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Scope and Method of Study: This study is composed of a short familiarization with ordinary Taylor series followed by a detailed development of a generalized Taylor's expansion about two points instead of the usual one. It is followed by the development of an interpolation correction formula arising as a result of the generalized expansion. The latter part of this study is devoted to the development of various approximation formulas for well known functions such as e^x, arc tan x, and ln(1+x). The results of these approximations are compared with results obtained from ordinary Taylor series and the well known Hasting's approximations for these same functions. Last of all, an approximation formula for modified Bessel functions of the second kind is developed and the results verified and tabulated by means of the IBM 650 computor.

Findings and Conclusions: The generalized Taylor's expansion with which we worked was found to converge much faster than the ordinary Taylor series for all functions expanded. The approximation formulas for the functions mentioned above, compared favorably with the Hasting's approximations over a limited range depending upon the points about which the function was expanded. The number of calculations necessary to evaluate a particular function was usually, however, more, in the approximation derived by the generalized expansion. The approximation formula derived for the previously mentioned Bessel functions proved to be accurate in general to seven significant figures over the ranges investigated.

The author would like to state in conclusion that only a few of the possible applications of this generalization have been investigated here. Besides the many other functions which might be approximated by this method, almost any application of ordinary Taylors series would bear investigation with respect to this generalization.

ADVISER'S APPROVAL O. H. Hamilton

A GENERALIZATION OF TAYLOR SERIES

By

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PREFACE

The ordinary Taylor series expansion has been in use for many years as a useful tool in applied mathematics. Among other things it has often been used as a means for approximating various functions for the purpose of numerical calculations. Oftentimes, however, a great number of terms need to be used in order that the desired degree of accuracy be obtained. It is this shortcoming of the ordinary Taylor series that the generalized Taylor series developed in this report will alleviate to a certain degree. The following study will demonstrate some of the possibilities of this generalized expansion with respect to the number of terms required to obtain a particular degree of accuracy.

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CHAPTER I

The idea of approximating a function with an n^{th} degree polynomial, which for a sufficiently large n represents the function to the degree of accuracy required for a particular application is usually first encountered in elementary calculus. In order that we might have a firm foundation for our later generalizations, we will exhibit a method for approximating a given function with a polynomial. This particular polynomial representation for a function is known as a Taylor series and was named after Brook Taylor, 1685-1731, an Englishman [1]. The ordinary Taylor series with remainder will not be derived rigorously in this paper.

There are several ways in which Taylor's formula may be proven, and consequently several forms which it may take. These differ principally in the representation of the remainder term. We shall examine only one form at this time, that being the form with the so-called Lagrangian form of the remainder. In order that this representation of a particular function be valid we require the following be true about the function in question.

The function $f(x)$ and its first $n-1$ derivatives are continuous in the closed interval $[\alpha, \beta]$ and its n^{th} derivative exists at every interior point of $[\alpha, \beta]$. These requirements are less restrictive on the function than usual and would limit our methods of proof were we to undertake a rigorous proof at this time [2]. Under the above requirements we obtain the following expansion known as the Taylor series expansion for a function f(x).

$$(1.1)\quad f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(\xi)$$

where $a < \xi < x$

It can be seen that all coefficients of the powers of x-a except the last one are constants. The coefficient of $(x-a)^n$ is a function of ξ, which in turn depends upon the magnitude of x [2]. It often happens that for a sufficiently large n, this last term may be negligibly small in comparison with the preceding ones and the function f(x) can be approximately represented by a polynomial of degree n-1 in x-a with constant coefficients.

The above equation is known as Taylor's formula and the last term,

$$R_n = \frac{(x-a)^n}{n!} f^{(n)}(\xi),$$

is called the Lagrangian form of the remainder after n terms. The infinite series resulting from this formula is known as Taylor's series. The above form of the remainder is only one of several forms in which it may be expressed. We shall however be primarily interested in this form for the purposes of this study.

Although as mentioned earlier the Englishman, Brook Taylor, is responsible for the infinite series known as Taylor's series which he first published in 1715, the previously derived form was first presented by the Frenchman, J. L. Lagrange, in 1797 [3].

A special case of the general Taylor series where a is taken to be 0 would take the following form.

$$(1.2) \qquad f(x) = f(a) + x f'(a) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{x^n}{n!} f^{(n)}(\xi)$$

This series expansion is commonly referred to as Maclaurin's series. We shall however, in any future reference to this series, call it simply a Taylor series expansion about zero.

Since our principal interest in this particular study of Taylor series is our ability to approximate some $f(x)$ in a neighborhood of x by the aforementioned series, it would be well that we mention some necessary properties of our remainder term.

In most applications of Taylor series it is important that for a particular value of n the approximate value of the remainder term be known. This will enable one to ascertain the size of the error occurring when a finite number of terms of a Taylor series expansion is taken to approximate the given function. In general the Lagrangian form of the remainder more readily lends itself to an approximation of this type, although other forms of the remainder may prove useful in particular applications.

Consider the remainder term in the previously exhibited form

$$R_n = f^{(n)}(\xi) \frac{(x-a)^n}{n!}$$

where $a < \xi < x$.

Since in general the exact value of ξ will not be known, neither will the exact value of $f^{(n)}(\xi)$ be known. This forces us then to estimate the value of $f^{(n)}(\xi)$. If we can establish bounds for $f^{(n)}(\xi)$ such that

$m - |f^{(n)}(\xi)| - M$ then we know that the following is true [2].

$$m \frac{(x-\alpha)^n}{n!} \leq |R_n| \leq M \frac{(x-\alpha)^n}{n!}$$

This further implies

$$|R_n| \leq M$$

where in general M depends on both x and n. If by further examination of R_n we see that R_n tends to zero uniformly as n increases we may conclude that the Taylor series expansion converges to the function f(x) in the region (α, x).

It would be well to state at this time in more precise language what exactly is required of the remainder term in order that a given series expansion is convergent. For this purpose we state the following necessary condition for convergence.

For every x in (α, β) and any $\epsilon > 0$, there exists a positive integer N, independent of the choice of ξ in (α, x) such that $|R_n| < \epsilon$ for all $n \geq N$.

As a final item in our preliminary study of Taylor series we will expand and examine some particular functions as follows. Consider first the function y = sin x about the point zero. Now,

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ &\cdots & &\cdots \\ f^{(n)}(x) &= \sin(x+\frac{n\pi}{2}) & f^{(n)}(0) &= \sin\frac{n\pi}{2} \end{aligned}$$

Using the general form previously derived for a Taylor series expansion about zero, we have

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{x^n}{n!} \sin(\xi + \frac{n\pi}{2})$$

Since $0 < \xi < x$ it follows that $\xi = \theta x$ where $0 < \theta < 1$. The remainder term is therefore

$$R_n = \frac{x^n}{n!} \sin(\theta x + \frac{n\pi}{2})$$

Since $|\sin x| \leq 1$ this implies $|R_n| \leq |\frac{x^n}{n!}|$ for all values of x. Examine the series, f(x) = sin x for convergence by means of the ratio test.

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^{n+2}}{(-1)^{n+1}} \quad \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right|$$

Since $\lim_{n \to \infty} \frac{x}{n+1} = 0 < 1$ for all values of x then the series converges. Furthermore, since $|R_n| \leq |\frac{x^n}{n!}|$ for all values of x. $R_n \to 0$ as n increases without bound. This would be true in this case since $|\frac{x^n}{n!}| = 0$ as n increases without bound since it is a necessary condition for the convergence of a series.

Hence the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$$

is $-\infty < x < \infty$ and we therefore conclude

$$\lim_{n \to \infty} |\frac{x^n}{n!}| = 0$$

for all finite x. We see then that the remainder term approaches zero for infinitely large n, which as mentioned before is a sufficient condition for our Taylor series to approximate the function in our region of convergence.

In order that we might illustrate the use of the remainder term in estimating the magnitude of the error involved in a particular calculation we will look at the following approximation based on the previous example.

Suppose that we wish to compute the numerical value of sin 10°. Since sin $10^{\circ} = \pi/18$ radians we examine the general form of the remainder as given previously

$$R_n = \frac{x^n}{n!} \sin(\theta x + n\pi/2)$$

where $0 < \theta < 1$.

Working with sin 10° we obtain

$$|R_n| = \left|\frac{1}{n!}\left(\frac{\pi}{18}\right) \sin\left(\theta\frac{\pi}{18} + \frac{n\pi}{2}\right)\right| < \frac{1}{n!}\left(\frac{\pi}{18}\right)^n$$

Suppose further that we take $n = 9$, then

$$\sin\frac{\pi}{18} = \frac{\pi}{18} - \left(\frac{\pi}{18}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{18}\right)^5 \frac{1}{5!} - \left(\frac{\pi}{18}\right)^7 \frac{1}{7!}$$

would have an error which is less than

$$\frac{1}{9!}\left(\frac{\pi}{18}\right)^9$$

As the final example in the preliminary study we will expand the following function about some point other than zero. We take the following,

$$f(x) = \frac{1}{2+x}$$

around -1 and consider the remainder when $n=3$.

Taking derivatives,

$$f(x) = \frac{1}{2+x} \qquad f(-1) = 1$$

$$f'(x) = \frac{-1}{(2+x)^2} \qquad f'(-1) = -1$$

$$f''(x) = \frac{2}{(2+x)^3} \qquad f''(-1) = 2$$

$$f'''(x) = \frac{-6}{(2+x)^4} \qquad f'''(-1) = -6$$

Using now the general form of a Taylor series expansion about a point other than zero, we have,

$$f(x) = \frac{1}{2+x} = 1-(x+1) + (x+1)^2 + R_3$$

where

$$R_3 = \frac{-6}{(2+\xi)^4} \frac{(x+1)^3}{3!} = - \frac{(x+1)^3}{(2+\xi)^4}$$

and $-1 < \xi < x$.

In order to estimate R_3 we observe $\frac{1}{2+\xi}$ lies between $\frac{1}{2+x}$ and 1, and further that $|R_3|$ lies between $\frac{|x+1|^3}{(2+x)^4}$ and $|x+1|^3$.

Taking for example $x = -0.9$, R_3 would be negative and $|R_3|$ would lie between $\frac{0.001}{(1.1)^4}$ and 0.001. This would then give us an estimate of the error involved, thus enabling us to decide whether or not this approximation was suitable for whatever purpose we had in mind.

The following chapter will introduce a generalized form of Taylor's series in one variable which will be somewhat different from the one studied here. The major part of the next chapter will be devoted to obtaining this generalization and some related properties.

