (\sum_{i=1}^{t} k_j f_j) a_l$$
$$\implies a_l' = [(\sum_{i=1}^{t} k_j f_j) + (m-l)g] a_l.$$

We observe that

$$a_l' = (\sum_{i=1}^{t+1} k_j f_j) a_l, \tag{2.7}$$

where $k_{t+1} := m-l$ and $f_{t+1} := g$.

We also know that $\prod_{i=1}^{t+1} \mathfrak{e}_i^{k_i}$ is also a nonzero solution of the equation 2.7 and therefore $\left(\frac{\prod_{i=1}^{t+1} \mathfrak{e}_i^{k_i}}{a_l}\right)' = 0$. Since $\mathbf{E}$ and $\mathbf{F}$ have the same field of constants, there is an $\alpha \in \mathbf{C} \setminus \{0\}$ such that $\prod_{i=1}^{t+1} \mathfrak{e}_i^{k_i} = \alpha a_l$. Now $a_l \in \mathbf{F}(S_t \setminus \{\mathfrak{e}\})$ will imply $\prod_{i=1}^{t+1} \mathfrak{e}_i^{k_i} \in \mathbf{F}(S_t \setminus \{\mathfrak{e}_{t+1}\})$, a contradiction to the maximality of t. Hence the theorem. □

The Kolchin-Ostrowski Theorem

Proof of theorem 2.3. Let us assume that $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$, $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ are algebraically dependent over $\mathbf{F}$ and also that $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ are algebraically independent over $\mathbf{F}$. (Note that if $\mathfrak{e}_1, \cdots, \mathfrak{e}_m$ are algebraically dependent over $\mathbf{F}$ we may apply theorem 2.7 to prove this theorem.) Let us prove that there are constants $c_i \in \mathbf{C}$ not all zero such that $\sum_{i=1}^{n} c_i \mathfrak{x}_i \in \mathbf{F}$.

It is clear from our assumption that $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ is algebraically dependent over

$\mathbf{K} := \mathbf{F}(\mathfrak{e}_1, \cdots, \mathfrak{e}_m)$. Since $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ are antiderivatives of $\mathbf{F}$ they are also antiderivatives of $\mathbf{K}$ and thus theorem 2.5 is applicable with $\mathbf{K}$ as the ground field. Thus there are constants $c_i \in \mathbf{C}$ not all zero such that $\sum_{i=1}^n c_i\mathfrak{x}_i \in \mathbf{K}$. Let $S \subset \{\mathfrak{e}_1, \cdots, \mathfrak{e}_m\}$ be a minimal subset such that $\sum_{i=1}^n c_i\mathfrak{x}_i \in \mathbf{F}(S)$ and for any subset S_1 of S, $\sum_{i=1}^n c_i\mathfrak{x}_i \notin \mathbf{F}(S_1)$.

We claim that such a set S is the empty set and this will prove that $\sum_{i=1}^n c_i\mathfrak{x}_i \in \mathbf{F}$. Suppose not. Then there is a $\mathfrak{e} \in S$ and we may write

$$\sum_{i=1}^{n} c_i\mathfrak{x}_i = \frac{P}{Q}, \tag{2.8}$$

where $P, Q \in \mathbf{F}(S \setminus \{\mathfrak{e}\})[\mathfrak{e}]$, $(P, Q) = 1$ and Q a monic polynomial. Let $f = (\sum_{i=1}^n c_i\mathfrak{x}_i)'$. Note that $f \in \mathbf{F}$ and if $f = 0$ then $(\sum_{i=1}^n c_i\mathfrak{x}_i)' = 0$ and since the extensions are NNC, we see that $\sum_{i=1}^n c_i\mathfrak{x}_i = \alpha \in \mathbf{C} \subset \mathbf{F}$ and we are done. So we assume $f \neq 0$ and note that this condition also says that $P \neq 0$. Now Differentiating the equation 2.8 we obtain

$$fQ^2 = P'Q - Q'P. \tag{2.9}$$

Hereafter one can complete the proof by precisely following the part of the proof of theorem 2.5 that follows after equation 2.2. Here I will give an alternate argument which is also applicable for the part of the proof of theorem 2.5 that follows after equation 2.2.

Note that $\deg(P'Q - Q'P) \leq r + s$ and $\deg(fQ^2)$ $=\deg Q^2 = 2s$.

Case 1: $\deg Q >\deg P$.

In this case we see that $r + s <\deg Q^2 = 2s$. Since the leading coefficient f of the LHS of 2.9 is nonzero, we obtain that $\mathfrak{e}$ is algebraic over $\mathbf{F}(S \setminus \{\mathfrak{e}\})$, a contradiction.

Case 2: deg$Q$ <deg$P$

Let $\frac{\mathfrak{e}'}{\mathfrak{e}} = g \in \mathbf{F}$, $P = \sum_{i=0}^{r} a_i \mathfrak{e}^i$, $a_r \neq 0$, $Q = \sum_{i=0}^{s} b_i \mathfrak{e}^i$ and $b_s = 1$. Note that $P'Q - Q'P = (a_r' - (r-s)a_r g)\mathfrak{e}^{r+s} + \cdots$. If $(a_r' - (r-s)a_r g) \neq 0$ then $r+s = deg(P'Q - Q'P)$ and since $s < r$, $\deg(Q^2) = 2s < r+s$, which implies $\mathfrak{e}$ is algebraic over $\mathbf{F}(S \setminus \{\mathfrak{e}\})$, a contradiction to our assumption that $\mathfrak{e}_i$'s are algebraically independent over $\mathbf{F}$. Thus $a_r' - (r-s)a_r g = 0$, that is $a_r' = (r-s)ga_r$. Note that $a_r \neq 0$ and since $(\mathfrak{e}^{r-s})' = (r-s)g\mathfrak{e}^{r-s}$ and $r \neq s$, we obtain $\left(\frac{\mathfrak{e}^{r-s}}{a_r}\right)' = 0$. Thus there is a constant $\alpha \in \mathbf{C} \setminus \{0\}$ such that $\mathfrak{e}^{r-s} = \alpha a_r \in \mathbf{F}(S \setminus \{\mathfrak{e}\})$ contradicting the algebraic independency of $\mathfrak{e}_i$'s over $\mathbf{F}$.

Case 3: deg$P$ =deg$Q$

Since $\deg Q^2 = 2s$, $\deg(P'Q - Q'P) \leq 2s$ and $f \neq 0$, we have $f = a_r' - (r-s)ga_r$ and this equation further reduces to $f = a_r'$ since $r = s$. Now the facts $(\sum_{i=1}^{n} c_i \mathfrak{x}_i)' = f$ and $\mathbf{K}$ is a NNC extension together will imply that $\sum_{i=1}^{n} c_i \mathfrak{x}_i = a_r + \alpha$ for some $\alpha \in \mathbf{C}$. Thus $\sum_{i=1}^{n} c_i \mathfrak{x}_i = a_r + \alpha \in \mathbf{F}(S \setminus \{\mathfrak{e}\})$, a contradiction to the minimality of S.

Thus S has to be the empty set and hence the theorem. $\square$

2.3 Extensions by Antiderivatives and by Exponentials of Integrals

Let $\mathbf{E} \supset \mathbf{F}$ be an extension by antiderivatives $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ of $\mathbf{F}$. We know from theorem 2.5 that the set of antiderivatives $\{\mathfrak{x}_i | 1 \leq i \leq n\}$ is either algebraically independent or there are constants $c_i \in \mathbf{C}$ not all zero such that $\sum_{i=1}^{n} c_i \mathfrak{x}_i \in \mathbf{F}$. Also note that if $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ is algebraically dependent over $\mathbf{F}$ then we may chose a transcendence base $S \subset \{\mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ of $\mathbf{E}$ over $\mathbf{F}$ and this makes $\mathbf{E}$ algebraic over $\mathbf{F}(S)$. But then each $\mathfrak{x} \in \{\mathfrak{x}_1, \cdots, \mathfrak{x}_n\} \setminus S$ becomes algebraic over $\mathbf{F}(S)$ and therefore from theorem 2.4 we obtain $\mathfrak{x} \in \mathbf{F}(S)$ which implies $\mathbf{E} = \mathbf{F}(S)$. In other words extensions by antiderivatives are purely transcendental. Thus, to study an extension by antiderivatives $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ of $\mathbf{F}$, we may very well assume that $\mathfrak{x}_1 \cdots, \mathfrak{x}_n$ are algebraically independent over $\mathbf{F}$.

In this section we will prove the following theorem

Theorem 2.8. *Let $\mathbf{E} = \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_n)$ be an extension by antiderivatives $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ of $\mathbf{F}$ and let $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ be algebraically independent over $\mathbf{F}$. Let $u \in \mathbf{E}$ and $u = \frac{P}{Q}$, $P, Q \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ and $(P, Q) = 1$. Then there is a $t \in \mathbb{N}$ and $\mathbf{F}-$linear forms $D_i \in Span_{\mathbf{F}}\{\mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ for $1 \leq i \leq t$ such that*

$$\mathbf{F}\langle u \rangle = \mathbf{F}(D_i | 1 \leq i \leq t).$$

Moreover these linear forms D_i can be explicitly computed from P and Q.

A much stronger result can be obtained using Galois theory and that is, if $\mathbf{K}$ is an intermediate differential subfield of $\mathbf{E}|\mathbf{F}$ then

$$\mathbf{K} = \mathbf{F}(L_i | 1 \leq i \leq t), \tag{2.10}$$

where the linear forms are over $\mathbf{C}$. That is $L_i \in Span_{\mathbf{C}}\{\mathfrak{x}_1 \cdots, \mathfrak{x}_n\}$. This follows immediately from the following three facts 1. The extension $\mathbf{E} \supset \mathbf{F}$ is a P-V extension with a differential Galois group $(\mathbf{C}, +)^n$. 2. There is a bijective correspondence between the algebraic subgroups of $(\mathbf{C}, +)^n$ and the intermediate differential subfields of $\mathbf{E}|\mathbf{F}$; see the fundamental theorem stated in chapter 1. 3. The algebraic subgroups of $(\mathbf{C}, +)^n$ are solution sets of linear forms over $\mathbf{C}$.

Though we know the structure of intermediate differential subfields of $\mathbf{E}|\mathbf{F}$, it is not clear how to obtain those linear forms for a given intermediate differential subfield. The theorem 2.8 shows that there is a way to figure out linear forms(not over $\mathbf{C}$ but over $\mathbf{F}$) for singly differentially generated subfields of $\mathbf{E}$ containing $\mathbf{F}$ and since a finitely differentially generated subfield is a compositum of singly differentially generated subfields of $\mathbf{E}$ containing $\mathbf{F}$, we may generalize the theorem 2.8 for any finitely differentially generated subfield of $\mathbf{E}$ containing $\mathbf{F}$. We will prove a similar result for extensions by exponentials of integrals and will also prove a similar structure theorem for NNC extensions of the form $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{e}_1, \cdots, \mathfrak{e}_m)$, where $\mathfrak{x}_i' \in \mathbf{F}$ and $\frac{\mathfrak{e}_i'}{\mathfrak{e}_i} \in \mathbf{F}$ and $\mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{e}_1, \cdots, \mathfrak{e}_m$ are algebraically independent over $\mathbf{F}$.

To prove theorem 2.8 we need some results about several variable polynomials over a commutative ring with unity, which will be dealt in the following section.

2.3.1 Multivariable Taylor formula

Let $\mathcal{R}$ be an integral domain with $\mathbb{Q} \subseteq \mathcal{R}$ and let $\mathcal{R}[y_1, \cdots, y_n]$ be the polynomial ring over $n-$indeterminates $y_1, \cdots, y_n$. Let $P := P(y_1, \cdots, y_n) \in \mathcal{R}[y_1, \cdots, y_n]$, $(r_1, \cdots, r_n) \in \mathcal{R}^n$ and denote $P(y_1 + r_1, \cdots, y_n + r_n)$ by $\tilde{P}$. Let $\frac{\partial}{\partial y_i}$ denote the standard partial derivation on the ring $\mathcal{R}[y_1, \cdots, y_n]$. From the Taylor series expansion of $\tilde{P}$, we have

$$\tilde{P} = P + \sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} + \frac{1}{2!} \sum_{j=1}^{n} \sum_{i=1}^{n} r_i r_j \frac{\partial^2 P}{\partial y_j \partial y_i} + \cdots \tag{2.11}$$

Proposition 2.9. *Let $P \in \mathcal{R}[y_1, \cdots, y_n]$ and for $1 \leq i \leq n$ let $r_i \in \mathcal{R}$. Suppose that P divides $\tilde{P} := P(y_1 + r_1, \cdots, y_n + r_n)$. Then $P = \tilde{P}$ and $\sum_{i=1}^{n} r_i \frac{\partial H_j}{\partial y_i} = 0$ for every homogeneous component H_j of total degree j of P. In particular $H_j = \tilde{H}_j$*

for every j and $\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} = 0$*.*

Proof. Rewrite the equation 2.11 as

$$\tilde{P} - P = \sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} + \frac{1}{2!} \sum_{j=1}^{n} \sum_{i=1}^{n} r_i r_j \frac{\partial^2 P}{\partial y_j \partial y_i} + \cdots \tag{2.12}$$

If $P \in \mathcal{R}$ then the proposition follows immediately. Assume that P has a monomial whose total degree is ≥ 1. Denote total degree by tdeg and observe that when the operator $\frac{\partial}{\partial y_i}$ is applied to a monomial M of P, tdeg $\left(\frac{\partial M}{\partial y_i}\right)$ =tdeg $M - 1$. Therefore $\sum_{i=1}^{n} r_i \frac{\partial M}{\partial y_i} = 0$, as cancellation may occur or tdeg$\left(\sum_{i=1}^{n} r_i \frac{\partial M}{\partial y_i}\right)$ =tdeg $M - 1$. Thus $\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} = 0$ or tdeg$\left(\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i}\right)$ = tdeg $P - 1$. Before we prove that $\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} = 0$, we observe from our above discussion that the total degree of the RHS of equation 2.12 is less than the total degree of P. Clearly, P divides $\tilde{P}$ implies P divides the LHS of equation 2.12 and therefore P divides the RHS whose total degree is less than that of P. Thus RHS of 2.12 equals 0, that is

$$\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} + \frac{1}{2!} \sum_{j=1}^{n} \sum_{i=1}^{n} r_i r_j \frac{\partial^2 P}{\partial y_j \partial y_i} + \cdots = 0 \tag{2.13}$$

and hence $\tilde{P} = P$.

Let $P = \sum_{j=0}^{k} H_j$, where H_j is the homogenous component of total degree j of P. Again we observe that if $\sum_{i=1}^{n} r_i \frac{\partial H_j}{\partial y_i} \neq 0$ then the total degree of $\sum_{i=1}^{n} r_i \frac{\partial H_j}{\partial y_i}$ $= j - 1$.

Now consider the homogeneous component H_k. We know that the $\sum_{i=1}^{n} r_i \frac{\partial H_k}{\partial y_i}$ $= 0$ or the total degree of $\sum_{i=1}^{n} r_i \frac{\partial H_k}{\partial y_i}$ is $k - 1$. Now we will show that he latter cannot happen. Since

$$\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} = \sum_{l=0}^{k} \sum_{i=1}^{n} r_i \frac{\partial H_l}{\partial y_i}$$

we may rewrite equation 2.13 and obtain

$$-\sum_{i=1}^{n} r_i \frac{\partial H_k}{\partial y_i} = \sum_{l=0}^{k-1}\sum_{i=1}^{n} r_i \frac{\partial H_l}{\partial y_i} + \frac{1}{2!}\sum_{j=1}^{n}\sum_{i=1}^{n} r_i r_j \frac{\partial^2 P}{\partial y_j \partial y_i} + \cdots \tag{2.14}$$

Since the total degree of the RHS is $\leq k-2$ we conclude that the total degree of LHS can not be $k-1$. Thus $\sum_{i=1}^{n} r_i \frac{\partial H_k}{\partial y_i} = 0$. Note that $\sum_{i=1}^{n} r_i \frac{\partial H_k}{\partial y_i} = 0$ implies $\sum_{j=1}^{n}\sum_{i=1}^{n} r_i r_j \frac{\partial^2 H_k}{\partial y_j \partial y_i} = 0$ and so on... Therefore from equation 2.11 we get $H_k = \tilde{H}_k$.

Now noting

$$\sum_{j=1}^{n}\sum_{i=1}^{n} r_i r_j \frac{\partial^2 P}{\partial y_j \partial y_i} = \sum_{l=0}^{k-1}\sum_{j=1}^{n}\sum_{i=1}^{n} r_i r_j \frac{\partial^2 H_l}{\partial y_j \partial y_i}$$

and $\sum_{i=1}^{n} r_i \frac{\partial H_k}{\partial y_i} = 0$, we may rewrite equation 2.14 and obtain

$$-\sum_{i=1}^{n} r_i \frac{\partial H_{k-1}}{\partial y_i} = \sum_{l=0}^{k-2}\sum_{i=1}^{n} r_i \frac{\partial H_l}{\partial y_i} + \frac{1}{2!}\sum_{l=0}^{k-1}\sum_{j=1}^{n}\sum_{i=1}^{n} r_i r_j \frac{\partial^2 H_l}{\partial y_j \partial y_i} + \cdots .$$

By comparing the total degrees of the LHS and RHS, we conclude that $\sum_{i=1}^{n} r_i \frac{\partial H_{k-1}}{\partial y_i} = 0$ and thus $H_{k-1} = \tilde{H}_{k-1}$. Similarly we can show that $\sum_{i=1}^{n} r_i \frac{\partial H_j}{\partial y_i} = 0$ for every j. From this equation it is easy to see that $H_i = \tilde{H}_i$ for each i and that $\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} = 0$. □

Proposition 2.10. *For every homogeneous polynomial $P \in \mathcal{R}[y_1, \cdots, y_n]$ there is a set $\{D_j | 1 \leq j \leq t\}$ of linear forms over $\mathcal{R}$ such that $P = \tilde{P}$ for some $(r_1, \cdots, r_n) \in \mathcal{R}^n$ if and only if $(r_1, \cdots, r_n) \in \mathcal{R}^n$ is a solution of the system $\{D_j | 1 \leq j \leq t\}$.*

Proof. Suppose that $P = \tilde{P}$ for some $(r_1, \cdots, r_n) \in \mathcal{R}^n$ then from proposition

2.9 we see that

$$\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i} = 0. \tag{2.15}$$

By grouping all the monomials, we could rewrite $\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i}$ as

$$\sum_{j=1}^{t} D_j(r_1, \cdots, r_n) X_{\omega_j},$$

where $\{D_j | 1 \leq j \leq t\}$ is a system of linear forms over $\mathcal{R}$ and X^{ω_j} represents a primitive monomial that appears in $\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i}$. Thus equation 2.15 becomes

$$\sum_{j=1}^{t} D_j(r_1, \cdots, r_n) X_{\omega_j} = 0$$

and clearly P satisfies equation 2.15 if and only if the the tuple $(r_1, \cdots, r_n) \in \mathcal{R}^n$ satisfies the system $\{D_j | 1 \leq j \leq t\}$. □

Proposition 2.11. *Let $\mathcal{R} := \mathbf{C}[x_1, \cdots, x_m]$ be a polynomial ring and let $D(y_1, \cdots, y_n)$ be a linear form over the ring $\mathcal{R}$ with variables $y_1, \cdots, y_n$. Then there is a system $\{L_j\}$ of linear forms over $\mathbf{C}$ such that $D(c_1, \cdots, c_n) = 0$ for $(c_1 \cdots, c_n) \in \mathbf{C}^n$ if and only if $(c_1 \cdots, c_n) \in \mathbf{C}^n$ is a solution of the system $\{L_j\}$*

Proof. By viewing the polynomial $D(y_1, \cdots, y_n) \in \mathcal{R}[y_1, \cdots, y_n]$ as a polynomial over the ring $\mathbf{C}[y_1, \cdots, y_n]$ with variables $x_1, \cdots, x_m$, we obtain vectors $\omega_j := (\omega_{j1}, \cdots \omega_{jm}) \in \mathbb{W}^m$, where $\mathbb{W} := \mathbb{N} \cup \{0\}$ and linear forms $L_j(y_1, \cdots, y_n) \in span_{\mathbf{C}}\{x_1, \cdots, x_m\}$ such that

$$D(y_1, \cdots, y_n) = \sum_{j=1}^{t} L_j(y_1, \cdots, y_n) X_{\omega_j},$$

where X_{ω_j} is the primitive monomial $x_1^{\omega_{j1}} \cdots x_m^{\omega_{jm}}$. Since primitive monomials are linearly independent over constants, we see that $D(c_1, \cdots, c_n) = 0$ if and only if $(c_1, \cdots, c_n)$ is a solution of the system $\{L_j | 1 \le j \le t\}$ of linear forms over $\mathbf{C}$. $\square$

Proof of theorem 2.8. Let $\mathbb{G} := \mathbb{G}(\mathbf{E}|\mathbf{F})$ and let $\mathbb{H} \le \mathbb{G}$ be the group of all automorphisms that fixes $\mathbf{F}\langle u\rangle$. Since $\phi(T) = T(\mathfrak{x}_1 + c_{1\phi}, \cdots, \mathfrak{x}_n + c_{n\phi})$ for any polynomial $T \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ and $\phi \in \mathbb{G}$, we observe that $\mathbb{G}$ keeps the ring $\mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ invariant.

For any $\sigma = (c_{1\sigma}, \cdots, c_{n\sigma}) \in \mathbb{H}$ we have $\sigma(u) = u$, that is

$$\frac{\sigma(P)}{\sigma(Q)} = \frac{P}{Q}$$

and thus

$$\sigma(P)Q = \sigma(Q)P. \tag{2.16}$$

Since $(P, Q) = 1$, from equation 2.16 we see that P divides $\sigma(P)$ and Q divides $\sigma(Q)$. Note that $\sigma(P) = P(\mathfrak{x}_1 + c_{1\sigma}, \cdots, \mathfrak{x}_n + c_{n\sigma})$ and $\sigma(Q) = Q(\mathfrak{x}_1 + c_{1\sigma}, \cdots, \mathfrak{x}_n + c_{n\sigma})$ and therefore from proposition 2.9 we obtain

$$\sigma(P) = P \qquad \text{and} \qquad \sigma(Q) = Q. \tag{2.17}$$

Thus every automorphism that fixes u also fixes P and Q and therefore from fundamental theorem $\mathbf{F}\langle P, Q\rangle = \mathbf{F}\langle u\rangle$. If both $P, Q \in \mathbf{F}$ then $\mathbb{G}$ fixes u and thus $\mathbf{F}\langle u\rangle = \mathbf{F}$. Let us assume $P \notin \mathbf{F}$ and consider $\phi(P) = P(\mathfrak{x}_1 + c_{1\phi}, \cdots, \mathfrak{x}_n + c_{n\phi})$ for $\phi \in \mathbb{G}$. Now apply propositions 2.9 and 2.10 with $R := \mathbf{F}$ to get linear forms $\{A_i | 1 \le i \le s\} \subset Span_{\mathbf{F}}\{\mathfrak{x}_1 \cdots, \mathfrak{x}_n\}$ such that $A_i(c_{1\phi}, \cdots, c_{n\phi}) = 0$ iff $\phi(P) = P$, where $\phi \in \mathbb{G}$. We also see that such A_i's are fixed by all $\phi \in \mathbb{G}$ that fixes P and vice versa. Therefore from the fundamental theorem we conclude that $\mathbf{F}\langle P\rangle = \mathbf{F}(A_i | 1 \le i \le s) \subseteq \mathbf{F}\langle u\rangle$. Similarly if $Q \notin \mathbf{F}$ then one can find these linear forms for Q say $\{B_i|1 \leq i \leq t\} \subset Span_{\mathbf{F}}\{\mathfrak{x}_1 \cdots, \mathfrak{x}_n\}$ such that $\mathbf{F}\langle Q\rangle = \mathbf{F}(B_i|1 \leq i \leq t)$ $\subseteq \mathbf{F}\langle u\rangle$. Now $u = \frac{P}{Q} \in \mathbf{F}(D_i|1 \leq i \leq r)$, where $\{D_i|1 \leq i \leq r\} = \{A_i|1 \leq i \leq s\}$ $\cup\{B_i|1 \leq i \leq t\}$. On the other hand, both the fields $\mathbf{F}(A_i|1 \leq i \leq s)$ and $\mathbf{F}(B_i|1 \leq i \leq t)$ are subfields of $\mathbf{F}\langle u\rangle$. Thus we see that

$$\mathbf{F}\langle u\rangle = \mathbf{F}(D_i|1 \leq i \leq t).$$

□

Remark 2.12. (**Algorithm**)

Let $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_n)$ be an extension by antiderivatives $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ of $\mathbf{F}$ and assume that $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ are algebraically independent over $\mathbf{F}$. Let $u = \frac{P}{Q}$, $P, Q \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ and $(P, Q) = 1$. To compute the differential field $\mathbf{F}\langle u\rangle$ we do the following:

1. Observe from equation 2.17 that $\sigma(u) = u$ if and only if $\sigma(P) = P$ and $\sigma(Q) = Q$.

2. To find all linear forms corresponding to $P = P(\mathfrak{x} + c_1, \cdots, \mathfrak{x}_n + c_n)$ and $Q = Q(\mathfrak{x} + c_1, \cdots, \mathfrak{x}_n + c_n)$ we perform steps 2a, 2b and 2c.

