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THE ENUMERATION PROBLEM ON NUMERICAL MONOIDS

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DEPARTMENT OF MATHEMATICS

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Contents

Abstract

Even though the problem of counting points with integer coordinates on a (rational) polytope has connections to sophisticated mathematical topics like Algebraic K-Theory, Fourier-Dedekind Sums, Heegard-Floer Homology, Symplectic Geometry and more, the basic (open) problem(s) are easy to describe. For example the following has been an open problem for over 60 years: If a, b and c are coprime positive integers how many ways are there of obtaining a given natural number n as a sum of (nonnegative integer) multiples of a, b and c? The problem is giving an effective computable formula for this number f(n). We are able to find this formula for a particular case. Furthermore, we use a variety of techniques to find the secondary asymptotic in any case, along with an effective computable formula for the McNugget Monoid and a couple of infinite families.

Chapter 1

Introduction

If chicken McNuggets come in boxes of 6, 9 or 20, what is the largest number of McNuggets that we can not get? More generally, given n coprime denominations, $a_1, a_2, \ldots, a_n$ (i.e. $\gcd(a_1, a_2, \ldots, a_n) = 1$), what is the the largest integer that can not be obtained as a sum of these (assuming an arbitrarily large supply of each)? This question is the Linear Diophantine Frobenius Problem (also referred to as the Frobenius coin exchange problem), named after Frobenius, who liked to ask this question in his lectures, in the late 1800's. However he never published anything regarding it. The first papers which referred to this as the Frobenius Problem were by Schur (a student of Frobenius) and Alfred Brauer (a student of Schur). They also named the largest number not attainable the Frobenius number. This question spawned other interesting, related questions:

- Genus: How many nonnegative integers can not be obtained? For 2 denominations, this was solved in 1882 by Sylvester [20]. This may have lead Frobenius to formulate his problem.

- Enumeration: For any $n \in \mathbb{N}$, how many ways can we express n in terms of the generators? Equivalently, the restricted partition problem: The number of partitions of n using only parts corresponding to the given denominations. This is the focus of this thesis.

The mathematical context of these questions is based on the concept of a numerical monoid: a cofinite (i.e., with finite compliment) subset of natural numbers, containing 0, which is closed under addition. Cofiniteness is a consequence of the fact that the denominations are coprime.

There are many other interesting related problems, such as the asymptotic distribution or the limit behavior of Frobenius numbers for 3 (or more) generators proposed by Arnold, see [10], [15]. These questions will not be addressed in this thesis.

The Enumeration Problem with 2 generators a and b, coprime positive integers, has a satisfactory solution given by the Popoviciu Theorem/Formula, for which we give a short geometric proof in Section 6.1:

$$\frac{n}{ab} - \left\{ \frac{tn}{a} \right\} - \left\{ \frac{sn}{b} \right\} + 1,$$

where $\{x\}$ is the fractional part of the real number x, and s, t are integers such that $sa + tb = 1$. This has been rediscovered several times in the literature, [8], [16], [19], [21]. However, an analogous result for 3 generators is not available. The example of the McNugget monoid with generators $6, 9$ and 20 will be analyzed in Section 6.2.2 below.

The modern approach to the Frobenius and the Genus Problems above is finding a short rational expression for the Hilbert series of the numerical monoid M:

$$H_M(z) := \sum_{m \in M} z^m.$$

(This is the Hilbert function of the algebra of regular functions on the monomial curve corresponding to the monoid M.) It is easy to see that $H_M(z)$ has a rational expression $\dfrac{P(z)}{1-z}$. However, the number of terms of $P(z)$ is exponential the data (to be explained below). So a modern solution to the Frobenius Problem involves making the denominator somewhat larger, while dramatically decreasing the number of terms of the numerator. When $H_M(z) = \dfrac{P(z)}{Q(z)}$, the Frobenius number of M is $\deg P - \deg Q$ and the genus of M is $\lim\limits_{z \longrightarrow 1} \left(\dfrac{1}{1-z} - H_M(z) \right)$, which can be computed by several applications of L'Hospital's rule.

A short rational expression for the Hilbert series of the monoid generated by coprime, positive integers a, b and c is given by the Morales-Denham formula below (Chapter 2) involving the positive integers R_a, R_b and R_c where R_a is the smallest positive multiple of a which can be expressed in terms of b and c (similarly, for R_b and R_c), [11]. There are 3 cases:

I. When $R_a a = R_b b = R_c c$, then

$$H_{<a,b,c>}(z) = \frac{1 - 2z^{R_a a} + z^{2R_a a}}{(1-z^a)(1-z^b)(1-z^c)}$$

II. When $R_a a = R_b b \neq R_c c$, then

$$H_{<a,b,c>}(z) = \frac{1 - z^{R_a a} - z^{R_c c} + z^{R_a a + R_c c}}{(1-z^a)(1-z^b)(1-z^c)}$$

III. When $R_a a$, $R_b b$, $R_c c$ are distinct, then

$$H_{<a,b,c>}(z) = \frac{1 - z^{R_a a} - z^{R_b b} - z^{R_c c} + z^{R_a a + s_{bc} c} + z^{R_c c + s_{ba} a}}{(1-z^a)(1-z^b)(1-z^c)},$$

where s_{ba}, s_{bc}, etc., are the (unique) natural numbers satisfying $R_a a = s_{ba} b + s_{ca} c$ and $R_c c = s_{ac} a + s_{bc} b$.

An analogous formula for 4 or more generators is not available. In fact, there are numerical monoids, generated by a, b, c and d, such that when the Hilbert series $H_M(z) = \dfrac{P(z)}{(1-z^a)(1-z^b)(1-z^c)(1-z^d)}$, the number of terms of $P(z)$ can be arbitrarily large.

If $f(n)$ is the answer to the Enumeration question (how many ways can we express n as a sum of nonnegative multiples of the generators $a_1, a_2, \ldots, a_d$), then

$$\sum_{n=0}^{\infty} f(n)z^n = \frac{1}{(1-z^{a_1})(1-z^{a_2})\dots(1-z^{a_d})}.$$

Using partial fractions, we can obtain a formula for $f(n)$ in terms of Fourier-Dedekind sums. This formula is useful for proving qualitative results like Ehrhart's Theorem stating that $f(n)$ is a *quasi-polynomial* in n (the coefficient are not constant, but periodic functions of n). However, this sum is not effectively computable because it has $a_1 + a_2 + \cdots + a_d$ terms, which is exponential in the data: the number of bits needed to express $a_1, a_2, \dots, a_d$ is $\log_2(a_1 a_2 \dots a_d)$.

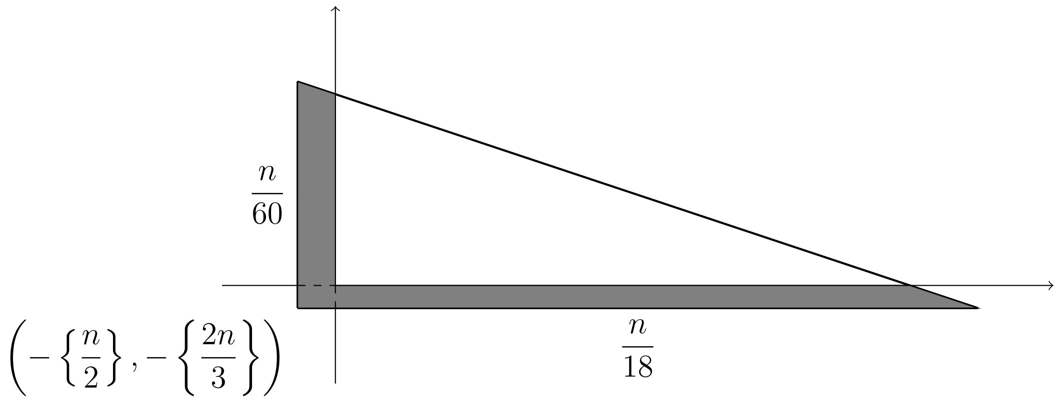
An algebraic/combinatorial approach to counting integer lattice points in polyhedra is provided by the theorems of Brion and Lawrence-Varchenko, [7], utilizing multivariable generating functions. In Chapter 4, we give a short geometric proof of these in 2 dimensions (the case that we use). We use these, partial fraction expansions (Chapter 5) and properties of the Discrete Fourier Transform (reviewed in Chapter 3), after applying a suitable geometric transformation. For instance, for the numerical monoid $< a, b, c >$, generated by coprime positive integers a, b and c, the Enumeration problem is equivalent to counting the integer lattice points on a rational triangle P scaled by n, in the plane: $f(n) = |nP \cap \mathbb{Z}^2|$.

If the vertices of P have integers coordinates then Pick's Theorem would provide the answer: $|nP \cap \mathbb{Z}^2| = \text{ Area}(P)n^2 + \frac{1}{2}|\partial P \cap \mathbb{Z}|n + 1$. This is the Ehrhart polynomial of P. Our geometric interpretation of the coefficients is that the leading coefficient of $f(n)$ is the *latticial area* while the linear coefficient is the *latticial length* of the boundary ∂P. These are the first two instances of the latticial measures we define in 2.1 below. They are some of the relevant invariants under the action of the group $\mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$.

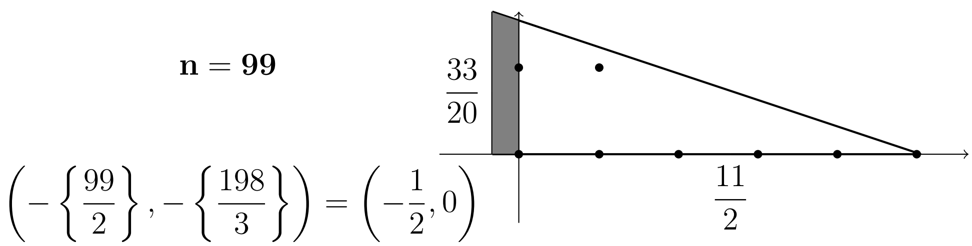
When the vertices have rational, but not integer, coordinates (which is always the case for the Enumeration Problem unless all the generators equal 1), other invariants (of pairs of faces of consecutive dimensions), which we call *margins* (see 2.1), are also relevant. This is already apparent in Popoviciu's Formula where the margins of the endpoints are $\left\{\frac{tn}{a}\right\}$ and $\left\{\frac{sn}{b}\right\}$, giving the latticial distance from an endpoint to the nearest lattice point.

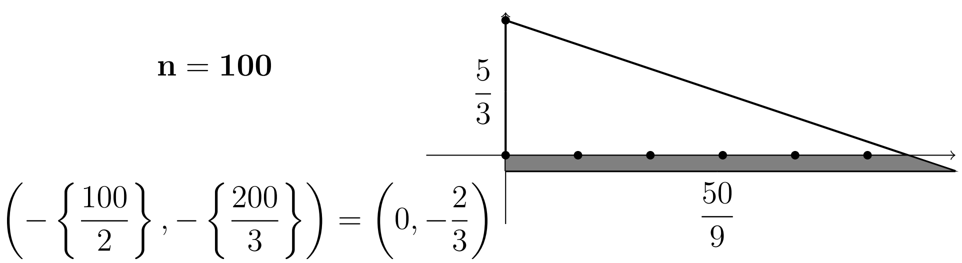
For example, via our geometric transformation, the McNugget Enumeration problem $f(n)$ is equivalent to the problem of counting points with integer coordinates (i.e., lattice points) in the unshaded triangle below in the plane (see Chapter 6).

Here the latticial lengths of the sides are $\frac{n}{18}$, $\frac{n}{60}$ and $\frac{n}{180}$ (for the hypotenuse). The margins of the sides are $\left\{\frac{2n}{3}\right\}$, $\left\{\frac{n}{2}\right\}$ and 0. In general, the latticial lengths of the side corresponding to the generators a and b is $\frac{n}{\operatorname{lcm}(a,b)}$ and its margin is $\left\{\frac{sn}{\gcd(a,b)}\right\}$ where s is the multiplicative inverse of c modulo $\gcd(a,b)$.



For the McNugget Problem, for instance, $f(99) = 8, f(100) = 7$.





The goal for this research is to find an effectively computable closed formula for the monoid $< a, b, c >$, which is better than the Fourier Dedekind Sum formula. By better, we mean a formula which is not exponential in the data. Ideally, a formula with a bounded number of terms, independent of the size of the generators a, b and c. There is currently an algorithm which is polynomial in the data, by Barvinok [2], in particular the number of steps grows with the size of the generators a, b and c; it is not bounded by an absolute constant.

We achieve this for Case I when $R_a a = R_b b = R_c c$:

$$f(n) = \frac{n^2}{2abc} + \frac{3 - 2\alpha(n)}{2R_a R_b R_c} n + \frac{(2 - \alpha(n))(1 - \alpha(n))}{2},$$

where $\alpha(n) := \left\{\frac{n}{R_a}\right\} + \left\{\frac{n}{R_b}\right\} + \left\{\frac{n}{R_c}\right\}$. We do not have an analogous solution for Cases II and III in general. We can do this for some infinite families, for instance, in 6.3.2 and 6.3.3 below. We also discuss the McNugget monoid in detail in 6.2.2 below.

Focusing on a numerical monoid M with generators a, b and c it is well known that $f_M(n) - \frac{n^2}{2abc}$ is $O(n)$, that is, the quadratic quasi-polynomial $f_M(n)$ has $\frac{1}{2abc}$ as its leading coefficient. (When M has generators $a_1, a_2, \ldots, a_k$ the leading term is $\frac{n^k}{(k-1)! a_1 a_2 \ldots a_k}$.) In Section 5.3 below we find the secondary asymptote, i.e., the (periodic) coefficient of n:

$$\frac{Aa + Bb + Cc - 2Aa\left\{\frac{s_1 n}{A}\right\} - 2Bb\left\{\frac{s_2 n}{B}\right\} - 2Cc\left\{\frac{s_3 n}{C}\right\}}{2abc},$$

where $\gcd(a, b) = C$, $\gcd(a, c) = B$, $\gcd(b, c) = A$ and $s_1 a + s_2 b + s_3 c = 1$. In Chapter 7, we conjecture that a similar formula is valid for an arbitrary polytope in any dimension.

Chapter 2

Basic Information

2.1 Notation, Terminology, etc.

Throughout this paper, we will be considering sets of the following type:

Definition 2.1.1. *A* **monoid** *is a set that is closed under an associative binary operation and has an identity element $I \in S$. Note that unlike a group, its elements need not have inverses. Another term that is used is semigroup, but when doing so, the statement about inverses is understood.*

Definition 2.1.2. *A* **numerical monoid** *is a set $M \subseteq \mathbb{N}$ such that*

1. *M is closed under addition*
2. *$\mathbb{N} \setminus M$ is finite*
3. *$0 \in M$*

Next note the following two facts about numerical monoids and their generators.

Lemma 2.1.3. *A set $\{a_1, a_2, \dots a_d\}$, of coprime positive integers generate a numerical monoid (we do not assume that this is the unique minimal set of generators).*

Lemma 2.1.4. *Any numerical monoid is finitely generated with a unique minimal set of generators $< a_1, a_2, \dots, a_d >$ where $a_1 = \min\{a \in M \mid a > 0\}$ and $a_k = \min\{a \in M \mid a \notin < a_1, a_2, \dots, a_{k-1} >\}$. Since a_i's are in different congruence classes $\mod a_1$ we see that $d \leq a_1$.*

Now lets define some characteristics, to a given numerical monoid M, or its compliment in $\mathbb{N}$:

Definition 2.1.5. *Given a monoid $M =< a_1, a_2, \dots, a_d >$, any number $a \in \mathbb{N}$ such that $a \notin M$ will be referred to as a* **gap**. *The* **Frobenius Number** *is the largest gap, that is the largest integer not in M. The total number of gaps is called the* **genus** *of M.*

Definition 2.1.6. *The* **generating function** *of the sequence $b_0, b_1, \dots$ is the formal power series*

$$f(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The generating function of the monoid $M =< a_1, a_2, \dots a_d >$ is:

$$F_M(z) = \sum_{n=0}^{\infty} f_M(n) z^n = \frac{1}{(1 - z^{a_1})(1 - z^{a_2}) \dots (1 - z^{a_d})},$$

(when order does not matter) where $f_M(n)$ is the number of ways to obtain n as a linear combination of $a_1, a_2, \ldots, a_d$, with nonnegative integer coefficients.

Definition 2.1.7. *Given a monoid M, the* **Hilbert series** *of M, denoted as $H_M(z)$ is defined as*

$$H_M(z) = \sum_{m \in M} z^m.$$

An example of this is if $M = \mathbb{N}$, then

$$H_{\mathbb{N}}(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}.$$

In later chapters, we will be using linear transformations in an effort to move triangles from a 3-space (x, y, z-plane) into a 2-space (u, v, w-plane, however $w = n$, a constant). Before we describe the specific transformation that we will use, we need to provide some notation:

Notation 2.1.8. *For the numerical monoid $< a_1, a_2, \ldots a_d >$, let $R_i a_i$ denote that smallest positive multiple of a_i which is in $< a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d >$, for all $1 \leq i \leq d$. Note that when $a_1, a_2, \ldots a_d$ are coprime, then the R_i's have to be pairwise coprime.*

Now we can define the relations and matrix used in our linear transformation:

Definition 2.1.9. *Given $a, b, c \in \mathbb{Z}$, the* **Johnson relations** *(Selmer Johnson* 1960*) are as follows:*

$$\begin{aligned} R_a a &= s_{ab}b + s_{ac}c \quad \textit{or} \quad 0 = -R_a a + s_{ab}b + s_{ac}c \\ R_b b &= s_{ba}a + s_{bc}c \quad \textit{or} \quad 0 = -R_b b + s_{ba}a + s_{bc}c \\ R_c c &= s_{ca}a + s_{cb}b \quad \textit{or} \quad 0 = -R_c c + s_{ca}a + s_{cb}b \end{aligned}$$

where $s_{ab}, s_{ac}, s_{ba}, s_{bc}, s_{ca}, s_{cb}$ *are in* $\mathbb{N}$*. If we make these coefficients the entries of columns in a matrix, we get the following* 3×3 *matrix, which will be referred to as the* **Johnson Matrix***, and computation:*

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -R_a & s_{ba} & s_{ca} \\ s_{ab} & -R_b & s_{cb} \\ s_{ac} & s_{bc} & -R_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

We will be using matrices similar to the Johnson Matrices. To construct a matrix for a **Johnson Transformation**, we need to consider any 2 (distinct) Johnson relations along with the following equation:

$$t_a a + t_b b + t_c c = \gcd(a, b, c)$$

where t_a, t_b, t_c are in $\mathbb{Z}$. We will use the coefficients of these three equations to create the following 3×3 matrix, which we will denote as A:

$$A = \begin{bmatrix} -R_a & s_{ba} & t_a \\ s_{ab} & -R_b & t_b \\ s_{ac} & s_{bc} & t_c \end{bmatrix},$$

such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where u, v, w are the new coordinates for $\mathbb{R}^3$.

Which matrix set up we will be using will depend upon which of the following 3 possibilities is applicable:

1. $R_a a, R_b b$ and $R_c c$ are all distinct,
2. two of these products are equal, for example $R_a a = R_b b \neq R_c c$, or
3. $R_a a = R_b b = R_c c$.

The following as a well-known result which answers the topic of this paper for a 1-dimensional polytope (a line segment or edge) in 2-space:

Theorem 2.1.10 (Pick's Theorem)**.** *Given a polygon, P, whose vertices are lattice points, if we denote the number of lattice points on the boundary of P as $|\partial P \cap \mathbb{Z}^2|$ and the number of lattice points on the interior of P as $|\mathring{P} \cap \mathbb{Z}^2|$, then we have that*

$$Area(P) = |\mathring{P} \cap \mathbb{Z}^2| + \frac{1}{2}|\partial P \cap \mathbb{Z}^2| - 1.$$

Note 2.1.11. *For the purposes of this topic, we will be considering a altered form of Pick's Thoerem, which writes the number of lattice points in a given polytope in terms of its area and the lattice points on its boundary:*

$$\begin{aligned} Area(P) &= |\mathring{P} \cap \mathbb{Z}^2| + \frac{1}{2}|\partial P \cap \mathbb{Z}^2| - 1 \\ Area(P) &= |P \cap \mathbb{Z}^2| - \frac{1}{2}|\partial P \cap \mathbb{Z}^2| - 1 \\ Area(P) + \frac{1}{2}|\partial P \cap \mathbb{Z}^2| + 1 &= |P \cap \mathbb{Z}^2| \end{aligned}$$

Since we will be counting lattice points in a polytope, we will be utilizing a length/measure which is not the Euclidean length/measure. It is defined as follows:

Definition 2.1.12. *Let P be a k-dimensional compact polytope. It's k-dimensional* **latticial measure**, $\mu_{\mathbb{Z}}^k$ *is*

$$\mu_{\mathbb{Z}}^k(P) = \lim_{n\to\infty} \frac{1}{n^k} \mid P \cap \frac{1}{n}\mathbb{Z}^d \mid = \lim_{n\to\infty} \frac{1}{n^k} \mid nP \cap \mathbb{Z}^d \mid.$$

Note that this is also the leading term of the Ehrhart quasi-polynomial as

$$\mathrm{Eh}(k) = \mid kP \cap \mathbb{Z}^d \mid = \mu_{\mathbb{Z}}^k(P)n^k + \dots.$$

Definition 2.1.13. *Given a polytope, P, the pair $F \subset G$, which are k and $k+1$ dimensional faces of P, respectively, have an associated* **margin** *in $[0, 1)$ which is*

$$\min\{\mu_{\mathbb{Z}}^1(x, y) \mid x \in F, y \in \overline{G} \cap \mathbb{Z}^d\}.$$

Note that there is an element of $H = \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$ which will send $\overline{F}$ and G into k and $k+1$ dimensional flats parallel to coordinate subspaces. Since H preserves all latticial measures and margins, the margin of F in G now is the Euclidian distance between F and $\overline{G} \cap \mathbb{Z}$.

Definition 2.1.14. *A* **polyhedron** *(in the Euclidean space $\mathbb{R}^d$) is the intersection of finitely many closed half spaces (given by linear inequalities).*

Definition 2.1.15. *A* **polytope** *is the convex hull of finitely many points $\{v_0, v_1, \ldots, v_k\}$, denoted hull$\{v_0, v_1, \ldots v_k\}$ in $\mathbb{R}^d$, equilvalently it is a compact polyhedron (by Fourier-Motzkin elimination [13]), where*

$$\begin{aligned} \textit{hull}\{v_0, v_1, \ldots, v_k\} &= \left\{\sum t_i v_i \mid t_i \geq 0, \sum t_i = 1\right\} \\ &\subset \textit{flat}\{v_0, v_1, \ldots v_k\} \\ \textit{where flat}\{v_0, v_1, \ldots v_k\} &:= \left\{\sum t_i v_i \mid \sum t_i = 1\right\} \end{aligned}$$

Note the following:

- $\hat{\text{hull}}\{v_0, v_1, \ldots v_k\} := \text{flat}\{v_0, v_1, \ldots v_k\}$
- $\text{flat}\{v_0, v_1, \ldots v_k\} = v_0 +$ span$\{v_1 - v_0, v_2 - v_0, \ldots v_k - v_0\}$
- dim hull$\{v_0, v_1, \ldots v_k\} :=$ dim flat$\{v_0, v_1, \ldots v_k\} =$ dim span$\{v_1 - v_0, v_2 - v_0, \ldots v_k - v_0\}$

Definition 2.1.16. *A* **rational polyhedron** *is given by linear inequalities with integer coefficients.*

Definition 2.1.17. *Given a rational polytope P, a k-dimensional face F of P is called* **latticial** *if the k-space F is contained in, also contains (infinitely many) lattice points, i.e. if $F \in T$, where T is a k-flat, and $T \cap \mathbb{Z}^k$ is nonempty, then F is latticial.*

For the purposes of simplifying expressions, we will use the following notation:

Definition 2.1.18. *The* **q-analog of** n*, denoted as* n_q *is the polynomial*

$$n_q = 1 + q + q^2 + \cdots + q^{n-1},$$

where $n \in \mathbb{Z}_{>0}$.

2.2 Morales - Denham Formula

Consider a monoid M with arbitrary many generators, i.e. $M =< a_1, a_2, \ldots, a_d >$, then the Hilbert series can be written as:

$$H_M(z) = \frac{P_M(z)}{(1 - z^{a_1})(1 - z^{a_2}) \ldots (1 - z^{a_d})},$$

where $P_M(z)$ is a polynomial. However, when $d \geq 4$, there is no bound on the number of terms for $P_M(z)$. On the other hand, for $d \leq 3$, the formula for $P_M(z)$ and in each case the polynomial is "short". For the purposes of this paper, we will be focusing on the cases where $d = 3$. So consider the monoid M with generators a, b and c, i.e. $M =< a, b, c >$. Then the Hilbert Series can be written as:

$$H_M(z) = \frac{P_M(z)}{(1 - z^a)(1 - z^b)(1 - z^c)}$$

where $P_M(z)$ depends on the relationship between $R_a a, R_b b$ and $R_c c$ listed above:

- When $R_a a, R_b b$ and $R_c c$ are all distinct, then

$$\begin{aligned} P_M(z) &= (1 - z^{R_a a})(1 - z^{R_c c}) - z^{R_b b}(1 - z^{s_{ca} a})(1 - z^{s_{ac} c}) \\ &= 1 - z^{R_a a} - z^{R_b b} - z^{R_c c} + z^{R_b b + s_{ac} c} + z^{R_b b + s_{ca} b} \end{aligned}$$

- When $R_a a = R_b b \neq R_c c$, then $P_M(z) = (1 - z^{R_a a})(1 - z^{R_c c}) = 1 - z^{R_a a} - z^{R_c c} + z^{R_a a + R_c c}$

- When $R_a a = R_b b = R_c c$, then $P_M(z) = (1 - z^{R_a a})^2 = 1 - 2z^{R_a a} + z^{2R_a a}$.

Definition 2.2.1. **Simplicial complex** Φ *on a (finite) set* X *is a* **hereditary** *collection of subsets of* X*, i.e.* $A \in \Phi$ *and* $B \subseteq A$*, then* $B \in \Phi$*. From the standpoint of a numerical monoid* $M =< a_1, a_2, \ldots, a_d >$ *where* $a_i \in \mathbb{Z}_{>0}$ *and* $\gcd(a_1, \ldots, a_d) = 1$*, for any* $n \in \mathbb{N}$*,*

$$\Phi_n := \left\{ \sigma \in \{a_1, \ldots, a_d\} \mid \left(n - \sum_{a_i \in \sigma} \right) \in M \right\}.$$

Remark 2.2.2. *The* **Reduced Euler Characteristic** *of a simplicial complex is*

$$\bar{X} = \sum_{\sigma \in \Phi} (-1)^{|\sigma|}$$

where $\dim \sigma = |\sigma| - 1$.

2.2.1 $R_a a = R_b b$

In this case, note that we have that $H_M(z) = \dfrac{(1 - z^{R_a a})(1 - z^{R_c c})}{(1 - z^a)(1 - z^b)(1 - z^c)}$. Thus since $(1 - z^{R_a a}) = (1 - z^a)(1 + z^a + \cdots + z^{(R_a - 1)a})$, then we have

$$H_M(z) = \frac{(1 - z^{R_a a})(1 - z^{R_c c})}{(1 - z^a)(1 - z^b)(1 - z^c)} = \frac{(1 + z^a + \cdots + z^{(R_a - 1)a})(1 + z^c + \cdots + z^{(R_c - 1)c})}{1 - z^b}.$$

Note that

$$\begin{aligned}
|\mathbb{N}\backslash M| &= \lim_{z\to 1}\left(\frac{1}{1-z} - H_M(z)\right) \\
&= \lim_{z\to 1}\left(\frac{1}{1-z} - \frac{(1+z^a+\cdots+z^{(R_a-1)a})(1+z^c+\cdots+z^{(R_c-1)c})}{1-z^b}\right) \\
&= \lim_{z\to 1}\left[\frac{(1+z+\cdots+z^{b-1})}{1-z^b}\right. \\
&\qquad \left. \frac{-(1+z^a+\cdots+z^{(R_a-1)a})(1+z^c+\cdots+z^{(R_c-1)c})}{1-z^b}\right]
\end{aligned}$$

But this limit has to exists as it is counting the number of gaps for the monoid M, hence $b - R_aR_c = 0$ or $b = R_aR_c$. Since a and b are symmetrical, we also have that $a = R_bR_c$. Further,

$$\begin{aligned}
R_c c &= s_{ca}a + s_{cb}b \\
R_c c &= R_bR_cs_{ca} + R_aR_cs_{cb} \\
c &= R_bs_{ca} + R_as_{cb},
\end{aligned}$$

since R_c does not equal 0.

