A STUDY OF THE INVENTION OF LOGARITHMS

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PREFACE

This report is primarily concerned with a study of the invention of logarithms by John Napier, of Scotland. I considered the original idea of logarithms as first discovered by Napier, and the method he used to construct the first tables of logarithms. Material for this report was not easily found, and I wish to express my appreciation for the help received from the Special Services Department of the Oklahoma State University Library. I especially thank Dr. James H. Zant, Director of the Academic Year Institute for his help and advice in preparing this report. I also want to acknowledge the financial assistance received from the National Science Foundation, without which, this year of study would not have been possible.

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CHAPTER I

INTRODUCTION

Having a knowledge of the history and development of a particular phase of mathematics may be a great help to a teacher in introducing and teaching it. It is the opinion of the writer that this is especially true in teaching a unit on logarithms. Too many students in high school algebra and trigonometry think of a logarithm as simply a number in a table.

This report is intended to be a source of additional information to the teacher who is preparing to introduce a unit on logarithms, and may also be used by the student who is interested in learning more about the subject than is usally covered in a high school algebra or trigonometry course.

The writer has found it needful to be able to give the students some idea of how logarithms may be computed. It is a mystery to high school students how the logarithm tables are formed. It takes only a few minutes to demonstrate a short method of computing the logarithm of some numbers, and if this will erase some of the question marks from the student's mind the time will be well spent. An individual needs to be curious to learn, and if this curiousity is not satisfied the person might stop seeking answers to questions. Many times in mathematics a student may study a unit and be left with some questions still not answered. It may not be possible for the teacher to answer all questions involved, since the background of the student may not make understanding the answer possible, but the learning situation is much better if questions of the students are answered with some degree of satisfaction.

A discussion of the invention of logarithms is covered in chapters two and three. The other chapters deal with methods of computing logarithms, and the bases of logarithms. It is hoped this report shall be of service to teachers and students alike.

CHAPTER II

THE INVENTION OF LOGARITHMS

It is probably true that no great mathematical invention, with one solitary exception, has resulted from the work of any one individual. The one solitary exception is the invention of logarithms. The normal development of mathematics follows a series of steps, starting with an idea of a mathematician which slowly grows as others add their thoughts to the original idea. The idea of logarithms was far removed from any other mathematical idea preceeding it, and may well be considered an original idea with the inventor and independent of the work of others.

In 1614 a small book of 147 pages, 90 of them filled with mathematical tables, was published. This book was the result of twenty years of tedious labor by John Napier of Scotland. The purpose of the book, in our present day language bearing the title, "A Description of an Admirable Table of Logarithms", is clearly pointed out in the opening words. To quote Napier

> "Seeing there is nothing (right well-beloved students of mathematics) that is so troublesome to mathematical practice, nor doth mere molest and hinder calculators, than the muliplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hinderances."[1]

[1]Alfred Hooper, Makers of Mathematics, (New York, 1948), p. 169.

During the sixteenth and seventeenth centuries trade between countries in Europe began to flourish, and it was easier and less expensive to transport goods by ship than by other means. Thus studies in navigation were carried out and required much calculating with large numbers. Also, during this span of time great advances were being made in astronomy. Both astronomy and navigation required work that involved many cumbersome and lengthy computations.

Wittich and Clavius (1584), two Danish mathematicians, suggested the use of trigonometrical tables for shortening calculations. For example in trigonometry the formulas (1) and (2) below are found.

(1) $\cos (A + B) = \cos A \cos B - \sin A \sin B$

(2) $\cos (A - B) = \cos A \cos B + \sin A \sin B$

Subtracting (2) from (1) gives

(3) $\cos (A + B) - \cos (A - B) = -2 \sin A \sin B$,

and upon dividing both members of (3) by -2 becomes

(4) $\sin A \sin B = 1/2 (\cos (A - B) - \cos (A + B))$.

Formula (4) transformed the task of finding the product of two sines into a problem of addition, subtraction, and division by two. In Napier's time the sine of an angle might be a number such as 9,934,321, which makes the finding of the product of two sines a lengthy multiplication problem. It may be that formula (4) suggested to Napier the possibility of finding some method whereby the process of multiplication could be reduced to that of addition.

In a short book called "The Constructio", published in 1619, two years after the death of Napier, an explanation of the conclusions reached by Napier and the method he used in calculating the tables of logarithms are given. The origin of his idea seems to have come about by considering arithmetic and geometric progressions.

An arithmetic progression is a series of numbers that increase or decrease in such a way that the difference between any term and the preceding term is always the same. For example, the numbers 0, 1, 2, 3, 4, ... form an arithmetic progression, and is the only one Napier made use of.

A geometric progression is a series of numbers such as 1, 3, 9, 27, 81, ... , which increase or decrease in such a way that the ratio between any term and the preceding term is a constant.

If the terms of the geometric progression 1, 3, 9, 27, 81, 243, 729, ... are labeled in numerical order, that is 0 for 1, 1 for 3, 2 for 9, 3 for 27, and so on, it will be seen that the product of a term labeled 2 and one labeled 3 will give the term labeled 5. To show this more clearly take the geometric progression

1, 2, 4, 8, 16, 32, 64, 128, 256, ...

or $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, \ldots\ldots$

label numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, ...

Note the label numbers are the exponents to which 2 must be raised to give the corresponding term in the geometric progression. It is clear that the product of the term labeled 3 and the term labeled 5 gives the term labeled 8, or 8 x 32 = 256. This fact was undoubtedly known by Napier, and had been known since the time of Archimedes and possibly before. The above discussion should make clear the possibility of a connection existing between terms forming a geometric progression and the label numbers that form an arithmetic progression.

Before considering Napier's method of forming a table of logarithms, it might be well to consider the meaning of the sine of an angle in the sixteenth century. The sine of an angle A (fig. 1) is the length of half the chord subtending twice the angle. In symbols, $\sin\angle A = PM$.