2a. From proposition 2.9, we see that $P = P(\mathfrak{x} + c_1, \cdots, \mathfrak{x}_n + c_n)$ if and only if

$$\sum_{i=1}^{n} c_i \frac{\partial P}{\partial y_i} = 0$$

and similarly $Q = Q(\mathfrak{x} + c_1, \cdots, \mathfrak{x}_n + c_n)$ if and only if

$$\sum_{i=1}^{n} c_i \frac{\partial Q}{\partial y_i} = 0.$$

2b. We rewrite the above equations as

$$\sum_{j=1}^{t} A_j(c_1, \cdots, c_n) X_{\omega_j} = 0$$

and

$$\sum_{j=1}^{s} B_j(c_1, \cdots, c_n) Y_{\omega_j} = 0,$$

where $\{A_j | 1 \leq j \leq t\} \subset Span_{\mathbf{F}}\{\mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is a system of linear forms over $\mathbf{F}$ and X^{ω_j} represents a primitive monomial that appears in $\sum_{i=1}^{n} r_i \frac{\partial P}{\partial y_i}$ and $\{B_j | 1 \leq j \leq s\} \subset Span_{\mathbf{F}}\{\mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is a system of linear forms over $\mathbf{F}$ and Y^{ω_j} represents a primitive monomial that appears in $\sum_{i=1}^{n} c_i \frac{\partial Q}{\partial y_i}$.

2c. Observe that the displayed equations from 2b holds if and only if $A_j(c_1, \cdots, c_n) = 0$ for all $1 \leq j \leq t$ and $B_j(c_1, \cdots, c_n) = 0$ for all $1 \leq j \leq s$. Thus $\sigma(u) = u$ if and only if $\sigma := (c_{1\sigma}, \cdots, c_{n\sigma})$ is a solution of the system $\{D_i | 1 \leq i \leq r\} = \{A_i | 1 \leq i \leq s\} \cup \{B_i | 1 \leq i \leq t\}$.

3. Thus the algebraic subgroup of all automorphisms of $\mathbb{G}$ that fixes u also fixes $\{D_j | 1 \leq j \leq r\}$ and vice versa. Therefore from the fundamental theorem we conclude that $\mathbf{F}\langle u \rangle$ equals the differential field $\mathbf{F}(D_j | 1 \leq j \leq r)$.

4. Finally, if $\mathbf{F}$ is a fraction field of a polynomial ring $\mathcal{R} := \mathbf{C}[x_1, \cdots, x_s] \subset \mathbf{F}$ then from proposition 2.11 we see that each of the D_j's can be reduced to a finite set of linear forms L_{ji}, $1 \leq i \leq m_j$ over $\mathbf{C}$ and thus $\mathbf{F}\langle u \rangle = \mathbf{F}(D_j | 1 \leq j \leq r) = \mathbf{F}(L_i | 1 \leq i \leq m)$, where $\{L_i | 1 \leq i \leq m\} = \cup_{j=1}^{r} \{L_{ji} | 1 \leq i \leq m_j\}$.

2.4 Extensions by Exponentials of Integrals

Let $\mathbf{F}$ be a differential field with an algebraically closed field of constants $\mathbf{C}$. Let $\mathbf{E} \supset \mathbf{F}$ be an extension by exponentials of integrals $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ of $\mathbf{F}$ and $\mathbb{G}$ the group of all differential automorphisms of $\mathbf{E}$ over $\mathbf{F}$. Since $f_i := \frac{\mathfrak{e}_i'}{\mathfrak{e}_i} \in \mathbf{F}$, $\mathfrak{e}_i$ satisfies the first order linear homogeneous differential equation $\mathfrak{e}_i' = f_i \mathfrak{e}_i$. For any $\sigma \in \mathbb{G}$, $\sigma(\mathfrak{e}_i)' = f_i \sigma(\mathfrak{e}_i)$ and thus $(\frac{\sigma(\mathfrak{e}_i)}{\mathfrak{e}_i})' = 0$. Since $\mathbf{E}$ is a NNC extension of $\mathbf{F}$, there is a $c_{i\sigma} \in \mathbf{C} \setminus \{0\}$ such that $\frac{\sigma(\mathfrak{e}_i)}{\mathfrak{e}_i} = c_{i\sigma}$. Thus $\sigma(\mathfrak{e}_i) = c_{i\sigma}\mathfrak{e}_i$. Also note that the action of σ on the elements $\mathfrak{e}_i$ completely determines the automorphism σ. For any $\phi, \sigma \in \mathbb{G}$,

$$\phi(\sigma(\mathfrak{e}_i)) = \phi(c_{i\sigma}\mathfrak{e}_i) = c_{i\phi}c_{i\sigma}\mathfrak{e}_i = c_{i\sigma}c_{i\phi}\mathfrak{e}_i = \sigma(\phi(\mathfrak{e}_i)). \tag{2.18}$$

Thus $\mathbb{G}$ is a commutative group and also the map $\sigma \mapsto (c_{1\sigma}, \cdots, c_{n\sigma})$ is an injective algebraic group homomorphism from $\mathbb{G}$ to $(\mathbf{C} \setminus \{0\}, \times)^n$.

If $\mathbf{E} = \mathbf{F}(\mathfrak{e})$, $\frac{\mathfrak{e}'}{\mathfrak{e}} \in \mathbf{F}$ then $\mathbb{G}$ is an algebraic subgroup of $(\mathbf{C} \setminus \{0\}, \times)$. Thus if $\mathbb{G}$ is non trivial then it has to be a finite subgroup of $(\mathbf{C} \setminus \{0\}, \times)$. Note that $\mathbb{G}$ could be a finite subgroup of $(\mathbf{C} \setminus \{0\}, \times)$; for example, let $\mathbf{F} = \mathbb{C}(x)$ and let $\mathbf{E} = \mathbf{F}(\sqrt[n]{x})$, $n \geq 2$. Then we have the equation

$$(\sqrt[n]{x})' = \frac{1}{nx}\sqrt[n]{x}.$$

Thus $\mathbf{E}$ is an extension by an exponential of an integral $\sqrt[n]{x}$ of $\mathbf{F}$. Clearly $\sqrt[n]{x} \notin \mathbf{F}$(therefore $\mathbb{G}$ is not the trivial group) and for any automorphism $\sigma \in \mathbb{G}$

$$\begin{aligned}
&\sigma(\sqrt[n]{x}) = c_\sigma \sqrt[n]{x} \\
\Longleftrightarrow\ &(\sigma(\sqrt[n]{x}))^n = c_\sigma^n (\sqrt[n]{x})^n \\
\Longleftrightarrow\ &\sigma(x) = c_\sigma^n x \\
\Longleftrightarrow\ &1 = c_\sigma^n
\end{aligned}$$

In fact one can also show that $\mathbb{G}$ is the group of nth roots of unity (follows from the fact that the ordinary Galois group and the differential Galois group are the same if the extension $\mathbf{E}$ of $\mathbf{F}$ is finite).
Let $\mathfrak{M} := \{\prod_{i=1}^{k} \mathfrak{e}_i^{m_i} | m_i \in \mathbb{Z}^*\}$, the set of all power products of $\{\mathfrak{e}_i | 1 \leq i \leq n\}$. We will now prove the following theorem

Theorem 2.13. *Let $\mathbf{E} = \mathbf{F}(\mathfrak{e}_1, \cdots, \mathfrak{e}_n)$ be an extension of $\mathbf{F}$ by exponentials of integrals $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ of $\mathbf{F}$ and let $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ are algebraically independent over $\mathbf{F}$. Let $u = \frac{P}{Q}$, $P, Q \in \mathbf{F}[\mathfrak{e}_1, \cdots, \mathfrak{e}_n]$ and $(P, Q) = 1$. Then there are power products $\mathfrak{p}_j \in \mathfrak{M}$, $1 \leq j \leq t$ such that*

$$\mathbf{F}\langle u \rangle = \mathbf{F}(\mathfrak{p}_1, \cdots, \mathfrak{p}_t).$$

Moreover, we may explicitly compute the power products $\mathfrak{p}_i$ from P and Q.

Proof. Let $\mathbb{G} := \mathbb{G}(\mathbf{E}|\mathbf{F})$ and let $\mathbb{H} \leq \mathbb{G}$ be the group of all automorphisms of $\mathbb{G}$ that fixes u. So, for $\sigma \in \mathbb{H}$ we have $\sigma(u) = u$ and therefore $\sigma(P)Q = \sigma(Q)P$. Thus P divides $\sigma(P)Q$ and since $(P, Q) = 1$, P divides $\sigma(P)$ and similarly Q divides $\sigma(Q)$.
We may assume either P or Q is not in $\mathbf{F}$; otherwise the differential field $\mathbf{F}\langle u \rangle = \mathbf{F}$. Assume that $P \notin \mathbf{F}$ and write

$$P = \sum_{i=1}^{r} f_{\mathfrak{m}_i} \mathfrak{m}_i, \tag{2.19}$$

where $\mathfrak{m}_i$ are primitive monomials and $f_{\mathfrak{m}_i} \in \mathbf{F}$. Note that

$$\sigma(P) = \sum_{i=1}^{r} f_{\mathfrak{m}_i} \mathfrak{m}_i(c_{1\sigma}, \cdots, c_{n\sigma}) \mathfrak{m}_i$$

and since $c_{i\sigma} \in \mathbf{C} \setminus \{0\}$, $\mathfrak{m}_i(c_{1\sigma}, \cdots, c_{n\sigma}) \neq 0$. Thus P and $\sigma(P)$ have the same number terms and every monomial that appears in P also appears in $\sigma(P)$ and vice versa. But P divides $\sigma(P)$ and therefore there is a $d_\sigma \in \mathbf{F}$ such that $\sigma(P) = d_\sigma P$. In fact $\mathfrak{m}_i(c_{\sigma 1}, \cdots, c_{\sigma n}) = d_\sigma$ for all i since $\mathfrak{m}_i$ are linearly independent over $\mathbf{F}$. Thus $d_\sigma \in \mathbf{C} \setminus \{0\}$.

This shows that $\frac{\mathfrak{m}_i}{\mathfrak{m}_1}$ is fixed by every $\sigma \in \mathbb{H}$. Thus, from fundamental theorem, we obtain $\mathbf{C}\langle u\rangle \supset \mathbf{F}(\frac{\mathfrak{m}_i}{\mathfrak{m}_1} | 1 \leq i \leq r)$. Since Q also divides $\sigma(Q)$, writing $Q = \sum_{j=1}^{s} g_{\mathfrak{n}_j}\mathfrak{n}_j$ similar to equation 2.19, we conclude that there is a $e_\sigma \in \mathbf{C} \setminus \{0\}$ such that $\sigma(Q) = e_\sigma Q$. Since $\sigma(\frac{P}{Q}) = \frac{P}{Q}$, we have $d_\sigma = e_\sigma$ and thus σ fixes $\frac{\mathfrak{n}_j}{\mathfrak{m}_1}$. Thus $\mathbf{C}\langle u\rangle \supset \mathbf{F}(\frac{\mathfrak{n}_j}{\mathfrak{m}_1} | 1 \leq j \leq t)$. Now we have

$$\mathbf{C}\langle u\rangle \supset \mathbf{F}\Big(\frac{\mathfrak{m}_i}{\mathfrak{m}_1}, \frac{\mathfrak{n}_j}{\mathfrak{m}_1} | 1 \leq i \leq r, 1 \leq j \leq s\Big).$$

On the other hand we could write

$$u = \frac{P}{Q} = \frac{\sum_{i=1}^{r} f_{\mathfrak{m}_i}\frac{\mathfrak{m}_i}{\mathfrak{m}_1}}{\sum_{j=1}^{r} g_{\mathfrak{n}_j}\frac{\mathfrak{n}_j}{\mathfrak{m}_1}}$$

for all $1 \leq i, j \leq r$. Hence from fundamental theorem it follows that

$$\mathbf{F}\langle u\rangle = \mathbf{F}(\mathfrak{p}_1, \cdots, \mathfrak{p}_t),$$

where $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_t\} = \{\frac{\mathfrak{m}_i}{\mathfrak{m}_1}, \frac{\mathfrak{n}_j}{\mathfrak{m}_1} | 1 \leq i \leq r, 1 \leq j \leq s\}$. $\square$

Now we will prove a theorem which is a combination of theorems 2.8 and 2.13. The following theorem also contains a procedure to compute the differential subfields of extensions by antiderivatives and exponentials of integrals.

Theorem 2.14. *Let* $\mathbf{E} \subset \mathbf{F}$ *be a NNC extension and let* $\mathbf{E} = \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_n,$

$\mathfrak{e}_1 \cdots, \mathfrak{e}_m)$, *where* $\mathfrak{x}_i' \in \mathbf{F}$, $\frac{\mathfrak{e}_i'}{\mathfrak{e}_i} \in \mathbf{F}$ *and* $\mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{e}_1 \cdots, \mathfrak{e}_m$ *are algebraically independent over* $\mathbf{F}$. *Let* $u \in \mathbf{E}$ *and suppose that* $u = \frac{P}{Q}$, *where* $P, Q \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{e}_1 \cdots, \mathfrak{e}_m]$ *and* $(P, Q) = 1$. *Then for* $i = 1, 2, \cdots, t$ *and* $j = 1, 2, \cdots s$ *there are* $\mathbf{F}-$ *linear forms* $\mathfrak{d}_i$ *over the set* $\{\mathfrak{x}_i | 1 \leq i \leq n\}$ *and power products* $\mathfrak{p}_j$ *over the set* $\{\mathfrak{e}_i | 1 \leq i \leq m\}$ *such that*

$$\mathbf{F}\langle u \rangle = \mathbf{F}(\mathfrak{d}_i, \mathfrak{p}_j | 1 \leq i \leq t, 1 \leq j \leq s).$$

Moreover these forms can be explicitly computed from the polynomials P *and* Q.

Proof. Let $u \neq 0$ and $u = \frac{P}{Q} \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{e}_1 \cdots, \mathfrak{e}_m]$, $(P, Q) = 1$. Rewrite P and Q as polynomials over the ring $\mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n][\mathfrak{e}_1 \cdots, \mathfrak{e}_m]$. That is $P = \sum_{i=0}^{k} a_{\mathfrak{m}_i} \mathfrak{m}_i$, $Q = \sum_{i=0}^{l} b_{\mathfrak{n}_i} \mathfrak{n}_i$, where $a_{\mathfrak{m}_i}, b_{\mathfrak{n}_i} \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ and $a_{\mathfrak{m}_k}$ and $b_{\mathfrak{n}_l}$ are non zero. Now divide through P and Q by $a_{\mathfrak{m}_k}$. Thus we obtain

$$u = \frac{P}{Q} = \frac{\sum_{i=0}^{k} \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}} \mathfrak{m}_i}{\sum_{i=0}^{l} \frac{b_{\mathfrak{n}_i}}{a_{\mathfrak{m}_k}} \mathfrak{n}_i} \tag{2.20}$$

and now the polynomials P, Q becomes polynomials over the ring $\mathbf{K}[\mathfrak{e}_1, \cdots, \mathfrak{e}_m]$, where $\mathbf{K} := \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_n)$. Hereafter we will call $\sum_{i=0}^{k} \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}} \mathfrak{m}_i$ as P and $\sum_{i=0}^{l} \frac{b_{\mathfrak{n}_i}}{a_{\mathfrak{m}_k}} \mathfrak{n}_i$ as Q. Note that P and Q are relatively prime in the ring $\mathbf{K}[\mathfrak{e}_1, \cdots, \mathfrak{e}_m]$.
We observe that $\mathbf{E} \supseteq \mathbf{F}$ is a P-V extension and let $\mathbb{G}$ be the group of differential automorphisms of $\mathbf{E} \supseteq \mathbf{F}$. Thus there is a subgroup $\mathbb{H} \leq \mathbb{G}$ such that $\mathbf{F}\langle u \rangle$ is the fixed field of $\mathbb{H}$. Let $\sigma \in \mathbb{H}$. Then $\sigma(u) = u$ and therefore we obtain

$$\sigma(P)Q = \sigma(Q)P. \tag{2.21}$$

Since $(P, Q) = 1$ in $\mathbf{K}[\mathfrak{e}_1, \cdots, \mathfrak{e}_m]$, P divides $\sigma(P)$ and Q divides $\sigma(Q)$.

We observe that

$$\sigma(P) = \sum_{i=0}^{k} \frac{\sigma(a_{\mathfrak{m}_i})}{\sigma(a_{\mathfrak{m}_k})} \mathfrak{m}_i(c_{1\sigma}, \cdots, c_{m\sigma})\mathfrak{m}_i,$$

$$\sigma(Q) = \sum_{i=0}^{l} \frac{\sigma(b_{\mathfrak{n}_i})}{\sigma(a_{\mathfrak{m}_k})} \mathfrak{n}_i(c_{1\sigma}, \cdots, c_{m\sigma})\mathfrak{n}_i$$

and that $\frac{\sigma(a_{\mathfrak{m}_i})}{\sigma(a_{\mathfrak{m}_k})} \in \mathbf{K}$ since $\mathbf{K}$ is a normal extension of $\mathbf{F}$, and $\mathfrak{m}_i(c_{1\sigma}, \cdots, c_{m\sigma}) \in \mathbf{C}$. Therefore $\sigma(P), \sigma(Q) \in \mathbf{K}[\mathfrak{e}_1, \cdots, \mathfrak{e}_n]$.

Claim: $\sigma(\frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}) = \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}$, $\sigma(\frac{b_{\mathfrak{n}_i}}{a_{\mathfrak{m}_k}}) = \frac{b_{\mathfrak{n}_i}}{a_{\mathfrak{m}_k}}$, $\sigma(\frac{\mathfrak{m}_i}{\mathfrak{m}_k}) = \frac{\mathfrak{m}_i}{\mathfrak{m}_k}$ and $\sigma(\frac{\mathfrak{n}_i}{\mathfrak{m}_k}) = \frac{\mathfrak{n}_i}{\mathfrak{m}_k}$.

From the facts that P divides $\sigma(P)$,

$$P = \sum_{i=0}^{k} \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}} \mathfrak{m}_i \quad \text{and} \quad \sigma(P) = \sum_{i=0}^{k} \frac{\sigma(a_{\mathfrak{m}_i})}{\sigma(a_{\mathfrak{m}_k})} \mathfrak{m}_i(c_{1\sigma}, \cdots, c_{m\sigma})\mathfrak{m}_i,$$

we see $\sigma(P) = \mathfrak{m}_k(c_{1\sigma}, \cdots, c_{m\sigma})P$. Since $\mathfrak{m}_i$ are linearly independent over $\mathbf{K}$, for each i, we have

$$\frac{\sigma(a_{\mathfrak{m}_i})}{\sigma(a_{\mathfrak{m}_k})} = \frac{\mathfrak{m}_k(c_{1\sigma}, \cdots, c_{m\sigma})}{\mathfrak{m}_i(c_{1\sigma}, \cdots, c_{m\sigma})} \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}.$$

Observe that $a_{\mathfrak{m}_i} \in \mathbf{F}[\mathfrak{x}_1 \cdots, \mathfrak{x}_n]$ and now replace $\frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}$ by $\frac{\alpha_{\mathfrak{m}_i}}{\beta_{\mathfrak{m}_i}}$, where $\alpha_{\mathfrak{m}_i} := \frac{a_{\mathfrak{m}_i}}{g_i}$, $g_i := (a_{\mathfrak{m}_i}, a_{\mathfrak{m}_k})$ and $\beta_{\mathfrak{m}_i} := \frac{a_{\mathfrak{m}_k}}{g_i}$. Thus we have

$$\sigma(\alpha_{\mathfrak{m}_i})\beta_{\mathfrak{m}_i} = \frac{\mathfrak{m}_k(c_{1\sigma}, \cdots, c_{m\sigma})}{\mathfrak{m}_i(c_{1\sigma}, \cdots, c_{m\sigma})} \alpha_{\mathfrak{m}_i}\sigma(\beta_{\mathfrak{m}_i}). \tag{2.22}$$

Clearly $(\alpha_{\mathfrak{m}_i}, \beta_{\mathfrak{m}_i}) = 1$ and since $\frac{\mathfrak{m}_k(c_{1\sigma}, \cdots, c_{m\sigma})}{\mathfrak{m}_i(c_{1\sigma}, \cdots, c_{m\sigma})} \in \mathbf{C}$, we have $\alpha_{\mathfrak{m}_i}$ divides $\sigma(\alpha_{\mathfrak{m}_i})$ and $\beta_{\mathfrak{m}_i}$ divides $\sigma(\beta_{\mathfrak{m}_i})$. Apply proposition 2.9 and obtain $\sigma(\alpha_{\mathfrak{m}_i}) = \alpha_{\mathfrak{m}_i}$, $\sigma(\beta_{\mathfrak{m}_i}) = \beta_{\mathfrak{m}_i}$ and thus from equation 2.22 we have

$$\frac{\mathfrak{m}_i(c_{1\sigma}, \cdots, c_{m\sigma})}{\mathfrak{m}_k(c_{1\sigma}, \cdots, c_{m\sigma})} = 1.$$

From this equation it is clear that

$$\sigma(\frac{\mathfrak{m}_i}{\mathfrak{m}_k}) = \frac{\mathfrak{m}_i}{\mathfrak{m}_k}. \tag{2.23}$$

Since $\sigma(\frac{\alpha_{\mathfrak{m}_i}}{\beta_{\mathfrak{m}_i}}) = \frac{\alpha_{\mathfrak{m}_i}}{\beta_{\mathfrak{m}_i}}$, we have $\sigma(\frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}) = \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}$ for each i. The claims $\sigma(\frac{b_{\mathfrak{n}_i}}{a_{\mathfrak{m}_k}}) = \frac{b_{\mathfrak{n}_i}}{a_{\mathfrak{m}_k}}$ and $\sigma(\frac{\mathfrak{n}_i}{\mathfrak{m}_k}) = \frac{\mathfrak{n}_i}{\mathfrak{m}_k}$ follows similarly.

We may apply theorem 2.8 for each $\alpha_{\mathfrak{m}_i}$ and $\beta_{\mathfrak{m}_i}$ and obtain $\mathbf{F}-$ linear forms over $\{\mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ so that the differential fields $\mathbf{F}\langle\alpha_{\mathfrak{m}_i}\rangle$ and $\mathbf{F}\langle\beta_{\mathfrak{m}_i}\rangle$ equals the field generated by their corresponding linear forms. Thus we have linear forms $\{D_{i1}, \cdots, D_{it_i}\}$ such that

$$\mathbf{F}\langle\frac{\alpha_{\mathfrak{m}_i}}{\beta_{\mathfrak{m}_i}}\rangle = \mathbf{F}\langle\alpha_{\mathfrak{m}_i}, \beta_{\mathfrak{m}_i}\rangle = \mathbf{F}(D_{i1}, \cdots, D_{it_i}).$$

Note that $\frac{\alpha_{\mathfrak{m}_i}}{\beta_{\mathfrak{m}_i}} = \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}$ and therefore

$$\mathbf{F}\langle\frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}}\rangle = \mathbf{F}(D_{i1}, \cdots, D_{it_i}).$$

Similarly we can obtain linear forms $\{E_{j1}, \cdots, E_{js_j}\}$ so that

$$\mathbf{F}\langle\frac{b_{\mathfrak{n}_j}}{a_{\mathfrak{m}_k}}\rangle = \mathbf{F}(E_{j1}, \cdots, E_{js_j}).$$

Let $\{\mathfrak{d}_i | 1 \leq i \leq t\} = \{D_{i1}, \cdots, D_{it_i}\} \cup \{E_{j1}, \cdots, E_{js_j}\}$, $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\} = \{\mathfrak{m}_i | 1 \leq i \leq k\} \cup \{\mathfrak{n}_j | 1 \leq j \leq l\}$ and $\mathfrak{p}_1 := \mathfrak{m}_k$. Then writing

$$u = \frac{\sum_{i=0}^{k} \frac{a_{\mathfrak{m}_i}}{a_{\mathfrak{m}_k}} \frac{\mathfrak{m}_i}{\mathfrak{m}_k}}{\sum_{i=0}^{l} \frac{b_{\mathfrak{n}_i}}{a_{\mathfrak{m}_k}} \frac{\mathfrak{n}_i}{\mathfrak{m}_k}},$$

we immediately see that

$$\sigma(u) = u \Leftrightarrow \sigma(\mathfrak{d}_i) = \mathfrak{d}_i, \sigma(\frac{\mathfrak{p}_i}{\mathfrak{p}_1}) = \frac{\mathfrak{p}_i}{\mathfrak{p}_1}.$$

Hence the theorem. □

Chapter 3

Tower of Extensions by Antiderivatives

Let $\mathbf{F}$ be a differential field with an algebraically closed field of Constants $\mathbf{C}$ and let $\mathbf{F}_\infty$ be a complete Picard-Vessiot closure of $\mathbf{F}$ (every homogeneous linear differential equation over $\mathbf{F}_\infty$ has a full set of solutions in $\mathbf{F}_\infty$ and it has $\mathbf{C}$ as its field of constants and $\mathbf{F}_\infty$ is minimal with respect to these properties). All the differential fields under consideration are subfields of $\mathbf{F}_\infty$.

A differential field $\mathbf{E}$ is called a *tower of extension by antiderivatives* (or *an extension by iterated antiderivatives*) of $\mathbf{F}$ if there are differential fields $\mathbf{E}_i$, $0 \leq i \leq n$ such that

$$\mathbf{E} := \mathbf{E}_n \supseteq \mathbf{E}_{n-1} \supseteq \cdots, \supseteq \mathbf{E}_1 \supseteq \mathbf{E}_0 := \mathbf{F}$$

and $\mathbf{E}_i$ is an extension by antiderivatives of $\mathbf{E}_{i-1}$ for each $1 \leq i \leq n$.

Theorem 3.1. *Let* $\mathbf{M} \supseteq \mathbf{F}$ *be differential fields and let*

$$\mathbf{E} := \mathbf{E}_n \supset \mathbf{E}_{n-1} \supset \cdots \supset \mathbf{E}_1 \supset \mathbf{E}_0 := \mathbf{F}$$

be a tower of extensions by antiderivatives. Then $u \in \mathbf{E}$ *is algebraic over* $\mathbf{M}$ *only if* $u \in \mathbf{M}$*.*

Proof. We will use an induction on n to prove this theorem. Consider the tower

$$\mathbf{M} \cdot \mathbf{E} := \mathbf{M} \cdot \mathbf{E}_n \supseteq \mathbf{M} \cdot \mathbf{E}_{n-1} \supseteq \cdots \supseteq \mathbf{M} \cdot \mathbf{E}_1 \supseteq \mathbf{M}.$$

Clearly, the above tower is a tower of extension by antiderivatives. Suppose that $u \in \mathbf{E}$ is algebraic over M.

