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AN ALGEBRAIC APPROACH TO BLOCKING AND CONFOUNDING

IN FACTORIAL ARRANGEMENTS

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TABLE OF CONTENTS

	Page
LIST OF TABLES	v
LIST OF ILLUSTRATIONS	vi
Chapter	
I. INTRODUCTION	1
II. DEVELOPMENT AND ANALYSIS OF DESIGNS INVOLVING COMPLETE REPLICATES OF FACTORIAL ARRANGEMENTS OF TREATMENTS	7
III. ESTIMATION AND ANALYSIS OF PARTITIONED FACTORIAL ARRANGEMENTS OF TREATMENTS	32
IV. EXAMPLES OF BLOCKING AND CONFOUNDING OF FACTORIAL ARRANGEMENTS OF TREATMENTS	54
V. A PRACTICAL EXAMPLE	79
VI. SUMMARY	85
LIST OF REFERENCES	88
APPENDIX I	91
APPENDIX II	92
APPENDIX III	93
APPENDIX IV	94

LIST OF TABLES

Table		Page
2.1	Abbreviated Analysis of Variance Table for One Replicate of an $n_1 \times n_2 \times \ldots \times n_m$ Factorial Arrangement of Treatments	24
2.2	Abbreviated Analysis of Variance Table for One Replicate of a 2 x 2 x 3 Factorial Arrangement of Treatments	25
2.3	Abbreviated Analysis of Variance for r Replicates of a C.R.D. of an $n_1 \times n_2 \times \ldots n_m$ Factorial Arrangement	29
2.4	An Abbreviated Analysis of Variance Table for b Blocks of a R.C.B.D. of an $n_1 \times n_2 \times \ldots \times n_m$ Factorial Arrangement of Treatments	31
4.1	An Analysis of Variance Table for Plan (C) in Figure 4.1	60
4.2	An Abbreviated Analysis of Variance Table for a Particular Plan in Figure 4.2	65
4.3	An Abbreviated Analysis of Variance Table for a Particular Plan of Figure 4.3	67
4.4	Abbreviated Analysis of Variance Table for the Plan Given in Figure 4.4	70
4.5	An Abbreviated Analysis of Variance Table for the Plan in Figure 4.5	72
4.6	An Abbreviated Analysis of Variance Table for the Plan $B_1 \cap B_2$ of a 2 x 2 x 3 x 3 Factorial Arrangement of Treatments	75
4.7	An Abbreviated Analysis of Variance Table of a 2 x 6 x 3 Factorial Arrangement of Treatments with the Plan $B_1 \cap B_2$ in Figure 4.7	78

LIST OF ILLUSTRATIONS

Figure		Page
2.1	The Lexicographic Order of the Design Points of a 2 x 2 x 3 Factorial Arrangement of Test	13
2.2	A Matrix L Defining the 8 Factorial Effects of a 2 x 2 x 3 Factorial Arrangement	19
3.1	The Incidence Matrices of Two Blocking Plans of a 2 x 2 x 3 Factorial	34
3.2	The Intersection of Plan (a) and Plan (b) of Figure 3.1 and Its Incidence Matrix	39
3.3	Extension of a Plan of a 2 x 2 Factorial Arrangement to a 2 x 2 x 3 Factorial Arrangement	51
4.1	The 24 Plans of a 4 x 4 Factorial Arrangement of Treatment Each of Which Confounds 3 Components of the A_1A_2 Effect	57
4.2	The 48 Plans of a 4 x 4 Factorial Each of Which Confounds 1 Component of the A_1A_2 Effect	62
4.3	The 3 Plans Confounding only the Mean and 1 Component of $A_1A_2A_3$ in a 2 x 2 x 4 Factorial Arrangement of Treatments	66
4.4	A Plan Confounding the Mean Effect and 1 Component each of $A_1A_2A_3$ and A_3 in a 2 x 2 x 3 Factorial Arrangement of Treatments	68
4.5	The Plan Confounding the Mean Effect and A_1A_2 in a 2 x 2 x 3 Factorial Arrangement of Treatments	71
4.6	Three Plans of a 2 x 2 x 3 x 3 Factorial Arrangement of Treatments	74
4.7	Three Plans in a 2 x 6 x 3 Factorial Arrangement of Treatments	77

AN ALGEBRAIC APPROACH TO BLOCKING AND CONFOUNDING IN FACTORIAL ARRANGEMENTS

CHAPTER I

INTRODUCTION

Factorial arrangements of treatments have been utilized many times in the designs of experiments. The factorial arrangement is a cross-classified arrangement with the classes being the factors. The chief advantage of the factorial arrangement is that in the absence of interactions of the factors the number of parameters describing the data can be reduced to the set of parameters describing the levels of each of the factors.

Factorial arrangements are customarily dichotomized into symmetrical factorial arrangements, where each factor has the same number of levels, and the asymmetrical factorial arrangements, where the number of levels differ in some two or more of the factors.

Yates (22) first introduced designs and analyses of symmetrical factorial arrangements of the types 2^m, 3^n and asymmetrical arrangements of the type 2^m3^n. Cochran's result concerning the joint distribution of the partition of the sum of squares of normal deviates, and Fisher's F ratio pertaining to the ratio of specified pairs of members of the partitioned sum of squares, made possible the analysis of variance. The theory of maximum likelihood yields estimates of parameters appearing in linear normal models and results by Gauss and Markoff show that these same estimates are valid in more general linear statistical models. The likelihood ratio approach to hypothesis testing confirmed that Fisher's F was a good statistic for testing hypotheses in factorial models.

As experimenters ran afoul of the assumptions of the linear models used in factorial arrangements, efforts were concentrated on refinement of the models so that assumptions could more nearly be met. Perhaps the most basic assumptions of the factorial model that demanded to be met were the assumptions of homogeneous and uncorrelated error terms. Experimenters frequently found that heterogeneous errors accompanied an increase in the size of the experimental plot.

To cope with this problem the treatment combinations comprising the factorial arrangement were partitioned and each member of the partition was subsequently assigned to a smaller experimental plot. By this scheme it was felt that the within-plot variation of the experimental units were smaller and more homogeneous than the variation of the experimental units in the replicate plot, the plot consisting of the union of the smaller plots.

By no means was the result rendered by this technique without liabilities. The price of smaller and more homogeneous error terms was the loss of information on certain treatment contrasts. Since the motivation for choosing smaller experimental plots was that the experimental plots differed in one or more characteristics which influenced treatment responses, it was recognized that comparisons between the responses of two treatments occurring in different experimental plots could not be made with any degree of confidence.

Another problem to overcome was the selection of the "best" partition of the treatments relative to the objectives of the experiment. Haphazard partitions of the treatments resulted in the possible confounding of the factorial effects deemed most important.

Out of the last problem the theory of confounded designs flourished.

The question of how to confound parts of desired factorial effects led Bose and Kishen (2) to develop a theory for the construction of confounded symmetrical designs through finite projective geometries. Later Bose (1) discussed the problem of finding the maximum number of factors that can be accommodated in a block of a given size without confounding an interaction unto a given order.

Fisher (7, 8) discussed this point in the s^m factorial where s is a prime power and found that the maximum number of factors that can be accomodated in a block of size s^r without confounding any main effect or first-order interaction is $\frac{s^{r-1}}{s-1}$. Bose has shown that with s=2 the maximum number of factors that can be accommodated in a block of size 2^r without confounding any interaction of less than third order is 2^{r-1}. Rao (18) also obtained the same results independently. Finney (6) found these methods suitable for the development of fractional factorials.

Nair (16) gave a method for getting confounded arrangements in the symmetrical factorial.

Kempthorne (12) systemized the technique used by Fisher and Finney and a detailed account of the theory appears in a later text (13).

The construction and analysis of confounded designs for asymmetrical factorial arrangements was given by Kishen and Srivastiva (14).

Das (4) developed an alternative approach for construction of symmetrical factorial arrangements and obtained a maximum number of factors. Sarma (20) extended the approach for the construction of symmetrical factorial arrangements.

White and Hultquist (21) gave methods for construction of confounded designs of the type $p^n q^m$, where p and q are distinct primes. Raktoe (17) extended their approach and developed a method of confounding in factorials where the levels of the factors are from distinct prime fields.

Sardana gave procedures for constructing blocks of size 4q in 2 replications of an asymmetrical factorial of the form $2q \times 2^2$ which provided mutually independent estimates of all the effects.

Separate texts by Winer and Mann (15), Federer (5), and Kempthorne (13) give methods of confounding utilizing the Galois field approach. Mann in addition gives a brief algebraic development of the analysis of factorial experiments and confounding factorial experiments.

The blocking plans given in the general theory are not necessary for confounding to exist. If the nature of a factorial arrangement is such that the confounding plans given by current methods cannot be followed then the researcher has to rely on a different analysis or alter his experiment to fit one of the available confounding plans. A wider selection of blocking plans would enable more latitude for designing and analyzing experiments that would otherwise have to be approached through different channels.

One of the objectives of this dissertation is to complement the selection of blocking plans now available. For example, in the 4^2 factorial arrangement, the Galois field theory approach yields only 3 blocking plans to confound part of the interaction in 4 blocks of size 4. A method will be developed that yields 24 blocking plans each of which confounds part of the interaction in a 4^2 factorial.

Using combinatorial properties of blocks rather than field properties, the generalization of this result will give necessary and sufficient conditions for confounding effects in an n^m factorial where n is not restricted to a prime power. This result will be generalized to factorials of the type $n_1 \times n_2 \times \ldots \times n_m$. If $n_1, n_2, \ldots, n_m$ in addition have a non-trivial common divisor d, then blocking plans will be constructed that confound d-1 components of a specified interaction of the factors. This result will be further generalized to include the construction of blocking plans in the $n_1 \times n_2 \times \ldots \times n_m \times q$. A blocking plan that confounds a specified set of orthogonal effects will be shown to be unique and the class of sets of orthogonal effects confoundable with a given blocking plan will be determined.

Numerous examples of blocking plans will be exhibited with an assortment of block sizes.

The second objective of this dissertation is to give a general algebraic approach to construction of factorial effects. Kronecker products of matrices will be used extensively to define factorial effects, to establish the independence of the various factorial effects and to present the sums of squares due to the various effects. Because the usual sums of squares appearing in an analysis of variance are quadratic forms of random variables, the Kronecker product will be used to show that the various quadratic forms are jointly independent and that the matrix of a quadratic form is idempotent with a particular rank.

Mathematical models for designs of both the factorial and confounded factorial will be given.

The class of estimable functions relative to each of the models will be exhibited. Confounding will be defined explicitly utilizing Hadamard products.

As in the designs of factorial arrangements, designs of confounded factorial arrangements will be discussed. The quadratic forms of the sums of squares will be examined and the mutually independent quadratic forms will be determined. The matrices of all quadratic forms appearing in an analysis of variance will be seen to be idempotent.

The comprehensive set of blocking plans of a fixed block size confounding parts of a desired interaction effect will be given whenever feasible.

Examples will serve to illustrate the theory. The analysis of each example will be given.

CHAPTER II

DEVELOPMENT AND ANALYSIS OF DESIGNS INVOLVING COMPLETE REPLICATES OF FACTORIAL ARRANGEMENTS OF TREATMENTS

One of the problems encountered in an algebraic approach to the analysis of a design involving a factorial arrangement of treatments is the definition and representation of the factorial effects of the design. Another related problem is the representation of the sums of squares or quadratic forms due to the various factorial effects. The quadratic forms to be used in the construction of F ratios must be independent and the matrices of the forms must be idempotent. The ranks of the idempotent matrices of the two forms in an F ratio are the parameters of the F ratio and thus must be known before a test of hypothesis can be made.

Kronecker and Tensor Products

The Kronecker and Tensor products readily lend themselves to the definition and construction of factorial effects and to the representation of the quadratic forms of the factorial effects.

Let $V_{n_1}(R)$, $V_{n_2}(R)$, ..., $V_{n_m}(R)$ be m vector spaces over the field of real numbers R where $V_{n_i}(R)$ is the space of all n_i dimensional vectors for $i = 1, 2, \ldots, m$.

For vectors X and Y in $V_{n_i}(R)$ and $V_{n_j}(R)$ respectively the tensor product of X and Y is the $n_i n_j$ dimensional vector defined by

$$X \otimes Y = \begin{pmatrix} x_1 Y \\ x_2 Y \\ \vdots \\ x_{n_i} Y \end{pmatrix}.$$

This definition is easily extended to the tensor product $X_1 \otimes X_2 \otimes \ldots \otimes X_m$ of m vectors where X_i is a vector in $V_{n_i}(R)$. The tensor $X_1 \otimes X_2 \otimes \ldots \otimes X_m$ is an $N = n_1 n_2 \ldots n_m$ dimensional vector in $V_N(R)$ and the set of such tensors span $V_N(R)$. Although a vector in $V_N(R)$ is not necessarily a tensor product of vectors, it is a sum of such tensors.

The Kronecker product of matrices relates the linear operators or matrices of the component spaces to a linear operator of the tensors. If B and C are matrices such that $B : V_{n_i}(R) \to V_{m_i}(R)$ and $C : V_{n_j}(R) \to V_{m_j}(R)$ then the Kronecker product of B and C is the $m_i m_j \times n_i n_j$ matrix

$$B \otimes C = \begin{pmatrix} b_{11} C & b_{12} C & \cdots & b_{1n_i} C \\ b_{21} C & b_{22} C & \cdots & b_{2n_i} C \\ \vdots & \vdots & & \vdots \\ b_{m_i1} C & b_{m_i2} C & \cdots & b_{m_in_i} C \end{pmatrix}.$$

If $X \otimes Y$ is the $n_i n_j$ dimensional vector defined previously, then $(B \otimes C)(X \otimes Y) = BX \otimes CY$.

By extending the definition we can define the Kronecker product of the matrices $C_1, C_2, \ldots, C_m$ where C_i is an $m_i \times n_i$ matrix. The linear operator C_i maps $V_{n_i}(R)$ into $V_{m_i}(R)$ by mapping X_i into $C_i X_i$ and $C_1 \otimes C_2 \otimes \ldots \otimes C_m$ maps the tensor $X_1 \otimes X_2 \otimes \ldots \otimes X_m$ into $(C_1 \otimes C_2 \otimes \ldots \otimes C_m)(X_1 \otimes X_2 \otimes \ldots \otimes X_m)$. From the definitions of

Kronecker and tensor products it follows that $(C_1 \otimes C_2 \otimes \ldots \otimes C_m)$ $(X_1 \otimes X_2 \otimes \ldots \otimes X_m) = C_1X_1 \otimes C_2X_2 \otimes \ldots \otimes C_mX_m$, and thus that the image of a tensor product of vectors is a tensor product of vectors.

It is instructive and sometimes convenient to notice that if the m x n matrix A is blocked into n columns $A_1, \ldots, A_n$, each m x 1 and the r x s matrix B is blocked into s columns $B_1, \ldots, B_s$, each r x 1, then the Kronecker product $A \otimes B$ is blocked naturally into ns columns $A_1 \otimes B_1, A_1 \otimes B_2, \ldots, A_1 \otimes B_s, A_2 \otimes B_1, \ldots, A_2 \otimes B_s, \ldots, A_n \otimes B_1,$ $\ldots, A_n \otimes B_s$, each a tensor product $A_i \otimes B_j$ of vectors and each or size mr x 1.

Although this discussion of tensor products, or Kronecker products, is geared strictly to matrices because this is how they are used in this study and the discussion is adequate for these uses, it should be pointed out that if the standard approach is used to assign matrices to linear operators then the matrix of the tensor product of linear operators is the Kronecker product of the matrices of these operators.

The following theorems are sufficient for some of the developments appearing later. The representative matrices are not necessarily square but are of the proper sizes to make the indicated operations meaningful. The inverse, transpose, rank and trace of a matrix C are denoted respectively by C^{-1}, C', $\rho(C)$ and $tr(C)$. A matrix or vector consisting of all zeroes is denoted by $\emptyset$. Scalars are denoted by small letters.

To facilitate typing, $F^*(C_1, C_2, \ldots, C_m)$ will denote the Kronecker product $C_1 \otimes C_2 \otimes \ldots \otimes C_m$ and will later be used to denote the natural blocking of this product into tensor products. The proofs of some of the theorems are Appendixes and the remainder is given by Halmos or Jacobson (10, 11).

Theorem 2.1: $C_1 \otimes (C_2 \otimes C_3) = (C_1 \otimes C_2) \otimes C_3$.

Theorem 2.2: $F^*(C_1, \ldots, C_{i-1}, C_i, C_{i+1}, \ldots, C_m) +$

$F^*(C_1, \ldots, C_{i-1}, B_i, C_{i+1}, \ldots, C_m) =$

$F^*(C_1, \ldots, C_{i-1}, C_i+B_i, C_{i+1}, \ldots, C_m)$.

Theorem 2.3: $F^*(a_1C_1, a_2C_2, \ldots, a_mC_m) = a_1a_2\ldots a_m\ F^*(C_1,C_2,\ldots,C_m)$.

Theorem 2.4: $F^*(C_1, C_2, \ldots, C_m)\ F^*(B_1, B_2, \ldots, B_m) =$

$F^*(C_1B_1, C_2B_2, \ldots, C_mB_m)$.

Theorem 2.5: $F^*(C_1, C_2, \ldots, C_m) = \emptyset$ if and only if $C_i = \emptyset$ for some i.

Theorem 2.6: $[F^*(C_1, C_2, \ldots, C_m)]' = F^*(C_1', C_2', \ldots, C_m')$.

Theorem 2.7: If C_i^{-1} exists for each C_i then

$[F^*(C_1, C_2, \ldots, C_m)]^{-1} = F^*(C_1^{-1}, C_2^{-1}, \ldots, C_m^{-1})$.

Theorem 2.8: $\rho[F^*(C_1, C_2, \ldots, C_m)] = \rho(C_1)\ \rho(C_2)\ \ldots\ \rho\ (C_m)$.

Theorem 2.9: $tr[F^*(C_1, C_2, \ldots, C_m)] = tr(C_1)\ tr(C_2)\ \ldots\ tr(C_m)$.

Theorem 2.10: If $C_i = C_i'$ for $i = 1, 2, \ldots, m$ then

$F^*(C_1, C_2, \ldots, C_m) = [F^*(C_1, C_2, \ldots, C_m)]'$.

Theorem 2.11: If $C_i' = C_i^{-1}$ for $i = 1, 2, \ldots, m$ then

$[F^*(C_1, C_2, \ldots, C_m)]' = [F^*(C_1, C_2, \ldots, C_m)]^{-1}$.

Theorem 2.12: If $D_1, D_2, \ldots, D_m$ are diagonal matrices then $F^*(D_1, D_2, \ldots, D_m)$ is diagonal.

Theorem 2.13: If $C_1, C_2, \ldots, C_{i-1}, C_{i+1}, \ldots, C_m$ are idempotent matrices, then $F^*(C_1, C_2, \ldots, C_m)$ is idempotent if and only if C_i is idempotent.

Hadamard Products

Another operation used in the factorial development is the Hadamard product. If X and Y are vectors in $V_n(R)$ then the Hadamard product of X and Y is defined by

$$X \theta Y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \theta \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ \vdots \\ x_ny_n \end{pmatrix}.$$

This definition extends easily to a Hadamard product of a finite number of vectors from $V_n(R)$.