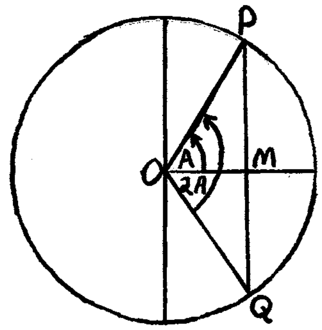


Figure 1

To avoid fractions the length PM was calculated in very small units. A person compiling values of trigonometric functions would choose a very large value, such as 10 million or more for the radius of a circle. By calculation the length of the half-chord, or sine, in terms of these units a close approximation could be tabulated without using fractions. When the sine of an angle is defined in this manner it is easily seen that the sine of 0° is 0, and the sine of 90° is 10,000,000, where 10 million is the value assigned to the radius.

The beginning point in the construction of the logarithms of the sines by Napier was the formation of a decreasing geometric progression, the first term being 10,000,000 and each successive term being 9,999,999/10,000,000 of the preceding term. This progression is formed most easily by subtracting from the first term the 1/10,000,000 the part of it to obtain the second term. The procedure for a few terms is shown below.

		term no.
10,000,000.0000000	....	1
1.0000000		
9,999,999.0000000	....	2
.9999999		
9,999,998.0000001	....	3
.9999998		
9,999,997.0000003	....	4
.9999997		
9,999,996.0000006	....	5
.		
.		
.		
9,999,900.0004950	....	101

Napier formed 100 proportionals in the preceding manner. The idea was to assign label numbers in arithmetic progression to each term in the geometric progression, but at this point it can be seen that it is a most difficult task to continue this progression until the last desired term of 1 is reached. Note in the above table, as the formation progresses the difference between any two successive terms becomes less and less showing that the formation of a geometric progression beginning with 10,000,000 for the first term and ending with 1 for the last term would require an almost endless number of calculations.

Abandoning the original idea Napier formed two other tables of numbers in geometric progression. The common ratio in the second table was 99,999/100,000, which is as near as possible to the ratio of the last term of the first table to the first term of the first table and is convenient to work with. The second table is formed in the same manner as the first one by subtracting from the first term of 10,000,000 its 100 thousandth part to obtain the second term. The procedure is as follows:

		term no.
10,000,000.000000	....	1
100.000000		
9,999,900.000000	....	2
99.999000		
9,999,800.001000	....	3
99.998000		
9,999,700.003000	....	4
.		
.		
.		
.		
9,995,001.224804	....	51

After forming the second table of 51 terms progressing in the ratio 99,999/100,000 a third table consisting of 69 columns, each column containing 21 rows or terms, was formed. The first of the 69 columns was formed using the ratio 9,995/10,000, which is near enough to the ratio of the second number of the second table that no significant error is introduced. Letting the first number of the first column be 10 million and subtracting from the first number its 2000th part the second number 9,995,000 is obtained. The last number of the first column is 9,900,473.57808, only five decimal places being kept since the ratio would introduce an error in the sixth place.

The first number of the second column is found by using the ratio 99/100, and subtracting from the first number in the first column its 100th part to obtain the first number in the second column. Subtracting from the second number in the first column its 100th part gives the second number in the second column. Continuing this procedure all of the numbers of the second column may be found. Subtracting from the first number of the second column its 100th part gives the first number of the third column, and subtracting from the second number of the second column its 100th part gives the second number in the third column. By continuing this procedure all the other columns may be constructed for the third table. An outline of the three tables formed by Napier is given below.

1st table 101 terms	2nd table 51 terms
10,000,000.0000000	10,000,000.000000
9,999,999.0000000	9,999,900.000000
9,999,998.0000001	9,999,800.001000
9,999,997.0000003	9,999,700.003000
.	.
.	.
9,999,900.0004950	9,995,001.224804

3rd table 69 columns 21 terms in each column

1st column	2nd column		69th column
10,000,000.0000	9,900,000.0000	up to	5,048,858.8900
9,995,000.0000	9,895,050.0000		5,046,334.4650
9,990,002.5000	9,890,102.4750		5,043,811.2932
9,985,007.4987	9,885,157.4237		5,041,289.3879
.	.		.
.	.		.
9,900,473.5781	9,801,468.8423		4,998,609.4034

It is obvious that the computation of the above tables was greatly facilitated by the decimal point, which was first used by Napier in the preparation of the tables of logarithms. The use of the decimal point made possible the computation of the tables without any error accumulating by frequent multiplications to the extent that it would be significant in the finished table of logarithms.

Just as the first and second tables were used in the formation of the third table, the third table will be used to form the logarithmic table. The problem then, is to assign to each sine or natural number in the third table a corresponding number or logarithm. These logarithms, or artificial numbers, as they were first called by Napier, are to be an arithmetic progression corresponding to the geometric progression formed by the sines.

To determine the logarithms for the third table it is necessary to have an understanding of the logarithm as thought of by Napier, and to know some of the relations concerning these logarithms. Consider a line AZ with a point P moving with decreasing velocity from A to Z, as shown in Figure 2 below.

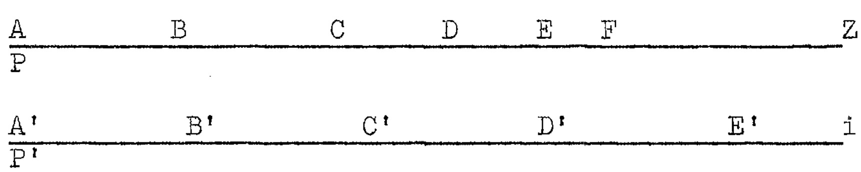


Figure 2

If the distances covered in equal spans of time are marked off on AZ then the distances these points are from Z will be in geometric progression. Now if a point P' is chosen to move with constant velocity equal to the initial velocity of the point P at A, and if P' moves from A' toward i during the same span of time that P moves from A toward Z an arithmetic progression will result from the lengths P' moves in units time along A'i. On AZ the lengths AZ, BZ, CZ, DZ, ... form the geometric progression while on A'i the lengths A'B', A'C', A'D', A'E', ... form the arithmetic progression.