CHAPTER II

From the preceding chapter we have seen that the standard Taylor series expansion is a powerful tool for the purpose of approximating the values of various functions in the region of convergence of their series expansions. Were one to attempt to apply this method of approximations to a large number of functions, it would become apparent that in many cases requiring a high degree of accuracy a large number of terms would have to be considered. It can further be seen, that if for the same degree of accuracy a fewer number of terms of a series expansion for the function in question could be considered, the work involved in approximating the function would be simplified. In the following pages we shall develop a series expansion of any function satisfying the conditions for a Taylor series expansion which can usually be made to converge about twice as rapidly as the corresponding Taylor series expansion [4]. We proceeded in the following manner.

Assume the following conditions on the function $f(x)$.

1. $f(x)$ is continuous in an interval which contains a and x.
2. All derivatives of $f(x)$ up to and including the one of order $m+n+1$, where m and n are positive integers or zero, are continuous in the previously mentioned interval.

Throughout this derivation the greek letter θ will be used to denote a value between a and x. Also we will denote the binomial coefficients by $\binom{p}{q}$ with the understanding that $\binom{p}{q} = 0$ if p is less than q.

Theorem 2.1: Under the conditions just stated

$$(2.1)\quad f(x) = f(a) + \sum_{k=1} \frac{(m+n-k)!}{(m+n)!} \left[\binom{m}{k} f^{(k)}(a) - (-1)^k \binom{n}{k} f^{(k)}(x) \right] (x-a)^k + R$$

where

$$R = (-1)^n \frac{m!\,n!\,(x-a)^{m+n+1}}{(m+n)!\,(m+n+1)!} f^{(m+n+1)}(\theta)$$

Proof: Let

$$(2.2)\qquad F(x) = \int_a^x f^{(m+n+1)}(t)\, g(t)\, dt$$

where $g(t) = (x-t)^m (t-a)^n$ and $a \leq t \leq x$.

By a well known formula for repeated integrations by parts

$$F(x) = \sum_{i=o}^{n+m} (-1)^i f^{(m+n-i)}(t)\, g^{(i)}(t) \Big]_a^x$$

We show by induction the following

$$(2.3)\qquad g^{(i)}(t) = \sum_{k=o}^{i} (-1)^{i-k}\, i!\, \binom{m}{i-k} \binom{n}{k} (x-t)^{m+k-i} (t-a)^{n-k}$$

where i represents the order of the derivative to be taken.

Taking $i = 1$

$$g^{(1)}(t) = -m(x-t)^{m-1} (t-a)^n + n(x-t)^m (t-a)^{n-1}$$

Using formal differentiation

$$g^{(1)}(t) = -m(x-t)^{m-1} (t-a)^n + n(x-t)^m (t-a)^{n-1}$$

The proposition is therefore true for $i = 1$.

Assume true for $i = h$, then

$$g^{(h)}(t) = \sum_{k=0}^{h} (-1)^{h-k}\, h!\binom{m}{h-k}\binom{n}{k} (x-t)^{m+k-h}(t-\alpha)^{n-k}$$

$$= (-1)^h h!\binom{m}{h}\binom{n}{0}(x-t)^{m-h}(t-\alpha)^n + (-1)^{h-1} h!\binom{m}{h-1}\binom{n}{1}(x-t)^{m+1-h}(t-\alpha)^{n-1}$$

$$+ \dots + (-1)^0 h!\binom{m}{0}\binom{n}{h}(x-t)^m(t-\alpha)^{n-h}$$

Consider $g^{(h+1)}(t)$ found by formally differentiating $g^{(h)}(t)$ with respect to t. We then have

$$g^{(h+1)}(t) = (-1)^h h!\binom{m}{h}\binom{n}{0}\left[n(x-t)^{m-h}(t-\alpha)^{n-1} - (m-h)(x-t)^{m-(h+1)}(t-\alpha)^n\right]$$

$$+ (-1)^{h-1} h!\binom{m}{h-1}\binom{n}{1}\left[(n-1)(x-t)^{m+1-h}(t-\alpha)^{n-2} - (m+1-h)(x-t)^{m-h}(t-\alpha)^{n-1}\right]$$

$$+ \dots + (-1)^0 h!\binom{m}{0}\binom{n}{h}\left[(n-h)(x-t)^m(t-\alpha)^{n-(h+1)} - m(x-t)^{m-1}(t-\alpha)^{n-h}\right]$$

$$= (-1)^{h+1}(h+1)!\ \frac{m!}{(h+1)!(m-h-1)!}\ (x-t)^{m-h-1}(t-\alpha)^n$$

$$+ (-1)^h h!\ \frac{m!}{h!(m-h)!} \cdot \frac{n!}{n!0!}\ n(x-t)^{m-h}(t-\alpha)^{n-1}$$

$$+ (-1)^h h!\ \frac{m!}{(h-1)!(m-h+1)!} \cdot \frac{n!}{(n-1)!1!}\ (m+1-h)(x-t)^{m-h}(t-\alpha)^{n-1}$$

$$+ \dots + (-1)^0 h!\ \frac{m!}{m!0!} \cdot \frac{n!}{h!(n-h)!}\ (n-h)(x-t)^m(t-\alpha)^{n-(h+1)}$$

Consider the second and third term of the above expansion as follows.

$$\left[(-1)^h h!\ \frac{m!}{h!(m-h)!} \cdot \frac{n!}{n!0!}\, n\right.$$

$$\left. + (-1)^h h!\ \frac{m!}{(h-1)!(m-h+1)!} \cdot \frac{n!}{(n-1)!1!}\,(m-h+1)\right]\left[(x-t)^{m-h}(t-\alpha)^{n-1}\right]$$

$$= (-1)^h h!m!n!\ \left(\frac{1}{h!(m-h)!(n-1)!} + \frac{1}{(h-1)!(m-h)!(n-1)!}\right)\ \left[(x-t)^{m-h}(t-\alpha)^{n-1}\right]$$

$$= \frac{(-1)^h h!m!n!}{(h-1)!(m-h)!(n-1)!}\ \left(\frac{1}{h} + 1\right)\ \left[(x-t)^{m-h}(t-\alpha)^{n-1}\right]$$

$$= \frac{(-1)^h (h+1)!m!n!}{h!(m-h)!(n-1)!}\ \left[(x-t)^{m-h}(t-\alpha)^{n-1}\right]$$

It can be seen at this point that the first term of our series expansion for $g^{(h+1)}(t)$ satisfies the general formula for $g^{(i)}(t)$ with $i=h+1$ if we take $k=0$. It can further be seen that the above combination of the second and third terms satisfy the general formula if $k=1$. By combining each succeeding pair of terms in a similar manner we can continue to satisfy our general formula for each successive value of k up to $k=h$. Due to the nature of the way in which we are combining successive terms, our last term must be considered alone and will have to satisfy the general formula for $k=h+1$ if our general formula is going to be proven. We precede to examine this last term in the following manner.

Consider the last term of our expansion for $g^{(h+1)}(t)$ obtained by differentiating $g^{(h)}(t)$. This term is as follows,

$$(-1)^0 h! \frac{m!}{m!0!} \frac{n!}{h!(n-h)!} (n-h)(x-t)^m (t-\alpha)^{n-h-1}$$

Rewriting the above we have

$$(-1)^0 \frac{h!m!n!(n-h)}{m!0!h!(n-h)!} (x-t)^m (t-\alpha)^{n-h-1}$$

Multiplying by $\frac{h+1}{h+1}$ and simplifying we have,

$$(-1)^0 \frac{(h+1)!m!n!}{m!0!(h+1)!(n-h-1)!} (x-t)^m (t-\alpha)^{n-h-1}$$

This satisfies our general formula when $k=h+1$ and our formula is proven.

Consider now, $g^{(i)}(\alpha)$ where $i < n$.

From the general formula for $g^{(i)}(t)$ we see that $(t-\alpha)^{n-k}$ would appear in each term and where $t=\alpha$ then $(t-\alpha)^{n-k} = 0$, and each term would vanish. Therefore $g^{(i)}(\alpha) = 0$ for $i < n$. Further consider $g^{(i)}(\alpha)$ where $i \geq n$.

Here we see that where $k = n$, $(t-a)^{n-k}$ would become 1 and for $k \quad n$, $\binom{n}{k} = 0$ and all succeeding terms would vanish. Therefore

$$(2.4) \qquad g^{(i)}(a) = (-1)^{i-n}\, i!\binom{m}{i-n}(x-a)^{m+n-i} \quad \text{for } i \geq n$$

Secondly consider $g^{(i)}(x)$ where $i \quad m$. Under this condition the factor $(x-t)^{m+k-i}$ would appear in each term and be equal to zero, therefore

$$g^{(i)}(x) = 0 \qquad \text{where } i < m$$

On the other hand, if $i - m$, the term $(x-t)^{m+k-i}$ would equal 1 where $k = i-m$ and this particular term would be, $(-1)^m i!\binom{n}{i-m}(x-a)^{m+n-i}$. Examining all terms where $k \quad i-m$, we see that the factor $(x-t)^{m+k-i}$ would appear in each term and therefore cause each of these terms to vanish when evaluated at $t = x$. Last of all we examine the terms where $k < i - m$ and see that $\binom{m}{i-k} = 0$ since $m < i - k$, thus causing all of these terms to vanish. We then conclude

$$(2.5) \qquad g^{(i)}(x) = (-1)^m i!\binom{n}{i-m}(x-a)^{m+n-i}$$

Using Equations (2.4) and (2.5) in (2.6) below,

$$(2.6) \qquad F(x) = \sum_{i=o}^{n+m} (-1)^i f^{(m+n-i)}(t)\, g^{(i)}(t) \Big]_a^x$$

we obtain the following difference.

$$F(x) = \sum_{i=o}^{n+m} (-1)^i f^{(m+n-i)}(x)(-1)^m i!\binom{n}{i-m}(x-a)^{m+n-i}$$

$$- \sum_{i=o}^{n+m} (-1)^i f^{(m+n-i)}(a)(-1)^{i-n}\, i!\binom{m}{i-n}(x-a)^{m+n-i}$$

Set m+n-i = k then,

$$F(x) = (-1)^{2i-n} \sum_{k=0} \left[(-1)^k f^{(k)}(x) i! \binom{n}{k} - f^{(k)}(a) i! \binom{m}{k}\right] (x-a)^k$$

Further simplifying, we have the following equation

$$(2.7) \quad (-1)^n F(x) = \sum_{k=0} (m+n-k)! \left[(-1)^k \binom{n}{k} f^{(k)}(x) - \binom{m}{k} f^{(k)}(a)\right] (x-a)^k$$

Once again consider Equation (2.2)

$$F(x) = \int_a^x f^{(m+n+1)}(t) g(t) dt$$

Examining g(t) we find that it is of constant sign through the interval of integration. This allows us to apply the mean value theorem for integrals which may be quoted as follows [2].