When $R_aa = R_bb = R_cc$ the converse of this result is true as well:

Proposition 2.2.3. *$R_aa = R_bb = R_cc \iff a = R_bR_c$, $b = R_aR_c$ and $c = R_aR_b$ for some (arbitrary) pairwise coprime positive integers R_a, R_b and R_c.*

2.2.2 R_aa, R_bb and R_cc are distinct

Lemma 2.2.4. *If R_aa, R_bb and R_cc are distinct, then all $s_{ij} > 0$, where $i, j \in \{a, b, c\}$ and $i \neq j$.*

Proof. Assume to the contrary that $s_{ab} = 0$. Then

$$0 < R_a a = s_{ab} b + s_{ac} c = s_{ac} c$$

$$\implies \quad s_{ac} \geq R_c.$$

If $R_c = s_{ac}$ then we have $R_a a = R_c c$, which is a contradiction. Thus $s_{ac} > R_c$.

So we have

$$R_a a = s_{ac} c = R_c c + (s_{ac} - R_c) c$$

$$R_a a = s_{ca} a + s_{cb} b + (s_{ac} - R_c) c. \text{ Then}$$

$$0 < (R_a - s_{ca}) a = s_{cb} b + (s_{ac} - R_c) c$$

So $s_{ca} = 0$ or we would have a contradiction of the definition R_a. Now

$$R_c c = s_{ca} a + s_{cb} b = s_{cb} b.$$

So we can say $s_{cb} > R_b$. So

$$\begin{aligned} R_c c &= R_b b + (s_{cb} - R_b) b \\ &= s_{ba} a + s_{bc} c + (s_{cb} - R_b) b. \text{ Then} \\ 0 < (R_c - s_{bc}) c &= s_{ba} a + (s_{cb} - R_b) b \end{aligned}$$

So. again, $s_{bc} = 0$, otherwise we have a contradiction. Now

$$R_b b = s_{bc} c + s_{ba} a = s_{ba} a.$$

So, now we can say $s_{ba} > R_a$. Hence we have

$$R_a a = s_{ac} c > R_c c = s_{cb} b > R_b b = s_{ba} a > R_a a,$$

which can not be true, hence we have a contradiction. Therefore, $s_{ab} \neq 0$. $\square$

Lemma 2.2.5. $R_a > s_{ba}$

Proof. Assume to the contrary that $s_{ba} \geq R_a$. Then we have

$$\begin{aligned} R_b b &= s_{ba} a + s_{bc} c \\ &= R_a a + (s_{ba} - R_a) a + s_{bc} c \\ &= s_{ab} b + s_{ac} c + (s_{ba} - R_a) a + s_{bc} c \end{aligned}$$

Then, since $s_{ab} > 0$ by the previous Lemma, we have

$$R_b b > (R_b - s_{ab}) b = (s_{ac} + s_{bc}) c + (s_{ba} - R_a) a.$$

However, note that by the previous Lemma and our assumption, we have a positive multiple of b written in terms of a and c that is strictly smaller than $R_b b$, which is a contradiction of the definition of R_b. Therefore, $R_a > s_{ba}$. (By symmetry, we have similar inequalities involving R_b and R_c. $\square$

Note 2.2.6. $\Phi_{R_a a + s_{bc} c} = 2^{\{a,b,c\}} - \{a, b, c\}$.

Proof. Let $n = R_a a + s_{bc} c$. See that

- $R_a, s_{bc} > 0 \implies n - a - c = (R_a - 1) a + (s_{bc} - 1) c \implies \{a, c\} \in \Phi_n$;
- $R_a a + s_{bc} c = s_{ab} b + s_{ac} c + s_{bc} c \implies \{b, c\} \in \Phi_n$, since s_{ab}, s_{ac} and $s_{bc} > 0$;
- $R_a a + s_{bc} c > s_{ba} a + (R_a - s_{ba}) a + s_{bc} c = (R_a - s_{ba}) a + R_b b \implies \{a, b\} \in \Phi_n$.

Therefore, all proper subsets of $\{a, b, c\}$ are contained in Φ_n. But is $\{a, b, c\} \in \Phi_n$, i.e there exists $k, l, m \in \mathbb{Z}_{>0}$ such that $n = ka + lb + mc$? Assume this is the case. Then

1. $s_{bc} > m$, otherwise, $s_{bc} \leq m$, so

$$
\begin{aligned}
n = R_a a + s_{bc} c &= ka + lb + mc \\
R_a a &= ka + lb + (m - s_{bc})c, \text{ then} \\
R_a a > (R_a - k)a &= lb + (m - s_{bc})c > 0,
\end{aligned}
$$

which contradicts the definition of R_a.

2. $R_a - s_{ba} > k$, otherwise $R_a - s_{ba} \leq k$, so

$$
\begin{aligned}
n = R_a a + s_{bc} c &= ka + lb + mc \\
R_a a + (R_b b - s_{ba} a) &= ka + lb + mc \\
R_b b + (R_a - s_{ba})a &= ka + lb + mc, \text{ then} \\
R_b b > (R_b - l)b &= (k - (R_a - s_{ba}))a + mc > 0
\end{aligned}
$$

which contradicts the definition of R_b.

Now $R_a a + s_{bc} c = ka + lb + mc$ then

$$
0 < (R_a - k)a + (s_{bc} - m)c = lb \implies l \geq R_b.
$$

Now note that since

$$
\begin{aligned}
& n = R_b b = (R_a - s_{ba})a = kl + lb + mc \\
\implies \quad & R_a a > (R_a - s_{ba} - k)a = (l - R_b)b + mc.
\end{aligned}
$$

So we have a contradiction to the definition of R_a. Hence $\{a, b, c\} \in \Phi_n$. $\square$

Let $M =< a, b, c >$ where $R_a a, R_b b$ and $R_c c$ are distinct. If $\Phi_n = 2^{\{a_1, \ldots, a_d\}} - \{a, b, c\}$, then

$$
\begin{aligned}
n \in \{R_a a + s_{bc} c, R_c c + s_{ba} a\} &= \{R_b b + s_{ac} c, R_c c + s_{ab} b\} \\
&= \{R_a a + s_{cb} b, R_b b + s_{ca} a\}
\end{aligned}
$$

Suppose $R_c c + s_{ba}a = R_c c + s_{ab}b$. This implies that $s_{ba} \geq R_a$ which is a contradiction. So

$$\begin{aligned}
R_c c + s_{ba}a &= R_b b + s_{ac}c \\
s_{ca}a + s_{cb}b + s_{ba}a &= s_{ba}a + s_{bc}c + s_{ac}c \\
s_{ca}a + s_{cb}b &= s_{bc}c + s_{ac}c \\
R_c c &= s_{bc}c + s_{ac}c \\
R_c &= s_{bc} + s_{ac}.
\end{aligned}$$

Since, in this case, a, b and c are symmetric, we also have that

- $R_b = s_{ab} + s_{cb}$;
- $R_a = s_{ba} + s_{ca}$.

In this case, following the same technique as the previous section,

$$\begin{aligned}
H_M(z) &= \frac{(1 - z^{R_a a})(1 - z^{R_c c}) - z^{R_b b}(1 - z^{s_{ca}a})(1 - z^{s_{ac}c})}{(1 - z^a)(1 - z^b)(1 - z^c)} \\
&= \frac{(1 + z^a + \ldots z^{(R_a - 1)a})(1 + z^c + \cdots + z^{(R_c - 1)c})}{1 - z^b} \\
&\quad - \frac{z^{R_b b}(1 + z^a + \cdots + z^{(s_{ca} - 1)a})(1 + z^c + \cdots + z^{(s_{ac} - 1)c})}{1 - z^b}
\end{aligned}$$

Thus we have that

$$
\begin{aligned}
|\mathbb{N}\backslash M| &= \lim_{z\to 1}\left(\frac{1}{1-z} - H_M(z)\right) \\
&= \lim_{z\to 1}\left[\frac{1}{1-z} - \frac{(1+z^a+\dots z^{(R_a-1)a})(1+z^c+\dots+z^{(R_c-1)c})}{1-z^b}\right. \\
&\quad \left. - \frac{z^{R_b b}(1+z^a+\dots+z^{(s_{ca}-1)a})(1+z^c+\dots+z^{(s_{ac}-1)c})}{1-z^b}\right] \\
&= \lim_{z\to 1}\left[\frac{1+z+\dots+z^{b-1}}{1-z^b}\right. \\
&\quad - \frac{(1+z^a+\dots z^{(R_a-1)a})(1+z^c+\dots+z^{(R_c-1)c})}{1-z^b} \\
&\quad \left. + \frac{z^{R_b b}(1+z^a+\dots+z^{(s_{ca}-1)a})(1+z^c+\dots+z^{(s_{ac}-1)c})}{1-z^b}\right]
\end{aligned}
$$

Once again, this counts the genus of the monoid, thus the limit exists, hence the numerator has to equal 0, i.e. $b - R_a R_c + s_{ca}s_{ac} = 0$ or $b = R_a R_c - s_{ca}s_{ac}$. Since a, b and c are all symmetric, we also have

- $a = R_b R_c - s_{cb}s_{bc}$
- $c = R_a R_b - s_{ab}s_{ba}$

2.3 The Johnson Transformation

Here we will be considering 2 cases:

1. $R_a a = R_b b$; and
2. $R_a a, R_b b$ and $R_c c$ are distinct

Case 1: $R_a a = R_b b$

In this case, after picking the 2 distinct Johnson relations, our matrix A will look like the following:

$$A = \begin{bmatrix} s_{ca} & -R_a & t_a \\ s_{cb} & R_b & t_b \\ -R_c & 0 & t_c \end{bmatrix}.$$

Consider the determinant of our matrix

$$\begin{aligned} \det(A) &= \begin{vmatrix} s_{ca} & -R_a & t_a \\ s_{cb} & R_b & t_b \\ -R_c & 0 & t_c \end{vmatrix} \\ &= t_a \begin{vmatrix} s_{cb} & R_b \\ -R_c & 0 \end{vmatrix} - t_b \begin{vmatrix} s_{ca} & -R_a \\ -R_c & 0 \end{vmatrix} + t_c \begin{vmatrix} s_{ca} & -R_a \\ s_{cb} & R_b \end{vmatrix} \\ &= t_a(R_bR_c) + t_b(R_aR_c) + t_c(s_{ca}R_b + s_{cb}R_a) \\ &= t_a a + t_b b + t_c c = 1. \end{aligned}$$

Therefore, in this case $A \in SL_2(\mathbb{Z})$.

Case 2: $R_a a, R_b b$ and $R_c c$ are distinct

In this case, after picking the 2 distinct Johnson relations, our matrix A will look like the following:

$$A = \begin{bmatrix} s_{ba} & s_{ca} & t_a \\ -R_b & s_{cb} & t_b \\ s_{bc} & -R_c & t_c \end{bmatrix}.$$

Consider the determinant of our matrix

$$\begin{aligned}
\det(A) &= \begin{vmatrix} s_{ba} & s_{ca} & t_a \\ -R_b & s_{cb} & t_b \\ s_{bc} & -R_c & t_c \end{vmatrix} \\
&= t_a \begin{vmatrix} -R_b & s_{cb} \\ s_{bc} & -R_c \end{vmatrix} - t_b \begin{vmatrix} s_{ba} & s_{ca} \\ s_{bc} & -R_c \end{vmatrix} + t_c \begin{vmatrix} s_{ba} & s_{ca} \\ -R_b & s_{cb} \end{vmatrix} \\
&= t_a(R_bR_c - s_{cb}s_{bc}) + t_b(s_{ba}R_c + s_{bc}s_{ca}) + t_c(s_{ba}s_{cb} + R_bs_{ca}) \\
&= t_aa + t_b(s_{ba}R_c + s_{bc}s_{ca}) + t_c(s_{ba}s_{cb} + R_bs_{ca}).
\end{aligned}$$

But note that in this case

$$\begin{aligned}
s_{ba}R_c + s_{bc}s_{ca} &= (R_a - s_{ca})R_c + s_{ca}s_{bc} \\
&= R_aR_c - s_{ca}R_c + s_{ca}s_{bc} \\
&= R_aR_c - s_{ca}(s_{ac} + s_{bc}) + s_{ca}s_{bc} \\
&= R_aR_c - s_{ca}s_{ac} - s_{ca}s_{bc} + s_{ca}s_{bc} \\
&= R_aR_c - s_{ca}s_{ac} = b,
\end{aligned}$$

$\because R_a = s_{ba} + s_{ca}$ or $s_{ba} = R_a - s_{ca}$ and $R_c = s_{ac} + s_{bc}$ in this case, as previously shown. Similarly, $s_{ba}s_{cb} + R_bs_{ca} = c$. Therefore, $\det(A) = t_aa + t_bb + t_cc = 1$. So once again, in this case, $A \in SL_2(\mathbb{Z})$.

Chapter 3

Discrete Fourier Transform

3.1 Basic Information and Properties

Let $b \in \mathbb{Z}_{>0}$. Then define

$$V_b := \{f : \mathbb{N} \text{ (or } \mathbb{Z}) \longrightarrow \mathbb{C} | f(n+b) = f(n), \forall n \in \mathbb{N}\},$$

i.e. the set of all b-periodic functions form $\mathbb{N}$ (or $\mathbb{Z}$) into $\mathbb{C}$. This is a vector space over $\mathbb{C}$ because $f + g$ and λf is in this set $\forall \lambda \in \mathbb{C}$ and $f, g \in V_b$. For convenience, we can also write a b-periodic function as $f(n) = f(n \text{ MOD } b)$ where $n \text{ MOD } b := b \left\{ \frac{n}{b} \right\}$. Now let's define ω_b where $\omega_b = e^{\frac{i2\pi}{b}}$, a primitive b^{th} root of uinity. Then note that

$$\omega_b^{n+b} = \omega_b^n \omega_b^b = \omega_b^n \text{ and } \omega_b^{k(n+b)} = \omega_b^{kn} \omega_b^{kb} = \omega_b^{kn},$$

and hence $f(n) := \omega_b^n \in V_b$. Futhermore, $\{1, \omega_b^n, \omega_b^{2n}, \ldots, \omega_b^{(b-1)n}\} = \{\omega_b^k\}_{k=0}^{b-1}$ is a basis for V_b, which is called the Fourier Basis. Note that this basis is also orthonormal with respect to the Hermitian Inner Product and hence it is linearly independent.

Now consider $f, g, h \in V_b$ and let define the **Hermitian Inner Product** $< f, g > := \frac{1}{b}\sum_{n=0}^{b-1} f(n)\bar{g(n)}$, where $\bar{g(n)}$ is the complex conjugate of $g(n)$.

Now, lets list some properties of the Hermitian Inner Product:

1. $< g, f > = \frac{1}{b}\sum_{n=0}^{b-1} g(n)\bar{f(n)} = \bar{< f, g >}$

2. $< f + g, h > = < f, h > + < g, h >$ and $< \lambda f, g > = \lambda < f, g >$

3. $< f, \lambda h > = \bar{\lambda} < f, h >$

4. $< f, f > = \frac{1}{b}\sum |f(n)|^2 \geq 0$ and $= 0$ iff $f \equiv 0$.

For all $f \in V_b$, $f(n) = c_0 1 + c_1\omega^n + c_2\omega^{2n} + \cdots + c_{b-1}\omega^{(b-1)n}$ where $\omega = \omega_b = e^{\frac{i2\pi}{b}}$.

$$
\begin{aligned}
< f, \omega^{kn} > &= c_0 < 1, \omega^{kn} > + c_1 < \omega^n, \omega^{kn} > + \cdots + c_{b-1} < \omega^{(b-1)n}, \omega^{kn} > \\
&= c_k < \omega^{kn}, \omega^{kn} > = c_k \text{ are called } \hat{f}(k) \text{ where } k = 0, 1, \ldots, b-1
\end{aligned}
$$

So $f(n) = \sum_{k=0}^{b-1} \hat{f}(k)\omega^{kn}$. Further, $f \longrightarrow \hat{f}$ is called the Discrete Fourier Transform (DFT). Therefore, for any $f \in V_b, f(n) = \sum_{k=0}^{b-1} \hat{f}(k)\omega_b^{kn}$ and $\hat{f}(k) = < f, \omega_b^{kn} > = \frac{1}{b}\sum_{n=0}^{b-1} f(n)\bar{\omega}_b{}^{kn}$. Finally note that $\hat{f}(0) = \frac{1}{b}\sum_{n=0}^{b-1} f(n) =$ average over 1 period of f or average(f). A basis for b-periodic ($\mathbb{C}$-valued) functions with average 0 is $\{\omega_b^k\}_{k=1}^{b-1}$.

Note 3.1.1. *Consider the following:* $1 - z^b = (1-z)(1+z+\cdots+z^{b-1})$*, hence* $0 = 1 - (\omega_b^k)^b = (1-\omega_b^k)(1+\omega_b+\omega_b^2+\cdots+\omega_b^{b-1})$ *and since* $1-\omega_b^k \neq 0$*, then* $1+\omega_b+\omega_b^2+\cdots+\omega_b^{b-1} = 0 \implies \dfrac{1-\omega_b^{kb}}{1-\omega_b^k} = 1+\omega_b+\omega_b^2+\cdots+\omega_b^{b-1}$.

Now lets look at three operations on V_b that will be used:

1. The first property we will discuss is called **translation**, which will be denoted as T_a, where $a \in \mathbb{Z}$. This is a linear property such that

$$\begin{aligned} T_a \quad &: \quad V \longrightarrow V \\ & \qquad f(n) \mapsto f(n-a) \text{ or} \\ & \qquad (T_a f)(n) = f(n-a) \end{aligned}$$

Note that this property is invertible as $T_a \circ T_{-a} = id_V = T_{-a} \circ T_a$. The next question that we need to ask is $\hat{T_a} f(k) = ?$

$$\begin{aligned} \hat{T_a} f(k) &= < T_a f, \omega^{kn} > \\ &= \frac{1}{b}\sum_{n=0}^{b-1} f(n-a)\bar{\omega}^{kn} \\ &= \frac{1}{b}\sum_{m=0}^{b-1} f(m)\bar{\omega}^{k(m+a)}, \text{ where we let } m = n-a \\ & \quad \text{(sum starts at } m=0 \text{ since } f \text{ is } b\text{-periodic)} \\ &= < f(m)\bar{\omega}^{ka}, \omega^{km} > \\ &= \bar{\omega}^{ka} < f(m), \omega^{km} > \\ &= \bar{\omega}^{ka}\hat{f}(k) \end{aligned}$$

Thus we have that $\hat{T_a} f(k) = \bar{\omega}^{ka}\hat{f}(k)$.

2. The next property we will discuss is called **dilation**, which will be denoted as U_s, where $s \in \mathbb{Z}$ and $\gcd(b, s) = 1$, i.e. $\exists a, t \in \mathbb{Z}$ such that $sa + tb = 1$ and hence $a \equiv s^{-1} \mod b$, i.e. $sa \equiv 1 \mod b$, such that:

$$\begin{aligned} U_s \quad : \quad & V \longrightarrow V \\ & f(n) \mapsto f(sn) \\ & U_s f(n) := f(sn) \end{aligned}$$

So, since $sa \equiv 1 \mod B$, then $U_a = U_s^{-1}$. Once again, we now need to ask $(\hat{U_s f})(k) =?$

$$(\hat{U_s f})(k) =< U_s f, \omega^{kn} >= \frac{1}{b} \sum f(sn) \bar{\omega}^{kn} = \frac{1}{b} \sum f(m) \bar{\omega}^{kam} = \hat{f}(ak),$$

where $m = sn$ and hence $am = n$. Thus we have that $(\hat{U_s f})(k) = \hat{f}(ak) = (U_a \hat{f})(k)$.

3. The final property we will use is called **modulation**. This will be denoted as M_d, where $d \in \mathbb{Z}$ such that:

$$\begin{aligned} M_d \quad : \quad & V \longrightarrow V \\ & f(n) \mapsto \omega^{nd} f(n) \\ & M_d f(n) := \omega^{nd} f(n). \end{aligned}$$

What is $\hat{M_d}$?

$$(\hat{M_d f})(k) =< M_d f, \omega^{kn} >= \frac{1}{b} \sum \omega^{nd} f(n) \bar{\omega}^{kn} = \frac{1}{b} \sum f(n) \bar{\omega}^{n(k-d)} = \hat{f}(k-d),$$

thus we have that $(\hat{M_d} f)(k) = \hat{f}(k-d)$.

Chapter 4

Theorem (Brion and Lawrence - Varchenko)

4.1 Description and Proof

In the paper written by Matthias Beck, Christian Haase and Frank Sottile named "Formulas of Brion Lawrence and Varchenko on Rational Generating Functions for Cones" [7], they discuss and prove two important thoerems which will be used in this research: the theorems of Brion and Lawrence & Varchenko. A general example of the motivation for each of these theorems (first Lawrence & Varchenko then Brion) is as follows: Consider the polytope $[a, b] \in \mathbb{R}$ where $a, b \in \mathbb{Z}$. First we can take the difference of 2 generating functions which list all integers less than a and b as follows: Consider $x^a + x^{a-1} + \cdots = \sum_{k \leq a} x^k = \frac{x^a}{1 - \frac{1}{x}}$

and $x^b + x^{b-1} + \cdots = \sum_{k \leq b} x^k = \frac{x^b}{1 - \frac{1}{x}}$. Now taking the difference of these we get:

$$x^b + x^{b-1} + \cdots - (x^a + x^{a-1} + \dots) = \sum_{k \leq b} x^k - \sum_{k \leq a} x^k = \frac{x^b}{1 - \frac{1}{x}} - \frac{x^a}{1 - \frac{1}{x}} = x^b + x^{b-1} + \cdots + x^a,$$

which is a sum of x-terms whose exponents are all integers in the interval $[a, b]$.

We can now approach this in a slightly different way, but end up with the same result. Let's list all of the integers **greater** than a in the form of a generating function:

$$x^a + x^{a+1} + \cdots = \sum_{k \geq a} x^k = \frac{x^a}{1 - x}.$$

We can also list all of the integers **less** than b in a similar way:

$$\cdots + x^{b-1} + x^b = \sum_{k \leq b} x^k = \frac{x^b}{1 - \frac{1}{x}}.$$

Now adding these together we get the following result:

$$\begin{aligned} \frac{x^a}{1-x} + \frac{x^b}{1 - \frac{1}{x}} &= \frac{x^a}{1-x} + \frac{x^{b+1}}{x-1} \\ &= \frac{x^a - x^{b+1}}{1-x} \\ &= x^a + x^{a+1} + \cdots + x^b, \end{aligned}$$

which is a sum of x-terms whose exponents are all integers in the interval $[a, b]$. These approaches can be expanded into higher dimensions; specifically for the purposes of this research, can be expanded into 2-dimensions. Consider the following arbitrary triangle on the x, y-plane in Figure 4.1:

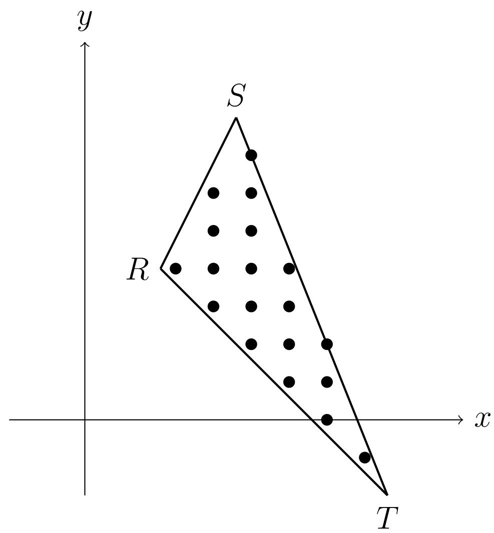


Figure 4.1: Arbitrary Triangle with lattice points indicated

Approaching this in a similar way as the previous example, we can add/subtract closed, half-open and open cones together to count precisely the lattice points in this triangle, as in Figure 4.2.

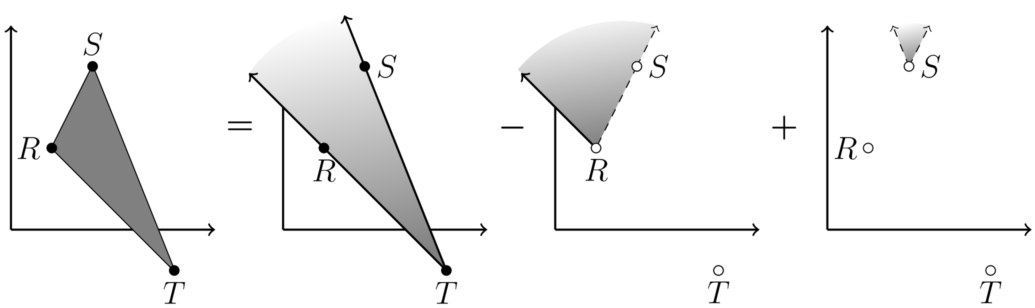


Figure 4.2: Cones for Lawrence-Varchenko Perspective

Note that this was an arbitrary triangle in 2-dimensions. Since this is the focus of this paper, for our purposes, this proves Lawrence-Verchenko whose statement is as follows:

Theorem 4.1.1. *Let P be a simple polytope, and for each vertex v of P choose a vector ϵ_v that is not perpendicular to any edge direction at v. Form the cone $K_{\epsilon_v,v}$. Then we have*

$$\sigma_P(x) = \sum_{v \text{ a vertex of } P} (-1)^{|E_v^-(\epsilon_v)|} \sigma_{K_{\epsilon_v,v}},$$

where

- $E_v^-(\epsilon)$ *is the edge directions w at vertex v with $w \cdot \epsilon < 0$.*

- $E_v^+(\epsilon)$ *is the edge directions w at vertex v with $w \cdot \epsilon > 0$.*

- $K_{\epsilon,v} := v + \sum_{w \in E_v^+(\epsilon)} \mathbb{R}_{\geq 0} w + \sum_{w \in E_v^-(\epsilon)} \mathbb{R}_{<0} w.$

- σ_K *is the generating function encoding the lattice points in the cone K.*

When looking at this from the perspective of Brion, at each vertex in Figure 4.1, we can create a cone with each vertex and its adjacent sides, as follows:

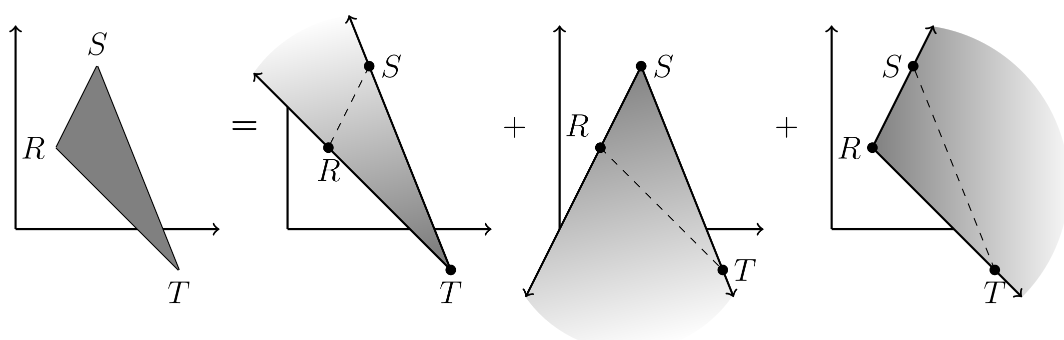


Figure 4.3: Cones for Brion Perspective

For each of the cones in Figure 4.3, we can once again list all of the lattice points, in the cone, in the form of a Hilbert Series, using the fundamental parallelograms which are generated by vectors, of latticial length 1, denoted by $\begin{bmatrix} a_{IJ} & b_{IJ} \end{bmatrix}$, where IJ is the line segment corresponding to a side of the cone. Let FP_V denote the fundamental parallelogram of cone K_V; $P_V(x)$ denote the polynomial, with exponents representing each lattice point in the fundamental parallelogram, of the cone K_V, which has vertex V.

- Lattice Points in C_R: $\displaystyle\sum_{(k,l)\in C_R\cap\mathbb{Z}} x^k y^l = \frac{P_R(x)}{(1-x^{a_{RS}}y^{b_{RS}})(1-x^{a_{RT}}y^{b_{RT}})}$, similarly
- Lattice Points in C_S: $\displaystyle\sum_{(k,l)\in C_S\cap\mathbb{Z}} x^k y^l = \frac{P_S(x)}{(1-x^{-a_{RS}}y^{-b_{RS}})(1-x^{a_{ST}}y^{b_{ST}})}$, and
- Lattice Points in C_T: $\displaystyle\sum_{(k,l)\in C_T\cap\mathbb{Z}} x^k y^l = \frac{P_T(x)}{(1-x^{-a_{RT}}y^{-b_{RT}})(1-x^{-a_{ST}}y^{-b_{ST}})}$.