If $B = (X_1, X_2, \ldots, X_r)$ and $C = (Y_1, Y_2, \ldots, Y_s)$ are matrices where a column of either is a vector in $V_n(R)$, then we define

$$B \theta C = (X_1 \theta C, X_2 \theta C, \ldots, X_r \theta C) \text{ where}$$

$$X_i \theta C = (X_i \theta Y_1, X_i \theta Y_2, \ldots, X_i \theta Y_s).$$

The set of columns of $B \theta C$ are defined to be the set of Hadamard products of the sets of vectors given by the columns of B and the columns of C.

The following theorems are used throughout this dissertation. Vectors are denoted by X and Y and B and C denote matrices of the proper sizes to make the indicated operations meaningful.

<u>Theorem 2.14</u>: $X \odot C = C \odot X$.

<u>Theorem 2.15</u>: $C_1 \odot (C_2 \odot C_3) = (C_1 \odot C_2) \odot C_3$.

<u>Theorem 2.16</u>: $X \odot C = \emptyset$ implies $X'C = \emptyset$.

<u>Theorem 2.17</u>: $J_n \odot C = C$ where J_n is the n dimensional vector each entry of which is 1.

<u>Theorem 2.18</u>: $(X \odot C)Y = X \odot CY$.

<u>Theorem 2.19</u>: There exists a permutation matrix P such that

$$F^*(B_1, B_2, \ldots, B_m) \odot F^*(C_1, C_2, \ldots, C_m) =$$
$$[F^*(B_1 \odot C_1, B_2 \odot C_2, \ldots, B_m \odot C_m)]P.$$

The remainder of this chapter is devoted to the definition and construction of the 2^m factorial effects of a design of a factorial arrangement of treatments as well as the partition of the total sums of squares into the sums of squares due to each factorial effect.

Both tensor and Hadamard products are utilized in the construction of the factorial effects and Kronecker products are used in the representation of sums of squares.

<u>Definition 2.1</u>: A set of treatments T* is said to be an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement of treatments if there exists a set of m factors ($m \geq 2$) such that each treatment is a combination of exactly one level from each of the factors and conversely each combination of exactly one level from each of the factors is a treatment in T*.

Let the set of n_i levels of the i^{th} factor be represented by $Z(n_i) = \{0, 1, \ldots, n_i -1\}$, the set of residue classes of the integers modulo n_i. We can represent T* as the Cartesian product

$$T = Z(n_1) \times Z(n_2) \times \ldots \times Z(n_m)$$

by associating the treatment consisting of the a_i level of the 1st factor, the a_2 level of the second factor, ..., and the a_m level of the m^{th} factor with the m-tuple $(a_1, a_2, \ldots, a_m)$ in T.

<u>Definition</u> <u>2.2</u>: The set $T = Z(n_1) \times Z(n_2) \times \ldots \times Z(n_m)$ representing the set of treatments T* is called the set of design points of the $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement of treatments.

The design points must be ordered in some structured way to utilize tensor products. The ordering most convenient is the lexicographic order. With this ordering the design point $(a_1, a_2, \ldots, a_m)$ is the $a_1(n_2n_3\ldots n_m) + a_2(n_3n_4\ldots n_m) + \ldots + a_{m-1}(n_m) + a_m$ ordinal. Figure 2.1 gives the lexicographic order of the design points of a $2 \times 2 \times 3$ factorial arrangement of treatments.

(000)
(001)
(002)
(010)
(011)
(012)
(100)
(101)
(102)
(110)
(111)
(112)

Figure 2.1--The lexicographic order of the design points of a $2 \times 2 \times 3$ factorial arrangement of test.

Having established the representation and ordering of the $N = n_1 n_2 \ldots n_m$ treatments in an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement, we can now define the simple observational model $Y = M+e$ where Y is an $N \times 1$ vector of observations of the responses of the treatments, M is an $N \times 1$ vector of treatment means and e is an $N \times 1$ vector of identically and independently distributed errors such that $E(e) = 0$ and $E(ee') = \sigma^2 I_N$. It is essential for later developments that Y and consequently M and e have the same ordering as T.

Definition 2.3: An effect in the model $Y = M+e$ is given by $\lambda' M$ where λ is an $N \times 1$ vector. The vector λ is said to define the effect $\lambda' M$.

Definition 2.4: The effects $\lambda_1' M$ and $\lambda_2' M$ are orthogonal if $\lambda_1' \lambda_2 = 0$.

Definition 2.5: The effect $\lambda' M$ is normalized if $\lambda' \lambda = 1$.

In general a set of N mutually orthogonal effects of M exists. Indeed infinitely many such sets exists. For the factorial arrangement the selection of a set of N orthogonal effects is crucial for estimation and analysis of the factorial effects. In the following development the mean effect and m main effects of an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement are defined and subsequently used to obtain all other factorial effects. The orthogonality of the 2^m factorial effects is also established.

The $n_i \times 1$ vector consisting of all ones, J_{n_i}, appears many times in tensor representation of factorial effects. It is convenient to suppress J_{n_i} whenever it occurs as the i^{th} argument of F^*. With this convention, for example,

$$F^*(C_j, C_k) = J_{n_1} \otimes \ldots \otimes J_{n_{j-1}} \otimes C_j \otimes \ldots \otimes J_{n_{j+1}} \otimes \ldots \otimes J_{n_{k-1}} \otimes C_k \otimes J_{n_{k+1}} \otimes \ldots \otimes J_{n_m} .$$

The subscript of lowest order of an argument depicts the position of the argument in the tensor representation. The order of the arguments in this abbreviated notation is assumed to agree with their relative order in the tensor representation. In the example, for instance, $j < k$ since C_j preceded C_k.

Definition 2.6: The mean effect is $J_N'M$ where

$$J_N = J_{n_1} \otimes J_{n_2} \otimes \ldots \otimes J_{n_m} .$$

Definition 2.7: The set of level totals of the i^{th} factor is

$$[F^*(I_{n_i})]'M.$$

The columns of $F^*(I_{n_i})$ are n_i mutually orthogonal N dimensional vectors and consequently span an n_i dimensional vector space. The subspace spanned by J_N is a subspace of the space spanned by the columns of $F^*(I_{n_i})$. The set of vectors which are orthogonal to J_N in the latter space also form a subspace of dimension n_i-1. This n_i-1 dimensional subspace is called the subspace orthogonal to J_N relative to the space spanned by the columns of $F^*(I_{n_i})$.

Definition 2.8: The i^{th} main effect A_i is defined by any orthogonal basis of the N_i-1 dimensional subspace orthogonal to J_N relative to the space spanned by the columns of $F^*(I_{n_i})$.

A_i is said to be defined by an $N \times n_i-1$ matrix L_i if the columns of L_i form a basis of the n_i-1 dimensional subspace. The n_i-1 effects defined by L_i are called the components of A_i. Two distinct bases, each of which defines A_i, yield two distinct sets of components of A_i.

<u>Theorem 2.20</u>: Let (J_{n_i}, U_i) be an $n_i \times n_i$ matrix such that the columns of (J_{n_i}, U_i) form an orthogonal basis of $V_{\bar{n}_i}(R)$. Then the columns of $F^*(U_i)$ defines A_i.

<u>Proof</u>: $F^*(I_{n_i})\ (J_{n_i}, U_i) = F^*((J_{n_i}, U_i)) = (J_N, F^*(U_i))$

$J_N' F^*(U_i) = [F^*(J_{n_i})]' F^*(U_i) =$

$n_1 \otimes \ldots \otimes n_{i-1} \otimes J_{n_i} U_i \otimes n_{i+1} \otimes \ldots \otimes n_m = \emptyset$

since $J_{n_i}' U_i = \emptyset$.

$[F^*(U_i)]' F^*(U_i) = n_1 \otimes \ldots \otimes n_{i-1} \otimes U_i' U_i \otimes n_{i+1} \otimes \ldots \otimes n_m =$

$n_1\, n_2 \cdots n_{i-1}\, n_{i+1} \cdots n_m\, U_i' U_i$.

Thus the columns of $F^*(U_i)$ and J_N are mutually orthogonal and the columns of $F^*(U_i)$ defines A_i.

Throughout this discussion U_i will always denote an $n_i \times n_i - 1$ matrix such that the columns of $F^*(U_i)$ defines A_i.

<u>Definition 2.9</u>: If $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are defined respectively by $F^*(U_{i_1}), F^*(U_{i_2}), \ldots, F^*(U_{i_k})$ where $i_1 < i_2 < \ldots < i_k$ and $1 < k \leq m$ then the $A_{i_1} A_{i_2} \ldots A_{i_k}$ interaction effect is defined by any orthogonal basis of the space spanned by the columns of $F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k})$.

<u>Theorem 2.21</u>: The space spanned by the columns of $F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k})$ has dimension $(n_{i_1}-1)(n_{i_2}-1)\ldots(n_{i_k}-1)$.

<u>Proof</u>: $[F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k})]'\ F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k}) =$

$$\frac{N}{n_{i_1} n_{i_2} \cdots n_{i_k}}\ D_{i_1} \otimes D_{i_2} \otimes \ldots \otimes D_{i_k} \text{ where } D_{i_j} = U_{i_j}'\, U_{i_j}.$$

Since D_{i_j} is a diagonal matrix for $j = 1, 2, \ldots, k$ then by Theorem 2.12 the Kronecker product is a diagonal matrix and the columns of $F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k})$ are orthogonal. Thus the dimension of the space spanned by the columns of $F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k})$ is the number of columns, $(n_{i_1}-1)(n_{i_2}-1)\ldots(n_{i_k}-1)$.

By definition, the mean effect is orthogonal to each main effect. The following theorems establish that a factorial effect is orthogonal to any other factorial effect.

<u>Theorem 2.22</u>: A_{i_1} is orthogonal to A_{i_2}.

<u>Proof</u>: $[F^*(U_{i_1})]'\ [F^*(U_{i_2})] = [F^*(U_{i_1}, J_{n_{i_2}})]'\ F^*(J_{n_{i_1}}, U_{i_2}) =$

$$\frac{N}{n_{i_1} n_{i_2}}\ (U_{i_1}'\, J_{n_{i_1}}) \otimes (J_{n_{i_2}}'\, U_{i_2}) = \emptyset$$

since $U_{i_1}'\, J_{n_{i_1}} = \emptyset$.

<u>Theorem 2.23</u>: The mean effect is orthogonal to the $A_{i_1} A_{i_2} \ldots A_{i_k}$ interaction effect.

<u>Proof</u>: $J_N'\ F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k}) =$

$$\frac{N}{n_{i_1} n_{i_2} \cdots n_{i_k}}\ J_{n_{i_1}}' U_{i_1} \otimes J_{n_{i_2}}' U_{i_2} \otimes \ldots \otimes J_{n_{i_k}}' U_{i_k} = \emptyset$$

since $J'_{n_{i_1}} U_{i_1} = \emptyset$.

Theorem 2.24: $\dot{A}_{i_1} \dot{A}_{i_2} \ldots \dot{A}_{i_r}$ is orthogonal to $\dot{A}_{j_1} \dot{A}_{j_2} \ldots \dot{A}_{j_s}$

if $\{i_1, i_2, \ldots, i_r\} \neq \{j_1, j_2, \ldots, j_s\}$.

Proof: Without loss of generality take $i_1 < j_1$. Then

$$[F*(U_{i_1}, U_{i_2}, \ldots, U_{i_r})]' F*(J_{n_{i_1}}, U_{j_1}, U_{j_2}, \ldots, U_{j_s}) = \emptyset$$

utilizing Theorem 2.5 and the fact that $U'_{i_1} J_{n_{i_1}} = \emptyset$.

The next theorem establishes the relationship between main effects and interaction effects.

Theorem 2.25: $F*(U_{i_1}, U_{i_2}, \ldots, U_{i_k}) = F*(U_{i_1}) \otimes F*(U_{i_2}) \otimes \ldots \otimes F*(U_{i_k})$.

Proof: The proof follows immediately by Theorems 2.17 and 2.19.

The matrix $L = (J_N, F*(U_1), \ldots, F*(U_m), F*(U_1,U_2), \ldots, F*(U_{m-1},U_m), \ldots, F*(U_1, U_2, \ldots, U_m))$ defines the 2^m factorial effects. L is a permutation of the columns of $F*((J_{n_1}, U_1), (J_{n_2},U_2), \ldots, (J_{n_m},U_m))$ and thus has N columns.

Definition 2.10: Given the simple linear model $Y = M+e$ and L as defined above, the model $L'Y = L'M + L'e$ is a factorial effects model. A factorial effect model is a normalized factorial effects model if $L'L = I_N$.

$L'M$ is a set of N orthogonal effects and these are partitioned into the 2^m factorial effects. A matrix defining the factorial effects of a 2 x 2 x 3 factorial arrangement is obtained from the expression

$$L = (J_N, F*(U_1), F*(U_2), F*(U_3), F*(U_1,U_2), F*(U_1,U_3), F*(U_2,U_3), F*(U_1,U_2,U_3))$$

by letting $N = 12$, $U_1 = U_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $U_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}$.

Figure 2.2 illustrates the resulting 12 x 12 matrix.

$$L = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -2 & 1 & 0 & -2 & 0 & -2 & 0 & -2 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 0 & -2 & -1 & 0 & -2 & 0 & 2 & 0 & 2 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -2 & -1 & 0 & 2 & 0 & -2 & 0 & 2 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 0 & -2 & 1 & 0 & 2 & 0 & 2 & 0 & -2 \end{bmatrix}$$

Figure 2.2--A Matrix L defining the 8 factorial effects of a 2 x 2 x 3 factorial arrangement.

An effect in the model $Y = M+e$ is $\lambda'M$ where λ is an $N \times 1$ vector. If an effect is not known then an estimate of that effect must be obtained before a confidence interval can be constructed.

<u>Definition 2.11</u>: An effect $\lambda'M$ in the model $Y = M+e$ is estimable if there exists an $N \times 1$ vector γ such that $E(\gamma'Y) = \lambda'M$.

Since $E(Y) = M$ in the model $Y = M+e$ then $E(\lambda'Y) = \lambda'M$ and any effect is estimable. In the factorial effects model $L'Y = L'M + L'e$, $E(L'Y) = L'M$ and thus $L'Y$ estimates $L'M$.

Many times the emphasis is not on estimation of the various factorial effects but is on the testing of hypotheses concerning the various factorial effects. In this situation the estimates of the factorial effects can be utilized to produce a concise expression of the usual sums of squares appearing in an A.O.V. table.

The usual assumption of the model $Y = M+e$ is that e is distributed as a multivariate normal random variable with $E(e) = 0$ and $E(ee') = \sigma^2 I_N$. Under these conditions Graybill (9) has shown that the quadratic form $Y'AY$ is distributed as a noncentral chi-squared variable with parameters k and $\frac{M'AM}{2\sigma^2}$ if and only if A is an idempotent matrix of rank k. Furthermore he has shown that the two quadratic forms $Y'AY$ and $Y'BY$ are independent if and only if $AB = \emptyset$. Since Fishers F statistic is the ratio of two independent chi-squared variables each divided by its degrees of freedom, we are interested in determining the ranks of idempotent matrices appearing in quadratic forms and in determining the independence of two or more quadratic forms.

<u>Definition 2.12</u>: Let $S'M$ be a set of r effects such that $S'S = I_r$. Then the quadratic form $Y'SS'Y$ is the sum of squares due to $S'M$ and SS' is the matrix of the quadratic form $Y'SS'Y$.

The following theorems establish the ranks and idempotent properties of the matrices of the quadratic forms that partition the total sum of squares into the sums of squares due to the factorial effects.

Let $(\tilde{J}_{n_i}, \tilde{U}_i)$ be an orthogonal $n_i \times n_i$ matrix. Then $(\tilde{J}_{n_i}, \tilde{U}_i)'$ $(\tilde{J}_{n_i}, \tilde{U}_i) = I_{n_i}$ and also $(\tilde{J}_{n_i}, \tilde{U}_i)\ (\tilde{J}_{n_i}, \tilde{U}_i)' = \tilde{J}_{n_i}\ \tilde{J}'_{n_i} + \tilde{U}_i\ \tilde{U}'_i = I_{n_i}$. Denoting $\tilde{J}_{n_i}\ \tilde{J}'_{n_i}$ by K_{n_i} we have $\tilde{U}_i\ \tilde{U}'_i = I_{n_i} - K_{n_i}$.

<u>Theorem 2.26</u>: K_{n_i} is an idempotent matrix of rank .

Proof: $K_{n_i} K_{n_i} = \tilde{J}_{n_i} \tilde{J}'_{n_i} \tilde{J}_{n_i} \tilde{J}'_{n_i} = \tilde{J}_{n_i} 1 \tilde{J}'_{n_i} = \tilde{J}_{n_i} \tilde{J}'_{n_i} = K_{n_i}$

$\rho(K_{n_i}) = \rho(\tilde{J}_{n_i} \tilde{J}'_{n_i}) = \rho(\tilde{J}_{n_i}) = 1.$

Theorem 2.27: If A is an n x n idempotent matrix of rank r then I_n-A is an idempotent matrix of rank n-r.

Proof: $(I_n-A)(I_n-A) = I_n-A -A + A^2 = I_n-A$ since $A^2 = A$. Since the rank and trace of an idempotent matrix are equal, then $\rho(I_n-A) = tr(I_n-A) = tr(I_n) - tr(A) = n-r.$

Corollary 2.1: $I_{n_i}-K_{n_i}$ is an idempotent matrix of rank n_i-1.

Theorem 2.28: $(I_{n_i} - K_{n_i}) K_{n_i} = K_{n_i}(I_{n_i} - K_{n_i}) = \emptyset$

Proof: $(I_{n_i} - K_{n_i}) K_{n_i} = K_{n_i} - K^2_{n_i} = K_{n_i} - K_{n_i} = K^2_{n_i} - K_{n_i} =$

$K_{n_i} (I_{n_i} - K_{n_i}) = \emptyset$.

Theorem 2.29: The matrix of the quadratic form of the mean effect is $K_N = \frac{1}{N} J_N J'_N$. Furthermore K_N is an idempotent matrix of rank 1.

Proof: The mean effect is given by $J'_N M$. Letting $\tilde{J}'_N M$ denote the normalized mean effect, we get $\tilde{J}_N = \frac{J_N}{\sqrt{N}}$ and

$$\frac{1}{N} J_N J'_N = \tilde{J}_N \tilde{J}'_N = K_N = F^*(\tilde{J}_{n_1}, \tilde{J}_{n_2}, \ldots, \tilde{J}_{n_m})$$

$$[F^*(\tilde{J}_{n_1}, \tilde{J}_{n_2}, \ldots, \tilde{J}_{n_m})] =$$

$$F^*(\tilde{J}_{n_1} \tilde{J}'_{n_1}, \tilde{J}_{n_2} \tilde{J}'_{n_2}, \ldots, \tilde{J}_{n_m} \tilde{J}'_{n_m}) = F^*(K_{n_1}, K_{n_2}, \ldots, K_{n_m}).$$

By Theorem 2.8 and Theorem 2.13 K_N is idempotent with rank 1.

Theorem 2.30: The matrix of the quadratic form of the A_i effect is

$$F^*(K_{n_1}, K_{n_2}, \ldots, K_{n_{i-1}}, I_{n_i} - K_{n_i}, K_{n_{i+1}}, \ldots, K_m).$$

Moreover this matrix is idempotent with rank n_i-1.