Observe that $u \in \mathbf{M} \cdot \mathbf{E}$ and assume that if $u \in \mathbf{M} \cdot \mathbf{E}_{n-1}$ then $u \in \mathbf{M}$(this is our induction hypothesis). Clearly, $\mathbf{M} \cdot \mathbf{E}$ is a Picard-Vessiot (extension by antiderivatives)extension of $\mathbf{M}\cdot\mathbf{E}_{n-1}$ and the differential Galois $\mathbb{G}(\mathbf{M}\cdot\mathbf{E}|\mathbf{M}\cdot\mathbf{E}_{n-1})$ is isomorphic to $(\mathbf{C}, +)^m$ for some $m \in \mathbb{N}$. Note that u is algebraic over $\mathbf{M} \cdot \mathbf{E}_{n-1}$ since $\mathbf{M}\cdot\mathbf{E}_{n-1} \supseteq \mathbf{M}$. We also observe that the index $[\mathbf{M}\cdot\mathbf{E}_{n-1}\langle u\rangle, \mathbf{M}\cdot\mathbf{E}_{n-1}] < \infty$. Then from the fundamental theorem we should have a finite algebraic subgroup of $\mathbb{G}(\mathbf{M} \cdot \mathbf{E}|\mathbf{M} \cdot \mathbf{E}_{n-1}) \backsimeq (\mathbf{C}, +)^m$ fixing $\mathbf{M} \cdot \mathbf{E}_{n-1}\langle u\rangle$. Since the only finite algebraic subgroup of $(\mathbf{C}, +)^m$ is the trivial group, we obtain $\mathbf{M} \cdot \mathbf{E}_{n-1}\langle u\rangle = \mathbf{M} \cdot \mathbf{E}_{n-1}$ and thus $u \in \mathbf{M} \cdot \mathbf{E}_{n-1}$. Now we apply our induction hypothesis to prove the theorem. $\square$

Note that we require $\mathbf{M}$ only to be a differential subfield of $\mathbf{F}_\infty$. We note a corollary of the above theorem here; if $\mathbf{E} \supseteq \mathbf{M} \supsetneq \mathbf{K} \supseteq \mathbf{F}$ are differentials fields and $\mathbf{E}$ is a tower of extension by antiderivatives of $\mathbf{F}$ then $\mathbf{M}$ is purely transcendental over $\mathbf{K}$.

3.1 Generating Algebraically Independent Antiderivatives

Proposition 3.2. *Let* $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l) \supset \mathbf{F}$ *be an extension by antiderivatives* $\mathfrak{x}_1, \cdots, \mathfrak{x}_l$ *of* $\mathbf{F}$ *and suppose that* $\mathfrak{x}_1, \cdots, \mathfrak{x}_l$ *are algebraically independent over* $\mathbf{F}$. *If* $R \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ *is an irreducible polynomial then the polynomials* R *and* R' *are relatively prime.*

Proof. Let $R \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ be an irreducible polynomial. Suppose that R' and R are not relatively prime. Then R, being irreducible, has to divide R'. Observe that the total degree of R' is $\leq$ the total degree of R and since R divides R', the total degree of R equals the total degree of R'. Thus

$$R' = \mathfrak{f}R$$

for some $\mathfrak{f} \in \mathbf{F}$. Let $\mathbb{G}$ be the differential Galois group of $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ over $\mathbf{F}$ and let $\sigma \in \mathbb{G}$. We observe that $\sigma(\mathfrak{x}_i) = \mathfrak{s}_i + c_{i\sigma}$, $c_{i\sigma} \in \mathbf{C}$ and therefore $\sigma(R) = R(\mathfrak{x}_1 + c_{1\sigma}, \cdots, \mathfrak{x}_l + c_{l\sigma})$. We also observe that $R' = \mathfrak{f}R$ implies $\sigma(R) = c_\sigma R$ for some $c_\sigma \in \mathbf{C}^\times$ see section 2.4. Then R divides $\sigma(R)$ and thus from proposition 2.9 we obtain $\sigma(R) = R$. Thus every automorphism of G has to fix R and since $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ is a Picard-Vessiot extension of $\mathbf{F}$, we obtain $R \in \mathbf{F}$, a contradiction. □

Theorem 3.3. *Let* $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l) \supset \mathbf{F}$ *be an extension by antiderivatives* $\mathfrak{x}_1, \cdots, \mathfrak{x}_l$ *of* $\mathbf{F}$. *Let* $S, T \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ *be relatively prime polynomials and assume that* T *has an irreducible factor* $R \in \mathbf{F}[x_1, \cdots, \mathfrak{x}_l]$ *such that* R^2 *does not divide* T. *Then there is no* $\mathfrak{y} \in \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ *such that* $\mathfrak{y}' = \frac{S}{T}$.

Proof. Suppose that there is a $\mathfrak{y} \in \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ such that $\mathfrak{y}' = \frac{S}{T}$. There are relatively prime polynomials $P, Q \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ such that $\mathfrak{y} = \frac{P}{Q}$. Thus taking the derivative we arrive at

$$Q^2 S = T(P'Q - Q'P). \tag{3.1}$$

Note that R is an irreducible factor of T and therefore from the above equation R divides $Q^2 S$. Since S and T are relatively prime, R has to divides Q^2, which implies R divides Q. Let n be the largest integer so that R^n divides Q. Then R^{n+1} divides Q^2 and again from the above displayed equation, R^{n+1} divides $T(P'Q - Q'P)$. Note that R divides T but R^2 does not and thus R^n divides $P'Q - Q'P$. Since R^n divides Q, and P and Q are relatively prime, we obtain R^n divides Q'. Let $H \in \mathbf{F}[\mathfrak{x}_1 \cdots, \mathfrak{x}_l]$ be a polynomial such that $Q = R^n H$. Note that R and H are relatively prime polynomials. Then R^n divides $Q' = nR^{n-1}R'H + R^n H'$, which implies R divides R', which contradicts proposition 3.2. □

Let $\mathbf{E} \supset \mathbf{F}$ be differential fields and let $\mathfrak{x}_1, \cdots, \mathfrak{x}_l \in \mathbf{E}$ be algebraically independent antiderivatives of $\mathbf{F}$.

Definition 3.4. An antiderivative $\mathfrak{y} \in \mathbf{F}_\infty$ of $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ is called an Irreducible-explicit(I-E)antiderivative if $\mathfrak{y}' = \frac{A}{CB}$, where $A, B, C \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$, $(A, B) = (B, C) = (C, A) = 1$ and C is an irreducible polynomial.

We also note from proposition 3.3 that such a $\mathfrak{y} \notin \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$.

Definition 3.5. For each $i = 1, 2, \cdots, m$ let $\mathfrak{y}_i \in \mathcal{U}$ be an antiderivative of $\frac{A_i}{C_i B_i}$, where $C_i, A_i, B_i \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$,$(A_i, B_i)$ =(B_i, C_i) =$(C_i, A_i) = 1$, and satisfying the following conditions;

C1: C_i is an irreducible polynomial, $C_i \nmid C_j$ if $i \neq j$ and $C_i \nmid B_j$ for any $1 \leq i, j \leq m$.

C2: for every $1 \leq i \leq m$ there is an element $\mathfrak{x}_{C_i} \in \{\mathfrak{x}_1, \cdots, \mathfrak{x}_l\}$ such that the partial $\frac{\partial C_i}{\partial \mathfrak{x}_{C_i}} \neq 0$ and $\frac{\partial A_i}{\partial \mathfrak{x}_{C_i}} = \frac{\partial B_i}{\partial \mathfrak{x}_{C_i}} = 0$.

We call $\mathfrak{y}_1, \cdots, \mathfrak{y}_m$ a J-I-E(Joint-Irreducible-Explicit) antiderivatives of $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$. We call the differential field $\mathbf{F}(\mathfrak{y}_1, \cdots, \mathfrak{y}_m, \mathfrak{x}_1, \cdots, \mathfrak{x}_l)$, a *2-tower* J-I-E *extension* of $\mathbf{F}$.

The following theorem shows any set of antiderivatives $\mathfrak{y}_1, \cdots, \mathfrak{y}_m$ of $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$, $\mathfrak{y}_i' = \frac{A_i}{C_i B_i}$, becomes algebraically independent over $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ once it satisfies C1 (see theorem 3.6) and thus J-I-E antiderivatives of $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ are algebraically independent $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$.

Theorem 3.6. *Let $\mathbf{E} \supseteq \mathbf{F}$ be differential fields, $\mathfrak{x}_1, \cdots, \mathfrak{x}_l \in \mathbf{E}$ be antiderivatives of $\mathbf{F}$ and assume that $\mathfrak{x}_1, \cdots, \mathfrak{x}_l$ are algebraically independent over $\mathbf{F}$. For each $i = 1, \cdots, m$ let $A_i, B_i, C_i \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$, $(A_i, B_i) = (A_i, C_i) = (B_i, C_i) = 1$ be polynomials satisfying the following condition*
C1: C_i is an irreducible polynomial, $C_i \nmid C_j$ if $i \neq j$ and $C_i \nmid B_j$ for any $1 \leq i, j \leq m$.
Let $\mathfrak{y}_1, \cdots, \mathfrak{y}_m \in \mathbf{F}_\infty$ be antiderivatives of $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ with $\mathfrak{y}_i' = \frac{A_i}{C_i B_i}$. Then $\mathfrak{y}_1, \cdots, \mathfrak{y}_m$ are algebraically independent over $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$.

Proof. Suppose that $\mathfrak{y}_1, \cdots, \mathfrak{y}_m$ are algebraically dependent over $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$. Then Kolchin-Ostrowski theorem guarantees constants $\alpha_1, \cdots, c_m \in \mathbf{C}$, not all zero, such that $\sum_{i=1}^m \alpha_i \mathfrak{y}_i \in \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$. Assume that $\alpha_1 \neq 0$.
First we note that if $\sum_{i=1}^m \alpha_i \mathfrak{y}_i \in \mathbf{C}$ then $\sum_{i=1}^m \alpha_i \frac{A_i}{C_i B_i} = 0$ and now writing

$\sum_{i=2}^{m} \alpha_i \frac{A_i}{C_i B_i} = \frac{F}{G}$, $F, G \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$, we obtain

$$\alpha_1 \frac{A_1}{C_1 B_1} = -\frac{F}{G}$$
$$\Longrightarrow \alpha_1 A_1 G = -F C_1 B_1.$$

Since $A_1 \neq 0$, we obtain $F \neq 0$ and thus we may assume F and G are relatively prime polynomials. Clearly, C_1 divides $A_1 G$ and since A_1 and C_1 are relatively prime, C_1 divides G. On the other hand $\sum_{i=2}^{m} \alpha_i \frac{A_i}{C_i B_i} = \frac{F}{G}$ implies G divides $\prod_{i=2}^{m} C_i B_i$, which implies C_1 divides $\prod_{i=2}^{m} C_i B_i$ contradicting the condition C1. Thus $\sum_{i=1}^{m} \alpha_i \mathfrak{y}_i \in \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l) \setminus \mathbf{C}$.

Let $P, Q \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ be relatively prime polynomials such that

$$\sum_{i=1}^{m} \alpha_i \mathfrak{y}_i = \frac{P}{Q}. \tag{3.2}$$

Let $S, T \in \mathbf{F}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ be polynomials such that $\frac{S}{T} = (\frac{P}{Q})' = \sum_{i=1}^{m} \alpha_i \frac{A_i}{C_i B_i}$. We know that $\sum_{i=1}^{m} \alpha_i \mathfrak{y}_i \notin \mathbf{C}$ and therefore $S \neq 0$ and thus we may assume S and T are relatively prime. Since $\alpha_1 \neq 0$, we see that C_1 divides T. And, T divides $\prod_{i=1}^{m} C_i B_i$ and that C_i, B_i satisfies condition C1 implies C_1^2 does not divide T. Thus P, Q, S and T satisfies the hypothesis of theorem 3.3. But, taking the derivative of equation 3.2 we obtain $(\frac{P}{Q})' = \frac{S}{T}$, which contradicts theorem 3.3. $\square$

Theorem 3.7. *Let $\mathbf{E} \supseteq \mathbf{F}$ be a NNC extension. If there is an $\mathfrak{x} \in \mathbf{E} \setminus \mathbf{F}$ such that $\mathfrak{x}' \in \mathbf{F}$ then for any $n \in \mathbb{N}$ and distinct $\alpha_1, \cdots, \alpha_n \in \mathbf{C}$, the elements $\mathfrak{y}_i \in \mathbf{F}_\infty$ such that $\mathfrak{y}'_{\alpha_i} = \frac{1}{\mathfrak{x} + \alpha_i}$ are algebraically independent over $\mathbf{F}(\mathfrak{x})$. Moreover, the differential field $\mathbf{F}(\mathfrak{y}_\alpha, \mathfrak{x})$, where $\mathfrak{y}'_\alpha = \frac{1}{\mathfrak{x} + \alpha}$ and $\alpha \in \mathbf{C}$ is not imbeddable in any Picard-Vessiot extension of $\mathbf{F}$.*

Proof. The algebraic independency of $\mathfrak{y}_i$'s follows immediatedly for theorem 3.6.

Suppose that there is an $\alpha \in \mathbf{C}$ such that $\mathbf{E} \supset \mathbf{F}(\mathfrak{y}_\alpha, \mathfrak{x}) \supset \mathbf{F}$ for some Picard-Vessiot extension $\mathbf{E}$ of $\mathbf{F}$. Note that $\mathbf{F}(\mathfrak{x})$ is a Picard-Vessiot sub-extension of $\mathbf{E} \supset \mathbf{F}$ with differential Galois group $\mathbb{G}(\mathbf{F}(\mathfrak{x})|\mathbf{F}) \backsimeq (\mathbf{C}, +)$, and every automorphism of $\mathbf{F}(\mathfrak{x})$ fixing $\mathbf{F}$ lifts to an automorphism of $\mathbf{E}$ over $\mathbf{F}$. In particular, there is an automorphism $\sigma \in \mathbb{G}(\mathbf{E}|\mathbf{F})$ such that $\sigma(\mathfrak{x}) = \mathfrak{x} + c$ for some $c \neq 0$. Observing that

$$\begin{aligned} \mathfrak{y}'_\alpha &= \frac{1}{\mathfrak{x} + \alpha} \\ \Longrightarrow \qquad \sigma^i(\mathfrak{y}_\alpha)' &= \frac{1}{\mathfrak{x} + \alpha + ic} \end{aligned}$$

and that $\sigma \in \mathbb{G}(\mathbf{E}|\mathbf{F})$, we obtain $\mathfrak{y}_{\alpha+ic} := \sigma^i(\mathfrak{y}_\alpha) \in \mathbf{E}$. Since $\alpha + ic$ are distinct for $1 = 1, 2, \cdots, m$, the elements $\mathfrak{y}_{\alpha+c}, \mathfrak{y}_{\alpha+2c}, \cdots \in \mathbf{E}$ are algebraically independent over $\mathbf{F}$. Thus we obtain a contradiction to the fact that a Picard-Vessiot extension has a finite transcendence degree over. $\square$

Remark 3.8. Thus if $\mathbf{E} \supseteq \mathbf{F}$ are differential fields such that $\mathfrak{x} \in \mathbf{E} \setminus \mathbf{F}$ and $\mathfrak{x}' \in \mathbf{F}$ then the differential field $\mathbf{F}(\mathfrak{y}_\alpha, \mathfrak{x})$, $\mathfrak{y}'_\alpha = \frac{1}{\mathfrak{x}+\alpha}$ and $\alpha \in \mathbf{C}$ is not imbeddable in any Picard-Vessiot extension of $\mathbf{F}$ and thus $\mathfrak{y}_\alpha \notin \mathbf{F}_1$. We may apply the above theorem again for the element $\mathfrak{y}_\alpha$ with $\mathbf{F}_1$ as the ground field. Then for any $\mathfrak{z}_\beta \in \mathbf{F}_\infty$ such that $\mathfrak{z}'_\beta = \frac{1}{\mathfrak{y}_\alpha + \beta}$, $\beta \in \mathbf{C}$, we obtain that the differential field $\mathbf{F}_1(\mathfrak{z}_\beta, \mathfrak{y}_\alpha)$ is not imbeddable in any Picard-Vessiot extension of $\mathbf{F}_1$ and thus $\mathfrak{z}_\beta \notin \mathbf{F}_2$. A repeated application of the theorem proves the following: If $\mathbf{F}$ is a differential field that has a proper extension by antiderivatives then for given any n, $\mathbf{F}_n$ has proper extensions by antiderivatives.

3.2 Differential Subfields of J-I-E Tower

In the next section we will prove a structure theorem for the differential subfields of a certain tower of extensions by antiderivatives, namely J-I-E extensions. These towers are made by adjoining J-I-E antiderivatives.

As usual, let $\mathbf{C}$ be an algebraically closed-characteristic zero field, $\mathbf{F}$ be a differential field with field of constants $\mathbf{C}$ and let $\mathbf{F}_\infty$ be a complete Picard-Vessiot closure with $\mathbf{C}$ as its field of constants.

3.2.1 Automorphisms of J-I-E towers

Let $\mathfrak{y}_{11}, \cdots, \mathfrak{y}_{1n_1}$ be algebraically independent antiderivatives of $\mathbf{F}$ and for $i = 1, 2, \cdots, k$, let $\mathbf{E}_i := \mathbf{E}_{i-1}(\mathfrak{y}_{i1}, \mathfrak{y}_{i2}, \cdots, \mathfrak{y}_{in_i})$, where $\mathbf{E}_0 := \mathbf{F}$ and for $i \geq 2$ $\mathfrak{y}_{i1}, \mathfrak{y}_{i2}, \cdots, \mathfrak{y}_{in_i}$ are I-E antiderivatives of $\mathbf{E}_{i-1}$, that is, $\mathfrak{y}'_{ij} = \frac{A_{ij}}{C_{ij}B_{ij}}$ and for each $2 \leq i \leq k$ and for all $1 \leq j \leq n_i$, $A_{ij}, B_{ij}, C_{ij} \in \mathbf{E}_{i-2}[\mathfrak{y}_{i-11}, \cdots, \mathfrak{y}_{i-1n_{i-1}}]$ are polynomials such that $(A_{ij}, B_{ij}) = (B_{ij}, C_{ij}) = (A_{ij}, C_{ij}) = 1$ and satisfying conditions C1 and C2. Let $I_i := \{\mathfrak{y}_{ij} | 1 \leq j \leq n_i\}$, $\Lambda_t := Span_{\mathbf{C}} \cup_{i=1}^{t} I_i$, $\Lambda_0 = \{0\}$ and $\mathbf{E} := \mathbf{E}_k$. We will also recall the conditions C1 and C2 here

- C1: C_{ij} is an irreducible polynomial for each i, j. For every i, $C_{is} \nmid C_{it}$ (that is, they are non associates)if $s \neq t$ and $C_{is} \nmid B_{it}$ for any $1 \leq s, t \leq n_i$.

- C2: for every $1 \leq j \leq n_i$ there is an element $\mathfrak{y}_{C_{ij}} \in \{\mathfrak{y}_{i-11}, \cdots, \mathfrak{y}_{i-1n_{i-1}}\}$ such that the partial $\frac{\partial C_{ij}}{\partial \mathfrak{y}_{C_{ij}}} \neq 0$ and $\frac{\partial A_{ij}}{\partial \mathfrak{y}_{C_{ij}}} = \frac{\partial B_{ij}}{\partial \mathfrak{y}_{C_{ij}}} = 0$.

Definition 3.9. We call

$$\mathbf{E} := \mathbf{E}_k \supset \mathbf{E}_{k-1} \supset \cdots \supset \mathbf{E}_2 \supset \mathbf{E}_1 \supset \mathbf{E}_0 := \mathbf{F} \tag{3.3}$$

a tower of extensions by J-I-E antiderivatives. Note that $\mathbf{E}_1$ is an ordinary antiderivative extension of $\mathbf{F}$.

Let $\mathbb{G}_\infty := \mathbb{G}(\mathbf{F}_\infty|\mathbf{F})$, the group of all differential automorphisms of the complete Picard-Vessiot closure $\mathbf{F}_\infty$ of $\mathbf{F}$. We will show that the group of differential automorphisms $\mathbb{G}(\mathbf{E}|\mathbf{F})$ is isomorphic to the additive group $(\mathbf{C}, +)^\delta$ for some $\delta \leq$ tr.d $\mathbf{E}|\mathbf{F}$. Moreover, the action of $\mathbb{G}(\mathbf{E}|\mathbf{F})$ on $\mathbf{E}$ is given by $\sigma(\mathfrak{y}_{ij}) = \mathfrak{y}_{ij} + c_{ij\sigma}$, $c_{ij\sigma} \in \mathbf{C}$.

Lemma 3.10. *For any $\sigma \in \mathbb{G}_\infty$ and $t \geq 2$ the elements of I_t, namely, $\mathfrak{y}_{t1}, \mathfrak{y}_{t2}, \cdots, \mathfrak{y}_{tn_t}$ are J-I-E antiderivatives of the differential field $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{t-1}))$, which is the compositum of differential fields $\mathbf{E}_{t-1}$ and $\sigma(\mathbf{E}_{t-1})$.*

Proof. We observe that $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_s)) = \mathbf{E}_{t-1}(\cup_{i=1}^{s}\sigma(I_i))$ and since $\sigma(\mathfrak{y}_{1j}) = \mathfrak{y}_{1j} + c_{j\sigma}$, $\mathbf{E}_{t-1}(\sigma(I_1)) = \mathbf{E}_{t-1}$. For $2 \leq s \leq t-1$, let $I_s^\sigma \subset \sigma(I_s)$ be a transcendence base of the differential field $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_s))$ over $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{s-1}))$. Note that $\sigma(I_s)$ consists of antiderivatives of $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{s-1}))$ and that $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{s-1}))(\sigma(I_s)) = \mathbf{E}_{t-1}(\sigma(\mathbf{E}_s))$. Thus, $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_s))$ is an extension by antiderivatives of $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{s-1}))$ and therefore, $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{s-1}))(I_s^\sigma) = \mathbf{E}_{t-1}(\sigma(\mathbf{E}_s))$ for each $1 \leq s \leq t-1$.

Thus $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{t-1})) = \mathbf{E}_{t-1}(\cup_{i=1}^{t-1} I_i^\sigma)$. Since $\mathbf{E}_{t-1} = \mathbf{E}_{t-2}(\mathfrak{y}_{t-11}, \cdots, \mathfrak{y}_{t-1n_{t-1}})$, $\mathfrak{y}_{t-11}, \cdots, \mathfrak{y}_{t-1n_{t-1}}$ are algebraically independent over $\mathbf{E}_{t-2}$(because they are J-I-E antiderivatives) and the set $\cup_{i=1}^{t-1} I_i^\sigma$ is algebraically independent over $\mathbf{E}_{t-1}$, we obtain that $\mathfrak{y}_{t-11}, \cdots, \mathfrak{y}_{t-1n_{t-1}}$ are algebraically independent over $\mathbf{E}_{t-2}(\cup_{i=1}^{t-1} I_i^\sigma)$. Also note that

$$\mathbf{E}_{t-2}(\cup_{i=1}^{t-1} I_i^\sigma) = \mathbf{E}_{t-2}\sigma(\mathbf{E}_{t-2})(I_{t-1}^\sigma)$$

and that the elements of I_{t-1}^σ are antiderivatives of $\mathbf{E}_{t-2}\sigma(\mathbf{E}_{t-2})$. Thus $\mathbf{E}_{t-2}(\cup_{i=1}^{t-1} I_i^\sigma)$ is a differential field which is also a fraction field of the polynomial ring $\mathbf{E}_{t-2}[\cup_{i=1}^{t-1} I_i^\sigma]$.

