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REPRESENTATIONS OF THE AUTOMORPHISM GROUP
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Contents

Abstract

In 2009 Grunewald and Lubotzky published a paper in which they defined a family of linear representations of the automorphism group of a free group. In this dissertation, we will use their ideas to construct a family of linear representation of the automorphism group of a right-angled Coxeter group. We will then use graph-theoretic properties of the defining graph to systematically decompose the image group into a group of block upper triangular matrices.

Chapter 1

Introduction

1.1 Main Results

In 2009 Grunewald and Lubotzky published a paper ([GL]) in which they constructed a family of linear representations of the automorphism group of a free group. The group $\mathrm{Aut}(F_n)$ is in some sense analagous to the mapping class group of a surface, which is the group of orientation-preserving homeomorphisms up to homotopy. In fact, the Dehn-Nielsen-Baer Theorem implies that the extended mapping class group of a surface is isomorphic to the outer automorphism group of the fundamental group of the surface, which is a 1-relator group. Both $\mathrm{Aut}(F_n)$ and the mapping class group are of much interest in geometric group theory. Grunewald and Lubotzky subsequently published a paper in conjunction with Larsen and Malestein ([GLLM]) in which they used the same idea to construct a family of linear representations of the mapping class group. They used these representations to show that the mapping class group has a rich family of arithmetic quotients.

Often a comparison is made between $\mathrm{Aut}(F_n)$/mapping class groups and lattices in Lie groups/arithmetic groups. They share many algebraic properties, such as being finitely generated, residually finite, and virtually torsion free. Furthermore, they both satisfy a Tits alternative (for the Tits alternative of a mapping class group see [M]). This comparison can be taken much further (see, e.g.[J]). This motivates the question of whether the mapping class group is linear, a question which still remains open. While it is known that $\mathrm{Aut}(F_n)$ is not linear, the analogy between $\mathrm{Aut}(F_n)$/mapping class groups and lattices in Lie groups has been fruitful.

The Grunewald Lubotzky representations can be used to prove other nice properties and extract useful information. Koberda showed that these same representations could detect the Nielsen-Thurston classification of automorphisms or mapping classes ([K2]). Following this, Hadari and Liu published papers in which they showed these representations can detect interesting dynamical properties of mapping classes and automorphisms of free groups ([H], [H2],[L]).

Mapping class groups are not the only groups that have similarities to $\mathrm{Aut}(F_n)$. The group F_n is an example of a graph product. A graph product is a group constructed from an underlying graph by letting the vertices of the graph be generators of the group and the edges of the graph represent relations. The group F_n corresponds to the graph with n vertices, but no edges. The automorphism groups of graph products were studied under various conditions in [CG], [CRSV], and [GPR]. In particular, F_n is a right-angled Artin group (RAAG), where the generators are of infinite order and the edges represent commutator relations. Right-angled Artin groups interpolate between the infinite abelian group $\mathbb{Z}^n$ and the free group F_n. A right-angled Coxeter group (RACG) is like a RAAG, but the generators are each of order 2. The similarities between free groups and RAAGs/RACGs are born out in their automorphism groups. In particular, it is shown in [CG] that both automorphism groups are generated by the same types of generators.

While the automorphism groups of both RAAGS and RACGs have been studied to various ends ([AC],[C],[CV],[GS],[KW],[SS]), not much work has been done on the representations of their automorphism groups, apart from specific cases like $\mathbb{Z}^n$ and F_n. Guirardel and Sale used the Grunewald Lubotzky construction to study automorphism groups of RAAGs ([GS]). However, no one has applied Grunewald and Lubotzky's ideas to construct representations of the automorphism group of a RACG. We do so in this dissertation.

Let Γ be a finite graph, and let W_Γ denote the right-angled Coxeter group associated to Γ. Let $\pi : W_\Gamma \to G$ be an epimorphism onto some finite group G. Following Grunewald and Lubotzky [GL], we construct a virtual representation $\rho_{\Gamma,G,\pi}$ of the automorphism group $\mathrm{Aut}(W_\Gamma)$. That is to say, we consider the finite index subgroup

$$\Gamma(G,\pi) := \{\varphi \in \mathrm{Aut}(W_\Gamma) \mid \pi \circ \varphi = \pi\}$$

of $\mathrm{Aut}(W_\Gamma)$ and construct a representation $\rho_{\Gamma,G,\pi} : \Gamma(G,\pi) \to \mathrm{GL}_t(\mathbb{Q})$ for some $t \in \mathbb{Z}$. This representation then induces a representation of $\mathrm{Aut}(W_\Gamma)$. While the construction works for any choice of G, π, we focus on a standard choice of G, π that depends on $|V(\Gamma)|$. In particular, we choose $G = (W_\Gamma)^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^{|V(\Gamma)|}$, and we choose π to be the abelianization map. Because we always make this choice, we suppress the G, π indices in our representation and write ρ_Γ. The goal of this paper is to better understand $\mathrm{Im}(\rho_\Gamma)$ as we vary Γ. We approach this goal by computing the isomorphism class of $\mathrm{Im}(\rho_\Gamma)$. In so doing, we give block matrix descriptions of the matrices in $\mathrm{Im}(\rho_\Gamma)$ as well as descriptions of the linear dependencies within each block. It turns out that $\mathrm{Im}(\rho_\Gamma)$ is the 2-congruence subgroup of the integer matrices in the integer points of a linear algebraic group; hence $\mathrm{Im}(\rho_\Gamma)$ is arithmetic.

Topologically, ρ_Γ can be constructed as follows. We first construct a certain $K(W_\Gamma, 1)$ space X. We take the cover $p : \hat{X} \to X$ corresponding to the finite-index subgroup $\ker(\pi)$. The group $\Gamma(G, \pi)$ acts on the first rational homology $H_1(\hat{X}; \mathbb{Q})$ of $\hat{X}$. This action is the representation ρ_Γ.

Using cellular homology, we can think of $H_1(\hat{X}; \mathbb{Q})$ as consisting of formal sums of edges of $\hat{X}$ (up to cellular 1-boundaries). Since G acts on $\hat{X}$ by deck transformations, we thus obtain an action of $\mathbb{Q}[G]$ on $H_1(\hat{X}; \mathbb{Q})$. As a $\mathbb{Q}[G]$-module, we may decompose $H_1(\hat{X}; \mathbb{Q})$ as a direct sum of irreducible $\mathbb{Q}[G]$-submodules. By grouping together the isomorphic irreducible submodules of $H_1(\hat{X}; \mathbb{Q})$, we obtain what is called the isotypic components of $H_1(\hat{X}; \mathbb{Q})$. We describe this in detail in Section 2.3. Due to our choice of G, π, the isotypic components can be indexed by subsets $J \subseteq V(\Gamma)$ of the generating set of W_Γ.

In Section 2.1 we show that $\Gamma(G, \pi)$ acts by $\mathbb{Q}[G]$-module automorphisms. Thus we may consider the action of $\Gamma(G, \pi)$ on the isotypic components of $H_1(\hat{X}; \mathbb{Q})$. This gives us a decomposition of ρ_Γ into sub-representations $\rho_{\Gamma,J}$ which are much easier to compute. It turns out that to compute $\rho_{\Gamma,J}$, we may restrict our attention to the subgraph Γ_J of Γ induced by J. This is made precise in the following lemma which we prove in Section 2.4. Here 1_J is an element of G that acts by projecting onto the J-isotypic component and $\hat{v}$ is a cellular 1-chain (up to cellular 1-boundaries) that corresponds in a natural way to $v \in W_\Gamma$. The precise meaning will be described later.

Lemma 1.1 (The Subgraph Lemma)**.** *Let Γ be a finite graph, and let $J \subseteq V(\Gamma)$. Then as $\mathbb{Q}[G]$-modules, $I_J(H_1(\hat{X}; \mathbb{Q})) = \langle 1_J(\hat{v} - \hat{w}) \mid v, w \in V(\Gamma_J)\rangle / \langle 1_J(\hat{v} -$*

$\hat{w}) \mid (v,w) \in E(\Gamma_J)\rangle \cong I_J(\mathbb{Q}[G])^{k_J-1}$, *where* k_J *is the number of components of* Γ_J.

In Chapter 3 we prove a number of decompositions that allow us to compute $\mathrm{Im}(\rho_{\Gamma,J})$. The idea for each of our decompositions is to write the matrices of $\mathrm{Im}(\rho_{\Gamma,J})$ in an upper triangular block matrix form where each of the entries in each block either come from $\mathrm{Im}(\rho_{\Gamma,A})$ for some $A \subsetneq J$ or from a known group. Thus repeated applications of the decompositions allow us to compute $\mathrm{Im}(\rho_{\Gamma,J})$.

Section 3.1 deals mainly with showing when elements of $\mathrm{Im}(\rho_{\Gamma,J})$ commute. In Section 3.2, we decompose $\mathrm{Im}(\rho_{\Gamma,J})$ by introducing a division of the set J. A *division* of J is a decomposition into subsets $J = \bigsqcup_{i=1}^{m} A_i$ such that each A_i is a union of components of Γ_J and contains a *special point* $a_i \in A_i$. The special point a_i provides some level of control over which components of $\Gamma \backslash \mathrm{st}(a_i)$ intersect A_i. We will formally define special points in Section 3.2. After some preliminary work, we get the following decomposition theorem.

Theorem 1.2 (TML Decomposition)**.** *Let* $J = \bigsqcup_{i=1}^{m} A_i$ *be a division of* J *with special points* $a_i \in A_I$. *Let* $J_0 := \{v \mid v \in [a_i]_J$ *for some* $1 \leqslant i \leqslant m\}$. *Then* $Im(\rho_{\Gamma,J}) \cong \Big(\Big(\prod_{i=1}^{m} Im(\rho_{\Gamma,A_i})\Big) \times Im(\rho_{\Gamma,J_0})\Big) \ltimes \mathbb{Z}^r$ *where* $r := |\{([v]_J, D) \mid v \in A_i \backslash [a_i]_J$ *for some* $1 \leqslant i \leqslant m, D$ *is a component of* $\Gamma \backslash st(v), D \cap A_i = \varnothing\}|$.

In Section 3.3 we define the notion of a splitting point. We then show that if a vertex set J contains a splitting point, then $\mathrm{Im}(\rho_{\Gamma,J})$ can be decomposed as a direct product (Proposition 3.22).

In Section 3.4 we introduce the notion of a separating set. Given two vertices v, w in distinct components of Γ_J, we define

$$\mathrm{sep}(v,w) := \{x \in J \mid \exists x^* \in [x]_J \text{ such that } D(x^*,v) \neq D(x^*,w)\}.$$

It is natural to explore this set because partial conjugations by these vertices act on $\hat{v} - \hat{w}$ as something other than an eigenvector. This set also has the nice property that for every element c outside of $\mathrm{sep}(v, w)$, the whole set $\mathrm{sep}(v, w)$ is in the same component of $\Gamma \backslash \mathrm{st}(c)$ (Proposition 3.24). We also prove another decomposition result (Lemma 3.26)

We take a slight detour in Section 3.5 to deal with a difficult case not covered by the hypotheses of Lemma 3.26. To that end, we introduce the notion of a compressible component. A *compressible component* is a component of Γ_J which can be compressed to a single vertex without affecting $\mathrm{Im}(\rho_{\Gamma,J})$. This will be formally defined in Section 3.5. This gives us a new method of computing $\mathrm{Im}(\rho_{\Gamma,J})$.

In Section 3.6, we finally prove how to compute the isomorphism class of $\mathrm{Im}(\rho_{\Gamma,J})$ for arbitrary Γ, J (Theorem 3.37).

Up until this point in the dissertation we have focused on how to compute $\mathrm{Im}(\rho_{\Gamma,J})$ for given Γ, J. One can also ask the following question. Which groups can be written as $\mathrm{Im}(\rho_{\Gamma,J})$ for some (Γ, J)? In Section 4.1 we show that a certain family of groups can be obtained in this way. To be more precise, consider the set

$$G_I := \{M \in \Gamma_n(2) \mid M_{i,j} = 0 \text{ for all } (i, j) \notin I\}$$

where $\Gamma_n(2)$ is the kernel of the map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/2\mathbb{Z})$ and $I \subseteq \{1, 2, \ldots, n\}^2$. We give a condition on the index set I that is equivalent to G_I being a group. Then every group G_I can be written as $\mathrm{Im}(\rho_{\Gamma,J})$ for some (Γ, J). In particular, we see that the intersection of the upper triangular matrix group with $\Gamma_n(2)$ can be written as the image of one of these representations.

1.2 Index of Notation

For the convenience of the reader, we collect here some of the more important notation that is consistent throughout the paper.

Γ: a finite graph

W_Γ: the right-angled Coxeter group associated to Γ

G: a finite group, usually $W_\Gamma^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^n$ where $n = |V(\Gamma)|$

π: an epimorphism from W_Γ to G

$\Gamma(G, \pi) := \{\varphi \in \mathrm{Aut}(W_\Gamma) \mid \pi \circ \varphi = \pi\}$

ρ_Γ: the virtual representation of $\mathrm{Aut}(W_\Gamma)$ constructed in Section 2.1

X: the $K(W_\Gamma, 1)$ space constructed in Section 2.1

$\hat{X}$: the cover of X corresponding to $\ker(\pi)$

$\mathrm{Aut}^0(W_\Gamma)$: the subgroup of $\mathrm{Aut}(W_\Gamma)$: generated by partial conjugations

$\mathrm{Out}^0(W_\Gamma)$: $\mathrm{Aut}^0(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$

$\sigma_{D,v}$: the partial conjugation by v on the component D of $\Gamma\backslash\mathrm{st}(v)$

J: a subset of $V(\Gamma)$

Γ_J: the subgraph of Γ induced by J

k_J: the number of components of Γ_J

$[v]_J$: the component of Γ_J containing v

$\rho_{\Gamma,J}$: the projection of ρ_Γ onto $\mathrm{Aut}(I_J(H_1(\hat{X}; \mathbb{Q})))$

$\mathcal{P}^{\Gamma,J} := \{\sigma_{D,v} \mid v \in J, D \cap J\backslash[v]_J \neq \varnothing\}$

$\bar{\sigma}$: the image of σ in $\mathrm{Out}(\Gamma)$

$\hat{\sigma} := \rho_{\Gamma,J}(\sigma)$

$D(v, w)$: the component of $\Gamma\backslash\mathrm{lk}(v)$ containing w

$\Gamma_n(2) := \ker(\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/2\mathbb{Z}))$

$$\mathrm{sep}(v,w) := \{x \in J \mid \exists x^* \in [x]_J \text{ such that } D(x^*,v) \neq D(x^*,w)\}$$
$$\mathrm{Int}(v,x) := \bigcap_{w \in [v]_J} D(w,x) \cap J$$
$$\mathfrak{D}(v) := \{D(w,y) \cap J \backslash [v]_J \mid w \in [v]_J, y \in J \backslash [v]_J\}.$$

Chapter 2

Setup

2.1 Constructing the representation

Let F_n be the free group with n generators. For each surjection $\pi : F_n \to G$ onto a finite group G, Grunewald and Lubotzky constructed a representation $\rho_{G,\pi}$ of $\mathrm{Aut}(F_n)$ [GL]. We follow Grunewald and Lubotzky's construction to get a representation $\rho_{\Gamma,G,\pi}$ for $\mathrm{Aut}(W_\Gamma)$, where W_Γ is the right angled Coxeter group associated to the graph Γ. That is to say,

$$W_\Gamma := \langle v \in V(\Gamma) \mid v^2 = 1, [v,w] = 1 \text{ for all } (v,w) \in E(\Gamma)\rangle.$$

Let Γ be a finite graph. Let G be a finite group and $\pi : W_\Gamma \to G$ be an epimorphism. Let $R = \ker(\pi)$ and $\bar{R} := R/[R,R]$. The action of W_Γ on R by conjugation leads to an action of G on $\bar{R}$. Let

$$\Gamma(G,\pi) := \{\varphi \in \mathrm{Aut}(W_\Gamma) \mid \pi \circ \varphi = \pi\}.$$

Then $\Gamma(G,\pi)$ is a finite index subgroup of $\mathrm{Aut}(W_\Gamma)$. Furthermore, every $\varphi \in$

$\Gamma(G, \pi)$ induces a G-equivariant linear automorphism $\bar{\varphi}$ of $\bar{R}$. Indeed, given $w \in W_\Gamma$ and $\varphi \in \Gamma(G, \pi)$, we have that $\varphi(w) = wr_w$ for some $r_w \in R$. Hence $\varphi(wrw^{-1}) = wr_w\varphi(r)r_w^{-1}w^{-1}$ holds for every $r \in R$. But this is equivalent to $w\varphi(r)w^{-1}$ modulo $[R, R]$, which implies that $\bar{\varphi} : \bar{R} \to \bar{R}$ is G-equivariant. Tensoring the domain and codomain of φ by $\mathbb{Q}$ induces a G-equivariant linear transformation $\hat{\varphi} : \bar{R} \otimes_{\mathbb{Z}} \mathbb{Q} \to \bar{R} \otimes_{\mathbb{Z}} \mathbb{Q}$. This yields a representation

$$\rho_{\Gamma,G,\pi} : \Gamma(G, \pi) \to \mathrm{GL}_t(\mathbb{Q}), \quad \varphi \mapsto \hat{\varphi}$$

for some $t \in \mathbb{Z}$. We wish to describe the image of this representation as we vary Γ for a standard choice of G, π that depends on Γ.

Let $V(\Gamma) = \{v_1, \ldots, v_n\}$. Then our choice of finite group G is the group $W_\Gamma^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^n$, and our epimorphism $\pi_\Gamma : W_\Gamma \to W_\Gamma^{ab}$ is the abelianization map. We write π when Γ is understood. Since we will always use this choice of G, π unless we say otherwise, we compress the G, π in our notation and write $\rho_\Gamma := \rho_{\Gamma,G,\pi}$.

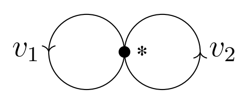
We can translate the above description of our representation into a topological description as follows. Let Γ be a finite graph, let G be a finite group, and let and $\pi : W_\Gamma \to G$ be an epimorphism. We construct a $K(W_\Gamma, 1)$ space X as outlined in [H3], Section 4.2. We start with the wedge sum $\bigvee_{v \in V(\Gamma)} S_v^1$. For each $v \in V(\Gamma)$ we attach a 2-cell to the loop vv. For each edge $(v, w) \in E(\Gamma)$, we attach a 2-cell to the loop $vw\bar{v}\bar{w}$. We then attach some higher dimensional cells in such a way that the higher order homotopy groups become trivial.