Let f(x) and g(x) be two functions which are continuous in the interval (α, β) and suppose that g(x) does not change sign in the interval. Then there exists at least one value θ, $\alpha \leq \theta \leq \beta$, such that

$$\int_\alpha^\beta f(x) g(x) dx = f(\theta) \int_\alpha^\beta g(x) dx$$

Applying this theorem directly to the function, we have,

$$F(x) = f^{(m+n+1)}(\theta) \int_a^x g(t) dt \qquad \text{where } a \leq \theta \leq x$$

To evaluate this integral we set,

$$t = (x-a)u + a \qquad dt = (x-a)du$$

therefore since $g(t) = (x-t)^m (t-a)^n$ we have

$$g(t) dt = \left[x - (x-a)u - a\right]^m \left[(x-a)u\right]^n (x-a) du$$

$$= \left[x(1-u)-a(1-u)\right]^m \left[u(x-a)\right]^n (x-a)\,du$$

$$= (x-a)^m (1-u)^m (x-a)^n u^n (x-a)\,du$$

$$= (x-a)^{m+n+1} (1-u)^m u^n\,du$$

We then determine the new limits as follows. Since $t=(x-a)u+a$ let $t=a$ then $a=(x-a)u+a$, this implies $0=(x-a)u$ which implies $u=0$. Now let $t=x$ then $x=(x-a)u+a$, this implies $x-a=(x-a)u$ which implies $u=1$. Therefore

$$F(x) = f^{(m+n+1)}(\theta)(x-a)^{m+n+1}\int_0^1 (1-u)^m u^n\,du$$

The integral on the right is a beta function and therefore,

$$\int_0^1 (1-u)^m u^n\,du = \frac{m!\,n!}{(m+n+1)!}$$

Then,

$$F(x) = \frac{m!\,n!\,(x-a)^{m+n+1}}{(m+n+1)!} f^{(m+n+1)}(\theta) \tag{2.8}$$

Considering Equations (2.7) and (2.8) we have

$$(-1)^n F(x) = \sum_{k=0} (m+n-k)!\left[(-1)^k \binom{n}{k} f^{(k)}(x) - \binom{m}{k} f^{(k)}(a)\right](x-a)^k$$

$$F(x) = \frac{m!\,n!\,(x-a)^{m+n+1}}{(m+n+1)} f^{(m+n+1)}(\theta).$$

Multiplying Equation (2.8) through by $(-1)^n$ we obtain

$$(-1)^n F(x) = (-1)^n \frac{m!\,n!\,(x-a)^{m+n+1}}{(m+n+1)!} f^{(m+n+1)}(\theta). \tag{2.9}$$

Subtracting (2.9) from (2.7) we further have

$$0 = \sum_{k=0} (m+n-k)!\left[(-1)^k \binom{n}{k} f^{(k)}(x) - \binom{m}{k} f^{k}(a)\right](x-a)^k - (-1)^n \frac{m!\,n!\,(x-a)^{m+n+1}}{(m+n+1)!} f^{(m+n+1)}(\theta) \tag{2.10}$$

Simplifying, we obtain the following equation

$$(2.11) \qquad 0 = \sum_{k=0} \frac{(m+n-k)!}{(m+n)!} \left[\binom{m}{k} f^{(k)}(a) - (-1)^k \binom{n}{k} f^{(k)}(x) \right] (x-a)^k$$

$$+ (-1)^n \frac{m!n!(x-a)^{m+n+1}}{(m+n+1)!} f^{(m+n+1)}(\theta).$$

Consider the term where $k=0$.

$$\frac{(m+n)!}{(m+n)!} \left[f(a) - f(x) \right] = f(a) - f(x)$$

we then have

$$f(x) = f(a) + \sum_{k=1} \frac{(m+n-k)!}{(m+n)!} \left[\binom{m}{k} f^{(k)}(a) - (-1)^k \binom{n}{k} f^{(k)}(x) \right] (x-a)^k + R$$

where

$$R = (-1)^n \cdot \frac{m!n!(x-a)^{m+n+1}}{(m+n)!(m+n+1)!} f^{(m+n+1)}(\theta) \quad \text{where } a < \theta < x.$$

The theorem is therefore proven.

It can readily be seen at this point, that by different choices of m and n, a wide variety of expansions are possible. Take notice here, that if we let $n = 0$ in our general expansion we have

$$(2.12) \qquad f(x) = f(a) + \sum_{k=1} \frac{(m-k)!}{m!} \left[\binom{m}{k} f^{(k)}(a) \right] (x-a)^k + R$$

where

$$R = \frac{m!(x-a)^{m+1}}{m!(m+1)!} f^{(m+1)}(\theta)$$

The above expansion is then recognized to be nothing more than the familiar Taylor finite expansion with remainder exhibited in the previous chapter.

An unusually interesting case and the one in which we are most interested because of its easily applied form is where $n = m$. This may be written as follows

(2.13) $$f(x) = f(a) + \sum_{k=1}^{n} \frac{(2n-k)!}{(2n)!} \binom{n}{k} \left[f^{(k)}(a) - (-1)^k f^{(k)}(x) \right] (x-a)^k + R$$

where

$$R = (-1)^n \frac{(n!)^2 (x-a)^{2n+1}}{(2n)!(2n+1)!} f^{(2n+1)}(\theta)$$

We may compute all coefficients C_k by setting

(2.14) $$C_k = \frac{(2n-k)!}{(2n)!} \binom{n}{k}$$

and establishing the following recursion formula

(2.15) $$C_{k+1} = \frac{n-k}{(2n-k)(k+1)} C_k \text{ and } C_1 = \frac{1}{2}$$

Direct substitution into the formula for C_k where $k=1$ gives us $C_1 = \frac{1}{2}$. In order to establish the above formula for C_{k+1}, we assume C_k as above and proceed as follows

(2.16)
$$C_{k+1} = \frac{\left[2n-(k+1)\right]!}{(2n)!} \binom{n}{k+1}$$

$$= \frac{(2n-k-1)!}{(2n)!} \cdot \frac{n!}{(k+1)!(n-k-1)!}$$

Multiplying through by $\frac{(n-k)(2n-k)}{(n-k)(2n-k)}$ we have

(2.17) $$C_{k+1} = \frac{n-k}{(2n-k)(k+1)}$$

which establishes our recursion formula for the coefficients of the above expansion.

Last of all we inspect the remainder term of the above expansion as a measure of rapidity of convergence. By using Sterlings approximations for factorials we see that

(2.18) $$|R| \leq 2(e/4n)^{2n+1} \left| (x-a)^{2n+1} f^{(2n+1)}(\theta) \right|$$

This indicates that for even a moderately small n the remainder term is usually small. This implies that convergence is usually rapid.

For purposes of completeness we shall conclude this discussion of our generalization by extending it to an infinite expansion. We proceed as follows.

Let p and q be non-negative real numbers not both zero and let m and n become infinite in such a way that $n/m \rightarrow p/q$. Under these conditions we do the following.

Consider the expressions

$$(2.19) \qquad \frac{(m+n-k)!}{(m+n)!}\binom{m}{k} \qquad \text{and} \qquad \frac{(m+n-k)}{(m+n)!}\binom{n}{k}.$$

Under the conditions just stated concerning the behavior of n/m as n and m become infinite we examine the first expression for its limit as n and m increase without bound,

$$(2.20) \qquad \begin{aligned} \frac{(m+n-k)!}{(m+n)!}\binom{m}{k} &= \frac{(m+n-k)!}{(m+n)!} \cdot \frac{m!}{k!(m-k)!} \\ &= \frac{(m-k+1)(m-k+2)\dots(m-k+k)}{(m+n-k+1)(m+n-k+2)\dots(m+n)} \cdot \frac{1}{k!}. \end{aligned}$$

Dividing numerator and denominator of the right side of Equation (2.20) by n we have

$$(2.21) \qquad \frac{(\frac{m}{n}-\frac{k}{n}+\frac{1}{n})(\frac{m}{n}-\frac{k}{n}+\frac{2}{n})\dots(\frac{m}{n}-\frac{k}{n}+\frac{k}{n})}{(\frac{m}{n}+1-\frac{k}{n}+\frac{1}{n})(\frac{m}{n}+1-\frac{k}{n}+\frac{2}{n})\dots(\frac{m}{n}+1-\frac{k}{n}+\frac{k}{u})} \cdot \frac{1}{k!}.$$

(2.21) can be made smaller term for term than the first expression in (2.19) by rewriting as follows

$$(2.22)\qquad \frac{(\frac{m}{n}-\frac{k}{n})(\frac{m}{n}-\frac{k}{n})\dots(\frac{m}{n}-\frac{k}{n})}{(\frac{m}{n}+1)(\frac{m}{n}+1)\dots(\frac{m}{n}+1)}\cdot\frac{1}{k!}=\frac{(\frac{m}{n}-\frac{k}{n})^k}{(\frac{m}{n}+1)^k}\cdot\frac{1}{k!}$$

Letting m and n increase without bound under our original conditions, we have

$$\lim_{(m,m)\to\infty}\frac{(\frac{m}{n}-\frac{k}{n})^k}{(\frac{m}{n}+1)^k}\cdot\frac{1}{k!}=\frac{(\frac{q}{p})^k}{(\frac{q}{p}+1)^k}\cdot\frac{1}{k!}=\frac{q^k}{(q+p)^k}\cdot\frac{1}{k!}$$

This gives us as a lower bound of the limit of our expression $\frac{q^k}{k!(q+p)^k}$

Consider again expression (2.21). This can be made larger term for term than the first expression in (2.19) by writing it as follows

$$(2.23)\qquad \frac{(\frac{m}{n}+\frac{k}{n})(\frac{m}{n}+\frac{k}{n})\dots(\frac{m}{n}+\frac{k}{n})}{(\frac{m}{n}+1-\frac{k}{n}+\frac{1}{n})(\frac{m}{n}+1-\frac{k}{n}+\frac{1}{n})\dots(\frac{m}{n}+1-\frac{k}{n}+\frac{1}{n})}\cdot\frac{1}{k!}=\frac{(\frac{m}{n}+\frac{k}{n})^k}{(\frac{m}{n}+1-\frac{k}{n}+\frac{1}{n})^k}\cdot\frac{1}{k!}$$

Taking the limit here as n and m increase without bound under the same conditions as above, we have

$$\lim_{(n,m)\to\infty}\frac{(\frac{m}{n}+\frac{k}{n})^k}{(\frac{m}{n}+1-\frac{k}{n}+\frac{1}{n})^k}\cdot\frac{1}{k!}=\frac{(\frac{q}{p})^k}{\frac{(q+p)^k}{p^k}}\cdot\frac{1}{k!}=\frac{q^k}{(q+p)^k}\cdot\frac{1}{k!}$$

This gives us as an upper bound of our expression $\frac{q^k}{(p+q)^k}\cdot\frac{1}{k!}$.