Now, when we add these rational expressions together, we get the following result:

$$\begin{aligned} & \frac{P_R(x)}{(1-x^{a_{RS}}y^{b_{RS}})(1-x^{a_{RT}}y^{b_{RT}})} + \frac{P_S(x)}{(1-x^{-a_{RS}}y^{-b_{RS}})(1-x^{a_{ST}}y^{b_{ST}})} \\ & + \frac{P_T(x)}{(1-x^{-a_{RT}}y^{-b_{RT}})(1-x^{-a_{ST}}y^{-b_{ST}})} \\ = \quad & \frac{P_R(x)}{(1-x^{a_{RS}}y^{b_{RS}})(1-x^{a_{RT}}y^{b_{RT}})} - \frac{x^{a_{RS}}y^{b_{RS}}P_S(x)}{(1-x^{a_{RS}}y^{b_{RS}})(1-x^{a_{ST}}y^{b_{ST}})} \\ & + \frac{x^{a_{RT}}y^{b_{RT}}x^{a_{ST}}y^{b_{ST}}P_T(x)}{(1-x^{a_{RT}}y^{b_{RT}})(1-x^{a_{ST}}y^{b_{ST}})} \end{aligned}$$

Note that the first rational expression remains unchanged, hence it still calculates the lattice points in K_R in Figure 4.3. The second rational expression is now calculating the lattice points in the shaded region of Figure 4.4. Similarly, the third rational expression is now calculating the lattice points in the shaded region of Figure 4.5.

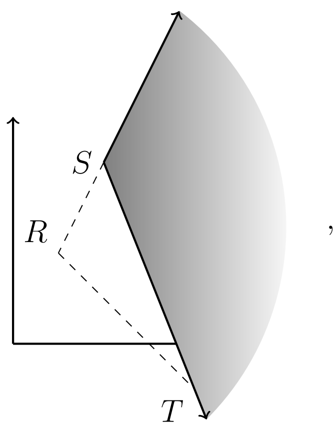


Figure 4.4: Region $\dfrac{x^{a_{RS}}y^{b_{RS}}P_S(x)}{(1-x^{a_{RS}}y^{b_{RS}})(1-x^{a_{ST}}y^{b_{ST}})}$ is counting lattice points.

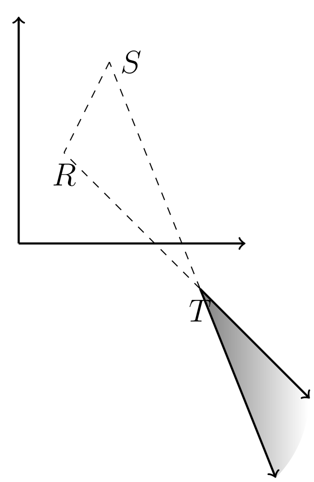


Figure 4.5: Region $\dfrac{x^{a_{RT}}y^{b_{RT}}x^{a_{ST}}y^{b_{ST}}P_T(x)}{(1-x^{a_{RT}}y^{b_{RT}})(1-x^{a_{ST}}y^{b_{ST}})}$ is counting lattice points.

Note that when performing the operations given in the regional expression above, this matches up exactly with the Lawrence-Varchenko approach to the same triangle. Further note that since this was an arbitrary triangle in 2-dimensions, which is the focus of this paper, for our purposes, this proves Brion, whose statement is as follows:

Theorem 4.1.2 (Brion's Theorem)**.** *Let P be a polytope with rational vertices $v_1, v_2, \ldots, v_d$. Let K_{v_i} denote the vertex cone for $i \in \{1, 2, \ldots, d\}$, and let $\sigma_{K_{v_i}}(x, y)$ be the rational function representing the integer points in the vertex cone K_{v_i}. Then*

$$\sigma_P(x, y) = \sum_{v_i \textit{ a vertex of } P} \sigma_{K_{v_i}}(x, y),$$

where $\sigma_P(x, y)$ is the polynomial encoding the integer points in P.

For proofs of these theorems with arbitrary dimension, please see [7].

Chapter 5

$f(n)$ using Partial Fraction

5.1 Quick example with monoid $< 1, b >$

Consider the monoid $< 1, b >$. Then

$$\sum_{n=0}^{\infty} f_{<1.b>}(n) z^n = \frac{1}{(1-z)(1-z^b)} = \frac{B_2}{(1-z)^2} + \frac{B_1}{1-z} + \sum_{\substack{\omega^b=1 \\ \omega\neq 1}} \frac{D^\omega}{(1-\omega z)}$$

$$\begin{aligned}
B_2 &= \lim_{z\longrightarrow 1} \frac{(1-z)^2}{(1-z)(1-z^b)} \\
&= \lim_{z\longrightarrow 1} \frac{1}{b_z} = \frac{1}{b}
\end{aligned}$$

$$\begin{aligned}
B_1 &= \lim_{z\longrightarrow 1} \left[\frac{(1-z)}{(1-z)(1-z^b)} - \frac{1}{b(1-z)} \right] \\
&= \lim_{z\longrightarrow 1} \left[\frac{b-b_z}{b(1-z^b)} \right] \\
&\overset{(LH)}{=} \lim_{x\longrightarrow 1} \frac{1+2z+3z^2+\cdots+(b-1)z^{b-2}}{b^2 z^{b-1}} \\
&= \frac{\frac{b(b-1)}{2}}{b^2} = \frac{b-1}{2b}
\end{aligned}$$

$$
\begin{aligned}
D^{\omega} &= \lim_{z \longrightarrow \bar{\omega}} \frac{1-\omega z}{(1-z)(1-z^b)} \\
&\overset{(*)}{=} \lim_{x \longrightarrow 1} \frac{1-x}{(1-(\bar{\omega}x))(1-(\bar{\omega}x)^b)} \\
&= \lim_{x \longrightarrow 1} \frac{1-x}{(1-\bar{\omega}x)(1-x^b)} \\
&= \lim_{x \longrightarrow 1} \frac{1-x}{(1-\bar{\omega}x)b_x(1-x)} \\
&= \lim_{x \longrightarrow 1} \frac{1}{b_x(1-\bar{\omega}x)} \\
&= \frac{1}{b(1-\bar{\omega})}
\end{aligned}
$$

(*) This is equal by substituting $x = \omega z$, then $z = \bar{\omega}x$ and when $z \longrightarrow \bar{\omega}$, then $x \longrightarrow 1$.

Thus we have that

$$
\begin{aligned}
\frac{1}{(1-z)(1-z^b)} &= \frac{1}{b(1-z)^2} + \frac{b-1}{2b(1-z)} + \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{(1-\bar{\omega}^k)(1-\omega^k z)} \\
& \quad \text{where } \omega = \omega_b = e^{\frac{i2\pi}{b}} \\
& \quad \text{and } \{\alpha \in \mathbb{C} | \alpha^b = 1, \alpha \neq 1\} = \{\omega, \omega^2, \dots, \omega^{b-1}\} \\
&= \sum_{n=0}^{\infty} \frac{1}{b}(n+1)z^n + \frac{b-1}{2b}\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{b} \sum_{k=1}^{b-1} \frac{\omega^{kn} z^n}{(1-\bar{\omega}^k)}.
\end{aligned}
$$

So

$$
\begin{aligned}
f_{<1,b>}(n) &= \frac{n+1}{b} + \frac{b-1}{2b} + \frac{1}{b}\sum_{k=1}^{b-1} \frac{\omega^{kn}}{1-\bar{\omega}^k} \\
&= \frac{n}{b} + \frac{1}{2b} + \frac{1}{2} + \frac{1}{b}\sum_{k=1}^{b-1} \frac{\omega^k}{1-\bar{\omega}^k} \\
& \quad \text{Note that } \frac{1}{2b} + \frac{1}{2} \text{ is constant and } \frac{1}{b}\sum_{k=1}^{b-1} \frac{\omega^k}{1-\bar{\omega}^k} \text{ is} \\
& \quad b\text{-periodic and 0-average.} \\
&\overset{(Popoviciu)}{=} \frac{n}{b} - \left\{\frac{n}{b}\right\} + 1, \text{ since } 1 \times 1 + 0 \times b = 1 \\
&= \frac{n}{b} + \frac{b-1}{2b} - \left\{\frac{n}{b}\right\} - \frac{b-1}{2b} + 1 \\
& \quad \text{Note that } \frac{b-1}{2b} - \left\{\frac{n}{b}\right\} \text{ is } b\text{-periodic with 0-average}
\end{aligned}
$$

Therefore, $\frac{1}{b}\sum_{k=1}^{b-1} \frac{\omega^k}{1-\bar{\omega}^k} = \frac{b-1}{2b} - \left\{\frac{n}{b}\right\}$.

5.2 Example with monoid $< a, A >$

Next consider the monoid $< a, A >$. Then

$$
\sum_{n=0}^{\infty} f_{<a,A>}(n) z^n = \frac{1}{(1-z^a)(1-z^A)} =
$$
$$
\frac{B_2}{(1-z)^2} + \frac{B_1}{1-z} + \sum_{\substack{w^a=1 \\ w\neq 1}} \frac{C^\omega}{(1-\omega z)} + \sum_{\substack{w^A=1 \\ w\neq 1}} \frac{D^\omega}{(1-\omega z)}
$$

$$
\begin{aligned}
B_2 &= \lim_{z\longrightarrow 1} \frac{(1-z)^2}{(1-z^a)(1-z^A)} \\
&= \lim_{z\longrightarrow 1} \frac{1}{a_z A_z} \\
&= \frac{1}{aA}
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= \lim_{z\longrightarrow 1}\left[\frac{(1-z)}{(1-z^a)(1-z^A)}-\frac{1}{aA(1-z)}\right]\\
&\overset{(*)}{=} \lim_{x\longrightarrow 1}\frac{x}{(1-(1-x)^a)(1-(1-x)^A)}-\frac{1}{aAx}\\
&= \lim_{x\longrightarrow 0}\frac{x}{a_{(1-x)}A_{(1-x)}x^2}-\frac{1}{aAx}\\
&= \lim_{x\longrightarrow 0}\frac{aA-a_{(1-x)}A_{(1-x)}}{a_{(1-x)}A_{(1-x)}xaA}\\
&= \frac{1}{(aA)^2}\lim_{x\longrightarrow 0}\frac{aA-a_{(1-x)}A_{(1-x)}}{x}\\
&\overset{(LH)}{=} \frac{1}{(aA)^2}\lim_{x\longrightarrow 0}-[a'_{(1-x)}A_{(1-x)}+a_{(1-x)}A'_{(1-x)}]\\
&= \frac{1}{(aA)^2}\left[\frac{(a-1)a}{2}A+a\frac{(a-1)A}{2}\right]\\
&= \frac{aA[(a-1)+(A-1)]}{2(aA)^2}\\
&= \frac{a+A-2}{2aA}
\end{aligned}
$$

(*) This is equal by substituting $x = 1-z$, i.e. $z = 1-x$. Note that when $z \longrightarrow 1$, then $x \longrightarrow 0$.

$$
\begin{aligned}
C^\omega &= \lim_{z\longrightarrow \bar{\omega}}\frac{1-\omega z}{(1-z^a)(1-z^A)}\\
&\overset{(**)}{=} \lim_{x\longrightarrow 1}\frac{1-x}{(1-(\bar{\omega}x)^a)(1-(\bar{\omega}x)^A)}\\
&= \lim_{x\longrightarrow 1}\frac{1-x}{(1-x^a)(1-\bar{\omega}^A x^A)}\\
&= \lim_{x\longrightarrow 1}\frac{1-x}{(1-x)a_x(1-\bar{\omega}^A x^A)}\\
&= \lim_{x\longrightarrow 1}\frac{1}{a_x(1-\bar{\omega}^A x^A)}\\
&= \frac{1}{a(1-\bar{\omega}^A)}
\end{aligned}
$$

(**) This is equal by substituting $x = \omega z$, then $z = \bar{\omega}x$ and when $z \longrightarrow \bar{\omega}$, then $x \longrightarrow 1$. Similarly, $D^\omega = \dfrac{1}{A(1-\bar{\omega}^a)}$.

So we have

$$\begin{aligned}\sum_{n=0}^{\infty} f_{<a,A>}(n) z^n &= \frac{1}{(1-z^a)(1-z^A)} \\ &= \frac{1}{aA(1-z)^2} + \frac{a+A-2}{2aA(1-z)} + \sum_{\substack{w^a=1\\ w\neq 1}} \frac{1}{a(1-\bar{\omega}^A)(1-\omega z)} \\ &\quad + \sum_{\substack{w^A=1\\ w\neq 1}} \frac{1}{A(1-\bar{\omega}^a)(1-\omega z)}\end{aligned}$$

Note that:

- $\frac{1}{1-z} = \sum_{n=0}^{\infty} \binom{n+0}{0} z^n = \sum_{n=0}^{\infty} z^n$
- $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} z^n = \sum_{n=0}^{\infty} (n+1) z^n$
- $\frac{1}{(1-\omega z)} = \sum_{n=0}^{\infty} \binom{n+0}{0} (\omega z)^n = \sum_{n=0}^{\infty} \omega^n z^n$

Thus

$$\begin{aligned}f_{<a,A>}(n) &= \frac{n}{aA} + \frac{a+A-2}{2aA} + \frac{1}{aA} + \sum_{\substack{w^a=1\\ w\neq 1}} \frac{\omega^n}{a(1-\bar{\omega}^A)} + \sum_{\substack{w^A=1\\ w\neq 1}} \frac{\omega^n}{A(1-\bar{\omega}^a)} \\ &= \frac{n}{aA} + \frac{a+A}{2aA} + \sum_{\substack{w^a=1\\ w\neq 1}} \frac{\omega^n}{a(1-\bar{\omega}^A)} + \sum_{\substack{w^A=1\\ w\neq 1}} \frac{\omega^n}{A(1-\bar{\omega}^a)} \\ &= \frac{n}{aA} + \frac{a+A}{2aA} + \frac{1}{a}\sum_{\substack{w^a=1\\ w\neq 1}} \frac{\omega^n}{(1-\bar{\omega}^A)} + \frac{1}{A}\sum_{\substack{w^A=1\\ w\neq 1}} \frac{\omega^n}{(1-\bar{\omega}^a)} \\ &= \frac{n}{aA} + \frac{1}{2A} + \frac{1}{2a} + \frac{1}{a}\sum_{\substack{w^a=1\\ w\neq 1}} \frac{\omega^n}{(1-\bar{\omega}^A)} + \frac{1}{A}\sum_{\substack{w^A=1\\ w\neq 1}} \frac{\omega^n}{(1-\bar{\omega}^a)}\end{aligned}$$

Also note, by using Popoviciu,

$f_{<a,A>}(n) \quad = \quad \frac{n}{aA} - \left\{\frac{t_A n}{a}\right\} - \left\{\frac{s_a n}{A}\right\} + 1$ where $s_a a + t_A A = 1$, i.e. $t_A A \equiv 1 \mod a$ and $s_a a \equiv 1 \mod A$.

Note that

$$\frac{1}{A}\sum_{n=0}^{A-1}\left\{\frac{s_a n}{A}\right\} = \frac{1}{A}\sum_{n=0}^{A-1}\frac{n}{A} = \frac{A(A-1)}{2A^2} = \frac{A-1}{2A} = \frac{1}{2} - \frac{1}{2A}$$

. Similarly,

$$\frac{1}{a}\sum_{n=0}^{a-1}\left\{\frac{t_A n}{a}\right\} = \frac{1}{a}\sum_{n=0}^{a-1}\frac{n}{a} = \frac{a(a-1)}{2a^2} = \frac{a-1}{2a} = \frac{1}{2} - \frac{1}{2a}$$

. Hence,

$$\begin{aligned} f_{<a,A>}(n) &= \frac{n}{aA} - \left\{\frac{t_A n}{a}\right\} + \left(\frac{1}{2} - \frac{1}{2a}\right) + \left(-\frac{1}{2} + \frac{1}{2a}\right) - \left\{\frac{s_1 n}{A}\right\} \\ &\quad + \left(\frac{1}{2} - \frac{1}{2A}\right) + \left(-\frac{1}{2} + \frac{1}{2A}\right) + 1. \end{aligned}$$

So,

$$\begin{aligned} f_{<a,A>}(n) &= \frac{n}{aA} - \left\{\frac{t_A n}{a}\right\} + \left(\frac{1}{2} - \frac{1}{2a}\right) - \left\{\frac{s_1 n}{A}\right\} + \left(\frac{1}{2} - \frac{1}{2A}\right) + \frac{1}{2a} \\ &\quad + \frac{1}{2A} \end{aligned}$$

Note that

- $-\left\{\frac{t_A n}{a}\right\} + \left(\frac{1}{2} - \frac{1}{2a}\right)$ is a-periodic, with 0-average and
- $-\left\{\frac{s_a n}{A}\right\} + \left(\frac{1}{2} - \frac{1}{2A}\right)$ is A-periodic, with 0-average.

Therefore, since they are both a-periodic and 0-average,

$$-\left\{\frac{t_A n}{a}\right\} + \left(\frac{1}{2} - \frac{1}{2a}\right) = \frac{1}{a}\sum_{\substack{w^a=1 \\ w\neq 1}}\frac{\omega^n}{1-\bar{\omega}^A} \text{ or } \sum_{\substack{w^a=1 \\ w\neq 1}}\frac{\omega^n}{1-\bar{\omega}^A} = -a\left\{\frac{t_A n}{a}\right\} + \frac{a}{2} - \frac{1}{2}$$

Similarly, since they are both A-periodic and 0-average,

$$-\left\{\frac{s_a n}{A}\right\} + \left(\frac{1}{2} - \frac{1}{2A}\right) = \frac{1}{A}\sum_{\substack{w^A=1 \\ w\neq 1}}\frac{\omega^n}{1-\bar{\omega}^a} \text{ or } \sum_{\substack{w^A=1 \\ w\neq 1}}\frac{\omega^n}{1-\bar{\omega}^a} = -A\left\{\frac{s_a n}{A}\right\} + \frac{A}{2} - \frac{1}{2}$$

5.3 Partial Fractions of $f_{<a,b,c>}$

When considering a numerical monoid $< a_1, a_2, \ldots, a_d >$, then we know that

$$\sum_{n=0}^{\infty} f_{<a_1,a_2,\ldots,a_d>}(n) z^n = \frac{1}{(1-z^{a_1})(1-z^{a_2})\ldots(1-z^{a_d})}$$

where $f_{<a_1,a_2,\ldots,a_d>}(n) = \#$ of ways to get n using $a_1, a_2, \ldots, a_d$. In our case, we will be considering this when $M =< a, b, c >$, i.e.

$$\sum_{n=0}^{\infty} f_{<a,b,c>}(n) z^n = \frac{1}{(1-z^a)(1-z^b)(1-z^c)}$$

where $f_{<a,b,c>}(n) = \#$ of ways to get n using a, b, c. Using partial fraction techniques, we can consider the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} f_{<a,b,c>}(n) z^n \quad &= \quad \frac{1}{(1-z^a)(1-z^b)(1-z^c)} = \frac{B_1}{1-z} + \frac{B_2}{(1-z)^2} + \frac{B_3}{(1-z)^3} \\
&+ \sum_{\substack{w^a=1 \\ w^b=1 \\ w^c\neq 1}} \left(\frac{C_2^\omega}{(1-\omega z)^2} + \frac{C_1^\omega}{(1-\omega z)} \right) \\
&+ \sum_{\substack{w^a=1 \\ w^c=1 \\ w^b\neq 1}} \left(\frac{D_2^\omega}{(1-\omega z)^2} + \frac{D_1^\omega}{(1-\omega z)} \right) \\
&+ \sum_{\substack{w^b=1 \\ w^c=1 \\ w^a\neq 1}} \left(\frac{E_2^\omega}{(1-\omega z)^2} + \frac{E_1^\omega}{(1-\omega z)} \right) + \sum_{\substack{w^a=1 \\ w^b\neq 1 \\ w^c\neq 1}} \frac{F_\omega}{(1-\omega z)} \\
&+ \sum_{\substack{w^b=1 \\ w^a\neq 1 \\ w^c\neq 1}} \frac{G_\omega}{(1-\omega z)} + \sum_{\substack{w^c=1 \\ w^a\neq 1 \\ w^b\neq 1}} \frac{H_\omega}{(1-\omega z)}
\end{aligned}$$

So, we will now calculate the coefficients:

- $$\begin{aligned}
B_3 \quad &= \quad \lim_{z\longrightarrow 1} \frac{(1-z)^3}{(1-z^a)(1-z^b)(1-z^c)} \\
&= \quad \lim_{z\longrightarrow 1} \frac{(1-z)^3}{(1-z)a_z(1-z)b_z(1-z)c_z} \\
&= \quad \lim_{z\longrightarrow 1} \frac{1}{a_z b_z c_z} = \frac{1}{abc}
\end{aligned}$$

$$
\begin{aligned}
\bullet\ B_2 \quad &= \lim_{z\longrightarrow 1}\left[\frac{(1-z)^2}{(1-z^a)(1-z^b)(1-z^c)}-\frac{1}{abc(1-z)}\right]\\
&= \lim_{z\longrightarrow 1}\left[\frac{1}{a_zb_zc_z(1-z)}-\frac{1}{abc(1-z)}\right]\\
&= \lim_{z\longrightarrow 1}\frac{abc-a_zb_zc_z}{a_zb_zc_zabc(1-z)}\\
&= \lim_{z\longrightarrow 1}\frac{abc-a_zb_zc_z}{a^2b^2c^2(1-z)}\\
&\overset{(*)}{=} \lim_{x\longrightarrow 0}\frac{abc-a_{(1-x)}b_{(1-x)}c_{(1-x)}}{a^2b^2c^2x}\\
&\overset{LH}{=} \lim_{x\longrightarrow 0}\frac{-[a'_{(1-x)}b_{(1-x)}c_{(1-x)}+a_{(1-x)}b'_{(1-x)}c_{(1-x)}+a_{(1-x)}b_{(1-x)}c'_{(1-x)}}{a^2b^2c^2}\\
&\overset{(**)}{=} \frac{\left(\frac{(a-1)a}{2}\right)bc+a\left(\frac{(b-1)b}{2}\right)c+ab\left(\frac{(c-1)c}{2}\right)}{a^2b^2c^2}\\
&= \frac{a+b+c-3}{2abc}
\end{aligned}
$$

(*) This is equal by substituting $x=1-z$, i.e. $z=1-x$. Note that when $z\longrightarrow 1$, then $x\longrightarrow 0$.

(**) This is equal since $a'_{(1-x)}\overset{(x\longrightarrow 0)}{=}-\dfrac{(a-1)a}{2}$, similarly for $b'_{(1-x)}$ and $c'_{(1-x)}$.

$$
\begin{aligned}
\bullet\ B_1 \quad &= \lim_{z\longrightarrow 1}\left[\frac{1-z}{(1-z^a)(1-z^b)(1-z^c)}-\frac{a+b+c-3}{2abc(1-z)}-\frac{1}{abc(1-z)^2}\right]\\
&= \lim_{z\longrightarrow 1}\left[\frac{1}{a_zb_zc_z(1-z)^2}-\frac{a+b+c-3}{2abc(1-z)}-\frac{1}{abc(1-z)^2}\right]\\
&= \lim_{z\longrightarrow 1}\frac{2abc-(a+b+c-3)a_zb_zc_z(1-z)-2a_zb_zc_z}{2a_zb_zc_zabc(1-z)^2}\\
&= \lim_{z\longrightarrow 1}\frac{2abc-(a+b+c-3)a_zb_zc_z(1-z)-2a_zb_zc_z}{2a^2b^2c^2(1-z)^2}\\
&\overset{(***)}{=} \lim_{x\longrightarrow 0}\frac{2abc+(a+b+c-3)a_{(1+x)}b_{(1+x)}c_{(1+x)}x-2a_{(1+x)}b_{(1+x)}c_{(1+x)}}{2a^2b^2c^2x^2}\\
&\overset{(LH)}{=} \lim_{x\longrightarrow 0}\frac{(a+b+c-3)\Big((a_{(1+x)}b_{(1+x)}c_{(1+x)})'x+a_{(1+x)}b_{(1+x)}c_{(1+x)}\Big)}{4a^2b^2c^2x}\\
&\qquad -\frac{2(a_{(1+x)}b_{(1+x)}c_{(1+x)})'}{4a^2b^2c^2x}\\
&\overset{(LH)}{=} \frac{1}{2a^2b^2c^2}\lim_{x\longrightarrow 0}\frac{(a+b+c-3)[(a_{(1+x)}b_{(1+x)}c_{(1+x)})''x+2(a_{(1+x)}b_{(1+x)}c_{(1+x)})']}{2}\\
&\qquad -\frac{2(a_{(1+x)}b_{(1+x)}c_{(1+x)})'')}{2}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{4a^2b^2c^2}\lim_{x\longrightarrow 0}(a+b+c-3)[(a_{(1+x)}b_{(1+x)}c_{(1+x)})''x+2(a_{(1+x)}b_{(1+x)}c_{(1+x)})']\\
&\quad -2(a_{(1+x)}b_{(1+x)}c_{(1+x)})''\\
&\overset{(4*)}{=} \frac{1}{4a^2b^2c^2}\Bigg[(a+b+c-3)\left[0+2\left(\frac{(a-1)a}{2}bc+a\frac{(b-1)b}{2}c+ab\frac{(c-1)c}{2}\right)\right]\\
&\quad -2\Big[\frac{(a-1)a(a+1)}{3}bc+2\left(\frac{(a-1)a}{2}\frac{(b-1)b}{2}c\right)+2\left(\frac{(a-1)a}{2}b\frac{(c-1)c}{2}\right)\\
&\quad +a\frac{(b-1)b(b+1)}{3}c+2\left(a\frac{(b-1)b}{2}\frac{(c-1)c}{2}\right)+ab\frac{(c-1)c(c+1)}{3}\Big]\Bigg]\\
&= \frac{1}{4abc}\Bigg[(a+b+c-3)(a+b+c-3)-\Big[\frac{2(a-1)(a+1)}{3}+(a-1)(b-1)\\
&\quad +(a-1)(c-1)+\frac{2(b-1)(b+1)}{3}+(b-1)(c-1)+\frac{2(c-1)(c+1)}{3}\Big]\Bigg]\\
&= \frac{1}{12abc}\Big[3(a+b+c-3)^2-(2(a-1)(a+1)+3(a-1)(b-1)\\
&\quad +3(a-1)(c-1)+2(b-1)(b+1)+3(b-1)(c-1)+2(c-1)(c+1))\Big]\\
&= \frac{1}{12abc}(a^2+b^2+c^2+3ab+3ac+3bc-12a-12b-12c+24)
\end{aligned}$$

(***) This is equal by substituting $x = z-1$, i.e. $z = x+1$. Note that when $z \longrightarrow 1$, then $x \longrightarrow 0$.

(4*) Note as $x \longrightarrow 0$ $a'_{(1+x)} = \dfrac{(a-1)a}{2}$ and $a''_{(1+x)} = \dfrac{(a-1)a(a+1)}{3}$.

- $$\begin{aligned}
C_2^{\omega} &= \lim_{z\longrightarrow\bar{\omega}}\frac{(1-\omega z)^2}{(1-z^a)(1-z^b)(1-z^c)}\\
&\overset{(5*)}{=} \lim_{x\longrightarrow 1}\frac{(1-x)^2}{(1-x^a)(1-x^b)(1-x^c)}\\
&= \lim_{x\longrightarrow 1}\frac{(1-x)^2}{(1-x)a_x(1-x)b_x(1-\bar{\omega}^c x^c)}\\
&= \frac{1}{ab}\lim_{x\longrightarrow 1}\frac{1}{(1-\bar{\omega}^c x^c)} = \frac{1}{ab(1-\bar{\omega}^c)}
\end{aligned}$$

(5*) This is equal by letting $x = \omega z$, then $z = \bar{\omega}x$ and when $z \longrightarrow \bar{\omega}$, then $x \longrightarrow 1$.

Similarly, $D_2^{\omega} = \dfrac{1}{ac(1-\bar{\omega}^b)}$ and $E_2^{\omega} = \dfrac{1}{bc(1-\bar{\omega}^a)}$.

$$
\begin{aligned}
\bullet\ C_1^\omega &= \lim_{z\longrightarrow\bar\omega}\left[\frac{1-\omega z}{(1-z^a)(1-z^b)(1-z^c)}-\frac{1}{ab(1-\bar\omega^c)(1-\omega z)}\right]\\
&\overset{(6*)}{=} \lim_{x\longrightarrow 1}\left[\frac{1-x}{(1-x^a)(1-x^b)(1-\bar\omega^c x^c)}-\frac{1}{ab(1-\bar\omega^c)(1-x)}\right]\\
&= \lim_{x\longrightarrow 1}\left[\frac{1}{a_x b_x(1-x)(1-\bar\omega^c x^c)}-\frac{1}{ab(1-\bar\omega^c)(1-x)}\right]\\
&= \lim_{x\longrightarrow 1}\frac{ab(1-\bar\omega^c)-a_x b_x(1-\bar\omega^c x^c)}{ab a_x b_x(1-x)(1-\bar\omega^c x^c)(1-\bar\omega^c)}\\
&= \frac{1}{a^2b^2(1-\bar\omega^c)}\lim_{x\longrightarrow 1}\frac{ab(1-\bar\omega^c)-a_x b_x(1-\bar\omega^c x^c)}{(1-x)(1-\bar\omega^c x^c)}\\
&= \frac{1}{a^2b^2(1-\bar\omega^c)^2}\lim_{x\longrightarrow 1}\frac{ab(1-\bar\omega^c)-a_x b_x(1-\bar\omega^c x^c)}{(1-x)}\\
&\overset{(LH)}{=} \frac{1}{a^2b^2(1-\bar\omega^c)^2}\lim_{x\longrightarrow 1}\frac{a_x' b_x(1-\bar\omega^c x^c)+a_x b_x'(1-\bar\omega^c)-a_x b_x c\bar\omega^c x^{c-1}}{1}\\
&= \frac{1}{a^2b^2(1-\bar\omega^c)^2}\left(\frac{(a-1)a}{2}b(1-\bar\omega^c)+a\frac{(b-1)b}{2}(1-\bar\omega^c)-abc\bar\omega^c\right)\\
&= \frac{1}{2ab(1-\bar\omega^c)^2}\left[(a-1)(1-\bar\omega^c)+(b-1)(1-\bar\omega^c)-2c\bar\omega^c\right]
\end{aligned}
$$

(6*) This is equal by letting $x=\omega z$, then $z=\bar\omega x$ and when $z\longrightarrow\bar\omega$, then $x\longrightarrow 1$.