If the whole sine or radius is thought of as the length AZ, then the first proportional or sine is the length BZ and the length A'B' is called the logarithm of the sine BZ. The logarithm of the whole sine AZ is zero, and the logarithms of CZ, DZ, and EZ are respectively A'C', A'D', A'E'. Notice the logarithm of CZ is twice that of BZ and the logarithm of DZ is three times the logarithm of BZ, since A'B' = B'C' = C'D'. From Figure 2 it is also clear that the logarithm of any given sine is greater than the difference between the radius and the given sine, i.e., A'B' $>$ AZ - BZ, or A'B' $>$ AB. ($>$ read, is greater than; $<$ read, is less than).

Now consider the moving point " on AZ moving in the opposite direction with velocity increasing at the same rate as it decreased while moving from A toward Z in Figure 2. In Figure 3 below let P move from A to O in the same time that P' moved from A' to B' and P moved from A to B in Figure 2. Then it is evident that OA $>$ A'B' $>$ AB, or A'B', and the logarithm of

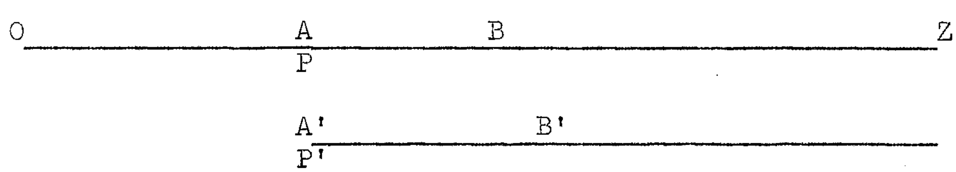


Figure 3

the first sine lies between two limits, the greater limit being OA and the less limit AB. In Figure 3 if a given sine say BZ is subtracted from the radius AZ the less limit AB remains. Also, since the marks on OZ are in proportion OA/AB = AZ/BZ, then OA = AB x AZ/BZ. The greater limit OA is the product of the radius and the lesser limit divided by the given sine BZ.

Taking the first term of the first table as AZ or 10,000,000 the first sine BZ = 9,999,999, then the lesser limit AB = AZ - BZ = 1.0000000. The greater limit OA = 10,000,000 x 1/9,999,999 or 1.000000010. Therefore the first proportional 9,999,999 in table 1 has OA and AB as its limits. Taking the average of the limits gives 1.00000005 as the logarithms of the first proportionals. The logarithm of the second proportional is twice that of the first, and the logarithm of the third is three times that of the first. Continuing in this manner all logarithms of the sines in the first tables are formed. This may be shown to be true by taking the familiar progressions

10	100	1000	10000	100000	. . . . .
1	2	3	4	5	. . . .,

from which it follows that the logarithm of the fourth term is four times the logarithm of the first term. Here recall the familiar logarithms to base 10.

The reader might conclude from the way the logarithms of the first table were formed that if the logarithm of the first sine was known in the second table the other logarithms would immediately be known. This is exactly the case; by finding the logarithm of the first sine in the second table it is easy to see the others would be found just as those of the first table.

Napier found the logarithm of the second sine by using the following rule: (1) The difference of the logarithms of two sines lies between two limits; the greater limit being to radius as the difference of the sines to the lesser sine, and the lesser limit being to radius as the difference of the sines to the greater sine.[2] The rule is proved by referring to the figure below.

[2]David Eugene Smith, A Source Book in Mathematics, (New York, 1929) p. 154

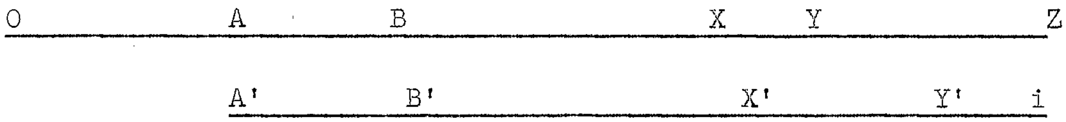


Figure 4

The lengths marked off on OZ are to be considered in geometric progression, and the lengths marked off on A'i are in arithmetic progression. As has already been shown, the logarithm A'B' of the sine BZ lies between the two limits OA and AB. The logarithms of sines XZ and YZ are respectively A'X' and A'Y'. Now A' Y' - A'X' = X'Y', but X'Y' = A'B' since these lengths are in arithmetic progression, then the difference of the logarithms of the two sines lies between the two limits OA and AB, or in mathematical terms OA $>$ A'Y' - A'X' $>$ AB. Since the ratios of any two corresponding terms are equal in a geometric progression, then OA/AZ = XY/YZ and AB/AZ = XY/XZ, which shows the relationship stated in the rule.

The rule just shown to be true will suffice to find the logarithm of any sine near or between those of the first table. Consider for example the sine 9,999,900 in the second table, which is near the last sine 9,999,900.0004950 in table one. By rule (1) the difference of the logarithms of these two sines lies between two limits, hence if these two limits are found the logarithm of the sine in the second table may be found since the logarithm of the other sine is known. Putting the information down in terms of symbols may be easier to follow than a description of the procedure, therefore the procedure is outlined below.

Let the two given sines be

$S_1 = 9{,}999{,}900.0004950$ and $S_2 = 9{,}999{,}900.0000000$

$\log S_1 = 100.0000050$ and $\log S_2 = ?$

The problem is to find the logarithm of S_2. Let A be the greater limit and B the lesser limit. The radius R = 10,000,000. Now rule (I) states

(5) $A/R = (S_1 - S_2)/S_2$

(6) $B/R = (S_1 - S_2)/S_1$.

In formulas (5) and (6) all quantities are known except A and B. Substitution of the known values in (5) and (6) gives A = 0.0004950 and B = 0.0004950. These limits are the same out to the seventh place, due to the small difference in the two sines, therefore the difference in the logarithms of the two sines may be taken as 0.0004950. Then adding this difference to the logarithm of S_1 gives 100.000500 for the logarithm of the sine in the second table. The logarithms of all the sines in the second table may now be found. If the limits differ in say the fourth or fifth decimal place then the average of the two limits should be taken as the difference between the logarithm of the two sines.