By covering space theory, there is a cover of X corresponding to each subgroup of $\pi_1(X) = W_\Gamma$. Let $p : \hat{X} \to X$ be the cover of X corresponding to the subgroup $R := \ker(\pi)$. Then G is the deck transformation group of $\hat{X}$. Fix a lift $\hat{*}$ of the vertex $*$ in X to the cover $\hat{X}$. Given a deck transformation $g \in G$, we let $g\hat{*}$ be the image of $\hat{*}$ under the deck transformation g. This gives us a labeling of the vertices of $\hat{X}$.

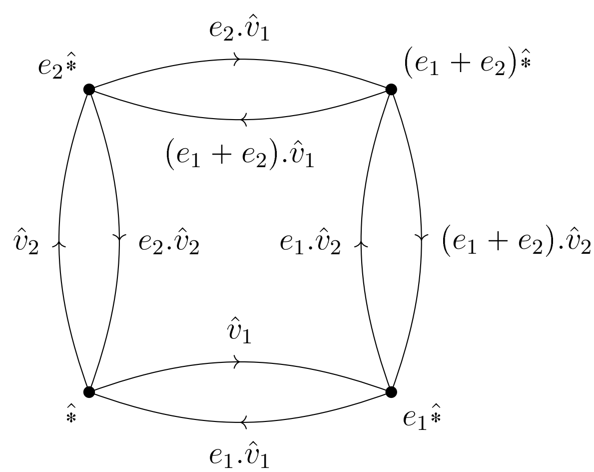
For each loop v in X, there is a lift $\hat{v}$ of v based at $*$. Note that the terminal vertex of v is the vertex $\pi(v)\hat{*}$. Given a deck transformation $g \in G$, we let $g\hat{v}$ denote the image of $\hat{v}$ under the deck transformation. This is precisely the lift of v based at the vertex $g\hat{*}$.

Example 1. Let Γ be the graph

Then the 1-skeleton of X is the graph



and the 1-skeleton of $\hat{X}$ is the graph



Let $\varphi \in \Gamma(G,\pi)$. Since X is a $K(W_\Gamma, 1)$ space, φ can be considered as an automorphism $\pi_1(X) \to \pi_1(X)$. This induces a unique homotopy equivalence $\varphi_X : (X,*) \to (X,*)$ up to based homotopy (see, e.g. [H3]).

The map $p \circ \varphi_X : \hat{X} \to X$ induces a map $(p \circ \varphi_X)_* : \pi_1(\hat{X}, \hat{*}) \to \pi_1(X, *)$. By definition of $\Gamma(G, \pi)$, the map φ stabilizes $\ker(\pi) = p_*(\pi_1(\hat{X}))$. It follows that $(p \circ \varphi_X)_*(\pi_1(\hat{X}, \hat{*})) = p_*(\pi_1(\hat{X}, \hat{*}))$. Therefore $(p \circ \varphi_X)$ lifts to a map $\hat{\varphi} : (\hat{X}, \hat{*}) \to (\hat{X}, \hat{*})$. It is not hard to see that $\hat{\varphi}$ is a lift of φ_X.

$$\begin{tikzcd}
(\hat{X}, \hat{*}) \arrow[r, dashed, "\hat{\varphi}"] \arrow[d, "p"'] \arrow[dr] & (\hat{X}, \hat{*}) \arrow[d, "p"'] \\
(X, *) \arrow[r, "\varphi_X"] & (X, *)
\end{tikzcd}$$

Since $\Gamma(G, \pi)$ acts trivially on $W_\Gamma / \ker(\pi) = G$, it follows that φ fixes the G-orbit of $\hat{*}$. But this is precisely $\hat{X}_0$. Consequently, $\hat{\varphi} : (\hat{X}, \hat{X}^0) \to (\hat{X}, \hat{X}^0)$ is a map of pairs. We thus get an induced homomorphism $\hat{\varphi}_* : H_1(\hat{X}, \hat{X}^0; \mathbb{Q}) \to H_1(\hat{X}, \hat{X}^0; \mathbb{Q})$.

Let $C_i^{cw}(\hat{X}; \mathbb{Q}) := H_1(\hat{X}_i, \hat{X}_{i-1}; \mathbb{Q})$. By Section 2.2 of [H3], there is a relative cellular chain complex

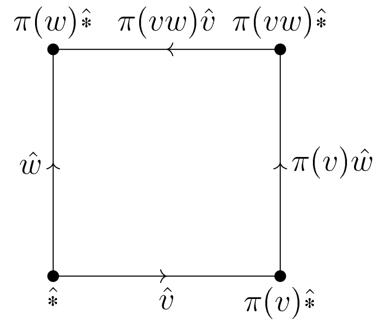
$$\cdots \to H_2(\hat{X}^2, \hat{X}^1 \cup \hat{X}^0; \mathbb{Q}) \to H_1(\hat{X}^1, \hat{X}^0 \cup \hat{X}^0; \mathbb{Q}) \to H_1(\hat{X}^0, \varnothing \cup \hat{X}^0; \mathbb{Q})$$

with homology groups isomorphic to $H_n(\hat{X}, \hat{X}^0; \mathbb{Q})$. But

- $H_2(\hat{X}^2, \hat{X}^1 \cup \hat{X}^0; \mathbb{Q}) = H_2(\hat{X}^2, \hat{X}^1; \mathbb{Q}) = C_2^{cw}(\hat{X}; \mathbb{Q})$.
- $H_1(\hat{X}^1, \hat{X}^0 \cup \hat{X}^0; \mathbb{Q}) = H_1(\hat{X}^1, \hat{X}^0; \mathbb{Q}) = C_1^{cw}(\hat{X}; \mathbb{Q})$.
- $H_0(\hat{X}^0, \varnothing \cup \hat{X}^0; \mathbb{Q}) = H_0(\hat{X}^0, \hat{X}^0; \mathbb{Q}) = 0$.

This shows that $H_1(\hat{X}, \hat{X}^0; \mathbb{Q}) \cong C_1^{cw}(\hat{X}; \mathbb{Q}) / \delta_2(C_2^{cw}(\hat{X}; \mathbb{Q}))$. Hence we have a well-defined action of $\Gamma(G, \pi)$ on cellular 1-chains mod cellular 1-boundaries. From this point on we let $\hat{v}$ represent the cellular 1-chain corresponding to the vertex v up to cellular 1-boundary.

Example 2. Consider the inner automorphism σ_v by the vertex v. By definition, for any $w \in E(\Gamma)$, we have that $\sigma_v(w) = vwv$. To compute the action of σ_v on $\hat{w}$, we look at the lift of the path vwv. Hence $\hat{\sigma}_v$ sends $\hat{w}$ to $\hat{v} + \pi(v)\hat{w} + \pi(vw)\hat{v}$.



2.2 Reduction to $\mathbf{Aut}^0(G, \pi)$

Let Γ be a finite graph, and let W_Γ be the right-angled Coxeter group associated to the graph Γ. The goal of this section is to show that for the standard choice of finite group G and epimorphism $\pi : W_\Gamma \to G$, the group $\Gamma(G, \pi)$ is just $\mathrm{Aut}^0(W_\Gamma)$, the group generated by partial conjugations of W_Γ. We begin by describing $\mathrm{Aut}(W_\Gamma)$. By [CG], $\mathrm{Aut}(W_\Gamma)$ is generated by three types of automorphisms:

1. Graph automorphisms

2. Dominated transvections: Choose distinct vertices u, v such that $\mathrm{st}(u) \subseteq \mathrm{st}(v)$. Then define
$$\tau_{u,v}(w) := \begin{cases} uv & \text{if } w = u \\ w & \text{if } w \neq u \end{cases}$$

3. Partial conjugations: Choose a vertex v. Let D be a connected component

of the graph $\Gamma\backslash\mathrm{st}(v)$. Then define

$$\sigma_{D,v}(w) := \begin{cases} vwv^{-1} & \text{if } w \in D \\ w & \text{if } w \notin D \end{cases}$$

We may then define the following two subgroups:

- $\mathrm{Aut}^0(W_\Gamma)$ is the group generated by partial conjugations
- $\mathrm{Aut}^1(W_\Gamma)$ is the group generated by graph automorphisms and dominated transvections.

Given a right-angled coxeter group W_Γ, a word is a finite sequence $w = (v_1^{\epsilon_1}, v_2^{\epsilon_2}, \ldots, v_n^{\epsilon_n})$ where for each $1 \leqslant i \leqslant n$ we have $v_i \in V(\Gamma)$ and $\epsilon_i \in \{\pm 1\}$. We call n the length of the word w. Each word represents an element of W_Γ, namely the element $v_1^{\epsilon_1} v_2^{\epsilon_2} \ldots v_n^{\epsilon_n}$ obtained by multiplying together the elements in the sequence. Note that multiple different words may represent the same group element. A word w is said to be in reduced form if there is no word representing the same group element with a smaller length than w. Note that the reduced form of a group element need not be unique. However, any two reduced words representing the same group element differ only by repeated swapping of the order of adjacent vertices (Lemma 2.3 of [GPR]).

The following lemma will help us to better understand $\mathrm{Aut}^1(W_\Gamma)$.

Lemma 2.1. *Let Γ be a finite graph. If $\varphi \in \mathrm{Aut}^1(W_\Gamma)$, then for all $v \in V(\Gamma)$ we have that $\varphi(v) = \prod_{i=1}^{k} v_i$ in reduced form for some $\{v_i \in V(\Gamma) \mid 1 \leqslant i \leqslant k\}$ which form a complete subgraph of Γ.*

Proof. We prove this by induction on the length of a word in $\mathrm{Aut}^1(W_\Gamma)$. The base case is trivial.

Now assume that for all $\varphi \in \mathrm{Aut}^1(W_\Gamma)$ of length $\leqslant k$ that the induction hypothesis holds. We can increase the length either by composing by a graph automorphism or a dominated transvection. Since graph automorphisms take complete subgraphs to complete subgraphs, we need only consider composing by a dominated transvection. Let $\varphi \in \mathrm{Aut}^1(W_\Gamma)$ such that $\varphi(v) = \prod_{i=1}^{k} v_i$ and let $\tau_{u,w}$ be a dominated transvection. If $u \notin \{v_i \mid 1 \leqslant j \leqslant k\}$, then $\tau_{u,w} \circ \varphi(v) = \prod_{i=1}^{k} v_i$ and we are done. If, on the other hand, $u \in \{v_i \mid 1 \leqslant i \leqslant k\}$, then for all $1 \leqslant i \leqslant k$ we have $v_i \in \mathrm{st}(u) \subseteq \mathrm{st}(w)$ since $\{v_i \mid 1 \leqslant i \leqslant k\}$ form a complete graph and $\tau_{u,w}$ is well-defined. Therefore $\{v_i \mid 1 \leqslant i \leqslant k\} \cup \{w\}$ form a complete subgraph, completing the induction. □

In order to prove that $\Gamma(G,\pi) = \mathrm{Aut}^0(W_\Gamma)$, we first show the following lemma.

Lemma 2.2. *If $\varphi \in \mathrm{Aut}^1(W_\Gamma)\backslash\{\mathrm{Id}\}$ then $\varphi \notin \Gamma(G,\pi)$.*

Proof. Let $\varphi \in \mathrm{Aut}^1(W_\Gamma)\backslash\{\mathrm{Id}\}$. If φ is a graph isomorphism, then it is not in $\Gamma(G,\pi)$. By this and Lemma 2.1, we may assume for some $v \in V(\Gamma)$ that $\varphi(v) = \prod_{i=1}^{k} v_i \neq v$ in reduced form and the v_i form a complete subgraph. But then $\pi\Big(\prod_{i=1}^{k} v_i\Big) \neq \pi(v)$. □

We now have all the tools we need to prove the main result of this section.

Lemma 2.3. $\Gamma(G,\pi) = \mathrm{Aut}^0(W_\Gamma)$

Proof. By proposition 5.3 of [CG], $\mathrm{Aut}^0(W_\Gamma)$ is the group of conjugating automorphisms, i.e. automorphisms where the images of generators are conjugates. Since G is abelian, and π is a homomorphism, for any $v \in V(\Gamma)$ the image of a conjugate of v under π is the same as for v. Thus $\mathrm{Aut}_0(W_\Gamma) \leqslant \Gamma(G,\pi)$.

To show the reverse inclusion, we will first show that $\mathrm{Aut}^1(W_\Gamma)$ normalizes $\mathrm{Aut}^0(W_\Gamma)$. Let $\sigma = \sigma_{D,v}$ be partial conjugation by v on the component D of $\Gamma \backslash \mathrm{st}(v)$ and γ be a graph isomorphism. Then for all $w \in V(\Gamma)$, we compute

$$\gamma^{-1} \circ \sigma \circ \gamma(w) = \gamma^{-1}(v^\epsilon \gamma(w) v^{-\epsilon}) = \gamma^{-1}(v)^\epsilon w \gamma^{-1}(v)^{-\epsilon}$$

where $\epsilon \in \{0,1\}$ depending on whether $\gamma(w) \in D$. This is a conjugate of w, so $\gamma^{-1} \circ \sigma \circ \gamma \in \mathrm{Aut}^0(W_\Gamma)$. Now let $\tau = \tau_{u,w}$ be a dominated transvection. For any vertex $v \neq u$, let $\sigma(v) = cvc^{-1}$; then $\tau^{-1} \circ \sigma \circ \tau(v) = \tau^{-1}(c) v \tau^{-1}(c)^{-1}$, which is a conjugate of v. Therefore to show that $\tau^{-1} \circ \sigma \circ \tau \in \mathrm{Aut}^0(W_\Gamma)$ it suffices to show that $\tau^{-1} \circ \sigma \circ \tau(u)$ is a conjugate of u. If $w \in \mathrm{st}(v)$, then w and v commute and $w \notin D$, so

$$\tau^{-1} \circ \sigma \circ \tau(u) = \tau^{-1}(v^\epsilon u v^{-\epsilon} w) = \tau^{-1}(v^\epsilon u w v^{-\epsilon}) = \tau^{-1}(v)^\epsilon u \tau^{-1}(v)^{-\epsilon}$$

is a conjugate of u. If $u \in \mathrm{st}(v)$, then $v \in \mathrm{st}(u) \subseteq \mathrm{st}(w)$. Hence $w \in \mathrm{st}(v)$ and we are in the case above. If $u, w \notin \mathrm{st}(v)$, then u and w are in the same component of $\Gamma \backslash \mathrm{st}(v)$ so $\tau^{-1} \circ \sigma \circ \tau(u) = \tau^{-1}(v^\epsilon u w v^{-\epsilon}) = \tau^{-1}(v)^\epsilon u \tau^{-1}(v)^{-\epsilon}$ is a conjugate of u. Since this covers all cases, $\tau^{-1} \circ \sigma \circ \tau \in \mathrm{Aut}^0(W_\Gamma)$. Therefore $\mathrm{Aut}^1(W_\Gamma)$ normalizes $\mathrm{Aut}^0(W_\Gamma)$. Since $\mathrm{Aut}^0(W_\Gamma), \mathrm{Aut}^1(W_\Gamma)$ generate $\mathrm{Aut}(W_\Gamma)$. this implies that any $\phi \in \mathrm{Aut}(W_\Gamma)$ can be written in the form $\sigma \circ \varphi$ for some $\sigma \in \mathrm{Aut}^0(W_\Gamma)$, $\varphi \in \mathrm{Aut}^1(W_\Gamma)$.

Now let $\phi \in \Gamma(G, \pi)$ and write $\phi = \sigma \circ \varphi$ as above. Then since $\sigma \in \mathrm{Aut}^0(W_\Gamma) \leqslant \Gamma(G, \pi)$, it follows that $\varphi = \sigma^{-1} \circ \phi \in \Gamma(G, \pi)$. By Lemma 2.2, this implies $\varphi = \mathrm{Id}$. Therefore $\phi = \sigma \in \mathrm{Aut}^0(C_\Gamma)$. $\square$

2.3 Isotypic components

The following information can be found in [CR]. Let G be a finite group. Then there are a finite number of irreducible left $\mathbb{Q}[G]$-modules. Let $M_1, M_2, \dots M_s$ be the irreducible left $\mathbb{Q}[G]$-modules. Every left $\mathbb{Q}[G]$-module M may be written as a finite direct sum of irreducible left $\mathbb{Q}[G]$-modules $M = \bigoplus_{i=1}^{s} M_i^{r_i}$ for some $r_i \in \mathbb{Z}_{\geqslant 0}$. Given any $1 \leqslant i \leqslant s$, the left $\mathbb{Q}[G]$-module $I_i(M) := M_i^{r_i}$ is called the M_i-isotypic component of the left $\mathbb{Q}[G]$-module M. We may therefore write $M = \bigoplus_{i=1}^{s} I_i(M)$ as a direct sum of its isotypic components.

Now $\mathbb{Q}[G]$ itself may be thought of as a left-$\mathbb{Q}[G]$ module, where the group action is multiplication on the left by elements of $\mathbb{Q}[G]$. Thus we may write $\mathbb{Q}[G] = \bigoplus_{i=1}^{s} I_i(\mathbb{Q}[G])$. Each $I_i(\mathbb{Q}[G])$ is not only a $\mathbb{Q}[G]$-module, it is also a ring with unity 1_i. From the theory of representations of finite groups, $Q[G]$ decomposes as a product of rings $B_1 \times B_2 \times \cdots \times B_s$ where B_i is the M_i-isotypic component. Moreover, each $(0, 0, \dots, 1_i, 0, \dots, 0)$ acts on M_i as identity and on M_j as 0 for each $j \neq i$. Thus, if we view B_i as a subring in this way, multiplying by 1_i is the same as projection to the M_i-isotypic component.

We now consider our standard finite group $G = W_\Gamma^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^n$. Since G is finite, abelian, and all elements are of order 2, its irreducible rational representations are all one-dimensional over $\mathbb{Q}$. In fact the irreducible rational representations of G are precisely $\rho_J : G \to \mathbb{Q}^\times$ where $J \subseteq \pi(V(\Gamma))$ and

$$\rho_J(\pi(v)) = \begin{cases} -1 & \text{if } \pi(v) \in J \\ 1 & \text{if } \pi(v) \notin J \end{cases}.$$

Let M_J denote the irreducible module corresponding to the representation ρ_J.