We will now show that $\mathfrak{y}_{t1}, \mathfrak{y}_{t,2}, \cdots, \mathfrak{y}_{t,n_t}$ are J-I-E antiderivatives of the compositum $\mathbf{E}_{t-1}\ \sigma(\mathbf{E}_{t-1})$. Since $\mathfrak{y}_{t1}, \mathfrak{y}_{t,2}, \cdots, \mathfrak{y}_{t,n_t}$ are J-I-E antiderivatives of $\mathbf{E}_{t-1}$, there are polynomials $A_{t-1j}, B_{t-1j}, C_{t-1j} \in \mathbf{E}_{t-2}[\mathfrak{y}_{t-11}, \mathfrak{y}_{t-1,2}, \cdots, \mathfrak{y}_{t-1,n_{t-1}}]$ such that $(A_{t-1j}, B_{t-1j}) = (B_{t-1j}, C_{t-1j}) = (A_{t-1j}, C_{t-1j}) = 1$ and satisfying conditions C1 and C2. We observe that all the above conditions on A_{t-1j}, B_{t-1j} and C_{t-1j} holds in the polynomial ring $\mathbf{E}_{t-2}[\cup_{i=1}^{t-1} I_i^{\sigma}, \mathfrak{y}_{t-11}, \mathfrak{y}_{t-1,2}, \cdots, \mathfrak{y}_{t-1,n_{t-1}}]$ as well and therefore by "Gauss' lemma" these conditions hold in the ring

$$\mathbf{E}_{t-2}(\cup_{i=1}^{t-1} I_i^{\sigma})[\mathfrak{y}_{t-11}, \mathfrak{y}_{t-1,2}, \cdots, \mathfrak{y}_{t-1,n_{t-1}}].$$

Thus $\mathfrak{y}_{t1}, \mathfrak{y}_{t,2}, \cdots, \mathfrak{y}_{t,n_t}$ become J-I-E antiderivatives of the field

$$\mathbf{E}_{t-2}(\cup_{i=1}^{t-1} I_i^{\sigma}, \mathfrak{y}_{t-11}, \mathfrak{y}_{t-1,2}, \cdots, \mathfrak{y}_{t-1,n_{t-1}}) = \mathbf{E}_{t-1}\sigma(\mathbf{E}_{t-1}).$$

□

Theorem 3.11. *Let* $\mathbf{M}$ *be a differential subfield of* $\mathbf{F}_\infty$, $\mathfrak{x}_1, \cdots, \mathfrak{x}_l \in \mathbf{F}_\infty$ *be algebraically independent antiderivatives of* $\mathbf{M}$ *and for* $i = 1, 2, \cdots, m$ *let* $\mathfrak{y}_i \in \mathbf{F}_\infty$ *be J-I-E antiderivatives of* $\mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ *(that is,* $\mathfrak{y}_i' = \frac{A_i}{B_i C_i}$*, where* A_i, B_i *and* C_i *satisfies conditions* $(A_i, B_i) = (B_i, C_i) = (A_i, C_i) = 1$, C1 *and* C2*). Suppose that there is a subgroup* $\mathbb{H}$ *of* $\mathbb{G}_\infty$ *of differential automorphisms fixing* $\mathbf{M}$ *and an element* $\mathfrak{s} := \sum_{i=1}^{e} \alpha_i \mathfrak{y}_i \in \mathbf{M}(\mathfrak{y}_1, \cdots, \mathfrak{y}_m, \mathfrak{x}_1, \cdots, \mathfrak{x}_l)$, $\alpha_i \in \mathbf{C}$ *such that for every* $\sigma \in \mathbb{H}$, $\sigma(\mathfrak{s}) \in \mathbf{M}(\mathfrak{y}_1, \cdots, \mathfrak{y}_m, \mathfrak{x}_1, \cdots, \mathfrak{x}_l)$*. Then every* $\sigma \in \mathbb{H}$ *fixes* A_i, B_i *and* $\mathbf{C}_i$ *whenever* $\alpha_i \neq 0$*, that is* $\sigma(\mathfrak{y}_i) = \mathfrak{y}_i + c_{i\sigma}$*, for some* $c_{i\sigma} \in \mathbf{C}$*. In particular, for every* $\sigma \in \mathbb{H}$ *there is a* $c_\sigma := \mathfrak{s}(c_{1\sigma}, \cdots, c_{l\sigma}) \in \mathbf{C}$ *such that* $\sigma(\mathfrak{s}) = \mathfrak{s} + c_\sigma$*.*

Proof. If $\mathbb{H}$ is the trivial group then the proof is trivial. Assume that $\mathbb{H}$ is a nontrivial group. Since $\mathfrak{x}_i' \in \mathbf{M}$, $\mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ is an extension by antiderivatives of $\mathbf{M}$ and thus the differential field $\mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ is preserved by $\mathbb{H}$. In particular $\sigma(\frac{A_i}{C_iB_i}) \in \mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$. Then $\mathbf{M}(\mathfrak{y}_1, \cdots, \mathfrak{y}_m, \mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ has $m+1$ antiderivatives $\sum_{i=1}^{e} \alpha_i \sigma(\mathfrak{y}_i)$, $\mathfrak{y}_1, \cdots, \mathfrak{y}_m$ of $\mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ and therefore the antiderivatives has to be algebraically dependent over $\mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$. Now from the Kolchin-Ostrowski theorem we have constants γ_i, $1 \leq i \leq m+1$ not all zero such that

$$\sum_{i=1}^{m} \gamma_i \mathfrak{y}_i + \gamma_{m+1} \sum_{i=1}^{e} \alpha_i \sigma(\mathfrak{y}_i) \in \mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l). \tag{3.4}$$

Note that if $\gamma_{m+1} = 0$ then $\mathfrak{y}_i$'s become algebraically dependent over $\mathbf{M}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$, which is not true. so $\gamma_{m+1} \neq 0$ and thus we may assume $\gamma_{m+1} = 1$(dividing through the equation 3.4 by γ_{m+1}).

First we will show that $\sigma(C_i) = C_i$ for all $\sigma \in \mathbb{H}$ whenever $\alpha \neq 0$. Then we will use this to show that $\mathbb{H}$ indeed fixes A_i as well as B_i whenever $\alpha_i \neq 0$.

Suppose that there is a $\rho \in \mathbb{H}$ and an $i, 1 \leq i \leq m$ such that $\alpha_i \neq 0$ and $\rho(C_i) \neq C_i$. For convenience, let us assume that $i = 1$. The automorphism ρ acts on the ring $\mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ by sending $\mathfrak{x}_i \to \mathfrak{x}_i + c_{i\rho}$ and if ρ is nontrivial then clearly ρ has an infinite order. Thus we have $\rho(C_1) = C_1(\mathfrak{x}_1 + c_{1\rho}, \cdots, \mathfrak{x}_l + c_{l\rho})$. From proposition 2.9 we see that C_1 divides $\rho(C_1)$ only if $C_1 = \rho(C_1)$ and thus $\rho(C_1)$ and C_1 are not associates (over $\mathbf{M}$). In fact, for any $i, j \in \mathbb{N} \cup \{0\}$, $i \neq j$, the elements $\rho^i(C_1)$ and $\rho^j(C_1)$ are non-associates. Since every polynomial in $\mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ has finitely many (non-associate) irreducibles and $\rho^i(C_1)$ is also an irreducible for each $i \in \mathbb{N}$, there is a $j \in \mathbb{N}$ such that

$$\rho^j(C_1) \nmid B_1 B_2 \cdots B_m.$$

We also note that $\rho^j(C_1) \nmid \rho^j(B_j)$ for any $1 \leq j \leq m$ and $\rho^j(C_1) \nmid \rho^j(C_i)$ for any

$i \neq 1$; otherwise $C_1|B_j$, or $C_1|C_i$ for some $i \neq 1$ and in either case, contradicts the condition **C**1. Thus

$$\rho^j(C_1) \text{ does not divide } B_1 \prod_{i=2}^{m} B_i \rho^j(B_i)\rho^j(C_i). \tag{3.5}$$

The equation 3.4 is true for all $\sigma \in \mathbb{H}$ and thus there are polynomials $A, B \in \mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$

$$\sum_{i=1}^{m} \gamma_i \mathfrak{y}_i + \sum_{i=1}^{e} \alpha_i \rho^j(\mathfrak{y}_i) = \frac{A}{B}.$$

Let $S, T \in \mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ be relatively prime polynomials such that

$$\alpha_1 \frac{\rho^j(A_1)}{\rho^j(C_1)\rho^j(B_1)} + \sum_{i=1}^{m} \gamma_i \frac{A_i}{C_i B_i} + \sum_{i=2}^{e} \alpha_i \frac{\rho^j(A_i)}{\rho^j(C_i)\rho^j(B_i)} = \frac{S}{T} \tag{3.6}$$

and let $F, G \in \mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ be relatively prime polynomials such that

$$\frac{F}{G} = -\sum_{i=1}^{m} \gamma_i \frac{A_i}{C_i B_i} + \sum_{i=2}^{e} \alpha_i \frac{\rho^j(A_i)}{\rho^j(C_i)\rho^j(B_i)}. \tag{3.7}$$

Note that

$$G \text{ divides } B_1 \prod_{i=2}^{m} B_i \rho^j(B_i)\rho^j(C_i) \tag{3.8}$$

Suppose that $S = 0$. Then

$$\alpha_1 \frac{\rho^j(A_1)}{\rho^j(C_1)\rho^j(B_1)} = \frac{F}{G}$$

$$\Longrightarrow \alpha_1 \rho^j(A_1) G = \rho^j(C_1)\rho^j(B_1) F. \tag{3.9}$$

Since A_1 is a non zero polynomial, so is $\rho^j(A_1)$ and thus $\alpha_1 \neq 0$ implies $F \neq 0$. From equation 3.9 we obtain $\rho^j(C_1)$ divides G and now equation 3.8 contradicts equation 3.5.

Thus $S \neq 0$. Substituting equation 3.7 in equation 3.6 we obtain

$$\alpha_1 \frac{\rho^j(A_1)}{\rho^j(C_1)\rho^j(B_1)} - \frac{F}{G} = \frac{S}{T}$$
$$\left(\alpha_1 \rho^j(A_1)G - \rho^j(C_1)\rho^j(B_1)F\right)T = SG\rho^j(C_1)\rho^j(B_1). \tag{3.10}$$

From the above equation 3.10 we obtain $\rho^j(C_1)$ divides $\alpha_1\rho^j(A_1)GT$. Again equations 3.8 and 3.5 guarantees $\rho^j(C_1)$ does not divide G and clearly $\rho^j(C_1)$ does not divide $\rho^j(A_1)$. Therefore $\rho^j(C_1)$ divides T, which implies that $\rho^j(C_1)$ is an irreducible factor of T. Thus we have produced polynomials $A, B, S, T \in \mathbf{M}[x_1, \cdots, \mathfrak{x}_n]$ contradicting theorem 3.3. Hence $\sigma(C_i) = C_i$ for all $\sigma \in \mathbb{H}$.

Now we will show that $\mathbb{H}$ fixes A_i and B_i for every i.

Assume that $\alpha_1 \neq 0$ and pick a $\sigma \in \mathbb{H}$. Note that $\sigma(C_1) = C_1$ and that σ is an automorphism, therefore $C_1 \neq \sigma(C_j)$ for any $j \neq 1$. If $P \in \mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ is a polynomial and C_1 divides $\sigma(P)$ then $\sigma^{-1}(C_1)$ divides P. But $\sigma(C_1) = C_1$ implies $\sigma^{-1}C_1 = C_1$ and therefore C_1 divides P. Hence we note that

$$C_1 \text{ does not divide } B_1 \prod_{i=2}^{m} B_i \sigma(B_i)\sigma(C_i). \tag{3.11}$$

Take the derivative of equation 3.4 to obtain

$$B^2 \left(\sum_{i=2}^{m} \gamma_i \frac{A_i}{C_i B_i} + \sum_{i=2}^{e} \alpha_i \frac{\sigma(A_i)}{\sigma(C_i)\sigma(B_i)} + \gamma_1 \frac{A_1}{C_1 B_1} + \alpha_1 \frac{\sigma(A_1)}{C_1 \sigma(B_1)} \right) = BA' - AB'. \tag{3.12}$$

Let $F, G \in \mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ be relatively prime polynomials such that

$$\sum_{i=2}^{m} \gamma_i \frac{A_i}{C_i B_i} + \sum_{i=2}^{e} \alpha_i \frac{\sigma(A_i)}{\sigma(C_i)\sigma(B_i)} = \frac{F}{G}, \tag{3.13}$$

and let $S, T \in \mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ be relatively prime polynomials such that

$$\gamma_1 \frac{A_1}{C_1 B_1} + \alpha_1 \frac{\sigma(A_1)}{C_1 \sigma(B_1)} + \frac{F}{G} = \frac{S}{T} \tag{3.14}$$

Note that $(\frac{A}{B})' = \frac{S}{T}$ and that

$$G \text{ divides } \prod_{i=2}^{m} B_i \sigma(B_i) \sigma(C_i). \tag{3.15}$$

We rewrite equation 3.14 as

$$TG\left(\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1\right) + TFC_1 \sigma(B_1) B_1 = SGC_1 \sigma(B_1) B_1 \tag{3.16}$$

Again, we will split our into two cases; $S \neq 0$ and $S = 0$. In both the cases, we will show that C_1 divides $\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1$. Assume for a moment that we proved C_1 divides $\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1$. Then from C2 we have $\mathfrak{x}_{C_1} \in \{\mathfrak{x}_1, \cdots, \mathfrak{x}_l\}$ such that $\frac{\partial C_1}{\partial \mathfrak{x}_{C_1}} \neq 0$ and $\frac{\partial A_1}{\partial \mathfrak{x}_{C_1}} = \frac{\partial B_1}{\partial \mathfrak{x}_{C_1}} = 0$. Since $\sigma(\mathfrak{x}_i) = \mathfrak{x}_i + c_{i\sigma}$ for some $c_{i\sigma} \in \mathbf{C}$, σ is an automorphism of the ring $\mathbf{M}[\{\mathfrak{x}_1, \cdots, \mathfrak{x}_l\} \setminus \{\mathfrak{x}_{C_1}\}]$ and therefore $\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1 \in \mathbf{M}[\{\mathfrak{x}_1, \cdots, \mathfrak{x}_l\} \setminus \{\mathfrak{x}_{C_1}\}]$. Thus C_1 divides $\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1$ implies $\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1 = 0$, that is, $\sigma\left(\frac{A_1}{B_1}\right) = -\frac{\gamma_1}{\alpha_1} \frac{A_1}{B_1}$. Then A_1 divides $\sigma(A_1)$ and B_1 divides $\sigma(B_1)$ and therefore from proposition 2.9 we obtain $\sigma(A_1) = A_1$ and $\sigma(B_1) = B_1$.

Let us show that C_1 divides $\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1$.

Case $S \neq 0$:

From equation 3.16 we observe that C_1 divides $TG\left(\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1\right)$ and from equations 3.15 and 3.11 that C_1 does not divide G and therefore C_1 has to divide $T(\gamma_1 A_1 \sigma(B_1) + \alpha_1 \sigma(A_1) B_1)$. If C_1 divides T then the polynomials $A, B, S, T \in \mathbf{M}[\mathfrak{x}_1, \cdots, \mathfrak{x}_l]$ contradicts theorem 3.3. Thus C_1 divides $\gamma_1 A_1 \sigma(B_1) +$

$\alpha_1\sigma(A_1)B_1$.

Case $S = 0$: From equation 3.16 we have

$$G\left(\gamma_1 A_1\sigma(B_1) + \alpha_1\sigma(A_1)B_1\right) = -FC_1\sigma(B_1)B_1.$$

As noted earlier, C_1 does not divide G and thus C_1 divides $\gamma_1 A_1\sigma(B_1)+\alpha_1\sigma(A_1)B_1$. Thus we see that for every $\sigma \in \mathbb{H}$, $\sigma(A_i) = A_i$, $\sigma(B_i) = B_1$ and $\sigma(C_i) = C_i$ and therefore

$$(\sigma(\mathfrak{y}_i))' = \sigma\left(\frac{A_i}{C_iB_i}\right) = \frac{A_i}{C_iB_i}.$$

Since $\mathfrak{y}_i' = \frac{A_i}{C_iB_i}$, we obtain $\sigma(\mathfrak{y}_i) = \mathfrak{y}_i + c_{i\sigma}$ for some $c_{i\sigma} \in \mathbf{C}$. Clearly, for every $\sigma \in \mathbb{H}$, $\sigma(\mathfrak{s}) = \mathfrak{s} + c_\sigma$ where $c_\sigma := \mathfrak{s}(c_{1\sigma}, \cdots, c_{l\sigma}) \in \mathbf{C}$. □

Before we classify the differential subfields of a general J-I-E tower we will first work with a two step tower.

Theorem 3.12. *Let* $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l) \supset \mathbf{F}$ *be an extension by algebraically independent antiderivatives* $\mathfrak{x}_1, \cdots, \mathfrak{x}_l$ *of* $\mathbf{F}$. *Let* $\mathfrak{y}_1, \cdots, \mathfrak{y}_m$ *be J-I-E antiderivatives of* $\mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$. *Then every differential subfield of* $\mathbf{F}(\mathfrak{y}_1, \cdots, \mathfrak{y}_m, \mathfrak{x}_1, \cdots, \mathfrak{x}_m)$ *is of the form* $\mathbf{F}(\mathrm{S}, \mathrm{T})$, *where* S *and* T *are finite subsets of* $span_{\mathbf{C}}\{\mathfrak{y}_1, \cdots, \mathfrak{y}_m, \mathfrak{x}_1, \cdots, \mathfrak{x}_l\}$ *and* $span_{\mathbf{C}}\{\mathfrak{x}_1, \cdots, \mathfrak{x}_m\}$ *respectively.*

Proof. Let $\mathbf{E} := \mathbf{F}(\mathfrak{y}_1, \cdots, \mathfrak{y}_m, \mathfrak{x}_1, \cdots, \mathfrak{x}_l)$, $\mathbf{L} := \mathbf{F}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l)$ and $\mathbf{E} \supseteq \mathbf{K} \supseteq \mathbf{F}$ be an intermediate differential field. Note that $\mathbf{L}$ is an extension by antiderivatives of $\mathbf{F}$ and $\mathbf{L} \supseteq \mathbf{K} \cap \mathbf{L} \supseteq \mathbf{F}$ is an intermediate subfield. Thus there is a finite set $\mathrm{T} \subset span_{\mathbf{C}}\{\mathfrak{x}_1, \cdots, \mathfrak{x}_l\}$, algebraically independent over $\mathbf{F}$ such that $\mathbf{K} \cap \mathbf{L} = \mathbf{F}(\mathrm{T})$. Let $\overline{\mathrm{T}} \subset \{\mathfrak{x}_1, \cdots, \mathfrak{x}_l\}$ be a transcendence base of $\mathbf{L}$ over $\mathbf{F}(\mathrm{T})$. We observe that $\mathbf{F}(\overline{\mathrm{T}}, \mathrm{T}) = \mathbf{L}$, $|T| + |\overline{T}| = l$, and $\overline{\mathrm{T}}$ is algebraically independent over $\mathbf{K}$; otherwise,

$\overline{\mathrm{T}}$ becomes algebraically dependent over $\mathbf{K} \cap \mathbf{L} = \mathbf{F}(\mathrm{T})$ which contradicts the choice of $\overline{\mathrm{T}}$.

Thus $\mathbf{K}(\mathfrak{x}_1, \cdots, \mathfrak{x}_l) = \mathbf{K}(\overline{\mathrm{T}})$. We observe that $\mathbf{E} \supseteq \mathbf{K}(\overline{\mathrm{T}}) \supset \mathbf{L}$ and that $\mathbf{E}$ is a (Picard-Vessiot) extension by antiderivatives of $\mathbf{L}$. Thus there is a finite set $\mathrm{S}^\sharp \subset span_{\mathbf{C}}\{\mathfrak{y}_1, \cdots, \mathfrak{y}_m\}$ such that $\mathbf{K}(\overline{\mathrm{T}}) = \mathbf{L}(\mathrm{S}^\sharp)$. We may also assume that $\mathrm{S}^\sharp$ is algebraically independent over $\mathbf{L}$. Since $\mathbf{K}(\overline{\mathrm{T}})$ is a (Picard-Vessiot) extension by antiderivatives of $\mathbf{K}$, for every $\mathfrak{s} \in \mathrm{S}^\sharp$ and $\rho \in \mathbb{G} := \mathbb{G}(\mathbf{K}(\overline{\mathrm{T}})|\mathbf{K})$, the element $\rho(\mathfrak{s}) \in \mathbf{K}(\overline{\mathrm{T}})$. Thus $\rho(\mathfrak{s}) \in \mathbf{E}$ for every $\rho \in \mathbb{G}$ and for every $\mathfrak{s} \in \mathrm{S}^\sharp$.

We have

$$\begin{array}{ccc} \mathbf{L} & \longrightarrow & \mathbf{K}(\overline{\mathrm{T}}) \\ \downarrow & & \downarrow \\ \mathbf{F} & \longrightarrow & \mathbf{K} \end{array}$$

where are arrows are inclusions. Thus there is a natural injective map $\phi : \mathbb{G}(\mathbf{K}(\overline{\mathrm{T}})|K) \rightarrow \mathbb{G}(\mathbf{L}|\mathbf{F})$ of algebraic groups such that $\rho(\mathfrak{x}_i) = \phi(\rho)(\mathfrak{x}_i)$ for all $\rho \in \mathbb{G}(\mathbf{K}(\overline{\mathrm{T}})|K)$, and there is an algebraic subgroup $\mathbb{H}$ of $\mathbb{G}(\mathbf{L}|\mathbf{F})$ such that the image $\phi(\mathbb{G}(\mathbf{K}(\overline{\mathrm{T}})|K)) = \mathbb{H}$. Note that the action of ρ on $\mathfrak{x}_i$ completely determines ρ for all $\rho \in \mathbb{G}(\mathbf{K}(\overline{\mathrm{T}})|K)$.

Thus $\sigma(\mathfrak{s}) \in \mathbf{E}$ for every $\sigma \in \mathbb{H}$ and for every $\mathfrak{s} \in \mathrm{S}^\sharp$. Now from theorem 3.11 we obtain $\sigma(\mathfrak{s}) = \mathfrak{s} + c_\sigma$ for all $\sigma \in \mathbb{H}$, $c_\sigma \in \mathbf{C}$. Thus $\mathfrak{s}' \in \mathbf{L}^{\mathbb{H}}$ and in particular $\sigma(\mathfrak{s}') = \mathfrak{s}'$ for all $\sigma \in \mathbb{H}$. Since $\rho(\mathfrak{x}_i) = \phi(\rho)(\mathfrak{x}_i)$ for all $\rho \in \mathbb{G}$ and ϕ is surjective, $\rho(\mathfrak{s}') = \mathfrak{s}'$ for every $\rho \in \mathbb{G}$ and therefore $\mathfrak{s}' \in \mathbf{K}^{\mathbb{G}} = \mathbf{K}$. Then $\mathfrak{s} \in \mathbf{K}(\overline{\mathrm{T}})$ is an antiderivative of $\mathbf{F}$ and therefore the set $\overline{\mathrm{T}} \cup \{\mathfrak{s}\}$ has to be algebraically dependent over $\mathbf{K}$. From The Kolchin-Ostrowski theorem, there is an element $\mathfrak{t}_\mathfrak{s} \in span_{\mathbf{C}}\overline{\mathrm{T}}$ such that $\mathfrak{s} + \mathfrak{t}_\mathfrak{s} \in \mathbf{K}$. We also observe that $\mathfrak{s}' \in \mathbf{K}$ and $\mathfrak{s}' \in \mathbf{L}$ and therefore $\mathfrak{s}' \in \mathbf{F}(\mathrm{T})$. Now we let $\mathrm{S} := \{\mathfrak{s} + \mathfrak{t}_\mathfrak{s} | \mathfrak{s} \in \mathrm{S}^\sharp\}$ and observe that $\mathbf{K} \supset \mathbf{F}(\mathrm{S}, \mathrm{T}) \supset \mathbf{F}(\mathrm{T}) \supset \mathbf{F}$. Let $\overline{\mathrm{S}} \subset \{\mathfrak{y}_1, \cdots, \mathfrak{y}_m\}$ be a transcendence base of $\mathbf{E}$

over $\mathbf{K}(\overline{\mathrm{T}}) = \mathbf{L}(\mathrm{S}^\sharp)$. Then $|\overline{\mathrm{S}}| + |\mathrm{S}^\sharp| = m$ and in particular $\mathbf{L}(\overline{\mathrm{S}} \cup \mathrm{S}^\sharp) = \mathbf{E}$.
We know that

$$\text{tr.d } \mathbf{E}|\mathbf{F} = \text{tr.d } \mathbf{E}|\mathbf{K} + \text{tr.d } \mathbf{K}|\mathbf{F}(S,T) + \text{tr.d } \mathbf{F}(S,T)|\mathbf{F} \tag{3.17}$$

$\text{tr.d}\mathbf{E}|\mathbf{K} = |\overline{\mathrm{S}}| + |\overline{\mathrm{T}}|$ and $\text{tr.d } \mathbf{F}(\mathrm{S},\mathrm{T})|\mathbf{F} = |\mathrm{S}| + |\mathrm{T}|$. Note that $|\mathrm{S}| = |\mathrm{S}^\sharp|$ and that $|\overline{\mathrm{S}}| + |\overline{\mathrm{T}}| + |\mathrm{S}^\sharp| + |\mathrm{T}| = \text{tr.d } \mathbf{E}|\mathbf{F} = l + m$. Thus $\text{tr.d } \mathbf{E}|\mathbf{F} = |\overline{\mathrm{S}}| + |\overline{\mathrm{T}}| + |\mathrm{S}| + |\mathrm{T}| = \text{tr.d } \mathbf{E}|\mathbf{K} + \text{tr.d } \mathbf{F}(S,T)|\mathbf{F}$ and therefore from equation 3.17 we obtain $\text{tr.d } \mathbf{K}|\mathbf{F}(S,T) = 0$. Thus $\mathbf{K}$ is algebraic over $\mathbf{F}(\mathrm{S},\mathrm{T})$. Now letting $\mathbf{M} := \mathbf{F}(\mathrm{S},\mathrm{T})$ and applying theorem 3.12, we obtain $\mathbf{K} = \mathbf{F}(\mathrm{S},\mathrm{T})$. □

Theorem 3.13. *If there is an* $\mathfrak{s} = \sum_{j=1}^{n_t} \alpha_{tj}\mathfrak{y}_{tj} + \sum_{i=1}^{t-1}\sum_{j=1}^{n_i} \alpha_{ij}\mathfrak{y}_{ij} \in \Lambda_t \setminus \Lambda_{t-1}$ *for some* $1 \le t \le k$ *and a subgroup* $\mathbf{H}$ *of* $\mathbb{G}(\mathbf{F}_\infty|\mathbf{F})$ *such that for every* $\sigma \in \mathbf{H}$, $\sigma(\mathfrak{s}) \in \mathbf{E}_k =: \mathbf{E}$ *then* $\sigma(\mathfrak{y}_{ij}) = \mathfrak{y}_{ij} + c_{ij\sigma}$ *for every* $\sigma \in \mathbb{H}$ *provided the coefficient* α_{ij} *of* $\mathfrak{y}_{ij}$ *in* $\mathfrak{s}$ *is nonzero.*

Proof. We will use an induction on t to prove this theorem.
$t = 1$: Then $\mathfrak{s}$ is a linear combination of antiderivatives $\mathfrak{y}_{11}, \cdots, \mathfrak{y}_{1n_1}$ of $\mathbf{F}$. Therefore for every $\sigma \in \mathbb{G}(\mathbf{F}_\infty|\mathbf{F})$ we have

$$\mathfrak{y}'_{1j} = \sigma(\mathfrak{y}'_{1j}) = \sigma(\mathfrak{y}_{1j})'.$$

Since $\mathbf{F}_\infty$ and $\mathbf{F}$ has the same field of constants, there is a $c_{1j\sigma} \in \mathbf{C}$ such that $\sigma(\mathfrak{y}_{1j}) = \mathfrak{y}_{1j} + c_{1j\sigma}$.
Assume that our theorem is true for $t-1$.
$t \ge 2$: For

$$\mathfrak{s} = \sum_{j=1}^{n_t} c_{tj}\mathfrak{y}_{tj} + \sum_{i=1}^{t-1}\sum_{j=1}^{n_i} c_{ij}\mathfrak{y}_{ij},$$

where $\alpha_{tj} \neq 0$ for some j, suppose that $\sigma(\mathfrak{s}) \in \mathbf{E}$. Then

$$\sigma(\mathfrak{s}) = \sum_{j=1}^{n_t} c_{tj}\sigma(\mathfrak{y}_{tj}) + \sum_{i=1}^{t-1}\sum_{j=1}^{n_i} c_{ij}\sigma(\mathfrak{y}_{ij}) \in \mathbf{E} \tag{3.18}$$

$$\implies \sum_{j=1}^{n_t} c_{tj}\sigma(\mathfrak{y}_{tj}) \in \mathbf{E}(\sigma(\mathbf{E}_{t-1})); \text{ since } \sum_{i=1}^{t-1}\sum_{j=1}^{n_i} c_{ij}\sigma(\mathfrak{y}_{ij}) \in \sigma(\mathbf{E}_{t-1}) \tag{3.19}$$

Suppose that for $i \geq t+1$, $\sigma(\mathfrak{s}) \in \mathbf{E}_i(\sigma(\mathbf{E}_{t-1}))$. Then note that $\mathbf{E}_i(\sigma(\mathbf{E}_{t-1}))$ is an extension by algebraically independent antiderivatives $\mathfrak{y}_{i1}, \cdots, \mathfrak{y}_{in_i}$ of $\mathbf{E}_{i-1}(\sigma(\mathbf{E}_{t-1}))$. Also note that $\sigma(\mathfrak{s})$ is an antiderivative of $\sigma(\mathbf{E}_{t-1})$ and therefore an antiderivative of $\mathbf{E}_i(\sigma(\mathbf{E}_{t-1}))$. Thus there are constants $\alpha_{i0}, \alpha_{ij} \in \mathbf{C}$, $1 \leq j \leq n_i$ not all zero such that

$$\alpha_{i0}\sigma(\mathfrak{s}) + \sum_{j=1}^{n_i} \alpha_{ij}\mathfrak{y}_{ij} \in \mathbf{E}_{i-1}(\sigma(\mathbf{E}_{t-1})).$$

But, if $\alpha_{ij} \neq 0$ for some $1 \leq j \leq n_i$ then from the above equation and from the facts that $\sigma(\mathfrak{s}) \in \sigma(\mathbf{E}_t)$ and $\sigma(\mathbf{E}_{t-1}) \subset \sigma(\mathbf{E}_t)$ we have

$$\sum_{j=1}^{n_i} \alpha_{ij}\mathfrak{y}_{ij} \in \mathbf{E}_{i-1}(\sigma(\mathbf{E}_t))$$

and since $t \leq i-1$, $\mathbf{E}_t \subset \mathbf{E}_{i-1}$, which implies

$$\sum_{j=1}^{n_i} \alpha_{ij}\mathfrak{y}_{ij} \in \mathbf{E}_{i-1}(\sigma(\mathbf{E}_{i-1})),$$

a contradiction to theorem...