Consider now the sine 9,995,000.0000 in the first column of the third table. If the logarithm of this sine is found in the manner that the logarithm of the sine in the second table was found an error will be introduced due to the greater difference in the two sines used in this case, so a fourth proportional is sought such that the ratio of it to radius is equal to the ratio of the lesser sine to the greater sine. If X is the fourth proportional, R the radius, S_1 the lesser sine and S_2 the greater sine, then $X/R = S_1/S_2$. Now since the logarithm of similarly proportioned sines are equidifferent; which means the difference of the logarithms of X and R is equal to the difference of the logarithms of S_1 and S_2, and since the logarithm of R is zero, then the difference of log X and log R is log X. Therefore if the logarithm of X is found then adding it to log S_2 gives log S_1. The limits of the logarithm of X are found by rule (I), and adding the average of these limits to the logarithm of S_2 gives the logarithm of S_1. All of the other logarithms of the sines in the first column are found just as those of table one and two were determined after the logarithm of the first sine was found.

The logarithm of the first sine of the second column of the third table is found by the same procedure used to find the logarithm of the first sine in the first column. The first sines in all the other columns are found in the same manner.

It is interesting to note the first and second tables were essential in forming the third table. After all the logarithms of the sines in the third table are determined it is used for the construction of the final logarithmic table.

All the logarithms of sines embraced within the third table may be found by the same procedure used to find the first sines in each column. The logarithms of all the sines embraced within the second table are found simply by subtracting the given sine from radius. It should be noted here that the logarithms in the third table are only given to the first decimal place and those in the final table are given to the nearest whole number.

The logarithms of sines less than the last sine in the third table are found by establishing a relationship between sines in a certain ratio and the differences of their logarithms. It can be shown for example that all sines in the ratio of two to one have 6,931,469.22 for the difference of their logarithms, and all sines in the ratio ten to one have 23,025,842.34 as the difference of their logarithms. Then it follows that all sines whose ratios are multiples of two to one or ten to one would have the same multiple times the difference of the logarithms of the sines in the ratio two to one and ten to one. For example, if two sines are in the ratio four to one the difference of their logarithms would be twice the difference of the logarithms of two sines in the ratio two to one. Using this relationship the following table was formed by Napier.

Short Table

Given Ratio of Sines	Corresponding Difference of Logarithms	Given Ratio of Sines	Corresponding Difference of Logarithms
2 to 1	6931469.22	8000 to 1	89871934.68
4 to 1	13862938.44	10000 to 1	92103369.36
8 to 1	20794407.66	20000 to 1	99034838.58
10 to 1	23025842.34	40000 to 1	105966307.80
20 to 1	29957311.56	80000 to 1	112897777.02
40 to 1	36888780.78	100000 to 1	115129211.70
80 to 1	43820250.00	200000 to 1	122060680.92
100 to 1	46051684.68	400000 to 1	128992150.14
200 to 1	52983153.90	800000 to 1	135923619.36
400 to 1	59914623.12	1000000 to 1	138155054.04
800 to 1	66846092.34	2000000 to 1	145086523.26
1000 to 1	69077527.02	4000000 to 1	152017992.48
2000 to 1	76008996.24	8000000 to 1	158949461.70
4000 to 1	82940465.46	10000000 to 1	161180896.38

As an example of how the logarithm of a sine outside the limits of the third table may be found take the sine 378064.00. Multiplying this sine by 20 gives 7561280.00, a sine within the third table. Now determine the logarithm of 7561280.00 by the procedure outlined for sines near or between those of the third table. The logarithm will be 2795444.9, and to this add 29957311.56 from the table above giving 32757556. as the logarithm of 378064.0.

The next problem in the construction of the logarithmic table is the determine the particular sine that corresponds to a given angle. It should be clear that the sine of $90^\circ\ 0'$ is 10 million and the sine of $30^\circ\ 0'$ is 5 million, and that the sines of $45^\circ 0'$ and $60^\circ 0'$ are easily found. Since the purpose of this report is to show how logarithms may be found for numbers in geometric progression, no attempt will be made to show how the sines for angles such as $63^\circ\ 40'$, $23^\circ\ 25'$, etc. are determined It should be clear to the reader that there will be 5400 terms in the final logarithmic table formed by Napier, since for each degree there are 60 minutes, and Napier was computing the logarithms for sines of angles to the minute. For each one of these 5400 different angles a particular sine would correspond, and from the third table these sines could be found. The rules which enabled Napier to find the particular sines corresponding to the angles are given below.

As half radius is to the sine of half a given arc, so is the sine of the complement of the half arc to the sine of the whole arc.

Double the logarithm of an arc of 45 degrees is the logarithm of half radius.

The sum of the logarithms of half radius and any given arc is equal to the sum of the logarithms of half the arc and the complement of the half arc. Hence, the logarithm of the half arc may be found if the logarithms of the other three be given.

When the logarithms of all arcs not less than 45 degrees are given the logarithms of all less arcs are very easily obtained.

The above rules should suffice to give the reader an idea of how the angles for the sines were found. The rules may be proved using geometric principles.

Below is given a portion of the third table with a few of the sines and their logarithms. This should help as a means of checking some of the computations that may be carried out by the reader who wants to examine some of the rules and relations discussed in this chapter.

THIRD TABLE

1st Column

Sines	Logarithms
10000000.0000	.0
9995000.0000	5001.2
9990002.5000	10002.5
9985007.4987	15003.7
9980014.9950	20005.0
.	.
.	.
.	.
9900473.5780	100025.0

2nd Column

Sines	Logarithms
9900000.0000	100503.3
9895050.0000	105504.6
9890102.4750	110505.8
9885157.4237	115507.1
9880214.8451	120508.3
.	.
.	.
.	.
9801468.8423	200528.2

69th Column

Sines	Logarithms
5048858.8900	6834225.8
5046334.4605	6839227.1
5043811.2932	6844228.3
5041289.3879	6849229.6
5028668.7435	6854230.8
.	.
.	.
.	.
4998609.4034	6934250.8

CHAPTER III

THE CONTRIBUTION OF BRIGGS AND OTHERS

Henry Briggs (1556 - 1631), Professor of Geometry at Oxford, received the work of Napier with such enthusiasm that a meeting with Napier was arranged. During this meeting both Napier and Briggs agreed that the tables would be more useful if they were altered so that the logarithm of 1 would be 0 and the logarithm of 10 would be an appropriate power of 10. Thus the Briggs or common system of logarithms of today was born, and Briggs devoted a considerable amount of time thereafter to the construction of tables of logarithms based upon the new plan.