Then for distinct M_{J_1} and M_{J_2} we have that $M_{J_1} \ncong M_{J_2}$ as left $\mathbb{Q}[G]$-modules. Therefore each M_J is an isotypic component of $\mathbb{Q}[G]$. We therefore index the isotypic components of $\mathbb{Q}[G]$-modules and their corresponding identity elements by subsets $J \subseteq \pi(V(\Gamma))$ rather than integers as above.

Because the standard epimorphism $\pi : W_\Gamma \to G$ induces a bijection between $V(\Gamma)$ and $\pi(V(\Gamma))$, we can think of J as indexing a subset of $V(\Gamma)$ rather than a subset of $\pi(V(\Gamma))$. We therefore identify J with a subset of $V(\Gamma)$. Hence the irreducible $\mathbb{Q}[G]$-modules are in bijective correspondence with $\mathcal{P}(V(\Gamma))$.

2.4 Subgraph Lemma

We now return to considering the representation ρ_Γ. Recall this arose by considering the action of $\Gamma(G, \pi)$ on the $\mathbb{Q}[G]$-module $H_1(\hat{X}; \mathbb{Q})$. It turns out to be much simpler to consider the action of $\Gamma(G, \pi)$ on the isotypic components of $H_1(\hat{X}; \mathbb{Q})$.

Let $J \subseteq V(\Gamma)$. Let Γ_J denote the subgraph of Γ induced by J, let k_J denote the number of connected components of Γ_J, and let $[v]_J$ denote the connected component of Γ_J containing v. It turns out that decomposing $H_1(\hat{X}; \mathbb{Q})$ into its isotypic components is simply a matter of understanding the subgraphs Γ_J (the subgraph induced by the vertices in J). To that end, we now prove the Subgraph Lemma (Lemma 1.1)

Lemma 1.1 (The Subgraph Lemma)**.** Let Γ be a finite graph, and let $J \subseteq V(\Gamma)$. Then as $\mathbb{Q}[G]$-modules, $I_J(H_1(\hat{X}; \mathbb{Q})) \cong \langle 1_J(\hat{v} - \hat{w}) \mid v, w \in V(\Gamma_J)\rangle / \langle 1_J(\hat{v} - \hat{w}) \mid (v, w) \in E(\Gamma_J)\rangle \cong I_J(\mathbb{Q}[G])^{k_J - 1}$.

Proof of Subgraph Lemma. Let $d_i : C_i^{cw}(\hat{X}; \mathbb{Q}) \to C_{i-1}^{cw}(\hat{X}; \mathbb{Q})$ be the boundary maps on the cellular chain complex and let $p_{i,J} : C_i^{cw}(\hat{X}; \mathbb{Q}) \to I_J(C_i^{cw}(\hat{X}; \mathbb{Q}))$

denote projection onto the isotypic component corresponding to J. Because $d_i(1_J\hat{v}) = 1_J d_i(\hat{v})$, the $p_{i,J}$ induce a boundary map on the chain complex $I_J(C_i^{cw}(\hat{X};\mathbb{Q}))$ such that the following diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r, "d_3"] & C_2^{cw}(\hat{X};\mathbb{Q}) \arrow[r, "d_2"] \arrow[d, "p_{2,J}"'] & C_1^{cw}(\hat{X};\mathbb{Q}) \arrow[r, "d_1"] \arrow[d, "p_{1,J}"'] & C_0^{cw}(\hat{X};\mathbb{Q}) \arrow[d, "p_{0,J}"'] \\
\dots \arrow[r, "\bar{d}_3"] & I_J(C_2^{cw}(\hat{X};\mathbb{Q})) \arrow[r, "\bar{d}_2"] & I_J(C_1^{cw}(\hat{X};\mathbb{Q})) \arrow[r, "\bar{d}_1"] & I_J(C_0^{cw}(\hat{X};\mathbb{Q}))
\end{tikzcd}$$

Furthermore $I_J(H_1^{cw}(\hat{X};\mathbb{Q})) \cong \ker(\bar{d}_1)/\mathrm{Im}(\bar{d}_2)$ is just the first homology of the induced chain complex.

Let $\hat{v}$ be an element of $C_1^{cw}(\hat{X};\mathbb{Q})$. Then $d_1(\hat{v}) = (\pi(v)-1)\hat{*}$. Projecting onto $I_J(C_i^{cw}(\hat{X};\mathbb{Q}))$ is the same as multiplying by 1_J. Thus we compute

$$\bar{d}_1(1_J\hat{v}) = 1_J(\pi(v)-1)\hat{*} = (\rho_J \circ \pi(v) - 1)1_J\hat{*} = \begin{cases} -2 \cdot 1_J\hat{*} & \text{if } v \in J \\ 0 & \text{if } v \notin J \end{cases}$$

Therefore $\ker(\bar{d}_1) = \langle 1_J(\hat{v} - \hat{w}) \mid v, w \in V(\Gamma_J)\rangle \bigoplus \langle 1_J\hat{v} \mid v \notin V(\Gamma_J)\rangle$.

We now wish to compute $\mathrm{Im}(\bar{d}_2)$. Let f_v denote the 2-cell in $\hat{X}$ attached to the lift of v^2 based at $\hat{*}$. Then $d_2(f_v) = (\pi(v)+1)\hat{v}$. Let $f_{v,w}$ denote the 2-cell in $\hat{X}$ attached to the lift of $[v,w]$ based at $\hat{*}$. Then $d_2(f_{v,w}) = (1-\pi(v))\hat{w} + (\pi(w)-1)\hat{v}$. Looking now at the induced boundary map, we see that

$$\bar{d}_2(1_J f_v) = (\rho_J \circ \pi(v) + 1)1_J\hat{v} = \begin{cases} 2 \cdot 1_J\hat{v} & \text{if } v \notin J \\ 0 & \text{if } v \in J \end{cases}$$

$$\bar{d}_2(1_J f_{v,w}) = (1 - \rho_J \circ \pi(v))1_J\hat{w} + (\rho_J \circ \pi(w) - 1)1_J\hat{v} = \begin{cases} 2 \cdot 1_J(\hat{w} - \hat{v}) & \text{if } v, w \in J \\ -2 \cdot 1_J\hat{w} & \text{if } v \in J, w \notin J \\ 2 \cdot 1_J\hat{w} & \text{if } v \notin J, w \in J \\ 0 & \text{if } v, w \notin J \end{cases}$$

It follows that $\mathrm{Im}(\bar{d}_2) = \langle 1_J(\hat{v} - \hat{w}) \mid (v,w) \in E(\Gamma_J)\rangle \bigoplus \langle 1_J\hat{v} \mid v \notin V(\Gamma_J)\rangle$. Therefore

$$I_J(H_1(\hat{X};\mathbb{Q})) \cong I_J(H_1^{cw}(\hat{X};\mathbb{Q})) = (\langle 1_J(\hat{v}-\hat{w}) \mid v,w \in V(\Gamma_J)\rangle \bigoplus \langle 1_J\hat{v} \mid v \notin V(\Gamma_J)\rangle)/$$

$$(\langle 1_J(\hat{v} - \hat{w}) \mid (v,w) \in E(\Gamma_J)\rangle \bigoplus \langle 1_J\hat{v} \mid v \notin V(\Gamma_J)\rangle).$$

This gives us the first isomorphism. To see the second one, let $\{v_1, \ldots, v_{k_J}\}$ be a set of representatives of the components of Γ_J. Then $\{1_J(\hat{v}_{i+1} - \hat{v}_i) \mid 1 \leqslant i < k_J\}$ is a basis for $I_J(H_1^{cw}(\hat{X};\mathbb{Q}))$. □

We now return to the automorphism group $\Gamma(G,\pi)$, which we showed is equal to $\mathrm{Aut}^0(W_\Gamma)$ (Lemma 2.3). Let $\rho_{\Gamma,J}$ denote the projection of ρ_Γ onto $\mathrm{Aut}(I_J(H_1(\hat{X};\mathbb{Q})))$. We will mostly be studying the images of these $\rho_{\Gamma,J}$. When there is no room for ambiguity, given any $\sigma \in \mathrm{Aut}^0(W_\Gamma)$, we let $\hat{\sigma}$ denote the map on $C_1^{cw}(\hat{X};\mathbb{Q})/\delta_2(C_2^{cw}(\hat{X};\mathbb{Q}))$ induced by σ described in Section 2.1. Since $\hat{\sigma}$ is linear, and elements of $I_J(H_1(\hat{X};\mathbb{Q}))$ are essentially linear combinations of elements of $C_1^{cw}(\hat{X};\mathbb{Q})/\delta_2(C_2^{cw}(\hat{X};\mathbb{Q}))$, we also let $\hat{\sigma}$ denote the matrix $\rho_{\Gamma,J}(\sigma)$. It will always be clear from context which definition of $\hat{\sigma}$ is being used.

The following corollary tells us that we can ignore partial conjugations by elements not in J. Recall that $[v]_J$ is the component of Γ_J containing v.

Corollary 2.4. *For any $v \notin J$ and any partial conjugation $\sigma = \sigma_{D,v}$ of D by v, we have that $\hat{\sigma} = Id$. Furthermore, for any partial conjugation $\sigma = \sigma_{D,v}$ such that $v \in J$ and $D \cap J \backslash [v]_J = \varnothing$, we have that $\hat{\sigma} = Id$.*

Proof. For the first statement, if $w \notin D$, then $\sigma(w) = w$, hence $\hat{\sigma}(1_J\hat{w}) = 1_J\hat{w}$. If $w \in D$, then $\sigma(w) = vwv^{-1} = vwv$. In this case, the induced automorphism on $C_1(\hat{X};\mathbb{Q})$ maps the 1-chain $\hat{w}$ to the 1-chain $\hat{v} + \pi(v)\hat{w} + \pi(vw)\hat{v}$. Note that $v \notin J \implies 1_J\hat{v} = 0$. Projecting onto $I_J(H_1(\hat{X};\mathbb{Q}))$, we get that $\hat{\sigma}$ maps $1_J\hat{w}$ to $1_J(\hat{v} + \pi(v)\hat{w} + \pi(vw)\hat{v}) = 1_J\hat{v} + \rho_J \circ \pi(v)1_J\hat{w} + \rho_J \circ \pi(vw)1_J\hat{v} = 1_J\hat{w}$. Therefore $\hat{\sigma}$ acts as identity on all the 1-chains mod boundaries in $I_J(C_1(\hat{X};\mathbb{Q}))$, so $\hat{\sigma} = \mathrm{Id}$.

For the second statement, let $1_J\hat{w} \in I_J(C_1(\hat{X};\mathbb{Q}))$ be non-zero. Then if $w \notin D$, we have that $\sigma(w) = w$. If $w \in D$, then by hypothesis $w \in [v]_J$. By the Subgraph Lemma (Lemma 1.1), $1_J\hat{v} = 1_J\hat{w}$ inside of $I_J(H_1(\hat{X};\mathbb{Q}))$. By the above calculation, $\hat{\sigma}(1_J\hat{w}) = 1_J\hat{v} + \rho_J \circ \pi(v)1_J\hat{w} + \rho_J \circ \pi(vw)1_J\hat{v} = 2(1_J\hat{v}) - 1_J\hat{w}$. Inside of $I_J(H_1(\hat{X};\mathbb{Q}))$ this equals $1_J\hat{w}$. Therefore $\hat{\sigma} = \mathrm{Id}$. $\square$

To make use of the above corollary we establish the following notation. Let Γ be a graph, $J \subseteq V(\Gamma)$. We define

$$\mathcal{P}^{\Gamma,J} := \{\sigma_{D,v} \mid v \in J, D \cap J\backslash[v]_J \neq \varnothing\}.$$

We also define the following subgroup of $\mathrm{Aut}^0(W_\Gamma)$.

$$\mathrm{Aut}^J(W_\Gamma) := \langle \mathcal{P}^{\Gamma,J} \rangle.$$

Given $\sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$, to compute $\hat{\sigma}_{D,v}$ it suffices to know how $\hat{\sigma}_{D,v}$ acts on vectors of the form $\hat{b} - \hat{a}$ for $a, b \in J$. That computation is the content of the following lemma.

Lemma 2.5. *Let $\sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$ and $a, b \in J$.*

1. *If $a, b \notin D$, then $\hat{\sigma}_{D,v}(1_J(\hat{b} - \hat{a})) = 1_J(\hat{b} - \hat{a})$.*

2. *If $a, b \in D$, then $\hat{\sigma}_{D,v}(1_J(\hat{b} - \hat{a})) = -1_J(\hat{b} - \hat{a})$.*

3. *If $b \in D, a \notin D$, then $\hat{\sigma}_{D,v}(1_J(\hat{b} - \hat{a})) = 2 \cdot 1_J(\hat{v} - \hat{b}) + 1_J(\hat{b} - \hat{a})$.*

4. *If $a \in D, b \notin D$, then $\hat{\sigma}_{D,v}(1_J(\hat{b} - \hat{a})) = 2 \cdot 1_J(\hat{v} - \hat{a}) - 1_J(\hat{b} - \hat{a})$.*

Proof. If $w \notin D$, then $\sigma_{D,v}(w) = w$, so $\hat{\sigma}_{D,v}(1_J\hat{w}) = 1_J\hat{w}$. If $w \in D$, then $\sigma_{D,v}(w) = vwv$, so $\hat{\sigma}_{D,v}(1_J\hat{w}) = 1_J\hat{v} + \rho_J \circ \pi(v) 1_J\hat{w} + \rho_J \circ \pi(vw) 1_J\hat{v} = 2 \cdot 1_J\hat{v} - 1_J\hat{w}$. The result follows by linearity. □

Knowing how partial conjugations act on vectors of the form $\hat{b} - \hat{a}$ for $a, b \in J$, we see that writing 1_J all the time isn't really necessary. Therefore we will abuse notation and will set $\hat{v} = 1_J\hat{v}$.

Corollary 2.4 allows us to restrict the domain of $\rho_{\Gamma,J}$ to $\mathrm{Aut}^J(W_\Gamma)$ when computing its image. The following lemma will allow us to restrict the codomain of $\rho_{\Gamma,J}$ as well.

Lemma 2.6. *For any choice of basis of $I_J(H_1(\hat{X}; \mathbb{Q}))$ where every vector is of the form $\hat{v} - \hat{w}$ for some $v, w \in V(\Gamma)$, we have $Im(\rho_{\Gamma,J}) \leqslant \Gamma_{k_J - 1}(2)$ where $\Gamma_n(2) := \ker(GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2\mathbb{Z}))$.*

Proof. Fix a basis $\{\hat{v}_i - \hat{v}_{i-1} \mid 1 < i \leqslant k_J\}$ of $I_J(H_1(\hat{X}; \mathbb{Q}))$. By the above commentary, we need only consider the images of elements of $\mathcal{P}^{\Gamma,J}$. Let $\sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$. Then by Lemma 2.5, we have

$$\hat{\sigma}_{D,v}(\hat{v}_{i+1} - \hat{v}_i) = a(\hat{v} - \hat{v}_{i+1}) + b(\hat{v} - \hat{v}_i) + c(\hat{v}_{i+1} - \hat{v}_i)$$

for some $a, b \in \{0, \pm 2\}$ and $c \in \{\pm 1\}$.

Now $\hat{v} - \hat{v}_i$ need not be a basis vector, but since $v, v_i \in J$ we can write $\hat{v} - \hat{v}_i$ as a linear combination of basis vectors with all coefficients equal to 1 or 0. If $v = v_j$ and $j < i$ then $\hat{v} - \hat{v}_i = \sum_{k=1}^{j-i} (\hat{v}_{j-k+1} - \hat{v}_{j-k})$ (similarly for $i < j$). If $i = j$, then $\hat{v} - \hat{v}_i = 0$. Thus each basis vector maps to a linear combination of basis vectors where the coefficients are all ± 2s or 0s except the coefficient on the original basis vector, which is ± 1. This shows that $\hat{\sigma} \in \Gamma_n(2)$. □

By virtue of Lemma 2.6 and Corollary 2.4, we adopt the standing assumption that $\rho_{\Gamma,J} : \mathrm{Aut}^J(W_\Gamma) \to \Gamma_{k_J - 1}(2)$. We now show that this is the most we can restrict the codomain in general. However, we first introduce the following notation which will be used henceforth. Given non-adjacent vertices, $v, w \in J$, let $D(v, w)$ denote the component of $\Gamma \backslash \mathrm{lk}(v)$ containing w. We use $\mathrm{lk}(v)$ in this definition rather than $\mathrm{st}(v)$ so that we have $D(v, v) = \{v\}$. This will be useful later.

Theorem 2.7. *Let $\varnothing \neq J \subseteq V(\Gamma)$. Let K be a set of representatives of the components of Γ_J. Assume that for each pair of distinct vertices $v, w \in K$ we have $D(v, w) \cap K = \{w\}$. Then Im$(\rho_{\Gamma,J}) = \Gamma_{k_J - 1}(2)$. In particular, if Γ is discrete, then Im$(\rho_{\Gamma,J}) = \Gamma_{|J|-1}(2)$ for every nonempty $J \subseteq V(\Gamma)$.*

Proof. Let $K = \{v_1, v_2, \ldots, v_{k_J}\}$. It is known that for all $n \geqslant 1$ the group $\Gamma_n(2)$ is generated by $\{E_{i,j}, F_i \mid 1 \leqslant i, j \leqslant n, i \neq j\}$ where $E_{i,j}$ is the matrix identical to the identity matrix except in the (i, j)-entry, which equals 2, and F_i is the matrix identical to the identity matrix except in the (i, i)-entry, which equals -1 (see, for example, [K]).

Fix the basis $\hat{v}_{k_J} - \hat{v}_1, \hat{v}_{k_J} - \hat{v}_2, \ldots, \hat{v}_{k_J} - \hat{v}_{k_J - 1}$ of $I_J(H_1(\hat{X}; \mathbb{Q}))$. By Lemma 2.6, we know that $\mathrm{Im}(\rho_{\Gamma,J}) \leqslant \Gamma_{k_J - 1}(2)$. Thus it suffices to show that each element of $\{E_{i,j}, F_i \mid 1 \leqslant i, j \leqslant k_J - 1, i \neq j\}$ is in $\mathrm{Im}(\rho_{\Gamma,J})$. Let $1 \leqslant i, j \leqslant k_J - 1$ with $i \neq j$. Then by direct calculation,

$$\hat{\sigma}_{D(v_i,v_j),v_i} \cdot \hat{\sigma}_{D(v_{k_J},v_j),v_{k_J}} = E_{i,j}$$

$$\hat{\sigma}_{D(k_J,v_i),v_{k_J}} = F_i.$$

This completes the proof.