Thus $\sigma(\mathfrak{s}) \in \mathbf{E}(\sigma(\mathbf{E}_{t-1}))$ implies $\sigma(\mathfrak{s}) \in \mathbf{E}_t(\sigma(\mathbf{E}_{t-1}))$. Let $\mathbf{M} := \mathbf{E}_{t-2}(\sigma(\mathbf{E}_{t-2}))$. We know that $I_{t-1} = \{\mathfrak{y}_{t-1,1}, \cdots, \mathfrak{y}_{t-1,n_{t-1}}\}$ is algebraically independent over $\mathbf{M}$. Now let $I^{\sigma}_{t-1} \subset \sigma(I_{t-1})$ be a transcendence base of $\mathbf{E}_{t-1}(\sigma(\mathbf{E}_{t-1}))$ over $\mathbf{M}(I_{t-1})$. Then $\mathbf{M}(I_{t-1}, I^{\sigma}_{t-1}) = \mathbf{E}_{t-1}(\sigma(\mathbf{E}_{t-1}))$ and $\mathbf{E}_t(\sigma(\mathbf{E}_{t-1})) = \mathbf{M}(I_t, I_{t-1}, I^{\sigma}_{t-1})$. Thus we have the following tower of antiderivatives

$$\mathbf{M}(I_t, I_{t-1}, I^{\sigma}_{t-1}) \supset \mathbf{M}(I_{t-1}, I^{\sigma}_{t-1}) \supset \mathbf{M}.$$

We also know that I_t consists of J-I-E antiderivatives of $\mathbf{M}(I_{t-1}, I^{\sigma}_{t-1})$. Now applying lemma 3.11 we obtain that $\sigma(\mathfrak{y}_{tj}) = \mathfrak{y}_{tj} + c_{tj\sigma}$ for every $\sigma \in \mathbb{H}$. Also note that $\sigma(\mathfrak{y}_{tj}) = \mathfrak{y}_{tj} + c_{tj\sigma}$ implies

$$\sigma(\sum_{j=1}^{n_t} \alpha_{tj}\mathfrak{y}_{tj}) = \sum_{j=1}^{n_t} \alpha_{tj}\mathfrak{y}_{tj} + \sum_{j=1}^{n_t} \alpha_{tj}c_{tj\sigma}.$$

Thus

$$\begin{aligned}
&\sigma(\mathfrak{s}) \in \mathbf{E} \\
\Longrightarrow\ &\sigma(\mathfrak{s}) - \sum_{j=1}^{n_t} \alpha_{tj}\mathfrak{y}_{tj} \in \mathbf{E} \\
\Longrightarrow\ &\sum_{j=1}^{n_t} \alpha_{tj}c_{tj\sigma} + \sigma(\sum_{i=1}^{t-1}\sum_{j=1}^{n_i} \alpha_{ij}\mathfrak{y}_{ij}) \in \mathbf{E} \\
\Longrightarrow\ &\sigma(\sum_{i=1}^{t-1}\sum_{j=1}^{n_i} \alpha_{ij}\mathfrak{y}_{ij}) \in \mathbf{E}.
\end{aligned}$$

Now we apply our induction hypothesis to the sum $\sum_{i=1}^{t-1}\sum_{j=1}^{n_i} \alpha_{ij}\mathfrak{y}_{ij}$ to prove our theorem.

Corollary 3.14. *The group of differential automorphisms of* $\mathbf{E}$ *over* $\mathbf{F}$ *is a subgroup of* $(\mathbf{C}, +)^n$, *where* $n = tr.d(\mathbf{E}|\mathbf{F})$.

From theorem we observe that if $\sigma(\mathfrak{y}_{ij}) \in \mathbf{E}$ then $\sigma(\mathfrak{y}_{ij}) = \mathfrak{y}_{ij} + c_{ij\sigma}$ for some $c_{ij\sigma} \in \mathbf{C}$. Thus $\mathbb{G}(\mathbf{E}|\mathbf{F})$ is a subgroup of $(\mathbf{C}, +)^n$. □

Now we will prove a generalization of the Ostrowski theorem for a tower of extensions by J-I-E antiderivatives.

Theorem 3.15. *Generalized Ostrowski Theorem*

Let $\mathbf{E}_k \supset \mathbf{K} \supset \mathbf{F}$ be an intermediate differential field and let $T_i \subseteq I_i$ be subsets such that T_i is a set of antiderivatives of $\mathbf{K}(\cup_{j=1}^{i-1} T_j)$ for each $1 \leq i \leq k$. If $\cup_{j=1}^{k} T_j$ is algebraically dependent over $\mathbf{K}$ then there is a nonzero element in $\mathbf{K} \cap \Lambda_k$.

Proof. Suppose that $\cup_{j=1}^{k} T_i$ is algebraically dependent over $\mathbf{K}$. Then there is a t such that T_t is algebraically dependent over $\mathbf{K}(\cup_{j=1}^{t-1} T_j)$. Then by The Kolchin-Ostrowski theorem, there is a non zero $\mathfrak{t}_t \in \mathbf{K}(\cup_{j=1}^{t-1} T_j) \cap \Lambda_t$. Let $\mathbb{H}_{t-1}$ be the group of all differential automorphisms of $\mathbf{K}(\cup_{j=1}^{t-2} T_j)(T_{t-1})$ over $\mathbf{K}(\cup_{j=1}^{t-2} T_j)$. Note that for every $\sigma \in \mathbb{H}_{t-1}$, $\sigma(y) \in \mathbf{K}(\cup_{j=1}^{t-2} T_j)(T_{t-1}) \subseteq \mathbf{E}_k$ for every $y \in \mathbf{K}(\cup_{j=1}^{t-2} T_j)(T_{t-1})$ and that there is a subgroup of $\mathbb{G}_\infty$ whose restriction upon the field $\mathbf{K}(\cup_{j=1}^{t-2} T_j)(T_{t-1})$ is the group $\mathbb{H}_{t-1}$, see [9]. Thus we may apply theorem 3.13 and obtain that $\sigma(\mathfrak{t}_t) = \mathfrak{t}_t + \alpha_{\mathfrak{t}_t \sigma}$ for some $\alpha_{\mathfrak{t}_t \sigma} \in \mathbf{C}$. This shows us that $\mathfrak{t}_t \in \mathbf{K}(\cup_{j=1}^{t-2} T_j)(T_{t-1})$ is an antiderivative of $\mathbf{K}(\cup_{j=1}^{t-2} T_j)$ and therefore the set $\{\mathfrak{t}_t\} \cup T_{t-1}$ is algebraically dependent over $\mathbf{K}(\cup_{j=1}^{t-2} T_j)$; observe that $\mathfrak{t}_t \notin \cup_{j=1}^{t-1} T_j$. Again by the Kolchin-ostrowski theorem there is a $\mathfrak{t}_{t-1} \in \Lambda_{t-1}$ and a constant $c_{\mathfrak{t}_t, t-1}$, where $\mathfrak{t}_{t-1}$ or $c_{\mathfrak{t}_t, t-1}$ is nonzero such that

$$c_{\mathfrak{t}_t, t-1} \mathfrak{t}_t + \mathfrak{t}_{t-1} \in \mathbf{K}(\cup_{j=1}^{t-2} T_j) \cap \Lambda_{t-1}.$$

Now a repeated application of thereom 3.13 and the Kolchin-Ostrowski theorem will prove the existence of a nonzero element in $\mathbf{K} \cap \Lambda_t$. $\square$

Theorem 3.16. *For every differential subfield $\mathbf{K}$ of $\mathbf{E} := \mathbf{E}_k$, the field generated* *by* $\mathbf{F}$ *and* $S_k := \mathbf{K} \cap \Lambda_k$ *equals the differential field* $\mathbf{K}$*. That is*

$$\mathbf{K} = \mathbf{F}(S_k).$$

Moreover $\mathbf{K}$ *itself is a tower of extensions by antiderivatives, namely*

$$\mathbf{K} = \mathbf{F}(S_k) \supset \mathbf{F}(S_{k-1}) \supset \mathbf{F}(S_{k-2}) \supset \cdots \supset \mathbf{F}(S_1) \supset \mathbf{F},$$

where $S_i := S_k \cap \Lambda_i$*.*

Proof. We will use an induction on k to prove this theorem. $k = 1$: Here $\mathbf{E} := \mathbf{E}_1$ is an extension by antiderivatives of $\mathbf{F}$ and therefore from theorem 2.8 our desired result follows immediately

$k \geq 2$: Assume that for any differential subfield of $\mathbf{E}_{k-1}$ our theorem is true. Let $S_i := \mathbf{K} \cap \Lambda_i$ and note that $S_i \supset S_{i-1}$ and the following containments

$$\mathbf{E} \supseteq \mathbf{K} \supseteq \mathbf{F}(S_k) \supseteq \mathbf{F}. \tag{3.20}$$

We will first show that $\mathbf{F}(S)$ is a differential field. Applying our induction hypothesis to the differential field $\langle \mathbf{F}(S_i') \rangle \subseteq \mathbf{E}_{k-1}$, where $S_i' = \{\mathfrak{y}' | \mathfrak{y} \in S_i\}$ we obtain that $\langle \mathbf{F}(S_i') \rangle = \mathbf{F}(T)$, where $T = \langle \mathbf{F}(S_i') \rangle \cap \Lambda_{i-1}$. Also note that $\langle \mathbf{F}(S_i') \rangle \subseteq \mathbf{K}$ and therefore

$$T = \langle \mathbf{F}(S_i') \rangle \cap \Lambda_{i-1} \subseteq K \cap \Lambda_{i-1} \subseteq K \cap \Lambda_i = S_i.$$

Thus $\mathbf{F}(S_i) \supseteq \mathbf{F}(T)$ and since $S_i' \subset \mathbf{F}(T)$ and $\mathbf{F}(T)$ is a differential field, $\mathbf{F}(S_i)$ is also a differential field. Hence $\mathbf{F}(S_k)$ is a differential field and

$$\mathbf{F}(S_k) \supset \mathbf{F}(S_{k-1}) \supset \mathbf{F}(S_{k-2}) \supset \cdots \supset \mathbf{F}(S_1) \supset \mathbf{F},$$

is a tower of extension by antiderivatives.

Let $\bar{S}_i \subset I_i$ be a transcendence base of $\mathbf{E}_i$ over the differential field $\mathbf{E}_{i-1}(S_i)$. Since $\mathbf{E}_i$ is purely transcendental over $\mathbf{E}_{i-1}$ it is also purely transcendental over $\mathbf{E}_{i-1}(\bar{S}_i)$ too and therefore $\mathbf{E}_{i-1}(S_i, \bar{S}_i) = \mathbf{E}_i$. We note that $\mathbf{F}(S_1, \bar{S}_1) = \mathbf{E}_1$, $\mathbf{F}(S_2, \bar{S}_1, \bar{S}_2) = \mathbf{E}_1(S_2, \bar{S}_2) = \mathbf{E}_2$ and in general we have $\mathbf{F}(S_t)(\cup_{i=1}^t \bar{S}_i) = \mathbf{E}_t$. Since $\mathbf{K} \supseteq \mathbf{F}(S_k)$ we have

$$\mathbf{E} = \mathbf{K}(\cup_{i=1}^k \bar{S}_i) \supseteq \mathbf{K}(\cup_{i=1}^{k-1} \bar{S}_i) \supseteq \cdots \supseteq \mathbf{K}(\bar{S}_2, \bar{S}_1) \supseteq \mathbf{K}(\bar{S}_1) \supseteq \mathbf{K} \supseteq \mathbf{F}(S_k) \supseteq \mathbf{F} \tag{3.21}$$

We know that $\cup_{i=1}^t \bar{S}_i$ is algebraically independent over $\mathbf{F}(S_t)$. Since $S_t = \mathbf{K} \cap \Lambda_k$ we obtain from theorem 3.15 that $\cup_{i=1}^t \bar{S}_i$ is algebraically independent over $\mathbf{K}$. Now from equation 3.21 we obtain

$$tr.d(\mathbf{E}|\mathbf{F}) = \sum_{i=1}^{k} |\bar{S}_i| + tr.d(\mathbf{K}|\mathbf{F}(S_k)) + tr.d(\mathbf{F}(S_k)|\mathbf{F}). \tag{3.22}$$

On the other hand we have

$$\mathbf{E} = \mathbf{F}(S_k)(\cup_{i=1}^k \bar{S}_i) \supseteq \mathbf{F}(S_k) \supseteq \mathbf{F}$$

and thus

$$tr.d(\mathbf{E}|\mathbf{F}) = \sum_{i=1}^{k} |\bar{S}_i| + tr.d(\mathbf{F}(S_k)|\mathbf{F}) \tag{3.23}$$

From equation 3.22 and 3.23 we obtain tr.d $(\mathbf{K}|\mathbf{F}(S_k)) = 0$, that is, $\mathbf{K}$ is algebraic over $\mathbf{F}(S_k)$. Now from theorem 3.1 we obtain $\mathbf{K} = \mathbf{F}(S_k)$. □

3.2.2 Example

Let $\mathbf{C} := \mathbb{C}$ denote the complex numbers, $\mathbf{C}_\infty$ the complete Picard-Vessiot closure of $\mathbf{C}$, $x \in \mathbf{C}_\infty$ be an element whose derivative is 1, $\tan^{-1} x \in \mathbf{C}_\infty$ be an element such that

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

and let $\tan^{-1}(\tan^{-1} x) \in \mathbf{C}_\infty$ be an element such that

$$\big(\tan^{-1}(\tan^{-1} x)\big)' = \frac{1}{(1+(\tan^{-1} x)^2)(1+x^2)}.$$

We will use theorem 3.16 to compute the differential field $\mathbf{M} := \mathbf{C}\langle \tan^{-1}(\tan^{-1}(x)$ $)\rangle$. First we observe that $(\tan^{-1}(x))' = \frac{1}{1+x^2} = \frac{1}{(x+i)(x-i)}$ and thus $\tan^{-1} x$ is an I-E (J-I-E) antiderivative of $\mathbf{C}(x)$. We also observe that $\tan^{-1}(\tan^{-1}(x))$ is an I-E (J-I-E) antiderivative of $\mathbf{C}(x, \tan^{-1}(x))$ (note that $(\tan^{-1}(\tan^{-1}(x)))' = \frac{1}{(1+(\tan^{-1} x)^2)(1+x^2)}$). Thus $x, \tan^{-1}(x), \tan^{-1}(\tan^{-1}(x))$ are algebraically independent over $\mathbf{C}$. Also from theorem 3.16 we see that there should be a linear combination of the form $c_1 \tan^{-1} x + c_2 x$, where c_1 is non zero (since $\frac{1}{(1+(\tan^{-1} x)^2)(1+x^2)}$ $\in \mathbf{M}$). Thus by differentiating $c_1 \tan^{-1} x + c_2 x$, we see that $\mathbf{M} \cap \mathbf{C}(x) \supsetneq \mathbf{C}$ and thus again by theorem 3.16, $x \in \mathbf{M}$. Therefore $\tan^{-1}(x) \in \mathbf{M}$ since $c_1 \tan^{-1} x + c_2 x \in \mathbf{M}$. Hence

$$\mathbf{M} := \mathbf{C}\langle \tan^{-1}(\tan^{-1} x)\rangle = \mathbf{C}(\tan^{-1}(\tan^{-1} x), \tan^{-1} x, x).$$

We observe that

$$\left(\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i)\right)' = \frac{1}{x^2+1}$$

and since $(\tan^{-1} x)' = \frac{1}{x^2+1}$ there is a $c \in \mathbf{C}$ such that

$$\tan^{-1} x = \frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c.$$

Also note that

$$\begin{aligned}
&\frac{1}{2i}\big(\ln(\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c - i) - (\ln(\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c + i)\big)' \\
&= \frac{1}{x^2+1}\left(\frac{1}{\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c - i} - \frac{1}{\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c + i}\right) \\
&= \frac{1}{(1+(\tan^{-1} x)^2)(x^2+1)}
\end{aligned}$$

and since $(\tan^{-1}(\tan^{-1} x))' = \dfrac{1}{(1+(\tan^{-1} x)^2)(x^2+1)}$, there is a constant $d \in \mathbf{C}$ such that

$$\begin{aligned}
\tan^{-1}(\tan^{-1}(x)) = &\frac{1}{2i}\big(\ln(\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c - i) \\
&- (\ln(\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c + i)\big) + d.
\end{aligned}$$

Hence

$$\tan^{-1}(\tan^{-1} x) \in \mathbf{C}(x, y_1, y_2, z),$$

where

$$\begin{aligned}
z :=&\frac{1}{2i}\Big(\ln(\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c - i) \\
&- \ln(\frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i) + c + i)\Big) \\
y_1 :=&\ln(x-i) \\
y_2 :=&\ln(x+i).
\end{aligned}$$

Clearly

$$\mathbf{C}\langle \tan^{-1}(\tan^{-1} x)\rangle = \mathbf{C}(z, \frac{1}{2i}(y_1 - y_2), x).$$

Remark 3.17. The J-I-E extensions may have non-elementary functions. For example; if $a_i \in \mathbf{C}$ are distinct constants for $i = 1, \cdots, n$ then the elements $\mathfrak{y}_i := \int \frac{1}{\ln(x-a_i)}$ are J-I-E antiderivatives of the differential field $\mathbf{C}\big(x, \ln(x - a_1), \cdots, \ln(x - a_n)\big)$ with $\mathfrak{y}_i' := \frac{A_i}{C_i B_i}$ where $A_i := 1$, $B_i := 1$ and $C_i := \ln(x - a_i)$. These $\mathfrak{y}_i$'s are non-elementary functions, see [3]. From theorem 3.6 we see that these $\mathfrak{y}_i$'s are algebraically independent over $\mathbf{C}\big(x, \ln(x-a_1), \cdots, \ln(x-a_n)\big)$ and from theorem 3.16 we see that any differential field $\mathbf{K}$ such that $\mathbf{C}(x, \ln(x - a_i), \mathfrak{y}_i | 1 \leq i \leq n) \supseteq \mathbf{K} \supseteq \mathbf{C}$ is of the form $\mathbf{C}(S)$, where $S \subset \text{span}_{\mathbf{C}}\{x, \ln(x - a_i), \mathfrak{y}_i | 1 \leq i \leq n\}$ is a finite set. Moreover $\mathbf{C}(S)$ itself is a tower of (Picard-Vessiot) extensions by antiderivatives.

Chapter 4

Extensions by Iterated Logarithms

In this chapter we will provide an example of a J-I-E tower namely, the extensions by iterated logarithms. Though many of the results for iterated logarithms setting can be deduced from the J-I-E tower setting from section 3.2, we will still prove those results here separately and this will help us in writing an algorithm for computing the finitely differentially generated subfields of the extensions by iterated logarithms.

4.1 Iterated Logarithms

Let $\mathbf{C}$ be an algebraically closed-characteristic zero differential field with a trivial derivation and let$\mathbf{C}_\infty$ be the complete Picard-Vessiot closure of $\mathbf{C}$. Let $\mathfrak{l}[0,0] \in \mathbf{C}_\infty$ be an element such that $\mathfrak{l}'[0,0] = 1$. We will often denote $\mathfrak{l}[0,0]$ by x. Given

$\vec{c} = (c_1, \cdots, c_n) \in \mathbf{C}^n$ let $\mathfrak{l}[\vec{c}, n] \in \mathbf{C}_\infty$ be an element such that

$$\mathfrak{l}'[\vec{c}, n] = \frac{\mathfrak{l}'[\pi(\vec{c}), n-1]}{\mathfrak{l}[\pi(\vec{c}), n-1] + \psi_n(\vec{c})}, \tag{4.1}$$

where $\psi_n : \mathbf{C}^n \to \mathbf{C}$ is the map $\psi_n(c_1, \cdots, c_n) = c_n$ and $\pi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ is the map

$$\begin{cases} \pi(c_1, \cdots, c_n) = (c_1, \cdots, c_{n-1}), & \text{when } n > 1; \\ \pi(c) = 0, & \text{when } n = 1. \end{cases}$$

Whenever we write $\mathfrak{l}[\vec{c}, n]$, it is understood that $\vec{c} \in \mathbf{C}^n$. We observe that for $\vec{c} = (c) \in \mathbf{C}$

$$\begin{aligned} \mathfrak{l}'[\vec{c}, 1] &= \frac{\mathfrak{l}'[\pi(\vec{c}), 0]}{\mathfrak{l}[\pi(\vec{c}), 0] + \psi_1(\vec{c})} \\ &= \frac{\mathfrak{l}'[0, 0]}{\mathfrak{l}[0, 0] + c} \\ &= \frac{1}{\mathfrak{l}[0, 0] + c}. \end{aligned}$$

Thus for $c \in \mathbf{C}$, the element $\mathfrak{l}[\vec{c}, 1]$ can be seen as the element $\ln(x + c)$. Similarly for $\vec{c} = (c_1, \cdots, c_n) \in \mathbf{C}^n$, the element $\mathfrak{l}[\vec{c}, n]$ can be seen as the element $\ln(\ln(\cdots(\ln(x + c_1) + c_2) \cdots + c_{n-1}) + c_n)$.

For $1 \leq k \leq n-1$, let $\pi^k : \mathbf{C}^n \to \mathbf{C}^{n-k}$ be the map $\pi^k(c_1, \cdots, c_n) = (c_1, \cdots, c_{n-k})$ and let $\pi^n : \mathbf{C}^n \to \mathbf{C}^0 := \{0\}$ be the zero map. For $1 \leq k \leq n$ let $\psi_k : \mathbf{C}^n \to \mathbf{C}$ be the map $\psi_k(c_1, \cdots, c_n) = c_k$. Under these notations, we can rewrite equation 4.1 as

$$\mathfrak{l}'[\vec{c}, n] = \Big(\prod_{i=1}^{n-1} \frac{1}{\mathfrak{l}[\pi^{i+1}(\vec{c}), n-(i+1)] + \psi_{n-i}(\pi^i(\vec{c}))}\Big) \frac{1}{\mathfrak{l}[\pi(\vec{c}), n-1] + \psi_n(\vec{c})}. \tag{4.2}$$

This above equation is obtained simply by clearing the derivative that appears in the numerator of the RHS of the equation 4.1. Note that

$$\mathfrak{l}'[\pi(\vec{c}), n-1] = \prod_{i=1}^{n-1} \frac{1}{\mathfrak{l}[\pi^{i+1}(\vec{c}), n-(i+1)] + \psi_{n-i}(\pi^i(\vec{c}))}. \tag{4.3}$$

Definition 4.1. When $n \in \mathbb{N}$ we will call $\mathfrak{l}[\vec{c}, n]$ an n^{th} *level iterated logarithm* or simply an *iterated logarithm*, without specifying its level.