In 1624 Briggs published "Arithmetica Logarithmica", containing a 14-place table of common logarithms of the numbers from 1 to 20,000 and from 90,000 to 100,000. The gap between 20,000 and 90,000 was later filled in with help, by Adriaen Vlacq (1600 - 1666), a Dutch bookseller and publisher.

In the formation of a table of common logarithms a great deal of computation is necessary as is shown by the following discussion. To calculate logarithms Briggs made use of the following facts. Consider the series below:

1	2	3	4	5	6	7	. . . .	. arithmetic series (logs)
2	4	8	16	32	64	128	. . . .	. geometric series (antilogs)

Now the arithmetic mean between any two numbers a and b is $1/2\ (a + b)$, and the geometric mean between two numbers A and B is $\sqrt{AB}$. As an example, take $a = 2$ and $b = 4$ in the arithmetic series, then $1/2\ (a + b) = 1/2\ (2 + 4) = 3$. Note the arithmetic mean of any three consecutive numbers in the arithmetic series is the middle one, and the same is true for three consecutive numbers in the geometric series. For example, consider $A = 4$ and $B = 16$ in the geometric series, then $\sqrt{AB} = \sqrt{4 \times 16} = \sqrt{64} = 8$. These properties hold true even if the difference between the terms in the arithmetic series is a fraction. The method Briggs used then, is essentially inserting arithmetic means in between geometric means to find the logarithm of prime numbers. After the logarithms of the prime numbers are known the other logarithms may be found from the relations of logarithms, i.e., $\log A/B = \log A - \log B$, $\log AB = A + \log B$. The calculation of the logarithm of 5 is given below as an example of the method Briggs used to form a table of logarithms.

Numbers or antilogarithms			Logarithms		
A =		1.000000	a =		0.0000000
B =		10.000000	b =		1.0000000
C =	$\sqrt{AB}$ =	3.162277	c =	1/2 (a + b) =	0.5000000
D =	$\sqrt{BC}$ =	5.623413	d =	1/2 (b + c) =	0.7500000
E =	$\sqrt{CD}$ =	4.216964	e =	1/2 (c + d) =	0.6250000
F =	$\sqrt{DE}$ =	4.869674	f =	1/2 (d + e) =	0.6875000
G =	$\sqrt{DF}$ =	5.232991	g =	1/2 (d + f) =	0.7187500
H =	$\sqrt{FG}$ =	5.048065	h =	1/2 (f + g) =	0.7031250
I =	$\sqrt{FH}$ =	4.958069	i =	1/2 (f + h) =	0.6953125
J =	$\sqrt{HI}$ =	5.002865	j =	1/2 (h + i) =	0.6992187
K =	$\sqrt{IJ}$ =	4.980416	k =	1/2 (i + j) =	0.6972656
L =	$\sqrt{JK}$ =	4.991627	l =	1/2 (j + k) =	0.6982421
M =	$\sqrt{JL}$ =	4.997240	m =	1/2 (j + l) =	0.6987304
N =	$\sqrt{JM}$ =	5.000052	n =	1/2 (j + m) =	0.6989745
O =	$\sqrt{MN}$ =	4.998647	o =	1/2 (m + n) =	0.6988525
P =	$\sqrt{NO}$ =	4.999350	p =	1/2 (n + o) =	0.6989135
Q =	$\sqrt{OP}$ =	4.999701	q =	1/2 (o + p) =	0.6989440
R =	$\sqrt{PQ}$ =	4.999876	r =	1/2 (p + q) =	0.6989592
S =	$\sqrt{NQ}$ =	4.999963	s =	1/2 (n + q) =	0.6989668
T =	$\sqrt{NS}$ =	5.000008	t =	1/2 (n + s) =	0.6989707
U =	$\sqrt{ST}$ =	4.999984	u =	1/2 (s + t) =	0.6989687
V =	$\sqrt{TU}$ =	4.999997	v =	1/2 (t + u) =	0.6989697
W =	$\sqrt{TV}$ =	5.000003	w =	1/2 (t + v) =	0.6989702
X =	$\sqrt{VW}$ =	5.000000	x =	1/2 (v + w) =	0.6989700

The logarithm of a prime number such as 41 would have to be found using 1 and 100, whose corresponding logarithms are 0 and 2.

Napier's only rival for priority of invention of logarithms was Jobst Bürgi (1552 - 1932), a Swiss instrument maker. Bürgi conceived and constructed a table of logarithms independently of Napier, but did not publish the results until 1620, six years after Napier announced the discovery of logarithms. The work of Bürgi was definitely based on the laws of exponents, as logarithms are now regarded to be. It is generally believed that Napier conceived the idea of logarithms before Bürgi.

The calculations in astronomy and navigation were not solely responsible for man seeking methods to shorten such calculations. The flourishing trade between nations during the 16th and 17th centuries called for quicker ways of calculating interest, and led to the construction of tables for such calculations. The calculation of compound interest is a practical application of the geometric series. If r is the rate of interest per dollar invested, then in one year \$1.00 invested grows to \$(1 + r), and in two years grows to $\$(1 + r)^2$, etc. For instance if r = 5%, then in one year \$1.00 grows to \$(1.05); at the end of two years it grows to $\$(1.05)^2 = \$1.1025$.

For interest compounded once a year the following series are given: The amount is

$(1 + r)^0 \quad (1 + r)^1 \quad (1 + r)^2 \quad (1 + r)^3 \quad (1 + r)^4 \quad (1 + r)^5$

at the end of

0 1 2 3 4 5 years.

The years making up the arithmetic series correspond to the exponents, or logarithms as they are now known to be, in the geometric series. If a table of logarithms was made from the two series above what would the base be? To make the above series more meaningful compare them with the arithmetic and geometric series below.