Since the upper bound is equal to the lower bound of the limit of our expression, we conclude the following:

Under the conditions previously stated

$$\lim_{(n,m)\to\infty}\frac{(m+n-k)!}{(m+n)!}\binom{m}{k}=\frac{q^k}{k!(p+q)^k}$$

Using a similar argument we may further show that under the same conditions on m and n

$$\lim_{(n,m)\to\infty} \frac{(m+n-k)!}{(m+n)!}\binom{n}{k} = \frac{p^k}{k!(p+q)^k}$$

Using the limits just derived, our general expansion

$$f(x) = f(a) + \sum_{k=1} \frac{(m+n-k)!}{(m+n)!}\left[\binom{m}{k}f^{(k)}(a) - (-1)^k\binom{n}{k}f^{(k)}(x)\right](x-a)^k + R$$

becomes

$$(2.24) \qquad f(x) = f(a) + \sum_{k=1}^{\infty}\left[q^k f^{(k)}(a) - (-p)^k f^{(k)}(x)\right]\frac{(x-a)^k}{k!(p+q)^k} .$$

If in this expansion we let $p = 0$ we have

$$(2.25) \qquad f(x) = f(a) + \sum_{k=1}^{\infty}\left[q^k f^{(k)}(a)\right]\frac{(x-a)^k}{k!(p+q)^k}$$

which is the familar Taylor expansion. Letting $p = q$ we get the following

$$(2.26) \qquad f(x) = f(a) + \sum_{k=1}^{\infty}\left[f^{(k)}(a) - (-1)^k f^{(k)}(x)\right]\frac{(x-a)^k}{k!\,2^k}$$

To prove the validity of our generalized infinite expansion

$$f(x) = f(a) + \sum_{k=1}^{\infty}\left[q^k f^{(k)}(a) - (-p)^k f^{(k)}(x)\right]\frac{(x-a)^k}{k!(p+q)^k}$$

we do the following.

Let $f(x)$ have the following Taylor series expansions

$$f(u) = \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(u-a)^k}{k!} \quad \text{and} \quad f(v) = \sum_{k=0}^{\infty} f^{(k)}(x)\frac{(v-x)^k}{k!}$$

with respective radii of convergence R_a, R_x, each greater than $|x-a|$.

Choose u and v such that $u-a = \dfrac{q(x-a)}{(p+q)}$ and $v-x = -\dfrac{p(x-a)}{(p+q)}$.

Since $\dfrac{q}{p+q}$ and $\dfrac{p}{p+q}$ both lie between 0 and 1, it follows that u and v as defined both lie in their respective intervals of convergence.

Since

$$(2.27) \qquad u = \frac{q(x-a) + a(p+q)}{(p+q)} = \frac{qx + ap}{(p+q)}$$

and

$$(2.28) \qquad v = \frac{-p(x-a) + x(p+q)}{(p+q)} = \frac{pa + xq}{(p+q)}$$

We may conclude that $u = v$. Therefore we may equate our Taylor expansions for $f(u)$ and $f(v)$ thusly

$$(2.29) \qquad \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(u-a)^k}{k!} = \sum_{k=0}^{\infty} f^{(k)}(x) \frac{(v-x)^k}{k!}$$

By substituting the values for u and v we obtain

$$(2.30) \qquad \sum_{k=0}^{\infty} \left[\frac{q}{p+q}(x-a)\right]^k \frac{f^{(k)}(a)}{k!} = \sum_{k=0}^{\infty} \left[\frac{p}{p+q}(a-x)\right]^k \frac{f^{(k)}(x)}{k!}$$

By slight rearrangement of terms we obtain

$$f(x) = f(a) + \sum_{k=1}^{\infty} \left[q^k f^{(k)}(a) - (-p)^k f^{(k)}(x)\right] \frac{(x-a)^k}{k!\,2^k}$$

which is precisely our generalized infinite expansion arrived at previously. Therefore we conclude that this infinite expansion is valid for all non-negative values of p and q not both zero provided $|x-a|$ is less than both R_a and R_x.

By choosing a finite number of terms of the infinite expansion, the accuracy is usually not as good as that obtained by using a finite expansion of the same order. Moreover it is difficult to obtain a bound for the error when the infinite expansion is used [4].

In the next chapter of this report we shall exhibit several examples of familiar functions expanded by this generalized method and compare them with the familiar Taylor expansions of the same functions. We shall further exhibit some approximation formulas derived by this method and compare them with some well known approximation formulas often used for this purpose. Finally we shall derive an interpolation correction formula arising from this generalized expansion and exhibit some of its possible applications.

CHAPTER III

Since this chapter is going to deal mainly with comparing the results obtained by our generalized scheme, with the results obtained by better known methods for the same purpose, we shall begin as follows.

Consider the ordinary Taylor series expansion around the point zero for the function ln(1+x).

$$(3.1) \qquad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

For the purposes of our approximation we take 8 terms of the above series and approximate ln 2 and ln 1.1. By this means we get

$$\ln 2 = .6345238090$$

$$\ln 1.1 = .0953101797$$

We can see ln 2 has an error in the second decimal place and ln 1.1 has an error in the tenth decimal place, illustrating the well known fact that Taylor series approximations are more accurate in a small neighborhood about the point of expansion.

Let us now consider our generalized expansion for the same function. By the general formula of the previous chapter, where, $n = m = 4$ and $\alpha = 0$, we obtain the following

$$(3.2) \qquad \ln(1+x) = \ln 1 + \frac{1}{2}\left[1 + \frac{1}{1+x}\right]x + \frac{3}{28}\left[-1 + \frac{1}{(1+x)^2}\right]x^2 + \frac{1}{84}\left[2 + \frac{2}{(1+x)^3}\right]x^3 + \frac{1}{1680}\left[-6 + \frac{6}{(1+x)^4}\right]x^4$$

Combining terms and simplifying we then have

$$(3.3)\quad \ln(1+x) = \frac{840x + 2940x^2 + 3640x^3 + 1750x^4 + 168x^5 - 28x^6 + 8x^7 - 3x^8}{840 + 3360x + 5040x^2 + 3360x^3 + 840x^4}$$

Factoring into a more usable form for the desk calculator or for programming we have

$$(3.4)\quad \ln(1+x) = \frac{x\left\{840 + x\left[2940 + x(3640 + x\left\{1750 + x\left[168 + x(-28 + x\,\overline{8-3x})\right]\right\})\right]\right\}}{840 + x\left\{3360 + x\left[5040 + x(3360 + 840x)\right]\right\}}$$

Using the above approximation for ln 2 and ln 1.1 we obtain

$$\ln 2 = .6930803571$$

$$\ln 1.1 = .09531017980$$

We see that ln 2 has an error in the fourth decimal place as compared with an error in the second place for the ordinary Taylor series. We further notice that ln 1.1 is accurate to ten decimal places as compared with nine places for the ordinary Taylor series. This is more significant when we remember that we used only half as many terms in our generalized expansion as we did in the Taylor series.

In order that we might more fully appreciate the capabilities of this approximation, we will examine the error term involved by taking the first four terms as our approximation. From our general form for the error term as given in Chapter II we have in this case

$$R = (-1)^4 \frac{(4!)^2 x^9}{8!9!} = \frac{x^9}{5\cdot 6\cdot 7\cdot 8\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}\, f^{(9)}(\theta)$$

where

$$a < \theta < x.$$

By inspecting the derivatives of $\ln(1+x)$ we see that $f^{(9)}(\theta) = \frac{8!}{(1+\theta)^9}$.

Taking a worst than possible value for θ, for the purpose of creating a larger error, we have $f^{(9)}(\theta) = 8!$, which in turn gives us

$$R \leq \frac{x^9}{630} = .0015.$$

We see then that when x=1 we have at most an error in the third decimal. We have already seen of course that our error actually occurs in the fourth decimal place. If we take x=.1, we remember that we obtained ten decimal accuracy. This, however, was limited by the number of places available on a desk calculator. By examining the error term when x=.1, and letting θ be such that our error is the worst possible, we obtain

$$R \leq \frac{(.1)^9}{630} = .000000000001$$

This indicates our error would not occur before the twelfth decimal place and further indicates a good approximation using only a few terms of the expansion. We want to keep in mind however, that as x becomes greater than 1 our error begins increasing in the order of $(x-\alpha)^9$, where $x-\alpha > 1$ and accuracy decreases rapidly. It is then apparent that in order to insure a high degree of accuracy, α must be taken in a reasonably small neighborhood of x. This would, of course, limit the range over which we could approximate the function $\ln(1+x)$ satisfactorily.

Having met with such success in our first comparison it seems appropriate that we make a comparison between our scheme and a well known and accepted method for that purpose. For our basis of comparison we take the Hasting's approximations [5]. These approximations are well known in industry for both desk calculator and digital computor applications. We want to compare these, not only on the basis of results, but also as to the number of operations necessary to obtain these results. For the function in question, namely $\ln(1+x)$, Hastings has several approximations over the range $0 \leq x \leq 1$ which vary only in the degree of accuracy, number of operations, and the value of constants used. Since our approximation used an eighth degree polynomial in the numerator, we will select the Hastings approximation which also employs an eighth degree polynomial. It is as follows [5]

(3.5) $\ln(1+x) = a_1 x + a_2 x^2 + \ldots + a_8 x^8$

where

$a_1 = .9999,9642,39$	$a_5 = .1676,5407,11$
$a_2 = -.4998,7412,38$	$a_6 = -.0953,2938,97$
$a_3 = .3317,9902,58$	$a_7 = .0360,8849,37$
$a_4 = -.2407,3380,84$	$a_8 = -.0064,5354,42$

An outstanding feature of the Hastings approximations is that over the range of consideration the error is cyclic. Due to this fact the approximation may be just as accurate at a point near the end of the range as at some point near the beginning of the range. This is not true of our approximation however, since the beginning of our range is taken to be the point at which we expanded the series and we lose accuracy as we move further away from this point.

Over the above range we find that Hastings' approximation has at most an error in the eighth decimal place. By investigating our error function for $x=.4$ we have

$$R \leq \frac{(.4)^9}{630} = .0000004$$

which indicates at most an error in the seventh place. Since our error is not cyclic, we see that from $.4 \leq x \leq 1$ our approximation will give an increasingly worse approximation than the Hastings approximation. Examining our error when $x = .3$, we see that our error term is as follows

$$R \leq \frac{(.3)^9}{630} = .00000003$$

which indicates at most an error in the eighth decimal place and therefore implies that our approximation over the range $0 \leq x \leq .3$ will give in general better results than the Hastings approximation. We need also observe at this time that the Hastings approximation requires the evaluation of an eighth degree polynomial, while our approximation requires the evaluation of an eighth degree polynomial and a fourth degree polynomial in order to obtain a result. It becomes then a matter of the required degree of accuracy and the range over which you have to work as to the relative values of the two approximations. It should also be observed that, by using more terms of our series expansion, the accuracy would increase over a greater range. It should also be noted that, with each increase of the number of terms used, the polynomials to be evaluated increase in degree in the ratio of 2 to 1. For example, by using five terms we would need to evaluate a tenth degree polynomial and a fifth degree polynomial.