Proof: Let $\tilde{F}^*(U_i)$ denote the normalized arguments of $F^*(U_i)$.

Then $\tilde{F}^*(U_i)\ [F^*(U_i)]' =$

$$F^*(\tilde{J}_{n_1}\tilde{J}'_{n_i}, \ldots, \tilde{J}_{n_{i-1}}\tilde{J}'_{n_{i-1}}, \tilde{U}_i\tilde{U}'_i, \tilde{J}_{n_{i+1}}\tilde{J}'_{n_{i+1}}, \ldots, \tilde{J}_{n_m}\tilde{J}'_{n_m}) =$$

$$F^*(K_{n_1}, \ldots, K_{n_{i-1}}, I_{n_i} - K_{n_i}, K_{n_{i+1}}, \ldots, K_{n_m})$$

and is by Theorem 2.8 and Theorem 2.13 an idempotent matrix of rank n_i-1.

Theorem 2.31: The matrix of the quadratic form of the $A_{i_1}A_{i_2}\ldots A_{i_k}$ effect is $F^*(K_{n_1}, \ldots, K_{n_{i_1-1}}, I_{n_{i_1}} - K_{n_{i_1}}, K_{n_{i_1+1}}, \ldots,$

$$K_{n_{i_k-1}}, I_{n_{i_k}} - K_{n_{i_k}}, K_{n_{i_k+1}}, \ldots, K_{n_m}),$$

an idempotent matrix of rank $(n_{i_1}-1)(n_{i_2}-1)\ldots(n_{i_k}-1)$.

Proof: $\tilde{F}^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k})[\tilde{F}^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k})]'$ is by definition the matrix of the quadratic form and by Theorem 2.4 we get the desired matrix.

Since the Kronecker product of idempotent matrices is idempotent we have the matrix of the quadratic form of the $A_{i_1}A_{i_2}\ldots A_{i_k}$ effect is idempotent with rank $(n_{i_1}-1)(n_{i_2}-1)\ldots(n_{i_k}-1)$.

Theorem 2.32: The product of the matrices of quadratic forms of two distinct factorial effects is the zero matrix.

Proof: The i^{th} argument of F^* representing the matrix of the quadratic form of any factorial effect is either K_{n_i} or $I_{n_i} - K_{n_i}$. Since the factorial effects are distinct, one of the arguments, say the j^{th}, is K_{n_j} for one of the quadratic forms while the j^{th} argument of the other quadratic form is $I_{n_j} - K_{n_j}$. Then by Theorem 2.5 and Theorem 2.28 we get the desired result.

The sum of the matrices of the quadratic form of the 2^m factorial effects is

$$\sum_{i=1}^{m} \sum_{W_i \varepsilon \{I_{n_i} - K_{n_i},\, K_{n_i}\}} F^*(W_1, W_2, \ldots, W_m) =$$

$$\sum_{i=1}^{m-1} \sum_{W_i \varepsilon \{I_{n_i} - K_{n_i},\, K_{n_i}\}} [F^*(W_1, W_2, \ldots, W_{m-1}, I_{n_{i_1}} - K_{n_{i_1}}) +$$

$$F^*(W_1, W_2, \ldots, W_{m-1}, K_{n_{i_1}})] =$$

$$\sum_{i=1}^{m-1} \sum_{W_i \varepsilon \{I_{n_i} - K_{n_i},\, K_{n_i}\}} F^*(W_1, W_2, \ldots, W_{m-1}, I_{n_i}) =$$

$$F^*(I_{n_1}, I_{n_2}, \ldots, I_{n_m}) = I_N.$$

Thus the sum of the quadratic forms of the 2^m factorial effects is the total sum of squares $Y'I_NY$.

Table 2.1 gives abbreviated A.O.V. of one replicate $n_1 \times n_2 \times \ldots \times n_m$ factorial. Since it is customary to call the rank of an idempotent matrix of a quadratic form the degrees of freedom of the quadratic form, the ranks of the matrices of the quadratic forms of the factorial effects will give the degrees of freedom (d.f.) column.

An abbreviated analysis of variance table for one replicate of a 2 x 2 x 3 factorial arrangement of treatments is given in Table 2.2.

TABLE 2.1

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR ONE REPLICATE OF AN $n_1 \times n_2 \times \ldots \times n_m$ FACTORIAL ARRANGEMENT OF TREATMENTS

Source	d.f.	S.S.
Total	$n_1 n_2 \ldots n_m$	$Y'Y$
Mean	1	$Y'K_N Y$
A_1	n_1-1	$Y'(I_{n_1}-K_{n_1} \otimes K_{n_2} \otimes \ldots \otimes K_{n_m})Y$
A_2	n_2-1	$Y'(K_{n_1} \otimes I_{n_2}-K_{n_2} \otimes K_{n_3} \otimes \ldots \otimes K_{n_m})Y$
$\vdots$	$\vdots$	$\vdots$
A_m	n_m-1	$Y'(K_{n_1} \otimes \ldots \otimes K_{n_{m-1}} \otimes I_{n_m}-K_{n_m})Y$
A_1A_2	$(n_1-1)(n_2-1)$	$Y'(I_{n_1}-K_{n_1} \otimes I_{n_2}-K_{n_2} \otimes K_{n_3} \otimes \ldots \otimes K_{n_m})Y$
$\vdots$	$\vdots$	$\vdots$
$A_{m-1}A_m$	$(n_{m-1}-1)(n_m-1)$	$Y'(K_{n_1} \otimes \ldots \otimes K_{n_{m-2}} \otimes I_{n_{m-1}}-K_{n_{m-1}} \otimes I_{n_m}-K_{n_m})Y$
$\vdots$	$\vdots$	$\vdots$
$A_1A_2\ldots A_{m-1}$	$(n_1-1)(n_2-1)\ldots(n_{m-1}-1)$	$Y'(I_{n_1}-K_{n_1} \otimes I_{n_2}-K_{n_2} \otimes \ldots \otimes I_{n_{m-1}}-K_{n_{m-1}} \otimes K_{n_m})Y$
$\vdots$	$\vdots$	$\vdots$
$A_2A_3\ldots A_m$	$(n_2-1)(n_3-1)\ldots(n_m-1)$	$Y'(K_{n_1} \otimes I_{n_2}-K_{n_2} \otimes I_{n_3}-K_{n_3} \otimes \ldots \otimes I_{n_m}-K_{n_m})Y$
$A_1A_2\ldots A_m$	$(n_1-1)(n_2-1)\ldots(n_m-1)$	$Y'(I_{n_1}-K_{n_1} \otimes I_{n_2}-K_{n_2} \otimes \ldots \otimes I_{n_m}-K_{n_m})Y$

TABLE 2.2

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR ONE REPLICATE OF A 2 x 2 x 3 FACTORIAL ARRANGEMENT OF TREATMENTS

Source	d.f.	S.S.
Total	12	$Y'Y$
Mean	1	$Y'K_{12}Y$
A_1	1	$Y'(I_2-K_2 \otimes K_2 \otimes K_3)Y$
A_2	1	$Y'(K_2 \otimes I_2-K_2 \otimes K_3)Y$
A_3	2	$Y'(K_2 \otimes K_2 \otimes I_3-K_3)Y$
A_1A_2	1	$Y'(I_2-K_2 \otimes I_2-K_2 \otimes K_3)Y$
A_1A_3	2	$Y'(I_2-K_2 \otimes K_2 \otimes I_3-K_3)Y$
A_2A_3	2	$Y'(K_2 \otimes I_2-K_2 \otimes I_3-K_3)Y$
$A_1A_2A_3$	2	$Y'(I_2-K_2 \otimes I_2-K_2 \otimes I_3-K_3)Y$

A situation frequently encountered is that only one replicate of a factorial arrangement of treatments is available. In this instance no estimate of experimental error is available from the data unless it is known that some factorial effect is zero.

A frequent practice is to assume that the interaction of highest order is negligible. Upon making this assumption we have

$$E[Y'(I_{n_1}-K_{n_1}\otimes I_{n_2}-K_{n_2}\otimes\ldots\otimes I_{n_m}-K_{n_m})Y] =$$

$$E[(M+e)'(I_{n_1}-K_{n_1}\otimes I_{n_2}-K_{n_2}\otimes\ldots\otimes I_{n_m}-K_{n_m})(M+e)] =$$

$$E[e'(I_{n_1}-K_{n_1}\otimes I_{n_2}-K_{n_2}\otimes\ldots\otimes I_{n_m}-K_{n_m})e] =$$

$$\sigma^2\ tr(I_{n_1}-K_{n_1}\otimes I_{n_2}-K_{n_2}\otimes\ldots\otimes I_{n_m}-K_{n_m} =$$

$$(n_1-1)(n_2-1)\ldots(n_m-1)\sigma^2\ .$$

If e is assumed to be distributed as a multivariate normal random variable, the quadratic form $\frac{1}{\sigma^2}\ Y'(I_{n_1}-K_{n_1}\otimes I_{n_2}-K_{n_2}\otimes\ldots\otimes I_{n_m}-K_{n_m})Y$ is distributed as a chi-squared random variable with parameters $(n_1-1)(n_2-1)\ldots(n_m-1)$. To test a hypothesis that some factorial effect other than the highest order interaction effect is zero, the ratio of the mean squares is formed and this ratio is compared to the critical value of the F of the appropriate degrees of freedom.

Although the assumption concerning the highest order interaction may be untenable, the proposed test is conservative in that the "true" Type I error is less than that used to obtain the critical F value.

A completely randomized design (C.R.D.) is a design in which the treatments are randomly assigned to the experimental units.

If r replicates of the N treatment of an $n_1 \times n_2 \times \ldots \times n_m$

factorial arrangements is desired than Nr experimental units are required. The Nr experimental units are partitioned in some random fashion into N sets, each set containing r units. The N treatments are now assigned at random to the N sets of experimental units.

We than have the r simple linear models $Y_i = M+e_i$ for $i=1,\ldots,r$. These can be combined into the simple model $Y^* = M^*+e^*$ where

$$Y^* = \begin{pmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_r \end{pmatrix}, \quad M^* = J_r \otimes M \quad \text{and} \quad e^* = \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_r \end{pmatrix}.$$

Y^* is an $Nr \times 1$ vector, M^* is the $Nr \times 1$ vector

$\begin{pmatrix} M \\ M \\ \cdot \\ \cdot \\ \cdot \\ M \end{pmatrix}$ and e^* is an $Nr \times 1$ error vector such that $E(e^*) = 0$ and $E(e^*e^{*\prime}) = \sigma^2 I_{Nr}$. For tests of hypotheses e^* is also assumed to be distributed as a multivariate normal random variable.

The Gauss-Markoff Theorem states that the best linear unbiased estimate of M is given by the least squares estimate and is

$$\hat{M} = \bar{Y} = \frac{1}{r} \sum_{i=1}^{r} Y_i .$$

Theorem 2.33: $Y^{*\prime}(I_r-K_r \otimes I_N)Y^*$ is an unbiased estimate of $N(r-1)\sigma^2$.

Proof: $E[Y^{*\prime}(I_r-K_r \otimes I_N)Y^*] = E[(J_r\otimes M+e^*)'(I_r-K_r\otimes I_N)(J_r\otimes M+e^*)] =$

$E[(J_r\otimes M)'(I_r-K_r\otimes I_N)(J_r\otimes M) + e^{*\prime}(I_r-K_r\otimes I_N)e^*] =$

$(J_r' (I_r-K_r)J_r) \otimes (M'I_N M) + E[e^{*\prime}(I_r-K_r\otimes I_N)e^*] =$

$$0 \otimes M'M \;+\; E[e^{*\prime}(I_r - K_r \otimes I_N)e^*] =$$

$$\sigma^2 \; tr(I_r - K_r \otimes I_N) \;=\; \sigma^2(r-1)N.$$

The total sum of squares is the sum of the sums of squares due to error and treatments. Thus $Y^{*\prime}Y^* = Y^{*\prime}(I_r - K_r \otimes I_N)Y^* + Y^{*\prime}(K_r \otimes I_N)Y^*$ and $Y^{*\prime}(K_r \otimes I_N)Y^*$ is the sum of squares due to treatments. In the simple model $Y^* = J_r \otimes M + e^*$, the estimate of $J_r \otimes M$ is $J_r \otimes \overline{Y}$. Thus the sum of squares due to treatments is $(J_r \otimes \overline{Y})'(J_r \otimes \overline{Y}) = r\ \overline{Y}'\overline{Y}$. Since $J_r \otimes \overline{Y} = (K_r \otimes I_N)Y^*$, then $r\ \overline{Y}'\overline{Y} = Y^{*\prime}(K_r \otimes I_N)'(K_r \otimes I_N)Y^* = Y^{*\prime}(K_r \otimes I_N)Y^*$.

The matrix $\tilde{J}_r \otimes L$ defines the factorial effects in the model $Y^* = J_r \otimes M + e^*$. If $S'M$ defines some factorial effect in the model $Y = M + e$ then we have seen that $S'Y$ estimates this effect. $(\tilde{J}_r \otimes S)'(J_r \otimes M)$ is the factorial effect in the model $Y^* = J_r \otimes M + e^*$ and the estimate of this effect is $(\tilde{J}_r \otimes S)'Y^*$. The sum of squares due to this effect is $Y^{*\prime}(\tilde{J}_r \otimes S)(\tilde{J}_r \otimes S)'Y^* =$ $Y^{*\prime}(K_r \otimes SS')Y^* = Y^{*\prime}(K_r \otimes I_N)'(K_r \otimes SS')(K_r \otimes I_N)Y^* = (J_r \otimes \overline{Y})'(K_r \otimes SS')(J_r \otimes \overline{Y}) =$ $J_r' K_r J_r \otimes \overline{Y}'SS'\overline{Y} = r\ \overline{Y}'SS'\overline{Y}$.

Table 2.3 gives an A.O.V. for a C.R.D. of an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement. A randomized complete block design (R.C.B.D.) is a design in which blocks of experimental material is available. The blocks may occur naturally or may be chosen. The basic motivation for the randomized complete block design is that blocks of homogeneous units may be chosen with the units in different blocks differing considerably. In this design it is desirable to account for the differences in block totals. The blocks of experiments are chosen randomly and the N units of a block are randomly assigned to the N treatments of an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement.

The randomized complete block design can be analyzed as a single

TABLE 2.3

ABBREVIATED ANALYSIS OF VARIANCE FOR r REPLICATES OF A C.R.D. OF AN $n_1 \times n_2 \times \ldots n_m$ FACTORIAL ARRANGEMENT

Source	d.f.	S.S.
Total	rN	$\sum_{i=1} Y_i'Y_i$
Mean	1	$r\overline{Y}'K_N\overline{Y}$
A_1	n_1-1	$r\overline{Y}'(I_{n_1}-K_{n_1}\otimes K_{n_2}\otimes\ldots\otimes K_{n_m})\overline{Y}$
.	.	.
.	.	.
.	.	.
A_m	n_m-1	$r\overline{Y}'(K_{n_1}\otimes\ldots\otimes K_{n_{m-1}}\otimes I_{n_m}-K_{n_m})\overline{Y}$
A_1A_2	$(n_1-1)(n_2-1)$	$r\overline{Y}'(I_{n_1}-K_{n_1}\otimes I_{n_2}-K_{n_2}\otimes K_{n_3}\otimes\ldots\otimes K_{n_m})\overline{Y}$
.	.	.
.	.	.
.	.	.
$A_{m-1}A_m$	$(n_{m-1}-1)(n_m-1)$	$r\overline{Y}'(K_{n_1}\otimes\ldots\otimes K_{n_{m-2}}\otimes I_{n_{m-1}}-K_{n_{m-1}}\otimes I_{n_m}-K_{n_m})\overline{Y}$
.	.	.
.	.	.
.	.	.
$A_1A_2\ldots A_m$	$(n_1-1)(n_2-1)\ldots(n_m-1)$	$r\overline{Y}'(I_{n_1}-K_{n_1}\otimes I_{n_2}-K_{n_2}\otimes\ldots\otimes I_{n_m}-K_{n_m})\overline{Y}$
error	$(r-1)N$	$\sum_{i=1}^{r} (Y_i-\overline{Y})'(Y_i-\overline{Y})$

replicate of a $b \times n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement of treatments, but instead of using only the highest order interaction for a measure of error, all interactions involving blocks are used for the measure of error.

The model is $Y^* = M^* + \alpha \otimes J_N + e^*$ where

$$Y^* = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_b \end{pmatrix}, \quad M^* = J_b \otimes M, \quad \alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_b \end{pmatrix} \quad \text{and} \quad e^* = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_b \end{pmatrix}$$

The sums of squares of this design are obtained in the same manner as in the completely randomized design with the exception that block sums of squares are taken from the error sums of squares.

Table 2.4 gives an analysis of variance for b blocks of a randomized complete blocks design of an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement of treatments.

TABLE 2.4

AN ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR b BLOCKS OF A R.C.B.D. OF AN $n_1 \times n_2 \times \ldots \times n_m$ FACTORIAL ARRANGEMENT OF TREATMENTS

Source	d.f.	S.S.
Total	bN	$\sum_{i=1}^{b} Y_i'Y_i$
Mean	1	$b\bar{Y}'K_N\bar{Y}$
Blocks	b-1	$Y^{*\prime}(I_b-K_b \otimes K_N)Y^*$
A_1	n_1-1	$b\bar{Y}'(I_{n_1}-K_{n_1} \otimes K_{n_2} \otimes \ldots \otimes K_{n_m})\bar{Y}$
.	.	.
.	.	.
.	.	.
A_m	n_m-1	$b\bar{Y}'(K_{n_1} \otimes \ldots \otimes K_{n_{m-1}} \otimes I_{n_m}-K_{n_m})\bar{Y}$
A_1A_2	$(n_1-1)(n_2-1)$	$b\bar{Y}'(I_{n_1}-K_{n_1} \otimes I_{n_2}-K_{n_2} \otimes K_{n_3} \otimes \ldots \otimes K_{n_m})\bar{Y}$
.	.	.
.	.	.
.	.	.
$A_{m-1}A_m$	$(n_{m-1}-1)(n_m-1)$	$b\bar{Y}'(K_{n_1} \otimes \ldots \otimes K_{n_{m-2}} \otimes I_{n_{m-1}}-K_{n_{m-1}} \otimes I_{n_m}-K_{n_m})\bar{Y}$
.	.	.
.	.	.
.	.	.
$A_1A_2 \ldots A_m$	$(n_1-1)(n_2-1)\ldots(n_m-1)$	$b\bar{Y}'(I_{n_1}-K_{n_1} \otimes I_{n_2}-K_{n_2} \otimes \ldots \otimes I_{n_m}-K_{n_m})\bar{Y}$
Residual	(b-1)(N-1)	$Y^{*\prime}(I_b-K_b \otimes I_N-K_N)Y^*$

CHAPTER III

ESTIMATION AND ANALYSIS OF PARTITIONED FACTORIAL ARRANGEMENTS OF TREATMENTS

Frequently a factorial arrangement of treatments is so large that it becomes difficult to get a replicate of homogeneous experimental units. From practical considerations it is often impossible to obtain large homogeneous replicates, especially if the replicate consists of litter mates of laboratory animals or hospital out-patients in a specific age-sex-race classification. It may also be that the units within a replicate are fairly homogeneous originally but change with time and the factorial arrangement is so large that all the treatments cannot be applied in a sufficiently small time span. Consequently time becomes a factor or "factor" and the heterogeneity of units results in larger errors and loss of power in tests of hypotheses.