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad . \ . \ . \ . \ . \ . \ . \ . \ .$

$2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5 \quad 2^6 \quad 2^7 \quad . \ . \ . \ . \ . \ . \ . \ . \ .$

Simon Stevin (1548 - 1620) of Bruges in Belgium prepared tables of compound interest for calculations in commercial arithmetic. These tables were actually logarithmic tables, but were not recognized as such at the time.

The series for the limiting value of e, the base of the natural logarithms, arises in the construction of compound interest tables. This series can be obtained from the expansion of $(1 + 1/n)^n$, and has a limiting value of e = 2.718281828459......... To see how this value for e is suggested by a study of compound interest a list of the amounts to which $1.00 has grown after n years at a rate r = 1/n is given.

Number of Years n	Rate of Interest 1/n	Amount
20	5 %	$2.653
25	4 %	2.666
40	2 1/2 %	2.685
50	2 %	2.682
100	1 %	2.705

If the table is continued for larger and larger values of n the amount gets closer and closer to the value e. A more detailed discussion of the base e will be made in the next chapter.

CHAPTER IV

BASES OF LOGARITHMS AND SERIES

COMPUTATION OF LOGARITHMS

The tremendous labor required for the construction of logarithmic tables naturally led mathematicians to search for quicker and easier methods of calculating logarithms of numbers. This search gave new impetus to the study of infinite series. As was shown in the preceding chapter, the base e of the Napierian or natural logarithms is related to infinite series. In this chapter e is shown to be a suitable base for a system of logarithms, and also gives a convenient method of calculating logarithms.

The Base e

To find e expand the expression $(1 + 1/n)^n$ by the binomial formula giving

$$(1) \quad (1 + 1/n)^n = 1 + n/n + \frac{n(n - 1)}{n^2 \times 2!} + \frac{n\,(n - 1)\,(n - 2)}{n^3 \times 3!} + \ldots.$$

$$= 2 + (1 - 1/n)\,1/2! + \left(1 - \frac{3n - 2}{n^2}\right) 1/3! + \ldots\ldots$$

Now as n takes on larger and larger values (1) approaches a limiting value, since for large values of n the value of the expressions $(1 - 1/n)$ and $(1 - (3n - 2)\,/n^2)$ are very close to 1. These expressions may be made as near 1 in value as desired by letting n assume a sufficiently large value. Therefore series (1) may be made to approach a limiting value by letting n approach an infinitely large value. Then the limit of (1) is

(2) $(1 + 1/n)^n = (2 + 1/2! + 1/3! + 1/4! + \ldots\ldots\ldots)$.

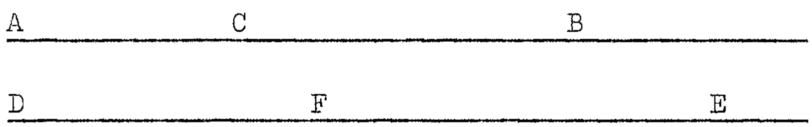
The series (2) has a limiting value between 2.7 and 2.8. It is denoted by the letter e. The value of e may be calculated as accurately as desired by taking a sufficient number of terms in the series. The number e serves as the base of the natural system of logarithms. It is a convenient base to use, since a logarithmic series may be developed for computing logarithms of numbers to base e. The base e logarithms also simplifies the differentiation of the logarithmic function in calculus. An approximate value of e may be computed from the series (2) as follows:

1 + 1			=	2.000000000
1 ÷ 2!			=	.500000000
1 ÷ 3!	=	(1 ÷ 2!) ÷ 3	=	.166666667
1 ÷ 4!	=	(1 ÷ 3!) ÷ 4	=	.041666667
1 ÷ 5!	=	(1 ÷ 4!) ÷ 5	=	.008333333
1 ÷ 6!	=	(1 ÷ 5!) ÷ 6	=	.001388889
1 ÷ 7!	=	(1 ÷ 6!) ÷ 7	=	.000198413
1 ÷ 8!	=	(1 ÷ 7!) ÷ 8	=	.000024802
1 ÷ 9!	=	(1 ÷ 8!) ÷ 9	=	.000002756
1 ÷ 10!	=	(1 ÷ 9!) ÷ 10	=	.000000276
1 ÷ 11!	=	(1 ÷ 10!) ÷ 11	=	.000000025
		e	=	2.7182818 to seven places.

The Base of Napier's Logarithms

Although the base e logarithms are called Napierian or natural logarithms the base e is not the base of the logarithms first formed by Napier. Note the logarithms of Napier increased as the numbers decreased, which is just opposite to the natural logarithms. Napier undoubtedly did not even think of a base in developing logarithms, and it was not until Euler pointed out the connection between logarithms, bases, and exponents that logarithmic bases were considered.

To see the relation that exists between the base of Napier's logarithms and the base <u>e</u> of the natural logarithms consider the figure below.



Let $AB = a = 10^7$, $x = DF$, and $y = BC$, then $AC = a - y$. Remember Napier considered AB as the whole radius, and a point moving along AB from A toward B with velocity decreasing geometrically. Using a bit of calculus; if the velocity of the point C is defined as

(1) $$\frac{d(a - y)}{dt} = y,$$

then separating the variables in (1) and integrating gives

$$\int \frac{d(a - y)}{dt} = \int dt,$$

or

(2) $-\text{nat. log } y = t + K$, where K is the constant of integration.

Now when $t = 0$, then $y = AB = a$ and

(3) $K = -\text{nat. log } a$

Now since in Napier's concept of logarithms the point F moves with a constant velocity equal to the initial velocity of point C when $t = 0$, that is, the velocity $y = a$, then the velocity of the point F is given by

(4) $$\frac{dx}{dt} = a$$

From (4) $dx = a\,dt$, which upon integrating each side gives

$$(5) \qquad x = at,$$

but by definition x = Nap. log y, hence by substitution Nap. log y = x = at, or

$$(6) \qquad \text{Nap. log } y = 10^7 (-\text{ nat. log } y - K),$$

which becomes upon substituting the value of K from (3)

$$(7) \qquad \text{Nap. log } y = 10^7 \text{ nat. log } 10^7/y.$$

Formula (7) gives the relation between the two systems.