As a further example of this generalized expansion we examine the function $f(x) = e^x$. We shall in this case speak only briefly of the expansion of e^x by the ordinary Taylor series, and deal mainly with the Hastings approximations for e^{-x}. We will obtain our approximation for e^{-x} simply by taking the reciprocal of our approximation for e^x.

First consider the generalized expansion for e^x with m=n=4 and $\alpha = 0$

$$(3.6) \qquad e^x = \frac{1+\frac{1}{2}x+ 3x^2/28 + x^3/84 + x^4/1680}{1-\frac{1}{2}x + 3x^2/28 - x^3/84 + x^4/1680}$$

For our brief comparison with the ordinary Taylor's expansion we set x=1 and calculate e by the above expansion, which gives $e = 2721/1001 = 2.71821718\ldots$ which is an error less than 1 in the seventh decimal place. Using a finite Taylor expansion with n=4 gives us an error in the second decimal place. In fact it is necessary to take n=12 in the Taylor expansion to obtain accuracy to eight decimal places [4].

Before we compare our expansion with Hastings' approximation we will again expand e^x taking n=5 and $\alpha=0$ for a more accurate approximation. Once again using our general expansion, and our recursion formula for coefficients, we obtain the following

$$(3.7) \qquad e^x = \frac{1+\frac{1}{2}x+ x^2/9 + x^3/72 + x^4/1008 + x^5/30,240}{1-\frac{1}{2}x+ x^2/9 - x^3/72 + x^4/1008 + x^5/30,240} .$$

Rewriting for easier calculation as we did for the function $f(x) = \ln(1+x)$ we obtain

$$(3.8) \qquad e^x = \frac{30240 + x\left\{15120 + x\left[3360 + x(420 + x\,\overline{30+x})\right]\right\}}{30240 + x\left\{-15120 + x\left[3360 + x(-420 + x\,\overline{30-x})\right]\right\}}$$

Taking x=1 we obtain $e = 49171/18089 = 2.718281828\ldots$ which is accurate to the capacity of the desk calculator used in computation. To obtain an estimate of its accuracy past the ninth decimal we examine the error term as follows

$$R = -\frac{5!5!x^{11}e^{\theta}}{10!11!(1 - \frac{x}{2} + \frac{x^2}{9} - \frac{x^3}{72} + \frac{x^4}{1008} - \frac{x^5}{39240})}$$

where

$$0 < \theta < 1$$

which will reduce to

$$R = \frac{39240\, x^{11} e^{\theta}}{10,059,033,600(39240 - 19620x + 4360x^2 - 540x^3 + 30x^4 - x^5)} .$$

Taking x=1 we obtain

$$R \leq .0000000002$$

which implies at worst an error in the tenth decimal place. Consider the results obtained for various other values of x as follows.

$x = 1.5$

$e^x = 4.481689110$ (Error in the seventh decimal place)

$e^{-x} = .223130158$ (Error in the eighth decimal place)

$x = 2$

$e^x = 7.389057750$ (Error in the sixth decimal place)

$e^{-x} = .135335252$ (Error in the eighth decimal place)

$x = 3$

$e^x = 20.0859728$ (Error in the fourth decimal place)

$e^{-x} = .4978598783$ (Error in the fifth decimal place)

We see then that the error is increasing as we move further away from x=0 and by an examination of the error term we see that it is increasing by an order of x^{11}, which causes us to restrict our range over which we will attempt to make a favorable comparison with the Hastings approximation. Before we restrict our range in any manner we will examine the Hastings approximation for e^{-x}. We examine first the Hastings approximation using a twelfth degree polynomial. We selected this one since it is the approximation using a polynomial of degree nearest to the fifth degree polynomial used in our approximation. The Hastings approximation is as follows [5]

$$(3.9) \qquad e^{-x} = \frac{1}{[1 + a_1 x + a_2 x^2 + a_3 x^3]^4}$$

where

$$a_1 = .2507,213$$
$$a_2 = .0292,732$$
$$a_3 = .0038,278$$

Using this approximation we find that over the range $0 \leq x \leq 3$ we can expect at least three place accuracy. We see then from our previous results that we can expect at least five place accuracy over the same range using the approximation derived from our generalized expansion with n=5. Since this is by no means the most accurate Hastings approximation for this function, we will now consider the most accurate Hastings approximation, as follows [5]

$$(3.10) \qquad e^{-x} = \frac{1}{[1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6]^4}$$

where

$$a_1 = .2499,9868,42 \qquad a_4 = .0001,7156,20$$
$$a_2 = .0312,5758,32 \qquad a_5 = .0000,0543,02$$
$$a_3 = .0025,9137,12 \qquad a_6 = .0000,0069,06$$

Using the above approximation over the same range as before we can expect at least six place accuracy. As a means of comparison, we consider our approximation over the range $0 \leq x \leq 2$, and find that we may expect at least seven place accuracy. We see then that as before our approximation over a selected range gives greater accuracy than the Hastings approximation. In the previous example it would, however, be well to note that our approximation requires the evaluation of two fifth degree polynomials followed by a division. By comparison the Hastings approximation requires the evaluation of a sixth degree polynomial, three multiplications, followed by a division. One might also note that the constants involved in the Hastings approximation are of such length that you would be unable to handle all significant digits on a computor using eight significant digits in its floating point system. This of course would reduce the accuracy of the results.

As in the previous example, we could achieve a satisfactory degree of accuracy over a different range by an appropriate choice of α. This would always be true of our generalized expansion, as with any Taylor series expansion, and will not be brought out specifically in any of our other examples.

As a last example of this type we will consider the function $f(x) = \arctan x$. Consider first the ordinary Taylor expansion for arc tan x around the point zero.

(3.11) $$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$$

Taking $n = 11$ and $x = 1$ in the above expansion we obtain arc tan 1 = .744011544, which has an error in the second decimal place [6]. We now consider our generalized expansion with $n = m = 4$ and $\alpha = 0$.

$$(3.12)\quad \text{arc tan } x = \text{arc tan } 0 + \frac{1}{2}\left[1 + \frac{1}{1+x^2}\right]x + \frac{3}{28}\left[0 + \frac{2x}{(1+x^2)^2}\right]x^2$$

$$+ \frac{1}{84}\left[-2 + \frac{2(3x^2-1)}{(1+x^2)^3}\right]x^3 + \frac{1}{1680}\left[0 + \frac{24x(x^2-1)}{(1+x^2)^4}\right]x^4 + R$$

Simplifying and combining terms we have

$$(3.13)\ \text{arc tan } x = \frac{-40x^{11} + 680x^9 + 4464x^7 + 8176x^5 + 6160x^3 + 1680x}{1680x^8 + 6720x^6 + 10080x^4 + 6720x^2 + 1680}.$$

Rewriting for easier calculation we obtain

$$(3.14)\ \text{arc tan } x = \frac{1680x + x^3\ \overline{6160 + x^2\left[8176 + x^2(4.464 + x^2\ \overline{680 - 40x^2})\right]}}{1680 + x^2\ \overline{6720 + x^2\left[10080 + x^2(6720 + 1680x^2)\right]}}$$

Taking x=1 we obtain, arc tan 1 = 21120/26880 which gives us arc tan 1 = .785714285, which has an error in the fourth decimal place [6]. By taking x=.1 we obtain arc tan.1 = .0996686505, which has an error in the ninth decimal place. We see then that our error is naturally increasing as we move further away from zero. Our error over the range $0 \leq x \leq 1$ would correspond to our error over the range $-1 \leq x \leq 0$ and would be equal in magnitude for points of equal distance either side of zero. Taking then for our range of consideration $-1 \leq 0 \leq 1$ we could expect our error curve over this range to be symmetric about the y axis passing through the point x=0.

For our final comparison, we consider the Hastings approximation for arc tan x [5].

$$(3.15)\quad \text{arc tan } x = C_1 x + C_3 x^3 + C_5 x^5 + C_7 x^7$$

where

$C_1 = .9992150$ $\qquad C_5 = .1462766$

$C_3 = -.3211819$ $\qquad C_7 = -.0389929$

This approximation over the previously stated range will have at least four place accuracy with a minimum number of operations necessary to arrive at a result. Therefore our approximation would be preferred over the Hastings approximation only in a limited neighborhood of zero and where the increased accuracy was of such a necessity to warrant the added calculation necessary.

It would then seem that over particular ranges this generalized expansion will produce an approximation formula which is superior to the Hastings approximations. One must keep in mind however that the Hastings approximations in general are applicable over a much wider range than any one particular approximation formula derived by our general method. Also the number of operations required to obtain a result with our generalized expansion is generally greater than with the Hastings approximation. We note however that the degree of accuracy obtained by approximation formulas obtained in this manner rely principally on the number of terms used and the choice of α, as long as the function in question will satisfy the conditions of the expansion.

We have by no means investigated all the possibilities of this expansion as a means of obtaining approximation formulas. We have instead tried to show its behavior in only a few cases, and we leave its total possibilities along this line relatively unexplored.

As a second possible use for this expansion we derive the following interpolation correction formula [7].

Theorem 3.1: $f(x) = \sum_{i=0}^{n} \prod_{j \neq i} \frac{x_j - x}{x_j - x_i} S_i + R$

where

$$(3.16) \qquad R = \sum_{i=0}^{n} \prod_{j \neq i} \frac{x_j - x}{x_j - x_i} R_i .$$

We develop the above as follows:

Let f(x) together with its first m+ n+ 1 derivatives be continuous in an interval containing $x_0, x_1 \ldots, x_n$, where $x_i \neq x_j$. We define the following

$$C_i' = \frac{(m+n-i)!\,m!}{(m+n)!\,(m-i)!\,i!} \qquad\qquad C_i = \frac{(m+n-i)!\,n!}{(m+n)!\,(n-i)!\,i!}$$

$$(3.17) \qquad S_k = f(x_k) + \sum_{i=1}^{m} C_i' f^{(i)}(x_k)(x-x_k)^i \qquad k = 0, 1, 2, \ldots, n$$

$$(3.18) \qquad R_k = (-1)^n \frac{m!\,n!\,(x-x_k)^{m+n+1}}{(m+n)!\,(m+n+1)!} \quad f^{(m+n+1)}(\theta_k)$$

where

$$x < \theta_k < x_k$$

Proof:

From the previous chapter we know that the function f(x) may be expanded in a generalized Taylor series expansion as follows

$$(3.19) \quad f(x) = f(x_k) + \sum_{i=1} \frac{(m+n-i)!}{(m+n)!} \left[\binom{m}{i} f^{(i)}(x_k) - (-1)^i \binom{n}{i} f^{(i)}(x) \right] (x-x_k)^i + R_k$$

By rearranging terms we may write (3.19) as follows

$$(3.20) \qquad f(x) + \sum_{i=1} \frac{(m+n-i)!}{(m+n)!} \binom{n}{i} f^{(i)}(x)(x_k-x)^i = S_k + R_k$$

or

$$(3.21) \qquad f(x) + \sum_{i=1} C_i f^{(i)}(x)(x_k-x)^i = S_k + R_k \qquad k = 0, 1, \ldots, n$$

These may be considered as n+1 linear equations in $f(x), f'(x), \ldots, f^{(n)}(x)$.