Similarly, $D_1^\omega = \dfrac{1}{2ac(1-\bar\omega^b)^2}\left[(a-1)(1-\bar\omega^b)+(c-1)(1-\bar\omega^b)-2b\bar\omega^b\right]$ and $E_1^\omega = \dfrac{1}{2bc(1-\bar\omega^a)^2}\left[(b-1)(1-\bar\omega^a)+(c-1)(1-\bar\omega^a)-2a\bar\omega^a\right]$.

$$
\begin{aligned}
\bullet\ F^\omega &= \lim_{z\longrightarrow\bar\omega}\frac{1-\omega z}{(1-z^a)(1-z^b)(1-z^c)}\\
&\overset{(7*)}{=} \lim_{x\longrightarrow 1}\frac{1-x}{(1-x^a)(1-\bar\omega^b x^b)(1-\bar\omega^c x^c)}\\
&= \lim_{x\longrightarrow 1}\frac{1}{a_x(1-\bar\omega^b x^b)(1-\bar\omega^c x^c)}=\frac{1}{a(1-\bar\omega^b)(1-\bar\omega^c)}
\end{aligned}
$$

(7*) This is equal by letting $x=\omega z$, then $z=\bar\omega x$ and when $z\longrightarrow\bar\omega$ then $x\longrightarrow 1$.

Similarly, $G^\omega=\dfrac{1}{b(1-\bar\omega^a)(1-\bar\omega^c)}$ and $H^\omega=\dfrac{1}{c(1-\bar\omega^a)(1-\bar\omega^b)}$.

Further note the following equalities:

- $\displaystyle \frac{1}{1-z}=\sum_{n=0}^{\infty}\binom{n+0}{0}z^n=\sum_{n=0}^{\infty}z^n$

- $\dfrac{1}{(1-z)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} z^n = \sum_{n=0}^{\infty} (n+1) z^n = \sum_{n=0}^{\infty} (nz^n + z^n)$
- $\dfrac{1}{(1-z)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} z^n = \dfrac{1}{2} \sum_{n=0}^{\infty} (n^2+3n+2) z^n$
- $\dfrac{1}{(1-\omega z)} = \sum_{n=0}^{\infty} \binom{n+0}{0} (\omega z)^n = \sum_{n=0}^{\infty} \omega^n z^n$
- $\dfrac{1}{(1-\omega z)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} (\omega z)^n = \sum_{n=0}^{\infty} (n+1)\omega^n z^n$

Let $\dfrac{1}{(1-z^a)(1-z^b)(1-z^c)} = \sum_{n=0}^{\infty} f(n) z^n$. If $f(n) = rn^2 + sn + t$, then

- $r \;=\; \dfrac{1}{2} B_3 = \dfrac{1}{2abc}$
- $$\begin{aligned} s &= \frac{3}{2} B_3 + B_2 + \sum_{\substack{w^a=1 \\ w^b=1 \\ w^c\neq 1}} C_2^{\omega} \omega^n + \sum_{\substack{w^a=1 \\ w^c=1 \\ w^b\neq 1}} D_2^{\omega} \omega^n + \sum_{\substack{w^b=1 \\ w^c=1 \\ w^a\neq 1}} E_2^{\omega} \omega^n \\ &= \frac{3}{2}\left(\frac{1}{abc}\right) + \frac{a+b+c-3}{2abc} \\ &\quad + \sum_{\substack{w^C=1 \\ w\neq 1}} \frac{\omega^n}{ab(1-\bar{\omega}^c)} + \sum_{\substack{w^B=1 \\ w\neq 1}} \frac{\omega^n}{ac(1-\bar{\omega}^b)} + \sum_{\substack{w^A=1 \\ w\neq 1}} \frac{\omega^n}{bc(1-\bar{\omega}^a)} \\ &= \frac{a+b+c}{2abc} + \frac{1}{ab}\left(-C\left\{\frac{s_3 n}{C}\right\} + \frac{C}{2} - \frac{1}{2}\right) + \frac{1}{ac}\left(-B\left\{\frac{s_2 n}{B}\right\} + \frac{B}{2} - \frac{1}{2}\right) \\ &\quad + \frac{1}{bc}\left(-A\left\{\frac{s_1 n}{A}\right\} + \frac{A}{2} - \frac{1}{2}\right) \\ &= \frac{Aa + Bb + Cc - 2Aa\left\{\frac{s_1 n}{A}\right\} - 2Bb\left\{\frac{s_2 n}{B}\right\} - 2Cc\left\{\frac{s_3 n}{C}\right\}}{2abc} \end{aligned}$$

where $\gcd(a,b) = C$, $\gcd(a,c) = B$, $\gcd(b,c) = A$ and $s_1 a + s_2 b + s_3 c = 1$.

$$\bullet\ t = B_1 + B_2 + 2B_3 + \sum_{\substack{w^a=1\\ w^b=1\\ w^c\neq 1}} (C_2^\omega + C_1^\omega)\omega^n + \sum_{\substack{w^a=1\\ w^c=1\\ w^b\neq 1}} (D_2^\omega + D_1^\omega)\omega^n$$

$$+ \sum_{\substack{w^b=1\\ w^c=1\\ w^a\neq 1}} (E_2^\omega + E_1^\omega)\omega^n + \sum_{\substack{w^a=1\\ w^b\neq 1\\ w^c\neq 1}} F_\omega \omega^n + \sum_{\substack{w^b=1\\ w^a\neq 1\\ w^c\neq 1}} G_\omega \omega^n + \sum_{\substack{w^c=1\\ w^a\neq 1\\ w^b\neq 1}} H_\omega \omega^n$$

$$= \frac{a^2+b^2+c^2+3ab+3ac+3bc-12a-12b-12c+24}{12abc} + \frac{a+b+c-3}{2abc}$$

$$+\frac{2}{abc} + \sum_{\substack{w^a=1\\ w^b=1\\ w^c\neq 1}} \frac{\omega^n}{ab(1-\bar{\omega}^c)} + \sum_{\substack{w^a=1\\ w^c=1\\ w^b\neq 1}} \frac{\omega^n}{ac(1-\bar{\omega}^b)} + \sum_{\substack{w^b=1\\ w^c=1\\ w^a\neq 1}} \frac{\omega^n}{bc(1-\bar{\omega}^a)}$$

$$+ \sum_{\substack{w^a=1\\ w^b=1\\ w^c\neq 1}} \frac{[(a-1)(1-\bar{\omega}^c) + (b-1)(1-\bar{\omega}^c) - 2c\bar{\omega}^c]\omega^n}{2ab(1-\bar{\omega}^c)^2}$$

$$+ \sum_{\substack{w^a=1\\ w^c=1\\ w^b\neq 1}} \frac{[(a-1)(1-\bar{\omega}^b) + (c-1)(1-\bar{\omega}^b) - 2b\bar{\omega}^b]\omega^n}{2ac(1-\bar{\omega}^b)^2}$$

$$+ \sum_{\substack{w^b=1\\ w^c=1\\ w^a\neq 1}} \frac{[(b-1)(1-\bar{\omega}^a) + (c-1)(1-\bar{\omega}^a) - 2a\bar{\omega}^a]\omega^n}{2bc(1-\bar{\omega}^a)^2}$$

$$+ \sum_{\substack{w^a=1\\ w^A\neq 1}} \frac{\omega^n}{a(1-\bar{\omega}^b)(1-\bar{\omega}^c)} + \sum_{\substack{w^b=1\\ w^B\neq 1}} \frac{\omega^n}{b(1-\bar{\omega}^a)(1-\bar{\omega}^c)}$$

$$+ \sum_{\substack{w^c=1\\ w^C\neq 1}} \frac{\omega^n}{c(1-\bar{\omega}^a)(1-\bar{\omega}^b)}$$

Chapter 6

Geometric Approach

6.1 $d = 2$ - Popoviciu

Lets first consider the numerical monoid, M, with two generators a, b which are coprime positive integers, i.e. $M = \mathbb{N}a + \mathbb{N}b$. Let $f_M(n) :=$ the number of ways of getting n as a linear combination of a's and b's, with positive integer coefficients. In other words:

$$f_M(n) = \left| \left\{ \begin{bmatrix} x \\ \\ y \end{bmatrix} \subset \mathbb{N}^2 \mid ax + by = n \right\} \right|.$$

Geometrically, this is the same questions as asking how many lattice points are on the hypothenuse in Figure 6.1.

Now we would like to transform the hypotenuse in Figure 6.1 to a horizontal line. To do this, we need a 2×2 matrix, M, with determinant 1 and integer

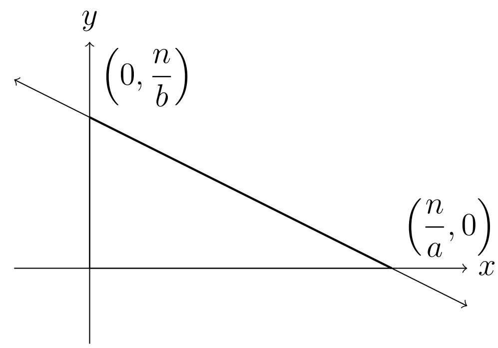


Figure 6.1: Line segment from $\left(0, \dfrac{n}{b}\right)$ to $\left(\dfrac{n}{a}, 0\right)$.

entries such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix} \text{ or } \begin{bmatrix} u \\ v \end{bmatrix} = M^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

which changes the basis from x, y to u, v. Note that since a, b are coprime, $\exists s, t \in \mathbb{Z}$ such that $as + bt = 1$ and if we let

$$M = \begin{bmatrix} b & s \\ -a & t \end{bmatrix}$$

then M has determinant 1, integer entries and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b & s \\ -a & t \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \text{ or } \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} t & -s \\ a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Further note that we have

$$n = ax + by = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} b & s \\ -a & t \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = v = n.$$

hence the endpoints of the hypotenuse will have the same v-coordinate under the new basis and hence will be a horizontal line. Performing this computation we get

$$M^{-1}\begin{bmatrix} 0 \\ \frac{n}{b} \end{bmatrix} = \begin{bmatrix} t & -s \\ a & b \end{bmatrix}\begin{bmatrix} 0 \\ \frac{n}{b} \end{bmatrix} = \begin{bmatrix} -\frac{sn}{b} \\ n \end{bmatrix} \text{ and}$$
$$M^{-1}\begin{bmatrix} \frac{n}{a} \\ 0 \end{bmatrix} = \begin{bmatrix} t & -s \\ a & b \end{bmatrix}\begin{bmatrix} \frac{n}{a} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{tn}{a} \\ n \end{bmatrix}.$$

Note, since the determinant is positive, we preserve the orientation, hence we know that $-\frac{sn}{b} < \frac{tn}{a}$. Thus the hypotenuse in Figure 6.1 has been transformed into

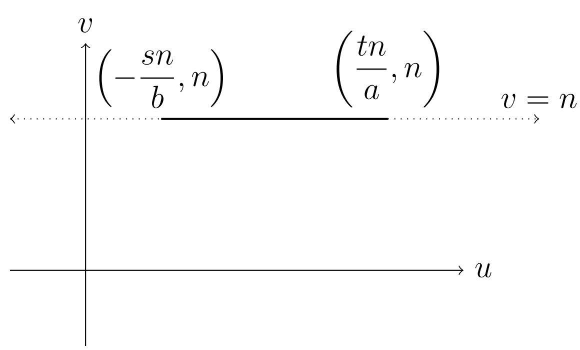


Figure 6.2: Transformed line segment from $\left(-\frac{sn}{b}, n\right)$ and $\left(\frac{tn}{a}, n\right)$.

Thus we now have

$$f_M(n) = \left\lfloor \frac{tn}{a} \right\rfloor - \left\lceil -\frac{sn}{b} \right\rceil + 1 = \left\lfloor \frac{tn}{a} \right\rfloor + \left\lfloor \frac{sn}{b} \right\rfloor + 1 = \frac{n}{ab} - \left\{ \frac{tn}{a} \right\} - \left\{ \frac{sn}{b} \right\} + 1,$$

which is the **Popoviciu Theorem/Formula**, which was found in the 1950's.

6.2 $d = 3$

Consider the numerical monoid $< a, b, c >$, where a, b, c are coprime. Our question is as follows: How many ways can we write a given value $n \in \mathbb{Z}$ as a linear combination of a, b, and c, such that $n = as_a + bs_b + cs_c$ where $s_a, s_b, s_c \in \mathbb{Z}_{\geq 0}$? Geometrically, we can reformulate this question to be: Given a value $n \in \mathbb{Z}$, how many integer points are in the triangle, in the x, y, z-plane shown in Figure 6.3.

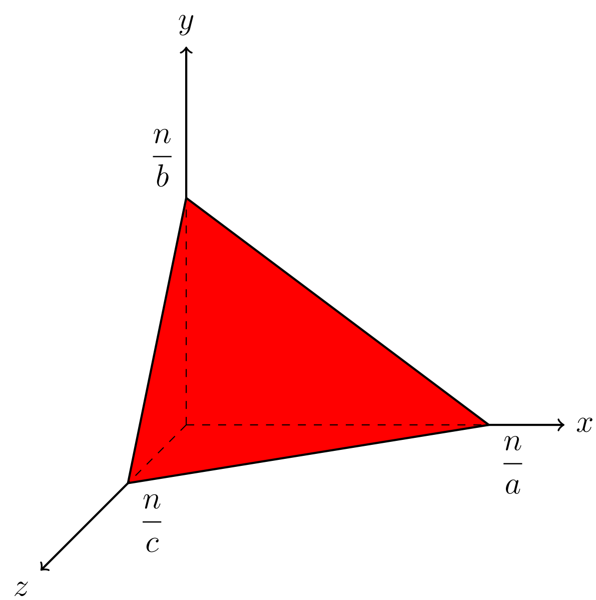


Figure 6.3: Plane in the first octant we are considering.

In order to answer this, the first thing we will to do is transform this triangle so that it fits in a 2-dimensional plane. To do this we will use the Johnson relations along with a linear combination for the gcd$(a, b, c) = 1$ in each of the cases:

- $R_a a = R_b b = R_c c$;

- $R_a a = R_b b \neq R_c c$;
- R_a, R_b, R_c are distinct.

Before we do this, it was mentioned earlier that there may be a choice to be made when constructing the matrix A. Each choice will result in a triangle in a 2-dimensional plane, but the shape will vary. With this in mind, we will be choosing the matrix A that sends our triangle in 3-dimensions, into a right triangle in 2-dimensions, when possible.

Proposition 6.2.1. *If there is a matrix A defining the Johnson Transformation with two entries in the first two columns are* 0*, then the triangle in Figure 6.3 will be transformed into a right triangle in* 2 *dimensions.*

Proof. Note that in the matrix A above, the entries s_{ab} and s_{ac} can not simultaneously be 0, as $R_a > 0$. Similarly for s_{ba} and s_{bc}. Also, s_{ac} and s_{bc} can not be simultaneously 0, otherwise $R_a a = s_{ab} b$ and $s_{ba} a = R_b b$. Then $R_a = s_{ba}$ and $R_b = s_{ab}$ by the definition of R_a and R_b, hence the first two columns are just multiples of each other, contradicting the construction of the matrix A. Thus, the two 0's in matrix A, have to be in different rows and columns.

Up to a permutation of the generators a, b and c the matrix A is:

$$A = \begin{bmatrix} -R_a & s_{ba} & t_a \\ s_{ab} & -R_b & t_b \\ s_{ac} & s_{bc} & t_c \end{bmatrix}.$$

Note

$$n = ax + by + cy = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -R_a & s_{ba} & t_a \\ s_{ab} & -R_b & t_b \\ s_{ac} & s_{bc} & t_c \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w$$

thus $w = n$ and hence we have that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -R_a & s_{ba} & t_a \\ s_{ab} & -R_b & t_b \\ s_{ac} & s_{bc} & t_c \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -R_a u + s_{ba} v + t_a n \\ s_{ab} u - R_b v + t_b n \\ s_{ac} u + s_{bc} v - t_c n \end{bmatrix}.$$

Thus we have that

- $x = -R_a u + s_{ba} v + t_a n$
- $y = s_{ab} u - R_b v + t_b n$
- $z = s_{ac} u + s_{bc} v - R_c n$

Next recall that by definition, R_a and R_b can not equal 0, hence two of the other entries have to be zero. Since they have to be on distinct rows and columns, that means that two of the equations listed above have a coefficient of zero. Lets investigate each:

- If $s_{ba} = 0$, then $x = -R_a u + t_a n$. Note that one of the edges from Figure 6.3 corresponds to $x = 0$, so to determine where that edge goes in out transformation, we will set $x = 0$ and solve for u, as follows:

$$\begin{aligned} 0 &= -R_a u + t_a n \\ R_a u &= t_a n \\ u &= \frac{t_a n}{R_a}, \text{ since } R_a \neq 0. \end{aligned}$$

  Hence the edge that corresponds to $x = 0$ in Figure 6.3 is sent to vertical line in our transformation.

  We get a similar result when $s_{bc} = 0$.

- If $s_{ab} = 0$, then $y = -R_b v + t_b n$. Similar to previous case, one of the edges from Figure 6.3 corresponds to $y = 0$, so to determine where that edge goes in out transformation, we will set $y = 0$ and solve for v, as follows:

$$\begin{aligned} 0 &= -R_b v + t_b n \\ R_b v &= t_b n \\ v &= \frac{t_b n}{R_b}, \text{ since } R_b \neq 0. \end{aligned}$$

  Hence the edge that corresponds to $y = 0$ in Figure 6.3 is sent to horizontal line in our transformation.

  We get a similar result when $s_{ac} = 0$.

Thus, with any combination of the entries being 0 allowed by the hypothesis, we will get one horizontal edge and one vertical edge. Therefore the triangle in Figure 6.3 will be transformed into a right triangle in 2-dimensions. □

Conjecture 6.2.2. *When $R_a a, R_b b$ and $R_c c$ are distinct, then using the Johnson Transformation will not result in a right triangle.*

Proof. Lemma 2.2.4 proved that in this case, the entries s_{ab}, s_{ac}, s_{ba} and $s_{bc} > 0$. Hence by Proposition 6.2.1, the triangle in Figure 6.3 will not be transformed into a right triangle. □

Now lets consider the cases where it is possible to transform the triangle from Figure 6.3 into a right triangle.

6.2.1 $R_a a = R_b b = R_c c$

In this case, we can use the fact that $R_a a = R_b b = R_c c$ to create our matrix:

$$A = \begin{bmatrix} -R_a & 0 & t_a \\ R_b & -R_b & t_b \\ 0 & R_c & t_c \end{bmatrix}.$$

Similar to above,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

and note

$$n = ax + by + cy = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -R_a & 0 & t_a \\ R_b & -R_b & t_b \\ 0 & R_c & t_c \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w$$

hence this triangle has been transformed into the 2-dimensional space $w = n$. Now lets plug in the vertices to see where they are mapped:

- $$\begin{bmatrix} \frac{n}{a} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -R_a & 0 & t_a \\ R_b & -R_b & t_b \\ 0 & R_c & t_c \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -R_a u + t_a w \\ R_b u + -R_b v + t_b w \\ R_c v + t_c w \end{bmatrix}.$$

  So we have that

  $$\begin{aligned} \frac{n}{a} &= -R_a u + t_a w & \implies \quad u &= \frac{t_a n - \frac{n}{a}}{R_a} = \frac{t_a a n - n}{R_a a} \\ 0 &= R_b u - R_b v + t_b w & \implies \quad u - v &= \frac{t_b n}{R_b} \\ 0 &= R_c v + t_c w & \implies \quad v &= -\frac{t_c n}{R_c} \\ w &= n \end{aligned},$$

  i.e. $\left(\frac{n}{a}, 0, 0\right) \longmapsto \left(\frac{t_a a n - n}{R_a a}, -\frac{t_c n}{R_c}, n\right)$.

- $$\begin{bmatrix} 0 \\ \frac{n}{b} \\ 0 \end{bmatrix} = \begin{bmatrix} -R_a & 0 & t_a \\ R_b & -R_b & t_b \\ 0 & R_c & t_c \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -R_a u + t_a w \\ R_b u + -R_b v + t_b w \\ R_c v + t_c w \end{bmatrix}.$$

  So we have that

  $$\begin{aligned} 0 &= -R_a u + t_a w & \implies && u &= \frac{t_a n}{R_a} \\ \frac{n}{b} &= R_b u - R_b v + t_b w & \implies && u - v &= \frac{\frac{n}{b} - t_b n}{R_b} = \frac{n - t_b b n}{R_b b} \\ 0 &= R_c v + t_c w & \implies && v &= -\frac{t_c n}{R_c} \\ w &= n \end{aligned},$$

  i.e. $\left(0, \frac{n}{b}, 0\right) \longmapsto \left(\frac{t_a n}{R_a}, -\frac{t_c n}{R_c}, n\right)$.

- $$\begin{bmatrix} 0 \\ 0 \\ \frac{n}{c} \end{bmatrix} = \begin{bmatrix} -R_a & 0 & t_a \\ R_b & -R_b & t_b \\ 0 & R_c & t_c \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -R_a u + t_a w \\ R_b u + -R_b v + t_b w \\ R_c v + t_c w \end{bmatrix}.$$

  So we have that

  $$\begin{aligned} 0 &= -R_a u + t_a w & \implies && u &= \frac{t_a n}{R_a} \\ 0 &= R_b u - R_b v + t_b w & \implies && u - v &= \frac{t_b n}{R_b} \\ \frac{n}{c} &= R_c v + t_c w & \implies && v &= \frac{\frac{n}{c} - t_c n}{R_c} = \frac{n - t_c c n}{R_c c} \\ w &= n \end{aligned},$$

i.e. $\left(0,0,\dfrac{n}{c}\right)\longmapsto\left(\dfrac{t_an}{R_a},\dfrac{n-t_ccn}{R_cc},n\right).$

Note that two of the three points lie on the same vertical line and have a distance of $\dfrac{n}{c}$ between them. Meanwhile, two of the three points lie on the same horizontal line and have a distance of $\dfrac{n}{a}$ between them. So we have the expected right triangle in 2-space. Further, note that the slope is

$$\begin{aligned}\text{Slope} &= \frac{\frac{\frac{n}{c}-t_cn}{R_c}-\left(-\frac{t_cn}{R_c}\right)}{\frac{t_an}{R_a}-\left(\frac{t_an-\frac{n}{a}}{R_a}\right)}\\ &= \frac{\frac{\frac{n}{c}}{R_c}}{\frac{\frac{n}{a}}{R_a}}=\frac{\frac{n}{c}R_a}{R_c\frac{n}{a}}\\ &= \frac{nR_aa}{nR_cc}=1,\end{aligned}$$

since $R_aa=R_cc$. Note that the sides of this triangle may or may not be lattical. This will be completely dependent on the endpoints of the hypotenuse. If they are lattice points, then all three sides are latticial. Thus, after some reflections, shifts of integer distance and shading the margins, we have Figure 6.4.

If margins are present, then note that there will not be any lattice points in the margins of our triangle, thus when counting the lattice points, we will focus on the non-shaded area of the triangle in Figure 6.4. Further note that since $R_aa=R_cc$ then this triangle is an isosceles right triangle. This combined with the fact the the hypotenuse has slope $=-1$ means that both the vertices on the hypotenuse and hence all vertices of the non-shaded triangle, are lattice points. We can now simply use Pick's Theorem to count the number of lattice points

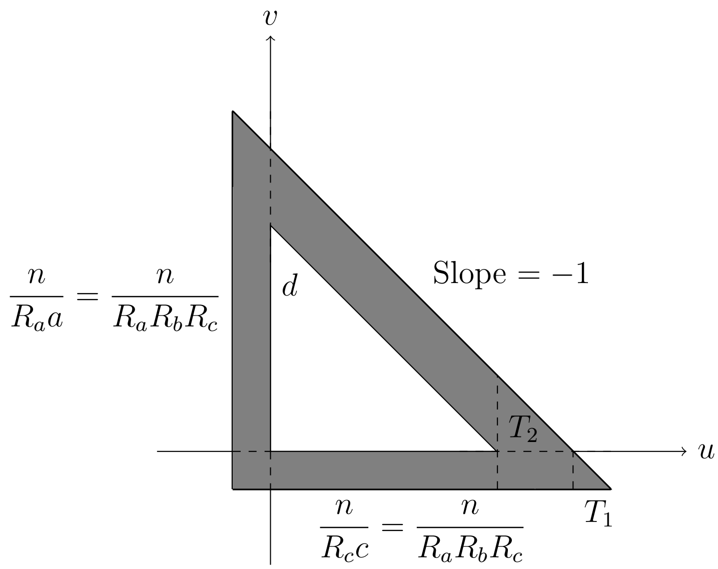


Figure 6.4: Isosceles Triangle after transformation from 3 to 2 dimensional space and integral translations

in our triangle. If the height of this isosceles right triangle is d, as indicated in Figure 6.4, then we have that

$$f(n) = \frac{d^2}{2} + 3\frac{d}{2} + 1 = \binom{d+2}{2}.$$

However, what is d in this case? To find that out, notice that T_1 and T_2 in Figure 6.4 are also isosceles right triangles. Further note that the latticial measure of the margin on the horizontal side of our triangle is $\left\{\frac{n}{\gcd(a,b)}\right\} = \left\{\frac{n}{R_c}\right\}$ and the latticial measure of the margin on the vertical side of our triangle is $\left\{\frac{n}{\gcd(a,c)}\right\} = \left\{\frac{n}{R_b}\right\}$. Thus the latticial measure of legs of $T_1 = \left\{\frac{n}{R_c}\right\}$ and the latticial measure of the legs of $T_2 = \left\{\frac{n}{R_a}\right\}$. Thus,

$$\text{the (latticial) length of } d = \frac{n}{R_a R_b R_c} - \left\{\frac{n}{R_a}\right\} - \left\{\frac{n}{R_b}\right\} - \left\{\frac{n}{R_c}\right\}.$$

So now we can write out $f(n)$:

$$\begin{aligned}
f(n) &= \frac{1}{2}\left(\frac{n}{R_aR_bR_c} - \left\{\frac{n}{R_a}\right\} - \left\{\frac{n}{R_b}\right\} - \left\{\frac{n}{R_c}\right\} + 2\right) \\
&\quad \times \left(\frac{n}{R_aR_bR_c} - \left\{\frac{n}{R_a}\right\} - \left\{\frac{n}{R_b}\right\} - \left\{\frac{n}{R_c}\right\} + 1\right) \\
&= \frac{n^2}{2R_a^2R_b^2R_c^2} - \frac{n}{R_aR_bR_c}\left(\left\{\frac{n}{R_a}\right\} + \left\{\frac{n}{R_b}\right\} + \left\{\frac{n}{R_c}\right\}\right) + \frac{n}{2R_aR_bR_c} \\
&\quad + \frac{1}{2}\left\{\frac{n}{R_a}\right\}^2 + \frac{1}{2}\left\{\frac{n}{R_b}\right\}^2 + \frac{1}{2}\left\{\frac{n}{R_c}\right\}^2 + \left\{\frac{n}{R_a}\right\}\left\{\frac{n}{R_b}\right\} + \left\{\frac{n}{R_a}\right\}\left\{\frac{n}{R_c}\right\} \\
&\quad + \left\{\frac{n}{R_b}\right\}\left\{\frac{n}{R_c}\right\} - \frac{1}{2}\left\{\frac{n}{R_a}\right\} - \frac{1}{2}\left\{\frac{n}{R_b}\right\} - \frac{1}{2}\left\{\frac{n}{R_c}\right\} + \frac{n}{R_aR_bR_c} \\
&\quad - \left\{\frac{n}{R_a}\right\} - \left\{\frac{n}{R_b}\right\} - \left\{\frac{n}{R_c}\right\} + 1
\end{aligned}$$

$$\begin{aligned}
= \ & \frac{n^2}{2abc} - \frac{n}{R_aR_bR_c}\left(\left\{\frac{n}{R_a}\right\} + \left\{\frac{n}{R_b}\right\} + \left\{\frac{n}{R_c}\right\} - \frac{3}{2}\right) + \frac{1}{2}\left\{\frac{n}{R_a}\right\}^2 + \frac{1}{2}\left\{\frac{n}{R_b}\right\}^2 \\
& + \frac{1}{2}\left\{\frac{n}{R_c}\right\}^2 + \left\{\frac{n}{R_a}\right\}\left\{\frac{n}{R_b}\right\} + \left\{\frac{n}{R_a}\right\}\left\{\frac{n}{R_c}\right\} + \left\{\frac{n}{R_b}\right\}\left\{\frac{n}{R_c}\right\} - \frac{3}{2}\left\{\frac{n}{R_a}\right\} \\
& - \frac{3}{2}\left\{\frac{n}{R_b}\right\} - \frac{3}{2}\left\{\frac{n}{R_c}\right\} + 1
\end{aligned}$$

6.2.2 $R_a a = R_b b \neq R_c c$

McNugget Problem - $R_6 6 = 18 = R_9 9 \neq R_{20} 20 = 60$

Consider the numerical monoid $< 6, 9, 20 >$. So our question for this example is, how many integer points are in the triangle, in the x, y, z-plane shown in Figure 6.5.