The Exponential Series

The exponential series is the development in ascending powers of x the xth power of a certain constant base. The series is derived by using the binomial formula as follows:

Expanding by the binomial formula the expression $(1 + 1/n)^{nx}$ gives

$$(1) \quad (1 + 1/n)^{nx} = 1 + \frac{nx}{n} + \frac{nx(nx-1)}{n^2 \times 2!} + \frac{nx(nx-1)(nx-2)}{n^3 \times 3!} + \ldots\ldots$$

$$= 1 + x + (x^2 - x/n)\, 1/2! + \left(x^3 - \frac{3nx^2 - 2x}{n^2}\right) 1/3! + \ldots\ldots\ldots$$

In (1) as n approaches an infinitely large value the expression $(x^2 - x/n)$ approaches x^2 as a limit, and the expression $(x^3 - (3nx^2 - 2x)/n^2)$ approaches x^3 as a limit. Hence for n approaching infinitely large values the limiting value of the series in (1) becomes

$$(2) \qquad (1 + 1/n)^{nx} = (1 + x + x^2/2! + x^3/3! + \ldots\ldots\ldots\ldots)$$

Now since from the rule of exponents $(x^m)^n = x^{mn}$, then it follows that

$$(3) \qquad ((1 + 1/n)^n)^x = (1 + 1/n)^{nx},$$

and substitution in (2) gives

(4) $((1 + 1/n)^n)^x = (1 + x + x^2/2! + x^3/3! + \ldots\ldots\ldots)$

But as n approaches a large value the expression $(1 + 1/n)^n$ approaches $(1 + 1 + 1/2! + 1/3! + \ldots\ldots)$ or e, and substitution in (4) gives

(5) $e^x = 1 + x + x^2/2! + x^3/3! + \ldots\ldots\ldots\ldots\ldots$

In (5) e^x is called the exponential function of x, and the series developed from e^x is called the exponential series. To derive a formula applicable to any positive constant base a, let $\log_e a = k$, then $a = e^k$ and $a^x = e^{kx} = e^{(\log_e a)x}$. Therefore by (5)

(6) $$a^x = 1 + (\log_e a)x + \frac{(\log_e a)^2\, x^2}{2!} + \frac{(\log_e a)^3 x^3}{3!} + \ldots\ldots$$

The convergency of the series derived in this chapter will not be discussed here, since the purpose is simply to show the series involved in computing logarithms and how they are derived. Formula (6) is called the exponential formula and will be used to derive the logarithmic series.

The Logarithmic Series

The logarithmic series is the expansion of $\log_e(1 + x)$ in ascending powers of x. It is derived as follows: By the exponential formula, when $(1 + x)$ is the base and y the exponent,

(1) $$(1 + x)^y = 1 + (\log_e (1 + x))y + \frac{(\log_e(1 + x))^2 y^2}{2!} + \ldots\ldots$$

By the binomial formula,

(2) $$(1 + x)^y = 1 + yx + \frac{y(y - 1)}{2!}x^2 + \frac{y(y - 1)\,(y - 2)}{3!}x^3 + \ldots\ldots$$

Now equating the second members of (1) and (2) gives

(3) $$1 + (\log_e (1 + x))y + \frac{(\log_e(1 + x))^2 y^2}{2!} + \ldots\ldots$$

$$= 1 + yx + \frac{y(y - 1)x^2}{2!} + \frac{y(y - 1)\,(y - 2)x^3}{3!} + \ldots\ldots$$

$$= 1 + xy + \frac{x^2y^2 - x^2y}{2!} + \frac{x^3y^3 - 3x^3y^2 + 2x^3y}{3!} + \ldots\ldots$$

$$= 1 + \quad x + \frac{-1x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots \quad y + \frac{x^2y^2}{2!} + \frac{-3x^3y^2}{3!} + \frac{x^3\,y^3}{3!} + \ldots$$

Now equation (3) is an identical equation since the two series have the same sum and both are absolutely convergent for all finite rational values of y, x being numerically less than 1. Therefore, equating the coefficients of y in the two series gives

$$(4)\quad \log_e(1 + x) = x + \frac{-1x^2}{2!} + \frac{(-1)\,(-2)x^3}{3!} + \frac{(-1)\,(-2)\,(-3)x^4}{4!} + \ldots\ldots$$

Simplifying the second member gives

$$(5)\quad \log_e(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots\ldots$$

This series is called the logarithmic series, and is absolutely convergent for $x < 1$, and conditionally convergent for $x = 1$.

To Compute Natural Logarithms

Since the logarithmic series

$$(1)\quad \log_e(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots\ldots$$

is not convergent for $x > 1$, it cannot be used to find the natural logarithm of any positive number, however great; and for ease in computation it is desirable that the series obtained be rapidly convergent.

Substituting -x for x in (1) gives

$$(2)\quad \log_e(1 - x) = -x - x^2/2 - x^3/3 - x^4/4 - \ldots\ldots$$

Subtracting (2) from (1)

(3) $\log_e(1 + x) - \log_e(1 - x) = 2(x + x^3/3 + x^5/5 + \ldots\ldots.),$

or

(4) $\log_e \dfrac{1 + x}{1 - x} = 2(x + x^3/3 + x^5/5 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

which is true when $x < 1$. Let n be a positive number when natural logarithm is known, and let m be a greater positive number whose natural logarithm is to be computed. Then, since $(m - n) / (m + n)$ is positive and less than 1, this value may be substituted for x in (4). If $x = (m - n) / (m + n)$

$\dfrac{1 + x}{1 - x} = \dfrac{1 + (m - n) / (m + n)}{1 - (m - n) / (m + n)} = \dfrac{2m}{2n} = \dfrac{m}{n}$; and in (4) since

$\log_e \dfrac{1 + x}{1 - x} = \log_e \dfrac{m}{n} = \log_e m - \log_e n$, then

(5) $\log_e m = \log_e n = 2\left(\dfrac{m - n}{m + n} + \dfrac{1}{3}\left(\dfrac{m - n}{m + n}\right)^3 + \dfrac{1}{5}\left(\dfrac{m - n}{m + n}\right)^5 + \ldots\ldots\right)$

This is the logarithmic formula for $m > n > 0$.