We note that $\mathfrak{l}[0, 0]$, whose derivative equals 1, is not an iterated logarithm under our definition. Hereafter we will call $\mathfrak{l}[0, 0]$ as x.

More notations

Let $\Lambda_0 := \{x\}$, $\Lambda_n := \{\mathfrak{l}[\vec{c}, n] | \vec{c} \in \mathbf{C}^n\}$ and $\Lambda_\infty = \cup_{i=0}^{\infty} \Lambda_i$ and let $\mathfrak{L}_0 = \mathbf{C}(\Lambda_0)$, $\mathfrak{L}_n := \mathbf{C}(\cup_{i=0}^{n} \Lambda_i)$ and $\mathfrak{L}_\infty = \mathbf{C}(\Lambda_\infty)$. Note that $\mathfrak{L}_0$, $\mathfrak{L}_n$ and $\mathfrak{L}_\infty$ are differential fields(follows from equation 4.9).

Let $\vec{c} \in \mathbf{C}^n$. We define $\pi^k(\mathfrak{l}[\vec{c}, n] := \mathfrak{l}[\pi^k(\vec{c}), n-k]$ whenever $k \leq n$. Note that $\pi^n(\mathfrak{l}[\vec{c}, n]) = \mathfrak{l}[0, 0] = x$. When $k > n$ we define $\pi^k(\mathfrak{l}[\vec{c}, n]) := x$ and $\pi^k(x) := x$ for any $k \in \mathbb{N}$. Now we may also define $\pi^k(S)$ for a non empty set $S \subset \Lambda_\infty$ as $\pi^k(S) = \{\pi^k(y) | y \in S\}$. Thus

$$\text{if } y \in \Lambda_n, \text{ then } \pi(y) \in \Lambda_{n-1}, \ \pi^2(y) \in \Lambda_{n-2}, \ \cdots, \ \pi^n(y) = x \in \Lambda_0. \tag{4.4}$$

We also see that if $E \subset \Lambda_\infty$ is a finite set, then there is an $n \in \mathbb{N}$ such that $\pi^n(E) = \{x\}$. Given a nonempty set $E \subset \Lambda_\infty$ it is not necessary that $\mathbf{C}(E)$ is a differential field. For example $\mathbf{C}(\mathfrak{l}[\vec{0}, 1])$, that is the field $\mathbf{C}(\ln(x))$ is not a differential field. whereas, $\mathbf{C}(\ln(x), x) = \mathbf{C}(\mathfrak{l}[\vec{0}, 1], \pi(\mathfrak{l}[\vec{0}, 1]) = x)$ is a differential field. (note that $x \notin \mathbf{C}(\ln(x))$; in fact x and $\ln(x)$ are algebraically independent over $\mathbf{C}$. We will later show that any collection of iterated logarithms is algebraically independent over $\mathbf{C}(x)$.) More in general we have the following propositions.

Proposition 4.2. *Let* $\mathfrak{l}[\vec{c}, n] \in \Lambda_\infty$ *be an iterated logarithm. Then*

$$\mathbf{C}(\mathfrak{l}[\vec{c}, n], \mathfrak{l}[\pi(\vec{c}), n-1], \mathfrak{l}[\pi^2(\vec{c}), n-2], \cdots, \mathfrak{l}[\pi^n(\vec{c}), 0] = x),$$

is a differential field

Proof. We will use an induction on n to prove our proposition.

$n = 1$. Note that $\mathfrak{l}'[c, 1] = \frac{1}{x+c}$ and $x' = 1$. Therefore $\mathbf{C}(\mathfrak{l}[c, 1], x)$ is also a differential field. We recall that if $\vec{v} \in \mathbf{C}^n$ then $\pi^n(\vec{v}) = 0$ and therefore $\mathfrak{l}[\pi^n(\vec{v}), n - n] = \mathfrak{l}[0, 0] = x$. Let us assume for any $\vec{v} \in \mathbf{C}^n$ that $\mathbf{C}(\mathfrak{l}[\vec{v}, n], \mathfrak{l}[\pi(\vec{v}), n-1], \cdots, x)$ is a differential field and let $\vec{c} \in \mathbf{C}^{n+1}$. From our induction hypothesis, we know that $\mathbf{F} := \mathbf{C}(\mathfrak{l}[\pi(\vec{c}), n-1], \mathfrak{l}[\pi^2(\vec{c}), n-2], \cdots, x]$ is a differential field since $\pi(\vec{c}) \in \mathbf{C}^{n-1}$. Thus

$$\mathfrak{l}'[\vec{c}, n] = \frac{\mathfrak{l}'[\pi(\vec{c}), n-1]}{\mathfrak{l}[\pi(\vec{c}), n-1] + \psi_n(\vec{c})} \in \mathbf{F}.$$

Hence $\mathbf{F}(\mathfrak{l}[\vec{c}, n]) = \mathbf{C}(\mathfrak{l}[\vec{c}, n], \mathfrak{l}[\pi(\vec{c}), n-1], \cdots, x)$ is a differential field. $\square$

Proposition 4.3. *Let* $E \subset \Lambda_\infty$ *be a finite set of iterated logarithms. Then*

$$\mathbf{C}(E, \pi(E), \pi^2(E), \cdots, x)$$

is a differential field

Proof. If $E = \emptyset$ then $\mathbf{C}(E, \pi(E), \pi^2(E), \cdots, x) = \mathbf{C}(x)$ which is a differential field and we are done. Let $E = \{y_j | 1 \leq j \leq s\}$. We know from proposition 4.2 that $\mathbf{K}_j := \mathbf{C}(y_j, \pi(y_j), \cdots, \pi^{n_j}(y_j) = x)$ is a differential field and since $\mathbf{C}(E, \pi(E), \pi^2(E), \cdots, x)$ is a compositum of differential fields $\mathbf{K}_j$, we see that $\mathbf{C}(E, \pi(E), \pi^2(E), \cdots, x)$ is also a differential field. $\square$

Definition 4.4. For $E \subset \Lambda_\infty$, we will call the field $\mathbf{C}(E)$ an *extension by iterated logarithms* if E contains at least one iterated logarithm, that is, if E has an element from Λ_∞ other than x. And, we will call the differential field $\mathbf{C}(E, \pi(E), \pi^2(E), \cdots, x)$ as the *Container Differential Field*[*CDF*] for the set E.

4.2 The Two Towers and a Structure Theorem for $\mathfrak{L}_n$

Let $E \subset \Lambda_\infty$ be a finite non empty set. Then there is a minimal $n \in \mathbb{N}$ such that $\pi^n(E) = \{x\}$. Once this minimal n is chosen, it is clear that E contains at least one element from Λ_n and no elements from Λ_i for any $i > n$. Hereafter we will use the symbol $\mathfrak{E}$ to denote $\cup_{i=0}^{n} \pi^i(E)$, where n satisfies the above minimality condition. Thus $\mathbf{C}(E, \pi(E), \pi^2(E), \cdots, x)$, the container differential field of E is the field $\mathbf{C}(\mathfrak{E})$. Note that $\pi(\mathfrak{E}) \subset \mathfrak{E}$ and let $T_i := \Lambda_i \cap \mathfrak{E}$ for all $1 \leq i \leq n$. Then the T_i's are disjoint and partitions $\mathfrak{E}$ in such a way that each T_i contains iterated logarithms only from level i. Clearly $E \subseteq \mathfrak{E}$, and $\mathfrak{E}$ may contain more elements than E, but those elements that are in $\mathfrak{E}$ but not in E has to come from $\cup_{i=0}^{n-1} \Lambda_i$. Thus $T_n := \Lambda_n \cap \mathfrak{E} = \Lambda_n \cap E$. Also we observe that $\mathbf{C}(\mathfrak{E})$ is a differential field and it contains $\mathbf{C}(E)$.

Definition 4.5. We will call this partition $T_0, T_1, \cdots, T_n$ of $\mathfrak{E}$ as the *levelled partition* of $\mathfrak{E}$.

We observe that

$$\begin{aligned}\pi(T_i) &= \pi(\Lambda_i \cap \mathfrak{E}) \\ &\subseteq \pi(\Lambda_i) \cap \pi(\mathfrak{E}) \\ &\subseteq \pi(\Lambda_i) \cap \mathfrak{E} \\ &\subseteq T_{i-1}.\end{aligned}$$

Thus $\pi(T_i) \subseteq T_{i-1}$ for all $1 \leq i \leq n$. We also note that $T_0 = \{x\}$ since E is non empty. We will use this partition of $\mathfrak{E}$ to prove that the iterated logarithms are algebraically independent over $\mathbf{C}(x)$ and this will be done in section 4.3.

Now we will construct a tower of Picard-Vessiot extensions by antiderivatives (iterated logarithms) to reach $\mathbf{C}(\mathfrak{E})$ from $\mathbf{C}$ using this *leveled partition* of $\mathfrak{E}$. (Note that this tower is not imbeddable in the Picard-Vessiot closure of $\mathbf{C}$.)

The construction of this tower is obvious. Let $\mathbf{K}_0 := \mathbf{C}(T_0) = \mathbf{C}(x)$ and let $\mathbf{K}_i := \mathbf{K}_{i-1}(T_i)$ for all $i \in \mathbb{N}$. That is $\mathbf{K}_i = \mathbf{C}(\cup_{j=0}^{i} T_j)$ for $0 \leq i \leq n$. Clearly $\mathbf{K}_0$ is an extension by antiderivatives of $\mathbf{C}$. Also, for $y \in T_i$, $\pi^j(y) \in \cup_{k=0}^{i-1} T_k$ for all $i, j \in \mathbb{N}$ and in fact, $\pi^j(y) = x$ for all $j \geq i$. Now from equation 4.9 we see that $y' \in \mathbf{K}_{i-1}$ and thus $\mathbf{K}_i$ is also an extension by antiderivatives of $\mathbf{K}_{i-1}$. Therefore we have a tower of P-V extensions by antiderivatives namely

$$\mathbf{C}(\mathfrak{E}) = \mathbf{K}_n \supset \mathbf{K}_{n-1} \supset \cdots \supset \mathbf{K}_1 \supset \mathbf{K}_0 \supset \mathbf{C}. \tag{4.5}$$

We will call this the *levelled partition tower* of $\mathbf{C}(\mathfrak{E})$.

There is another useful way of dividing the set $\mathfrak{E} = \cup_{i=0}^{n} \pi^i(E)$. Let

$$\mathcal{P} = E \setminus \cup_{i=1}^{n} \pi^i(E).$$

We claim that $\cup_{i=0}^{n}\pi^{i}(\mathcal{P}) = \mathfrak{E}$. Before we prove this claim, we note that $\mathcal{P} \cap \pi^{i}(\mathcal{P}) = \emptyset$ for all i, $1 \leq i \leq n$ and this statement immediately follows from the definition of $\mathcal{P}$. We also note that $\mathcal{P} \subseteq E$.

Now we will use an induction argument to show that $\cup_{i=0}^{n}\pi^{i}(\mathcal{P}) = \mathfrak{E}$. First we observe that $\mathfrak{E} = \cup_{i=0}^{n}(\Lambda_i \cap \mathfrak{E})$. From the choice of n it is clear that $\Lambda_n \cap E \neq \emptyset$. From equation 4.4 we see that for every $y \in \Lambda_n \cap E$, $y \notin \cup_{i=1}^{n}\pi^{i}(E)$. Thus $\Lambda_n \cap E \subseteq \mathcal{P}$ and therefore $\Lambda_n \cap E \subseteq \mathcal{P} \subseteq \cup_{i=0}^{n}\pi^{i}(\mathcal{P})$. Assume that there is a $k \leq n$ such that for any i, $k \leq i \leq n$, $\Lambda_i \cap \mathfrak{E} \subseteq \cup_{i=0}^{n}\pi^{i}(\mathcal{P})$. We will show that $\Lambda_{k-1} \cap \mathfrak{E} \subseteq \cup_{i=0}^{n}\pi^{i}(\mathcal{P})$. Let $y \in \Lambda_{k-1} \cap \mathfrak{E}$. If $y \in \mathcal{P}$, we are done. So, we suppose that $y \notin \mathcal{P}$. Then $y \in \cup_{i=1}^{n}\pi^{i}(E)$ and therefore there is a $z \in E$ and a $j \in \mathbb{N}$ such that $\pi^{j}(z) = y$. Clearly such a $z \in \cup_{i=k}^{n}\Lambda_i \cap E$. That is, z has to be a higher level iterated logarithm than y is (see equation 4.4). Now from our induction hypothesis we obtain $z \in \cup_{i=0}^{n}\pi^{i}(\mathcal{P})$ and since $\cup_{i=0}^{n}\pi^{i}(\mathcal{P})$ is invariant under π, we obtain $y \in \cup_{i=0}^{n}\pi^{i}(\mathcal{P})$. Thus $\cup_{i=0}^{n}\pi^{i}(\mathcal{P}) = \mathfrak{E}$.

Definition 4.6. We will call the set $\mathcal{P} \subset E$ as the $\pi-$*base* of $\mathfrak{E}$.

We may also construct a tower of Picard-Vessiot extension by antiderivatives(by iterated logarithms) to reach $\mathbf{C}(\mathfrak{E})$ by defining $\mathbf{P}_i := \mathbf{P}_{i-1}(\pi^{n-i}(\mathcal{P}))$ for $1 \leq i \leq n$, where $\mathbf{P}_0 := \mathbf{C}(x)$. Then $\mathbf{P}_i = \mathbf{C}(\cup_{j=0}^{i}\pi^{n-j}(\mathcal{P}))$ for $0 \leq i \leq n$ and clearly, $\mathbf{P}_i$ is a differential field. Thus we see that

$$\mathbf{C}(\mathfrak{E}) = \mathbf{P}_n \supset \mathbf{P}_{n-1} \supset \cdots \supset \mathbf{P}_1 \supset \mathbf{P}_0 \supset \mathbf{C}. \tag{4.6}$$

We will call the above tower as the $\pi-$*tower* of $\mathbf{C}(\mathfrak{E})$.

We observe that $\mathcal{P} \subset \cup_{i=0}^{n}\Lambda_i$ and therefore $\pi(\mathcal{P}) \subset \cup_{i=0}^{n-1}\Lambda_i$, $\pi^2(\mathcal{P}) \subset \cup_{i=0}^{n-2}\Lambda_i$ and in general $\pi^{j}(\mathcal{P}) \subset \cup_{i=0}^{n-j}\Lambda_i$. Thus $\pi^{n-j}(\mathcal{P}) \subset \cup_{i=0}^{j}\Lambda_i$ and from this fact we also obtain $\cup_{j=0}^{m}\pi^{n-j}(\mathcal{P}) \subset \cup_{i=0}^{m}\Lambda_i$ for any $m, 0 \leq m \leq n$. Since $\cup_{i=0}^{n}\pi^{i}(\mathcal{P}) = \mathfrak{E}$,

$$\begin{aligned}\cup_{i=0}^{m}\pi^{n-j}(\mathcal{P}) &\subseteq \cup_{i=0}^{m}\Lambda_i \cap \mathfrak{E}\\ &= \cup_{i=0}^{m}T_i\end{aligned}$$

and thus $\cup_{i=0}^{m}\pi^{n-j}(\mathcal{P}) \subseteq \cup_{i=0}^{m}T_i$. This shows that $\mathbf{P}_m \subseteq \mathbf{K}_m$ for every $0 \leq m \leq n$. Nonetheless the inequality could be strict and we will now provide an example for the same.

Let $\mathbf{C} := \mathbb{C}$ and let $E = \{\ln(\ln(x+e)+5), \ln(\ln(x)), \ln(x), \ln(x+1)\}$. In our notation, the set $E = \{\mathfrak{l}[\vec{v}_1, 2], \mathfrak{l}[\vec{v}_2, 2], \mathfrak{l}[\vec{v}_3, 1], \mathfrak{l}[\vec{v}_4, 1]\}$, where $\vec{v}_1 = (\exp, 5)$, $\vec{v}_2 = (0, 0)$, $\vec{v}_3 = (0)$ and $\vec{v}_4 = (1)$. Then we immediately see that $\pi(\ln(\ln(x+\exp)+5)) = \ln(x+e)$, $\pi^2(\ln(\ln(x+\exp)+5)) = \pi(\ln(x+e))= x$, $\pi(\ln(\ln(x))) = \ln(x)$, $\pi^2(\ln(\ln(x))) = x$, $\pi(\ln(\ln(x+1))) = \ln(x+1)$, $\pi(\ln(x+1)) = x$ and $\pi(x) = x$. Thus the set $\mathfrak{E} = \{\ln(\ln(x+e)+5), \ln(\ln(x)), \ln(x), \ln(x+1), \ln(x+e), x\}$.

Let us obtain the *levelled partition* of $\mathfrak{E}$. The set $T_0 = \mathfrak{E}\cap\Lambda_0 = \{x\}$, $T_1 = \Lambda_1\cap\mathfrak{E} = \{\ln(x), \ln(x+1), \ln(x+e)\}$ and the set $T_2 = \mathfrak{E}\cap\Lambda_2 = \{\ln(\ln(x)), \ln(\ln(x+e)+5)\}$. Therefore the *levelled partition tower* would be

$$\mathbf{C}(\mathfrak{E}) \supset \mathbf{C}(\ln(x), \ln(x+1), \ln(x+e), x) \supset \mathbf{C}(x) \supset \mathbf{C}.$$

Note that the $\pi-$base $\mathcal{P}$ of $\mathfrak{E}$ is given by $\mathcal{P} = E \setminus \cup_{i=1}^{2}\pi^{i}(E)$. Since $\cup_{i=1}^{2}\pi^{i}(E) = \{\ln(x+e), \ln(x), x\}$ we see that $\mathcal{P} = \{\ln(\ln(x+e)+5), \ln(\ln(x)), \ln(x+1)\}$. Thus the $\pi-$*partition tower* of $\mathbf{C}(\mathfrak{E})$ is

$$\mathbf{C}(\mathfrak{E}) \supset \mathbf{C}(\ln(x), \ln(x+e), x) \supset \mathbf{C}(x) \supset \mathbf{C}.$$

Therefore, if we assume that the iterated logarithms are algebraically independent over $\mathbf{C}(x)$ then $\ln(x+1) \notin \mathbf{C}(\ln(x), \ln(x+e), x)$ and thus the two towers are distinct.

Structure theorem for $\mathfrak{L}_n$: Here we will assume that the iterated logarithms are algebraically independent over $\mathbf{C}(x)$. That is, the set Λ_∞ is algebraically independent over $\mathbf{C}$. A proof for this fact is provided in section 4.3, theorem 4.13. Thus $\mathfrak{L}_\infty$ is the field of fractions of the polynomial ring $\mathbf{C}[\Lambda_\infty]$. For $y \in \Lambda_\infty$ let $\frac{\partial}{\partial y}$ denote the standard partial derivation on the polynomial ring $\mathbf{C}[\Lambda_\infty]$.

Let $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$. Then there is a finite non empty set $S \subset \cup_{i=0}^{n} \Lambda_i$ such that $u = \frac{P}{Q}$, $P, Q \in \mathbf{C}[S]$ and $(P, Q) = 1$(that is the G.C.D of P and Q in the polynomial ring $\mathbf{C}[S]$ is 1). It is conceivable that some of the elements of S may not be necessary to express u. So, we define a set E_u as

$$E_u := \Big\{ y \in S \Big| \frac{\partial P}{\partial y} \neq 0 \ or \ \frac{\partial Q}{\partial y} \neq 0 \Big\}. \tag{4.7}$$

Definition 4.7. The set E_u is called the set of all *essential elements* of u

We observe that $u \in \mathbf{C}(E_u)$ and that if $u \in \mathbf{C}(S)$ for some set $S \subset \Lambda_\infty$ then $E_u \subset S$. Sometimes we drop the suffix u and simply write E instead of E_u. Since $\mathbf{C}[\Lambda_\infty]$ is a polynomial ring (over a field), the set E_u is unique. The following theorem proves the uniqueness of E_u.

Theorem 4.8. *(Uniqueness of E_u) Let $u \in \mathfrak{L}_\infty$ and let E_u be a set of essential elements of u. Then $u \in \mathbf{C}(S)$ for some $S \subset \Lambda_\infty$ only if $E_u \subseteq S$ and thus the set E_u is unique for a given u.*

Proof. Let $S \subset \Lambda_\infty$ and let $u \in \mathbf{C}(S)$. Then

$$u = \frac{P}{Q} = \frac{A}{B}, \tag{4.8}$$

for some $A, B \in \mathbf{C}[S]$ and $P, Q \in \mathbf{C}[E_u]$, where $(P, Q) = 1$. Since $(P, Q) = 1$, from the above equation it is clear that P divides A and Q divides B in the polynomial ring $\mathbf{C}[S \cup E_u]$. Thus there are $R, T \in \mathbf{C}[S \cup E_u]$ such that $PR = A$ and $QT = B$. Note that if $y \in E_u$ then $\frac{\partial P}{\partial y} \neq 0$ *or* $\frac{\partial Q}{\partial y} \neq 0$. Suppose that there is a $y \in E_u$ such that $\frac{\partial P}{\partial y} \neq 0$. Consider the equation $PR = A$. Then $\deg_y(P) \geq 1$ and note that $PR = A$ implies $\deg_y(P) + \deg_y(R) = \deg_y(A)$. Thus $\deg_y(A) \geq 1$. Hence $y \in S$. Similarly if $\frac{\partial Q}{\partial y} \neq 0$ and $\frac{\partial P}{\partial y} = 0$, we may use the equation $QT = B$ to show that $y \in S$ and thus $E_u \subset S$. □

The following corollary is a direct consequence of the above theorem.

Corollary 4.9. *Let $S \subset \Lambda_\infty$ be any nonempty set and for $1 \leq j \leq s$ let $y_j \in \Lambda_\infty$ be distinct. Then for any constants $a_j \in \mathbf{C}^*$ such that $\sum_{j=1}^{s} a_j y_j \in \mathbf{C}(S)$, the element $y_j \in S$ for each j.*

Proof. Suppose that there are $a_j \in \mathbf{C}^*$ and such that $\sum_{j=1}^{s} a_j y_j \in \mathbf{C}(S)$. Since $a_j \in \mathbf{C}^*$, the essential elements of $\sum_{j=1}^{s} a_j y_j$ is the set $E := \{y_j | 1 \leq j \leq s\}$. Now from theorem 4.8 we obtain $E \subset S$. □

Now we will state the structure theorem for singly generated differential subfields of $\mathfrak{L}_n$.

Theorem 4.10. *Let $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$, E the essential elements of u and $\mathbf{C}(\mathfrak{E})$ the container differential field of E. Let $\mathcal{P} \subseteq E$ be the $\pi-$ base of $\mathfrak{E}$. Then the differential field*

$$\mathbf{C}\langle u \rangle = \mathbf{C}(\mathcal{S}, \pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x),$$

where $\mathcal{S}$ is a finite nonempty subset of $span_{\mathbf{C}}\mathcal{P}$. Moreover for every $y \in \mathcal{P}$, $\mathcal{S}$ contains at least one linear combination in which y appears nontrivially.

The above structure theorem is proved in subsection 4.3.2. There we will also generalize this theorem to finitely generated differential subfields of $\mathfrak{L}_n$ and give an algorithm to find the set $\mathcal{S}$ and $\mathcal{P}$ that appears in the above structure theorem.

Remark 4.11. Given a $u \in \Lambda_\infty$, there is a set finite $E \subset \cup_{i=0}^{n} \Lambda_i$ and we may also choose a minimal n such that the above inclusion holds. Then $\mathbf{C}\langle u \rangle$ becomes a subfield of the container differential field $\mathbf{C}(\mathfrak{E})$ of E. The field $\mathbf{C}(\mathfrak{E})$ is an *elementary extension* of $\mathbf{C}$. The above stated theorem(and its generalized version) shows that every differential subfield of $\mathbf{C}(\mathfrak{E})$, more in general, a finitely differentially generated subfields of $\mathfrak{L}_\infty$ has to be a *generalized elementary extension* of a special form. For a definition of elementary and generalized elementary extension and results related to our theorem in a more general context, one may refer to the following papers [12], [13] and [14].

4.3 Algebraic Independence of Iterated logarithms

Here we will show that the set Λ_∞ is algebraically independent over $\mathbf{C}$. For $i = 1, 2, \cdots, n$ let $c_i \in \mathbf{C}$ be distinct constants. By choosing $C_i := x + c_i$, $A_i = B_i = 1$, we see that $\mathbf{C}(x, \mathfrak{l}[\vec{c}_1, 1], \cdots, \mathfrak{l}[\vec{c}_n, 1])$, where $\vec{c}_i := (c_i)$ is an extension by J-I-E antiderivatives of $\mathbf{C}(x)$ and thus $\mathfrak{l}[\vec{c}_1, 1], \cdots, \mathfrak{l}[\vec{c}_n, 1]$ are algebraically independent over $\mathbf{C}(x)$. Assume that every finite subset of Λ_{t-1}, $t \geq 2$ consists of J-I-E antiderivatives of $\mathbf{C}(\cup_{j=0}^{t-2} \Lambda_j)$. For $i = 1, 2, \cdots, n$ let $\vec{c}_i := (c_{1i}, c_{2i}, \cdots, c_{ti})$

$\in \mathbf{C}^t \setminus \{0\}$ be distinct vectors. Note that

$$\mathfrak{l}'[\vec{c}_i, t] = \Big(\prod_{j=1}^{n-1} \frac{1}{\mathfrak{l}[\pi^{j+1}(\vec{c}_i), n-(j+1)] + \psi_{n-j}(\pi^j(\vec{c}_i))}\Big) \frac{1}{\mathfrak{l}[\pi(\vec{c}_i), t-1] + \psi_t(\vec{c}_i)} \tag{4.9}$$

and therefore choosing $A_i = 1$, $B_j := \mathfrak{l}[\pi^{j+1}(\vec{c}_i), t-(j+1)] + \psi_{t-j}(\pi^j(\vec{c}_i))$ and $C_j := \mathfrak{l}[\pi(\vec{c}_i), t-1] + \psi_t(\vec{c}_i)$ we see that $\mathbf{C}(\cup_{j=0}^{t-1}\Lambda_j,\ \mathfrak{l}[\vec{c}_1, t], \cdots, \mathfrak{l}[\vec{c}_n, t])$ is an extension by J-I-E antiderivatives of $\mathbf{C}(\cup_{j=0}^{t-1}\Lambda_j)$ and thus Λ_t is algebraically independent over $\mathbf{C}(\cup_{j=0}^{t-1}\Lambda_j)$. Now we will give a proof for the algebraic independence of the iterated logarithms without appealing to results from section3.2.