First thing we need to do is transform this triangle so that it fits in a 2-dimensional plane. To do this we will use the Johnson relations along with a linear combination for the $\gcd(6, 9, 20) = 1$. Note the following:

- $R_6 = 3$ as $6 \times 3 = 9 \times 2 = 18 \in < 9, 20 >$,

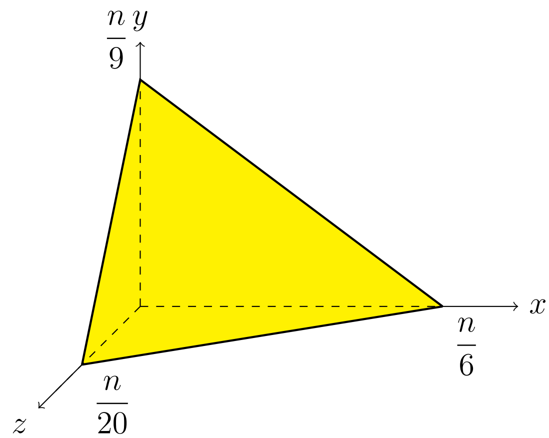


Figure 6.5: Plane in the first octant we are considering.

- $R_9 = 2$ as $9 \times 2 = 6 \times 3 = 18 \in < 6, 20 >$,

- $R_{20} = 3$ as $20 \times 3 = 6 \times 10 = 60 \in < 6, 9 >$,

- $6(2) + 9(1) + 20(-1) = 1$.

Using these items, we will create our matrix:

$$A = \begin{bmatrix} 10 & -3 & 2 \\ 0 & 2 & 1 \\ -3 & 0 & -1 \end{bmatrix},$$

such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Note that A has integer entries and $\det(A) = 1$, i.e. $A \in SL_2(\mathbb{Z})$ and hence

$$A^{-1} = \begin{bmatrix} -2 & -3 & -7 \\ -3 & -4 & -10 \\ 6 & 9 & 20 \end{bmatrix}.$$

Further note that

$$n = 6x + 9y + 20y = \begin{bmatrix} 6 & 9 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 & 9 & 20 \end{bmatrix} \begin{bmatrix} 10 & -3 & 2 \\ 0 & 2 & 1 \\ -3 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w$$

hence the vertices of the triangle will have the same w-coordinate under the new basis and hence will all lay on the plane $w = n$, i.e. this triangle has been transformed into a 2-dimensional space. Performing the transformation on the vertices we get the following in our new basis u, v, w:

$$A^{-1} \begin{bmatrix} \frac{n}{6} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{n}{3} \\ -\frac{n}{2} \\ n \end{bmatrix}, A^{-1} \begin{bmatrix} 0 \\ \frac{n}{9} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{n}{3} \\ -\frac{4n}{9} \\ n \end{bmatrix} \text{ and } A^{-1} \begin{bmatrix} 0 \\ 0 \\ \frac{n}{20} \end{bmatrix} = \begin{bmatrix} -\frac{7n}{20} \\ -\frac{n}{2} \\ n \end{bmatrix}.$$

Note that the first two points lie on the same vertical line and have a distance of $\frac{n}{18}$ between them. Meanwhile, the first and third point lie on the same horizontal line and have a distance of $\frac{n}{60}$ between them. Thus we have a right triangle

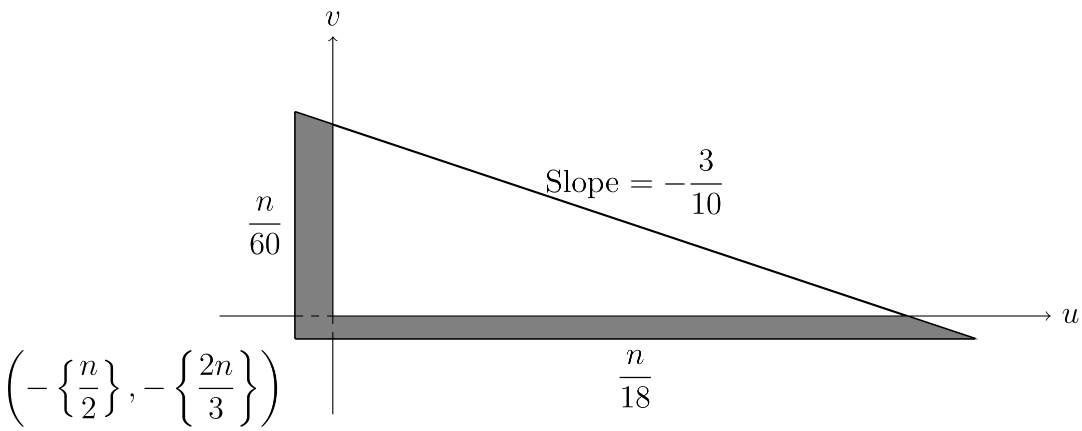


Figure 6.6: Triangle after transformation from 3 to 2 dimensional space and integral translations

in 2-dimensions. With some elementary reflections and translations by integral distances, we have Figure 6.6.

Notice that $0 \leq \left\{\frac{n}{2}\right\} < 1, 0 \leq \left\{\frac{2n}{3}\right\} < 1$. Therefore, there will never be a lattice point in the shaded area of Figure 6.6, hence we can just focus on the non-shaded area of the triangle, which we will now refer to as P, to find our lattice points. So the first thing we need to do is identify the location of the vertices of P, denoted at R and S, aside from the obvious vertex, which is at the origin, $O = (0, 0)$. In order to do this, we also need to vertices of the larger triangle (which includes the shaded area), which will be denoted as P', aside from the obvious vertex, which is $\left(-\left\{\frac{n}{2}\right\}, -\left\{\frac{2n}{3}\right\}\right)$. Lets denote these vertices of P' as R' and S'. Since the lengths of the vertical side and the horizontal side of P' are known, $\frac{n}{60}$ and $\frac{n}{18}$ respectively, then we can determine that $R' = \left(-\left\{\frac{n}{2}\right\}, \frac{n}{60} - \left\{\frac{2n}{3}\right\}\right)$ and $S' = \left(\frac{n}{18} - \left\{\frac{n}{2}\right\}, -\left\{\frac{2n}{3}\right\}\right)$. Thus we can calculate the vertices of P, $R = \left(0, \frac{n}{60} - \left\{\frac{2n}{3}\right\} - \frac{3}{10}\left\{\frac{n}{2}\right\}\right)$ and $S = \left(\frac{n}{18} - \left\{\frac{n}{2}\right\} - \frac{10}{3}\left\{\frac{2n}{3}\right\}, 0\right)$.

We will now proceed using the theorem of Brion. Therefore, we will be breaking our triangle from Figure 6.6 into the following cones:

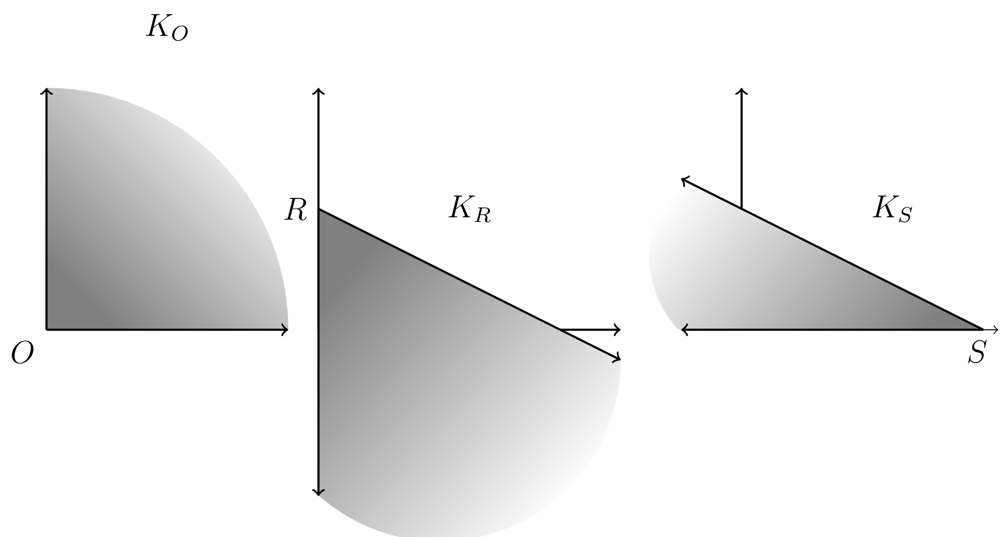


Figure 6.7: Cones used for McNugget Problem

However, before we fully use Brion, we need to investigate the fundamental parallelograms for each of the cones in Figure 6.7. Note that the fundamental parallelograms will be created by vectors, of latticial length 1, which are generating vectors of the cone, and will be used to tile the entire cone. For K_O, the latticial length will be the same as the Euclidean length so the fundamental parallelogram, which we will denote as FP_O, will actually be a half open unit square, as in Figure 6.8.

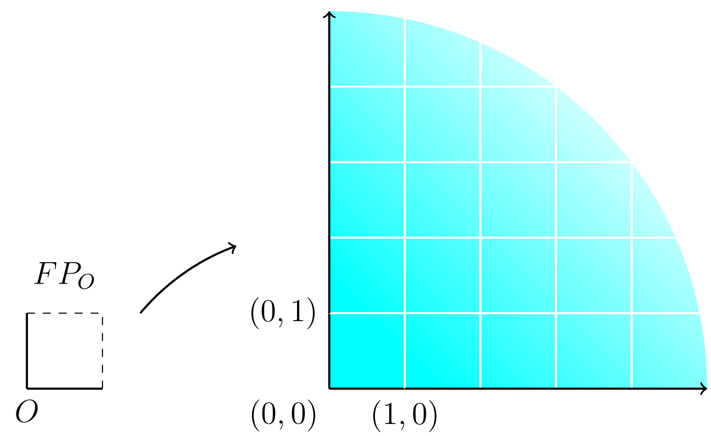


Figure 6.8: Fundamental Parallelogram of C_O

Therefore, we can represent all the lattice points in K_O using a Hilbert Series, written as a rational expression. However, the ability to tile this cone with FP_O makes this easier, as the numerator of this rational expression will correspond to the lattice points in FP_O, which, in this case, is just the origin O. Further, the factors of the denominator correspond to the basis vectors of K_O. So the Hilbert Series representing the lattice points in K_O is:

$$\frac{1}{(1-x)(1-y)}, \text{ since } x^0y^0 = 1.$$

Similarly, the FP_R, the latticial length of the vertical side will be the usual Euclidean length, however the side corresponding to the hypothenuse of P, will not. The vector creating that side of FP_R will the vector $\begin{bmatrix} 10 & -3 \end{bmatrix}$ translated up to R, as in Figure 6.9.

Note that in FP_R we have 10 lattice point, therefore the numerator of our

.

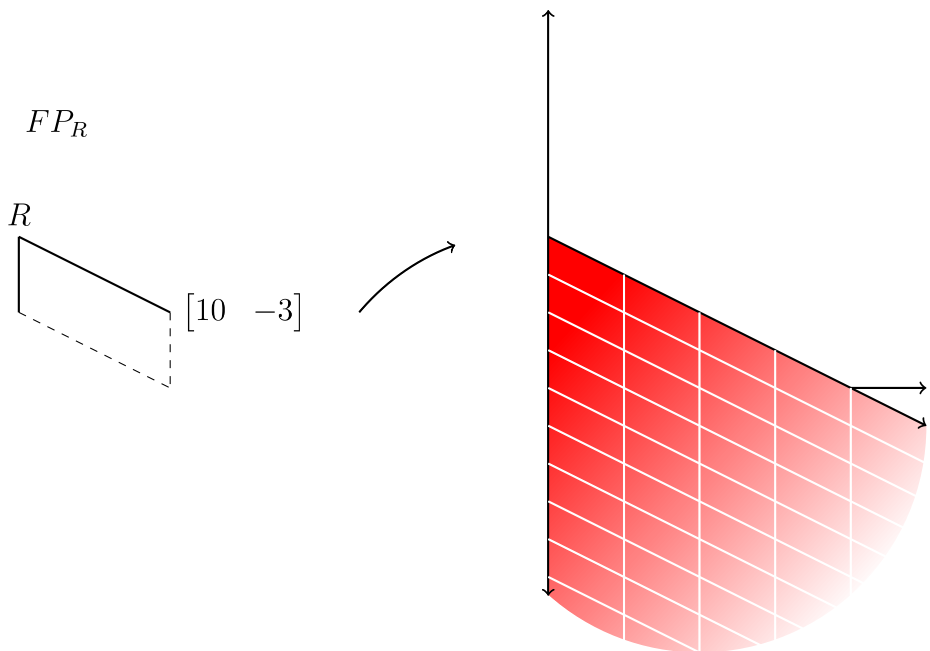


Figure 6.9: Fundamental Parallelogram of C_R

Hilbert Series, written as a rational expression, will have 10 terms (what those 10 terms are will depend on n). Lets denote these 10 terms as the polynomial $P_R(x)$. So the Hilbert Series, written as a rational expression will have the following form:

$$\frac{P_R(x)}{(1-y^{-1})(1-x^{10}y^{-3})}.$$

However, note that the y coordinates of the lattice points in FP_R are either $\lfloor y(R) \rfloor$, $\lfloor y(R) \rfloor - 1$, $\lfloor y(R) \rfloor - 2$ or $\lfloor y(R) \rfloor - 3$, where $y(R)$ denotes the y-coordinate of R.

Finally, the FP_S, the latticial length of the horizontal side will be the usual Euclidean length, however the side corresponding to the hypothenuse of P, will not. The vector creating that side of FP_S will the vector $\begin{bmatrix}-10 & 3\end{bmatrix}$ translated

.

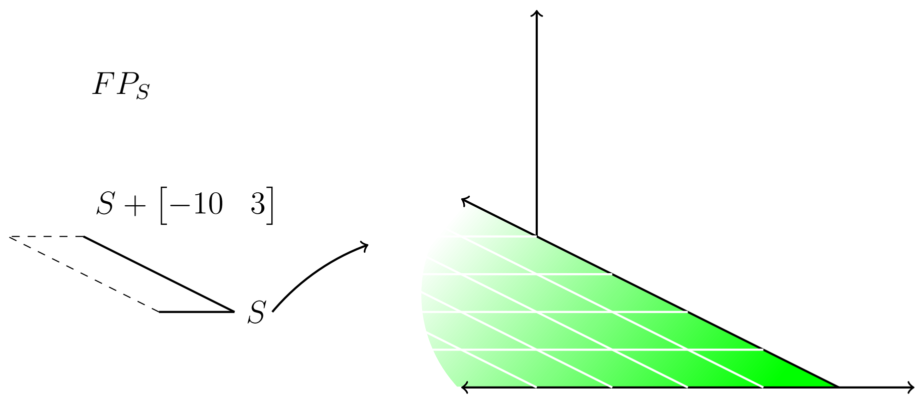


Figure 6.10: Fundamental Parallelogram of C_S

over to S, as in Figure 6.10.

Finally, note that in FP_S we have 3 lattice points, therefore the numerator of our Hilbert Series, written as a rational expression, will have 3 terms (what those 3 terms are will depend on n). Lets denote these 3 terms as the polynomial $P_S(x) = x^a + x^b y + x^c y^2$, where $a, b, c \in \mathbb{Z}_{\geq 0}$ and are distinct. So the generating Hilbert Series, written as a rational expression will have the following form:

$$\frac{P_S(x)}{(1 - x^{-1})(1 - x^{-10}y^3)} = \frac{x^a + x^b y + x^c y^2}{(1 - x^{-1})(1 - x^{-10}y^3)}.$$

Now consider the sum of these three rational expressions:

$$\frac{1}{(1-x)(1-y)} + \frac{P_R(x)}{(1 - y^{-1})(1 - x^{10}y^{-3})} + \frac{P_S(x)}{(1 - x^{-1})(1 - x^{-10}y^3)}.$$

Now trying to write these with the same denominator, we get

$$
\begin{aligned}
& \frac{1}{(1-x)(1-y)}+\frac{P_R(x)}{(1-y^{-1})(1-x^{10}y^{-3})}+\frac{P_S(x)}{(1-x^{-1})(1-x^{-10}y^{3})} \\
= \quad & \frac{(1-x^{10}y^{-3})}{(1-x)(1-y)(1-x^{10}y^{-3})}+\frac{y(P_R(x))}{(y-1)(1-x^{10}y^{-3})}+\frac{(x)(x^{10}y^{-3})(P_S(x))}{(x-1)(x^{10}y^{-3}-1)} \\
= \quad & \frac{(1-x^{10}y^{-3})}{(1-x)(1-y)(1-x^{10}y^{-3})}-\frac{y(1-x)(P_R(x))}{(1-x)(1-y)(1-x^{10}y^{-3})} \\
& +\frac{(1-y)(x^{11}y^{-3})(P_S(x))}{(1-x)(1-y)(1-x^{10}y^{-3})}.
\end{aligned}
$$

Now we need to look at $y(1-x)(P_R(x))$ and $(1-y)(x^{11}y^{-3})(P_S(x))$ a little closer.

$\underline{\mathbf{(1-y)(x^{11}y^{-3})(P_S(x))}}$

Recall that $P_S(x) = x^a + x^b y + x^c y^2$, thus

$$(x^{11}y^{-3})(P_S(x)) = x^{a'}y^{-3} + x^{b'}y^{-2} + x^{c'}y^{-1},$$

where $a' = a+11, b' = b+11$ and $c' = c+11$. So we can leave this product as

$$(1-y)(x^{11}y^{-3})(P_S(x)) = (1-y)(x^{a'}y^{-3} + x^{b'}y^{-2} + x^{c'}y^{-1}).$$

$\underline{\mathbf{y(1-x)(P_R(x))}}$

Note that since the y coordinates of the lattice points in FP_R are either $\lfloor y(R)\rfloor, \lfloor y(R)\rfloor - 1, \lfloor y(R)\rfloor - 2$ or $\lfloor y(R)\rfloor - 3$ and there are 10 lattice points in FP_R, then we have 2 cases to consider: $\underline{(\mathbf{1})}$ the points will have 3 distinct y coordinates values or $\underline{(\mathbf{2})}$ the points will have 4 distinct y coordinate values.

$\underline{(\mathbf{1})}$ If the points have 3 distinct y coordinate values, then $(1-x)(P_R(x)) = P_R(x) - x(P_R(x))$ will have some cancelling as multiplying by $-x$ will shift each point to the right one units but will be negative. Any overlap will result in a cancellation. Hence, $(1-x)(P_R(x))$ will have only 6 terms in this case, 3 inside FP_R and 3 above FP_R. Further, the points will be arranged in the following manner:

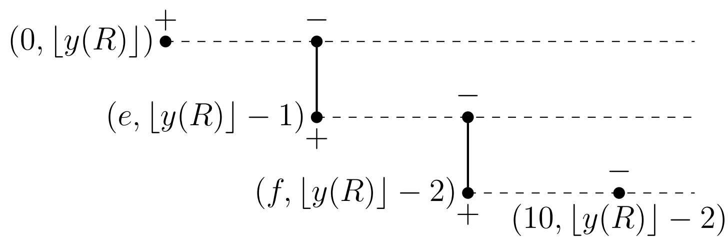


Figure 6.11: Layout of lattice points in/around FP_R in 6 term case

where $e, f \in \mathbb{Z}$ and $0 < e, f < 10$. Note that for point in Figure 6.11 which lay on the same vertical line can be written as (using the 2 points with f as the x-coordinate):

$$x^f y^{\lfloor y(R) \rfloor - 2} - x^f y^{\lfloor y(R) \rfloor - 1} = x^f y^{\lfloor y(R) \rfloor - 2}(1-y).$$

So using Figure 6.11, we can see that

$$\begin{aligned} (1-x)(P_R(x)) &= y^{\lfloor y(R) \rfloor} + x^e y^{\lfloor y(R) \rfloor - 1}(1-y) + x^f y^{\lfloor y(R) \rfloor - 2}(1-y) - x^{10} y^{\lfloor y(R) \rfloor - 2} \\ y(1-x)(P_R(x)) &= y^{\lfloor y(R) \rfloor + 1} + x^e y^{\lfloor y(R) \rfloor}(1-y) + x^f y^{\lfloor y(R) \rfloor - 1}(1-y) - x^{10} y^{\lfloor y(R) \rfloor - 1} \end{aligned}$$

Hence

$$(1 - x^{10} y^{-3}) - y(1-x)(P_R(x)) =$$
$$(1 - y^{\lfloor y(R) \rfloor + 1}) - x^e y^{\lfloor y(R) \rfloor}(1-y) - x^f y^{\lfloor y(R) \rfloor - 1}(1-y) - x^{10} y^{-3} + x^{10} y^{\lfloor y(R) \rfloor - 1}.$$

Going back to the sum of the three rational expressions, one for each cone, we have

$$\frac{P(x)}{(1-x)(1-y)(1-x^{10}y^{-3})}$$

where

$$\begin{aligned} P(x) \quad = \quad & (\lfloor y(R) \rfloor + 1)_y(1-y) - x^e y^{\lfloor y(R) \rfloor}(1-y) - x^f y^{\lfloor y(R) \rfloor - 1}(1-y) \\ & -x^{10}y^{-3}(\lfloor y(R) \rfloor + 2)_y(1-y) + (1-y)[x^{a'}y^{-3} + x^{b'}y^{-2} + x^{c'}y^{-1}]. \end{aligned}$$

We can now cancel (1-y) from this rational expression, so we have

$$\frac{(\lfloor y(R) \rfloor + 1)_y - x^e y^{\lfloor y(R) \rfloor} - x^f y^{\lfloor y(R) \rfloor - 1} - x^{10}y^{-3}(\lfloor y(R) \rfloor + 2)_y + x^{a'}y^{-3} + x^{b'}y^{-2} + x^{c'}y^{-1}}{(1-x)(1-x^{10}y^{-3})}.$$

Now, since we are counting lattice points, we can simplify this expression further by setting $y = 1$, hence we now have

$$\frac{(\lfloor y(R) \rfloor + 1) - x^e - x^f - x^{10}(\lfloor y(R) \rfloor + 2) + x^{a'} + x^{b'} + x^{c'}}{(1-x)(1-x^{10})}.$$

Now consider the limit of this expression as $x \longrightarrow 1$.

$$\begin{aligned} & \lim_{x \longrightarrow 1} \frac{(\lfloor y(R) \rfloor + 1) - x^e - x^f - x^{10}(\lfloor y(R) \rfloor + 2) + x^{a'} + x^{b'} + x^{c'}}{(1-x)(1-x^{10})} \\ \overset{(LH)}{=} \; & \lim_{x \longrightarrow 1} \frac{-ex^{e-1} - fx^{f-1} - 10x^9(\lfloor y(R) \rfloor + 2) + a'x^{a'-1} + b'x^{b'-1} + c'x^{c'-1}}{-(1-x^{10})(-10x^9)(1-x)} \\ \overset{(LH)}{=} \; & \lim_{x \longrightarrow 1} \frac{-e(e-1)x^{e-2} - f(f-1)x^{f-2} - 90x^8(\lfloor y(R) \rfloor + 2)}{10x^9 + 10x^9 + (1-x)(-90x^8)} \\ & + \frac{a'(a'-1)x^{a'-2} + b'(b'-1)x^{b'-2} + c'(c'-1)x^{c'-2}}{10x^9 + 10x^9 + (1-x)(-90x^8)} \\ = \; & \frac{-e(e-1) - f(f-1) - 90(\lfloor y(R) \rfloor + 2) + a'(a'-1) + b'(b'-1) + c'(c'-1)}{20}. \end{aligned}$$

Note that after the first use of L'Hopital's Rule, we get a new identity:

$$-e - f - 10(\lfloor y(R) \rfloor + 2) + a' + b' + c' = 0 \text{ or}$$

$$-10(\lfloor y(R) \rfloor + 2) = e + f - a' - b' - c'.$$

(**2**) If the points have 4 distinct y coordinate values, then the cancellation will work the same way as the previous case. Hence, $(1-x)(P_R(x))$ will have only 8 terms in this case, 4 inside FP_R and 4 above FP_R. Further, the points will be arranged in the following manner:

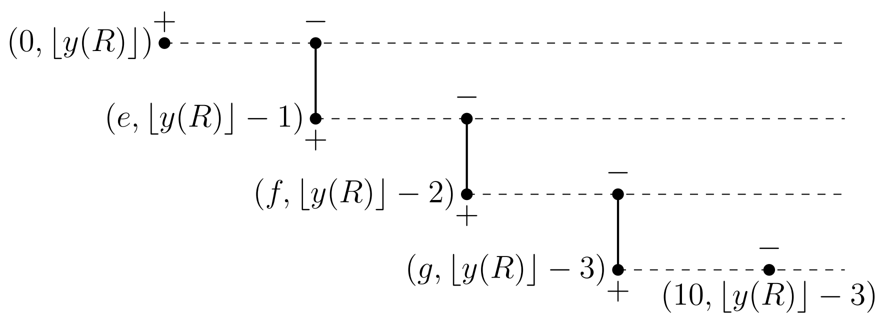


Figure 6.12: Layout of lattice points in/around FP_R in 8 term case

where $e, f, g \in \mathbb{Z}$ and $0 < e, f, g < 10$. Once again, note that the sum of points on the same vertical line in Figure 6.12 can be written as:

$$x^f y^{\lfloor y(R) \rfloor - 2} - x^f y^{\lfloor y(R) \rfloor - 1} = x^f y^{\lfloor y(R) \rfloor - 2}(1-y).$$

So using Figure 6.12, we can see that

$$\begin{aligned}
(1-x)(P_R(x)) &= y^{\lfloor y(R) \rfloor} + x^e y^{\lfloor y(R) \rfloor - 1}(1-y) + x^f y^{\lfloor y(R) \rfloor - 2}(1-y) \\
&\quad + x^g y^{\lfloor y(R) \rfloor - 3}(1-y) - x^{10} y^{\lfloor y(R) \rfloor - 3} \\
y(1-x)(P_R(x)) &= y^{\lfloor y(R) \rfloor + 1} + x^e y^{\lfloor y(R) \rfloor}(1-y) + x^f y^{\lfloor y(R) \rfloor - 1}(1-y) \\
&\quad + x^g y^{\lfloor y(R) \rfloor - 2}(1-y) - x^{10} y^{\lfloor y(R) \rfloor - 2}
\end{aligned}$$

Hence

$$(1 - x^{10}y^{-3}) - y(1-x)(P_R(x)) = (1 - y^{\lfloor y(R)\rfloor+1}) - x^e y^{\lfloor y(R)\rfloor}(1-y) -$$
$$x^f y^{\lfloor y(R)\rfloor-1}(1-y) - x^g y^{\lfloor y(R)\rfloor-2}(1-y) - x^{10}y^{-3} + x^{10}y^{\lfloor y(R)\rfloor-2}.$$

Going back to the sum of the three rational expressions, one for each cone, we have

$$\frac{P(x)}{(1-x)(1-y)(1-x^{10}y^{-3})}$$

where

$$\begin{aligned} P(x) &= (\lfloor y(R)\rfloor+1)_y(1-y) - x^e y^{\lfloor y(R)\rfloor}(1-y) - x^f y^{\lfloor y(R)\rfloor-1}(1-y) \\ &\quad - x^g y^{\lfloor y(R)\rfloor-2}(1-y) - x^{10}y^{-3}(d+1)_y(1-y) \\ &\quad + (1-y)[x^{a'}y^{-3} + x^{b'}y^{-2} + x^{c'}y^{-1}]. \end{aligned}$$