Since $\log_e 1 = 0$, by substituting 1 for n and 2 for m, $\log_e 2$ may be found; then by substituting 2 for n and 3 for m $\log_e 3$ may be found; etc. Hence, a table of logarithms may be constructed by substituting for n in (5) the successive values 1, 2, 3, 4, ... and for m values greater by one in each instance.

Substituting $n + 1$ for m in (5) gives the more convenient formula

(6) $\log_e(n + 1) = \log_e n = 2\left(\dfrac{1}{2n + 1} + \dfrac{1}{3(2n + 1)^3} + \ldots\ldots\right),$

which is true for all positive values of n. By taking a value of n and substituting in (6) it can be seen that (6) converges very rapidly. To see how formula (6) is used consider the following example:

Find the natural logarithm of 2 to the nearest sixth decimal place.

Solution. - - Substituting 1 for n and 0 for $\log_e 1$ in the formula for $\log_e(n + 1)$ gives

$$\log_e 2 = 0 + 2(1/3 + 1/3\cdot3^3 + 1/5\cdot3^5 + 1/7\cdot3^7 + \ldots\ldots\ldots\ldots)$$

$$= 2/3 + 2/3\cdot3^3 + 2/5\cdot3^5 + 2/7\cdot3^7 + \ldots\ldots\ldots\ldots\ldots\ldots$$

Since $2/3^3 = 2/3 \div 9$, $2/3^5 = 2/3^3 \div 9$, $2/3^7 = 2/3^5 \div 9$, etc., and since these quotients are divided by 1, 3, 5, 7, . . . respectively, the computation may be neatly arranged as follows:

3	2.00000000			
9	.66666667	÷ 1	=	0.66666667
9	.07407407	÷ 3	=	.02469136
9	.00823045	÷ 5	=	.00164609
9	.00091449	÷ 7	=	.00013064
9	.00010161	÷ 9	=	.00001129
9	.00001129	÷ 11	=	.00000103
	.00000125	÷ 13	=	.00000010

adding gives $\log_e 2 = 0.69314718 = 0.693147$ to six places.

Computing Base Ten Logarithms

The base e arises naturally in the process of finding a formula for computing logarithms. Natural logarithms are more convenient to use in theoretical work, but in numerical calculations common or Briggs logarithms are the most convenient to use because the base of the common logarithms is the same as the base of the decimal system of notation. Hence, the next problem is the see how natural logarithms can be changed to common logarithms.

Let N be the number whose logarithm to base 10 is sought. From formula (6) of the previous section $\log_e N$ and $\log_e 10$ may be found. Suppose

(1) $\log_e N = p$, or $N = e^p$, and that

(2) $\log_e 10 = q$, or $10 = e^q$.

Now let r be the multiplier, as yet unknown, by which $\log_e N$ is multiplied to produce $\log_{10} N$; that is, let

(3) $\log_{10} N = r \log_e N = rp$, then

by (3) $N = 10^{rp}$, and by (2)

(4) $N = (e^q)^{rp} = e^{qrp}$

by (4) and (1) $e^{qrp} = e^p$, from which

(5) $qrp = p$, and $r = 1/q$, a constant.

Now by (2) $q = \log_e 10$ and upon substituting this in (5)

(6) $r = 1/\log_e 10$.

By formula (6) of the previous section it is found that $\log_e 10 =$ 2.30258509. . ., from which r = .43429448. . . Then the logarithm of any number to base 10 may be found by formula (3) of this section, by using r equal to the value given above. In general formula (6) of this section holds for any base a so that $r = 1/\log_e a$. A discussion of the relation of base e to other bases can usually be found in college algebra and trigonometry textbooks.

CHAPTER V

CONCLUSION

The problem involved in this report was simply to study the invention of logarithms; being principally concerned with the construction of the logarithmic tables, and writing the results in a form suitable for study by a high school mathematics teacher or student.

In summary the main points considered in this report were discussions concerned with John Napier's invention of logarithms, the work of Briggs and others on the improvement of logarithmic tables, and a rather detailed account of the development of series for computing logarithms. It is difficult to find material for any further study of the historical account of the invention of logarithms, but a further study of series may be pursued by those interested, and much information on this subject may be found.

There is little doubt that John Napier was the true inventor of logarithms and no doubt at all that his ideas were original. This study should give one the feeling that great things may be accomplished by the invention of such mathematical ideas as logarithms. It would be impossible to estimate the worth of logarithms to mankind. The slide rules carried by the many engineering and science students are good examples of the value of Napier's invention.

Although many wonder about the value of some mathematics taught just because immediate applications may not come to mind, it should always be kept in mind that mathematics has thrived because of its many applications. A study of an invention in the past should give one a feeling that such efforts on the part of mathematicians are not in vain; and even if some of the mathematics being stressed now doesn't make too much sense to those who are not well versed in the subject it should not be condemmed in view of the great value past mathematical ideas have been to man.

BIBLIOGRAPHY

Bakst, Aaron, _Mathematics its Magic and Mastery_, Princeton: D. Van Nostrand Co., Inc. 1952.

Cajori, Florion, _A History of Mathematics_, New York: The Macmillian Company, 1893.

Hogben, Lancelot, _Mathematics for the Million_, New York: W. W. Norton and Company, Inc., 1951.

Hooper, Alfred, _Makers of Mathematics_, New York: Random House, Inc., 1948.

Kasner, Edward, and James Newman, _Mathematics and the Imagination_, New York: Simon and Schuster, 1958.

Macdonald, William Rae, _The Construction of the Wonderful Canon of Logarithms_, London: William Blockwood and Sons, 1620.

Milne, William J., _Advanced Algebra for Colleges and Schools_, New York: America Book Company, 1901.

Scott, J. F., _A History of Mathematics_, London: Taylor and Francis Ltd. 1958.

Smith, David E. _A Source Book in Mathematics_, New York: McGraw-Hill Book Co., Inc., 1929.

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