To circumvent the problem of heterogeneous errors the technique of blocking is employed. By this technique the units of a replicate are partitioned into a number of blocks of units where the units within each block are more homogeneous than units within the replicate. The blocks may occur naturally as in the case of litter mates of laboratory animals or they may be determined by actually selecting a homogeneous group of units.

While a judicious choice of a blocking plan yields a set of homogeneous blocks of units, the incorporation of a blocking plan in a factorial arrangement of treatments produces an inherent loss in the number of orthogonal estimable effects. The blocks are selected to be homogeneous within themselves and thus the blocks differ considerably. Any comparison between treatments applied in different blocks reflects both block differences and treatment differences. Thus we say that comparisons between treatments in different blocks are confounded with blocks.

The theory of blocking is simplified by partitioning the set of design points T and applying the treatments corresponding to a block of T to a homogeneous set of experimental units of the exact size to accomodate the treatments.

Definition 3.1: The collection of subsets $B = \{\beta_i : i=1, 2, \ldots, b\}$ of T is a blocking plan or partition of T if

1) $\bigcup_{i=1}^{b} \beta_i = T$ and

2) $\beta_i \cap \beta_j$ is null for $i \neq j$.

Since comparisons among treatments in different blocks have little meaning, only plans in which the size (number of treatments) of each block is larger than 1 will be considered.

Definition 3.2: The N x 1 vector χ_i is the incidence matrix of the block β_i of a plan B and is defined by the characteristic function χ_i^* where

$$\chi_i = \chi_i^*(T) = \begin{pmatrix} \chi_{i_1}^*(1) \\ \chi_{i_2}^*(2) \\ \vdots \\ \chi_{i_N}^*(N) \end{pmatrix} \quad \text{where} \quad \chi_{i_j}^*(j) = \begin{cases} 1 \text{ if } j \in \beta_i \\ 0 \text{ otherwise.} \end{cases}$$

T has the lexicographic ordering.

<u>Definition 3.3</u>: $X = (x_1, x_2, \ldots, x_b)$ is the incidence matrix of the blocking plan $B = \{\beta_i : i=1, 2, \ldots, b\}$.

It follows from the definitions of blocking plan and characteristic matrix that $X J_b = J_N$ and $x_i \odot x_j = \delta_{ij} x_i$ where

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} .$$

Figure 3.1 gives the incidence matrices of two plans of a 2 x 2 x 3 factorial.

Plan (a)

000	010
001	011
002	012
110	100
111	101
112	102

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Plan (b)

000	010	001
002	102	011
012	110	100
101	112	
111		

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Figure 3.1--The incidence matrices of two blocking plans of a 2 x 2 x 3 factorial.

With a given blocking plan B the model for the factorial arrangement of treatments is assumed to be $Y = M + X\alpha + e$ where Y is the observational vector, M is the vector of treatment means,

$$\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_b \end{pmatrix} \quad \text{where } a_i$$

is the additive effect due to β_i and e is an N x 1 vector of independent and identically distributed errors with a zero mean and a variance of σ^2. This model is equivalent to the model

$$\sum_{i=1}^{b} \chi_i \ \theta\ Y = \sum_{i=1}^{b} (\chi_i \ \theta\ M + a_i\chi_i + \chi_i \ \theta\ e)$$ and it is easy

to see that the observational model for the units in β_i is $\chi_i \ \theta\ Y = \chi_i \ \theta\ M + a_i\chi_i + \chi_i \ \theta\ e$.

In the model $Y = M + e$ we have seen that $\lambda'Y$ is the estimate of $\lambda'M$. However in the model $Y = M + X\alpha + e$, $E(\lambda'Y) = \lambda'M + \lambda'X\alpha$ and thus $\lambda'M$ is estimable if $\lambda'X\alpha = 0$.

Mann (15) gave a brief discussion of the technique of blocking and confounding. His definition of confounding is equivalent to the following definition although he chose not to use the concept of the Hadamard product.

Definition 3.4: An effect $\lambda'M$ is confounded with the block β_i of a plan B if $\chi_i \ \theta\ \lambda = c\chi_i$ where c is a scalar.

A set of r effects $S'M$ is confounded with the block β_i if $\chi_i \ \theta\ S = (c_1\chi_i, c_2\chi_i, \ldots, c_r\chi_i)$ and $S'M$ is confounded with each block of a plan B if

$X \Theta S = (c_{11}\chi_1, c_{12}\chi_1, \ldots, c_{1r}\chi_1, c_{21}\chi_2, c_{22}\chi_2, \ldots, c_{2r}\chi_2, \ldots,$
$c_{b1}\chi_b, c_{b2}\chi_b, \ldots, c_{br}\chi_b)$.

<u>Definition 3.5</u>: An effect $\lambda'M$ is orthogonal to the block β_i of a plan B if $\lambda'\chi_i = 0$.

A set of r effects $S'M$ is orthogonal to β_i if $S'\chi_i = \emptyset$ and $S'M$ is orthogonal to each block of a plan B if $S'X = \emptyset$.

<u>Definition 3.6</u>: An effect $\lambda'M$ is partially confounded with the block β_i of a plan B if $\lambda'M$ is neither orthogonal to β_i nor confounded with β_i.

The two theorems that follow are due to Mann (15).

<u>Theorem 3.1</u>: The mean effect $J_N'M$ is confounded with each block of a plan B.

<u>Proof</u>: $X \Theta J_N = X$ by Theorem 2.17.

<u>Theorem 3.2</u>: If a set of r effects $S'M$ is confounded with β_i and γ is an r x 1 vector, then $(S\gamma)'M$ is confounded with β_i.

<u>Proof</u>: $\chi_i \Theta S\gamma = (\chi_i \Theta S)\gamma = (c_1\chi_i, c_2\chi_i, \ldots, c_r\chi_i)\gamma =$

$(c_1, c_2, \ldots, c_r)\gamma\ \chi_i = c\ \chi_i$ by Theorem 2.18.

<u>Theorem 3.3</u>: If $S'M$ is a set of r effects orthogonal to each block of a plan B and A is an r x s matrix, then $(SA)'M$ is orthogonal to each block of B.

<u>Proof</u>: $\emptyset = S'X = A'(S'X) = A'S'X = (SA)'X$.

<u>Theorem 3.4</u>: Let B be a plan of b blocks and let $S'M$ be a set of b normalized orthogonal effects. Then S is confounded in each block of B if and only if $S = \tilde{X}C$ where C is an orthogonal matrix and $\tilde{X}$ is the matrix resulting from the

normalization of the columns of X.

Proof: Since S'M is confounded in each block, then

$$S = \left(\sum_{i=1}^{b} c_{i1}\tilde{x}_i, \sum_{i=1}^{b} c_{i2}\tilde{x}_i, \ldots, \sum_{i=1}^{b} c_{ib}\tilde{x}_i \right) = \tilde{X}C \quad \text{and}$$

$I_b = S'S = (\tilde{X}C)'(\tilde{X}C) = C'\tilde{X}'\tilde{X}C = C' \; I_b \; C = C'C$. Thus C is an orthogonal matrix. Conversely, if $S = \tilde{X}C$, then S'M is confounded in each block of B by Theorem 3.2.

Theorem 3.5: Let (S, S*) define a set of N normalized orthogonal effects where S is N x b and let B be a plan of b blocks. Then S'M is confounded in each block of B if and only if S*'M is orthogonal to each block of B.

Proof: By Theorem 3.4, $S = \tilde{X}C$ where $C'C = I_b$, and from $S^{*\prime}S = \emptyset$ we obtain $S^{*\prime}\tilde{X}C = \emptyset$ whence we get $S^{*\prime}\tilde{X} = \emptyset = S^{*\prime}X$. Conversely if S*'M is orthogonal to each block of B, then $S^{*\prime}X = \emptyset = S^{*\prime}\tilde{X}$. Now $\tilde{X}'M$ is a set of b normalized orthogonal effects confounded with blocks. Since the columns of $\tilde{X}$ and S respectively are orthonormal bases of the same subspace then there exists an orthogonal matrix C such that $S = \tilde{X}C$. Since $\tilde{X}'M$ is confounded in each block then S is confounded in each block by Theorem 3.2.

Theorem 3.6: If $\lambda_1'M$ and $\lambda_2'M$ are confounded in a block of β_i of a plan B then $(\lambda_1 \ominus \lambda_2)'M$ is confounded in β_i.

Proof: $\chi_i \ominus(\lambda_1 \ominus \lambda_2) = (\chi_i \ominus \lambda_1) \ominus \lambda_2 = c_1\chi_i \ominus \lambda_2 =$

$$c_1(\chi_i \ominus \lambda_2) = c_1c_2\chi_i \; .$$

Theorem 3.7: If a plan B of b blocks confounds the b normalized orthogonal effects S'M, then B is unique.

Proof: Let B_* be a plan of b blocks that confounds S'M. Then by Theorem 3.4, $S = \tilde{X}C = \tilde{X}_*C_*$ where $\tilde{X}'\tilde{X} = I_b = \tilde{X}_*'\tilde{X}_*$ and $C'C = I_b = C_*'C_*$. Thus $\tilde{X} = \tilde{X}_*C_*C' = \tilde{X}_*P$ and P is orthogonal. From $\tilde{X} = \tilde{X}_*P$ we obtain $\tilde{\chi}_i = \tilde{X}_* \rho_i$ where $P = (\rho_1, \rho_2, \ldots, \rho_b)$ and notice that each entry of ρ_i must be non-negative and at least one entry must be positive. Since $\rho_i'\rho_j = 0$ for $i \neq j$ we have that $\rho_j \Theta \rho_i = \emptyset$ for $i = 1, 2, \ldots, b$ and $i \neq j$ and furthermore that

$$\rho_j \Theta \sum_{\substack{i=1\\i\neq j}}^{b} \rho_i = \sum_{\substack{i=1\\i\neq j}}^{b} (\rho_j \Theta \rho_i) = \emptyset .$$ The vector $\sum_{\substack{i=1\\i\neq j}}^{b} \rho_i$ has at least b-1 positive entries which implies the ρ_j has at most one positive entry. Thus ρ_j has exactly one positive entry and thus b-1 entries of zero. Therefore P is a permutation matrix and the plans B and B_* are identical.

Definition 3.2: Let B_1 and B_2 be plans consisting of b_1 and b_2 blocks respectively. Then $B_1 \cap B_2$ is the set

$$B^* = \{\beta^*_{ij} : \beta^*_{ij} = \beta_{1i} \cap \beta_{2j}, i=1, 2, \ldots, b_1, \\ j = 1, 2, \ldots, b_2\}.$$

B^* is called the intersection of B_1 and B_2 and is a blocking plan. The incidence matrix of β^*_{ij} is $\chi_{1i} \Theta \chi_{2j}$ where χ_{1i} and χ_{2j} are the respective incidence matrices of β_{1i} and β_{2j}. Letting the members of B^* assume the lexicographic order, we see that the incidence matrix of B^* is $X^* = X_1 \Theta X_2$.

Figure 3.2 gives the incidence matrix of the intersection of Plan (a) and Plan (b) of Figure 3.1.

000	110	001	012	010	011
002	112		101	102	100
111					

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3.2-- The intersection of Plan (a) and Plan (b) of Figure 3.1 and its incidence matrix.

From practical considerations it is desirable to have blocks of equal size. The assumption of homogeneous errors is rarely met in most biological situations. It is intuitively obvious that the units of a small block can be chosen to be more homogeneous than the units of a large block. Blocks of equal size also are advantageous from a theoretical standpoint.

Theorem 3.8: Let B be a plan consisting of b blocks of size r. Then B confounds the set of b orthogonal effects $S'M$ if and only if there exists a matrix C satisfying $S = XC$ and $C'C = D$, a diagonal matrix.

Proof: By Theorem 3.4 B confounds $\tilde{S}$ if and only if there exist an orthogonal matrix C_* such that $\tilde{S} = \tilde{X}C_*$. Since

$$\tilde{X} = \frac{1}{\sqrt{r}} X$$

then B confounds S and $S'S = rD$ if and only if there exists a C such that $C'C = D$ and $S = XC$.

Theorem 3.9: Let B_1 and B_2 be two plans of b_1 and b_2 blocks of equal size respectively and let $(\tilde{J}_{b_1}, W_1)$ and $(\tilde{J}_{b_2}, W_2)$ be orthogonal matrices. If the effects $(X_1W_1)'M$ are mutually orthogonal to the effects $(X_2W_2)'M$, then

$X_1'X_2 = \frac{N}{b_1b_2} J_{b_1}J_{b_2}'$ and thus the blocks of the plan

$B_1 \cap B_2$ are of equal size.

<u>Proof</u>: Since the effects $(X_1W_1)'M$ are mutually orthogonal to the effects $(X_2W_2)'M$, we have

$$\emptyset = (X_1W_1)'X_2W_2 = W_1' X_1' X_2 W_2 = W_1W_1'X_1'X_2W_2W_2' =$$

$$(I_{b_1}-K_{b_1}) X_1'X_2 (I_{b_2}-K_{b_2}) = X_1'X_2 - K_{b_1}X_1'X_2 - X_1'X_2K_{b_2} +$$

$$K_{b_1}X_1'X_2K_{b_2}$$

whence $$X_1'X_2 = K_{b_1}X_1'X_2 + X_1'X_2K_{b_2} - K_{b_1}X_1'X_2K_{b_2} \quad . \tag{3.1}$$

From $J_{b_1}' X_1' X_2 = (X_1J_{b_1})'X_2 = J_N' X_2 = \frac{N}{b_2} J_{b_2}'$ and

$X_1'X_2J_{b_2} = X_1' J_N = \frac{N}{b_1} J_{b_1}$ we obtain

$\frac{N}{b_2} J_{b_1} J_{b_2}' = J_{b_1}J_{b_1}'X_1'X_2 = b_1K_{b_1}X_1'X_2$ and

$\frac{N}{b_1} J_{b_1}J_{b_2}' = X_1'X_2J_{b_2}J_{b_2}' = b_2X_1'X_2K_{b_2}$ respectively.

Since $K_{b_1}J_{b_1} = J_{b_1}$, the last equality yields

$$b_2K_{b_1}X_1'X_2K_{b_2} = \frac{N}{b_1} J_{b_1}J_{b_2}' \quad .$$

Substituting for the quantities in (3.1) we obtain

$X_1'X_2 = \frac{N}{b_1b_2} J_{b_1}J_{b_2}'$ which implies that each block of

of the plan $B_1 \cap B_2$ is of size $\frac{N}{b_1b_2}$.

Since the blocks of $B_1 \cap B_2$ are of equal size and $C'C$ is diagonal where $C = (J_{b_1} \otimes J_{b_2},\ W_1 \otimes J_{b_2},\ J_{b_1} \otimes W_2,\ W_1 \otimes W_2)$, then by Theorem 3.8 $(X_1 \otimes X_2)C$ defines a set of $b_1 b_2$ orthogonal effects.

<u>Theorem 3.10</u>: $(X_1 \otimes X_2)(J_{b_1} \otimes J_{b_2},\ W_1 \otimes J_{b_2},\ J_{b_1} \otimes W_2) = (J_N,\ X_1 W_1,\ X_2 W_2)$

<u>Proof</u>: $(X_1 \otimes X_2)(J_{b_1} \otimes J_{b_2}) = (X_1 \otimes X_2) J_{b_1 b_2} = J_N$

Let $\omega_i = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{b_i} \end{pmatrix}$ denote a column of W_i.

Then for $i = 1$ $(X_1 \otimes X_2)(\omega_1 \otimes J_{b_2}) = (X_{11} \otimes X_2,\ X_{12} \otimes X_2,\ \ldots,\ X_{1b_2} \otimes X_2) \begin{pmatrix} w_1 J_{b_2} \\ w_2 J_{b_2} \\ \vdots \\ w_{b_1} J_{b_2} \end{pmatrix} =$

$$\sum_{j=1}^{b_1} w_j (X_{1j} \otimes X_2) J_{b_2} = \sum_{j=1}^{b_1} w_j X_{1j} \otimes X_2 J_{b_2} =$$

$$\sum_{j=1}^{b_1} w_j X_{1j} \otimes J_N = \sum_{j=1}^{b_1} w_j X_{1j} = X_1 \omega_1$$

thus $(X_1 \otimes X_2)(W_1 \otimes J_{b_2}) = X_1 W_1$

For $i = 2$ we have

$$(X_1 \otimes X_2)(J_{b_1} \otimes \omega_2) = (X_{11} \otimes X_2,\ X_{12} \otimes X_2,\ \ldots,\ X_{1b_1} \otimes X_2) \begin{pmatrix} \omega_2 \\ \omega_2 \\ \vdots \\ \omega_2 \end{pmatrix} =$$

$$\sum_{j=1}^{b_1} (X_{1j} \otimes X_2)\omega_2 = \sum_{j=1}^{b_1} X_{1j} \otimes X_2 \omega_2 = J_N \otimes X_2 \omega_2 = X_2 \omega_2$$ and

$$(X_1 \otimes X_2)(J_{b_1} \otimes W_2) = X_2W_2. \quad \text{Thus}$$

$$(X_1 \otimes X_2)(J_{b_1} \otimes J_{b_2},\ W_1 \otimes J_{b_2},\ J_{b_1} \otimes W_2) = (J_N,\ X_1W_1,\ X_2W_2).$$

Thus we see that $B_1 \cap B_2$ confounds the effects confounded by either B_1 or B_2. The $(b_1-1)(b_2-1)$ orthogonal effects defined by $(X_1 \otimes X_2)(W_1 \otimes W_2)$ are also confounded.

Definition 3.3: If the respective blocks of B_1 and B_2 are of equal size and B_i confounds the orthogonal effects defined by $X_i(J_{b_i}, W_i)$ and $(X_1W_1)'(X_2W_2) = \emptyset$, the set of $(b_1-1)(b_2-1)$ effects $[(X_1 \otimes X_2)(W_1 \otimes W_2)]'M$ is the generalized interaction of $(X_1W_1)'M$ and $(X_2W_2)'M$.

A blocking plan B determines the model $Y = M + X\alpha + e$. The following theorem gives a sufficient condition for the estimability of a set of effects.

Theorem 3.11: $S'M$ is estimable in the model $Y = M + X\alpha + e$ if $S'M$ is orthogonal to each block of the plan determined by X.

Proof: If $S'X = \emptyset$, then $E(S'Y) = S'M + S'X\alpha = S'M$.

In most experimental situations α is not known. For the case in which α is not known it is extremely unlikely that $S'X\alpha = \emptyset$ unless $S'X = \emptyset$ and for practical purposes one can say that $S'M$ is estimable only if $S'X = \emptyset$.

Definition 3.4: In the model $Y = M + X\alpha + e$ the block sum of squares is $Y'\tilde{X}\tilde{X}'Y$.

Since $S = \tilde{X}C$ with C orthogonal defines a set of normalized orthogonal effects then $Y'SS'Y$ is also the block sum of squares. The mean effect is confounded in each block of a plan and the sum of squares due to the mean is $Y'K_NY$. The quadratic form $Y'(\tilde{X}\tilde{X}'-K_N)Y$ is called the between all blocks (B.A.B.) sum of squares.