We can now cancel (1-y) from this rational expression, so we have

$$\frac{(\lfloor y(R)\rfloor+1)_y - x^e y^{\lfloor y(R)\rfloor} - x^f y^{\lfloor y(R)\rfloor-1} - x^g y^{\lfloor y(R)\rfloor-2} - x^{10}y^{-3}(\lfloor y(R)\rfloor+1)_y}{(1-x)(1-x^{10}y^{-3})}$$
$$+\frac{x^{a'}y^{-3} + x^{b'}y^{-2} + x^{c'}y^{-1}}{(1-x)(1-x^{10}y^{-3})}$$

Now, since we are counting lattice points, we can simplify this expression further by setting $y = 1$, hence we now have

$$\frac{(\lfloor y(R)\rfloor+1) - x^e - x^f - x^g - x^{10}(\lfloor y(R)\rfloor+1) + x^{a'} + x^{b'} + x^{c'}}{(1-x)(1-x^{10})}.$$

Now consider the limit of this expression as $x \longrightarrow 1$.

$$
\begin{aligned}
&\lim_{x\longrightarrow 1}\frac{(\lfloor y(R)\rfloor+1)-x^e-x^f-x^g-x^{10}(\lfloor y(R)\rfloor+1)+x^{a'}+x^{b'}+x^{c'}}{(1-x)(1-x^{10})}\\
\overset{(LH)}{=}\;&\lim_{x\longrightarrow 1}\frac{-ex^{e-1}-fx^{f-1}-gx^{g-1}-10x^9(\lfloor y(R)\rfloor+1)+a'x^{a'-1}+b'x^{b'-1}+c'x^{c'-1}}{-(1-x^{10})(-10x^9)(1-x)}\\
\overset{(LH)}{=}\;&\lim_{x\longrightarrow 1}\frac{-e(e-1)x^{e-2}-f(f-1)x^{f-2}-g(g-1)x^{g-2}-90x^8(\lfloor y(R)\rfloor+1)}{10x^9+10x^9+(1-x)(-90x^8)}\\
&+\frac{a'(a'-1)x^{a'-2}+b'(b'-1)x^{b'-2}+c'(c'-1)x^{c'-2}}{10x^9+10x^9+(1-x)(-90x^8)}\\
=\;&\frac{-e(e-1)-f(f-1)-g(g-1)-90(\lfloor y(R)\rfloor+1)}{20}\\
&+\frac{a'(a'-1)+b'(b'-1)+c'(c'-1)}{20}
\end{aligned}
$$

Note that after the first use of L'Hopital's Rule, we get a new identity:

$$-e-f-g-10(\lfloor y(R)\rfloor+1)+a'+b'+c'=0 \text{ or}$$

$$-10(\lfloor y(R)\rfloor+1)=e+f+g-a'-b'-c'.$$

Now lets focus the possibilities for FP_R. Note that the lattice point on the y-axis in FP_R is $(0,\lfloor y(R)\rfloor)$, then all of the possible number of lattice points at heights $\lfloor y(R)\rfloor$, $\lfloor y(R)\rfloor-1$, $\lfloor y(R)\rfloor-2$ and $\lfloor y(R)\rfloor-3$, respectively, in the fundamental parallelogram are:

- when $\{y(R)\}=0$, then the number of integer points at each height is $1,3,3,3$;
- when $\{y(R)\}=\frac{1}{10}$, then the number of integer points at each height is $1,3,4,2$;
- when $\{y(R)\}=\frac{2}{10}$, then the number of integer points at each height is $1,4,3,2$;

- when $\{y(R)\} = \frac{3}{10}$, then the number of integer points at each height is $2, 3, 3, 2$;

- when $\{y(R)\} = \frac{4}{10}$, then the number of integer points at each height is $2, 3, 4, 1$;

- when $\{y(R)\} = \frac{5}{10}$, then the number of integer points at each height is $2, 4, 3, 1$;

- when $\{y(R)\} = \frac{6}{10}$, then the number of integer points at each height is $3, 3, 3, 1$;

- when $\{y(R)\} = \frac{7}{10}$, then the number of integer points at each height is $3, 3, 4, 0$;

- when $\{y(R)\} = \frac{8}{10}$, then the number of integer points at each height is $3, 4, 3, 0$;

- when $\{y(R)\} = \frac{9}{10}$, then the number of integer points at each height is $4, 3, 3, 0$;

so the 6-term case happens when $\{\lfloor y(R) \rfloor\} \geq \frac{7}{10}$ and the 8-term case happens when $0 \leq \{\lfloor y(R) \rfloor\} < \frac{7}{10}$ (or $0 \leq \{\lfloor y(R) \rfloor\} \leq \frac{6}{10}$ since these values always have to form $\frac{i}{10}$ where $0 \leq i \leq 9$ and $i \in \mathbb{Z}_{\geq 0}$).

Now lets focus on the possibilities for FP_S. Note that the lattice point on the x-axis in FP_S is $\left(\frac{10}{3}\lfloor y(R)\rfloor, 0\right)$, then all of the possible location of lattice points in the fundamental parallelogram are:

- when $\left\{\frac{10}{3}\lfloor y(R)\rfloor\right\} = 0$, then the integer points are at $\left(\frac{10}{3}\lfloor y(R)\rfloor, 0\right)$, $\left(\frac{10}{3}\lfloor y(R)\rfloor - 4, 1\right)$ and $\left(\frac{10}{3}\lfloor y(R)\rfloor - 7, 2\right)$;
- when $\left\{\frac{10}{3}\lfloor y(R)\rfloor\right\} = \frac{1}{3}$, then the integer points are at $\left(\left\lfloor\frac{10}{3}\lfloor y(R)\rfloor\right\rfloor, 0\right)$, $\left(\left\lfloor\frac{10}{3}\lfloor y(R)\rfloor\right\rfloor - 3, 1\right)$ and $\left(\left\lfloor\frac{10}{3}\lfloor y(R)\rfloor\right\rfloor - 7, 2\right)$;
- when $\left\{\frac{10}{3}\lfloor y(R)\rfloor\right\} = \frac{2}{3}$, then the integer points are at $\left(\left\lfloor\frac{10}{3}\lfloor y(R)\rfloor\right\rfloor, 0\right)$, $\left(\left\lfloor\frac{10}{3}\lfloor y(R)\rfloor\right\rfloor - 3, 1\right)$ and $\left(\left\lfloor\frac{10}{3}\lfloor y(R)\rfloor\right\rfloor - 6, 2\right)$;

Recall from the 6-term case, after factoring out and cancelling $(1-y)$, we have

$$\frac{(\lfloor y(R)\rfloor + 1)_y - x^e y^{\lfloor y(R)\rfloor} - x^f y^{\lfloor y(R)\rfloor - 1} - x^{10}y^{-3}(\lfloor y(R)\rfloor + 2)_y + x^{a'}y^{-3} + x^{b'}y^{-2} + x^{c'}y^{-1}}{(1-x)(1-x^{10}y^{-3})}.$$

Note that since we are counting lattice points with this rational expression, once simplified, this will be a polynomial, thus we know that the numerator is divisible by $(1 - x^{10}y^{-3})$. What this means is that when the points represented are plotted, they all have to be able to be mapped to another point by a multiple of the vector $\begin{bmatrix} 10 & -3 \end{bmatrix}$, further, they should cancel based on the sign associated with each point. This is represented visually in Figure 6.13.

Note that the equation for the line segment of the hypothenuse is $3x + 10y = 10(y(R))$. Then the location for the remaining unknown coordinates from Figure 6.13 are as follows:

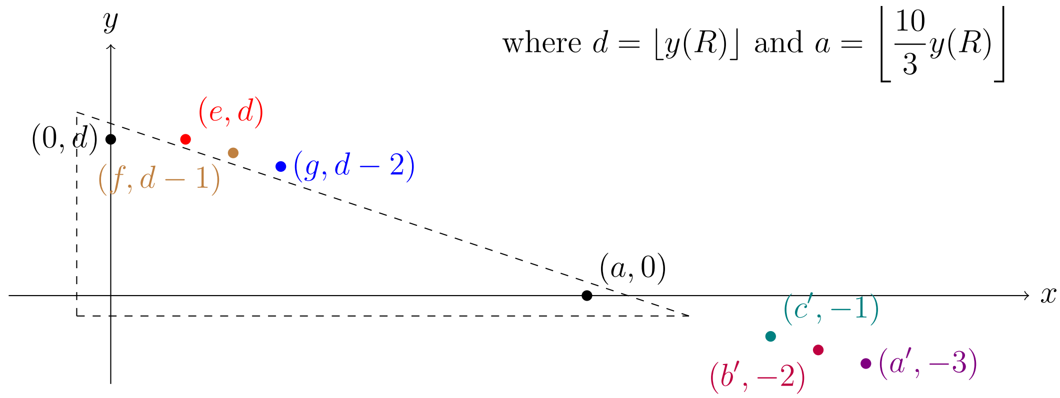


Figure 6.13: Dashed McNugget Triangle with important lattice points indicated 8 term case (for the 6 term case, $g = 10$)

- $e = \left\lfloor \frac{10}{3}\{y(R)\} \right\rfloor + 1;$
- $f = \left\lfloor \frac{10}{3}(\{y(R)\} + 1) \right\rfloor + 1;$
- $g = \left\lfloor \frac{10}{3}(\{y(R)\} + 2) \right\rfloor + 1;$
- $a = \left\lfloor \frac{10}{3} y(R) \right\rfloor;$
- $b = \left\lfloor \frac{10}{3}(y(R) - 1) \right\rfloor;$
- $c = \left\lfloor \frac{10}{3}(y(R) - 2) \right\rfloor.$

Now lets look at the difference in the y-coordinates of the points with these x-coordinates.

<u>**6 term case**</u>

Lets consider $[(\lfloor y(R) \rfloor - 2) - (-3)] \mod 3 = \lfloor y(R) \rfloor + 1 \mod 3$.

- When $\lfloor y(R) \rfloor + 1 \mod 3 \equiv 0$ then the points $(10, \lfloor y(R) \rfloor - 2)$ and $(a', -3)$

and hence also $(f, \lfloor y(R) \rfloor - 1)$ and $(b', -2)$; and $(e, \lfloor y(R) \rfloor)$ and $(c', -1)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

* $10 + 10N = a' \Longrightarrow 10(N+1) = a'$;
* $f + 10N = b'$;
* $e + 10N = c'$;
* $\lfloor y(R) \rfloor - 2 - 3N = -3 \Longrightarrow \lfloor y(R) \rfloor = 3N - 1$;
* $\lfloor y(R) \rfloor - 1 - 3N = -2 \Longrightarrow \lfloor y(R) \rfloor = 3N - 1$;
* $\lfloor y(R) \rfloor - 3N = -1 \Longrightarrow \lfloor y(R) \rfloor = 3N - 1$;

Now substituting the results for a', b' and c' into the numerator of the rational expression for this case we get

$$\begin{aligned}
& -e(e-1) - f(f-1) - 90(\lfloor y(R) \rfloor + 2) + a'(a'-1) + b'(b'-1) \\
& +c'(c'-1) \\
= \quad & -e^2 + e - f^2 + f - 90[(3N-1)+2] + [10(N+1)][10(N+1)-1] \\
& +(f+10N)(f+10N-1) + (e+10N)(e+10N-1) \\
= \quad & -e^2 + e - f^2 + f - 270N - 90 + 100N^2 + 190N + 90 \\
& +f^2 + 20Nf - f + 100N^2 - 10N + e^2 + 20Ne - e + 100N^2 - 10N \\
= \quad & 300N^2 - 100N + 20Ne + 20Nf
\end{aligned}$$

So, since the denominator of the rational expression for this case is 20, then we have

$$15N^2 - 5N + Ne + Nf.$$

Since $N = \dfrac{\lfloor y(R) \rfloor + 1}{3}$, then we have

$$\begin{aligned}
& 15\left(\frac{\lfloor y(R)\rfloor + 1}{3}\right)^2 - 5\left(\frac{\lfloor y(R)\rfloor + 1}{3}\right) + e\left(\frac{\lfloor y(R)\rfloor + 1}{3}\right) \\
& +f\left(\frac{\lfloor y(R)\rfloor + 1}{3}\right) \\
= \;& \frac{5}{3}(\lfloor y(R)\rfloor^2 + 2\lfloor y(R)\rfloor + 1) + \frac{e+f-5}{3}(\lfloor y(R)\rfloor + 1) \\
= \;& \frac{5}{3}\lfloor y(R)\rfloor^2 + \frac{e+f+5}{3}\lfloor y(R)\rfloor + \frac{e+f}{3} \\
= \;& \frac{5}{3}(y(R)^2 - 2y(R)\{y(R)\} + \{y(R)\}^2) \\
& +\frac{e+f+5}{3}(y(R) - \{y(R)\}) + \frac{e+f}{3} \\
= \;& \frac{5}{3}y(R)^2 + \left(\frac{e+f+5}{3} - \frac{10}{3}\{y(R)\}\right) y(R) + \frac{5}{3}\{y(R)\}^2 \\
& -\frac{e+f+5}{3}\{y(R)\} + \frac{e+f}{3}
\end{aligned}$$

Recall when simplifying the partial fractions, we wanted to write the generating function in the form $f(x) = rx^2 + sx + t$. Recall the linear coefficient, s, was

$$\frac{Aa + Bb + Cc - 2Aa\left\{\frac{s_1 n}{A}\right\} - 2Bb\left\{\frac{s_2 n}{B}\right\} - 2Cc\left\{\frac{s_3 n}{C}\right\}}{2abc},$$

thus for $< 6, 9, 20 >$ we have

$$\begin{aligned}
& \frac{6 + 18 + 60 - 12\left\{\frac{2n}{1}\right\} - 36\left\{\frac{n}{2}\right\} - 120\left\{\frac{-1n}{3}\right\}}{2*6*9*20} \\
= \;& \frac{6 + 18 + 60 - 36\left\{\frac{n}{2}\right\} - 120\left\{\frac{2n}{3}\right\}}{2*6*9*20} \\
= \;& \frac{1}{2}\left(\frac{1}{3*60} + \frac{1}{60} + \frac{1}{18}\right) - \frac{\left\{\frac{n}{2}\right\}}{60} - \frac{\left\{\frac{2n}{3}\right\}}{18}.
\end{aligned}$$

Recall that $y(R) = \frac{n}{60} - \left\{\frac{2n}{3}\right\} - \frac{3}{10}\left\{\frac{n}{2}\right\}$ and hence

$$\frac{10}{3}y(R)^2 = \frac{n^2}{360*6} - \left(\frac{1}{18}\left\{\frac{2n}{3}\right\} + \frac{1}{6}\left\{\frac{n}{2}\right\}\right)n + \ldots.$$

Thus we have

$$\begin{aligned}
& \frac{1}{60}\left(\frac{e+f+5}{3} - \frac{10}{3}\{y(R)\}\right) - \left(\frac{1}{18}\left\{\frac{2n}{3}\right\} + \frac{1}{6}\left\{\frac{n}{2}\right\}\right) \\
= \quad & \frac{1}{2}\left(\frac{1}{3*60} + \frac{1}{60} + \frac{1}{18}\right) - \frac{\left\{\frac{n}{2}\right\}}{60} - \frac{\left\{\frac{2n}{3}\right\}}{18} \\
\Longrightarrow \quad & \frac{1}{60}\left(\frac{e+f+5}{3} - \frac{10}{3}\{y(R)\}\right) = \frac{1}{2}\left(\frac{1}{3*60} + \frac{1}{60} + \frac{1}{18}\right) = \frac{7}{180} \\
\Longrightarrow \quad & e + f + 5 - 10\{y(R)\} = 7 \\
\Longrightarrow \quad & e + f = 2 + 10\{y(R)\}
\end{aligned}$$

- When $\lfloor y(R) \rfloor + 1 \mod 3 \equiv 1$ then the points $(10, \lfloor y(R) \rfloor - 2)$ and $(b', -2)$ and hence also $(f, \lfloor y(R) \rfloor - 1)$ and $(c', -1)$; and $(e, \lfloor y(R) \rfloor)$ and $(a', -3)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

  * $10 + 10N = b' \Longrightarrow 10(N+1) = b'$;

  * $f + 10N = c'$;

  * $e + 10(N+1) = a'$;

  * $\lfloor y(R) \rfloor - 2 - 3N = -2 \Longrightarrow \lfloor y(R) \rfloor = 3N$;

  * $\lfloor y(R) \rfloor - 1 - 3N = -1 \Longrightarrow \lfloor y(R) \rfloor = 3N$;

  * $\lfloor y(R) \rfloor - 3(N+1) = -3 \Longrightarrow \lfloor y(R) \rfloor = 3N$;

Now substituting the results for a', b' and c' into the numerator of the rational expression for this case we get

$$\begin{aligned}
& -e(e-1) - f(f-1) - 90(\lfloor y(R) \rfloor + 2) + a'(a'-1) + b'(b'-1) \\
& +c'(c'-1) \\
= \quad & -e^2 + e - f^2 + f - 90[3N+2] + [e+10N+10][e+10N+9] \\
& +(10N+10)(10N+9) + (f+10N)(f+10N-1) \\
= \quad & -e^2 + e - f^2 + f - 270N - 180 + e^2 + 20Ne + 19e + 100N^2 + 190N \\
& +100N^2 + 190N + 180 + f^2 + 20Nf - f + 100N^2 - 10N \\
= \quad & 300N^2 + 100N + 20Ne + 20Nf + 20e
\end{aligned}$$

So, since the denominator of the rational expression for this case is 20, then we have

$$15N^2 + 5N + Ne + Nf + e.$$

Since $N = \dfrac{\lfloor y(R) \rfloor}{3}$, then we have

$$\begin{aligned}
& 15\left(\frac{\lfloor y(R) \rfloor}{3}\right)^2 + 5\left(\frac{\lfloor y(R) \rfloor}{3}\right) + e\left(\frac{\lfloor y(R) \rfloor}{3}\right) + f\left(\frac{\lfloor y(R) \rfloor}{3}\right) + e \\
= \quad & \frac{5}{3}\lfloor y(R) \rfloor^2 + \frac{e+f-5}{3}(\lfloor y(R) \rfloor) + e \\
= \quad & \frac{5}{3}(y(R)^2 - 2y(R)\{y(R)\} + \{y(R)\}^2) + \frac{e+f+5}{3}(y(R) - \{y(R)\}) + e \\
= \quad & \frac{5}{3}y(R)^2 + \left(\frac{e+f+5}{3} - \frac{10}{3}\{y(R)\}\right) y(R) + \frac{5}{3}\{y(R)\}^2 \\
& -\frac{e+f+5}{3}\{y(R)\} + e
\end{aligned}$$

Similarly to previous case, we have

$$\begin{aligned}
& \frac{1}{60}\left(\frac{e+f+5}{3}-\frac{10}{3}\{y(R)\}\right)-\left(\frac{1}{18}\left\{\frac{2n}{3}\right\}+\frac{1}{6}\left\{\frac{n}{2}\right\}\right) \\
= \quad & \frac{1}{2}\left(\frac{1}{3*60}+\frac{1}{60}+\frac{1}{18}\right)-\frac{\left\{\frac{n}{2}\right\}}{60}-\frac{\left\{\frac{2n}{3}\right\}}{18} \\
\Longrightarrow \quad & e+f=2+10\{y(R)\}
\end{aligned}$$

- When $\lfloor y(R)\rfloor+1 \mod 3 \equiv 2$ then the points $(10, \lfloor y(R)\rfloor-2)$ and $(c', -1)$ and hence also $(f, \lfloor y(R)\rfloor-1)$ and $(a', -3)$; and $(e, \lfloor y(R)\rfloor)$ and $(b', -2)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

  * $10+10N=c' \Longrightarrow 10(N+1)=c'$;

  * $f+10(N+1)=a'$;

  * $e+10(N+1)=b'$;

  * $\lfloor y(R)\rfloor-2-3N=-1 \Longrightarrow \lfloor y(R)\rfloor=3N+1$;

  * $\lfloor y(R)\rfloor-1-3(N+1)=-3 \Longrightarrow \lfloor y(R)\rfloor=3N+1$;

  * $\lfloor y(R)\rfloor-3(N+1)=-2 \Longrightarrow \lfloor y(R)\rfloor=3N+1$;

Now substituting the results for a', b' and c' into the numerator of the rational expression for this case we get

$$\begin{aligned}
& -e(e-1) - f(f-1) - 90(\lfloor y(R) \rfloor + 2) + a'(a'-1) + b'(b'-1) \\
& +c'(c'-1) \\
= \quad & -e^2 + e - f^2 + f - 90[3N+3] + [f+10N+10][f+10N+9] \\
& +(e+10N+10)(e+10N+9) + (10N+10)(10N+9) \\
= \quad & -e^2 + e - f^2 + f - 270N - 270 + f^2 + 20Nf + 19f + 100N^2 + 190N \\
& +e^2 + 20Ne + 19e + 100N^2 + 190N + 100N^2 + 190N + 270 \\
= \quad & 300N^2 + 300N + 20Ne + 20Nf + 20e + 20f
\end{aligned}$$

So, since the denominator of the rational expression for this case is 20, then we have

$$15N^2 + 15N + Ne + Nf + e + f.$$

Since $N = \dfrac{\lfloor y(R) \rfloor - 1}{3}$, then we have

$$\begin{aligned}
& 15\left(\frac{\lfloor y(R) \rfloor - 1}{3}\right)^2 + 15\left(\frac{\lfloor y(R) \rfloor - 1}{3}\right) + e\left(\frac{\lfloor y(R) \rfloor - 1}{3}\right) \\
& +f\left(\frac{\lfloor y(R) \rfloor - 1}{3}\right) + e + f \\
= \quad & \frac{5}{3}(\lfloor y(R) \rfloor^2 - 2\lfloor y(R) \rfloor + 1) + \frac{15}{3}(\lfloor y(R) \rfloor - 1) + \frac{e+f}{3}(\lfloor y(R) \rfloor - 1) \\
& +e + f
\end{aligned}$$

$$\begin{aligned}
&= \frac{5}{3}(\lfloor y(R)\rfloor^2 - 2\lfloor y(R)\rfloor + 1) + \frac{e+f+15}{3}(\lfloor y(R)\rfloor - 1) + e + f \\
&= \frac{5}{3}\lfloor y(R)\rfloor^2 + \left(\frac{e+f+5}{3}\right)\lfloor y(R)\rfloor - \left(\frac{e+f+10}{3}\right) + e + f \\
&= \frac{5}{3}(y(R)^2 - 2y(R)\{y(R)\} + \{y(R)\}^2) + \left(\frac{e+f+5}{3}\right)(y(R) - \{y(R)\}) \\
&\quad + \frac{2e+2f-10}{3} \\
&= \frac{5}{3}y(R)^2 + \left(\frac{e+f+5}{3} - \frac{10}{3}\{y(R)\}\right)y(R) + \frac{5}{3}\{y(R)\}^2 \\
&\quad - \left(\frac{e+f+5}{3}\right)\{y(R)\} + \frac{2e+2f-10}{3}
\end{aligned}$$

Similarly to previous case, we have

$$\begin{aligned}
&\frac{1}{60}\left(\frac{e+f+5}{3} - \frac{10}{3}\{y(R)\}\right) - \left(\frac{1}{18}\left\{\frac{2n}{3}\right\} + \frac{1}{6}\left\{\frac{n}{2}\right\}\right) \\
= \;& \frac{1}{2}\left(\frac{1}{3*60} + \frac{1}{60} + \frac{1}{18}\right) - \frac{\left\{\frac{n}{2}\right\}}{60} - \frac{\left\{\frac{2n}{3}\right\}}{18} \\
\implies \;& e + f = 2 + 10\{y(R)\}
\end{aligned}$$

<u>8 **term case**</u>

Recall that the numerator for this case is:

$$-e(e-1) - f(f-1) - g(g-1) - 90(\lfloor y(R)\rfloor + 1) + a'(a'-1) + b'(b'-1) + c'(c'-1),$$

and that we have the identity $-10(\lfloor y(R)\rfloor + 1) = e + f + g - a' - b' - c'$. So when simplifying, we have

$$
\begin{aligned}
&-e(e-1)-f(f-1)-g(g-1)-90(\lfloor y(R)\rfloor+1)\\
&+a'(a'-1)+b'(b'-1)+c'(c'-1)\\
=\ &-e^2+e-f^2+f-g^2+g-90\lfloor y(R)\rfloor-90+(a')^2-a'+(b')^2-b'\\
&+(c')^2-c'\\
=\ &-e^2-f^2-g^2-90\lfloor y(R)\rfloor-90+(a')^2+(b')^2+(c')^2+e+f+g-a'\\
&-b'-c'\\
=\ &-e^2-f^2-g^2-90\lfloor y(R)\rfloor-90+(a')^2+(b')^2+(c')^2-10(\lfloor y(R)\rfloor+1)\\
=\ &-e^2-f^2-g^2-100(\lfloor y(R)\rfloor+1)+(a')^2+(b')^2+(c')^2.
\end{aligned}
$$

Now, like the previous case, consider $\lfloor y(R)\rfloor+1 \mod 3$.