The determinant of the coefficients would then be

$$D = \begin{vmatrix} 1 & C_1(x_0-x) & C_2(x_0-x)^2 & \cdots & C_n(x_0-x)^n \\ 1 & C_1(x_1-x) & C_2(x_1-x)^2 & \cdots & C_n(x_1-x)^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & C_1(x_n-x) & C_2(x_n-x)^2 & \cdots & C_n(x_n-x)^n \end{vmatrix}$$

We recognize D as a modified form of a Vandermonde determinant and since any Vandermonde determinant in x_i can be expressed as $\prod_{i\,j}(x_i-x_j)$ we write D as follows

$$D = \left(\prod_{i=1}^{n} C_i\right)\left[\prod_{i>j}(x_i-x_j)\right]$$

which does not vanish since all x's are distinct and the C's are different from zero. Hence we may solve the system of equations for f(x) by Kramer's rule and obtain

$$D\cdot f(x) = \begin{vmatrix} S_0+R_0 & C_1(x_0-x) & C_2(x_0-x) & \cdots & C_n(x_0-x) \\ S_1+R_1 & C_1(x_1-x) & C_2(x_1-x) & \cdots & C_n(x_1-x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_n+R_n & C_1(x_n-x) & C_2(x_n-x) & \cdots & C_n(x_n-x) \end{vmatrix}$$

Expanding the above determinant by the elements of the first column we obtain

$$(3.22)\qquad D\cdot f(x) = \sum_{k=0}^{n} (-1)^k (S_k+R_k)M_k$$

where

$$M_k = \begin{vmatrix} C_1(x_0-x) & C_2(x_0-x)^2 & \cdots & C_n(x_0-x)^n \\ C_1(x_1-x) & C_2(x_1-x)^2 & \cdots & C_n(x_1-x)^n \\ & \text{kth row missing} & & \\ C_1(x_n-x) & C_2(x_n-x)^2 & \cdots & C_n(x_n-x)^n \end{vmatrix}$$

Expanding M_k we obtain

$$(3.23) \qquad M_k = \Big[\prod_{i=0}^{n} C_i\Big]\Big[\prod_{i\neq k}(x_i-x)\Big]\Big[\prod_{i>j;\, j\neq k}(x_i-x_j)\Big]$$

Dividing (3.22) through by D we obtain

$$(3.24) \qquad f(x) = \sum_{k=0}^{n} (-1)^k (S_k+R_k) \frac{\Big[\prod_{i=0}^{n} C_i\Big]\Big[\prod_{i\neq k}(x_i-x)\Big]\Big[\prod_{i>j;\, i,j\neq k}(x_i-x_j)\Big]}{\Big[\prod_{i=1}^{n} C_i\Big]\Big[\prod_{i>j}(x_i-x_j)\Big]}$$

Since $\prod_{k>j}(x_k-x_j)$ can be written as $(-1)^k \prod_{j \; k}(x_j-x_k)$ the above becomes as follows.

$$f(x) = \sum_{k=0}^{n} \frac{\prod_{i=k}(x_i-x)}{\Big[\prod_{i>k}(x_i-x_k)\Big]\Big[\prod_{j<k}(x_j-x_k)\Big]} \quad (S_k + R_k)$$

Setting i=j and k=i in the above expression we have

$$(3.25) \qquad f(x) = \sum_{i=0}^{n} \frac{\prod_{j\neq i} x_j-x}{\prod_{j>i}(x_j-x_i) \prod_{j<i}(x_j-x_i)} \quad (S_k + R_k)$$

$$= \sum_{i=0}^{n} \prod_{j\neq i} \frac{x_j-x}{x_j-x_i} S_i + R$$

where

$$R = \sum_{j \neq i} \frac{x_j - x}{x_j - x_i} R_i$$

and the theorem is proven.

In order that we might obtain the previously mentioned interpolation correction formula in a readily usable form, we do the following.

Let n=m=1 in Equation (6) and obtain the following

(3.26) $$f(x) = \frac{x_1 - x}{x_1 - x_0} S_0 + \frac{x_0 - x}{x_0 - x_1} S_1$$

where

$$S_0 = f(x_0) + \frac{1}{2} f'(x_0)(x - x_0)$$

and

$$S_1 = f(x_1) + \frac{1}{2} f'(x_1)(x - x_1).$$

We then have

$$f(x) = \frac{x_1 - x}{x_1 - x_0} f(x_0) + \frac{x_1 - x}{x_1 - x_0} f'(x_0) \frac{1}{2}(x - x_0)$$

$$+ \frac{x_0 - x}{x_0 - x_1} f(x_1) + \frac{x_0 - x}{x_0 - x_1} f'(x_1) \frac{1}{2}(x - x_1)$$

$$= \frac{x_1 - x}{x_1 - x_0} f(x_0) + \frac{x_1 - x}{x_1 - x_0} f'(x_0) \frac{1}{2}(x - x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$- \frac{x - x_0}{x_1 - x_0} f'(x_1) \frac{1}{2}(x_1 - x)$$

which then gives us the following

(3.27) $$f(x) = I + C + R$$

where

$$(3.28) \qquad I = \frac{(x_1-x)f(x_0) + (x-x_0)f(x_1)}{x_1-x_0}$$

and

$$(3.29) \qquad C = \frac{(x_1-x)(x-x_0)}{2(x_1-x_0)} \left[f'(x_0) - f'(x_1)\right]$$

and

$$(3.30) \qquad R = \frac{(x_1-x)(x-x_0)}{12(x_1-x_0)} \left[(x_1-x)^2 f'''(\theta_1) - (x-x_0)^2 f'''(\theta_0)\right]$$

with θ, between x_1 and x and θ_0 between x_0 and x. It may well be observed here that I is ordinary linear interpolation, while C is an additive correction to be applied to the interpolated value obtained by I. R may be used to establish bounds for the remaining error after the correction has been applied. The accuracy of this corrective factor may be improved by the use of additional derivatives in C.

To illustrate the previously derived interpolation correction formula, we present the following examples.

Calculating cos 40° by the previously described method and taking $x_0 = \pi/6$ and $x_1 = \pi/4$, we proceed as follows.

Take cos x = f(x). Then f(x) = cos x = I + C + R.

Calculating I, which is actually Lagrangian interpolation, we have

$$I = \frac{(\pi/4 - .69813)(\sqrt{3}/2) + (.69813 - \pi/6)(\sqrt{2}/2)}{\pi/4 - \pi/6}$$

where .69813 is the value of 40° in radians. Evaluating the expression, we obtain

$$I = \frac{.198986}{.261799} = .76007 \quad \text{(Error in third decimal place).}$$

Then calculating C, we have

$$C = \frac{(\pi/4 - .69813)(.69813 - \pi/6)}{2(\pi/9 - \pi/6)} \left[f'(x_0) - f'(x_1)\right]$$

where the value of $f'(x_0)$ and $f'(x_1)$ are found by taking the value of $-\sin x$ at x_0 and x_1. We then have

$$C = \frac{.0031545}{.523598} = .00602$$

Therefore

$$f(x) = I + C \approx .76007 + .00602 = .76609$$

where $\cos 40^\circ = .76604$ to five places.

In order that we might determine the error to be expected in this sort of correction we examine R, where

$$R = \frac{(x_1-x)(x-x_0)}{12(x_1-x_0)} \left[(x_1-x)^2 f'''(\theta_1) - (x-x_0)^2 f'''(\theta_0)\right]$$

and θ_0 lies between x_0 and x, and θ_1 lies between x and x_1.

Taking the worst possible values for θ_0 and θ_1, we have

$$R < .000085$$

which implies that we may expect no worse than an error in the fifth decimal place. This is precisely what we had in the above example.

As a second and final example of what we might use this correction for, we cite the following.

Suppose we wish to approximate the compound interest formula $(1.031)^{50}$. Taking $f(x) = (1+x)^{50}$ we do the following.

Take n=m=2 in our formula for f(x) and setting $x_0 = 0.0275$, $x_1 = 0.03$, $x_2 = 0.035$, and $x = 0.031$, we have

$$(1.031)^{50} = -\frac{16}{75} S_0 + \frac{84}{75} S_1 + \frac{7}{25} S_2 + R$$

where

$$S_0 = f(x_0) + \sum_{i=1}^{m} \frac{(m+n-i)!\,m!}{(m+n)!\,(m-i)!\,i!} f^{(i)}(x_0)(x-x_0)^i$$

Since $m = n = 2$ and $x_0 = .0275$ we have

$$S_0 = f(x_0) + \sum_{i=1}^{2} \frac{(4-i)!\,2!}{4!\,(2-i)!\,i!} f^{(i)}(x_0)(x-x_0)^i$$

Then

$$S_0 = (1.0275)^{50} + \frac{1}{2}50(1.0275)^{49}(.0035) + \frac{1}{12}\, 49.50(1.0275)^{48}(.0035)^2$$

$$= 3.88232177 + 0.33061125 + 0.00919714$$

$$= 4.22213016.$$

By a similar process we obtain

$$S_1 = 4.49115519$$

$$S_2 = 5.06235147$$

and we then obtain

$$(1.031)^{50} = 4.6018588 + R.$$

Using Equation (3.18) from this chapter and

$$R = \sum_{i=0}^{n} \prod_{j \neq i} \frac{x_j - x}{x_j - x_i} R_i$$

we obtain the following

$$-0.00000032 \leq R \leq -0.00000026$$

which implies that we have seven decimal accuracy. When this value is approximated by means of ordinary linear interpolation between the rates 0.03 and 0.035, the error exceeds 2 in the second decimal place.

We see then, that the accuracy of this means of interpolation is limited only by the number of derivatives used. This would, of course, be determined by the needed accuracy and the desired ease of calculation.

As a final and concluding example of how this generalized Taylor series might be applied, we will formulate a scheme for the purpose of approximating modified Bessel functions of the second kind. These are usually denoted by the notation $K_n(x)$. For our example we will consider only the case where $n = 0$ and to a lesser extent the case where $n = 1$.