Theorem 3.12: $\tilde{X}\tilde{X}'$ is an idempotent matrix of rank b where b is the number of blocks in the plan defined by X.

Proof: $\tilde{X}\tilde{X}'\tilde{X}\tilde{X}' = \tilde{X}\ I_b\ \tilde{X}' = \tilde{X}\tilde{X}'$. Since $\tilde{X}'\tilde{X} = I_b$, the rank of $\tilde{X}\tilde{X}'$ is b.

Theorem 3.13: $K_N\ \tilde{X}\tilde{X}' = \tilde{X}\tilde{X}'K_N = K_N$

Proof: Let the size of β_i be r_i for $i = 1, 2, \ldots, b$. Then $\tilde{X} = XD$, where D is a diagonal matrix with $d_{ii} = \dfrac{1}{\sqrt{r_i}}$, and

$\tilde{X}\tilde{X}' = XD^2X'$. So $K_N\tilde{X}\tilde{X}' = \frac{1}{N}\ J_NJ_N'XD^2X' = \frac{1}{N}\ J_N(r_1,r_2,\ldots,r_b)D^2X' =$ $\frac{1}{N}\ J_N\ J_b'\ X' = \frac{1}{N}\ J_N\ J_N' = K_N$. Also

$K_N = K_N' = (K_N\ \tilde{X}\tilde{X}')' = \tilde{X}\tilde{X}'K_N$.

Definition 3.5: In the model $Y = M + X\alpha + e$, $Y'(I_N - \tilde{X}\tilde{X}')Y$ is the within all blocks (W.A.B.) sum of squares.

Theorem 3.14: $I_N - \tilde{X}\tilde{X}'$ and $\tilde{X}\tilde{X}' - K_N$ are idempotent matrices of rank N-b and b-1 respectively.

Proof: Since $\tilde{X}\tilde{X}'$ is idempotent of rank b, $I_N - \tilde{X}\tilde{X}'$ is idempotent of rank N-b by Theorem 2.27.

$(\tilde{X}\tilde{X}'-K_N)(\tilde{X}\tilde{X}'-K_N) = \tilde{X}\tilde{X}'\tilde{X}\tilde{X}' - K_N\tilde{X}\tilde{X}' - \tilde{X}\tilde{X}'K_N+K_N^2 =$

$\tilde{X}\tilde{X}' - K_N$ and $\rho(\tilde{X}\tilde{X}'-K_N) = tr(\tilde{X}\tilde{X}'-K_N) =$

$tr(\tilde{X}\tilde{X}') - tr(K_N) = tr(\tilde{X}'\tilde{X})-1 = b-1$

We have seen that a plan B determines the model $Y = M + X\alpha + e$. If L defines the 2^m factorial effects $L'M$, then we have the model $L'Y = L'M + L'X\alpha + L'e$. The following results are due to plans which confound one or more factorial effects. The $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement is assumed unless otherwise indicated.

Theorem 3.15: If $(\tilde{J}_n, U)$ is an orthogonal matrix and $n > 2$, then the columns of $U \ominus U$ span $V_n(R)$.

Proof: It suffices to show that the rows of $U \ominus U$ are linearly independent. Let

$$U \ominus U = \begin{pmatrix} \rho_1^* \\ \rho_2^* \\ \vdots \\ \rho_n^* \end{pmatrix} \quad \text{where } \rho_i^* = (u_{i1}\rho_i, u_{i2}\rho_i, \ldots, u_{in-1}\rho_i)$$

and $\rho_i = (u_{i1}, u_{i2}, \ldots, u_{in-1})$ is the i^{th} row of U.

Then $\sum_{i=1}^{n} d_{ii}\rho_i^* = \emptyset$ if and only if $\sum_{i=1}^{n} d_{ii}\rho_i'\rho_i = \emptyset$

if and only if $U'DU = \emptyset$ where D is the diagonal matrix with d_{ii} as the i^{th} diagonal element. $U'DU = \emptyset$ implies $(I_n - K_n)D(I_n - K_n) = \emptyset$. Thus we have $D = DK_n + K_nD - K_nDK_n$ and

$$d_{ii} = \frac{d_{ii}}{n} + \frac{d_{ii}}{n} - \frac{tr(D)}{n^2} \quad \text{which yields} \quad d_{ii} = \frac{-tr(D)}{n(n-2)} .$$

The diagonal elements are equal and

$$\sum_{i=1}^{n} d_{ii} = tr(D) = \frac{-tr(D)}{n-2}$$ implies that $D = \emptyset$. Therefore the rows of $U \ominus U$ are linearly independent and consequently the columns of $U \ominus U$ span $V_n(R)$.

Theorem 3.16: If B confounds A_iA_j and $n_i > 2$, $n_j > 2$, then B confounds A_i and A_j.

Proof: By Theorem 3.6 B confounds $[F^*(U_i,U_j) \Theta F^*(U_i,U_j)]'M$. $F^*(U_i,U_j) \Theta F^*(U_i,U_j) = F^*(U_i \Theta U_i, U_j \Theta U_j)P$ by Theorem 2.19. Since the columns of $U_i \Theta U_i$ span $V_{n_i}(R)$ then there exists a matrix H_i such that $(U_i \Theta U_i)H_i = (J_{n_i}, U_i)$. Similarly there exists a matrix H_j such that $(U_j \Theta U_j)H_j = (J_{n_j}, U_j)$. Thus $F^*(U_i \Theta U_i, U_j \Theta U_j)(H_i \otimes H_j) = F^*((J_{n_i}, U_i), (J_{n_j}, U_j))$ is confounded by Theorem 3.2. Therefore the effects defined by $(J_N, F^*(U_i), F^*(U_j), F^*(U_i,U_j))$ are confounded with B.

Theorem 3.17: If B confounds $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ and $n_i > 2$ for for $i = 1, 2, \ldots, k$ then B confounds $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ and any interaction involving only these effects.

Proof: The proof follows that of the last theorem. There exists matrices $H_{i_1}, \ldots, H_{i_k}$ such that

$$F^*(U_{i_1} \Theta U_{i_1}, U_{i_2} \Theta U_{i_2}, \ldots, U_{i_k} \Theta U_{i_k})(H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_k}) =$$

$$F^*((J_{n_{i_1}}, U_{i_1}), (J_{n_{i_2}}, U_{i_2}), \ldots, (J_{n_{i_k}}, U_{i_k})).$$

Thus the effects defined by $(J_N, F^*(U_{i_1}), \ldots, F^*(U_{i_k}),$

$F^*(U_{i_1}, U_{i_2}), \ldots, F^*(U_{i_{k-1}}, U_{i_k}), \ldots, F^*(U_{i_1}, U_{i_2}, \ldots, U_{i_k}))$

are confounded with B.

Theorem 3.18: The plan B determined by $X = F^*(I_{n_i})$ uniquely confounds only the mean effect and A_i.

<u>Proof</u>: By Theorem 3.8, $X(J_{n_i}, U_i) = F^*(I_{n_i})(J_{n_i}, U_i) = F^*((J_{n_i}, U_i)) = (J_N, F^*(U_i))$ defines a set of effects confounded with B. By Theorem 3.7 X is unique.

<u>Theorem 3.19</u>: If B confounds $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ then B confounds any interaction involving only these effects.

<u>Proof</u>: Let $\{j_1, j_2, \ldots, j_r\}$ be a subset of $\{i_1, i_2, \ldots, i_k\}$. Then by Theorem 3.6 $F^*(U_{j_1})\Theta F^*(U_{j_2})\Theta\ldots\Theta F^*(U_{j_r}) = F^*(U_{j_1}, U_{j_2}, \ldots, U_{j_r})$ defines a set of effects confounded with B.

$F^*(I_{n_{i_1}}, I_{n_{i_2}}, \ldots, I_{n_{i_k}})$ is the incidence matrix of the plan which confounds the effects given in the last theorem. This plan is the intersection of the plans defined by $F^*(I_{n_{i_1}})$, $F^*(I_{n_{i_2}})$, ..., and $F^*(I_{n_{i_k}})$.

<u>Definition 3.3</u>: The number of design points in β_i whose $j_1, j_2, \ldots, j_s$ entries are respectively $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ is $h_i(j_1,j_2,\ldots,j_s;\ a_{j_1},a_{j_2},\ldots,a_{j_s})$.

<u>Theorem 3.20</u>: A blocking plan B of b blocks confounds the mean effect and b-1 components of $A_1A_2\ldots A_m$ if and only if for each set $\{j_1,j_2,\ldots,j_s\}$, where $1 \le s < m$, $h_i(j_1,j_2,\ldots,j_s\ ;\ 0,0,\ldots,0) = h_i(j_1,j_2,\ldots,j_s\ ;\ c_{j_1},c_{j_2},\ldots,c_{j_s})$ for each

$(c_{j_1},c_{j_2},\ldots,c_{j_s})$ in $Z(N_{j_1}) \times Z(N_{j_2}) \times \ldots \times Z(N_{j_s})$.

<u>Proof</u>: By Theorem 3.5, any factorial effect other than $A_1A_2\ldots A_m$ or the mean effect is orthogonal to B.

Thus $[F^*((J_{n_{j_1}}, U_{j_1}), (J_{n_{j_2}}, U_{j_2}), \ldots, (J_{n_{j_{r-1}}}, U_{j_{r-1}}), \tilde{U}_{j_r}, (J_{n_{j_{r+1}}}, U_{j_{r+1}}), \ldots, (J_{n_{j_s}}, U_{j_s}))]'\ X = \emptyset$ for

r=1, 2, ..., s. Multiplication on the left of the last equality by $[(J_{n_{j_1}}, U_{j_1})']^{-1} \otimes [(J_{n_{j_2}}, U_{j_2})']^{-1} \otimes \ldots \otimes$

$[(J_{n_{j_{r-1}}}, U_{j_{r-1}})']^{-1} \otimes \tilde{U}_{j_r} \otimes [(J_{n_{j_{r+1}}}, U_{j_{r+1}})']^{-1} \otimes \ldots \otimes$

$[(J_{n_{j_s}}, U_{j_s})']^{-1}$ yields

$$F^*(I_{n_{j_1}}, I_{n_{j_2}}, \ldots, I_{n_{j_{r-1}}}, I_{n_{j_r}} - K_{n_{j_r}}, I_{n_{j_{r+1}}}, \ldots, I_{n_{j_s}})'\chi_i = \emptyset \quad (3.2)$$

for i=1,2,...,b and r=1,2,...,s. Upon choosing the first column of each of the arguments we have

$$(1 - \frac{1}{n_{j_r}})\ h_i(j_1, j_2, \ldots, j_s\ ;\ 0, 0, \ldots, 0) =$$

$$\frac{1}{n_{j_r}} \sum_{a_{j_r}=1}^{n_{j_r}-1} h_i(j_1,j_2,\ldots,j_r,\ldots,j_s;\ 0,0,\ldots,0,a_{j_r},0,\ldots,0)$$

which simplifies to the equality

$$h_i(j_1,j_2,\ldots,j_s;0,0,\ldots,0) = \frac{1}{n_{j_r}} \sum_{a_{j_r}=0}^{n_{j_r}-1} h_i(j_1,j_2,\ldots,j_r,\ldots,j_s;$$
$$0,0,\ldots,0,a_{j_r},0,\ldots,0)$$

for $i = 1, 2, \ldots, b$ and $r = 1, 2, \ldots, s$.

Choosing the $c_{j_r}+1$ st columns of $I_{n_{j_r}}-K_{n_{j_r}}$ we obtain

$$h_\ell(j_1,j_2,\ldots,j_r,\ldots,j_s;0,0,\ldots,c_{j_r},\ldots,0) =$$

$$\frac{1}{n_{j_r}} \sum_{a_{j_r}=0}^{n_{j_r}} h_\ell(j_1,j_2,\ldots,j_r,\ldots,j_s;0,0,\ldots,a_{j_r},\ldots,0).$$

Thus $h_\ell(j_1,j_2,\ldots,j_s;0,0,\ldots,0) = h_\ell(j_1,j_2,\ldots,j_r,\ldots,j_s;$
$$0,0,\ldots,c_{j_r},\ldots,0)$$

for $c_{j_r} = 0,1,\ldots,n_{j_r}$. Letting $r=1$ we have

$$h_\ell(j_1,j_2,\ldots,j_s;0,0,\ldots,0) = h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},0,\ldots,0).$$

When $r=2$ we have from (3.2)

$$[F^*(I_{n_{j_1}},I_{n_{j_2}}-K_{n_{j_2}},I_{n_{j=1}},\ldots,I_{n_{j_s}})]'\ \chi_\ell=\emptyset.$$

By choosing the $c_{j_1}+1$ st column of $I_{n_{j_1}}$ and the first column of each of the remaining arguments we obtain

$$h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},0,\ldots,0) = \frac{1}{n_{j_2}} \sum_{a_{j_2}=0}^{n_{j_2}} h_\ell(j_1,j_2,\ldots,j_s; c_{j_1},a_{j_2},0,\ldots,0).$$

Choosing the $c_{j_2}+1$ st column of $I_{n_{j_2}} - K_{n_{j_2}}$ yields

$$h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},c_{j_2},0,\ldots,0) = \frac{1}{n_{j_2}} \sum_{a_{j_2}=0}^{n_{j_2}} h_\ell(j_1,j_2,\ldots,j_s; c_{j_1},a_{j_2},0,\ldots,0).$$

Thus $h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},0,\ldots,0) = h_\ell(j_1,j_2,\ldots,j_s; c_{j_1},c_{j_2},0,\ldots,0).$

Continuing in this manner we obtain the following equalities corresponding to the value that r assumes:

$r=1$, $h_\ell(j_1,j_2,\ldots,j_s;0,0,\ldots,0) = h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},0,\ldots,0)$;

$r=2$, $h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},0,\ldots,0) = h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},c_{j_2},0,\ldots,0)$;

$r=3$, $h_\ell(j_1,j_2,j_3,\ldots,j_s;c_{j_1},c_{j_2},0,\ldots,0) =$

$h_\ell(j_1,j_2,j_3,\ldots,j_s;c_{j_1},c_{j_2},c_{j_3},0,\ldots,0)$;

$\vdots$

$r=s-1$, $h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},c_{j_2},\ldots,c_{j_{s-2}},0,0) =$

$h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},c_{j_2},\ldots,c_{j_{s-2}},c_{j_{s-1}},0)$;

$r=s$, $h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},c_{j_2},\ldots,c_{j_{s-1}},0) =$

$h_\ell(j_1,j_2,\ldots,j_s;c_{j_1},c_{j_2},\ldots,c_{j_{s-1}},c_{j_s})$.

Hence the equality $h_\ell(j_1,j_2,\ldots,j_s;0,0,\ldots,0) =$

$h_\ell(j_1,j_2,\ldots j_s;c_{j_1},c_{j_2},\ldots,c_{j_s})$ holds for each

$(c_{j_1},c_{j_2},\ldots,c_{j_s})$ in $Z(n_{j_1}) \times Z(n_{j_2}) \times \ldots \times Z(n_{j_s})$.

The converse follows by reversing the steps of the proof.

<u>Theorem 3.21</u>: Let B be a plan of b blocks confounding $\tilde{X}(\tilde{J}_b,C)$, where $\tilde{X}C$ defines b-1 normalized components of $A_{i_1},A_{i_2},\ldots,A_{i_k}$, and let $A = \tilde{F}^*(U_{i_1},U_{i_2},\ldots,U_{i_k})[\tilde{F}^*(U_{i_1},U_{i_2},\ldots,U_{i_k})]'$. Then $A-(\tilde{X}\tilde{X}'-K_N)$ is idempotent and $[A-(\tilde{X}\tilde{X}'-K_N)]X = \emptyset$ if and only if the blocks of B are of equal size r.

<u>Proof</u>: $\tilde{X}K_b\tilde{X}' = \tilde{X}\tilde{J}_b\tilde{J}_b\tilde{X}' = \tilde{J}_N\tilde{J}_N' = K_N$ if and only if $\tilde{X}\tilde{J}_b = \tilde{J}_N$ if and only if the blocks of B are of equal size. Let (W,W^*) be an orthogonal matrix such that $\tilde{F}^*(U_{i_1},U_{i_2},\ldots,U_{i_k})W = \tilde{X}C$.

Then $\tilde{X}\tilde{X}'A = \tilde{X}\tilde{X}'(\tilde{X}C,\tilde{F}^*(U_{i_1},U_{i_2},\ldots,U_{i_k})W^*$

$(\tilde{X}C,\tilde{F}^*(U_{i_1},U_{i_2},\ldots,U_{i_k})W^*)' = \tilde{X}CC'\tilde{X}' = \tilde{X}(I_b-K_b)\tilde{X}' = A\tilde{X}\tilde{X}'$.

Since A and $\tilde{X}\tilde{X}-K_N$ are idempotent and $AK_N = \emptyset$ and

$\tilde{X}\tilde{X}'K_N = K_N$ then $A-(\tilde{X}\tilde{X}-K_N)$ is idempotent if and only if

$\tilde{X}\tilde{X}'A = \tilde{X}\tilde{X}'-K_N$ if and only if $\tilde{X}(I_b-K_b)\tilde{X}' = \tilde{X}\tilde{X}'-K_N$ if and only

$\tilde{X}K_b\tilde{X}' = K_N$ if and only if the blocks of B are of equal size.

$(A-(\tilde{X}\tilde{X}'-K_N))\tilde{X} = \emptyset = A\tilde{X} - \tilde{X} + K_N\tilde{X} = \tilde{X}(I_b-K_b)-\tilde{X} + K_N\tilde{X} =$

$-\tilde{X}K_b + K_N\tilde{X}$. $K_N\tilde{X} - \tilde{X}K_b = \emptyset$ if and only if $\tilde{X}K_b\tilde{X}' = K_N$

if and only if the blocks of B are of equal size.

Definition 3.4: The extension of a plan B* of b blocks of an $n_1 \times n_2 \times \ldots \times n_k$ factorial arrangement of treatments to an $n_1 \times n_2 \times \ldots \times n_k \times n_{k+1} \times \ldots \times n_m$ factorial arrangement of treatments is $B = \{\beta_i : i = 1,2,\ldots,b\}$ where $\beta_i = \beta_i^* \times Z(n_{k+1}) \times \ldots \times Z(n_m)$.

B is a plan of b blocks in an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement and the incidence matrix of B is $X = X^* \otimes J_q$ where X* is the incidence matrix of B* and $q = n_{k+1}, n_{k+2}, \ldots, n_m$. Figure 3.3 gives the extension of a plan of a 2 x 2 factorial arrangement to a 2 x 2 x 3 factorial arrangement.

B*:
$$\begin{array}{|c|c|}\hline 0\,0 & 0\,1 \\ 1\,1 & 1\,0 \\ \hline\end{array}$$

B:
$$\begin{array}{|c|c|}\hline 0\,0\,0 & 0\,1\,0 \\ 0\,0\,1 & 0\,1\,1 \\ 0\,0\,2 & 0\,1\,2 \\ 1\,1\,0 & 1\,0\,0 \\ 1\,1\,1 & 1\,0\,1 \\ 1\,1\,2 & 1\,0\,2 \\ \hline\end{array}$$

$$X^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Figure 3.3--Extension of a plan of a 2 x 2 factorial arrangement to a 2 x 2 x 3 factorial arrangement.