- When $\lfloor y(R)\rfloor+1 \mod 3 \equiv 0$ then the points $(g, \lfloor y(R)\rfloor-2)$ and $(a', -3)$ and hence also $(f, \lfloor y(R)\rfloor-1)$ and $(b', -2)$; and $(e, \lfloor y(R)\rfloor)$ and $(c', -1)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

  * $g+10N=a'$;
  * $f+10N=b'$;
  * $e+10N=c'$;
  * $\lfloor y(R)\rfloor-2-3N=-3 \Longrightarrow \lfloor y(R)\rfloor=3N-1$;
  * $\lfloor y(R)\rfloor-1-3N=-2 \Longrightarrow \lfloor y(R)\rfloor=3N-1$;
  * $\lfloor y(R)\rfloor-3N=-1 \Longrightarrow \lfloor y(R)\rfloor=3N-1$;

Now substituting the results for a', b' and c' into the numerator of the rational expression for this case we get

$$\begin{aligned} & -e^2 - f^2 - g^2 - 100(3N) + (g + 10N)^2 + (f + 10N)^2 + (e + 10N)^2 \\ = \; & -e^2 - f^2 - g^2 - 300N + (g^2 + 20gN + 100N^2) \\ & +(f^2 + 20fN + 100N^2) + (e^2 + 20eN + 100N^2) \\ = \; & 300N^2 - 300N + 20eN + 20fN + 20gN \end{aligned}$$

So, since the denominator of the rational expression for this case is 20, then we have

$$15N^2 - 15N + eN + fN + gN.$$

Since $N = \dfrac{\lfloor y(R) \rfloor + 1}{3}$, then we have

$$\begin{aligned} & 15\left(\frac{\lfloor y(R) \rfloor + 1}{3}\right)^2 - 15\left(\frac{\lfloor y(R) \rfloor + 1}{3}\right) + e\left(\frac{\lfloor y(R) \rfloor + 1}{3}\right) \\ & +f\left(\frac{\lfloor y(R) \rfloor + 1}{3}\right) + g\left(\frac{\lfloor y(R) \rfloor + 1}{3}\right) \\ = \; & \frac{5}{3}(\lfloor y(R) \rfloor^2 + 2\lfloor y(R) \rfloor + 1) + \frac{e + f + g - 15}{3}(\lfloor y(R) \rfloor + 1) \\ = \; & \frac{5}{3}\lfloor y(R) \rfloor^2 + \frac{e + f + g - 5}{3}\lfloor y(R) \rfloor + \frac{e + f + g - 10}{3} \\ = \; & \frac{5}{3}(y(R)^2 - 2y(R)\{y(R)\} + \{y(R)\}^2) + \frac{e + f + g - 5}{3}(y(R) - \{y(R)\}) \\ & +\frac{e + f + g - 10}{3} \\ = \; & \frac{5}{3}y(R)^2 + \left(\frac{e + f + g - 5}{3} - \frac{10}{3}\{y(R)\}\right) y(R) + \frac{5}{3}\{y(R)\}^2 \\ & -\frac{e + f + g - 5}{3}\{y(R)\} + \frac{e + f + g - 10}{3} \end{aligned}$$

Now using the same technique for comparing the linear coefficients used in the 6 term case, we have

$$\begin{aligned}
& \frac{1}{60}\left(\frac{e+f+g-5}{3}-\frac{10}{3}\{y(R)\}\right)-\left(\frac{1}{18}\left\{\frac{2n}{3}\right\}+\frac{1}{6}\left\{\frac{n}{2}\right\}\right) \\
= \quad & \frac{1}{2}\left(\frac{1}{3*60}+\frac{1}{60}+\frac{1}{18}\right)-\frac{\left\{\frac{n}{2}\right\}}{60}-\frac{\left\{\frac{2n}{3}\right\}}{18} \\
\implies \quad & \frac{1}{60}\left(\frac{e+f+g-5}{3}-\frac{10}{3}\{y(R)\}\right)=\frac{1}{2}\left(\frac{1}{3*60}+\frac{1}{60}+\frac{1}{18}\right)=\frac{7}{180} \\
\implies \quad & e+f+g-5-10\{y(R)\}=7 \\
\implies \quad & e+f+g=12+10\{y(R)\}.
\end{aligned}$$

Now if we plug this result into our counting function we get:

$$\begin{aligned}
= \quad & \frac{5}{3}y(R)^2+\left(\frac{e+f+g-5}{3}-\frac{10}{3}\{y(R)\}\right)y(R)+\frac{5}{3}\{y(R)\}^2 \\
& -\frac{e+f+g-5}{3}\{y(R)\}+\frac{e+f+g-10}{3} \\
= \quad & \frac{5}{3}y(R)^2+\left(\frac{12+10\{y(R)\}-5}{3}-\frac{10}{3}\{y(R)\}\right)y(R)+\frac{5}{3}\{y(R)\}^2 \\
& -\frac{12+10\{y(R)\}-5}{3}\{y(R)\}+\frac{12+10\{y(R)\}-10}{3} \\
= \quad & \frac{5}{3}y(R)^2+\frac{7}{3}y(R)+\frac{5}{3}\{y(R)\}^2-\frac{7+10\{y(R)\}}{3}\{y(R)\}+\frac{2+10\{y(R)\}}{3} \\
= \quad & \frac{5}{3}y(R)^2+\frac{7}{3}y(R)-\frac{5}{3}\{y(R)\}^2+\{y(R)\}+\frac{2}{3}
\end{aligned}$$

- When $\lfloor y(R)\rfloor+1 \mod 3\equiv 1$ then the points $(g,\lfloor y(R)\rfloor-2)$ and $(b',-2)$ and hence also $(f,\lfloor y(R)\rfloor-1)$ and $(c',-1)$; and $(e,\lfloor y(R)\rfloor)$ and $(a',-3)$ are cancelling each other out in our calculations. Then there is a $N\in\mathbb{Z}_{>0}$ such that

  * $g+10N=b'$;

* $f + 10N = c'$;
* $e + 10(N+1) = e + 10N + 1 = a'$;
* $\lfloor y(R) \rfloor - 2 - 3N = -2 \Longrightarrow \lfloor y(R) \rfloor = 3N$;
* $\lfloor y(R) \rfloor - 1 - 3N = -1 \Longrightarrow \lfloor y(R) \rfloor = 3N$;
* $\lfloor y(R) \rfloor - 3(N+1) = -3 \Longrightarrow \lfloor y(R) \rfloor = 3N$;

Now substituting the results for a', b' and c' into the numerator of the rational expression for this case we get

$$\begin{aligned}
& -e^2 - f^2 - g^2 - 100(3N+1) + (e + 10N + 10)^2 + (g + 10N)^2 \\
& +(f + 10N)^2 \\
= \; & -e^2 - f^2 - g^2 - 300N - 100 + e^2 + 20eN + 20e + 100N^2 + 200N \\
& +100 + g^2 + 20gN + 100N^2 + f^2 + 20fN + 100N^2 \\
= \; & 300N^2 - 100N + 20eN + 20fN + 20gN + 20e
\end{aligned}$$

So, since the denominator of the rational expression for this case is 20, then we have

$$15N^2 - 5N + eN + fN + gN + e.$$

Since $N = \dfrac{\lfloor y(R) \rfloor}{3}$, then we have

$$\begin{aligned}
& 15\left(\frac{\lfloor y(R) \rfloor}{3}\right)^2 - 5\left(\frac{\lfloor y(R) \rfloor}{3}\right) + (e + f + g)\left(\frac{\lfloor y(R) \rfloor}{3}\right) + e \\
= \; & \frac{5}{3}\lfloor y(R) \rfloor^2 - \frac{5}{3}\lfloor y(R) \rfloor + \frac{e + f + g}{3}\lfloor y(R) \rfloor + e
\end{aligned}$$

$$\begin{aligned}
&= \frac{5}{3}(y(R)^2 - 2y(R)\{y(R)\} + \{y(R)\}^2) + \frac{e+f+g-5}{3}(y(R) - \{y(R)\}) \\
&\quad +e \\
&= \frac{5}{3}y(R)^2 + \left(\frac{e+f+g-5}{3} - \frac{10}{3}\{y(R)\}\right) y(R) + \frac{5}{3}\{y(R)\}^2 \\
&\quad -\frac{e+f+g-5}{3}\{y(R)\} + e
\end{aligned}$$

Similarly to previous case, we have

$$\begin{aligned}
&\frac{1}{60}\left(\frac{e+f+g-5}{3} - \frac{10}{3}\{y(R)\}\right) - \left(\frac{1}{18}\left\{\frac{2n}{3}\right\} + \frac{1}{6}\left\{\frac{n}{2}\right\}\right) \\
= \ &\frac{1}{2}\left(\frac{1}{3*60} + \frac{1}{60} + \frac{1}{18}\right) - \frac{\left\{\frac{n}{2}\right\}}{60} - \frac{\left\{\frac{2n}{3}\right\}}{18} \\
\implies \ &e+f+g = 12 + 10\{y(R)\}
\end{aligned}$$

Now if we plug this result into our counting function we get:

$$\begin{aligned}
&= \frac{5}{3}y(R)^2 + \left(\frac{e+f+g-5}{3} - \frac{10}{3}\{y(R)\}\right) y(R) + \frac{5}{3}\{y(R)\}^2 \\
&\quad -\frac{e+f+g-5}{3}\{y(R)\} + e \\
&= \frac{5}{3}y(R)^2 + \left(\frac{12+10\{y(R)\}-5}{3} - \frac{10}{3}\{y(R)\}\right) y(R) + \frac{5}{3}\{y(R)\}^2 \\
&\quad -\frac{12+10\{y(R)\}-5}{3}\{y(R)\} + e \\
&= \frac{5}{3}y(R)^2 + \frac{7}{3}y(R) + \frac{5}{3}\{y(R)\}^2 - \frac{7+10\{y(R)\}}{3}\{y(R)\} + e \\
&= \frac{5}{3}y(R)^2 + \frac{7}{3}y(R) - \frac{5}{3}\{y(R)\}^2 - \frac{7}{3}\{y(R)\} + e
\end{aligned}$$

- When $\lfloor y(R) \rfloor + 1 \mod 3 \equiv 2$ then the points $(g, \lfloor y(R) \rfloor - 2)$ and $(c', -1)$ and hence also $(f, \lfloor y(R) \rfloor - 1)$ and $(a', -3)$; and $(e, \lfloor y(R) \rfloor)$ and $(b', -2)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

* $g + 10N = c'$;

* $f + 10(N+1) = f + 10N + 10 = a'$;

* $e + 10(N+1) = e + 10N + 1 = b'$;

* $\lfloor y(R) \rfloor - 2 - 3N = -1 \Longrightarrow \lfloor y(R) \rfloor = 3N + 1$;

* $\lfloor y(R) \rfloor - 1 - 3(N+1) = -3 \Longrightarrow \lfloor y(R) \rfloor = 3N + 1$;

* $\lfloor y(R) \rfloor - 3(N+1) = -2 \Longrightarrow \lfloor y(R) \rfloor = 3N + 1$;

Now substituting the results for a', b' and c' into the numerator of the rational expression for this case we get

$$\begin{aligned}
& -e^2 - f^2 - g^2 - 100(3N+2) + (f + 10N + 10)^2 \\
& +(e + 10N + 10)^2 + (g + 10N)^2 \\
= \quad & -e^2 - f^2 - g^2 - 300N - 200 + f^2 + 20fN + 20f + 100N^2 + 200N \\
& +100 + e^2 + 20eN + 20e + 100N^2 + 200N + 100 + g^2 + 20gN \\
& +100N^2 \\
= \quad & 300N^2 + 100N + 20eN + 20fN + 20gN + 20e + 20f
\end{aligned}$$

So, since the denominator of the rational expression for this case is 20, then we have

$$15N^2 + 5N + eN + fN + gN + e + f.$$

Since $N = \dfrac{\lfloor y(R) \rfloor - 1}{3}$, then we have

$$
\begin{aligned}
& 15\left(\frac{\lfloor y(R)\rfloor-1}{3}\right)^2+5\left(\frac{\lfloor y(R)\rfloor-1}{3}\right)+(e+f+g)\left(\frac{\lfloor y(R)\rfloor-1}{3}\right) \\
& +e+f \\
= \; & \frac{5}{3}(\lfloor y(R)\rfloor^2-2\lfloor y(R)\rfloor+1)+\left(\frac{e+f+g+5}{3}\right)(\lfloor y(R)\rfloor-1)+e+f \\
= \; & \frac{5}{3}\lfloor y(R)\rfloor^2+\frac{e+f+g-5}{3}\lfloor y(R)\rfloor+\frac{2e+2f-g}{3} \\
= \; & \frac{5}{3}(y(R)^2-2y(R)\{y(R)\}+\{y(R)\}^2)+\frac{e+f+g-5}{3}(y(R)-\{y(R)\}) \\
& +\frac{2e+2f-g}{3} \\
= \; & \frac{5}{3}y(R)^2+\left(\frac{e+f+g-5}{3}-\frac{10}{3}\{y(R)\}\right)y(R)+\frac{5}{3}\{y(R)\}^2 \\
& -\frac{e+f+g-5}{3}\{y(R)\}+\frac{2e+2f-g}{3}
\end{aligned}
$$

Similarly to previous case, we have

$$
\begin{aligned}
& \frac{1}{60}\left(\frac{e+f+g-5}{3}-\frac{10}{3}\{y(R)\}\right)-\left(\frac{1}{18}\left\{\frac{2n}{3}\right\}+\frac{1}{6}\left\{\frac{n}{2}\right\}\right) \\
= \; & \frac{1}{2}\left(\frac{1}{3*60}+\frac{1}{60}+\frac{1}{18}\right)-\frac{\left\{\frac{n}{2}\right\}}{60}-\frac{\left\{\frac{2n}{3}\right\}}{18} \\
\implies \; & e+f+g=12+10\{y(R)\}
\end{aligned}
$$

Now if we plug this result into our counting function we get:

$$
\begin{aligned}
= \; & \frac{5}{3}y(R)^2+\left(\frac{e+f+g-5}{3}-\frac{10}{3}\{y(R)\}\right)y(R)+\frac{5}{3}\{y(R)\}^2 \\
& -\frac{e+f+g-5}{3}\{y(R)\}+\frac{2e+2f-g}{3} \\
= \; & \frac{5}{3}y(R)^2+\left(\frac{12+10\{y(R)\}-5}{3}-\frac{10}{3}\{y(R)\}\right)y(R)+\frac{5}{3}\{y(R)\}^2 \\
& -\frac{12+10\{y(R)\}-5}{3}\{y(R)\}+\frac{2e+2f-g}{3} \\
= \; & \frac{5}{3}y(R)^2+\frac{7}{3}y(R)+\frac{5}{3}\{y(R)\}^2-\frac{7+10\{y(R)\}}{3}\{y(R)\}+\frac{2e+2f-g}{3} \\
= \; & \frac{5}{3}y(R)^2+\frac{7}{3}y(R)-\frac{5}{3}\{y(R)\}^2-\frac{7}{3}\{y(R)\}+\frac{2e+2f-g}{3}
\end{aligned}
$$

Looking back at the 3 results for both cases, notice once again that the 6 term case is just a special case for the more general 8 term case, as when $g = 10$ we get precisely the 6 term results.

Next, note from this result from the general case $e + f + g - 10\{y(R)\} = 12$, and the previous identity $a' + b' + c' = e + f + g + 10(\lfloor y(R) \rfloor + 1)$, then we have

$$
\begin{aligned}
a' + b' + c' &= 12 + 10\{y(R)\} + 10(\lfloor y(R) \rfloor + 1) \\
\implies a + b + c + 33 &= 12 + 10\{y(R)\} + 10(y(R) - \{y(R)\}) + 10 \\
\implies a + b + c + 33 &= 10y(R) + 22 \\
\implies a + b + c &= 10y(R) - 11 = \left(\frac{n}{6} - 10\left\{\frac{2n}{3}\right\} - 3\left\{\frac{n}{2}\right\}\right) - 11
\end{aligned}
$$

Now lets compare the 3 results of the general case against each other by taking their differences:

$$
\begin{aligned}
& \left(\frac{5}{3}y(R)^2 + \frac{7}{3}y(R) - \frac{5}{3}\{y(R)\}^2 - \frac{7}{3}\{y(R)\} + \frac{2e + 2f - g}{3}\right) \\
& - \left(\frac{5}{3}y(R)^2 + \frac{7}{3}y(R) - \frac{5}{3}\{y(R)\}^2 - \frac{7}{3}\{y(R)\} + e\right) \\
= \; & \frac{-e + 2f - g}{3}; \\
& \left(\frac{5}{3}y(R)^2 + \frac{7}{3}y(R) - \frac{5}{3}\{y(R)\}^2 - \frac{7}{3}\{y(R)\} + e\right) \\
& - \left(\frac{5}{3}y(R)^2 + \frac{7}{3}y(R) - \frac{5}{3}\{y(R)\}^2 + \{y(R)\} + \frac{2}{3}\right) \\
= \; & -\frac{10}{3}\{y(R)\} - \frac{2}{3} + e; \text{ and} \\
& \left(\frac{5}{3}y(R)^2 + \frac{7}{3}y(R) - \frac{5}{3}\{y(R)\}^2 - \frac{7}{3}\{y(R)\} + \frac{2e + 2f - g}{3}\right) \\
& - \left(\frac{5}{3}y(R)^2 + \frac{7}{3}y(R) - \frac{5}{3}\{y(R)\}^2 + \{y(R)\} + \frac{2}{3}\right) \\
= \; & \frac{2e + 2f - g - 2 - 10\{y(R)\}}{3}
\end{aligned}
$$

Let's denote $\alpha_0 = \dfrac{2e + 2f - g - 2 - 10\{y(R)\}}{3}$ and $\alpha_1 = -\dfrac{10}{3}\{y(R)\} - \dfrac{2}{3} + e$.

Now lets look at all the possible values for $\{y(R)\}$ and see what the results for these differences are

	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$
e	1	1	1	2	2	2	3	3	3	4
f	4	4	5	5	5	6	6	6	7	7
g	7	8	8	8	9	9	9	10	10	10
$\frac{-e+2f-g}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0
$-\frac{10}{3}\{y(R)\} - \frac{2}{3} + e$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$
$\frac{2e+2f-g-2-10\{y(R)\}}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$

Therefore, the only difference amongst these 3 results is either $0, \frac{1}{3}$ or $-\frac{1}{3}$. Using the table above we have the following formulas for e, f and g:

$$
\begin{aligned}
e &= 1 + \left\lfloor \frac{10}{3}\{y(R)\} \right\rfloor \\
f &= 4 + \left\lfloor \frac{1}{3}(10\{y(R)\} + 1) \right\rfloor \\
g &= 7 + \left\lfloor \frac{1}{3}(10\{y(R)\} + 2) \right\rfloor .
\end{aligned}
$$

When plugging these formulas into the differences calculated above we get:

$$
\begin{aligned}
\alpha_0 &= \frac{2}{3}\left(1+\left\lfloor\frac{10}{3}\{y(R)\}\right\rfloor\right)+\frac{2}{3}\left(4+\left\lfloor\frac{1}{3}(10\{y(R)\}+1)\right\rfloor\right)\\
&\quad -\frac{1}{3}\left(7+\left\lfloor\frac{1}{3}(10\{y(R)\}+2)\right\rfloor\right)-\frac{2}{3}-\frac{10}{3}\{y(R)\}\\
&= \frac{1}{3}\left(1-2\left\{\frac{10}{3}\{y(R)\}\right\}-2\left\{\frac{10}{3}\{y(R)\}+\frac{1}{3}\right\}+\left\{\frac{10}{3}\{y(R)\}+\frac{2}{3}\right\}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_1 &= -\frac{10}{3}\{y(R)\}-\frac{2}{3}+\left(1+\left\lfloor\frac{10}{3}\{y(R)\}\right\rfloor\right)\\
&= \frac{1}{3}-\left\{\frac{10}{3}\{y(R)\}\right\}.
\end{aligned}
$$

Therefore, for the McNugget Problem, we have

$$
f(n)=\frac{5}{3}y(R)^2+\frac{7}{3}y(R)-\frac{5}{3}\{y(R)\}^2+\{y(R)\}+\frac{2}{3}+
$$
$$
\left\lfloor\left\lfloor\frac{y(R)}{3}+1\right\rfloor-\frac{y(R)+2}{3}\right\rfloor\alpha_0+\left\lfloor\left\lfloor\frac{y(R)+2}{3}+1\right\rfloor-\frac{y(R)}{3}\right\rfloor\alpha_1,
$$

$$
\left(\text{Recall that } y(R)=\frac{n}{60}-\left\{\frac{2n}{3}\right\}-\frac{3}{10}\left\{\frac{n}{2}\right\}\right).
$$

6.3 Other Families

Now let's consider a monoid with positive integer generators $< 1, p, q >$, where $\gcd(p, q) = 1$. Luckily, at this point, we have a lot of information about the general case along with a geometric approach from $< 6, 9, 20 >$ that we can, at least attempt, to replicate.

6.3.1 $< \mathbf{1}, \mathbf{p}, \mathbf{q} >$

Consider $< 1, p, q >$ where $\gcd(p, q) = 1$. Using the same technique that we used for the $< 6, 9, 20 >$, we can find a matrix $A \in SL_2(\mathbb{Z})$ which transforms this triangle from a 3 dimensional space into a 2 dimensional space. The resulting space is again a right triangle, and since the $\gcd(p, q) = 1$, there is no margins. Hence we have the following:

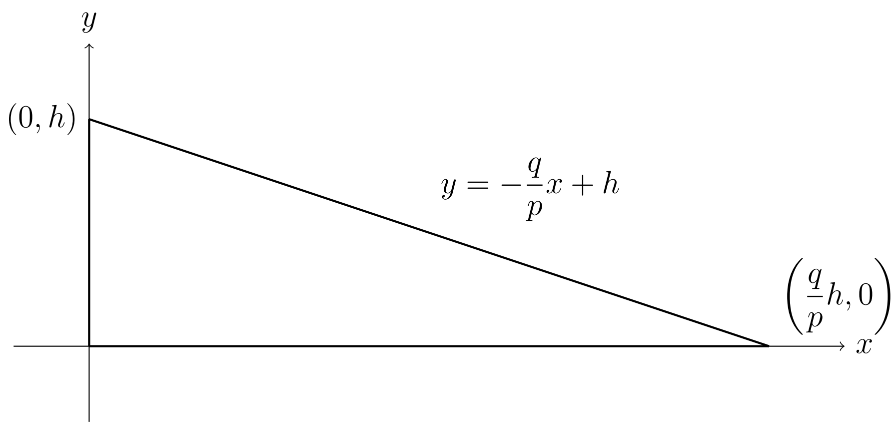


Figure 6.14: Transformed and Translated $< 1, p, p + 1 >$ triangle after changing basis back to x, y.

Now, using Brion, we know that the number of lattice points can be found using

$$\begin{aligned}
& \frac{1}{(1-x)(1-y)} + \frac{q \text{ terms}}{(1-y^{-1})(1-x^q y^{-p})} + \frac{p \text{ terms}}{(1-x^{-1})(1-x^{-q}y^p)} \\
= \quad & \frac{(1-x^q y^{-p})}{(1-x)(1-y)(1-x^q y^{-p})} - \frac{y(1-x)[q \text{ terms}]}{(1-x)(1-y)(1-x^q y^{-p})} \\
& + \frac{(1-y)(x^{q+1}y^{-p})[p \text{ terms}]}{(1-x)(1-y)(1-x^q y^{-p})} \\
= \quad & \frac{(1-x^q y^{-p}) - y[2p \text{ or } 2(p+1) \text{ terms}] + (1-y)(x^{q+1}y^{-p})[p \text{ terms}]}{(1-x)(1-y)(1-x^q y^{-p})}.
\end{aligned}$$

Let $d = \lfloor h \rfloor$. Note that in the $2(p + 1)$ case above, we have

$$(1-x)[q \text{ terms}] = y^d + x^{e_1}y^{d-1}(1-y) + x^{e_2}y^{d-2}(1-y) + \cdots + x^{e_p}y^{d-p}(1-y) - x^q y^{d-p},$$

and in the $2p$ case above, we have

$$(1-x)[q \text{ terms}] =$$
$$y^d + x^{e_1}y^{d-1}(1-y) + x^{e_2}y^{d-2}(1-y) + \cdots + x^{e_{p-1}}y^{d-p+1}(1-y) - x^q y^{d-p}.$$

<u>$2(p+1)$ **terms case**</u>

Using the same reasoning from the $< 6, 9, 20 >$, we know that the $2(p+1)$ case for this monoid is the general case and hence we will focus on that. Looking at the numerator of the above rational expression, we can rewrite it as follows:

$$\begin{aligned}
& (1 - x^q y^{-p}) - y(y^d + x^{e_1}y^{d-1}(1-y) + \cdots + x^{e_p}y^{d-p}(1-y) - x^q y^{d-p}) \\
& +(1-y)(x^{q+1}y^{-p})[p \text{ terms}] \\
= \; & (1 - x^q y^{-p}) - y^{d+1} - x^{e_1}y^d(1-y) - \cdots - x^{e_p}y^{d-p+1}(1-y) \\
& +x^q y^{d-p+1} + (1-y)(x^{q+1}y^{-p})[p \text{ terms}] \\
= \; & (1 - y^{d+1}) - x^{e_1}y^d(1-y) - \cdots - x^{e_p}y^{d-p+1}(1-y) + (x^q y^{d-p+1} - x^q y^{-p}) \\
& +(1-y)(x^{q+1}y^{-p})[p \text{ terms}] \\
= \; & (d+1)_y(1-y) - x^{e_1}y^d(1-y) - \cdots - x^{e_p}y^{d-p+1}(1-y) \\
& -x^q y^{-p}(d+1)_y(1-y) + (1-y)(x^{q+1}y^{-p})[p \text{ terms}]
\end{aligned}$$

Now writing the complete rational expression we have

$$\begin{aligned}
& \frac{(d+1)_y(1-y) - x^{e_1}y^d(1-y) - \cdots - x^{e_p}y^{d-p+1}(1-y)}{(1-x)(1-y)(1-x^q y^{-p})} \\
& \frac{-x^q y^{-p}(d+1)_y(1-y) + (1-y)(x^{q+1}y^{-p})[p \text{ terms}]}{(1-x)(1-y)(1-x^q y^{-p})} \\
= \; & \frac{(d+1)_y - x^{e_1}y^d - \cdots - x^{e_p}y^{d-p+1} - x^q y^{-p}(d+1)_y + (x^{q+1}y^{-p})[p \text{ terms}]}{(1-x)(1-x^q y^{-p})}.
\end{aligned}$$

Since we are counting lattice points, we can now set $y = 1$ and hence we have

$$\frac{(d+1) - x^{e_1} - \cdots - x^{e_p} - x^q(d+1) + x^{b_1} + x^{b-2} + \cdots + x^{b_p}}{(1-x)(1-x^q)},$$

where $b_i = a_{p-i} + q + 1$. Now consider the limit of this expression as $x \longrightarrow 1$.

$$\begin{aligned}
& \lim_{x \longrightarrow 1} \frac{(d+1) - x^{e_1} - \cdots - x^{e_p} - x^q(d+1) + x^{b_1} + x^{b-2} + \cdots + x^{b_p}}{(1-x)(1-x^q)} \\
\overset{(LH)}{=} & \lim_{x \longrightarrow 1} \frac{-e_1 x^{e_1-1} - \cdots - e_p x^{e_p-1} - (d+1)q x^{q-1} + b_1 x^{b_1-1} + \cdots + b_p x^{b_p-1}}{-(1-x^q) - q x^{q-1}(1-x)} \\
\overset{(LH)}{=} & \lim_{x \longrightarrow 1} \frac{-e_1(e_1-1)x^{e_1-2} - \cdots - e_p(e_p-1)x^{e_p-2} - (d+1)q(q+1)x^{q-2}}{2qx^{q-1} - q(q-1)x^{q-2}(1-x)} \\
& + \frac{b_1(b_1-1)x^{b_1-2} + \cdots + b_p(b_p-1)x^{b_p-2}}{2qx^{q-1} - q(q-1)x^{q-2}(1-x)} \\
= & \frac{-e_1(e_1-1) - \cdots - e_p(e_p-1) - (d+1)q(q-1)}{2q} \\
& + \frac{b_1(b_1-1) + \cdots + b_p(b_p-1)}{2q}.
\end{aligned}$$

Note that after the first use of L'Hopital's Rule, we get a new identity:

$$-e_1 - \cdots - e_p - q(d+1) + b_1 + \cdots + b_p = 0 \text{ or}$$

$$b_1 + \cdots + b_p = e_1 + \cdots + e_p + q(d+1).$$

Now consider all possibilities for $(d - (p-1) - (-p)) \mod p = d + 1 \mod p$.

- When $d + 1 \mod p \equiv 0$ then the points (e_1, d) and $(b_1, -1)$ and hence also $(e_2, d-1)$ and $(b_2, -2), \ldots, (e_p, d-p+1)$ and $(b_p, -p)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

  * $e_1 + qN = b_1$;

  * $e_2 + qN = b_2$;

  $\vdots$

* $e_p + qN = b_p$;
* $d + 1 - pN = 0 \Longrightarrow d = pN - 1$ or $N = \dfrac{d+1}{p} = \dfrac{\lfloor h \rfloor + 1}{p}$.

Now going back to the numerator of the rational expression for this case, we can simplify and substituting in for d and b_i's as follows:

$$\begin{aligned}
& -e_1(e_1 - 1) - \cdots - e_p(e_p - 1) - (d+1)q(q-1) \\
& +b_1(b_1 - 1) + \cdots + b_p(b_p - 1) \\
= \quad & -e_1^2 + e_1 - \cdots - e_p^2 + e_p - (d+1)(q^2 - q) + b_1^2 - b_1 + \cdots + b_p^2 - b_p \\
= \quad & -e_1^2 + e_1 - \cdots - e_p^2 + e_p - (dq^2 - dq + q^2 - q) \\
& +b_1^2 - b_1 + \cdots + b_p^2 - b_p \\
= \quad & -e_1^2 - \cdots - e_p^2 - dq^2 + dq - q^2 + q + b_1^2 + \cdots + b_p^2 \\
& +(e_1 + \cdots + e_p - b_1 - \cdots - b_p) \\
= \quad & -e_1^2 - \cdots - e_p^2 - dq^2 + dq - q^2 + q + b_1^2 + \cdots + b_p^2 - q(d+1) \\
= \quad & -e_1^2 - \cdots - e_p^2 - dq^2 - q^2 + b_1^2 + \cdots + b_p^2 \\
= \quad & -e_1^2 - \cdots - e_p^2 - (pN-1)q^2 - q^2 + (e_1 + qN)^2 + \cdots + (e_p + qN)^2 \\
= \quad & -e_1^2 - \cdots - e_p^2 - pNq^2 + q^2 - q^2 + (e_1^2 + 2e_1 qN + q^2N^2) \\
& + \cdots + (e_p^2 + 2e_p qN + q^2 N^2) \\
= \quad & pq^2N^2 - q^2pN + 2qN(e_1 + \ldots e_p).
\end{aligned}$$

Recall that the denominator of the rational expression was $2q$, so we have

$$= \frac{pq}{2}N^2 - \frac{pq}{2}N + N(e_1 + \cdots + e_p).$$

Since $N = \dfrac{\lfloor h \rfloor + 1}{p}$:

$$
\begin{aligned}
&= \frac{pq}{2}\left(\frac{\lfloor h\rfloor+1}{p}\right)^2 - \frac{pq}{2}\left(\frac{\lfloor h\rfloor+1}{p}\right) + \left(\frac{\lfloor h\rfloor+1}{p}\right)(e_1+\cdots+e_p).\\
&= \frac{q}{2p}(\lfloor h\rfloor^2+2\lfloor h\rfloor+1) - \frac{q}{2}(\lfloor h\rfloor+1) + \left(\frac{\lfloor h\rfloor+1}{p}\right)(e_1+\cdots+e_p).\\
&= \frac{q}{2p}\lfloor h\rfloor^2 + \left(\frac{q}{p}-\frac{q}{2}+\frac{e_1+\cdots+e_p}{p}\right)\lfloor h\rfloor + \left(\frac{q}{2p}-\frac{q}{2}+\frac{e_1+\cdots+e_p}{p}\right)\\
&= \frac{q}{2p}(h^2-2h\{h\}+\{h\}^2) + \left(\frac{q}{p}-\frac{q}{2}+\frac{e_1+\cdots+e_p}{p}\right)(h-\{h\})\\
&\quad + \left(\frac{q}{2p}-\frac{q}{2}+\frac{e_1+\cdots+e_p}{p}\right)\\
&= \frac{q}{2p}h^2 + \left(\frac{q}{p}-\frac{q}{2}+\frac{e_1+\cdots+e_p}{p}-\frac{q}{p}\{h\}\right)h\\
&\quad +\frac{q}{2p}\{h\}^2 - \left(\frac{q}{p}-\frac{q}{2}+\frac{e_1+\cdots+e_p}{p}\right)\{h\} + \frac{q}{2p}-\frac{q}{2}+\frac{e_1+\cdots+e_p}{p}\\
&= \frac{q}{2p}h^2 + \left(\frac{e_1+\cdots+e_p}{p}+\frac{2q-pq-2q\{h\}}{2p}\right)h\\
&\quad +\frac{q}{2p}\{h\}^2 - \left(\frac{2q-pq}{2p}+\frac{e_1+\cdots+e_p}{p}\right)\{h\} + \frac{q-pq}{2p}+\frac{e_1+\cdots+e_p}{p}
\end{aligned}
$$

- When $d+1 \mod p \equiv 1$ then the points (e_1, d) and $(b_p, -p)$ and hence also $(e_2, d-1)$ and $(b_1, -1), \ldots, (e_p, d-p+1)$ and $(b_{p-1}, -p+1)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

  * $e_1 + q(N+1) = b_p$;

  * $e_2 + qN = b_1$;

    $\vdots$

  * $e_p + qN = b_{p-1}$;

  * $d+1-pN = 1 \Longrightarrow d = pN$ or $N = \dfrac{d}{p} = \dfrac{\lfloor h\rfloor}{p}$.