□

We conclude this section with a couple of simple observations that will be used throughout

Lemma 2.8. *For any $J \subseteq V(\Gamma)$ the following hold.*

1. *For any partial conjugation $\sigma \in \mathcal{P}^{\Gamma,J}$, we have $\hat{\sigma}$ is of order 2.*

2. *For any inner automorphism σ relative to J (i.e. σ conjugates all of J by some element v), we have $\hat{\sigma} = -Id$.*

Proof. For the first statement, this follows immediately from the fact that σ is of order 2, which follows from the fact that every generator $v \in V(\Gamma)$ of W_Γ is of order 2. For the second statement, this follows from the second case calculation in the proof of Lemma 2.6. □

Chapter 3

Computing Im($\rho_{\Gamma,J}$)

3.1 Commutation relations

In this section we prove a strong connection between commuting matrices in Im($\rho_{\Gamma,J}$) and commuting outer automorphisms in $\mathrm{Out}^0(W_\Gamma) := \mathrm{Aut}^0(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$. To understand when two outer automorphisms commute, we must first recall the definition of a separating intersection of links (SIL). We do so here.

Definition 3.1. Let Γ be a graph. A *Separating Intersection of Links(SIL)* is a triple $(u, v \mid w)$ of vertices $u, v, w \in V(\Gamma)$ such that u and v are non-adjacent and w is not in the same component of $\Gamma \backslash (\mathrm{lk}(u) \cap \mathrm{lk}(v))$ as u or v. One may also write $(u, v \mid D)$ where D is the component of $\Gamma \backslash (\mathrm{lk}(u) \cap \mathrm{lk}(v))$ containing w.

One reason that SILs are useful is that if $(u, v \mid D)$ is a SIL, then D is not only a component of $\Gamma \backslash (\mathrm{lk}(u) \cup \mathrm{lk}(v))$, it is also a component of $\Gamma \backslash \mathrm{lk}(u)$ and $\Gamma \backslash \mathrm{lk}(v)$. This will be the content of Lemma 3.4. However, before we can prove that, we need the following lemma from [GPR] (Lemma 4.3).

Lemma 3.2 (GPR)**.** *Let $\sigma_{C,v}, \sigma_{D,w}$ be partial conjugations. If v, w are non-adjacent and $w \notin C$, then $C \cap D = \varnothing$ or $C \subseteq D$.*

For our purposes, the following form of the above lemma will be more useful. Recall from Section 2.4 the following notation. Given non-adjacent vertices, $v, w \in J$, let $D(v, w)$ denote the component of $\Gamma \backslash \mathrm{lk}(v)$ containing w.

Corollary 3.3. *Let $u, v, w \in J$ be pairwise non-adjacent. Then if $v \notin D(u, w)$ then $D(u, w) \subseteq D(v, w)$.*

Proof. Consider the partial conjugations $\sigma_{D(u,w),u}$ and $\sigma_{D(v,w),v}$. By Lemma 3.2, either $D(u, w) \cap D(v, w) = \varnothing$ or $D(u, w) \subseteq D(v, w)$. But $w \in D(u, w) \cap D(v, w)$. Therefore $D(u, w) \subseteq D(v, w)$. □

Lemma 3.4. *Let v, w be non-adjacent vertices. Then, $(v, w \mid D)$ is a SIL $\iff$ for all $d \in D$, we have $D(v, d) = D(w, d) = D$.*

Proof. By the definition of a SIL, D is a component of $\Gamma \backslash (\mathrm{lk}(v) \cap \mathrm{lk}(w))$. Let $d \in D$. Then by the definition of a SIL, we have that $w \notin D(v, d)$ and $v \notin D(w, d)$. Since $D(v, d) \cap D(w, d) \neq \varnothing$, Corollary 3.3 implies that $D(v, d) = D(w, d)$.

Let $u \in D(v, d)$. Then there is a path α from u to d that does not pass through $\mathrm{st}(v)$. It follows that α does not pass through $\mathrm{lk}(v) \cap \mathrm{lk}(w)$, hence $u \in D$. This shows that $D(v, d) \subseteq D$.

Now let $u \in D(v, d)^{\complement}$. Let α be a path from d to u. We claim that α must pass through $\mathrm{lk}(v) \cap \mathrm{lk}(w)$. Clearly it passes through $\mathrm{lk}(v)$. If α passed through $\mathrm{lk}(w)$ prior to passing through $\mathrm{lk}(v)$, then we would have a path β from d to w that did not pass through $\mathrm{lk}(v)$. However $(v, w \mid d)$ is a SIL, so this is not the case. Therefore α does not pass through $\mathrm{lk}(w)$ prior to passing through $\mathrm{lk}(v)$. Similarly, α does not pass through $\mathrm{lk}(v)$ prior to passing through $\mathrm{lk}(w)$. Since α passes through $\mathrm{lk}(v)$, it follows that α passes through $\mathrm{lk}(v) \cap \mathrm{lk}(w)$. This shows that $D(v,d)^{\complement} \subseteq D^{\complement}$ and hence $D(v,d) = D$. By symmetry, $D(w,d) = D = D(v,d)$.

Next, assume that $D(v,d) = D(w,d) = D$ where D is the component of $\Gamma \backslash (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ containing d. Then $v \notin D(v,d) = D$ and $w \notin D(w,d) = D$, so $(v, w \mid D)$ is a SIL. □

We now turn our attention to determining when matrices in $\rho_{\Gamma,J}(\mathcal{P}^{\Gamma,J})$ commute. We establish the following notation. Given an automorphism $\sigma \in \mathrm{Aut}^0(W_\Gamma)$, we let $\bar{\sigma}$ denote the image of σ in $\mathrm{Out}^0(W_\Gamma)$. The following lemma, though proved in [GPR] is stated in our preferred form in [SS] (Lemma 1.4). It tells us when two outer automorphisms commute.

Lemma 3.5 (SS)**.** *Given two partial conjugations $\sigma_1 := \sigma_{C,u}$ and $\sigma_2 := \sigma_{D,v}$, we have that $\bar{\sigma}_1, \bar{\sigma}_2$ do not commute $\iff$ there exists some $w \in V(\Gamma)$ such that $(u, v \mid w)$ is a SIL and one of the following conditions is met:*

1. $w \in C = D$
2. $u \in D, v \in C$
3. $v \in C, w \in D$
4. $u \in D, w \in C$

Recall that by Lemma 2.8, every inner automorphism maps to $\pm\mathrm{Id}$. Combining this with Lemma 2.6, we get the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Aut}^J(W_\Gamma) \arrow[r] \arrow[d, "\rho_{\Gamma,J}"'] & \mathrm{Out}^J(W_\Gamma) \arrow[d, "\bar{\rho}_{\Gamma,J}"] \\
\Gamma_{k_J-1}(2) \arrow[r] & \Gamma_{k_J-1}(2)/\{\pm\mathrm{Id}\}
\end{tikzcd}$$

We use this diagram to prove the following result.

Proposition 3.6. *Let $\sigma_{C,u}, \sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$. If $\bar{\sigma}_{C,u}$ and $\bar{\sigma}_{D,v}$ commute then $\hat{\sigma}_{C,u}$ and $\hat{\sigma}_{D,v}$ commute.*

However before we prove this, we need the following lemma:

Lemma 3.7. *There is no $A \in \Gamma_n(2)$ such that $A^2 = -Id$*

Proof of Lemma 3.7. Let $A \in \Gamma_n(2)$. Then either $A_{1,1} \equiv_4 1$ or $A_{1,1} \equiv_4 3$. In either case, we see that $A_{1,1}^2 \equiv_4 1$. Furthermore, for all $2 \leqslant i \leqslant n$, we have that $A_{1,i}A_{i,1} \equiv_4 0$. Thus $(A^2)_{1,1} = A_{1,1}^2 + \sum_{i=2}^{n} A_{1,i}A_{i,1} \equiv_4 1$. But $(-\mathrm{Id})_{1,1} \not\equiv_4 1$. Therefore $A^2 \neq -\mathrm{Id}$. □

Proof of Proposition 3.6. If $\bar{\sigma}_1, \bar{\sigma}_2$ commute, then the above diagram implies that $[\hat{\sigma}_1, \hat{\sigma}_2] = \pm\mathrm{Id}$. By the first statement of Lemma 2.8, we have $([\sigma_1, \sigma_2]) = (\hat{\sigma}_1 \circ \hat{\sigma}_2)^2$. Then by Lemma 3.7, we have $(\hat{\sigma}_1 \circ \hat{\sigma}_2)^2 \neq -\mathrm{Id}$. Therefore $[\hat{\sigma}_1, \hat{\sigma}_2] = \mathrm{Id}$, so $\hat{\sigma}_1$ and $\hat{\sigma}_2$ commute. □

The reverse direction to Proposition 3.6 does not hold in general. However it is easy to state when it holds. First we need the following definition.

Definition 3.8. Let Γ be a finite graph and let $J \subseteq V(\Gamma)$. We say $(u, v \mid w)$ is a <u>*SIL relative to J*</u> if $(u, v \mid w)$ is a SIL, $u, v, w \in J$, and $[u]_J \neq [v]_J$. We also say $(u, v \mid D)$ is a <u>*SIL relative to J*</u> if $(u, v \mid w)$ is a SIL relative to J for some $w \in D$.

We require $[u]_J \neq [v]_J$ in the above definition because if $[u]_J = [v]_J$ then $\hat{u} = \hat{v}$. Thus this requirement is in some sense analogous to the requirement for SILs that u and v be non-adjacent. We now show that partial conjugations by elements $u, v \in J$ such that $[u]_J = [v]_J$ commute.

Proposition 3.9. *Let $\sigma_{C,u}, \sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$ be such that $[u]_J = [v]_J$. Then $\hat{\sigma}_{C,u}$ and $\hat{\sigma}_{D,v}$ commute.*

Proof. Fix a basis $\mathcal{B}$ of $I_J(H_1(\hat{X};\mathbb{Q}))$ such that each element of $\mathcal{B}$ is of the form $\hat{u}-\hat{w}$ for some $w \notin [u]_J$. Then for each $\hat{u}-\hat{w} \in \mathcal{B}$ we have that $\hat{\sigma}_{C,u}(\hat{u}-\hat{w}) = \pm(\hat{u}-\hat{w})$. Therefore $\hat{\sigma}_{C,u}$ is a diagonal matrix. Similarly, since $\hat{u}=\hat{v}$, we get that $\hat{\sigma}_{D,v}$ is a diagonal matrix. Therefore $\hat{\sigma}_{C,u}$ and $\hat{\sigma}_{D,v}$ commute. $\square$

Corollary 3.10. *Given $u, v \in J$, if u and v do not form a SIL relative to J, then $\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}$ commute for all $\sigma_{C,u}, \sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$.*

Proof. Let $\sigma_{C,u}, \sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$. Assume that u and v do not form a SIL relative to J. If $[u]_J = [v]_J$, then $\hat{\sigma}_{D,v}$ commutes with $\hat{\sigma}_{C,u}$ by Proposition 3.9. If not, then by Proposition 3.6, $\hat{\sigma}_{D,v}$ commutes with $\hat{\sigma}_{C,u}$ unless $\bar{\sigma}_{D,v}$ does not commute with $\bar{\sigma}_{C,u}$. By Lemma 3.5, this implies that $(u, v \mid w)$ is a SIL for some $w \in V(\Gamma)$ and one of the four conditions is met. Since u and v do not form a SIL relative to J, it follows from Lemma 3.4 that $w \notin C$ and $w \notin D$. Thus we must have that $u \in D, v \in C$.

We now show by contradiction that $(C \cup D) \cap J = J$. Let $j \in J \backslash (C \cup D)$. Then since $j \notin C = D(u,v)$, it follows that $v \notin D(u,j)$. Corollary 3.3 implies that $D(u,j) \subseteq D(v,j)$. Similarly, since $j \notin D = D(v,u)$, we get that $D(v,j) \subseteq D(u,j)$. This implies that $D(u,j) = D(v,j)$. But then by Lemma 3.4, we have that $(u, v \mid D(u,j))$ is a SIL relative to J. This is a contradiction. Therefore $(C \cup D) \cap J = J$.

By direct computation, for any $w \in J$ we have $[\sigma_{C,u}, \sigma_{D,v}](w) = uvuvwvuvu$ so $[\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}] = \mathrm{Id}$. This shows that $\hat{\sigma}_{C,u}$ and $\hat{\sigma}_{D,v}$ commute. $\square$

We now have all the tools we need to say exactly when the images of two partial conjugations commute.

Proposition 3.11. *Let $\sigma_{C,u}, \sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$. Then $\hat{\sigma}_{C,u}$ and $\hat{\sigma}_{D,v}$ do not commute*

$\iff$ *there exists some $w \in J$ such that $(u, v \mid w)$ is a SIL relative to J and one of the following conditions is met:*

1. $w \in C = D$
2. $u \in D, v \in C$
3. $v \in C, w \in D$
4. $u \in D, w \in C$

Proof. Assume that $\hat{\sigma}_{C,u}$ and $\hat{\sigma}_{D,v}$ do not commute. Then by the contrapositive of Proposition 3.6, we see that $\bar{\sigma}_{C,u}$ and $\bar{\sigma}_{D,v}$ do not commute. By Lemma 3.5, this implies that there exists some $w \in J$ such that $(u, v \mid w)$ is a SIL and one of the four conditions is met. Note that the contrapositive of Proposition 3.9 implies that $[u]_J \neq [v]_J$. If $D(u, w) = C$, then $(u, v \mid C)$ is a SIL relative to J and one of the four conditions is met. Similarly, if $D(u, w) = C$, then $(u, v \mid C)$ is a SIL relative to J and one of the four conditions is met. Thus we can assume that $u \in D$ and $v \in C$. In this case, it remains to show that u and v form a SIL relative to J. This follows from the contrapositive to Corollary 3.10.

Now assume that there exists some $w \in J$ such that $(u, v \mid w)$ is a SIL relative to J and one of the four conditions holds. We check case by case that $\hat{\sigma}_{C,u}$ and $\hat{\sigma}_{D,v}$ do not commute by finding a vector in $I_J(H_1(\hat{X}, \mathbb{Q}))$ which $[\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}]$ does not fix.

1. $(w \in C = D)$: Then $[\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}](\hat{w} - \hat{u}) = 4\hat{v} - 4\hat{u} + \hat{w} - \hat{u}$.
2. $(u \in D, v \in C)$: In this case, neither $(u, v \mid C)$ nor $(u, v \mid D)$ is a SIL. Therefore $w \notin C \cup D$, It follows that $[\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}](\hat{w} - \hat{v}) = 4\hat{v} - 4\hat{u} + \hat{w} - \hat{v}$.

3. $(v \in C, w \in D)$: Since $w \in D$, we have that $(u, v \mid D)$ is a SIL. Hence $u \notin D$. It follows that $[\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}](\hat{w} - \hat{u}) = 4\hat{v} - 4\hat{u} + \hat{w} - \hat{u}$.

4. $(u \in D, w \in C)$: Then $[\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}](\hat{w} - \hat{v}) = 4\hat{v} - 4\hat{u} + \hat{w} - \hat{v}$.

Since $[u]_J \neq [v]_J$, we have that $4\hat{v} - 4\hat{u} \neq 0$ in $I_J(H_1(\hat{X}; \mathbb{Q}))$. Thus each of the above calculations show that $[\hat{\sigma}_{C,u}, \hat{\sigma}_{D,v}] \neq \mathrm{Id}$. □

3.2 TML Decomposition

The goal of this section is to prove a decomposition theorem that will allow us to reduce the size of our vertex set J subject to certain conditions. The basic idea of the argument is to partition J into "components" $J = \bigsqcup_{i=1}^{m} A_i$ in a nice way. We use this partition to break $\mathrm{Im}(\rho_{\Gamma,J})$ into three different subgroups.

- The subgroup $\mathcal{T}$ is generated by images of partial conjugations by elements of A_i on subsets of A_i for some i. Thus, choosing the appropriate basis, you almost end up with block upper diagonal matrices with one block corresponding to each A_i. Unfortunately, to make a complete basis, we need to include vectors which are a difference of vectors from distinct A_i. We group these all into a block at the end. After some slight modifications, you end up with matrices that look like this:

$$\begin{bmatrix} M_1 & 0 & 0 & \dots & 0 \\ 0 & M_2 & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & M_m & 0 \\ 0 & 0 & 0 & \dots & \mathrm{Id} \end{bmatrix}.$$

To clarify how $\mathcal{T}$ relates to our other groups, we compress the non-identity blocks and think of $\mathcal{T}$ as consisting of block 2x2 matrices

$$\begin{bmatrix} M & 0 \\ 0 & \mathrm{Id} \end{bmatrix}.$$

The blocks for our other subgroups correspond to the blocks here.

- The subgroup $\mathcal{L}$ is generated by images of partial conjugations by elements of A_i on things outside of A_i. We choose our A_i in such a way that such elements can only act as $\pm\mathrm{Id}$ on the blocks corresponding to A_i. This can be easily modified so as to act as Id on these blocks. In so doing, we end up with matrices of the following form:

$$\begin{bmatrix} \mathrm{Id} & M \\ 0 & \mathrm{Id} \end{bmatrix}.$$

- In making the modifications for the subgroup $\mathcal{L}$, some partial conjugations end up getting left out. These get grouped together to form the subgroup $\mathcal{M}$. Again modifications are made to ensure that $\mathcal{M}$ behaves nicely on the A_i blocks, and we end up with matrices of the form

$$\begin{bmatrix} \mathrm{Id} & 0 \\ 0 & M \end{bmatrix}.$$

When considering the block diagonal forms of these matrices, it seems natural to expect that $\mathrm{Im}(\rho_{\Gamma,J}) = (\mathcal{T} \times \mathcal{M}) \ltimes \mathcal{L}$. Furthermore, each of these subgroups is either isomorphic to $\mathrm{Im}(\rho_{\Gamma,J'})$ for some $J' \subsetneq J$ or to a known group.

Before we define our partition condition, we must first make a different definition.