Theorem 3.22: If B^* confounds the set of b normalized orthogonal defined by S in an $n_1 \times n_2 \times \ldots \times n_k$ factorial arrangement, then the extension of B^* confounds the set of b normalized orthogonal effects defined by $S \otimes J_q$ in an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangment.

Proof: By Theorem 3.4 there exist an orthogonal matrix C such that $S = \tilde{X}^*C$. Thus $S \otimes \tilde{J}_q = \tilde{X}^*C \otimes \tilde{J}_q = (\tilde{X}^* \otimes \tilde{J}_q)C = \tilde{X}C$ and by Theorem 3.4 the effects defined by $S \otimes \tilde{J}_q$ are confounded with the extension of B^*.

Corollary 3.2: If B^* confounds the mean effect and A_i^* in an $n_1 \times n_2 \times \ldots \times n_k$ factorial arrangement, then the extension of B^* confounds the mean effect and A_i in an $n_1 \times n_2 \times \ldots \times n_k$ factorial arrangement.

Corollary 3.3: If B^* confounds the mean effect and b-1 components of $A_{i_1}^* A_{i_2}^* \ldots A_{i_r}^*$ in an $n_1 \times n_2 \times \ldots \times n_k$ factorial arrangement, then the extension of B^* confounds the mean effect and b-1 components of $A_{i_1} A_{i_2} \ldots A_{i_r}$ in an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement.

Theorem 3.23: If B is a plan of b blocks and confounds the mean effect and b-1 components of $A_1 A_2 \ldots A_k$ then B is the extension of a plan that confounds the mean effect and b-1 components of $A_1^* A_2^* \ldots A_k^*$ in an $n_1 \times n_2 \times \ldots \times n_k$ factorial arrangement.

Proof: Let $q = n_1 n_2 \ldots n_k$, $p = (n_1-1)(n_2-1)\ldots(n_k-1)$ and $r = N/q$. By hypothesis, there exists a $p \times b-1$ matrix W with normalized orthogonal columns such that B confounds the b normalized orthogonal effects defined by $(\tilde{J}_N, (\tilde{U}_1 \otimes \tilde{U}_2 \otimes \ldots \otimes \tilde{U}_k \otimes \tilde{J}_r)W)$. By Theorem 3.4 there exists an orthogonal matrix C such that $\tilde{X} = (\tilde{J}_N, (\tilde{U}_1 \otimes \tilde{U}_2 \otimes \ldots \otimes \tilde{U}_k \otimes \tilde{J}_r)W)C = [(\tilde{J}_q, (\tilde{U}_1 \otimes \tilde{U}_2 \otimes \ldots \otimes \tilde{U}_k)W)C] \otimes \tilde{J}_r =$ $\tilde{X}^* \otimes \tilde{J}_r$. That X* is an incidence matrix follows from the fact that X is an incidence matrix. Thus $\tilde{X}^*$ is a normalized incidence matrix of a plan B* that confounds the mean effect and b-1 components of $A_1^* A_2^* \ldots A_k^*$ in an $n_1 \times n_2 \times \ldots \times n_k$ factorial arrangement of treatments.

CHAPTER IV

EXAMPLES OF BLOCKING AND CONFOUNDING OF FACTORIAL ARRANGEMENTS OF TREATMENTS

The purpose of this chapter is to illustrate how the development in the preceding chapters can be utilized in obtaining plans and their analyses of variance.

In light of Theorem 3.21, only plans with equal block sizes will be considered. It is evident from the results in CHAPTER III that one cannot confound an arbitrary set of b orthogonal effects with b blocks. However, a plan of b blocks does confound at least one set of b orthogonal effects.

If there exists a plan of b blocks that confounds the mean effect and b-1 components of $A_1A_2...A_k$ in an $n_1 \times n_2 \times ... \times n_k$ factorial arrangement, then it follows from Theorem 3.20 that b divides n_i for $i=1, 2, ..., k$. Conversely, if $b>1$ and b divides n_i for $i=1, 2, ..., k$, then the condition of Theorem 3.20 can be met and thus there exists a plan of b blocks that confounds the mean effect and b-1 components of the highest-order interaction.

In the analysis of any confounding plan, it should be remembered that differences in responses due to blocks is eliminated in the W.A.B. analysis. The price of eliminating these differences is the loss of information on the mean effect and b-1 orthogonal effects.

If a researcher has no preference as to what effects to confound, then components of the highest-order interaction is often a good choice since this interaction is the most difficult to interpret. Since the highest-order interaction is usually taken as the error term in the single replicate experiments, the confounding of b-1 components of this interaction will reduce the error degrees of freedom. This loss in error degrees of freedom alone is not necessarily a liability since the same situation occurs whenever one chooses a randomized complete block design in lieu of a completely randomized design. However, if this interaction has very few degrees of freedom, then it is perhaps better to confound components of an interaction of little interest so as not to reduce the degrees of freedom for error.

The examples which follow illustrate the use of Theorem 3.20 in obtaining plans that confound components of the highest-order interaction. The first example is given in somewhat greater detail than the others and it is hoped that the reader can see how the other examples can be similarly developed.

Example 4.1: Suppose that a researcher is interested in the responses of mice upon administration of 16 treatments comprising a 4 x 4 factorial arrangement of treatments. The factors are taken as 4 levels of different drugs and for the purposes of this example the factors are considered fixed. All the possible combinations of one level from each of the factors comprise the set of design points $T = Z(4) \times Z(4)$. Let the levels within each factor be naturally ordered so that, for example, the design point

(2, 3) represents the third level of the first drug and the fourth level of the second drug.

The researcher has at his disposal 4 strains of mice with 4 mice in each strain and he has good reason to believe that differences in strains will influence the responses to the treatments. He desires information on the main effects and also wants to assess the magnitude of the contrast $\lambda_0'M = \mu_{02} - \mu_{13} - \mu_{21} + \mu_{30}$.

Since differences among the strains of mice are thought to influence the responses, then strains should be confounded with blocks. The need for information on the main effects indicates that the main effects A_1 and A_2 should be orthogonal to strains (blocks).

The largest block size that permits these considerations is four. With a block size of 4 we can satisfy the conditions of Theorem 3.20 and thus obtain a plan of 4 blocks that confounds the mean effect and 3 components of the A_1A_2 interaction effect. The conditions are: $h_i(j, c_i) = 1$ for $i = 1, 2, 3, 4$, $j = 1, 2$, and $c_j = 0, 1, 2, 3$. A plan satisfies these conditions if and only if A_1 and A_2 are orthogonal to blocks. Since the mean effect and A_1A_2 are orthogonal to both A_1 and A_2 and since the mean effect is confounded in any plan, then 3 components of A_1A_2 are confounded with blocks. Appearing in Figure 4.1 are the 24 plans, each of which confounds only the mean effect and 3 components of A_1A_2. That no two plans

(a)	00	01	02	03	(b)	00	01	02	03	(c)	00	01	02	03	(d)	00	01	02	03
	11	12	13	10		11	10	13	12		11	10	13	12		11	13	10	12
	22	23	20	21		22	23	20	21		22	23	21	20		22	20	23	21
	33	30	31	32		33	32	31	30		33	32	30	31		33	32	31	30

(e)	00	01	02	03	(f)	00	01	02	03	(g)	00	01	02	03	(h)	00	01	02	03
	12	12	11	10		13	10	11	12		13	12	10	11		13	12	11	10
	23	20	23	22		21	22	23	20		21	20	23	22		21	23	20	22
	31	33	30	31		32	33	30	31		32	33	31	30		32	30	33	31

(i)	00	01	02	03	(j)	00	01	02	03	(k)	00	01	02	03	(l)	00	01	02	03
	12	13	10	11		12	13	10	11		12	13	11	10		12	10	13	11
	23	20	21	22		23	22	21	20		23	22	20	21		23	22	21	20
	31	32	33	30		31	30	33	32		31	30	33	32		31	33	30	32

(m)	00	01	02	03	(n)	00	01	02	03	(o)	00	01	02	03	(p)	00	01	02	03
	12	13	10	11		12	13	10	11		12	13	11	10		12	10	13	11
	21	22	23	20		21	20	23	22		21	20	23	22		21	23	20	22
	33	30	31	32		33	32	31	30		33	32	30	31		33	32	31	30

(q)	00	01	02	03	(r)	00	01	02	03	(s)	00	01	02	03	(t)	00	01	02	03
	11	12	13	10		11	10	13	12		11	10	13	12		11	13	10	12
	23	20	21	22		23	22	21	20		23	22	20	21		23	22	21	20
	32	33	30	31		32	33	30	31		32	33	31	30		32	30	33	31

(u)	00	01	02	03	(v)	00	01	02	03	(w)	00	01	02	03	(x)	00	01	02	03
	13	10	11	12		13	12	11	10		13	12	10	11		13	12	11	10
	22	23	20	21		22	23	20	21		22	23	21	20		22	20	23	21
	31	32	33	30		31	30	33	32		31	30	33	32		31	33	30	32

Figure 4.1--The 24 plans of a 4 x 4 factorial arrangement of treatment each of which confounds 3 components of the A_1A_2 effect.

confound the same 3 components follows from Theorem 3.7.

In order to assess the magnitude of the contrast $\lambda_0'M$ the set of design points {(0,2), (1,3), (2,1), (3,0)} must appear in the same block and thus these treatments must be given to the same strain of mice. An inspection of Figure 4.1 reveals that these design points comprise a block in plans (c), (q), and (r). Each of these three plans confounds 3 components of A_1A_2 and leaves A_1, A_2 and the contrast $\lambda_0'M$ free of strain effects. These three plans are also good choices for obtaining information on the contrasts $\mu_{02} - 2\mu_{13} - \mu_{30}$ and $\mu_{02} - \mu_{30}$ since these contrasts are intra-block and thus are free of block effects.

The plans appearing in Figure 4.1 exclusive of plans (c), (q) and (r) are of dubious value for estimating the contrast $\lambda_0'M$ because the usual estimate of $\lambda_0'M$ involves differences in blocks.

To proceed farther with this example let us choose plan (c) as the design plan. Strains are then randomly assigned to blocks and the 4 mice within a strain are randomly assigned to the treatments within a block.

The model for plan (c) is

$$Y = M + X\alpha + e$$

or more explicitly

$$\begin{bmatrix} y_{00} \\ y_{01} \\ y_{02} \\ y_{03} \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{20} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{30} \\ y_{31} \\ y_{32} \\ y_{33} \end{bmatrix} = \begin{bmatrix} \mu_{00} \\ \mu_{01} \\ \mu_{02} \\ \mu_{03} \\ \mu_{10} \\ \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{20} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \\ \mu_{30} \\ \mu_{31} \\ \mu_{32} \\ \mu_{33} \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_2 \\ a_1 \\ a_4 \\ a_3 \\ a_4 \\ a_3 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_2 \\ a_1 \end{bmatrix} + e.$$

In this model both M and α are unknown and a contrast $\lambda'M$ is estimable if and only if $X'\lambda = \emptyset$. Since $X'\lambda_0 = \emptyset$ then $\lambda_0'M$ is estimable. An analysis of variance for plan (c) is given in Table 4.1.

The 5 components of A_1A_2 in the W.A.B. analysis are called residual and their mean square is used as the denominator of the mean square ratio to test $\lambda_0'M$. The mean square for residual is also used as the denominator for testing the mean square ratios of A_1 and A_2. In the event that one accepts the hypothesis that $\lambda_0'M = 0$ then the sum

TABLE 4.1

AN ANALYSIS OF VARIANCE TABLE FOR PLAN (C)
IN FIGURE 4.1

Source	d.f.	S.S.	F
Total	16	$\sum\limits_{i,j} y_{ij}^2$	
Mean	1	$16\bar{y}^2$	
B.A.B. (A_1A_2)	3	$\frac{1}{4} \sum\limits_{k} \left[\sum\limits_{(i,j)\varepsilon\beta_k} y_{ij} \right]^2 -16\bar{y}^2$	
W.A.B.	12		
A_1	3	$\frac{1}{4} \sum\limits_{i} (\sum\limits_{j} y_{ij})^2 -16\bar{y}^2$	$\frac{\text{M.S.R.}(A_1)}{\text{M.S.R.(Res.)}}$
A_2	3	$\frac{1}{4} \sum\limits_{j} (\sum\limits_{i} y_{ij})^2 -16\bar{y}^2$	$\frac{\text{M.S.R.}(A_2)}{\text{M.S.R.(Res.)}}$
$\lambda_0'M$	1	$\frac{1}{4} (y_{02} - y_{13} - y_{21} + y_{30})^2$	$\frac{\text{M.S.R.}(\lambda_0'M)}{\text{M.S.R.(Res.)}}$
Residual (A_1A_2)	5	S.S.(W.A.B.)-S.S.(A_1)-S.S.(A_2)-S.S.$(\lambda_0'M)$	

of squares for $\lambda_0'M$ may be pooled with the sum of square for residual and the mean square of this pool may be used as the denominator for testing the mean square ratios of A_1 and A_2.

A test of B.A.B. is futile since the B.A.B. sum of squares reflects both differences among strains and differences among the sets of treatments administered to the different strains. One would intuitively hope that the B.A.B. be relatively large but a fortuitous selection of the sets of treatments assigned to the strains could produce a relatively small mean square for B.A.B. One should therefore avoid the practice of pooling the sum of squares for B.A.B. with the sum of squares for residual. This situation differs from the practice of pooling the block sum squares with the sum of squares for residual in the randomized complete block design for in the latter design each treatment appears in each block and a preliminary test of the mean square for blocks can be made.

This concludes the discussion of confounding 3 components of a 4 x 4 factorial arrangement of treatments with a plan consisting of 4 blocks. With a block size of 8 in a 4 x 4 factorial arrangment of treatments we have 2 blocks and to confound 1 component of A_1A_2 a plan must satisfy the following conditions: $h_i(j,c_j) = 2$ for $i = 1, 2$, $j = 1, 2$ and $c_j = 0, 1, 2, 3$. The set of 48 plans satisfying these conditions are given in Figure 4.2. An

00	01	00	02	00	01	00	02	00	01	00	01
02	03	01	03	03	02	01	03	03	02	02	03
11	10	11	10	10	12	10	12	11	10	10	12
13	12	12	13	11	13	11	13	12	13	11	13
20	21	22	20	21	20	22	20	21	20	22	20
22	23	23	21	22	23	23	21	22	23	23	21
31	30	30	32	32	30	32	30	30	31	31	30
33	32	31	33	33	31	33	31	33	32	33	32

00	01	00	01	00	02	00	01	00	02	00	01
02	03	03	02	01	03	02	03	01	03	03	02
11	10	11	10	11	10	10	11	12	10	11	10
13	12	12	13	13	12	12	13	13	11	12	13
21	20	20	21	20	21	21	20	21	20	20	22
22	23	22	23	22	23	23	22	22	23	21	23
30	31	31	30	32	30	31	30	30	32	32	30
33	32	33	32	33	31	33	32	31	33	33	31

00	02	00	01	00	01	00	01	00	02	00	02
01	03	02	03	02	03	03	02	01	03	01	03
12	10	12	10	11	10	10	11	10	11	10	12
13	11	13	11	12	13	12	13	12	13	11	13
20	22	20	22	21	20	21	20	21	20	22	20
21	23	21	23	23	22	22	23	23	22	23	21
32	30	31	30	30	31	31	30	32	30	31	30
33	31	33	32	33	32	33	32	33	31	32	33

00	01	00	02	00	01	00	01	00	01	00	01
03	02	01	03	03	02	02	03	02	03	03	02
12	10	12	10	10	11	11	10	10	11	11	10
13	11	13	11	13	12	13	12	13	12	13	12
21	20	22	20	21	20	22	20	21	20	20	21
22	23	23	21	22	23	23	21	22	23	22	23
30	32	30	32	31	30	30	32	31	30	31	30
31	33	31	33	32	33	31	33	33	32	32	33

Figure 4.2--The 48 plans of a 4 x 4 factorial each of which confounds 1 component of the A_1A_2 effect.

00	02	00	01	00	02	00	01	00	01	00	01
01	03	02	03	01	03	03	02	03	02	02	03
12	10	11	10	11	10	10	12	11	10	10	12
13	11	13	12	12	13	11	13	12	13	11	13
20	21	21	20	20	22	22	20	20	21	21	20
22	23	23	22	21	23	23	21	23	22	23	22
31	30	30	31	32	30	31	30	31	30	32	30
33	32	32	33	33	31	32	33	32	33	33	31

00	01	00	01	00	02	00	02	00	01	00	01
02	03	03	02	01	03	01	03	03	02	03	02
11	10	11	10	11	10	12	10	11	10	11	10
13	12	12	13	13	12	13	11	12	13	12	13
20	21	21	20	22	20	20	22	22	20	20	21
23	22	23	22	23	21	21	23	23	21	23	22
31	30	30	31	30	31	31	30	30	32	31	30
32	33	32	33	32	33	32	33	31	33	32	33

00	01	00	01	00	01	00	02	00	02	00	01
02	03	02	03	03	02	01	03	01	03	03	02
12	10	11	10	10	11	10	11	10	12	12	10
13	11	12	13	12	13	12	13	11	13	13	11
21	20	20	21	21	20	22	20	21	20	20	22
23	22	23	22	23	22	23	21	22	23	21	23
30	32	31	30	31	30	31	30	32	30	31	30
31	33	33	32	32	33	33	32	33	31	32	33

00	01	00	01	00	01	00	01	00	02	00	01
03	02	02	03	02	03	03	02	01	03	02	03
10	11	11	10	10	11	11	10	12	10	10	11
13	12	13	12	13	12	13	12	13	11	12	13
21	20	20	22	21	20	21	20	21	20	21	20
22	23	21	23	23	22	22	23	23	22	23	22
31	30	32	30	31	30	30	31	30	31	31	30
32	33	33	31	32	33	32	33	32	33	33	32

Figure 4.2--Continued

abbreviated analysis of variance of a particular plan is given in Table 4.2.

Example 4.2: In the 2 x 2 x 4 factorial arrangement a plan that confounds only the mean effect and components of $A_1A_2A_3$ must have a block size of 8. Such a plan must satisfy the following conditions:

$$h_i(j;c_j) = 4,$$

$$h_i(3;c_3) = 2,$$

$$h_i(1,2;c_1,c_2) = 2 \text{ and}$$

$$h_i(j,3;c_j,c_3) = 1, \text{ for } i=1,2,\ j=1,2, c_j = 0,1 \text{ and}$$

$$c_3 = 0,1,2,3.$$

The three plans that confounds only the mean effect and 1 component of $A_1A_2A_3$ are given in Figure 4.3. An abbreviated analysis of variance appears in Table 4.3.

If the numbers of levels of the factors do not have a common divisor other than unity, then there is no plan with equal block sizes that confounds only the mean effect and components of the highest-order interaction. In this case the confounding of components of the highest-order interaction results in the confounding or partial confounding of components of other factorial effects.