  Now by substituting back into the numerator of the rational expression above, we have

$$
\begin{aligned}
& -e_1^2 - \cdots - e_p^2 - (pN)q^2 - q^2 + (e_2 + qN)^2 + \cdots + (e_p + qN)^2 \\
& +(e_1 + q(N+1))^2 \\
= \quad & -e_1^2 - \cdots - e_p^2 - pq^2N - q^2 + (e_2^2 + 2e_2qN + q^2N^2) \\
& + \cdots + (e_p^2 + 2e_pqN + q^2N^2) + (e_1 + 2e_1q(N+1) + q^2(N+1)^2) \\
= \quad & -pq^2N - q^2 + 2e_2qN + q^2N^2 + \cdots + 2e_pqN + q^2N^2 + 2e_1q + q^2N^2 \\
& +2q^2N + q^2 \\
= \quad & pq^2N^2 - pq^2N + 2qN(e_1 + \cdots + e_p) + 2e_1q + 2q^2N.
\end{aligned}
$$

Recall that the denominator of the rational expression was $2q$, so we have

$$
= \frac{pq}{2}N^2 - \frac{pq}{2}N + N(e_1 + \cdots + e_p) + e_1 + qN.
$$

Since $N = \dfrac{\lfloor h \rfloor}{p}$:

$$
\begin{aligned}
= \quad & \frac{pq}{2}\left(\frac{\lfloor h \rfloor}{p}\right)^2 - \frac{pq}{2}\left(\frac{\lfloor h \rfloor}{p}\right) + \left(\frac{\lfloor h \rfloor}{p}\right)(e_1 + \cdots + e_p) + e_1 + q\frac{\lfloor h \rfloor}{p} \\
= \quad & \frac{q}{2p}\lfloor h \rfloor^2 - \frac{q}{2}\lfloor h \rfloor + \left(\frac{\lfloor h \rfloor}{p}\right)(e_1 + \cdots + e_p) + e_1 + q\frac{\lfloor h \rfloor}{p} \\
= \quad & \frac{q}{2p}\lfloor h \rfloor^2 + \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1 + \cdots + e_p}{p}\right)\lfloor h \rfloor + e_1 \\
= \quad & \frac{q}{2p}(h^2 - 2h\{h\} + \{h\}^2) + \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1 + \cdots + e_p}{p}\right)(h - \{h\}) + e_1 \\
= \quad & \frac{q}{2p}h^2 + \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1 + \cdots + e_p}{p} - \frac{q}{p}\{h\}\right)h \\
& + \left(\frac{q}{2p}\{h\}^2 - \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1 + \cdots + e_p}{p}\right)\{h\} + e_1\right) \\
= \quad & \frac{q}{2p}h^2 + \left(\frac{2q - pq - 2q\{h\}}{2p} + \frac{e_1 + \cdots + e_p}{p}\right)h \\
& + \left(\frac{q}{2p}\{h\}^2 - \left(\frac{2q - pq}{2p} + \frac{e_1 + \cdots + e_p}{p}\right)\{h\} + e_1\right)
\end{aligned}
$$

- Now just consider when $(d+1) \mod p \equiv i$, where $0 \le i < p$. Then the points (e_1, d) and $(b_{p-i+1}, -p+i+1), \ldots, (e_i, d-i-1)$ and $(b_p, -p), (e_{i+1}, d-$

i) and $(b_1, -1) \ldots, (e_{p-1}, d-p+2)$ and $(b_{p-i-1}, -p+i+1), (e_p, d-p+1)$ and $(b_{p-i}, -p+i)$ are cancelling each other out in our calculations. Then there is a $N \in \mathbb{Z}_{>0}$ such that

* $e_1 + q(N+1) = b_{p-i+1}$;

  $\vdots$

* $e_i + q(N+1) = b_p$;

* $e_{i+1} + qN = b_1$;

  $\vdots$

* $e_{p-i} + qN = b_{p-i-1}$;

* $e_p + qN = b_{p-i}$;

* $d + 1 - pN = i \Longrightarrow d = pN + i - 1$ or $N = \dfrac{d+1-i}{p} = \dfrac{\lfloor h \rfloor + 1 - i}{p}$.

Now by substituting back into the numerator of the rational expression above, we have

$$
\begin{aligned}
& -e_1^2 - \cdots - e_p^2 - (pN+i-1)q^2 - q^2 + (e_{i+1}+qN)^2 + \ldots \\
& +(e_p+qN)^2 + (e_1+q(N+1))^2 + \cdots + (e_i+q(N+1))^2 \\
= \quad & -e_1^2 - \cdots - e_p^2 - pq^2N - iq^2 + q^2 - q^2 + (e_{i+1}^2 + 2e_{i+1}qN + q^2N^2) \\
& + \cdots + (e_p^2 + 2e_pqN + q^2N^2) + (e_1^2 + 2e_1q(N+1) + q^2(N+1)^2) \\
& + \cdots + (e_i^2 + 2e_iq(N+1) + q^2(N+1)^2) \\
= \quad & -pq^2N - iq^2 + 2e_{i+1}qN + q^2N^2 + \cdots + 2e_pqN + q^2N^2 + 2e_1qN \\
& +2e_1q + q^2N^2 + 2q^2N + q^2 + \cdots + 2e_iqN + 2e_iq + q^2N^2 + 2q^2N + q^2 \\
= \quad & pq^2N^2 - pq^2N + 2qN(e_1+\cdots+e_p) + 2q(e_1+\cdots+e_i) + 2iq^2N.
\end{aligned}
$$

Recall that the denominator of the rational expression was $2q$, so we have

$$
= \frac{pq}{2}N^2 - \frac{pq}{2}N + N(e_1+\cdots+e_p) + e_1 + \cdots + e_i + iqN.
$$

Since $N = \dfrac{\lfloor h \rfloor + 1 - i}{p}$:

$$
\begin{aligned}
= \quad & \frac{pq}{2}\left(\frac{\lfloor h \rfloor + 1 - i}{p}\right)^2 - \frac{pq}{2}\left(\frac{\lfloor h \rfloor + 1 - i}{p}\right) \\
& + \left(\frac{\lfloor h \rfloor + 1 - i}{p}\right)(e_1+\cdots+e_p) + e_1 + \cdots + e_i + iq\frac{\lfloor h \rfloor + 1 - i}{p}. \\
= \quad & \frac{q}{2p}(\lfloor h \rfloor^2 + 2\lfloor h \rfloor(1-i) + (1-i)^2) - \frac{q}{2}(\lfloor h \rfloor + 1 - i) \\
& + \left(\frac{\lfloor h \rfloor + 1 - i}{p}\right)(e_1+\cdots+e_p) + e_1 + \cdots + e_i + iq\frac{\lfloor h \rfloor + 1 - i}{p} \\
= \quad & \frac{q}{2p}\lfloor h \rfloor^2 + \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1+\cdots+e_p}{p}\right)\lfloor h \rfloor \\
& + \frac{q(1-i)^2}{2p} - \frac{q(1-i)}{2} + \frac{(e_1+\cdots+e_p)(1-i)}{p} + e_1 \\
& + \cdots + e_i + iq\frac{1-i}{p}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{q}{2p}(h^2 - 2h\{h\} + \{h\}^2) + \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1 + \cdots + e_p}{p}\right)(h - \{h\}) \\
& + \frac{q(1-i)^2}{2p} - \frac{q(1-i)}{2} + \frac{(e_1 + \cdots + e_p)(1-i)}{p} + e_1 \\
& + \cdots + e_i + iq\frac{1-i}{p} \\
= \quad & \frac{q}{2p}h^2 + \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1 + \cdots + e_p}{p} - \frac{q}{p}\{h\}\right) h \\
& + \frac{q}{2p}\{h\}^2 - \left(\frac{q}{p} - \frac{q}{2} + \frac{e_1 + \cdots + e_p}{p}\right)\{h\} \\
& + \frac{q(1-i)^2}{2p} - \frac{q(1-i)}{2} + \frac{(e_1 + \cdots + e_p)(1-i)}{p} + e_1 \\
& + \cdots + e_i + iq\frac{1-i}{p} \\
= \quad & \frac{q}{2p}h^2 + \left(\frac{2q - pq - 2q\{h\}}{2p} + \frac{e_1 + \cdots + e_p}{p}\right) h \\
& + \frac{q}{2p}\{h\}^2 - \left(\frac{2q - pq}{2p} + \frac{e_1 + \cdots + e_p}{p}\right)\{h\} \\
& + \frac{q(1-i)^2}{2p} - \frac{q(1-i)}{2} + \frac{(e_1 + \cdots + e_p)(1-i)}{p} + e_1 \\
& + \cdots + e_i + iq\frac{1-i}{p}
\end{aligned}
$$

Now lets calculate the location for $e_1, e_2, \ldots, e_p$ for all the possible values for $\{h\}$:

$\{h\}$	0	$\frac{1}{q}$	$\frac{2}{q}$	$\ldots$	$\frac{q-1}{q}$
e_1	1	$\left\lfloor \frac{1+p}{p} \right\rfloor$	$\left\lfloor \frac{2+p}{p} \right\rfloor$	$\ldots$	$\left\lfloor \frac{q+p-1}{p} \right\rfloor$
e_2	$\left\lfloor \frac{q+p}{p} \right\rfloor$	$\left\lfloor \frac{q+p+1}{p} \right\rfloor$	$\left\lfloor \frac{q+p+2}{p} \right\rfloor$	$\ldots$	$\left\lfloor \frac{2q+p-1}{p} \right\rfloor$
e_3	$\left\lfloor \frac{2q+p}{p} \right\rfloor$	$\left\lfloor \frac{2q+p+1}{p} \right\rfloor$	$\left\lfloor \frac{2q+p+2}{p} \right\rfloor$	$\ldots$	$\left\lfloor \frac{3q+p-1}{p} \right\rfloor$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
e_p	$\left\lfloor \frac{q(p-1)+p}{p} \right\rfloor$	$\left\lfloor \frac{q(p-1)+p+1}{p} \right\rfloor$	$\left\lfloor \frac{q(p-1)+p+2}{p} \right\rfloor$	$\ldots$	$\left\lfloor \frac{qp+p-1}{p} \right\rfloor$

Thus $e_i = \left\lfloor \frac{q(i-1)+j}{p} + 1 \right\rfloor$, where $\frac{j}{q} = \{h\}$ and $1 \leq i \leq p$. So note that

$$
\begin{aligned}
e_1 + e_2 + \cdots + e_p &= \sum_{i=1}^{p} \left(\frac{q(i-1)+j}{p} + 1 \right) - \sum_{i=1}^{p} \left\{ \frac{q(i-1)+j}{p} + 1 \right\} \\
&= p + \sum_{i=1}^{p} \left(\frac{qi}{p} + \frac{j-q}{p} \right) - \sum_{i=1}^{p} \left\{ \frac{q(i-1)+j}{p} \right\} \\
&= p + j - q + \sum_{i=1}^{p} \left(\frac{qi}{p} \right) - \sum_{i=1}^{p} \left\{ \frac{qi+j-q}{p} \right\} \\
&= p + j - q + \frac{q}{p} \sum_{i=1}^{p} (i) - \sum_{i=1}^{p} \left(\frac{i}{p} \right) \\
&= p + j - q + \frac{q}{p} \left(\frac{p(p+1)}{2} \right) - \frac{1}{p} \left(\frac{p(p-1)}{2} \right) \\
&= p + j - q + \frac{q(p+1)}{2} - \frac{p-1}{2} \\
&= p + j - q + \frac{q(p+1)}{2} - \frac{(p+1)-2}{2} \\
&= p + j - q + 1 + \frac{(q-1)(p+1)}{2}.
\end{aligned}
$$

So going back to the results from when $(d+1) \mod p \equiv i$ and subbing this in for $e_1 + e_2 + e_3 + \cdots + e_p$, then we have:

$$
\begin{aligned}
& \frac{q}{2p}h^2 + \left(\frac{2q - pq - 2q\{h\}}{2p} + \frac{p + j - q + 1 + \frac{(q-1)(p+1)}{2}}{p} \right) h + \frac{q}{2p}\{h\}^2 \\
& - \left(\frac{2q - pq}{2p} + \frac{p + j - q + 1 + \frac{(q-1)(p+1)}{2}}{p} \right) \{h\} + \frac{q(1-i)^2}{2p} - \frac{q(1-i)}{2} \\
& + \frac{(p + j - q + 1 + \frac{(q-1)(p+1)}{2})(1-i)}{p} + e_1 + \cdots + e_i + iq\frac{1-i}{p} \\
= \; & \frac{q}{2p}h^2 + \left(\frac{q + p + 2j - 2q\{h\} + 1}{2p} \right) h + \frac{q}{2p}\{h\}^2 - \left(\frac{q + p + 2j + 1}{2p} \right) \{h\} \\
& + \frac{q(1-i)^2}{2p} - \frac{q(1-i)}{2} + \frac{(2p + 2j - 2q + 2 + 2iq + (q-1)(p+1))(1-i)}{2p} \\
& + e_1 + \cdots + e_i
\end{aligned}
$$

However, we now need to find what $e_1 + e_2 + \cdots + e_i$ equals, where $1 \leq i < p$. We will now do this for a couple of special cases of $< 1, p, q >$.

6.3.2 $< \mathbf{1, p, p+1} >$

Consider $< 1, p, p+1 >$ where $p \geq 2$. Note that $\gcd(p, p+1) = 1$ as $1(p+1) + (-1)(p) = 1$. Now consider $x + py + (p+1)z = n$ and want to find a matrix A such that

$$\begin{bmatrix} 1 & p & p+1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & p & p+1 \end{bmatrix} A \begin{bmatrix} u \\ v \\ w \end{bmatrix} = n.$$

First note that $R_1 = p, R_p = 1, R_{p+1} = 1$ and lets consider $s_1 + s_2 p + s_3(p+1) = 1$ where $s_1 = 0, s_2 = -1$ and $s_3 = 1$. So let

$$A = \begin{bmatrix} -p & -(p+1) & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

So if we have that

$$\begin{bmatrix} 1 & p & p+1 \end{bmatrix} \begin{bmatrix} -p & -(p+1) & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = n,$$

then note that

$$\begin{aligned} x &= -pu - (p+1)v & \qquad x &= -pu - (p+1)v \\ y &= u - w \quad \text{or} & \qquad y &= u - n \\ z &= v + w & \qquad z &= v + n \\ w &= n & & \end{aligned}$$

So when

- $x = 0$, then $v = -\dfrac{p}{p+1}u$
- $y = 0$, then $u = n$
- $z = 0$, then $v = -n$,

hence we have a right triangle in the u, v-plane with no margin, as the hypothenuse is latticial and the horizontal and vertical side have integer y-coordinates and integer x-coordinates, respectively. Following a couple of shifts of integer length, we have Figure 6.15.

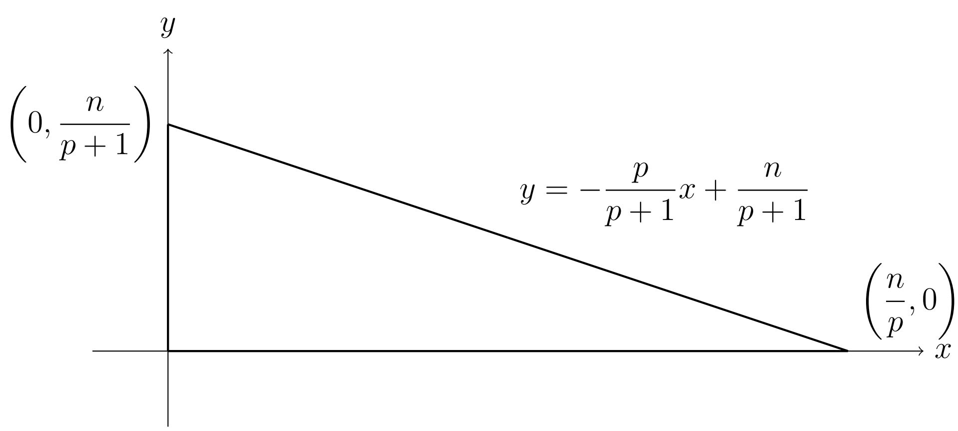


Figure 6.15: Transformed and Translated $< 1, p, p+1 >$ triangle after changing basis back to x, y.

Now following the results from $< 1, p, q >$, we have

$$\begin{aligned}
&= \frac{p+1}{2p}\left(\frac{n}{p+1}\right)^2 + \left(\frac{(p+1)+p+2j-2(p+1)\left\{\frac{n}{p+1}\right\}+1}{2p}\right)\left(\frac{n}{p+1}\right) \\
&\quad + \frac{p+1}{2p}\left\{\frac{n}{p+1}\right\}^2 - \left(\frac{(p+1)+p+2j+1}{2p}\right)\left\{\frac{n}{p+1}\right\} + \frac{(p+1)(1-i)^2}{2p} \\
&\quad - \frac{(p+1)(1-i)}{2} + \frac{(2p+2j-2(p+1)+2+2i(p+1)+p(p+1))(1-i)}{2p} \\
&\quad + e_1 + \cdots + e_i \\
&= \frac{n^2}{2p(p+1)} + \left(\frac{p+j-(p+1)\left\{\frac{n}{p+1}\right\}+1}{p(p+1)}\right) n + \frac{p+1}{2p}\left\{\frac{n}{p+1}\right\}^2 \\
&\quad - \left(\frac{p+j+1}{p}\right)\left\{\frac{n}{p+1}\right\} + \frac{(p+1)(1-i)^2}{2p} - \frac{(p+1)(1-i)}{2} \\
&\quad + \frac{(p^2+p+2ip+2i+2j)(1-i)}{2p} + e_1 + \cdots + e_i
\end{aligned}$$

Now we need to determine what $\sum_{j=1}^{i} e_j$ equals, where $i \in \{1, 2, \ldots, p\}$. This is determined by $\left\{\frac{n}{p+1}\right\}$:

- When $\left\{\frac{n}{p+1}\right\} = 0$, then $\sum_{l=1}^{i} e_l = 1 + 2 + \cdots + i = \frac{i(i+1)}{2}$
- When $\left\{\frac{n}{p+1}\right\} = \frac{1}{p+1}$, then

$$\sum_{l=1}^{i} e_l = \begin{cases} 1 + 2 + \cdots + i = \dfrac{i(i+1)}{2} & \text{for } 1 \le i \le p-1 \\[2ex] (1 + 2 + \cdots + p) + 1 = \dfrac{p(p+1)}{2} + 1 & \text{for } i = p \end{cases}$$

$$\vdots$$

$$\vdots$$

- When $\left\{\frac{n}{p+1}\right\} = \frac{r}{p+1}$ where $r \in \{0, 1, 2, \ldots, p\}$, then

$$\sum_{l=1}^{i} e_l =$$

$$\begin{cases} 1+2+\cdots+i = \dfrac{i(i+1)}{2}; & 1 \leq i \leq p-r \\ (1+2+\cdots+i)+(i-(p-r)) = \dfrac{i(i+1)}{2}+i+r-p; & p-r < i \leq p \end{cases}$$

6.3.3 $< \mathbf{1}, \mathbf{p}, \mathbf{kp+1} >$

Remark 6.3.1. *When considering the monoid* $< 1, p, kp+1 >$ *where* $p \geq 2$*, there will be many similarities as the monoid we considered in the previous section. The key to the similarities will be that* $kp+1 \mod p \equiv 1$*, for any* $k \in \mathbb{Z}_{>0}$*.*

Consider $< 1, p, kp+1 >$ where $p \geq 2$. Note that $\gcd(p, kp+1) = 1$ as $1(kp+1) + (-k)(p) = 1$. Now consider $x + py + (kp+1)z = n$ and want to find a matrix A such that

$$\begin{bmatrix} 1 & p & kp+1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & p & kp+1 \end{bmatrix} A \begin{bmatrix} u \\ v \\ w \end{bmatrix} = n.$$

First note that $R_1 = p, R_p = 1, R_{kp+1} = 1$ and lets consider $s_1 + s_2 p + s_3(p+1) = 1$ where $s_1 = 0, s_2 = -k$ and $s_3 = 1$. So let

$$A = \begin{bmatrix} -p & -(kp+1) & 0 \\ 1 & 0 & -k \\ 0 & 1 & 1 \end{bmatrix}.$$

So if we have that

$$\begin{bmatrix} 1 & p & kp+1 \end{bmatrix} \begin{bmatrix} -p & -(kp+1) & 0 \\ 1 & 0 & -k \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = n,$$

then note that

$$\begin{aligned} x &= -pu - (kp+1)v & \qquad x &= -pu - (kp+1)v \\ y &= u - kw \quad \text{or} & y &= u - kn \\ z &= v + w & z &= v + n \\ w &= n \end{aligned}$$

So when

- $x = 0$, then $v = -\dfrac{p}{kp+1}u$
- $y = 0$, then $u = kn$
- $z = 0$, then $v = -n$,

hence we have a right triangle in the u, v-plane with no margin, as the hypothenuse is latticial and the horizontal and vertical side have integer y-coordinates and integer x-coordinates, respectively. Following a couple of shifts of integer length, we have Figure 6.16.

Now following the results from $< 1, p, q >$, we have

$$= \frac{kp+1}{2p}\left(\frac{n}{kp+1}\right)^2 + \left(\frac{(kp+1) + p + 2j - 2(kp+1)\left\{\frac{n}{kp+1}\right\} + 1}{2p}\right)\left(\frac{n}{kp+1}\right)$$

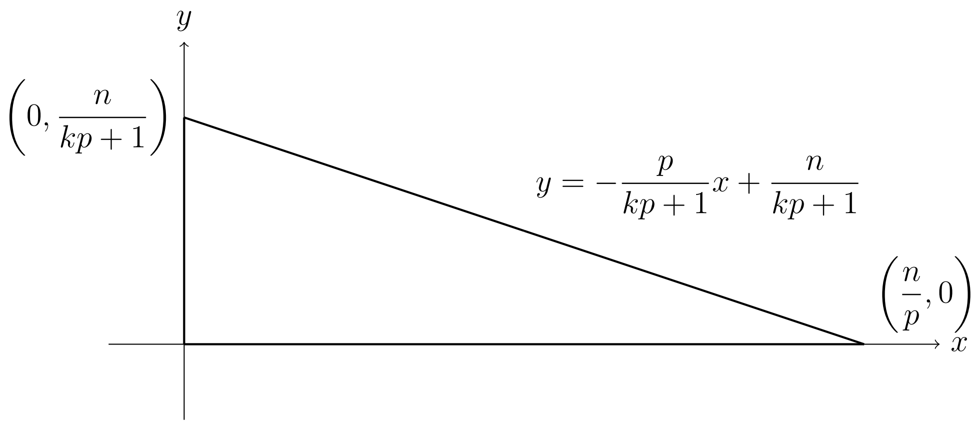


Figure 6.16: Transformed and Translated $< 1, p, kp+1 >$ triangle after changing basis back to x, y.

$$+\frac{kp+1}{2p}\left\{\frac{n}{kp+1}\right\}^2 - \left(\frac{(kp+1)+p+2j+1}{2p}\right)\left\{\frac{n}{kp+1}\right\} + \frac{(kp+1)(1-i)^2}{2p}$$
$$-\frac{(kp+1)(1-i)}{2} + \frac{(2p+2j-2(kp+1)+2+2i(kp+1)+kp(p+1))(1-i)}{2p}$$
$$+e_1 + \cdots + e_i$$
$$= \frac{n^2}{2p(kp+1)} + \left(\frac{(k+1)p+2j-2(kp+1)\left\{\frac{n}{kp+1}\right\}+2}{2p(kp+1)}\right)n + \frac{kp+1}{2p}\left\{\frac{n}{kp+1}\right\}^2$$
$$-\left(\frac{(k+1)p+2j+2}{2p}\right)\left\{\frac{n}{kp+1}\right\} + \frac{(kp+1)(1-i)^2}{2p} - \frac{(kp+1)(1-i)}{2}$$
$$+\frac{(kp^2+2p+2j-kp+2kpi+2i)(1-i)}{2p} + e_1 + \cdots + e_i$$

Now we need to determine what $\sum_{j=1}^{i} e_j$ equals, where $i \in \{1, 2, \ldots, p\}$. This is determined by $\left\{\frac{n}{p+1}\right\}$:

- When $\left\{\frac{n}{p+1}\right\} = 0$, then $\sum_{j=1}^{i} e_j = 1+(k+1)+(2k+1)\cdots+((i-1)k+1) =$

$k\dfrac{i(i-1)}{2}+i$

- When $\left\{\dfrac{n}{p+1}\right\}=\dfrac{1}{p+1}$, then

$$\sum_{j=1}^{i} e_j =$$

$$\begin{cases} 1+(k+1)+\cdots+((i-1)k+1)=k\dfrac{i(i-1)}{2}+i; & 1\le i<p \\ 1+(k+1)+\cdots+((p-1)k+1)+1=k\dfrac{p(p-1)}{2}+p+1; & i=p \end{cases}$$

$\vdots$

$\vdots$

- When $\left\{\dfrac{n}{kp+1}\right\}=\dfrac{r}{kp+1}$ where $r\in\{0,1,2,\ldots,kp\}$, then

$$\sum_{j=1}^{i} e_j = \begin{cases} k\dfrac{i(i-1)}{2}+i & \text{for } 1\le i\le kp-r \\ k\dfrac{i(i-1)}{2}+i+r-p & \text{for } kp-r+1\le i\le kp \end{cases}$$

Chapter 7

Conclusions and Future Work

The novel contributions of this thesis to the literature are:

1. The secondary asymptotic of the Enumeration Problem for arbitrary 3 generator numerical monoids. (Section 5.3)
2. The complete solution of the Enumeration Problem for 3 generator numerical monoids of the third kind ($R_a a = R_b b = R_c c$). (Section 6.2.1)
3. The complete solution of the Enumeration Problem for the McNugget Monoid ($M =< 6, 9, 20 >$). (Section 6.2.2)
4. The complete solution of the Enumeration Problem for a couple of infinite families outside the cases mentioned above ($< 1, p, p+1 >$ and $< 1, p, kp+1 >$). (Section 6.3.2 and 6.3.3)

Another novel aspect of our approach is the geometric techniques employed, for instance, the Johnson Transformation (named after Selmer Johnson because of its connection to the Johnson Equations), which was not previously available in the literature.

The ultimate goal of an effectively computable formula with a bounded number of terms for $f_M(n)$ when M is a numerical monoid with 3 generators for the cases II and III seems very difficult. It may not even be possible. If possible, clearly it will be quite complicated (based on the examples worked out in this thesis). For 3 generators what is missing is the (periodic) constant term of the quadratic quasi-polynomial $f_M(n)$.

However, for a general polygon, P, based on the second asymptote found for the Frobenius Problem, we believe that the second asymptote is given by the following expression:

$$\begin{aligned} f(n) &= \text{Area } n^2 + \left(\frac{1}{2}\mu_{\mathbb{Z}}^1(\partial P) - \sum_{\text{e:edge}} m_e \mu_{\mathbb{Z}}^1(e)\right) n + \text{ periodic function} \\ &= \text{Area } n^2 + \left(\sum_{\text{e:edge}} \left(\frac{1}{2} - m_e\right) \mu_{\mathbb{Z}}^1(e)\right) n - \frac{3}{2} + \text{ periodic function} \end{aligned}$$

Similarly, for a general d-dimensional polytope, we believe that the second asymptotic is also given in the follow expression:

$$f(n) = |nP| = \mu_{\mathbb{Z}}^d(P) n^d + \left(\sum_{\text{f:facets}} \left(\frac{1}{2} - m_f\right) \mu_{\mathbb{Z}}^{d-1}(f)\right) n^{d-1} + \dots.$$

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