Definition 3.12. Let $A \subseteq J$ be a non-empty union of components of Γ_J. Then a point $a \in A$ is a special point if the following hold:

1. For each $v \in [a]_J$ and each component D of $\Gamma \backslash \mathrm{st}(v)$, either $D \cap J \backslash [v]_J \subseteq A$ or $D \cap J \backslash [v]_J \subseteq A^{\complement}$

2. For each $v \in A^{\complement}$, we have that $A \subseteq D(v, a)$.

The reason a special point is special is that, for any $v \in J$, it provides some level of control over which components of $\Gamma \backslash \mathrm{lk}(v)$ intersect A. If $v \in A^{\complement}$, we see by the second condition that only one component of $\Gamma \backslash \mathrm{lk}(v)$ intersects A. If $v \in [A]_J$, the first condition ensures that any component that intersects A (up to elements of $[v]_J$) is in fact a subset of A. To see what happens to elements in $A \backslash [a]_J$, we prove the following lemma.

Lemma 3.13. *Let $A \subseteq J$ be a non-empty union of components of Γ_J, Let $a \in A$ be such that for each $v \in [a]_J$ and each component D of $\Gamma \backslash st(v)$ we have that either $D \cap J \backslash [v]_J \subseteq A$ or $D \cap J \backslash [v]_J \subseteq A^{\complement}$. Then for all $b \in A \backslash [a]_J$ and every component D of $\Gamma \backslash st(b)$ the following hold.*

1. *If $a \notin D$, then either $D \cap J \subseteq A$ or $D \cap J \subseteq A^{\complement}$.*

2. *If $D \subseteq A^{\complement}$, we have that D is a component of $\Gamma \backslash st(a)$.*

Proof. 1. Let $d \in D$ so that $D = D(b, d)$. Then by Corollary 3.3, we have that $D(b, d) \subseteq D(a, d)$. But by hypothesis, we have that $D(a, d) \cap J \backslash [a]_J \subseteq A$ or $D(a, d) \cap J \backslash [a]_J \subseteq A^{\complement}$. Since $[a]_J \cap D = \varnothing$, this shows that $D \cap J \subseteq A$ or $D \cap J \subseteq A^{\complement}$.

2. Let $d \in D$ so that $D = D(b,d)$. Since $a \notin D(b,d)$, Corollary 3.3 implies that $D(b,d) \subseteq D(a,d)$. Furthermore, since $d \in A^{\complement}$, by hypothesis we have that $D(a,d) \cap J \backslash [a]_J \subseteq A^{\complement}$. In particular, we see that $b \notin D(a,d)$. Therefore Corollary 3.3 implies that $D(a,d) \subseteq D(b,d)$. This shows that $D = D(b,d) = D(a,d)$.

□

We now define the partition condition necessary for the TML decomposition to hold.

Definition 3.14. Let $J \subseteq V(\Gamma)$. We say $J = \bigsqcup_{i=1}^{m} A_i$ is a <u>division</u> of J if the following hold.

1. Each A_i is a union of components of Γ_J.

2. Each A_i has a special point a_i.

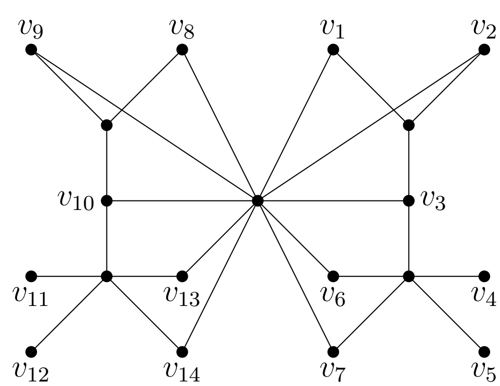
To get a sense of this definition we state a few examples.

Example 3. Let $J = \bigsqcup_{i=1}^{m} [v_i]_J$. Then define $A_i := [v_i]_J$. Then it is trivial that $J = \bigsqcup_{i=1}^{m} A_i$ is a division of J. Thus every J has a trivial division. However, this division will not be useful to us.

Example 4. Let Γ have m components $D_1, D_2, \ldots, D_m$ that intersect J for some $m > 1$. For each $1 \leqslant i \leqslant m$, let $A_i := D_i \cap J$. Since each component of Γ_J lies in a component of Γ, each A_i is a union of components of Γ_J. For each $1 \leqslant i \leqslant m$, fix some $a_i \in A_i$. Then for any $1 \leqslant i \leqslant m$, any $v \in [a_i]$, and any $w \in A_j$ with $j \neq i$, we have $D(v, a_j) \cap J = A_j$ and $D(a_j, a_i) \cap J = A_i$. Thus each a_i is a special point, and $J = \bigsqcup_{i=1}^{m} A_i$ is a division of J.

Example 5. Let Γ be a tree, and let J be the set of leaves of Γ. Given $v, w \in J$, we say $v \sim w$ if $\mathrm{lk}(v) = \mathrm{lk}(w)$. Then $\sim$ is an equivalence relation. Assume that there is more than one equivalence class under this relation, and let $A_1, A_2, \ldots, A_m$ be the equivalence classes. Since J consists of pairwise non-adjacent vertices, each A_i is a union of components of Γ_J. For each $1 \leqslant i \leqslant m$, fix some $a_i \in A_i$. Then for any $v \sim w$ we have that $D(v, w) = \{w\}$. Furthermore, if $v \nsim w$, then $\mathrm{lk}(v) \cap \mathrm{lk}(w) = \varnothing$. Therefore for any $v \in A_i, w \in A_i^{\complement}$, we have that $D(w, v) \supseteq A_i$. This shows that $J = \bigsqcup_{i=1}^{m} A_i$ is a division of J.

Example 6. In the previous examples, every point was a special point relative to A_i. This need not be the case. Consider, for example the following graph:



Let $J = \{v_1, v_2, \ldots, v_{14}\}$. Let $A_1 = \{v_1, v_2, \ldots, v_7\}$, and let $A_2 = \{v_8, v_9, \ldots, v_{14}\}$. Then $J = A_1 \sqcup A_2$ is a division of J. However, v_4 and v_5 are not special points with respect to A_1 and v_{11} and v_{12} are not special points with respect to A_2.

To simplify notation, we extend our partial conjugation notation as follows. We write $\sigma_{D \cap J, v} := \sigma_{D,v}$ whenever $\sigma_{D,v}$ is defined. Furthermore, since conjugating any element of $[v]_J$ by v acts as identity in the homology, we may add or remove elements of $[v]_J$ to the set being conjugated and still have the same automorphism. Thus for an arbitrary subset $D \subseteq J$, we define $\sigma_{D,v} := \sigma_{D',v}$ for any D' such that $\sigma_{D',v}$ is defined and $D \backslash [v]_J = D' \backslash [v]_J$. Thus, for example, we may write $\sigma_{J,v}$ to represent an inner automorphism relative to J.

Recall that the group $\mathcal{T}$ consists of block diagonal matrices that are non-identity on a number of distinct blocks. It will be beneficial to us later on to look at these blocks individually. Therefore, before working with the full subgroup T, we consider a subgroup T_1 consisting of a single block. Recall that $\mathcal{P}^{\Gamma,J} := \{\sigma_{D,v} \mid v \in J, D \cap J \backslash [v]_J \neq \varnothing\}$.

Lemma 3.15. *Let $J \subseteq V(\Gamma)$ be an arbitrary vertex set. Let $A \subsetneq J$ be a non-empty union of components of Γ_J. Assume there exists an $a \in A$ such that for each $v \in [a]_J$ and each component D of $\Gamma \backslash st(v)$, either $D \cap J \backslash [v]_J \subseteq A$ or $D \cap J \backslash [v]_J \subseteq A^{\complement}$. We define the following subgroup of $Im(\rho_{\Gamma,J})$.*

$$\mathcal{T}_1 := \langle \hat{\sigma}_{D,v} \in \rho_{\Gamma,J}(\mathcal{P}^{\Gamma,A}) \mid a \notin D \rangle.$$

Then with respect to the appropriate bases,

$$\mathcal{T}_1 = \left\{ \begin{bmatrix} M & 0 \\ 0 & Id \end{bmatrix} \;\middle|\; M \in Im(\rho_{\Gamma,A}) \right\} \cong \mathcal{T}_1 \cong Im(\rho_{\Gamma,A}).$$

Proof. Let $\mathcal{B}_A$ be an ordered basis of $\langle \hat{v} - \hat{w} \mid v, w \in A \rangle \leqslant I_J(H_1(\hat{X};\mathbb{Q}))$ consisting of vectors of the form $\hat{v} - \hat{w}$. Let S be an ordered list of representatives of the components of $\Gamma_{J \backslash A}$. We extend $\mathcal{B}_A$ to an ordered basis $\mathcal{B}$ of $I_J(H_1(\hat{X};\mathbb{Q}))$ so that $\mathcal{B} = \mathcal{B}_A \cup \{\hat{s} - \hat{a} \mid s \in S\}$.

Let $G := \{\sigma_{D,v} \in \mathcal{P}^{\Gamma,A} \mid a \notin D \backslash [v]_J\}$. Let $\sigma_{D,v} \in G$. By the definition of a special point, combined with Lemma 3.13, we have that $D \cap J \backslash [v]_J \subseteq A$. Therefore the matrix $\hat{\sigma}_{D,v}$ only differs from identity on the block corresponding to $\mathcal{B}_A$.

Let $\pi : \mathcal{T}_1 \to \Gamma_{|\mathcal{B}_A|}(2)$ be the projection map onto the block corresponding to $\mathcal{B}_A$. By the above argument π is both well-defined and injective, hence an isomorphism onto its image.

Because $A \subseteq J$, we have a linear embedding

$$T : I_A(H_1(\hat{X};\mathbb{Q})) \to I_J(H_1(\hat{X};\mathbb{Q})), \quad T(1_A(\hat{v} - \hat{w})) = 1_J(\hat{v} - \hat{w}).$$

Fix the basis $T^{-1}(\mathcal{B}_A)$ of $I_A(H_1(\hat{X};\mathbb{Q}))$. Then for all $\sigma \in \rho_{\Gamma,J}^{-1}(\mathcal{T}_1)$ we have that $\rho_{\Gamma,A}(\sigma) = \pi \circ \rho_{\Gamma,J}(\sigma)$. Thus it suffices to show that $\rho_{\Gamma,A}(\mathcal{P}^{\Gamma,A}) \in \pi(\mathcal{T}_1)$. By the hypothesis, for each $v \in [a]_J$ and each component D of $\Gamma \backslash \mathrm{st}(v)$, either $D \cap J\backslash[v]_J \subseteq A$ or $D \cap J\backslash[v]_J \subseteq A^{\complement}$. Thus we need only show that $\rho_{\Gamma,A}(\sigma_{D(v,a),v}) \in \pi(\mathcal{T}_1)$ for each $v \in A\backslash[a]_J$.

Note that $\sigma_{A,a} \in \langle G \rangle$, since $\sigma_{D,a} \in G$ for all components of $\Gamma\backslash\mathrm{st}(a)$ which intersect $A\backslash[a]_J$. Since $\sigma_{A,a}$ is inner relative to A, we have that $\rho_{\Gamma,A}(\sigma_{A,a}) = -\mathrm{Id}$ (Lemma 2.8). It follows that for any $v \in A\backslash[a]_J$ we have

$$\rho_{\Gamma,A}(\sigma_{D(v,a),v}) = \rho_{\Gamma,A}\Big(\sigma_{A,a} \circ \prod_{a \notin D} \sigma_{D,v}\Big) \in \pi(\mathcal{T}_1).$$

□

The following corollary will be useful in proving other similar decomposition theorems later.

Corollary 3.16. *Let $J \subseteq V(\Gamma)$ be an arbitrary vertex set. Fix some $v \in J$. Then $\{\hat{\sigma}_{D,w} \mid v \notin D\}$ generates $Im(\rho_{\Gamma,J})$.*

Proof. This follows immediately from Lemma 3.15 by letting $A = J$. □

With Lemma 3.15 in hand, understanding the subgroup $\mathcal{T}$ is fairly straightforward.

Corollary 3.17. *Let* $J = \bigsqcup_{i=1}^{m} A_i$ *be a division of* J *with special points* $a_i \in A_I$. *We define the following subgroup of* $Im(\rho_{\Gamma,J})$.

$$\mathcal{T} := \langle \hat{\sigma}_{D,v} \mid v \in A_i, 1 \leqslant i \leqslant m, D \cap A_i \backslash [v]_J \neq \varnothing, a_i \notin D \rangle.$$

Then $\mathcal{T} \cong \prod_{i=1}^{m} Im(\rho_{\Gamma,A_i})$.

Proof. For each $1 \leqslant i \leqslant m$, let $\mathcal{B}_{A_i}$ be an ordered basis of $\langle \hat{v} - \hat{w} \mid v, w \in A_i \rangle \leqslant I_J(H_1(\hat{X};\mathbb{Q}))$ consisting of vectors of the form $\hat{v} - \hat{w}$. Fix the following ordered basis of $I_J(H_1(\hat{X};\mathbb{Q}))$.

$$\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m, \hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_m - \hat{a}_1.$$

Consider the set of automorphisms

$$G_i := \{\sigma_{D,v} \mid v \in A_i, D \cap A_i \backslash [v]_J \neq \varnothing, a_i \notin D\}.$$

Then $\bigsqcup_{i=1}^{m} \rho_{\Gamma,J}(G_i)$ is a generating set for $\mathcal{T}$. Furthermore, by Lemma 3.13, each $\hat{\sigma}_{D,v} \in \rho_{\Gamma,J}(G_i)$ only acts non-trivially on the block corresponding to $\mathcal{B}_i$. Note that our choice of basis corresponds to the choice of basis in Lemma 3.15. Therefore by Lemma 3.15, we have that

$$\mathcal{T} = \left\{ \begin{bmatrix} M_1 & & & & \\ & M_2 & & & \\ & & \ddots & & \\ & & & M_m & \\ & & & & \mathrm{Id}_{m-1} \end{bmatrix} \;\middle|\; M_i \in \mathrm{Im}(\rho_{\Gamma,A_i}) \right\} \cong \prod_{i=1}^{m} \mathrm{Im}(\rho_{\Gamma,A_i}).$$

□

Next, we turn our attention to the subgroup $\mathcal{M}$.

Lemma 3.18. *Let $J = \bigsqcup_{i=1}^{m} A_i$ be a division of J with special points $a_i \in A_i$. Let*

$$\mathcal{M} := \left\langle \Big(\prod_{A_i \subseteq D} (\hat{\sigma}_{A_i,a_i}) \Big) \hat{\sigma}_{D,v} \mid v \in [a_j]_J \text{ for some } 1 \leqslant j \leqslant m, D \cap J \backslash [v]_J \subseteq A_j^{\complement} \right\rangle.$$

Let $J_0 := \{v \mid v \in [a_i]_J$ for some $1 \leqslant i \leqslant m\}$. Then with respect to the appropriate bases

$$\mathcal{M} = \left\{ \begin{bmatrix} Id_{|J|-m} & 0 \\ 0 & M \end{bmatrix} \;\middle|\; M \in Im(\rho_{\Gamma,J_0}) \right\} \cong Im(\rho_{\Gamma,J_0}).$$

Proof. For each $1 \leqslant i \leqslant m$, let $\mathcal{B}_i$ be an ordered basis of $\langle \hat{v} - \hat{w} \mid v, w \in A_i \rangle \leqslant I_J(H_1(\hat{X};\mathbb{Q}))$ consisting of vectors of the form $\hat{v} - \hat{w}$. Fix the following ordered basis of $I_J(H_1(\hat{X};\mathbb{Q}))$.

$$\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m, \hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_m - \hat{a}_1.$$

Let $\Big(\prod_{A_i \subseteq D} (\hat{\sigma}_{A_i,a_i}) \Big) \cdot \hat{\sigma}_{D,v}$ be a generator of $\mathcal{M}$. Then for each A_i such that $i \neq j$ and $A_i \cap D \neq \varnothing$, by definition of special point we have that $A_i \subseteq D$. It follows that $D \cap J \backslash [v]_J = \bigcup_{A_i \subseteq D} A_i$. Therefore for each $d \in D \cap J \backslash [v]_J$, there exists an $i \neq j$ such that $d \in A_i \subseteq D$. It follows that $\Big(\prod_{A_i \subseteq D} (\sigma_{A_i,a_i}) \Big) \circ \sigma_{D,v}(d) = va_i da_i v$. Hence $\Big(\prod_{A_i \subseteq D} (\hat{\sigma}_{A_i,a_i}) \Big) \hat{\sigma}_{D,v}$ fixes $\mathcal{B}_i$ for each $1 \leqslant i \leqslant m$.

Let $\hat{V}_{end}$ be the vector space spanned by $\hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_m - \hat{a}_1$. Let $\pi : \mathcal{M} \to \Gamma_{m-1}(2)$ be the projection onto $\hat{V}_{end}$. By the above argument, π is both well-defined and injective. Since $J_0 \subsetneq J$, we have a linear embedding

$$T : I_{J_0}(H_1(\hat{X}, \mathbb{Q})) \to I_J(H_1(\hat{X}, \mathbb{Q})), \quad T(1_{J_0}(\hat{v} - \hat{w})) = 1_J(\hat{v} - \hat{w}).$$

Fix the basis $T^{-1}(\{1_J(\hat{a}_i - \hat{a}_1) \mid 2 \leqslant i \leqslant m\})$ of $I_{J_0}(H_1(\hat{X}, \mathbb{Q}))$. Then for each $\sigma \in \rho_{\Gamma,J}^{-1}(\mathcal{M})$, we have that $\rho_{\Gamma,J_0}(\sigma) = \pi \circ \rho_{\Gamma,J}(\sigma)$.

Note that $\rho_{\Gamma,J_0}(\sigma_{A_i,a_i}) = \mathrm{Id}$. Thus for each generator $\Big(\prod_{A_i \subseteq D} (\hat{\sigma}_{A_i,a_i}) \Big) \hat{\sigma}_{D,v}$ of $\mathcal{M}$, we have that $\pi(\Big(\prod_{A_i \subseteq D} (\hat{\sigma}_{A_i,a_i}) \Big) \hat{\sigma}_{D,v}) = \rho_{\Gamma,J_0}(\hat{\sigma}_{D,v})$.