Example 4.3: In the 2 x 2 x 3 factorial arrangement no plan with equal block sizes exists that confounds only the mean effect and components of $A_1A_2A_3$. The plan B in Figure 4.4 confounds 1 component each of $A_1A_2A_3$ and A_3 in addition

TABLE 4.2

AN ABBREVIATED ANALYSIS OF VARIANCE TABLE
FOR A PARTICULAR PLAN IN FIGURE 4.2

Source	d.f.	S.S.
Total	16	$Y'Y$
Mean	1	$Y'K_{16}Y$
B.A.B. (A_1A_2)	1	$Y'(\tilde{X}\tilde{X}-K_{16})Y$
W.A.B.	14	$Y'(I_{16}-\tilde{X}\tilde{X}')Y$
A_1	3	$Y'(I_4-K_4\otimes K_4)Y$
A_2	3	$Y'(K_4\otimes I_4-K_4)Y$
A_1A_2	8	$Y'(I_4-K_4\otimes I_4-K_4-\tilde{X}\tilde{X}'+K_{16})Y$

0 0 0	0 0 2
0 0 1	0 0 3
0 1 2	0 1 0
0 1 3	0 1 1
1 0 2	1 0 0
1 0 3	1 0 1
1 1 0	1 1 2
1 1 1	1 1 3

0 0 0	0 0 1
0 0 2	0 0 3
0 1 1	0 1 0
0 1 3	0 1 2
1 0 1	1 0 0
1 0 3	1 0 2
1 1 0	1 1 1
1 1 2	1 1 3

0 0 0	0 0 1
0 0 3	0 0 2
0 1 1	0 1 0
0 1 2	0 1 3
1 0 1	1 0 0
1 0 2	1 0 3
1 1 0	1 1 1
1 1 3	1 1 2

Figure 4.3--The 3 plans confounding only the mean and 1 component of $A_1A_2A_3$ in a 2 x 2 x 4 factorial arrangement of treatments.

TABLE 4.3

AN ABBREVIATED ANALYSIS OF VARIANCE TABLE
FOR A PARTICULAR PLAN OF FIGURE 4.3

Source	d.f	S.S.
Total	16	$Y'Y$
Mean	1	$Y'K_{16}Y$
B.A.B. $(A_1A_2A_3)$	1	$Y'(\tilde{X}\tilde{X}'-K_{16})Y$
W.A.B.	14	$Y'(I_{16}-\tilde{X}\tilde{X}')Y$
A_1	1	$Y'(I_2-K_2 \otimes K_2 \otimes K_4)Y$
A_2	1	$Y'(K_2 \otimes I_2-K_2 \otimes K_4)Y$
A_3	3	$Y'(K_2 \otimes K_2 \otimes I_4-K_4)Y$
A_1A_2	1	$Y'(I_2-K_2 \otimes I_2-K_2 \otimes K_4)Y$
A_1A_3	3	$Y'(I_2-K_2 \otimes K_2 \otimes I_4-K_4)Y$
A_2A_3	3	$Y'(K_2 \otimes I_2-K_2 \otimes I_4-K_4)Y$
$A_1A_2A_3$	2	$Y'((I_2-K_2 \otimes I_2-K_2 \otimes I_4-K_4) - \tilde{X}\tilde{X}'+ K_{16})Y$

0 0 0	0 0 1	0 0 2
0 1 1	0 1 0	0 1 2
1 0 0	1 0 1	1 0 2
1 1 1	1 1 0	1 1 2

Figure 4.4--A plan confounding the mean effect and 1 component each of $A_1A_2A_3$ and A_3 in a 2 x 2 x 3 factorial arrangement of treatments.

to the mean effect. The latter statement follows from the application of Theorem 3.8 with the matrix $C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = (J_3, \gamma_2, \gamma_3)$,

$$XC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = \left(J_{12} , \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right)$$

and thus the mean effect, 1 component of $A_1A_2A_3$ and 1 component of A_3 are confounded with B. The plan B is not a good choice if information on A_3 is paramount. Table 4.4 gives an abbreviated analysis of variance for the plan given in Figure 4.4.

For the situation in which we cannot sacrifice information on the highest-order interaction we can confound components of another interaction or main effect.

<u>Example 4.4</u>: In a 2 x 2 x 3 factorial arrangement A_1A_2 has 1 component. A plan confounding A_1A_2 is unique by Theorem 3.7. By Theorem 3.23 the plan is the extension of a plan confounding $A_1^*A_2^*$ in a 2 x 2 factorial arrangement. The plan is given in Figure 4.5 and an abbreviated analysis of variance appears in Table 4.5.

<u>Example 4.5</u> : Using Corollary 3.2 and Theorem 3.23 we see that there exists a total of 24 plans that confounds the mean effect

TABLE 4.4

ABBREVIATED ANALYSIS OF VARIANCE TABLE
FOR THE PLAN GIVEN IN FIGURE 4.4

Source	d.f.		S.S.
Total	12		$Y'Y$
Mean	1		$Y'K_{12}Y$
B.A.B.	2		$Y'(\tilde{X}\tilde{X}'-K_{12})Y$
A_3		1	$Y'(\tilde{X}\tilde{Y}_3\tilde{Y}_3'\tilde{X}')Y$
$A_1A_2A_3$		1	$Y'(\tilde{X}\tilde{Y}_2\tilde{Y}_2'\tilde{X}')Y$
W.A.B.	9		$Y'(I_{12}-\tilde{X}\tilde{X})Y$
A_1		1	$Y'(I_2-K_2 \otimes K_2 \otimes K_3)Y$
A_2		1	$Y'(K_2 \otimes I_2-K_2 \otimes K_3)Y$
A_3		1	$Y'((K_2 \otimes K_2 \otimes I_3-K_3) - \tilde{X}\tilde{Y}_3\tilde{Y}_3'\tilde{X}')Y$
A_1A_2		1	$Y'(I_2-K_2 \otimes I_2-K_2 \otimes K_3)Y$
A_1A_3		2	$Y'(I_2-K_2 \otimes K_2 \otimes I_3-K_3)Y$
A_2A_3		2	$Y'(K_2 \otimes I_2-K_2 \otimes I_3-K_3)Y$
$A_1A_2A_3$		1	$Y'((I_2-K_2 \otimes I_2-K_2 \otimes I_3-K_3) - \tilde{X}\tilde{Y}_2\tilde{Y}_2'\tilde{X}')Y$

0 0 0	0 1 0
0 0 1	0 1 1
0 0 2	0 1 2
1 1 0	1 0 0
1 1 1	1 0 1
1 1 2	1 0 2

Figure 4.5--The plan confounding the mean effect and A_1A_2 in a 2 x 2 x 3 factorial arrangement of treatments.

TABLE 4.5

AN ABBREVIATED ANALYSIS OF VARIANCE TABLE
FOR THE PLAN IN FIGURE 4.5

Source	d.f.	S.S.
Total	12	$Y'Y$
Mean	1	$Y'K_{12}Y$
B.A.B. (A_1A_2)	1	$Y'(\tilde{X}\tilde{X}'-K_{12})Y$
W.A.B.	0	$Y'(I_{12}-\tilde{X}\tilde{X}')Y$
A_1	1	$Y'(I_2-K_2 \otimes K_2 \otimes K_3)Y$
A_2	1	$Y'(K_2 \otimes I_2-K_2 \otimes K_3)Y$
A_3	2	$Y'(K_2 \otimes K_2 \otimes I_3-K_3)Y$
A_1A_3	2	$Y'(I_2-K_2 \otimes K_2 \otimes I_3-K_3)Y$
A_2A_3	2	$Y'(K_2 \otimes I_2-K_2 \otimes I_3-K_3)Y$
$A_1A_2A_3$	2	$Y'(I_2-K_2 \otimes I_2-K_2 \otimes I_3-K_3)Y$

and 3 components of A_1A_2 in any 4 x 4 x q factorial arrangement. An analysis of variance is straightforward. The W.A.B. analysis of A_1A_2 has 6 degrees of freedom and the matrix of its quadratic form is $I_4-K_4 \otimes I_4-K_4 - \tilde{X}\tilde{X}' + K_{16}$.

If B_1 confounds only the mean effect and components X_1C_1 of $A_1A_2 \ldots A_k$ and B_2 confounds only the mean effect and components X_2C_2 of $A_{k+1}A_{k+2} \ldots A_r$ then by Theorem 3.9 and Theorem 3.6 $B = B_1 \cap B_2$ is a plan with blocks of equal size and confounds the effects defined by $(J_N, X_1C_1, X_2C_2, (X_1 \otimes X_2)(C_1 \otimes C_2)$. Furthermore $(X_1 \otimes X_2)(C_1 \otimes C_2) = X_1C_1 \otimes X_2C_2$ defines components of $A_1A_2 \ldots A_r$ since there exist W_1 and W_2 such that $X_1C_1 = \tilde{F}^*(U_1, U_2, \ldots, U_k)W_1$ and $X_2C_2 = \tilde{F}^*(U_{k+1}, U_{k+2}, \ldots, U_r)W_2$ and $X_1C_1 \otimes X_2C_2 = [(\tilde{U}_1 \otimes \tilde{U}_2 \otimes \ldots \otimes \tilde{U}_k)W_1 \otimes$

$$J_q \otimes \tilde{J}_{\frac{N}{pq}}] \otimes$$

$$J_p \otimes (\tilde{U}_{k+1} \otimes \tilde{U}_{k+2} \otimes \ldots \otimes \tilde{U}_r)W_2 \otimes \tilde{J}_{\frac{N}{pq}} =$$

$$(\tilde{U}_1 \otimes \tilde{U}_2 \otimes \ldots \otimes \tilde{U}_r)(W_1 \otimes W_2) \otimes \tilde{J}_{\frac{N}{pq}} .$$

Example 4.6: In a 2 x 2 x 3 x 3 factorial arrangement the plans B_1 and B_2 in Figure 4.6 confound respectively the mean effect and A_1A_2 and the mean effect and 2 components of A_3A_4. $B = B_1 \cap B_2$ confounds the mean effect, A_1A_2, 2 components of A_3A_4 and 2 components of $A_1A_2A_3A_4$ with its 6 blocks of size 6. An abbreviated analysis of variance appears in Table 4.6

B_1:

0 0 0 0	0 1 0 0
0 0 0 1	0 1 C 1
0 0 0 2	0 1 0 2
0 0 1 0	0 1 1 0
0 0 1 1	0 1 1 1
0 0 1 2	0 1 1 2
0 0 2 0	0 1 2 0
0 0 2 1	0 1 2 1
0 0 2 2	0 1 2 2
1 1 0 0	1 0 0 0
1 1 0 1	1 0 0 0
1 1 0 2	1 0 0 2
1 1 1 0	1 0 1 0
1 1 1 1	1 0 1 1
1 1 1 2	1 0 1 2
1 1 2 0	1 0 2 0
1 1 2 1	1 0 2 1
1 1 2 2	1 0 2 2

B_2:

0 0 0 0	0 0 0 1	0 0 0 2
0 0 1 1	0 0 1 2	0 0 1 0
0 0 2 2	0 0 2 0	0 0 2 1
0 1 0 0	0 1 0 1	0 1 0 2
0 1 1 1	0 1 1 2	0 1 1 0
0 1 2 2	0 1 2 0	0 1 2 1
1 0 0 0	1 0 0 1	1 0 0 2
1 0 1 1	1 0 1 2	1 0 1 0
1 0 2 2	1 0 2 0	1 0 2 1
1 1 0 0	1 1 0 1	1 1 0 2
1 1 1 1	1 1 1 2	1 1 1 0
1 1 2 2	1 1 2 0	1 1 2 1

$B_1 \cap B_2$:

0 0 0 0	0 0 0 1	0 0 0 2	0 1 0 0	0 1 0 1	0 1 0 2
0 0 1 1	0 0 1 2	0 0 1 0	0 1 1 1	0 1 1 2	0 1 1 0
0 0 2 2	0 0 2 0	0 0 2 1	0 1 2 2	0 1 2 0	0 1 2 1
1 1 0 0	1 1 0 1	1 1 0 2	1 0 0 0	1 0 0 1	1 0 0 2
1 1 1 1	1 1 1 2	1 1 1 0	1 0 1 1	1 0 1 2	1 0 1 0
1 1 2 2	1 1 2 0	1 1 2 1	1 0 2 2	1 0 2 0	1 0 2 1

Figure 4.6--Three plans of a 2 x 2 x 3 x 3 factorial arrangement of treatments.

TABLE 4.6

AN ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR THE PLAN $B_1 \cap B_2$ OF A 2 x 2 x 3 x 3 FACTORIAL ARRANGEMENT OF TREATMENTS

Source	d.f.	S.S.
Total	36	$Y'Y$
Mean	1	$Y'K_{36}Y$
B.A.B.	5	$Y'(\tilde{X}\tilde{X}'-K_{36})Y$
A_1A_2	1	$Y'(\tilde{X}_1\tilde{X}_1'-K_{36})Y$
A_3A_4	2	$Y'(\tilde{X}_2\tilde{X}_2'-K_{36})Y$
$A_1A_2A_3A_4$	2	$Y'(\tilde{X}\tilde{X}'-\tilde{X}_1\tilde{X}_1'-\tilde{X}_2\tilde{X}_2' + K_{36})Y$
W.A.B.	30	
A_1	1	$Y'(I_2-K_2 \otimes K_2 \otimes K_3 \otimes K_3)Y$
A_2	1	$Y'(K_2 \otimes I_2-K_2 \otimes K_3 \otimes K_3)Y$
A_3	2	$Y'(K_2 \otimes K_2 \otimes I_3-K_3 \otimes K_3)Y$
A_4	2	$Y'(K_2 \otimes K_2 \otimes K_3 \otimes I_3-K_3)Y$
A_1A_3	2	$Y'(I_2-K_2 \otimes K_2 \otimes I_3-K_3 \otimes K_3)Y$
A_1A_4	2	$Y'(I_2-K_2 \otimes K_2 \otimes K_3 \otimes I_3-K_3)Y$
A_2A_3	2	$Y'(K_2 \otimes I_2-K_2 \otimes I_3-K_3 \otimes K_3)Y$
A_2A_4	2	$Y'(K_2 \otimes I_2-K_2 \otimes K_3 \otimes I_3-K_3)Y$
A_3A_4	2	$Y'((K_2 \otimes K_2 \otimes I_3-K_3 \otimes I_3-K_3)-\tilde{X}_2\tilde{X}_2' + K_{36})Y$
$A_1A_2A_3$	2	$Y'(I_2-K_2 \otimes I_2-K_2 \otimes I_3-K_3 \otimes K_3)Y$
$A_1A_2A_4$	2	$Y'(I_2-K_2 \otimes I_2-K_2 \otimes K_3 \otimes I_3-K_3)Y$
$A_1A_3A_4$	4	$Y'(I_2-K_2 \otimes K_2 \otimes I_3-K_3 \otimes I_3-K_3)Y$
$A_2A_3A_4$	4	$Y'(K_2 \otimes I_2-K_2 \otimes I_3-K_3 \otimes I_3-K_3)Y$
$A_1A_2A_3A_4$	2	$Y'((I_2-K_2 \otimes I_2-K_2 \otimes I_3-K_3 \otimes I_3-K_3 - \tilde{X}\tilde{X}' + \tilde{X}_1\tilde{X}_1' + \tilde{X}_2\tilde{X}_2' - K_{36})Y$

<u>Example 4.7</u>: In the 2 x 6 x 3 factorial arrangement B_1 and B_2 given in Figure 4.7 respectively confound the mean effect and 1 component of A_1A_2 and the mean effect and 2 components of A_2A_3. The component of A_1A_2 confounded is defined by

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes J_3 \right] \otimes J_3 \quad \text{and}$$

the two components of A_2A_3 are defined by

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \\ 0 & -2 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$$

By Theorem 3.6 the Hadamard product of these two sets of effects is confounded and defines a set of effects also confounded with $B = B_1 \cap B_2$. The Hadamard product is

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \\ 0 & -2 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix} \quad \text{and thus 2 components}$$

of $A_1A_2A_3$ are confounded. An abbreviated analysis of variance is given in Table 4.7.

B_1:

0 0 0	0 3 0
0 0 1	0 3 1
0 0 2	0 3 2
0 1 0	0 4 0
0 1 1	0 4 1
0 1 2	0 4 2
0 2 0	0 5 0
0 2 1	0 5 1
0 2 2	0 5 2
1 3 0	1 0 0
1 5 1	1 0 1
1 3 2	1 0 2
1 4 0	1 1 0
1 4 1	1 1 1
1 4 2	1 1 2
1 5 0	1 2 0
1 5 1	1 2 1
1 5 2	1 2 2

B_2:

0 0 0	0 0 1	0 0 2
0 1 1	0 1 2	0 1 0
0 2 2	0 2 0	0 2 1
0 3 0	0 3 1	0 3 2
0 4 1	0 4 2	0 4 0
0 5 2	0 5 0	0 5 1
1 0 0	1 0 1	1 0 2
1 1 1	1 1 2	1 1 0
1 2 2	1 2 0	1 2 1
1 3 0	1 3 1	1 3 2
1 4 1	1 4 2	1 4 0
1 5 2	1 5 0	1 5 1

$B_1 \cap B_2$:

0 0 0	0 0 1	0 0 2	0 3 0	0 3 1	0 3 2
0 1 1	0 1 2	0 1 0	0 4 1	0 4 2	0 4 0
0 2 2	0 2 0	0 2 1	0 5 2	0 5 0	0 5 1
1 3 0	1 3 1	1 3 2	1 0 0	1 0 1	1 0 2
1 4 1	1 4 2	1 4 0	1 1 1	1 1 2	1 1 0
1 5 2	1 5 0	1 5 1	1 2 2	1 2 0	1 2 1

Figure 4.7--Three plans in a 2 x 6 x 3 factorial arrangement of treatments.

TABLE 4.7

AN ABBREVIATED ANALYSIS OF VARIANCE TABLE OF A 2 x 6 x 3 FACTORIAL ARRANGEMENT OF TREATMENTS WITH THE PLAN $B_1 \cap B_2$ IN FIGURE 4.7

Source	d.f.	S.S.
Total	36	$Y'Y$
Mean	1	$Y'K_{36}Y$
B.A.B.	5	$Y'(\tilde{X}\tilde{X}'-K_{36})Y$
A_1A_2	1	$Y'(\tilde{X}_1\tilde{X}_1'-K_{36})Y$
A_2A_3	2	$Y'(\tilde{X}_2\tilde{X}_2'-K_{36})Y$
$A_1A_2A_3$	2	$Y'(\tilde{X}\tilde{X}'-\tilde{X}_1\tilde{X}_1'-\tilde{X}_2\tilde{X}_2'+K_{36})Y$
W.A.B.	30	$Y'(I_{36}-\tilde{X}\tilde{X}')Y$
A_1	1	$Y'(I_2-K_2 \otimes K_6 \otimes K_3)Y$
A_2	5	$Y'(K_2 \otimes I_6-K_6 \otimes K_3)Y$
A_3	2	$Y'(K_2 \otimes K_6 \otimes I_3-K_3)Y$
A_1A_2	4	$Y'((I_2-K_2 \otimes I_6-K_6 \otimes K_3)-\tilde{X}_1\tilde{X}_1' + K_{36})Y$
A_1A_3	2	$Y'(I_2-K_2 \otimes K_6 \otimes I_3-K_3)Y$
A_2A_3	8	$Y'((K_2 \otimes I_6-K_6 \otimes I_3-K_3) - \tilde{X}_2\tilde{X}_2' + K_{36})Y$
$A_1A_2A_3$	8	$Y'((I_2-K_2 \otimes I_6-K_6 \otimes I_3-K_3)-\tilde{X}\tilde{X}'+\tilde{X}_1\tilde{X}_1'+\tilde{X}_2\tilde{X}_2'+K_{36})Y$

CHAPTER V

A PRACTICAL EXAMPLE

This chapter is intended to illustrate how the development in CHAPTERS III and IV can be utilized to design a practical plan that is appropriate for data analysis. Emphasis is placed upon those aspects where the design or analysis was either impossible or more difficult before. In the past, designs have been artificially forced into patterns where all factors had the same number of levels and even these numbers were restricted.