Lemma 4.12. *Let $S_{n-1} \subset \Lambda_{n-1}$ be a finite set of antiderivatives of a differential field $\mathbf{F}$ and let $S_n \subset \Lambda_n$ be such that $\pi(S_n) \subseteq S_{n-1}$. Suppose that S_{n-1} is algebraically independent over $\mathbf{F}$. Then S_n is algebraically independent over $\mathbf{F}(S_{n-1})$.*

Proof. Note that $\mathbf{F}(S_{n-1})$ is a differential field and since $\pi(S_n) \subseteq S_{n-1}$, from equations 4.1 and 4.3 it is clear that $\mathbf{F}(S_{n-1})(S_n)$ is also a differential field. Let $S_n = \{\mathfrak{l}[\vec{c}_i, n] | 1 \le i \le s\}$, $\vec{c}_i = (c_{1i}, c_{2i}, \cdots, c_{ni})$ and Suppose that S_n is algebraically dependent over $\mathbf{F}(S_{n-1})$. Then by theorem 2.3 there are constants $\alpha(\vec{c}_i) \in \mathbf{C}$ not all zero such that $\sum_{i=1}^{s} \alpha(\vec{c}_i)\mathfrak{l}[\vec{c}_i, n] \in \mathbf{F}(S_{n-1})$. We may assume that $\alpha(\vec{c}_1) \neq 0$ and rewrite the sum as $X + \sum_{j=1}^{t} \alpha(\vec{b}_j)\mathfrak{l}[\vec{b}_j, n]$ where $\{\vec{b}_j\} \subseteq \{\vec{c}_i\}$ is the set of all vectors such that $\pi(\vec{b}_j) = \pi(\vec{c}_1) = (c_{11}, c_{21}, \cdots, c_{n-11})$ and $X = \sum_{i=1}^{s} \alpha(\vec{c}_i)\mathfrak{l}[\vec{c}_i, n] - \sum_{j=1}^{t} \alpha(\vec{b}_j)\mathfrak{l}[\vec{b}_j, n]$. We may order the set $\{\vec{b}_j\}$ so that $\vec{b}_1 = \vec{c}_1$. Let $\mathbf{K} := \mathbf{F}(S_{n-1} \setminus \{\mathfrak{l}[\pi(\vec{c}_1), n-1]\})$ and let $X + \sum_{j=1}^{t} \alpha(\vec{b}_j)\mathfrak{l}[\vec{b}_j, n] = \frac{P}{Q}$, where $P, Q \in \mathbf{K}[\mathfrak{l}[\pi(\vec{c}_1), n-1]]$, $(P, Q) = 1$ and Q a monic polynomial. Then

$$X' + \sum_{j=1}^{t} \frac{\alpha(\vec{b}_j)\mathfrak{l}'[\pi(\vec{c}_1), n-1]}{(\mathfrak{l}[\pi(\vec{c}_1), n-1] + c_{jn})} = \frac{QP' - PQ'}{Q^2}.$$

Let $f := \mathfrak{l}'[\pi(\vec{c_1}), n-1]$ and let $\frac{F}{G} = \sum_{j=1}^{t} \frac{\alpha(\vec{b}_j)}{\mathfrak{l}[\pi(\vec{c}_1),n]+c_{jn}}$, where F and G are obtained by clearing the denominator of the sum $\sum_{j=1}^{t} \frac{\alpha(\vec{b}_j)}{\mathfrak{l}[\pi(\vec{c}_1),n]+c_{jn}}$. Note that $(F, G) = 1$. Now we have

$$Q^2(GX' + fF) = G(QP' - PQ'). \tag{4.10}$$

From the definition of X, it is clear that $X = \sum_{j=1}^{t} \alpha(\vec{a}_j)\mathfrak{l}[\vec{a}_j, n]$ where $\{\vec{a}_j\} \subset \{\vec{c}_j\}$ is the set of all vectors such that $\pi(\vec{a}_j) \neq \pi(\vec{c}_1)$. Therefore $X' \in \mathbf{K}$. Thus equation 4.10 is a polynomial in $\mathfrak{l}[\pi(\vec{c_1}), n-1]$ over the field $\mathbf{K}$. Let $y := \mathfrak{l}[\pi(\vec{c}_1), n-1]+c_{1n}$. Since y divides G and $(F, G) = 1$, y does not divide F. Thus y does not divide $GX' + fF$ and therefore from 4.10 y divides Q^2. Hence y divides Q. Let $l \in \mathbb{N}$ be the greatest positive integer such that y^l divides Q. Then y^{2l} divides Q^2 and therefore y^{l+1} divides Q^2, which implies y^{l+1} divides $G(QP' - PQ')$. Since y divides G and y^2 does not divide G, y^l divides $QP' - PQ'$. But y^l divides Q and therefore y^l divides PQ'. Since $(P, Q) = 1$, we see that y^l divides Q'. Write $Q = y^l H$ and consider $Q' = ly^{l-1}y'H + y^l H'$. Note that y^l divides Q' implies y^l divides $ly^{l-1}y'H$ and since $y' \in \mathbf{K}$, y divides H. Thus y^{l+1} divides Q, contradicting the maximality of l. □

Theorem 4.13. *Let $E \subset \Lambda_\infty$ be a nonempty finite set. Then E is algebraically independent over* $\mathbf{C}$.

Proof. As usual, let $\mathfrak{E} := \cup_{i=0}^{n}\pi^i(E)$ where n is the least positive integer such that $E \subset \cup_{i=0}^{n}\Lambda_i$ and let $\{T_i | 0 \leq i \leq n\}$ be the *levelled partition* of $\mathfrak{E}$. As we noted earlier $\pi(T_i) \subseteq T_{i-1}$, $T_n \neq \emptyset$ and $\pi^n(T_n) = \{x\} = T_0$. Clearly, T_0 is algebraically independent over $\mathbf{C}$ (see theorem 2.4) and since $\pi(T_1) \subset T_0$, from lemma 4.12 we get T_1 is algebraically independent over $\mathbf{C}(T_0)$. Since $\pi(T_i) \subset T_{i-1}$, a repeated application of lemma 4.12 will show us that $\mathfrak{E} = \cup_{j=0}^{n}T_j$ is algebraically independent over $\mathbf{C}$. Since $E \subset \mathfrak{E}$, E is also algebraically independent over $\mathbf{C}$. □

4.3.1 Normality of $\mathfrak{L}_n$ and Some Consequences

Let $\mathbf{C}_\infty$ be the complete Picard-Vessiot closure of $\mathbf{C}$ and let $\Phi \in \mathbb{G}(\mathbf{C}_\infty|\mathbf{C})$. Let $(v_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbf{C}$ and let $\vec{v}_n := (v_1, \cdots, v_n)$ for all $n \in \mathbb{N}$ (the vector $\vec{v}_1 = (v_1)$). Thus in our notation $\pi(\vec{v}_n) = \vec{v}_{n-1}$. We observe that $\Phi(x) = x+\alpha_\Phi$ for some $\alpha_\Phi \in \mathbf{C}$. Since $\mathfrak{l}'[\vec{v}_1, 1] = \frac{1}{x+v_1}$ we see that $\Phi\big(\mathfrak{l}[\vec{v}_1, 1]\big)' = \frac{1}{\Phi(x)+v_1} = \frac{1}{x+\alpha_\Phi+v_1}$ $= \mathfrak{l}'[\Phi(\vec{v}_1), 1]$, where $\Phi(\vec{v}_1) := (v_1) + (\alpha_\Phi)$. Since any two antiderivatives differ by a constant, $\Phi\big(\mathfrak{l}[v_1, 1]\big) = \mathfrak{l}[\Phi(\vec{v}_1), 1] + \alpha_{\Phi(\vec{v}_1)}$, for some $\alpha_{\Phi(\vec{v}_1)} \in \mathbf{C}$. Assume that $\Phi\big(\mathfrak{l}[\vec{v}_{n-1}, n-1]\big) = \mathfrak{l}[\Phi(\vec{v}_{n-1}), n-1] + \alpha_{\Phi(\vec{v}_{n-1})}$ where $\Phi(\vec{v}_{n-1}) = (v_1 + \alpha_\Phi, v_2 + \alpha_{\Phi(\vec{v}_1)}, \cdots, v_{n-1} + \alpha_{\Phi(\vec{v}_{n-2})})$ and $\alpha_{\Phi(\vec{v}_{n-1})} \in \mathbf{C}$. Since

$$\mathfrak{l}'[\vec{v}_n, n] = \frac{\mathfrak{l}'[\vec{v}_{n-1}, n-1]}{\mathfrak{l}[\vec{v}_{n-1}, n-1] + v_n},$$

we see that

$$\begin{aligned}\Phi\big(\mathfrak{l}[\vec{v}_n, n]\big)' &= \frac{\mathfrak{l}'[\Phi(\vec{v}_{n-1}), n-1]}{\mathfrak{l}[\Phi(\vec{v}_{n-1}), n-1] + v_n + \alpha_{\Phi(\vec{v}_{n-1})}} \\ &= \mathfrak{l}'[\Phi(\vec{v}_n), n]\end{aligned}$$

where $\Phi(\vec{v}_n) = (v_1 + \alpha_\Phi, v_2 + \alpha_{\Phi(\vec{v}_1)}, \cdots, v_n + \alpha_{\Phi(\vec{v}_{n-1})})$. Since any two antiderivatives differ by a constant, we obtain

$$\Phi\big(\mathfrak{l}[\vec{v}_n, n]\big) = \mathfrak{l}[\Phi(\vec{v}_n), n] + \alpha_{\Phi(\vec{v}_n)} \tag{4.11}$$

for some $\alpha_{\Phi(\vec{v}_n)} \in \mathbf{C}$.

From equation4.11, we see that for every $\Phi \in \mathbb{G}(\mathbf{C}_\infty|\mathbf{C})$,

$$\Phi(\Lambda_i) \subseteq \Lambda_i + \mathbf{C} \tag{4.12}$$

for all $i \in \mathbb{N}$. Thus $\mathfrak{L}_n$ is a normal differential subfield of $\mathbf{C}_\infty$.

Remark 4.14. Let $\Phi \in \mathbb{G}(\mathfrak{L}_\infty|\mathbf{C})$ and for $n \in \mathbb{N} \cup \{0\}$ let

$$\Phi\big(\mathfrak{l}[\vec{v}_n, n]\big) = \mathfrak{l}[\vec{v}_n, n] + \alpha_{\Phi(\vec{v}_n)},$$

with $\alpha_{\Phi(\vec{v}_n)} \in \mathbf{C}^*$. Then from the above discussion, we see that for any $m < n$

$$\Phi\big(\mathfrak{l}[\vec{v}_m, m]\big) = \mathfrak{l}[\vec{v}_m, m].$$

For any $m > n$ and $k \in \mathbb{N}$

$$\Phi^k\big(\mathfrak{l}[\vec{v}_m, m]\big) = \mathfrak{l}[\Phi^k(\vec{v}_m), m] + \alpha_{\Phi^k(\vec{v}_m)},$$

where $\Phi^k(\vec{v}_m) = (v_1, \cdots, v_n, v_{n+1} + k\alpha^{\Phi}_{\vec{v}_n}, \cdots, v_m + k\alpha^{\Phi}_{\vec{v}_{m-1}})$. Since $\alpha^{\Phi}_{\vec{v}_n} \neq 0$, $\Phi^i(\vec{v}_m) \neq \Phi^j(\vec{v}_m)$ when $i \neq j$. Thus $\mathfrak{l}[\Phi^i(\vec{v}_m), m] \neq \mathfrak{l}[\Phi^j(\vec{v}_m), m]$ for any $i \neq j$ and for any $m > n$. Hence the set $\{\mathfrak{l}[\vec{v}_m, m], \mathfrak{l}[\Phi^j(\vec{v}_m), m] | i \in \mathbb{N}\}$ is algebraically independent over $\mathbf{C}$ for any $m > n$(follows from theorem 4.13).

Now we will prove a theorem which will help us to prove the structure theorem for the differential subfields of $\mathfrak{L}_n$.

Theorem 4.15. *Let* $\mathbf{F}$ *be a differential field finitely generated over its constants* $\mathbf{C}$, $\mathbf{E}$ *be a Picard-Vessiot extension of* $\mathbf{F}$, *and let* $\mathbf{F} \subset \mathbf{E} \subset \mathfrak{L}_\infty$. *If* $\sum_{j=1}^{s} a_j y_j \in \mathbf{E}$ *for some* $a_j \in \mathbf{C} \setminus \{0\}$, $y_j \in \cup_{i=0}^{\infty} \Lambda_i$ *and* $s \in \mathbb{N}$ *then* $\pi^i(y_j) \in \mathbf{F}$ *for all* $i \in \mathbb{N}$ *and thus* $y_j' \in \mathbf{F}$.

Proof. Let there be $y_j \in \cup_{i=0}^{\infty} \Lambda_i$ and $a_j \in \mathbf{C}^*$ such that $\sum_{j=1}^{s} a_j y_j \in \mathbf{E}$. Note that $\mathbf{E}$ is finitely generated over $\mathbf{F}$ and $\mathbf{F}$ is finitely generated over $\mathbf{C}$ and thus $\mathbf{E}$ is finitely generated over $\mathbf{C}$. Let $u_1, \cdots, u_t \in \mathbf{E}$ such that $\mathbf{C}(u_1, \cdots, u_t) = \mathbf{E}$,

E_{u_i} be the set of *essential elements* of u_i, and let $S := \cup_{i=1}^{t} E_{u_i} \cup \{y_j | 1 \leq j \leq s\}$. From the definition of S it is quite clear that we have the following containments

$$\mathbf{C}(S) \supseteq \mathbf{E}(y_1, \cdots, y_s) \supseteq \mathbf{E} \supseteq \mathbf{F} \supseteq \mathbf{C}. \tag{4.13}$$

Since $\mathfrak{L}_n$ and $\mathbf{E}$ are normal differential subfields of the complete Picard-Vessiot closure $\mathbf{F}_\infty$ of $\mathbf{F}$, every automorphism $\phi \in \mathbb{G}(\mathbf{E}|\mathbf{F})$ extends to an automorphism $\Phi \in \mathbb{G}(\mathfrak{L}_n|\mathbf{F})$ and every automorphism $\Phi \in \mathbb{G}(\mathfrak{L}_n|\mathbf{F})$ restricts to an automorphism $\phi \in \mathbb{G}(\mathbf{E}|\mathbf{F})$.

Let $\Phi \in \mathbb{G}(\mathfrak{L}_n|\mathbf{F})$. Since $\mathbf{E}$ is a normal differential subfield of $\mathfrak{L}_n|\mathbf{F}$, $\Phi(\mathbf{E}) \subseteq \mathbf{E}$ and therefore

$$\sum_{j=1}^{s} a_j \Phi^k(y_j) \in \mathbf{E}. \tag{4.14}$$

Let $y_j = \mathfrak{l}[\vec{v}_{jm_j}, m_j]$, where $\vec{v}_{jm_j} = (v_{j1}, \cdots, v_{jm_j})$. Then $\Phi^k(y_j) = \mathfrak{l}[\Phi^k(\vec{v}_{jm_j}), m_j] + \alpha_{\Phi^k(\vec{v}_{jm_j})}$, where $\alpha_{\Phi^k(\vec{v}_{jm_j})} \in \mathbf{C}$. Therefore

$$\sum_{j=1}^{s} a_j \mathfrak{l}[\Phi^k(\vec{v}_{jm_j}), m_j] + \sum_{j=1}^{s} a_j \alpha_{\Phi^k(\vec{v}_{jm_j})} \in \mathbf{E}$$

and thus

$$\sum_{j=1}^{s} a_j \mathfrak{l}[\Phi^k(\vec{v}_{jm_j}), m_j] \in \mathbf{E} \subseteq \mathbf{C}(S).$$

Now from corollary 4.9 we see that

$$\mathfrak{l}[\Phi^k(\vec{v}_{jm_j}), m_j] \in S$$

for every $j, k \in \mathbb{N}$. For a fixed j, consider the set $T := \{\mathfrak{l}[\vec{v}_{jm_j}, m_j], \mathfrak{l}[\Phi^k(\vec{v}_{jm_j}), m_j] \,\big|\, k \in \mathbb{N}\}$. From the action of Φ on $\vec{v}_{jm_j}$, it is clear that if $\Phi(\vec{v}_{jm_j}) \neq \vec{v}_{jm_j}$ then T

is infinite. But T cannot be infinite because it sits inside the finite set S. Hence $\Phi(\vec{v}_{jm_j}) = \vec{v}_{jm_j}$ and therefore

$$\begin{aligned}\Phi(\mathfrak{l}[\vec{v}_{jm_j}, m_j]) &= \mathfrak{l}[\Phi(\vec{v}_{jm_j}), m_j] + \alpha_{\Phi(\vec{v}_{jm_j})} \\ &= \mathfrak{l}[\vec{v}_{jm_j}, m_j] + \alpha_{\vec{v}_{jm_j}}.\end{aligned}$$

Now from the remark 4.14 it follows that $\Phi(\pi^i(y_j)) = \pi^i(y_j)$ for all $i \in \mathbb{N}$. This shows that $\pi^i(y_j) \in \mathfrak{L}_n^{\mathbb{G}(\mathfrak{L}_n|\mathbf{F})} = \mathbf{F}$. □

4.3.2 Differential Subfields of Λ_∞

In this section we will classify the finitely generated differential subfields of $\mathfrak{L}_n$. First we will point out an interesting property that every differential subfield $\mathbf{F} \neq \mathbf{C}$ of Λ_n possesses, which is that $x \in \mathbf{F}$ and this result is a consequence of the structure theorem.

Proposition 4.16. *Let $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}, n \in \mathbb{N}$, E be the set of essential elements of u, $\mathfrak{E} := \cup_{j=0}^{n} \pi^i(E)$ and let $\{T_i | 0 \leq i \leq n\}$ be the levelled partition of $\mathfrak{E}$. Then u is not algebraic over $\mathbf{C}(\cup_{j=0}^{i} T_j)$ for any $0 \leq i \leq n-1$.*

Proof. Let $u = \frac{P}{Q}$, $P, Q \in \mathbf{C}(\mathfrak{E} \setminus \{y\})[y]$, where $y \in T_n$. The *levelled partition* of $\mathfrak{E}$ is constructed in such a way that $T_n \neq \emptyset$ and $T_n \subseteq E$. Since E consists of essential elements of u and $y \in E$, $u \notin \mathbf{F} := \mathbf{C}(\mathfrak{E} \setminus \{y\})$. Let $\mathbf{K}_i := \mathbf{C}(\cup_{j=0}^{i} S_j)$ for each $i, 1 \leq i \leq n-1$. Then $\mathbf{K}_i \subset \mathbf{F}$. Since $y' \in \mathbf{F}$ and $y \notin \mathbf{F}$, $\mathbf{E} = \mathbf{F}(y)$ is a Picard-Vessiot extension of $\mathbf{F}$ with a differential Galois group $\mathbb{G} := (\mathbf{C}, +)$. Note that $\mathbb{G}$ has no non trivial algebraic subgroups(in particular no nontrivial finite subgroups). Since $\mathbf{F}\langle u \rangle \supsetneq \mathbf{F}$, $\mathbf{F}\langle u \rangle = \mathbf{E}$, which implies u is not algebraic over $\mathbf{F}$. Thus u is not algebraic over $\mathbf{K}_i$ for any $0 \leq i \leq n-1$. □

Thus we have just shown that if

$$\mathbf{C}(\mathfrak{E}) = \mathbf{K}_n \supset \mathbf{K}_{n-1} \supset \cdots \supset \mathbf{K}_1 \supset \mathbf{K}_0 \supset \mathbf{C}.$$

is the *levelled partition* tower of $\mathfrak{E}$, where $\mathfrak{E} := \cup_{j=0}^{n} \pi^i(E)$ and E is the set of essential elements of an element $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$ then u is not algebraic over $\mathbf{K}_i$ for any $0 \leq i \leq n-1$.

Note that if $u \in \mathbf{C}(x)$ then $\mathbf{C}\langle u \rangle = \mathbf{C}$ or $\mathbf{C}(x)$ depending whether u is a constant or not. Thus if $\mathbf{F}$ is a differential subfield(need not be finitely generated) of $\mathbf{C}(x)$ then $\mathbf{F} = \mathbf{C}(x)$ or $\mathbf{C}$ depending whether $\mathbf{F}$ contains a nonconstant or not. Thus it is enough to state the structure theorem only for elements in $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$.

Theorem 4.17. *Let $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$, E the essential elements of u and $\mathbf{C}(\mathfrak{E})$ the container differential field of E. Let $\mathcal{P} \subseteq E$ be the $\pi-$ base of $\mathfrak{E}$. Then the differential field*

$$\mathbf{C}\langle u \rangle = \mathbf{C}(\mathcal{S}, \pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x),$$

where $\mathcal{S}$ is a finite nonempty subset of $span_{\mathbf{C}}\mathcal{P}$. Moreover, for every $y \in \mathcal{P}$, $\mathcal{S}$ contains at least one linear combination in which y appears nontrivially.

Proof. For $i \geq 1$ let $\mathbf{P}_{n-i}$ denote the differential field $\mathbf{C}(\pi^i(\mathcal{P}), \pi^{i+1}(\mathcal{P}), \cdots, x)$ and let $\mathbf{P}_{n-i}\langle u \rangle$ be the differential field generated by $\mathbf{P}_{n-i}$ and u. Note that $\mathbf{C}(\mathfrak{E}) = \mathbf{P}_n = \mathbf{C}(\mathcal{P}, \pi(\mathcal{P}), \pi^2(\mathcal{P}) \cdots, x)$ is a Picard-Vessiot extension of $\mathbf{P}_{n-1} = \mathbf{C}(\pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x)$ with Galois group $\mathbb{G} := (\mathbf{C}, +)^m$. Note that the transcendence degree of $\mathbf{C}(\mathfrak{E}) = \mathbf{P}_n$ over $\mathbf{P}_{n-1}$ is $|\mathcal{P}|$ since $\mathcal{P} \cap \pi^j(\mathcal{P}) = \emptyset$ for any $1 \leq j \leq n$ and therefore $m = |\mathcal{P}|$. Clearly $\mathbf{P}_{n-1}\langle u \rangle$ is an intermediate differential field. Since $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$, we see that $\mathbf{P}_{n-1}\langle u \rangle \neq \mathbf{P}_{n-1}$. Let $\mathbb{H} \leq \mathbb{G}$ be the group of all automorphisms that fixes $\mathbf{P}_{n-1}$ and let $\{L_i(x_1, \cdots, x_m) | 1 \leq i \leq t\}$ be the system of polynomials for which $\mathbb{H}$ is the set of solutions. Then it is easy to see that

$$\mathbf{P}_{n-1}\langle u\rangle = \mathbf{P}_{n-1}(L_i(y_1,\cdots,y_m)), \tag{4.15}$$

where $y_j \in \mathcal{P}$. Note that $L_i(y_1,\cdots,y_m)' \in \mathbf{P}_{n-1}$ and thus $\mathbf{P}_{n-1}(L_i(y_1,\cdots,y_m))$ is a differential field.

Let D_i be the set of *essential elements* of $L_i(y_1,\cdots,y_m)$. Then from equation 4.15 $u \in \mathbf{C}(U)$, where $U = (\cup_{i=1}^{t} D_i) \cup (\cup_{i=1}^{n} \pi^i(E))$. Since E is the *essential elements* of u, we obtain $\mathcal{P} \subset E \subset U$. Now, $\mathcal{P} \cap (\cup_{i=1}^{n} \pi^i(E)) = \emptyset$ will imply $\mathcal{P} \subset \cup_{i=1}^{t} D_i$. Hence for every $y_j \in \mathcal{P}$ there is an $L_i(y_1,\cdots,y_m)$ such that the coefficient of y_j is nonzero. Let us denote the set $\{L_i(y_1,\cdots,y_m) | 1 \leq i \leq t\}$ by $\mathcal{S}$.