We now show that the condition $D \cap J \backslash [v]_J \subseteq A_j^{\complement}$ in the definition of $\mathcal{M}$ can be replaced by the condition $D \cap J_0 \backslash [v]_J \neq \varnothing$. From this it follows that $\mathcal{M} \cong \pi(\mathcal{M}) = \langle \rho_{\Gamma,J_0}(\mathcal{P}^{\Gamma,J_0}) \rangle = \mathrm{Im}(\rho_{\Gamma,J_0})$, which proves the lemma.

Let $\Big(\prod_{A_i \subseteq D} (\hat{\sigma}_{A_i,a_i}) \Big) \circ \hat{\sigma}_{D,v}$ be a generator of $\mathcal{M}$. If $v \in A_j$ and $D \cap J \backslash [v]_J \subseteq A_j^{\complement}$, then for some $i \neq j$ we must have that $D \cap A_i \neq \varnothing$. Since a_i is a special point and $v \notin A_i$, we have that $D \supseteq A_i$. In particular, $a_i \in D$. This shows that $D \cap J_0 \backslash [v]_J \neq \varnothing$.

Now assume that $D \cap J_0 \backslash [v]_J \neq \varnothing$. Then for some $i \neq j$, we have that $a_i \in D$. In particular, this shows that $D \cap A_i^{\complement} \neq \varnothing$. But since $v \in [a_j]_J$, the definition of a special point implies that $D \cap J \backslash [v]_J \subseteq A_j^{\complement}$. This completes the proof. $\square$

Just as $\mathcal{T}$ can be divided into distinct subgroups with similar properties to

$\mathcal{T}$, the same can be said of the group $\mathcal{L}$. It will be beneficial to us later to work with these smaller subgroups, so we prove their properties first. The same results about $\mathcal{L}$ will follow from them.

Lemma 3.19. *Let $J \subseteq V(\Gamma)$ be an arbitrary vertex set. Let $A \subsetneq J$ be a non-empty union of components of Γ_J containing a special point a. We define the following subgroup of Im$(\rho_{\Gamma,J})$.*

$$\mathcal{L}_1 := \langle \hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a} \mid v \in A \backslash [a]_J, D \text{ is a component of } \Gamma \backslash lk(v), D \cap J \subseteq A^{\complement} \rangle.$$

Then for the appropriate basis

$$\mathbb{Z}^r \cong \mathcal{L}_1 \leqslant \left\{ \begin{bmatrix} Id_{k_J-|S|} & M \\ 0 & Id_{|S|-1} \end{bmatrix} \,\middle|\, M \in M_{k_J-|S|,|S|-1}(2\mathbb{Z}) \right\}$$

where $r := |\{([v]_J, D) \mid v \in A \backslash [a]_J, D \text{ is a component of } \Gamma \backslash st(v), D \cap J \subseteq A^{\complement}\}|$. Furthermore $\mathcal{L}_1$ is normal in Im$(\rho_{\Gamma,J})$.

Proof. Let $\mathcal{B}_A$ be an ordered basis of $\langle \hat{v} - \hat{w} \mid v, w \in A_i \rangle \leqslant I_J(H_1(\hat{X}; \mathbb{Q}))$ consisting of vectors of the form $\hat{v} - \hat{w}$. Let S be an ordered list of representatives of the components of $\Gamma_{J \backslash A}$. We extend $\mathcal{B}_A$ to an ordered basis $\mathcal{B}$ of $I_J(H_1(\hat{X}; \mathbb{Q}))$ so that $\mathcal{B} = \mathcal{B}_A \cup \{\hat{s} - \hat{a} \mid s \in S\}$.

Fix a generator $\hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a}$ of $\mathcal{L}_1$. Let $d \in D$. First note that by Lemma 3.13 this generator is well-defined. This also implies that D is independent of the choice of representative of $[v]_J$. By direct computation, the only vectors that $\hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a}$ does not fix are vectors of the form $\hat{w} - \hat{a}$ where $w \in D$. It maps these vectors to $-2(\hat{v} - \hat{a}) + (\hat{w} - \hat{a})$. Therefore

$$\mathcal{L}_1 \leqslant \left\{ \begin{bmatrix} \mathrm{Id}_{k_J - |S|} & M \\ 0 & \mathrm{Id}_{|S|-1} \end{bmatrix} \;\Big|\; M \in M_{k_J - |S|, |S|-1}(2\mathbb{Z}) \right\}.$$

It follows that each generator $\hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a}$ of $\mathcal{L}_1$ is of infinite order and commutes with every other generator. Furthermore, since the entries where distinct generators differ from identity are distinct to those generators, $\mathcal{L}_1 \cong \mathbb{Z}^r$.

We now show that $\mathcal{L}_1$ is normal in $\mathrm{Im}(\rho_{\Gamma,J})$. Let $L := \hat{\sigma}_{C,v} \cdot \hat{\sigma}_{C,a} \in \mathcal{L}_1$. Let $M := \hat{\sigma}_{D,w} \in \rho_{\Gamma,J}(\mathcal{P}^{\Gamma,J})$. We break into a number of cases:

- $(w \in A, D \cap J \subseteq A)$: In this case, $D \cap C \cap J = \varnothing$. In particular, $D \neq C$. Furthermore, $w \notin C$. By Proposition 3.11, M and L commute unless $(v, w \mid C)$ is a SIL relative to J (this is equivalent to $(a, w \mid C)$ being a SIL relative to J) and either $v \in D$ or $a \in D$. By Lemma 3.4, this implies that C is a component of $\Gamma \backslash \mathrm{st}(w)$, so $\sigma_{C,w}$ is well-defined.

  - If both $v, a \in D$, then

$$\sigma_{D,w} \circ \sigma_{C,v} \circ \sigma_{C,a} \circ \sigma_{D,w}(u) = \begin{cases} wavwuwvaw & \text{if } u \in C \\ u \quad \text{else} \end{cases}$$

    so that

$$MLM^{-1} = L^{-1} \in \mathcal{L}_1$$

  - If $v \in D$, $a \notin D$, then

$$\sigma_{D,w} \circ \sigma_{C,v} \circ \sigma_{C,a} \circ \sigma_{D,w}(u) = \begin{cases} awvwuwvwa & \text{if } u \in C \\ u \quad \text{else} \end{cases}$$

so that

$$MLM^{-1} = (\hat{\sigma}_{C,w} \cdot \hat{\sigma}_{C,a}) \cdot (\hat{\sigma}_{C,v} \cdot \hat{\sigma}_{C,a})^{-1} \cdot (\hat{\sigma}_{C,w} \cdot \hat{\sigma}_{C,a}) \in \mathcal{L}_1$$

– If $v \notin D$, $a \in D$, then

$$\sigma_{D,w} \circ \sigma_{C,v} \circ \sigma_{C,a} \circ \sigma_{D,w}(u) = \begin{cases} wawvuvwaw & \text{if } u \in C \\ u & \text{else} \end{cases}$$

so that

$$MLM^{-1} = (\hat{\sigma}_{C,v} \cdot \hat{\sigma}_{C,a}) \cdot (\hat{\sigma}_{C,w} \cdot \hat{\sigma}_{C,a})^{-2} \in \mathcal{L}_1$$

- $(w \in A, D \cap J \not\subseteq A, a \notin D)$: If $D = C$, then by direct computation $MLM^{-1} = L^{-1} \in \mathcal{L}_1$. If not, let $d \in D \cap J \backslash A$. Then Corollary 3.3 implies that $D = D(w,d) \subseteq D(a,d)$. The definition of special point implies that $D(a,d) \backslash [a]_J \subseteq A^{\complement}$. This shows that $v \notin D$. Since $w \notin C$, Proposition 3.11 implies that M and L commute.

- $(w \in A, D \cap J \not\subseteq A, a \in D)$: By Lemma 2.8, every inner automorphism relative to J maps to $-\mathrm{Id}$. Since the image of every partial conjugation is of order 2 (Lemma 2.8) and every pair of partial conjugations by v commute, $M = \hat{\sigma}_{J,w} \cdot \prod_{D' \neq D} \hat{\sigma}_{D',w} = -\prod_{D' \neq D} \hat{\sigma}_{D',w}$. This reduces this case to the previous two cases.

- $(w \notin A, D \cap A = \varnothing)$: If $D = C$, then by direct calculation $MLM^{-1} = L^{-1} \in \mathcal{L}_1$. If not, then since $v, a \notin D$, Proposition 3.11 implies that M and L commute unless $w \in C$. In addition, we must have that $(v, w \mid D)$ is a SIL relative to J, or $(a, w \mid D)$ is a SIL relative to J. We show that the existence of one of these SILs implies the existence of the other.

Assume that $(v, w \mid D)$ is a SIL relative to J. Let $d \in D \cap J$. By Lemma 3.13, we have that $D(a, d) = D(v, d) = D$. Then Lemma 3.4 implies that $(a, w \mid D)$ is a SIL relative to J.

Now assume that $(a, w \mid D)$ is a SIL relative to J. Let $d \in D \cap J$. Since $D \cap A = \varnothing$, we have that $v \notin D = D(a, d)$. Therefore by Corollary 3.3, we have that $D(a, d) \subseteq D(v, d)$. Since $w \in C$, we have that $C = D(v, w)$. Since $a \notin C$, it follows that $w \notin D(v, a)$. Applying Corollary 3.3, we see that $D(v, a) \subseteq D(w, a)$. Then since $a \notin D = D(w, d)$, we have that $d \notin D(w, a)$. It follows that $d \notin D(v, a)$, which implies that $a \notin D(v, d)$. Applying Corollary 3.3, we get that $D(v, d) \subseteq D(a, d)$. Thus $D = D(a, d) = D(v, d)$. Finally, Lemma 3.4 implies that $(v, w \mid D)$ is a SIL relative to J.

We can now assume that $w \in C$ and that both $(v, w \mid D)$ and $(a, w \mid D)$ are SILs relative to J. Since $D(v, d) = D \neq C = D(v, w)$, we have that $D \cap C = \varnothing$. By direct computation,

$$\sigma_{D,w} \circ \sigma_{C,v} \circ \sigma_{C,a} \circ \sigma_{D,w}(u) = \begin{cases} avuva & \text{if } u \in C \\ vawavwuwvawaw & \text{if } u \in D \\ u & \text{else} \end{cases}$$

so that

$$MLM^{-1} = (\hat{\sigma}_{C,v} \cdot \hat{\sigma}_{C,a}) \cdot (\hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a})^{-2} \in \mathcal{L}_1$$

- $(w \notin A, D \cap A \neq \varnothing)$: Let $d \in D \cap A$. By definition of a special point, we have that $A \subseteq D(w, a) = D(w, d) = D$. Then $M = -\prod_{D' \neq D} \hat{\sigma}_{D',w}$ and this reduces to the previous case.

This shows that $\mathcal{L}_1$ is normal in $\mathrm{Im}(\rho_{\Gamma,J})$. $\square$

Corollary 3.20. *Let* $J = \bigsqcup_{i=1}^{m} A_i$ *be a division of* J *with special points* $a_i \in A_I$. *Let*

$$\mathcal{L} := \langle \hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a_i} \mid 1 \leqslant i \leqslant m, v \in A_i \backslash [a_i]_J, D \textit{ is a component of } \Gamma \backslash st(v), D \cap J \subseteq A_i^{\complement} \rangle.$$

Then for the appropriate basis,

$$\mathbb{Z}^r \cong \mathcal{L} \leqslant \left\{ \begin{bmatrix} Id_{|J|-m} & M \\ 0 & Id_{m-1} \end{bmatrix} \;\middle|\; M \in M_{|J|-m,m-1}(2\mathbb{Z}) \right\}$$

where $r := |\{([v]_J, D) \mid v \in A_i \backslash [a_i]_J$ *for some* $1 \leqslant i \leqslant m, D$ *is a component of* $\Gamma \backslash st(v)$, $D \cap J \subseteq A_i^{\complement}\}|$. *Furthermore* $\mathcal{L}$ *is normal in* $Im(\rho_{\Gamma,J})$.

Proof. For each $1 \leqslant i \leqslant m$, let $\mathcal{B}_i$ be an ordered basis of $\langle \hat{v} - \hat{w} \mid v, w \in A_i \rangle \leqslant I_J(H_1(\hat{X}; \mathbb{Q}))$ consisting of vectors of the form $\hat{v} - \hat{w}$. Fix the following ordered basis of $I_J(H_1(\hat{X}; \mathbb{Q}))$.

$$\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m, \hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_m - \hat{a}_1.$$

By direct computation,

$$\mathcal{L} \leqslant \left\{ \begin{bmatrix} \mathrm{Id}_{|J|-m} & M \\ 0 & \mathrm{Id}_{m-1} \end{bmatrix} \;\middle|\; M \in M_{|J|-m,m-1}(2\mathbb{Z}) \right\}.$$

Let

$$\mathcal{L}_i := \langle \hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a_i} \mid v \in A_i \backslash [a_i]_J, D \cap A_i \backslash [v]_J = \varnothing, a_i \notin D \rangle.$$

Note that our choice of basis corresponds to the choice of basis made in Lemma

3.19. Then by Lemma 3.19, for each $1 \leqslant i \leqslant m$ we have that $\mathcal{L}_i \cong \mathbb{Z}^{r_i}$ where $r_i := |\{([v]_J, D) \mid v \in A_i \backslash [a_i]_J, D \text{ is a component of } \Gamma \backslash \mathrm{st}(v), D \cap J \subseteq A_i^{\complement}\}|$. Since each $\mathcal{L}_i$ only differs from identity on rows corresponding to vectors in $\mathcal{B}_i$, it follows that $\mathcal{L} = \prod_{i=1}^{m} \mathcal{L}_i \cong \mathbb{Z}^r$. Furthermore, since each $\mathcal{L}_i$ is normal in $\mathrm{Im}(\rho_{\Gamma,J})$, we have that $\mathcal{L}$ is normal in $\mathrm{Im}(\rho_{\Gamma,J})$. $\square$

We are now ready to prove the TML decomposition (Theorem 1.2).

Theorem 1.2 (TML Decomposition)**.** Let $J = \bigsqcup_{i=1}^{m} A_i$ be a division of J with special points $a_i \in A_I$. Let $J_0 := \{v \mid v \in [a_i]_J \text{ for some } 1 \leqslant i \leqslant m\}$. Then $\mathrm{Im}(\rho_{\Gamma,J}) \cong \Big(\Big(\prod_{i=1}^{m} \mathrm{Im}(\rho_{\Gamma,A_i})\Big) \times \mathrm{Im}(\rho_{\Gamma,J_0})\Big) \ltimes \mathbb{Z}^r$ where $r := |\{([v]_J, D) \mid v \in A_i \backslash [a_i]_J \text{ for some } 1 \leqslant i \leqslant m, D \text{ is a component of } \Gamma \backslash \mathrm{st}(v), D \cap J \subseteq A_i^{\complement}\}|$.

Proof. For each $1 \leqslant i \leqslant m$, let $\mathcal{B}_i$ be an ordered basis of $\langle \hat{v} - \hat{w} \mid v, w \in A_i \rangle \leqslant I_J(H_1(\hat{X}; \mathbb{Q}))$ consisting of vectors of the form $\hat{v} - \hat{w}$. Fix the following ordered basis of $I_J(H_1(\hat{X}; \mathbb{Q}))$.

$$\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m, \hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_m - \hat{a}_1.$$

Let $\hat{V}_{end}$ be the vector space spanned by $\hat{a}_2 - \hat{a}_1, \hat{a}_3 - \hat{a}_1, \ldots, \hat{a}_m - \hat{a}_1$. Consider the following subgroups of $\mathrm{Im}(\rho_\delta)$

- $\mathcal{T} := \langle \hat{\sigma}_{D,v} \mid v \in A_i, 1 \leqslant i \leqslant m, D \cap A_i \backslash [a_i]_J \neq \varnothing \rangle$.
- $\mathcal{M} := \Big\langle \Big(\prod_{A_i \subseteq D} (\hat{\sigma}_{A_i,a_i}) \Big) \hat{\sigma}_{D,v} \mid v \in [a_j]_J \text{ for some } 1 \leqslant j \leqslant m, D \cap J \backslash [v]_J \subseteq A_j^{\complement} \Big\rangle$.
- $\mathcal{L} := \langle \hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a_i} \mid 1 \leqslant i \leqslant m, v \in A_i \backslash [a_i]_J, D \text{ is a component of } \Gamma \backslash \mathrm{lk}(v), D \cap J \subseteq A_i^{\complement} \rangle$.

Our choice of basis corresponds to the choices of bases made in Corollary 3.17, Lemma 3.18 and Corollary 3.20. Thus by Corollary 3.17, Lemma 3.18, and Corollary 3.20, we have that

$$\prod_{i=1}^{m} \operatorname{Im}(\rho_{\Gamma,A_i}) \cong \mathcal{T} \leqslant \left\{ \begin{bmatrix} M & 0 \\ 0 & \mathrm{Id}_{m-1} \end{bmatrix} \right\}.$$

$$\mathcal{M} = \left\{ \begin{bmatrix} \mathrm{Id}_{|J|-m} & 0 \\ 0 & M \end{bmatrix} \;\middle|\; M \in \operatorname{Im}(\rho_{\Gamma,J_0}) \right\} \cong \operatorname{Im}(\rho_{\Gamma,J_0}).$$

$$\mathbb{Z}^r \cong \mathcal{L} \leqslant \left\{ \begin{bmatrix} \mathrm{Id}_{|J|-m} & M \\ 0 & \mathrm{Id}_{m-1} \end{bmatrix} \;\middle|\; M \in M_{|J|-m,m-1}(2\mathbb{Z}) \right\}.$$

We also know from Corollary 3.20 that $\mathcal{L}$ is normal in $\operatorname{Im}(\rho_{\Gamma,J})$. From this and the above block matrix descriptions of $\mathcal{T}, \mathcal{M}$, and $\mathcal{L}$, we see that $\mathcal{T}\mathcal{M}\mathcal{L} = (\mathcal{T} \times \mathcal{M}) \ltimes \mathcal{L}$. It remains to show that $\mathcal{T}\mathcal{M}\mathcal{L} = \operatorname{Im}(\rho_{\Gamma,J})$. It suffices to show that $\rho_{\Gamma,J}(\mathcal{P}^{\Gamma,J}) \in \mathcal{T}\mathcal{M}\mathcal{L}$.