Also, many times researchers have well designed experiments, but unforeseen events or a lack of facilities or time forced them to compromise their analyses. Such is the scope of the following experimental example and although the results developed earlier are not necessarily restricted to this type of shortcoming, it is felt that the chosen example will provide some of the reasons as to why the usual analysis is not appropriate and illustrate one type of situation where it is advantageous to use these results. It is hoped that the reader can easily imagine that these applications can be made to a wide variety of situations in which the response to be measured depends on the levels of several factors, some kind of blocking is advantageous, and large numbers of replicates are not feasible.

Suppose that a researcher has collected a sample of blood from each of 24 dogs in order to determine the effects of 24 diets comprising a 2 x 2 x 6 factorial arrangement of treatments upon the total blood lipids in dogs.

The three factors are carbohydrate, protein and fat. The two levels of carbohydrate are 5 and 10 grams per kilogram of body weight, the two levels of protein are 20 and 40 grams per kilogram of body weight and the six levels of fat are 8, 16, 24, 32, 40, 48 grams per kilogram of body weight. The treatments are the 24 combinations of carbohydrate-protein-fat and the response he wishes to measure is the number of micrograms of total blood lipids per milliliter of whole blood. The high and low levels of both carbohydrate and protein can be indicated by 0 and 1 and the six levels of fat can be indicated by 0, 1, 2, 3, 4, and 5. Thus a three tuple such as (0, 1, 4) represents the diet consisting of the low level of carbohydrate, the high level of protein and 40 g/Kg of fat.

Because of situations beyond his control, the researcher must utilize two different laboratories for the assays. He realizes that the use of different laboratories might introduce bias into responses because of different techniques or technicians.

In his investigation he would like to ascertain if the two levels of carbohydrate are different relative to the measured response. Also, he desires to know if the two levels of protein influence the measured responses and the six levels of fat influence the measured responses. In statistical terms these statements are equivalent to the evaluation of the three main effects.

Also of interest to the experimenter is whether or not the pattern of responses for one factor is different at each level of another factor when the remaining factor is collapsed. Thus the researcher wants to investigate the carbohydrate x protein, carbohydrate x fat, and the protein x fat interactions. The experimenter is not interested in the carbohydrate x protein x fat interaction.

The experimenter knows that any assignment of the blood samples to the two laboratories will invalidate any comparisons of assays from different laboratories. That is, he is unable to attribute differences in blood samples assayed in different laboratories to a difference in treatments because of the bias introduced by the difference in laboratories. He also feels that he should assign 12 blood samples to each of the laboratories.

In order to obtain all the information desired by the experimenter, we can construct a plan using the results of CHAPTER III. We can immediately discard any plan which confounds any main effect or first order interaction since the researcher desires information on these effects. Thus we desire to confound part of the highest-order interaction. A plan that confounds part of the carbohydrate x protein x fat interaction is easy to construct. Since each laboratory is to receive 12 samples then we must have blocks of size 12.

The allocation of the blood samples to the laboratories is dependent upon the interest of the experimenter. For example, since he wants information on the main effect of carbohydrate, it would be very undesirable to assign all samples at the low level of carbohydrate to one laboratory and the remaining samples at the high level of carbohydrate to the other laboratory. Such a practice would invalidate or bias the usual estimate of the carbohydrate main effect and corresponding sum of squares.

Thus we see that the individual levels of each of the factors must be balanced in each laboratory. For similar reasons all combinations of levels from any two must be balanced in order for the estimates and sums of squares of the 3 first-order interaction to exist. These conditions are easy to satisfy in the construction of an allocation plan.

Ten allocation plans exist which will give the experimenter the desired estimates. The following plan is one of the ten allocation plans that confounds only the highest-order interaction.

Lab 1	Lab 2
000	003
001	004
002	005
013	010
014	011
015	012
103	100
104	101
105	102
110	113
111	114
112	115

The sum of squares for all effects other than the carbohydrate x protein x fat interaction are computed in the usual manner and have the usual rules governing the degrees of freedom. The sum of squares for the highest-order interaction is computed by subtracting the sum of squares for main effects first-order interactions and laboratories from the total (corrected for the mean) sum of squares. This sum of square can be used as the residual sum of squares and has only (1)(1)(5)-1 = 4 degrees of freedom because one degree of freedom due to the laboratory sum of squares is subtracted from the usual 5 degrees of freedom for this interaction.

We should be aware of the implication of the last computation. All the main effects and first-order interactions are intra-laboratory or sums of intra-laboratory comparisons and thus does not involve differences in laboratories or inter-laboratory comparisons. This is why the inter-laboratory in the form of the laboratory sum of squares was removed from the usual sum of squares due to the carbohydrate x protein x fat interaction. Since the highest-order interaction is used many times to test the significance of the first-order interactions and possibly main effects, then by not removing the laboratory sum of squares from the usual sum of squares for the highest-order interaction, we would be testing intra-laboratory comparisons with a residual error consisting of both intra-laboratory and inter-laboratory comparisons. Thus the inter-laboratory comparison is eliminated from the highest-order interaction and correspondingly one degree of freedom is lost.

Failure to eliminate the inter-laboratory comparison would tend to inflate the residual sum of squares by the inclusion of the square of bias due to the different laboratories. The researcher can follow the allocation plan and still get the usual sums of squares of the effects of interest at a loss of one degree of freedom of the highest-order interaction. If bias due to the difference in laboratories really exists, then the loss of the degree of freedom is welcome since the inter-laboratory sum of squares is substantial.

The usual tests of significance can be made in the manner appropriate to the 2 x 2 x 6 factorial arrangement of treatments with the exception that the residual sum of squares now has only 4 degrees of freedom associated with it. That is, the highest-order interaction is used as the error term in testing each first-order interaction for its effect. If no significance is found the main effects are tested using the highest-order interaction as the error term or the pooling of the non-significant interaction terms to obtain a new measure of error. Since a large number of textbooks of both methods and experimental design cover the tests of hypotheses for these types of situations, a detailed discussion of the tests to be employed in this example would be redundant and therefore is not undertaken.

CHAPTER VI

SUMMARY

This dissertation provided a method of construction of a set of orthogonal effects in an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement of treatments and a partition of this set into the 2^m factorial effects. A canonical representation of the 2^m factorial effects was established by utilizing tensor products and the set of tensors defining an interaction effect was related to the Hadamard product of sets of vectors defining the main effects. The matrix of the quadratic form of a factorial effect was established and was seen to be invariant of the choice of the orthogonal set defining the factorial effect. This matrix was also the Kronecker product of idempotent matrices and therefore idempotent by a preliminary theorem.

The preceding developments provided a simple expression for the partition of the total sum of squares into the sums of squares due to the factorial effects. The ranks of the matrices of the quadratic forms were determined and were related to parameters of non-central chi-squared random variables. Analyses of variance were presented in general and for selected simple examples.

Definitions, methods of construction, and analyses of variance were given for the randomized complete block design and the completely randomized design with factorial arrangements of treatments.

This dissertation also gave an algebraic treatment of blocking and confounding of a single replicate of a factorial arrangement of treatments. The set of treatments of an $n_1 \times n_2 \times \ldots \times n_m$ factorial arrangement was represented by the Cartesian product (in the respective order) of the residue classes of the respective moduli. The incidence matrices of the blocks of a plan were defined and the Hadamard product was used to explicitly define confounding of an effect with a block of a plan.

The Hadamard product of tensors that define confounded effects was seen to reproduce tensors which also defined confounded effects. Linear combinations of confounded effects also are confounded. An effect confounded in each block of a plan was seen to be defined by some linear combination of the incidence matrices of the blocks of the plan. The number of mutually orthogonal effects confounded in every block of a plan is equal to the number of blocks of the plan. The class of orthogonal effects confoundable with a given plan was determined and a plan that confounds only a given set of effects was shown to be unique.

Necessary and sufficient conditions are established for confounding only the mean effect and components of the highest-order interaction with the blocks of a plan. This result leads to necessary and sufficient conditions for the existence of such a plan and is extended to apply to lower-order interactions or main effects.

The effects confounded in the intersection of two plans are related to the effects confounded in the separate plans. Necessary and sufficient conditions for estimability of an effect are given.

It was established that blocks must be of equal size if only the mean effect and components of an interaction effect are confounded.

Aside from the mean effect if the effects confounded by one plan are orthogonal to the effects confounded by a second plan and the blocks of both plans have common sizes then the intersection of the two plans yields a plan whose blocks are of equal size. The latter plan confounded the effects confounded by either plan and the generalized interaction of the set of effects of one plan with the set of effects of the other.

It was shown that the generalized interaction of components of $A_1A_2\ldots A_k$ with components of $A_rA_{r+1}\ldots A_s$ is a set of components of $A_1A_2\ldots A_kA_r\ldots A_s$ provided $k<r$. This result can be extended to the case where $k\geq r$ if judicious choices of the two sets of components are made. However the actual construction of the two plans is difficult and it is easier to use the methods that have resulted from the theories of Galois field, and projective geometries.

This dissertation has attempted to provide broad insight into the construction of factorial effects and the representation of the quadratic forms thereof in a factorial arrangement of treatments.

It is hoped that the results concerning blocking and confounding will lead to an understanding as to when and why confounding is a worthwhile procedure and will make for easier construction of blocking plans.

LIST OF REFERENCES

1. Bose, R. C. (1947). Mathematical theory of the symmetrical factorial design. Sankhya, 8:107-166.

2. Bose, R. C. and Kishen, K. (1940). On the problem of confounding in the general symmetrical factorial design. Sankhya, 5:21-36.

3. Cochran, Wm. G. and Cox, G. M. (1957). Experimental Designs. John Wiley and Sons, Inc., New York.

4. Das, M. N. (1964). A somewhat alternative approach for construction of symmetrical factorial designs and obtaining maximum number of factors. Calcutta Statist. Assoc. Bull. 13:1-17.

5. Federer, W. T. (1955). Experimental Design. Macmillan Company, New York.

6. Finney, D. J. (1945). The factorial replication of factorial arrangements. Ann. Eugen., London, 12:291-301.

7. Fisher, R. A. (1942). The theory of confounding in factorial experiments in relation to the theory of groups. Ann. Eugen., London, 11:341-353.

8. Fisher, R. A. (1945). A system of confounding for factors with more than two alternatives, giving completely orthogonal cubes and higher powers. Ann. Eugen., London, 12:283-290.

9. Graybill, F. A. (1961). An Introduction to Linear Statistical Models. McGraw-Hill Book Co., New York.

10. Halmos, P. R. (1958). Finite-Dimensional Vector Spaces. D. VanNorstrand Co., Inc., Princeton, N. J.

11. Jacobson, N. (1953). Lectures in Abstract Algebra Vol. II. D. VanNostrand Co., Inc., New York.

12. Kempthorne, O. (1947). A simple approach to confounding fractional replication in factorial experiments. Biometrika, 34:255-272.

13. Kempthorne, O. (1952). The Design and Analysis of Experiments. John Wiley and Sons, Inc., New York.

14. Kishen, K. and Srivastava, J. N. (1959a). Confounding in asymetrical factorial designs in relation to finite geometries. Current Sci., 28:98-100.

15. Mann, H. B. (1952). Design and Analysis of Experiments. The Dover Press.

16. Nair, K. R. (1938). On a method of getting confounded arrangements in the general symmetrical type of experiment. Sankhya, 4: 121-138.

17. Raktoe, B. L. (1969). Combining Elements from Distinct Prime Fields in Mixed Factorials. Ann. of Math. Statist., 40:498-504.

18. Rao, C. R. (1947). Factorial experiments derivable from combinatorial arrangements of arrays. J. Royal Stat. Soc., Supp., 9:128-139.

19. Sardana, M. G. (1965). Construction and analysis of $2q \times 2^2$ asymmetrical factorial designs in two replications. J. Indian Soc. Agric. Statist., 17:111-115.

20. Sarma, T. C. S. R. (1967). On construction of symmetrical fractional factorial designs. J. Indian Soc. Agric. Statist., 19:83-91.

21. White, D. and Hultquist, R. A. (1965). Construction of confounding plans for mixed factorial designs. Ann. Math. Stat., 36: 1256-1271.

22. Yates, F. (1937). The design and analysis of factorial experiments. Tech. Commun. Imp. Bur. Soil Sci., No. 35.

APPENDIXES

APPENDIX I

$B \otimes C = \emptyset$ if and only if $B = \emptyset$ or $C = \emptyset$.

<u>Proof</u>: Necessity follows from definition.

If $B \otimes C = \emptyset$ then $b_{ij}C = \emptyset$ for each b_{ij} in B.

Thus $B = \emptyset$ or $C = \emptyset$.

Proceeding inductively, if $B_1 \otimes \ldots \otimes B_{m-1} = \emptyset$ implies $B_i = \emptyset$ for some i then $B_1 \otimes \ldots \otimes B_m = \emptyset$ implies either $B_1 \otimes \ldots \otimes B_{m-1} = \emptyset$ or $B_m = \emptyset$. Thus $B_i = \emptyset$ for some i.

APPENDIX II

If $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_m$ are idempotent and non-zero then $F^*(C_1, \ldots, C_m)$ is idempotent if and only if C_i is idempotent.

<u>Proof</u>: Necessity follows by Theorem 2.4. If $F^*(C_1, \ldots, C_m)$ is idempotent then

$[F^*(C_1,\ldots,C_m)]^2 = F^*(C_1^2,\ldots,C_{i-1}^2, C_i^2, C_{i+1}^2, \ldots, C_m^2) =$

$F^*(C_1,\ldots,C_{i-1}, C_i^2, C_{i+1},\ldots,C_m) = F^*(C_1, \ldots, C_m)$. Thus

$F^*(C_1, \ldots, C_{i-1}, C_i-C_i^2, C_{i+1}, \ldots, C_m) = \emptyset$ and by Theorem 2.5 $C_i-C_i^2 = \emptyset$. Thus C_i is idempotent.

APPENDIX III

If X is an n-dimensional vector, y is an r-dimensional vector and A is an nxr matrix, then $(X \ominus A)Y = X \ominus AY$.

<u>Proof</u>: $(X \ominus A)Y = (X \ominus \alpha_1, X \ominus \alpha_2, \ldots X \ominus \alpha_r)Y =$

$(X \ominus \alpha_1)y_1 + (X \ominus \alpha_2)y_2 + \ldots + (X \ominus \alpha_r)y_r =$

$X \ominus (\alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_r y_r) = X \ominus AY$ where

$$A = (\alpha_1, \alpha_2, \ldots, \alpha_r) \text{ and } Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{pmatrix}$$

APPENDIX IV

If A, B, C, D are matrices of dimension nxr, mxs, nxq, mxp respectively then there exists a permutation matrix P such that $(A \otimes B) \odot (C \otimes D) = [(A \odot C) \otimes (B \odot D)]P$.

<u>Proof:</u> $(A \otimes B) \odot (C \otimes D) = (\alpha_1 \odot \beta_1, \ldots, \alpha_1 \odot \beta_s, \alpha_2 \odot \beta_1, \ldots, \alpha_2 \odot \beta_s, \ldots, \alpha_r \odot \beta_1, \ldots, \alpha_r \odot \beta_s) \odot (\gamma_1 \odot \delta_1, \ldots, \gamma_1 \odot \delta_p, \gamma_2 \odot \delta_1, \ldots, \gamma_2 \odot \delta_p, \ldots, \gamma_q \odot \delta_p$ and

$(A \odot C) \otimes (B \odot D) = (\alpha_1 \odot \gamma_1, \ldots, \alpha_1 \odot \gamma_q, \alpha_2 \odot \gamma_1, \ldots, \alpha_2 \odot \gamma_q, \ldots, \alpha_r \odot \gamma_1, \ldots, \alpha_r \odot \gamma_q) \otimes (\beta_1 \odot \delta_1, \ldots, \beta_1 \odot \delta_p, \beta_2 \odot \delta_1, \ldots, \beta_2 \odot \delta_p, \ldots, \beta_s \odot \delta_p)$ where α_i, β_j, γ_k, δ_ℓ are the i^{th}, j^{th}, k^{th}, ℓ^{th} column of A, B, C, D respectively. Since both matrices are of dimension nm x rsqp and

$$(\alpha_i \otimes \beta_j) \odot (\gamma_k \otimes \delta_\ell) = \begin{pmatrix} a_{1i}\beta_j \\ a_{2i}\beta_j \\ \vdots \\ a_{mi}\beta_j \end{pmatrix} \begin{pmatrix} c_{1k}\delta_\ell \\ c_{2k}\delta_\ell \\ \vdots \\ c_{mk}\delta_\ell \end{pmatrix} \begin{pmatrix} a_{1i}\beta_j \odot c_{1k}\delta_\ell \\ a_{2i}\beta_j \odot c_{2k}\delta_\ell \\ \vdots \\ a_{mi}\beta_j \odot c_{mk}\delta_\ell \end{pmatrix} =$$

$$\begin{pmatrix} a_{1i}c_{1k}\beta_j \odot \delta_\ell \\ a_{2i}c_{2k}\beta_j \odot \delta_\ell \\ \vdots \\ a_{mi}c_{mk}\beta_j \odot \delta_\ell \end{pmatrix} = (\alpha_i \odot \gamma_k) \otimes (\beta_j \odot \delta_\ell)$$

then there exists a permutation matrix P such that

$$(A \otimes B) \oplus (C \otimes D) = [(A \oplus C) \otimes (B \oplus D)]P.$$

Proceeding inductively, if there exists a P_{m-1} such that

$$(B_1 \otimes B_2 \otimes \ldots \otimes B_{m-1}) \oplus (C_1 \otimes C_2 \otimes \ldots \otimes C_{m-1}) = [(B_1 \oplus C_1) \otimes (B_2 \oplus C_2) \otimes \ldots \otimes (B_{m-1} \oplus C_{m-1})]P_{m-1}$$ then

$$(B_1 \otimes \ldots \otimes B_{m-1} \otimes B_m) \oplus (C_1 \otimes \ldots \otimes C_{m-1} \otimes C_m) = [[(B_1 \otimes \ldots \otimes B_{m-1}) \oplus (C_1 \otimes \ldots \otimes C_{m-1})] \otimes (B_m \oplus C_m)]P_2 = [[(B_1 \oplus C_1) \otimes \ldots \otimes (B_{m-1} \oplus C_{m-1})] P_{m-1} \otimes (B_m \oplus C_m) I]P_2 = [(B_1 \oplus C_1) \otimes \ldots \otimes (B_m \oplus C_m)](P_{m-1} \otimes I)P_2 .$$

$(P_{m-1} \otimes I)P_2$ is a permutation matrix and thus there exists a permutation matrix P_m such that

$$F^*(B_1, \ldots, B_m) \oplus F^*(C_1, \ldots, C_m) = F^*(B_1 \oplus C_1, \ldots, B_m \oplus C_m)P_m.$$