As a preparation for the scheme which we are going to use for these approximations, we wish to show that our previously derived generalization is valid for $(x-\alpha_1) < 0$ as well as $(x-\alpha_1) > 0$. Under the same conditions on the function as imposed previously, we proceed as follows. Define

$$(3.31) \qquad F(x) = \int_x^{\alpha_1} f^{(m+n-i)}(t)q^{(i)}(t)\,dt$$

where

$$q(t) = (\alpha_1-t)^m (t-x)^n .$$

We notice here that the only difference from our previous derivation is that the roles of α_1 and x have been interchanged.

By the previously used formula for repeated integration by parts, we have

$$F(x) = \sum_{k=0}^{n+m} (-1)^i f^{(m+n-i)}(t)\, q^{(i)}(t)\Big]_x^{\alpha_1}$$

We obtain as before, by induction,

$$(3.32) \qquad q^{(i)}(t) = \sum_{k=0}^{i} (-1)^{i-k} i!\binom{m}{i-k}\binom{n}{k}(\alpha_1-t)^{m+k-i}(t-x)^{n-k}$$

Therefore $q^{(i)}(x) = 0$ if $i < n$ and all terms vanish for $i \geq n$ except where $k = n$. Therefore

(3.33) $$q^{(i)}(x) = (-1)^{i-n} i\binom{m}{i-n}(a_1-x)^{m+n-i}$$

Similarly $q^{(i)}(a_1) = 0$ if $i < m$ and for $i \geq m$

(3.34) $$q^{(i)}(a_1) = (-1)^m i!\binom{n}{i-m}(a_1-x)^{m+n-i}$$

Using (3.33) and (3.34) in (3.32) and changing indices by setting $k = m+n-i$ we obtain

(3.35) $$(-1)^n F(x) = \sum_{k=0} (m+n-k)! \left[(-1)^k\binom{n}{k}f^{(k)}(a_1) - \binom{m}{k}f^{(k)}(x)\right](a_1-x)^k$$

Consider once more Equation (3.31). Since q(t) is still of constant sign throughout the interval we may as before apply the mean value theorem for integrals and obtain

$$F(x) = f^{(m+n+1)}(\theta)\int_x^{a_1} q(t)dt \qquad \text{where} \quad x < \theta < a_1$$

To evaluate this integral we set $t = (a_1-x)u + x$ and obtain

$$F(x) = f^{(m+n+1)}(\theta)(a_1-x)^{m+n+1}\int_0^1 (1-u)^m u^n du$$

Recognizing as before that this last integral is a beta function and equals $m!n!/(m+n+1)!$, we obtain

(3.36) $$F(x) = \frac{m!n!(a_1-x)^{m+n+1}}{(m+n+1)!} f^{(m+n+1)}(\theta)$$

The proposition is proved by eliminating F(x) from Equation (3.35) and (3.36) and obtaining

$$(3.37) \quad f(x) = f(a_1) - \sum_{k=1} \frac{(m+n-k)!}{(m+n)!} \left[\binom{m}{k} f^{(k)}(x) - (-1)^k \binom{n}{k} f^{(k)}(a_1)\right](a_1 - x)^k - R$$

where

$$R = (-1)^n \frac{m!\,n!\,(a_1 - x)^{m+n+1}}{(m+n)!\,(m+n+1)!} f^{(m+n+1)}(\theta)$$

Taking $a_1 = a$ and comparing this expansion with our expansion derived in Chapter II, we see that for identical x these expansions are the same. We are therefore able to pick our point of expansion other than x with no regard as to its position relative to x. We need only be sure that the function satisfies the condition of our theorem in whatever interval selected.

Since we have the previously derived expansion verified, we are able to write for any function a general expansion about x and any general point a. Using this general expansion we are able to obtain an expansion for any x in terms of $(x-a)$ for any desired a. We find also that we are able to obtain our expansion for any desired function in a more readily calculable form by using the previously defined function

$$F(x) = \int_a^x f^{(m+n+1)}(t)\, g(t)\, dt$$

where

$$g(t) = (x-t)^m (t-a)^m$$

and performing the indicated integration. We will use this scheme for deriving an approximation formula for the Bessel function $K_0(x)$, using three derivatives. We will refer to all approximations for $K_0(x)$ obtained by using this formula as $K_0^3(x)$, in order to differentiate these values from any obtained using higher order derivatives. We proceed as follows.

$$F(x) = \int_{a}^{x} f^{(m+n+1)}(t)\ g(t)dt$$

$$g(t) = (x-t)^m (t-a)^n$$

Setting m = n = 3

$$F(x) = \int_{a}^{x} f^{vii}(t)\ g(t)dt$$

By either formal integration by parts applied seven times or by the previously exhibited formula for repeated integration by parts, we obtain

$$(3.38)\qquad F(x) = f(a) - f(x) + \frac{1}{2}\left[f'(a) + f'(x)\right](x-a) + \frac{1}{10}\left[f''(a)-f''(x)\right](x-a)^2 + \frac{1}{120}\left[f'''(a)+f'''(x)\right](x-a)^3 + R$$

Since the expansion obtained by this means is the same as obtained by substituting into the general formula, we incur the same error term. Therefore,

$$R = (-1)^n \frac{(n!)^2 (x-a)^{2n+1}}{(2n)!(2n+1)!} f^{(2n+1)}(\theta)$$

where θ lies between a and x.

Setting $F(x) = 0$ in the above expansion we obtain

$$(3.39)\qquad -\frac{1}{120}f'''(x)(x-a)^3 + \frac{1}{10}f''(x)(x-a)^2 - \frac{1}{2}f'(x)(x-a) + f(x) = \frac{1}{120}f'''(a)(x-a)^3 + \frac{1}{10}f''(a)(x-a)^2 + \frac{1}{2}f'(a)(x-a) + f(a) + R_a$$

Further setting $a = \beta$ in this expansion we have

$$(3.40)\qquad -\frac{1}{120}f'''(x)(x-\beta)^3 + \frac{1}{10}f''(x)(x-\beta)^2 - \frac{1}{2}f'(x)(x-\beta) + f(x) = \frac{1}{120}f'''(\beta)(x-\beta)^3 + \frac{1}{10}f''(x)(x-\beta)^2 + \frac{1}{2}f'(x)(x-\beta) + f(\beta) + R_\beta$$

where R_α and R_β are R, with the points of expansion other than x, α and β respectively.

Using a well known identity for differentiating Bessel functions of this type [8], we obtain

$$K_0'(x) = -K_1(x) \tag{3.41}$$

$$K_0''(x) = \frac{1}{x}K_1(x) + K_0(x)$$

$$K_0'''(x) = -\frac{1}{x}K_0(x) - K_1(x)(1+2/x^2)$$

Substituting into Equations (3.39) and (3.40) the value of the derivatives of $K_0(x)$ we obtain

$$P_0K_0(x) + P_1K_1(x) = P_2 + R_\alpha \tag{3.42}$$

$$Q_0K_0(x) + Q_1K_1(x) = Q_2 + R_\beta \tag{3.43}$$

where P_0, P_1 and Q_0, Q_1 are respectively the rational coefficients of $K_0(x)$, $K_1(x)$, in the expansions about α and β respectively. P_2 and Q_2 are all terms in the respective expansions which do not have as factors either $K_0(x)$ or $K_1(x)$. These terms would be as follows.

$$P_0 = \left[(x-\alpha)\frac{1}{x}\cdot\frac{1}{120} + \frac{1}{10}\right](x-\alpha)^2 + 1$$

$$P_1 = \left\{\left[(x-\alpha)(1+2/x^2)\frac{1}{120} + \frac{1}{10}\cdot\frac{1}{x}\right](x-\alpha) + \frac{1}{2}\right\}(x-\alpha)$$

$$P_2 = \left\{\left[(-\frac{1}{\alpha}K_0(\alpha) - K_1(\alpha)(1+2/\alpha^2))(x-\alpha)\frac{1}{120} + \frac{1}{10}(K_0(\alpha) + \frac{1}{\alpha}K_1(\alpha)\right](x-\alpha) - \frac{1}{2}K_1(\alpha)\right\}(x-\alpha) + K_0(\alpha)$$

$$Q_0 = \left[(x-\beta)\frac{1}{x}\cdot\frac{1}{120} + \frac{1}{10}\right](x-\beta)^2 + 1$$

$$Q_1 = \left\{\left[(x-\beta)(1+2/x^2)\frac{1}{120} + \frac{1}{10}\cdot\frac{1}{x}\right](x-\beta) + \frac{1}{2}\right\}(x-\beta)$$

$$Q_2 = \left\{\left[(-\frac{1}{\beta}K_0(\beta) - K_1(\beta)(1+2/\beta^2))(x-\beta)\frac{1}{120} + \frac{1}{10}(K_0(\beta) + \frac{1}{\beta}K_1(\beta))\right](x-\beta) - \frac{1}{2}K_1(\beta)\right\}(x-\beta) + K_0(\beta)$$

Multiplying Equation (3.42) by Q_1 and Equation (3.43) by $-P_1$, then adding, we have

$$(3.44) \quad (Q_1P_0 - P_1Q_0)K_0(x) = Q_1P_2 - P_1Q_2 + Q_1R_\alpha - P_1R_\beta$$

Therefore

$$(3.45) \quad K_0(x) = \frac{Q_1P_2-P_1Q_2}{Q_1P_0-P_1Q_0} + \frac{QR_\alpha-P_1R_\beta}{Q_1P_0-P_1Q_0}$$

where $\frac{Q_1R_\alpha-P_1R_\beta}{Q_1P_0-P_1Q_0}$ is the error term involved in using this approximation.

Using this approximation we obtained the following results. Computations were made on an IBM 650 computor using floating point and therefore eight decimal accuracy.

As shown on Table I on the following page, it will be noticed that we used $\beta-\alpha = 1$, since a somewhat greater difference was tried with poor results. Even with $\beta-\alpha = 2$ the error was undesirably large. It was also found that the expansion very quickly became intolerably inaccurate outside the interval $[\alpha, \beta]$. We have therefore restricted ourselves to a difference of 1 and attempted to approximate only between the values of α and β.

Over the range of our table you will note the following. Between 1 and 2, we obtained no worse than 4 place accuracy. Between 3 and 4, we obtained no worse than 6 place accuracy. Over any of the other ranges covered in the table we obtained no worse than 5 place accuracy, although often as good as 7. Although not shown on the table, over the range of 11 to 12 we obtained no worse than 6 place accuracy.