First, consider $\sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$ where $v \in [a_i]_J$ for some $1 \leqslant i \leqslant m$. By the definition of divisibility, either $D \cap J \backslash [v]_J \subseteq A_i$ or $D \cap J \backslash [v]_J \subseteq A_i^{\complement}$. In the first case, $\hat{\sigma}_{D,v} \in \mathcal{T}$. In the second case, since $\hat{\sigma}_{A_k,a_k} \in \mathcal{T}$, we see that

$$\hat{\sigma}_{D,v} = \Big(\prod_{A_k \subseteq D} \hat{\sigma}_{A_k,a_k} \Big) \cdot \Big(\Big(\prod_{A_k \subseteq D} (\hat{\sigma}_{A_k,a_k}) \Big) \hat{\sigma}_{D,v} \Big) \in \mathcal{T}\mathcal{M}.$$

In particular, we see that $\hat{\sigma}_{J,v} = -\mathrm{Id} \in \mathcal{T}\mathcal{M}\mathcal{L}$.

Next, consider $\sigma_{D,v} \in \mathcal{P}^{\Gamma,J}$ for $v \notin J_0$. Let $v \in A_i$. If $D \cap J \backslash [v]_J \subseteq A_i^{\complement}$, then $\hat{\sigma}_{D,v} = (\hat{\sigma}_{D,v} \cdot \hat{\sigma}_{D,a_i}) \cdot \hat{\sigma}_{D,a_i} \in \mathcal{T}\mathcal{M}\mathcal{L}$. If $D \cap A_i \backslash [a_i]_J \neq \varnothing$, then $\hat{\sigma}_{D,v} \in \mathcal{T}$. Finally, if $a_i \in D$, then $\hat{\sigma}_{D,v} = -\prod_{C \neq D} \hat{\sigma}_{D,v} \in \mathcal{T}\mathcal{M}\mathcal{L}$ by the previous arguments. This completes the proof. $\square$

3.3 Splitting Points

In this section we give a condition that will allow us to decompose $\mathrm{Im}(\rho_{\Gamma,J})$ as a direct sum of the images of representations with restricted domains.

Definition 3.21. We say that $v \in J$ is a splitting point if there exist non-empty disjoint subsets $A, B \subseteq J$, such that $J = A \sqcup B \sqcup [v]_J$ and v is a special point of both $A \sqcup [v]_J$ and $B \sqcup [v]_J$.

Proposition 3.22. *Let Γ be a graph, J a set of pairwise non-adjacent vertices, and $v \in J$ a splitting point. Let A, B be as in the definition of splitting set. Then we have*

$$Im(\rho_{\Gamma,J}) = Im(\rho_{\Gamma,A\cup[v]_J}) \times Im(\rho_{\Gamma,B\cup[v]_J}).$$

Proof. Let $\mathcal{B}_A$ be a basis of $\langle \hat{v} - \hat{w} \mid v, w \in A\rangle \leqslant I_J(H_1(\hat{X};\mathbb{Q}))$, and let $\mathcal{B}_B$ be a basis of $\langle \hat{v} - \hat{w} \mid v, w \in B\rangle \leqslant I_J(H_1(\hat{X};\mathbb{Q}))$. Fix some $a \in A, b \in B$. Then we use the ordered basis

$\mathcal{B} = \mathcal{B}_A, \hat{v} - \hat{a}, \hat{v} - \hat{b}, \mathcal{B}_B.$

We define the following subgroups of $\mathrm{Im}(\rho_{\Gamma,J})$:

- $\mathcal{A} := \langle \hat{\sigma}_{D,w} \in \mathcal{P}^{\Gamma,A\sqcup[v]_J} \mid v \notin D\backslash[w]_J\rangle$
- $\mathcal{B} := \langle \hat{\sigma}_{D,w} \in \mathcal{P}^{\Gamma,B\sqcup[v]_J} \mid v \notin D\backslash[w]_J\rangle$

Then by the definition of a splitting point, for any generator $\hat{\sigma}_{D,w}$ of $\mathcal{A}$, we must have that $D \cap J \subseteq A$. Similarly, for any generator $\hat{\sigma}_{D,w}$ of $\mathcal{B}$, we must have that $D \cap J \subseteq B$. By the definitions of splitting point, v is a special point with respect to the subsets $A \sqcup [v]_J$ and $B \sqcup [v]_J$. Note that our choice of basis corresponds to the choice of basis in Lemma 3.15. Hence by Lemma 3.15 we have

$$\mathcal{A} = \left\{ \begin{bmatrix} M & 0 \\ 0 & \mathrm{Id}_{|\mathcal{B}_B|+1} \end{bmatrix} \;\middle|\; M \in \mathrm{Im}(\rho_{\Gamma, A \sqcup [v]_J}) \right\} \cong \mathrm{Im}(\rho_{\Gamma, A \sqcup [v]_J})$$

$$\mathcal{B} = \left\{ \begin{bmatrix} \mathrm{Id}_{|\mathcal{B}_A|+1} & 0 \\ 0 & M \end{bmatrix} \;\middle|\; M \in \mathrm{Im}(\rho_{\Gamma, B \sqcup [v]_J}) \right\} \cong \mathrm{Im}(\rho_{\Gamma, B \sqcup [v]_J})$$

By the block diagonal descriptions above, it follows that $\mathcal{AB} = \mathcal{A} \times \mathcal{B} \cong \mathrm{Im}(\rho_{\Gamma, A \sqcup [v]_J}) \times \mathrm{Im}(\rho_{\Gamma, B \sqcup [v]_J})$. Thus it remains to show that $\mathrm{Im}(\rho_{\Gamma, J}) = \mathcal{AB}$. By Corollary 3.16, it suffices to show that $\hat{\sigma}_{D,w} \in \mathcal{AB}$ for all $\hat{\sigma}_{D,w}$ such that $v \notin D \backslash [w]_J$. But this is immediate from our definitions of $\mathcal{A}$ and $\mathcal{B}$. $\square$

3.4 Separating Sets

Consider a vector of the form $\hat{v} - \hat{w}$ in $I_J(H_1(\hat{X}; \mathbb{Q}))$ where $[v]_J \neq [w]_J$. For any partial conjugation $\sigma_{D,u}$, the action of $\hat{\sigma}_{D,u}$ on $\hat{v} - \hat{w}$ depends upon whether or not v or w are in D. If neither v nor w are in D, then $\hat{\sigma}_{D,u}$ acts as identity on $\hat{v} - \hat{w}$. If both v and w are in D, then $\hat{\sigma}_{D,u}$ acts as $-\mathrm{Id}$ on $\hat{v} - \hat{w}$. However, if D separates v and w, i.e. exactly one of these two vertices is in D, then $\hat{v} - \hat{w}$ is not an eigenvector (unless $u \in [v]_J \cup [w]_J$). This motivates us to make the following definition.

Definition 3.23. Let $v, w \in J$ be such that $[v]_J \neq [w]_J$. Then

$$\mathrm{sep}(v, w) := \{x \in J \mid \exists x^* \in [x]_J \text{ such that } D(x^*, v) \neq D(x^*, w)\}.$$

Right away from the definition, we see that $[v]_J \cup [w]_J \subseteq \text{sep}(v, w)$. Besides determining which partial conjugations induce homomorphisms for which $\hat{v} - \hat{w}$ is not an eigenvector, $\text{sep}(v, w)$ has a couple of nice properties that we outline below.

Proposition 3.24. *If $c \notin sep(v, w)$ then $D(c, v) \supseteq sep(v, w)$.*

Proof. Let $x \in \text{sep}(v, w)$. Then there exists a $x^* \in [x]_J$ such that $D(x^*, v) \neq D(x^*, w)$. If $x^* \notin D(c, v)$, then by Corollary 3.3, we have $D(c, v) \subseteq D(x^*, v)$. But $w \in D(c, v)$ and $w \notin D(x^*, v)$. This is a contradiction, therefore $x^* \in D(c, v)$. It follows that $x \in D(c, v)$. This proves the proposition. $\square$

Proposition 3.25. *If $a, b \in sep(v, w)$ then $sep(a, b) \subseteq sep(v, w)$*

Proof. Let $a, b \in \text{sep}(v, w)$. Then by definition of $\text{sep}(v, w)$, there exist $a^* \in [a]_J$ and $b^* \in [b]_J$ such that $D(a^*, v) \neq D(a^*, w)$ and $D(b^*, v) \neq D(b^*, w)$. Next, let $c \in \text{sep}(a, b)$. By definition, of $\text{sep}(a, b)$, there exists $c^* \in [c]_J$ such that $D(c^*, a^*) = D(c^*, a) \neq D(c^*, b) = D(c^*, b^*)$. Corollary 3.3 implies that $D(c^*, a^*) \subseteq D(b^*, a^*)$. Since $D(b^*, v) \neq D(b^*, w)$, it follows that either $D(b^*, a^*) \neq D(b^*, v)$ or $D(b^*, a^*) \neq D(b^*, w)$. Assume without loss of generality that $D(b^*, a^*) \neq D(b^*, v)$. Then since $D(c^*, a^*) \subseteq D(b^*, a^*)$, we have that $D(c^*, a^*) \neq D(c^*, v)$. It follows from Corollary 3.3 that $D(c^*, v) \subseteq D(a^*, v)$. But $w \notin D(a^*, v)$. Therefore $w \notin D(c^*, v)$, so $D(c^*, v) \neq D(c^*, w)$. This shows that $c^* \in \text{sep}(v, w)$, hence $c \in \text{sep}(v, w)$. The statement follows. $\square$

Earlier we observed that $[v]_J \cup [w]_J \subseteq \text{sep}(v, w)$. If we have that $\text{sep}(v, w) = [v]_J \cup [w]_J$, then $\hat{v} - \hat{w}$ is always an eigenvector. This motivates us to explore what happens in this case. We prove the following decomposition result.

Lemma 3.26. *If $sep(v,w) = \{v\} \cup [w]_J$, then $Im(\rho_{\Gamma,J}) \cong (Im(\rho_{\Gamma,A\cup[v]_J}) \times Im(\rho_{\Gamma,A^{\complement}})) \ltimes \mathbb{Z}^r$ where $A := D(v,w)$ and $r = |\{([a]_J, D) \mid a \in A, D$ is a component of $\Gamma \backslash st(a), D \cap J \subseteq (A \sqcup [v]_J)^{\complement}\}|$.*

Proof. We define the following subgroups:

- $\mathcal{A} := \langle \hat{\sigma}_{D,a} \in \mathcal{P}^{\Gamma, A\cup[v]_J} \mid v \notin D \rangle$
- $\mathcal{B} := \langle \hat{\sigma}_{D,b} \in \mathcal{P}^{\Gamma, A^{\complement}} \mid v \notin D \rangle$.
- $\mathcal{L} := \langle \hat{\sigma}_{D,a} \cdot \hat{\sigma}_{D,v} \mid a \in A, D$ is a component of $\Gamma\backslash\mathrm{st}(v), D \cap J \subseteq (A \sqcup [v]_J)^{\complement} \rangle$.

Let $\{a_1, \ldots, a_s\}$ be a set of representatives of the components of Γ_A and let $\{b_1, \ldots, b_t\}$ be a set of representatives of the components of $\Gamma_{A^{\complement}\backslash[v]_J}$. Consider the basis

$$\hat{v} - \hat{a}_1, \hat{v} - \hat{a}_2, \ldots, \hat{v} - \hat{a}_s, \hat{v} - \hat{b}_1, \hat{v} - \hat{b}_2, \ldots, \hat{v} - \hat{b}_t.$$

We show that v is a special point with respect to $A \sqcup [v]_J$. First note that by hypothesis, $[v]_J = \{v\}$. Thus if $a \in A = D(v,w)$, then $D(v,a) \cap J\backslash[v]_J = D(v,w) \cap J\backslash[v]_J = A \subseteq A \sqcup [v]_J$, and if $b \in A^{\complement}$, then $D(v,b) \cap D(v,w) = \varnothing$, so $D(v,b) \cap J\backslash[v]_J \subseteq (A \cup [v]_J)^{\complement}$. Furthermore, if $b \in (A \cup [v]_J)^{\complement}$, then $b \notin D(v,w) \cap J = A$. By Corollary 3.3, this implies that $D(v,w) \subseteq D(b,w)$. But since $b \notin \mathrm{sep}(v,w)$, we have that $D(b,w) = D(b,v)$. This shows that v is a special point with respect to $A \sqcup [v]_J$.

The above argument also shows that for all $j \in J\backslash[v]_J$ we have that either $D(v,j) \cap J\backslash[v]_J \subseteq A^{\complement}$ or $D(v,j) \cap J\backslash[v]_J \subseteq A$. Note that our choice of basis corresponds to the choice of basis in Lemma 3.15. Applying Lemma 3.15, we have that

$$\mathcal{A} = \left\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; A \in \mathrm{Im}(\rho_{\Gamma,A\cup[v]_J}) \right\} \cong \mathrm{Im}(\rho_{\Gamma,A\cup[v]_J})$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \;\middle|\; B \in \mathrm{Im}(\rho_{\Gamma, A^{\complement}}) \right\} \cong \mathrm{Im}(\rho_{\Gamma, A^{\complement}})$$

We now turn to the subgroup $\mathcal{L}$. Note that our choice of basis corresponds to the choice of basis in Lemma 3.19. Since v is a special point with respect to $A \sqcup [v]_J$, Lemma 3.19 implies that

$$\mathcal{L} \cong \mathbb{Z}^r \leqslant \left\{ \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \mid L \in M_{s,t}(2\mathbb{Z}) \right\}$$

and $\mathcal{L}$ is normal in $\mathrm{Im}(\rho_{\Gamma,J})$. Therefore $\mathcal{ABL} = (\mathcal{A} \times \mathcal{B}) \ltimes \mathcal{L}$. By Corollary 3.16, it suffices to show that $\mathcal{ABL}$ contains $\{\hat{\sigma}_{D,u} \mid v \notin D\}$.

First note that $\hat{\sigma}_{D,v} \in \mathcal{AB}$ for any choice of D. In particular, this shows that $-\mathrm{Id} \in \mathcal{ABL}$.

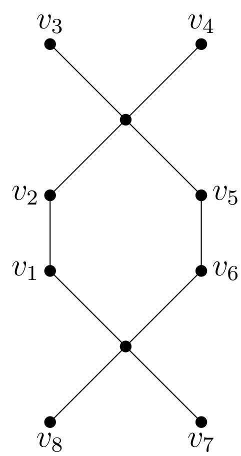
Let $a \in A$ and let $x \in J$ be such that $x \notin [a]_J$ and $v \notin D(a,x)$. By Corollary 3.3, we have that $D(a,x) \subseteq D(v,x)$. If $x \in A$, this implies that $D(a,x) \subseteq A$, so that $\hat{\sigma}_{D(a,x),a} \in \mathcal{A}$. If $x \in A^{\complement}$, then $D(a,x) \subseteq A^{\complement}$. Thus $\hat{\sigma}_{D(a,x),v} \cdot \hat{\sigma}_{D(a,x),a} \in \mathcal{L}$. Since $\hat{\sigma}_{D(a,x),v} \in \mathcal{B}$, we have that $\hat{\sigma}_{D(a,x),a} \in \mathcal{BL}$. Thus $\mathcal{ABL}$ includes $\hat{\sigma}_{D,a}$ for all $a \in A$ and all D such that $v \notin D$.

Finally, let $b \in A^{\complement} \backslash [v]_J$. In particular, we have that $b \notin [v]_J \cup [w]_J = \mathrm{sep}(v,w)$. Since $b \notin A = D(v,w)$, Corollary 3.3 implies that $A = D(v,w) \subseteq D(b,w)$. But $D(b,w) = D(b,v)$. Therefore $D(b,v) \supseteq A \sqcup [v]_J$. Now let $x \in J$ be such that $x \notin [b]_J$ and $v \notin D(b,x)$. By the above statement, it follows that $D(b,x) \subseteq A^{\complement} \backslash [v]_J$. Therefore $\hat{\sigma}_{D(b,x),b} \in \mathcal{B}$. Thus $\mathcal{ABL}$ includes $\hat{\sigma}_{D,b}$ for all D such that $v \notin D$. This shows that $\mathcal{ABL}$ contains a complete set of generators of $\mathrm{Im}(\rho_{\Gamma,J})$. □

3.5 Compressible Components

We now take a slight detour to deal with the hypotheses of Lemma 3.26. Ideally, we would like a similar statement to Lemma 3.26 but without requiring that $[v]_J = \{v\}$. To see that the statement of Lemma 3.26 does not hold in this case, we provide a counter-example.

Example 7. Let Γ be the graph

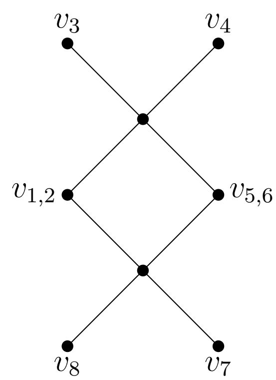


and let J be the vertex set $\{v_1, v_2, \ldots, v_8\}$. Then Γ has a unique minimal separating set, namely $\mathrm{sep}(v_1, v_5) = \{v_1, v_2, v_5, v_6\} = [v_1]_J \sqcup [v_5]_J$. However $D(v_2, v_5) \cap J \backslash [v_2]_J \nsubseteq D(v_1, v_5) \cap J \backslash [v_1]_J$ and $D(v_6, v_1) \cap J \backslash [v_6]_J \nsubseteq D(v_5, v_1) \cap J \backslash [v_5]_J$. Thus if $A := D(v_1, v_5) \cap J$, then $A \sqcup [v_1]_J$ does not have a special point. Similarly, if $A := (D(v_5, v_1) \cap J) \sqcup [v_5]_J$, then A does not have a special point. This was a crucial point in the proof of Lemma 3.26 because it allowed us to apply Lemma 3.19.

It is possible that the conclusion of Lemma 3.26 could still hold. If this were the case, letting $A = D(v_1, v_5) \cap J$, then applying Theorem 2.7, we would see that

$$\operatorname{Im}(\rho_{\Gamma,J}) \cong (\operatorname{Im}(\rho_{\Gamma,\{v_1,v_2,v_3,v_4,v_5,v_6\}}) \times \operatorname{Im}(\rho_{\Gamma,\{v_7,v_8\}})) \ltimes \mathbb{Z}^2 \cong (\Gamma_3(2) \times \mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}^2.$$

There is, however, an alternate way to solve this problem. Consider the slightly modified graph Γ'



with vertex set $J' := \{v_{1,2}, v_3, v_4, v_{5,6}, v_7, v_8\}$ obtained by compressing the edges (v_1, v_2) and (v_5, v_6) to a point. This graph is useful because $\operatorname{Im}(\rho_{\Gamma,J}) \cong \operatorname{Im}(\rho_{\Gamma',J'})$. We give a sketch of why this is true. We will give a more formal argument later.