Since $\mathbf{P}_{n-1}$ is a Picard-Vessiot extension of $\mathbf{P}_{n-2}$, we see that $\mathbf{P}_{n-1}\langle u\rangle$ is a Picard-Vessiot extension of $\mathbf{P}_{n-2}\langle u\rangle$. Also, $L_i(y_1,\cdots,y_m) \in \mathbf{P}_{n-1}$ for each i. Thus from theorem 4.15 we see that for each $y_j \in \mathcal{P}$, $\pi(y_j) \in \mathbf{P}_{n-2}\langle u\rangle$ and thus $\pi(\mathcal{P}) \subset \mathbf{P}_{n-2}\langle u\rangle$. This shows that $\mathbf{P}_{n-1}\langle u\rangle = \mathbf{P}_{n-2}\langle u\rangle$. Since $\mathbf{P}_{n-2}\langle u\rangle$ is a Picard-Vessiot extension of $\mathbf{P}_{n-3}\langle u\rangle$, again applying theorem 4.15 we see that $\pi^2(\mathcal{P}) \subset \mathbf{P}_{n-3}\langle u\rangle$ and therefore $\mathbf{P}_{n-2}\langle u\rangle = \mathbf{P}_{n-3}\langle u\rangle$. Thus $\mathbf{P}_{n-1}\langle u\rangle = \mathbf{P}_{n-2}\langle u\rangle = \mathbf{P}_{n-3}\langle u\rangle$. Assume that $\mathbf{P}_{n-(i-1)}\langle u\rangle = \mathbf{P}_{n-i}\langle u\rangle$. Then $\pi^{i-1}(\mathcal{P}) \subset \mathbf{P}_{n-i}\langle u\rangle$ and therefore applying theorem 4.15 to the Picard-Vessiot extension $\mathbf{P}_{n-i}\langle u\rangle |\ \mathbf{P}_{n-(i+1)}\langle u\rangle$, we see that $\pi^i(\mathcal{P}) \subset \mathbf{P}_{n-(i+1)}\langle u\rangle$. This shows us that $\mathbf{P}_{n-i}\langle u\rangle = \mathbf{P}_{n-(i+1)}\langle u\rangle$. Thus the above induction argument shows

$$\mathbf{P}_{n-1}\langle u\rangle = \mathbf{C}\langle u\rangle \tag{4.16}$$

and therefore from equation 4.15 we obtain

$$\mathbf{C}\langle u\rangle = \mathbf{C}(\mathcal{S}, \pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x),$$

where $\mathcal{S} = \{L_i(y_1, \cdots, y_m) | 1 \leq i \leq t\} \subset span_{\mathbf{C}}\mathcal{P}$. □

As we noted earlier, $\mathfrak{E} = \cup_{i=0}^{n} \pi^i(\mathcal{P})$ and therefore $\mathcal{P} \subseteq E$ implies $\pi(E) \subset \cup_{i=1}^{n} \pi^i(\mathcal{P})$. Thus $\cup_{i=1}^{n} \pi^i(\mathcal{P}) = \cup_{i=1}^{n} \pi^i(E)$ and hence we also have

$$\mathbf{C}\langle u \rangle = \mathbf{C}(\mathcal{S}, \pi(E), \pi^2(E), \cdots, x).$$

Remark 4.18. From theorem 4.17 we also see that, if $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$ and E the set of *essential elements* of u then

$$\mathbf{C}(\mathfrak{E}) \supseteq \mathbf{C}\langle u \rangle \supset \mathbf{P}_{n-1} \supset \cdots \mathbf{P}_1 \supset \mathbf{P}_0 \supset \mathbf{C}. \tag{4.17}$$

In particular, if $u \in \mathfrak{L}_\infty \setminus \mathbf{C}$ then $x \in \mathbf{C}\langle u \rangle$.

Now we will generalize theorem 4.17 to any finitely generated differential subfield of $\mathfrak{L}_n$.

Theorem 4.19. *Let* $\mathbf{K} := \mathbf{C}\langle u_1, \cdots, u_m \rangle$ *be a finitely differentially generated subfield of* $\mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$ *and let* $E := \cup_{i=1}^{m} E_i$*, where* E_i *is the set of* essential elements *of* u_i*. For each* i*, let* $n_i \in \mathbb{N}$ *be minimal such that* $E_i \subset \cup_{j=0}^{n_i} \Lambda_j$ *and let* $\mathcal{P}_i \subset E_i$ *be the* $\pi-$*base of* $\mathfrak{E}_i := \cup_{j=0}^{n_i} \pi^j(E_i)$*. Then there are finite sets* $\mathcal{S}_i \subset span_{\mathbf{C}}\mathcal{P}_i$ *such that*

$$\mathbf{K} = \mathbf{C}(\mathcal{S}, \pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x),$$

where $\mathcal{S} = \cup_{i=1}^{m} \mathcal{S}_i$ *and* $\mathcal{P} = \cup_{i=1}^{m} \mathcal{P}_i$*. Moreover, for every* $y \in \mathcal{P}$*,* $\mathcal{S}$ *contains at least one linear combination in which* y *appears nontrivially.*

Proof. Since $\mathbf{K}$ is a compositum of singly generated differential fields, the proof follows from theorem 4.17. □

Theorem 4.20. *Every finitely generated differential subfield of $\mathfrak{L}_\infty$ is singly generated.*

Proof. Let $\mathbf{K}$ be a finitely generated differential subfield of $\mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$. Then from theorem 4.19 there are sets $\mathcal{S}$ and $\mathcal{P}$ such that

$$\mathbf{K} = \mathbf{C}(\mathcal{S}, \pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x).$$

Let $\mathcal{S} = \{L_i | 1 \leq i \leq m\}$, $u = \sum_{i=1}^{n} x^i L_i$, $\mathbf{E} := \mathbf{C}(\mathcal{P}, \pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x)$ and let $\mathbf{F} := \mathbf{C}(\pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x)$. We see that $\mathbf{E}|\mathbf{F}$ is a Picard-Vessiot extension(antiderivative extension), and since $L_i \in span_{\mathbf{C}}\mathcal{P}$ we obtain $L_i' \in \mathbf{F}$ and thus $\mathbf{K}$ is an intermediate Picard-Vessiot sub-extension of $\mathbf{E}|\mathbf{F}$. Consider the Picard-Vessiot extension $\mathbf{K}|\mathbf{F}$. Since $\mathbf{F}(\mathcal{S})$=$\mathbf{K}$ is an antiderivative extension of $\mathbf{F}$ and $u \in \mathbf{K}$, we see that for any $\Phi \in \mathbb{G}(\mathbf{K}|\mathbf{F})$

$$\begin{aligned}\Phi(u) &= \sum_{i=1}^{n} x^i \Phi(L_i) \\ &= \sum_{i=1}^{n} x^i (L_i + c_i) \\ &= \sum_{i=1}^{n} x^i L_i + \sum_{i=1}^{n} c_i x^i \\ &= u + \sum_{i=1}^{n} c_i x^i,\end{aligned}$$

where $c_i \in \mathbf{C}$. Thus if Φ fixes u, we obtain $\sum_{i=1}^{n} c_i x^i = 0$ and therefore Φ has to be the identity. Thus $\mathbf{F}\langle u \rangle = \mathbf{K}$. Consider

$$\frac{\partial u}{\partial y} = \sum_{i=1}^{n} x^i \frac{\partial L_i}{\partial y}.$$

We observe from theorem 4.19 that for $y \in \mathcal{P}$ there is an i such that $\frac{\partial L_i}{\partial y} \neq 0$, and we also recall that $\mathcal{P} \cup \{x\}$ is algebraically independent over $\mathbf{C}$. Thus $\frac{\partial u}{\partial y} \neq 0$ for any $y \in \mathcal{P}$ and we also obtain that $E := \mathcal{P} \cup \{x\}$ is the set of *essential elements* of u. It can be easily seen that the $\pi-$base of $\mathfrak{E} := \cup_{i=0}^{n} \pi^i(E)$ is again $\mathcal{P}$ and therefore applying theorem 4.17, we see that $\pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x \subset \mathbf{C}\langle u \rangle$. Thus $\mathbf{F} \subset \mathbf{C}\langle u \rangle$ and therefore $\mathbf{K} = \mathbf{F}\langle u \rangle = \mathbf{C}\langle u \rangle$ and we are done. □

An Algorithm to Compute the Differential field $\mathbf{C}\langle u \rangle$

Theorem 4.21. *Let $u \in \mathfrak{L}_n \setminus \mathfrak{L}_{n-1}$ and let $P, Q \in \mathbf{C}[E]$, where E is the set of essential elements of u, $(P, Q) = 1$ and $u = \frac{P}{Q}$. Then the set $\mathcal{S}$ and $\mathcal{P}$ from theorem 4.17 can be computed from P and Q.*

Proof. Since $\mathcal{P} = E \setminus \cup_{i=1}^{n} \pi^i(E)$, we see that the set $\mathcal{P}$ can be computed once the set E of essential elements is known. From equation 4.16 we see that $\pi(\mathcal{P}), \pi^2(\mathcal{P}) \cdots \pi^n(\mathcal{P}) = \{x\} \subset \mathbf{C}\langle u \rangle$. That is $\mathbf{P}_{n-1} \subset \mathbf{C}\langle u \rangle$ and thus $\mathbf{C}\langle u \rangle$ is an intermediate differential field of the Picard-Vessiot extension $\mathbf{C}(\mathfrak{E})|\mathbf{P}_{n-1}$. That is

$$\mathbf{C}(\mathfrak{E}) \supseteq \mathbf{C}\langle u \rangle \supset \mathbf{P}_{n-1}. \tag{4.18}$$

Also note that $\mathbf{C}(\mathfrak{E})$ is an extension by antiderivatives of $\mathbf{P}_{n-1}$ and that $\mathbf{C}(\mathfrak{E}) = \mathbf{P}_{n-1}(\mathcal{P})$ and $\mathcal{P} \cap \mathbf{P}_{n-1} = \emptyset$ since $\mathfrak{E}$ is algebraically independent over $\mathbf{C}$. Thus $\mathbf{C}(\mathfrak{E})|\mathbf{P}_{n-1}$ is a pure transcendental extension of transcendence degree $|\mathcal{P}|$. Now we may apply theorem 2.8 to obtain the set $\mathcal{S}$. Thus from equation 4.16, we see that $\mathbf{C}\langle u \rangle = \mathbf{P}_{n-1}(\mathcal{S})$. □

Algorithm: Write out two polynomial expressions, say A, B, over $\mathbf{C}$ with elements from Λ_∞ as indeterminates. The following steps will find the differential field $\mathbf{C}\langle u\rangle$, where $u = \frac{A}{B}$, in the form of a finitely generated field expressed in theorem 4.17.

Step 0 First we form a finite set S by picking elements from Λ_∞ that appear in the expression of A or B. Then compute the set E of essential elements of u. That is, find the set

$$E := \left\{ y \in S \middle| \frac{\partial P}{\partial y} \neq 0 \; or \; \frac{\partial Q}{\partial y} \neq 0 \right\}.$$

Also find the set $\mathcal{P} = E \setminus \cup_{i=1}^{n} \pi^i(E)$, where n is the least positive integer such that $\pi^n(E) = \{x\}$ and let $\mathfrak{E} := \cup_{i=0}^{n} \pi^i(E)$.

Step 1 From equation 4.18, we obtain $\mathbf{P}_{n-1} \subset \mathbf{C}\langle u\rangle$. In particular $\pi(\mathcal{P}), \pi^2(\mathcal{P}) \cdots \pi^n(\mathcal{P}) = \{x\} \subset \mathbf{C}\langle u\rangle$. Since $\mathbf{C}(\mathfrak{E})$ is an antiderivative extension of $\mathbf{P}_{n-1}$, we obtain that $\mathbf{C}\langle u\rangle$ is an intermediate differential subfield of the Picard-Vessiot extension $\mathbf{C}(\mathfrak{E})$ of $\mathbf{P}_{n-1}$.

Step 2 We replace A, B by some $P, Q \in \mathbf{C}[E]$ such that $(P, Q) = 1$. This can be done in two ways. We may use $MATHEMATICA$ 5.2 and compute the GCD of A, B and divide A, B by the GCD to get P, Q such that $\frac{A}{B} = \frac{P}{Q}$ and GCD of P, Q is 1. In case, when *MATHEMATICA* 5.2 fails to compute the GCD, we way compute the Gröbner basis [1] for the Ideal $< A, B >$ generated over $\mathbf{C}[S]$ and use Gaydar's formula [2] to compute the GCD and then use the multivariable division algorithm [1] to find out P, Q such that $\frac{A}{B} = \frac{P}{Q}$ and GCD of P, Q is 1.

Thus we note that finding a relatively prime polynomials for a given pair of polynomial from $\mathbf{C}[\Lambda_\infty]$ is a finite process.

Now we have $u = \frac{P}{Q}$, $P, Q \in \mathbf{C}[E]$ and $(P, Q) = 1$.

Step 3 Write P and Q as polynomials over $\mathcal{R} := \mathbf{C}[\pi(\mathcal{P}),\ \pi^2(\mathcal{P}), \cdots, x]$ with elements of $\mathcal{P}$ as variables. Then $\mathbf{P}_{n-1}$ becomes the fraction field of $\mathcal{R}$. Note that $\mathbf{C}(\mathfrak{E})|\mathbf{P}_{n-1}$ is a Picard-Vessiot extension(by antiderivatives) of transcendence degree $p := |\mathcal{P}|$ and thus if $\sigma \in \mathbb{G}(\mathfrak{E}|\mathbf{K})$ then $\sigma(P) = P(y_1 + c_{1\sigma}, \cdots, y_p + c_{p\sigma})$ and $\sigma(Q) = Q(y_1 + d_{1\sigma}, \cdots, y_p + d_{p\sigma})$ where $c_{i\sigma}$, $d_{j\sigma} \in \mathbf{C}$ and $y_i \in \mathcal{P}$. Also from theorem 2.8, we see that $\sigma(u) = u$ if and only if $\sigma(P) = P$ and $\sigma(Q) = Q$.

Step 4 From proposition 2.9 we obtain that if σ fixes P and Q then it fixes each of the homogeneous components of P and Q and from this fact (following the proof of proposition 2.9) we obtain linear forms over $\mathcal{R}$ such that the field generated by $\mathbf{P}_{n-1}$ and the linear forms equals the field $\mathbf{C}\langle u\rangle$. Thus, we compute a system of linear forms $\{D_j\}$ over $\mathcal{R}$ such that $\sigma(P) = P$ and $\sigma(Q) = Q$ if and only if $D_j(c_{1\sigma}, \cdots, c_{p\sigma}) = 0$.

Step 5 Since $\mathcal{R}$ is a polynomial ring, using proposition 2.11, we could compute a system of linear forms $\{L_j\}$ over $\mathbf{C}$ from the system $\{D_j\}$ such that the set of solutions of L_j and D_j over $\mathbf{C}^p$ are the same.

Step 6 Finally, from theorem 4.17 we see that the field

$$\mathbf{C}\langle u\rangle = \mathbf{C}(\mathcal{S}, \pi(\mathcal{P}), \pi^2(\mathcal{P}), \cdots, x),$$

where $\mathcal{S} = \{L_j(y_1, \cdots, y_p) | y_i \in \mathcal{P}\}$.

4.4 Examples

In this section we will apply our algorithm to compute the differential fields generated by an element of $\mathfrak{L}_\infty$ and $\mathbf{C}$. Also we assume $\mathbf{C} := \mathbb{C}$, the field of complex numbers.

Example 1 Consider the field $\mathfrak{L}_1$ and Let

$$u = \frac{5x^3 \ln(x+1) + \ln(x+e) + 27x^3 \ln(x+\sqrt{2})}{\ln(x) + x\big(\ln(x+2) - 17\ln(x+3)\big)^2} \in \mathfrak{L}_1.$$

Step 0 Let $A := 5x^3 \ln(x+1) + \ln(x+e) + 27x^3 \ln(x+\sqrt{2})$ and $B := \ln(x) + x\big(\ln(x+2) - 17\ln(x+3)\big)^2$. We observe that $u \in \mathbf{C}(S)$, where $S = \{x, \ln(x), \ln(x+1), \ln(x+2), \ln(x+3), \ln(x+e), \ln(x+\sqrt{2})\}$. We easily see that the essential elements E equals the set S. The set $\mathfrak{E} = \cup_{i=0}^{1} \pi(E)$ and in this case, we see that $\mathfrak{E} = E$. The $\pi-$base of $\mathcal{P}$ of $\mathfrak{E}$ is the set $\mathcal{P} = \{\ln(x), \ln(x+1), \ln(x+2), log(x+3), \ln(x+e), \ln(x+\sqrt{2})\}$.

Step 1 Since $u \in \mathfrak{L}_1$, we have $n = 1$ and thus $\mathbf{C}(\mathfrak{E}) \supseteq \mathbf{C}\langle u\rangle \supset \mathbf{P}_0 = \mathbf{C}(x)$. The differential field $\mathbf{C}(\mathfrak{E})$ is an antiderivative extension of $\mathbf{C}(x)$ and therefore $\mathbf{C}\langle u\rangle$ is an intermediate differential subfield of the Picard-Vessiot extension $\mathbf{C}(\mathfrak{E})$ of $\mathbf{C}(x)$.

Step 2 We note that A and B are relatively prime and thus we may choose $P := A$ and $Q := B$.

Step 3 We rewrite P and Q as polynomials over $\mathcal{R} := \mathbf{C}[x]$. Then $P = x^3\big(5\ln(x+1) + 27\ln(x+\sqrt{2})\big) + \ln(x+e)$ and $Q = \ln(x) + x\big(\ln(x+2) - 17\ln(x+3)\big)^2$. Let $y_1 := \ln(x+1)$, $y_2 := \ln(x+\sqrt{2})$, $y_3 := \ln(x+e)$, $y_4 := \ln(x)$, $y_5 := \ln(x+2)$ and $y_6 := \ln(x+3)$. We observe that if $\sigma \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{P}_0)$, then $\sigma(y_i) = y_i + c_{i\sigma}$ for each $y_i \in \mathcal{P}$ and we also observe that for any $\sigma \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{C}(x))$, $\sigma(u) = u$ if and only if $\sigma(P) = P$ and $\sigma(Q) = Q$.

Step 4 Note that P is a homogeneous polynomial of total degree 1 over $\mathbf{C}[x]$. If σ fixes P then

$$\begin{aligned} &\sigma(P) = P \\ \iff &\sum_{i=1}^{3} c_{i\sigma}\frac{\partial P}{\partial y_i} = 0 \\ \iff &x^3\big(5c_{1\sigma} + 27c_{2\sigma}\big) + c_{3\sigma} = 0. \end{aligned}$$

Let $D_1 := x^3\big(5y_1 + 27y_2\big) + y_3$. Then we see that for any $\sigma \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{C}(x))$, $\sigma(D_1) = D_1$ if and only if $x^3\big(5c_{1\sigma} + 27c_{2\sigma}\big) + c_{3\sigma} = 0$.

If σ fixes Q then σ fixes the homogeneous components of Q and thus σ fixes

$y_4 := \ln(x)$ and $x\big(y_5 - 17y_6\big)^2$. Now

$$\begin{aligned} &\sigma(x(y_5 - 17y_6)^2) = x(y_5 - 17y_6)^2 \\ \iff &\sum_{i=5}^{6} c_{i\sigma} \frac{\partial Q}{\partial y_i} = 0 \\ \iff &x(c_{5\sigma} - 17c_{6\sigma})(y_5 - 17y_6) = 0 \\ \iff &c_{5\sigma} - 17c_{6\sigma} = 0. \end{aligned}$$

Let $D_2 := y_5 - 17y_6$. Then for any $\sigma \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{C}(x))$, $\sigma(D_2) = D_2$ if and only if $c_{5\sigma} - 17c_{6\sigma} = 0$.

Step 5 Note that $x^3\big(5c_{1\sigma} + 27c_{2\sigma}\big) + c_{3\sigma} = 0$ if and only if $c_{3\sigma} = 0$ and $5c_{1\sigma} + 27c_{2\sigma} = 0$. That is, σ fixes P if and only if it fixes y_3 and $5y_1 + 27y_2$. We also observe that the linear form D_2 is already over $\mathbf{C}$.

Thus we have proved that for any $\sigma \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{C}(x))$, σ fixes u if and only if σ fixes $x, y_3, y_4, 5y_1 + 27y_2$ and $y_5 - 17y_6$.

Step 6

$$\mathbf{C}\langle u\rangle = \mathbf{C}(x, \ln(x+e), \ln(x), 5\ln(x+1)+27\ln(x+\sqrt{2}), \ln(x+2)-17\ln(x+3))$$

□

Example 2

Let $y_1 := \ln(\ln(\ln(x-i)+2)+3)$, $y_2 := \ln(\ln(x+i)+\sqrt{3})$, $y_3 := \ln(x+\frac{5}{6})$, $y_4 := \ln(\ln(x+\frac{1}{2})+\frac{1}{2})$, $y_5 := \ln(x+\sqrt{5})$, $y_6 := \ln(x+5+i)$, $y_7 := \ln(\ln(\ln(x)+i))$

and let

$$u = \frac{\ln(x+i)^2 \ln(x-i)(y_1-y_3)^5 + x^3 \ln(x)(y_2-y_5)^2}{\ln(\ln(x)+i)^2(y_5-y_7)^7 + x\ln(x-i)^3 \ln(\ln(x-i)+2)^2(y_6-y_4)^{12}} \in \mathfrak{L}_3.$$

We will apply the algorithm to compute the differential field generated by $\mathbf{C}$ and u.

Step 0 Let $A := \ln(x+i)^2 \ln(x-i)(y_1-y_3)^5 + x^3 \ln(x)(y_2-y_5)^2$, $B := \ln(\ln(x)+i)^2(y_5-y_7)^7 + x\ln(x-i)^3 \ln(\ln(x-i)+2)^2(y_6-y_4)^{12}$ and $S := \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, \ln(x-i), \ln(x+i), \ln(\ln(x)+i), \ln(x), x, \ \ln(\ln(x-i)+2)\}$. We observe that the set of essential elements E of u equals the set S. Since $\pi(E) = \{\ln(x+\frac{1}{2}), \ln(\ln(x)+i), \ln(\ln(x-i)+2), \ln(x+i)\}$, $\pi^2(E) := \{\ln(x-i), x, \ln(x)\}$ and $\pi^3(E) = \{x\}$, we see that $\mathfrak{E} = \cup_{i=0}^3 \pi^i(E) = E \cup \{\ln(x+\frac{1}{2})\}$. Then the $\pi-$base $\mathcal{P}$ of E is the set $E \setminus \cup_{i=1}^3 \pi^i(E) = \{y_1, y_2, \cdots, y_7\}$.

Step 1 We know that $\cup_{i=1}^3 \pi^i(\mathcal{P}) = \{\ln(x-i), \ln(\ln(x-i)+2), \ln(x+i), \ln(x+\frac{1}{2}), \ln(\ln(x)+i), \ln(x), x\}$ and that $\mathbf{P}_2 = \mathbf{C}(\cup_{i=1}^3 \pi^i(\mathcal{P})) \subset \mathbf{C}\langle u\rangle$. Thus $\mathbf{C}\langle u\rangle$ is an intermediate subfield of the Picard-Vessiot extension(antiderivative extension) $\mathbf{P}_3 := \mathbf{C}(\mathfrak{E})$ of $\mathbf{P}_2$. Also note that $\mathbf{P}_3 = \mathbf{P}_2(y_1, y_2, \cdots, y_7)$.

Step 2 One can easily see that A and B are relatively prime and thus choose $P := A$ and $Q := B$.

Step 3 The polynomials P and Q are already presented as polynomials over the field $\mathbf{C}(\cup_{i=1}^3 \pi^i(\mathcal{P}))$ with $y_1, y_2, \cdots, y_7$ as variables. We note that if $\sigma \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{P}_2)$, then $\sigma(y_i) = y_i + c_{i\sigma}$ for each $y_i \in \mathcal{P}$ and we also observe that for any $\sigma \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{P}_2)$ such that $\sigma(u) = u$ then P divides $\sigma(P)$ and Q divides $\sigma(Q)$. Then from proposition 2.9 we have $\sigma(u) = u$ if and only if $\sigma(P) = P$ and $\sigma(Q) = Q$.

Step 4 Let $\sigma = (c_{1\sigma}, \cdots, c_{7\sigma}) \in \mathbb{G}(\mathbf{C}(\mathfrak{E})|\mathbf{P}_2)$ be an automorphism such that $\sigma(u) = u$. Then $\sigma(P) = P$ and $\sigma(Q) = Q$ and now we shall use proposition 2.9 to compute the linear forms. Note that σ fixes u if and only if it fixes $H_8 := \ln(x+i)^2 \ln(x-i)(y_1 - y_3)^5$, $H_6 = x^3 \ln(x)(y_2 - y_5)^2$ $H_{18} = x\ln(x-i)^3 \ln(\ln(x-i)+2)^2(y_6 - y_4)^{12}$ and $H_9 = \ln(\ln(x))^2(y_5 - y_7)^7$. Thus $\sum_{i=1}^{7} c_{i\sigma} \frac{\partial H_j}{\partial y_i} = 0$ for $j = 6, 8, 9$ and 18, which gives us the following equations

$$\ln(x+i)^2 \ln(x-i)(c_{1\sigma} - c_{3\sigma}) = 0,$$

$$x^3 \ln(x)(c_{2\sigma} - c_{5\sigma}) = 0,$$

$$\ln(\ln(x)+i)^2(c_{5\sigma} - c_{7\sigma}) = 0,$$

$$x\ln(x-i)^3 \ln(\ln(x-i)+2)^2(c_{6\sigma} - c_{4\sigma}) = 0.$$

We also observe that the $\mathbf{P}_2-$linear forms of the field $\mathbf{C}\langle u\rangle$ are H_j, $j = 6, 8, 9$ and 16. That is $\mathbf{C}\langle u\rangle = \mathbf{P}_2(H_6, H_8, H_9, H_{18})$.

Step 5 From the above displayed equations, it is clear that $\sigma(u) = u$ if and only if $c_{1\sigma} - c_{3\sigma} = 0$, $c_{2\sigma} - c_{5\sigma} = 0$, $c_{5\sigma} - c_{7\sigma} = 0$ and $c_{6\sigma} - c_{4\sigma} = 0$.

Step 6

$$\begin{aligned}\mathbf{C}\langle u\rangle =& \mathbf{C}(\ln(x-i), \ln(x+i), \ln(\ln(x)+i), \ln(x+\frac{1}{2}), \ln(x), x,\\ &\ln(\ln(x-i)+2), y_1 - y_3, y_2 - y_5, y_6 - y_4, y_5 - y_7).\end{aligned}$$

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