TABLE I

Table of Approximations for $K_0^3(x)$

x	$K_0^3(x)$ approx	α	β	$K_0(x)$ actual (8 places)
1.1	0.36562712	1.0	2.0	0.36560239
1.3	0.27825417	1.0	2.0	0.27824765
1.5	0.21379998	1.0	2.0	0.21380556
1.7	0.16547156	1.0	2.0	0.16549632
1.9	0.12881401	1.0	2.0	0.12884598
3.1	0.030954774	3.0	4.0	0.03095471
3.3	0.024610657	3.0	4.0	0.02461063
3.5	0.019598893	3.0	4.0	0.01959890
3.7	0.015630615	3.0	4.0	0.01563066
3.9	0.012482250	3.0	4.0	0.01248232
5.1	0.003308130	5.0	6.0	0.00330831
5.3	0.0026591080	5.0	6.0	0.00265911
5.5	0.0021387084	5.0	6.0	0.00213871
5.7	0.0017212083	5.0	6.0	0.00172121
5.9	0.0013860020	5.0	6.0	0.00138601
7.1	0.00038173961	7.0	8.0	0.00038174
7.3	0.00030836232	7.0	8.0	0.00030836
7.5	0.00024917761	7.0	8.0	0.00024918
7.7	0.00020141991	7.0	8.0	0.00020142
7.9	0.00016286744	7.0	8.0	0.00016287
9.1	0.000045792001	9.0	10.0	0.00004579
9.3	0.000037095919	9.0	10.0	0.00003710
9.5	0.000030057883	9.0	10.0	0.00003006
9.7	0.000024360288	9.0	10.0	0.00002436
9.9	0.000019746700	9.0	10.0	0.00001975

In order that we might see what accuracy could be expected by this scheme, we investigate a typical error term.

$$R_{\alpha\beta} = \frac{Q_1 R_\alpha - P_1 R_\beta}{Q_1 P_0 - P_1 Q_0} \quad \text{where, with } n = 3$$

$$R_\alpha = \frac{(-1)^3 (3!)^2 (x-\alpha)^7}{6!7!} f^{(vii)}(\theta_1) \quad \text{where } \theta_1 \text{ lies between } x \text{ and } \alpha$$

$$R_\beta = \frac{(-1)^3 (3!)^2 (x-\beta)^7}{6!7!} f^{(vii)}(\theta_2) \quad \text{where } \theta_2 \text{ lies between } x \text{ and } \beta$$

Evaluating successive derivatives in the same manner as we did to obtain (3.41), we have

$$f^{(vii)}(\theta) = -\frac{3}{\theta}(1+17/\theta^2 + 120/\theta^4)K_0(\theta) - (1+15/\theta^2 + 192/\theta^9 + 720/\theta^6)K_1(\theta)$$

If, for example, we attempt to find the expected error where $\alpha = 3$, $\beta = 4$, and $x = 3.4$, we have the following data available for computation.

$Q_1 = -.29152318$ $\quad P_0 = 1.0161569$ $\quad (x-\alpha)^7 = .0016384$

$P_1 = .20533149$ $\quad Q_0 = 1.0354706$ $\quad (x-\beta)^7 = -.0279936$

Evaluating $f^{(vii)}(\theta)$ using the worst possible values for θ_1 and θ_2 we obtain

$$R_{\alpha,\beta} < .00000002$$

which implies that we may expect no worse than seven place accuracy with $x = 3.4$, which is exactly what we obtained.

By examining the derivative of $K_0(\theta)$ we see that for larger values of θ this derivative would tend to decrease in size and reduce the size of our error. This does not become immediately effective however, since our error term is not a function of $f^{(vii)}(\theta)$ alone. By examining our table we see that overall accuracy is better for larger values of x, although for particular values of x the error may be greater than for smaller values of x in lower ranges.

In order that one might see what effect on accuracy the taking of further derivatives in our approximation formula might have, we do the following.

Integrate successively as before the following integral

$$F(x) = \int_a^x f^{(ix)}(t)\, g(t)dt$$

where $g(t) = (x-t)^m(t-a)^n$ with $n = m = 4$ obtaining by a similar scheme as before

$$\frac{1}{1680}(x-a)^4 f^{iv}(x) - \frac{1}{84}(x-a)^3 f'''(x) + \frac{3}{28}(x-a)^2 f''(x) - \frac{1}{2}(x-a)f'(x) + f(a)$$

$$= \frac{1}{1680}(x-a)^4 f^{iv}(a) + \frac{1}{84}(x-a)^3 f'''(a) + \frac{3}{28}(x-a)^2 f''(a) + \frac{1}{2}(x-a)f'(a) + f(a)$$

Differentiating $K_0'''(x)$ we have

$$K_0^{iv}(x) = (1 + 3/x^2)(K_0(x) + 2/x\ K_1(x))$$

Substituting in the above expansion and a similar expansion for $(x-\beta)$, we have, using the same notation as before, the following.

$$P_0 = \left\{ \left[\left(\frac{1}{840}\ \frac{1}{x}(1+6/x^2)(x-a) + \frac{1}{84}(1+3/x^2) \right)(x-a) + 3/28 \cdot \frac{1}{x} \right](x-a) + \frac{1}{2} \right\}(x-a)$$

$$P_1 = \left\{ \left[\left(\frac{1}{1680}(1+7/x^2+24/x^4)(x-a) + \frac{1}{42}\ \frac{1}{x}(1+3/x^2) \right)(x-a) + \frac{3}{28}(1+2/x^2) \right](x-a) + \frac{1}{2} \cdot \frac{1}{x} \right\}(x-a) + 1$$

$$P_2 = K_0(a) - \frac{1}{2}(x-a)K_1(a) + \frac{3}{28}(x-a)^2(K_0(a) + \frac{1}{a}K_1(a)) - \frac{1}{84}(x-a)^3\left[\frac{1}{a}K_0(a) + (1+2/a^2)K_1(a)\right] + \frac{1}{1680}(x-a)^4(1+3/a^2)(K_0(a) + 2/a\ K_1(a))$$

and similar expressions for Q_0, Q_1, and Q_2.

Obtaining and solving the same equations as (3.42) and (3.43), we have

$$K_0(x) = \frac{Q_1P_2 - P_1Q_2}{Q_1P_0 - P_1Q_0} + \frac{QR_\alpha - P_1R_\beta}{Q_1P_0 - P_1Q_0}$$

We then have the same general expression as before only with different values for the components.

Using our new approximation and again computing our values to eight places on the IBM 650 computor we obtain the values found in Table II on the following page.

Observing Table II we see that overall accuracy has been improved on the average of 1 to 2 places. Although on particular values we improved by as much as three places and on some none at all. It might be well at this point to examine the ninth derivative of $K_0(x)$ since it would be a factor in our error term. Differentiating our seventh derivative twice we obtain

$$K_0^{ix}(\theta) = -\frac{4}{\theta}(1+\frac{23}{\theta^2}+\frac{675}{\theta^4}+\frac{5040}{\theta^6})K_0(\theta) - (1+\frac{26}{\theta^2}+\frac{569}{\theta^4}+\frac{10440}{\theta^6}+\frac{40320}{\theta^8})K_1(\theta)$$

We observe here that although this factor is getting larger, it is not getting larger at an extremely rapid rate. This is particularly true for large values of θ.

Although the results obtained by our latest approximation gives seven place accuracy or better in all ranges covered by our table except between 1 and 2, better accuracy could possibly be obtained by using higher order approximations. We shall not at this time attempt to demonstrate this possibility, however.

TABLE II

Table of Approximations for $K_0^4(x)$

x	$K_0^4(x)$ approx	α	β	$K_0^4(x)$ actual (8 places)
1.1	0.36560041	1.0	2.0	0.36560239
1.3	0.27824736	1.0	2.0	0.27824765
1.5	0.21380578	1.0	2.0	0.21380556
1.7	0.16549786	1.0	2.0	0.16549632
1.9	0.12884889	1.0	2.0	0.12884598
3.1	0.030954709	3.0	4.0	0.03095471
3.3	0.024610631	3.0	4.0	0.02461063
3.5	0.019598897	3.0	4.0	0.01959890
3.7	0.015630660	3.0	4.0	0.01563066
3.9	0.012482324	3.0	4.0	0.01248232
5.1	0.0033083103	5.0	6.0	0.00330831
5.3	0.0026591070	5.0	6.0	0.00265911
5.5	0.0021387086	5.0	6.0	0.00213871
5.7	0.0017212101	5.0	6.0	0.00172121
5.9	0.0013860050	5.0	6.0	0.00138601
7.1	0.00038173938	7.0	8.0	0.00038174
7.3	0.00030836222	7.0	8.0	0.00030836
7.5	0.00024917762	7.0	8.0	0.00024918
7.7	0.00020142004	7.0	8.0	0.00020142
7.9	0.00016286767	7.0	8.0	0.00016287
9.1	0.000045791979	9.0	10.0	0.00004579
9.3	0.000037095912	9.0	10.0	0.00003710
9.5	0.000030057885	9.0	10.0	0.00003006
9.7	0.000024360302	9.0	10.0	0.00002436
9.9	0.000019746725	9.0	10.0	0.00001975

As a concluding example of Bessel function approximation we will exhibit the results obtained by using this type of approximation for the Bessel function $K_1(x)$ using 3 derivatives.

While Table II on the following page only covers the range between 1 and 2, we observe that the accuracy although good to five places for some x drops off to as poor as 3 place accuracy in the upper half of the interval. This is not as good as the approximation for $K_0(x)$ over this range. The effect of the derivative on the error term of this approximation would be similar to the effect encountered in approximating $K_0(x)$. This is because the derivatives of these two functions are related in the following manner for any number of successive derivatives.

$$K_0'(x) = -K_1(x)$$
$$K_0''(x) = -K_1'(x)$$
$$K_0'''(x) = -K_1''(x)$$

As a concluding remark about this entire group of approximations for Bessel functions, we feel it necessary to point out some of its obvious shortcomings. As pointed out before it will not extrapolate and therefore restricts one to interpolation between the points of one's two expansions. Furthermore, these points must be such that $(\beta-\alpha) \cong 1$ in order that reasonable accuracy be maintained over the entire range. Last of all, it is extremely hard to predict at what range and at what point in that range your greatest accuracy will be obtained. It has been observed, however, that as good as or better accuracy is obtained near the midpoint of the range as opposed to any point near the endpoints.

TABLE III

Table of Approximations for $K_1^3(x)$

x	$K_1^3(x)$ approx	α	β	$K_1^3(x)$ actual (8 places)
1.1	0.50987719	1.0	2.0	0.50976003
1.2	0.43466297	1.0	2.0	0.43459239
1.3	0.37257468	1.0	2.0	0.37254750
1.4	0.32083388	1.0	2.0	0.32083590
1.5	0.27735495	1.0	2.0	0.27738780
1.6	0.24055654	1.0	2.0	0.24063391
1.7	0.20922836	1.0	2.0	0.20936249
1.8	0.18244227	1.0	2.0	0.18262310
1.9	0.15949405	1.0	2.0	0.15966015

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