Any possible issue lies with partial conjugations by $v_{1,2}$ (or symmetrically $v_{5,6}$). The partial conjugations $\sigma_{\{v_3\},v_{1,2}}, \sigma_{\{v_4\},v_{1,2}}, \sigma_{\{v_7\},v_{1,2}}$, and $\sigma_{\{v_8\},v_{1,2}}$ have clear analogues in $\mathcal{P}^{\Gamma,J}$. However, the partial conjugation $\sigma_{\{v_{5,6}\},v_{1,2}}$ does not. By considering instead the product of partial conjugations $\sigma := \sigma_{\{v_3,v_4,v_5,v_6\},v_1} \circ \sigma_{\{v_3\},v_2} \circ \sigma_{\{v_4\},v_2}$, we are able to remedy this problem. The corresponding matrix $\hat{\sigma}$ acts on the the symmetric difference of the sets being conjugated $\{v_3, v_4, v_5, v_6\} \triangle \{v_3\} \triangle \{v_4\} = \{v_5, v_6\}$ by the 1-chain $\hat{v}_1 = \hat{v}_2$.

Now $\operatorname{sep}(v_{1,2}, v_{5,6}) = \{v_{1,2}, v_{5,6}\}$ is a minimal separating set. Applying Lemma 3.26, we get that

$$\operatorname{Im}(\rho_{\Gamma,J}) \cong (\operatorname{Im}(\rho_{\Gamma,\{v_{1,2},v_{5,6}\}}) \times \operatorname{Im}(\rho_{\Gamma,J\setminus\{v_{1,2}\}})) \ltimes \mathbb{Z}^4 \cong (\mathbb{Z}/2\mathbb{Z} \times \operatorname{Im}(\rho_{\Gamma,J\setminus\{v_{1,2}\}})) \ltimes \mathbb{Z}^4.$$

We see that $v_{1,2}$ is a splitting point in $\Gamma\backslash\{v_{5,6}\}$, Proposition 3.22 and Theorem 2.7 imply that

$$\mathrm{Im}(\rho_{\Gamma,J}) \cong (\mathbb{Z}/2\mathbb{Z}\times\mathrm{Im}(\rho_{\Gamma,\{v_{1,2},v_3,v_4\}})\times\mathrm{Im}(\rho_{\Gamma,\{v_{1,2},v_7,v_8\}}))\ltimes\mathbb{Z}^4 \cong (\mathbb{Z}/2\mathbb{Z}\times\Gamma_2(2)^2)\ltimes\mathbb{Z}^4.$$

As $(\mathbb{Z}/2\mathbb{Z} \times \Gamma_2(2)^2) \ltimes \mathbb{Z}^4 \ncong (\Gamma_3(2) \times \mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}^2$, we see that the conclusion of Lemma 3.26 does not hold for this example.

We now formalize the above method of compressing components of Γ_J.

Definition 3.27. Let Γ be a finite graph and let $J \subseteq V(\Gamma)$ be a vertex set. Given $v, x \in J$ such that $[v]_J \neq [x]_J$, we say that x is a <u>*v-expander*</u> if the set

$$\mathrm{Int}(v,x) := \bigcap_{w\in[v]_J} D(w,x) \cap J$$

cannot be written as a symmetric difference of elements of

$$\mathfrak{D}(v) := \{D(w,y) \cap J\backslash[v]_J \mid w \in [v]_J, y \in J\backslash[v]_J\}.$$

We say $[v]_J$ is a <u>*compressible component*</u> relative to J if no $x \in J\backslash[v]_J$ is a v-expander. A component that is not compressible is <u>*incompressible*</u>.

Proposition 3.28. *If J contains a compressible component $[v]_J$, then there exists a graph Γ' with vertex set $J' = (J\backslash[v]_J \sqcup \{v'\})$ such that $Im(\rho_{\Gamma,J}) \cong Im(\rho_{\Gamma',J'})$.*

Proof. Let Γ' be the graph where the subgraph $\Gamma_{[v]_J}$ induced by $[v]_J$ is replaced by a single point v'. Then the edges incident to v' in Γ' are precisely the edges incident to only one vertex in $[v]_J$ in Γ. Fix a basis $\mathcal{B}$ of $I_J(H_1(\hat{X};\mathbb{Q}))$. Let $\mathcal{B}'$ be the basis of $I_J(H_1(\hat{X};\mathbb{Q}))$ where for every $w \in [v]_J$, every occurrence of $\hat{w}$ is replaced by $\hat{v}'$.

Consider the map

$$f : J \to J', \quad f(j) = \begin{cases} v' & \text{if } j \in [v]_J \\ j & \text{else} \end{cases}.$$

Note that f induces a bijection between the components of Γ_J and the components of $\Gamma'_{J'}$. We show that for each $a, b \in J$ such that $[a]_J \neq [b]_J$ we have that $D(f(a), f(b)) \cap J' \subseteq f(D(a, b) \cap J)$, with equality when $a \notin [v]_J$.

If $a \in [v]_J$, then $f(a) = v'$. But $\Gamma' \backslash \mathrm{st}(v')$ is a subgraph of $\Gamma \backslash \mathrm{st}(a)$. Since $f|_{J \backslash [v]_J} = \mathrm{Id}$, the result follows. If $a \notin [v]_J$, then $f(a) = a$. Let $x \in D(f(a), f(b)) \cap J'$ and let α_x be a path in $\Gamma' \backslash \mathrm{st}(f(b))$ from $f(b)$ to x. If α_x does not pass through v', then α_x only passes through points in $J \backslash [v]_J$. It follows that α_x induces a path β_x in $\Gamma \backslash [v]_J$ from b to x such that $f(\beta_x) = \alpha_x$. The result follows. Assume that α_x passes through v'. Without loss of generality, we may assume that α_x does not contain a loop. Then we may write $\alpha_x = (\alpha_1, v', \alpha_2)$ for some (possibly empty) paths α_1, α_2 that do not pass through v'. Then α_1, α_2 induce paths β_1, β_2 in $\Gamma \backslash [v]_J$ such that $f(\beta_1) = \alpha_1$ and $f(\beta_2) = \alpha_2$. Furthermore, there exist $u, w \in [v]_J$ such that the terminal vertex of β_1 is adjacent to u and the initial vertex of β_2 is adjacent to w. By definition of $[v]_J$, there is a path β in Γ from u to w that only passes through vertices in $[v]_J$. Then $f((\beta_1, \beta, \beta_2)) = \alpha_x$. The result follows.

Next, we show that $\mathrm{Im}(\rho_{\Gamma', J'})$ contains the images of all partial conjugations. Let $\sigma_{D,a} \in \mathcal{P}^{\Gamma, J}$. Let $b \in D$ so that $D = D(a, b)$. By the previous claim, we have that $D(f(a), f(b)) \cap J' \subseteq f(D(a, b) \cap J)$. We may therefore write

$$f(D \cap J) = \bigsqcup_{w \in I} D(f(a), f(w)) \cap J'$$

for some index set I. It follows that

$$\rho_{\Gamma,J}(\sigma_{D,a}) = \prod_{w \in I} \rho_{\Gamma',J'}(\sigma_{D(f(a),f(w)),f(a)}) \in \mathrm{Im}(\rho_{\Gamma',J'}).$$

Finally, we show that $\rho_{\Gamma',J'}(\mathcal{P}^{\Gamma',J'}) \subseteq \mathrm{Im}(\rho_{\Gamma,J})$. First, let $\sigma_{D,v'} \in \mathcal{P}^{\Gamma',J'}$. Then D is a component of $\Gamma' \backslash \mathrm{st}(v')$. By construction, we have that

$$\Gamma' \backslash \mathrm{st}(v') = \Gamma \backslash \Big(\bigcup_{w \in [v]_J} \mathrm{st}(w) \Big) = \bigcap_{w \in [v]_J} \Gamma \backslash \mathrm{st}(w).$$

Let $x \in D$, so that $D = D(v', x)$. Then by the above observation,

$$D \cap J = D(v', x) \cap J = \mathrm{Int}(v, x).$$

By the definition of compressible, we may write $D \cap J = D_1 \triangle D_2 \triangle \ldots \triangle D_m$ where for each $1 \leqslant i \leqslant m$ we have $D_i \in \mathfrak{D}(v)$. Let $D_i = D(w_i, y_i)$. Then

$$\rho_{\Gamma',J'}(\sigma_{D,v'}) = \prod_{i=1}^{m} \rho_{\Gamma,J}(\sigma_{D_i,w_i}) \in \mathrm{Im}(\rho_{\Gamma,J}).$$

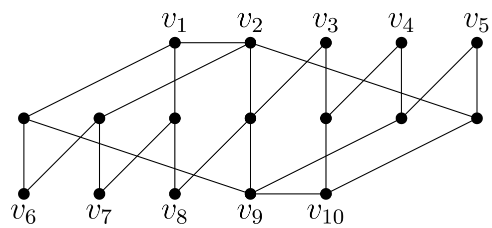
Now let $\sigma_{D,a} \in \mathcal{P}^{\Gamma',J'}$ for some $a \neq v'$. It follows that $f^{-1}(a) = \{a\}$. Let $b \in J$ be such that $f(b) \in D$. Then $D = D(f(a), f(b)) \cap J' \subseteq f(D(a,b) \cap J)$. Let $x \in D(a,b) \cap J \backslash [b]_J$, and let α_x be a path in $\Gamma \backslash \mathrm{st}(a)$ from b to x. If α_x does not intersect $[v]_J$, then α_x induces a path β_x in $\Gamma' \backslash \mathrm{st}(a)$ from $f(b) = b$ to x. If α_x intersects $[v]_J$, we may write $\alpha_x = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ where each of the even-indexed α_i lies entirely in $[v]_J$ and each of the odd-indexed α_i lies entirely outside $[v]_J$ (α_1 might be empty). Then each of the odd-indexed α_i induces a path β_i in $\Gamma' \backslash \mathrm{st}(a)$ and $f(\alpha) = (\beta_1, v', \beta_3, v', \ldots)$ is a path in $\Gamma' \backslash \mathrm{st}(a)$ from $f(b)$ to x. This shows that $D = f(D(a,b) \cap J)$. Therefore $\rho_{\Gamma',J'}(\sigma_{D,a}) = \rho_{\Gamma,J}(\sigma_{D,a}) \in \mathrm{Im}(\rho_{\Gamma,J})$.

This completes the proof.

□

Unfortunately, there exist vertex sets J with incompressible components of Γ_J, as the following example shows.

Example 8. Let Γ be the graph



and let J be the vertex set $\{v_1, v_2, \ldots, v_{10}\}$. Then $[v_1]_J$ and $[v_9]_J$ are both incompressible. Furthermore, the separating set $\text{sep}(v_1, v_9) = [v_1]_J \sqcup [v_9]_J$ fails the hypothesis of Lemma 3.26. As this is not the only separating set in this graph, we can get around this fact. However, it is not inconceivable that every separating set could fail in this way.

To avoid the issue presented in the previous example, we will show that for every finite graph Γ and $J \subseteq V(\Gamma)$ such that Γ_J has at least 2 components, there must be a minimal separating set $\text{sep}(v, w)$ such that either $[v]_J$ or $[w]_J$ is compressible. We first prove a couple of lemmas. The first lemma gives us a nice consequence of a vertex being a v-expander. The second lemma tells us that every incompressible component $[v]_J$ has at least two distinct vertices x and y, both of which are v-expanders, and that are separated by an element of $[v]_J$.

Lemma 3.29. *Let $v, x \in J$ be such that $[v]_J \neq [x]_J$. If x is a v-expander, then for all $y \in J \backslash ([v]_J \cup [x]_J)$ we have that $D(v, x) \neq D(y, x)$.*

Proof. We prove the contrapositive. Let $y \in J \backslash ([v]_J \cup [x]_J)$ be such that $D(v,x) = D(y,x)$. Then for all $w \in [v]_J$, we have $D(y,x) \neq D(y,v) = D(y,w)$. Hence by Corollary 3.3, we have that $D(y,x) \subseteq D(w,x)$ for all $w \in [v]_J$. It follows that $D(v,x) \cap J = D(y,x) \cap J \subseteq \mathrm{Int}(v,x)$. But $\mathrm{Int}(v,x) \subseteq D(v,x)$. Hence $D(v,x) = \mathrm{Int}(v,x)$ which shows that x is not a v-expander. $\square$

Lemma 3.30. *Let $v \in J$ be such that $[v]_J$ is incompressible. Then there exist $x, y \in J \backslash [v]_J$ such that Int$(v,x) \neq$ Int(v,y) and both x and y are v-expanders.*

Proof. First note that for any $x, y \in J \backslash [v]_J$, if $y \notin \mathrm{Int}(v,x)$, then there exists a $w \in [v]_J$ such that $D(w,x) \neq D(w,y)$. Hence $x \notin \mathrm{Int}(v,y)$. It follows that $\{\mathrm{Int}(v,x)\ x \in J \backslash [v]_J\}$ forms a partition of $J \backslash [v]_J$.

By definition, there exists some $x \in J \backslash [v]_J$ such that x is a v-expander. Assume that y is a v-expander only if $\mathrm{Int}(v,x) = \mathrm{Int}(v,y)$. Since x is a v-expander, it follows that $\mathrm{Int}(v,x) \neq J \backslash [v]_J$.

Let $c \in \mathrm{Int}(v,x)^{\complement} \backslash [v]_J$. Then by assumption $\mathrm{Int}(v,c)$ can be written as a symmetric difference of elements of $\mathfrak{D}(v)$. Since $\{\mathrm{Int}(v,j) \mid j \in J \backslash [v]_J\}$ forms a partition of $J \backslash [v]_J$, it follows that $\bigcup\limits_{c \in (\mathrm{Int}(v,x))^{\complement} \backslash [v]_J} \mathrm{Int}(v,c)$ can be written as a symmetric difference of elements of $\mathfrak{D}(v)$. Let $\{v_1, v_2, \ldots, v_n\} \subseteq J \backslash [v]_J$ be such that $\bigsqcup\limits_{i=1}^{n} D(v,v_i) \cap J \backslash [v]_J = J \backslash [v]_J$. Then

$$\mathrm{Int}(v,x) = (D(v,v_1) \cap J \backslash [v]_J) \triangle (D(v,v_2) \cap J \backslash [v]_J) \triangle \ldots \triangle (D(v,v_n) \cap J \backslash [v]_J) \triangle \bigcup_{c \in (\mathrm{Int}(v,x))^{\complement} \backslash [v]_J} \mathrm{Int}(v,c).$$

This contradicts the fact that x is a v-expander. Therefore there exists $y \in J \backslash [v]_J$ and $w \in [v]_J$ such that $D(w,x) \neq D(w,y)$ and y is a v-expander. $\square$

We are now ready to prove the Proposition.

Proposition 3.31. *There is no finite graph Γ and $J \subseteq V(\Gamma)$ such that Γ_J contains at least 2 components and every minimal separating set consists solely of incompressible components.*

Proof. If J contains less than 2 incompressible components then this is trivial. Assume that J contains at least two incompressible component.

Claim 1. *There exist $v, w \in J$ such that $[v]_J$ is an incompressible component, w is not a v-expander, and $D(v, w) \cap J$ consists solely of compressible components.*

Let $v_1, v_2 \in J$ be such that $[v_1]_J \neq [v_2]_J$ and $[v_1]_J, [v_2]_J$ are both incompressible components. Then by Lemma 3.30, there exists a $w_3 \in J \backslash ([v_1]_J \cup [v_2]_J)$ such that $\text{Int}(v_2, v_1) \neq \text{Int}(v_2, w_3)$ and w_3 is a v_2-expander. Without loss of generality, we may assume that $D(v_2, v_1) \neq D(v_2, w_3)$. If $D(v_2, w_3)$ consists solely of compressible components, then we are done. If not, there exists a $v_3 \in D(v_2, w_3)$ such that $[v_3]_J$ is incompressible. By Corollary 3.3, we have that $D(v_2, v_3) = D(v_2, w_3) \subseteq D(v_1, w_3)$. It follows that $D(v_1, w_3) = D(v_1, v_3)$ and $D(v_2, v_3) \subseteq D(v_1, v_3)$. By Lemma 3.29, we have that $D(v_1, v_3) = D(v_1, w_3) \neq D(v_2, w_3) = D(v_2, v_3)$. Therefore $D(v_1, v_3) \supsetneq D(v_2, v_3)$.

Assume that we have constructed $v_1, \ldots, v_n$ such that $[v_1]_J$, $[v_2]_J$, $\ldots, [v_n]_J$ are incompressible and $D(v_1, v_n) \supsetneq D(v_2, v_n) \supsetneq \cdots \supsetneq D(v_{n-1}, v_n)$. Then by Lemma 3.30, there exists a $w_{n+1} \in J \backslash [v_n]_J$ such that $\text{Int}(v_n, v_{n-1}) \neq \text{Int}(v_n, w_{n+1})$ and w_{n+1} is a v_n-expander. Without loss of generality, we may assume that $D(v_n, v_{n-1}) \neq D(v_n, w_{n+1})$. If $D(v_n, w_{n+1})$ consists solely of compressible components, then we are done. If not, there exists a $v_{n+1} \in D(v_n, w_{n+1})$ such that $[v_{n+1}]_J$ is incompressible. By Corollary 3.3, we have that $D(v_n, v_{n+1}) = D(v_n, w_{n+1}) \subseteq D(v_{n-1}, w_{n+1})$. It follows that $D(v_{n-1}, w_{n+1}) = D(v_{n-1}, v_{n+1})$ and $D(v_n, v_{n+1}) \subseteq D(v_{n-1}, v_{n+1})$. By Lemma 3.29, we have that $D(v_{n-1}, v_{n+1}) = D(v_{n-1}, w_{n+1}) \neq$

$D(v_n, w_{n+1}) = D(v_n, v_{n+1})$. Therefore $D(v_{n-1}, v_{n+1}) \supsetneqq D(v_n, v_{n+1})$. Furthermore, by the contrapositive of Corollary 3.3, it follows that $D(v_{n-1}, v_{n+1}) = D(v_{n-1}, v_n)$. Hence $D(v_1, v_{n+1}) \supsetneqq D(v_2, v_{n+1}) \supsetneqq \cdots \supsetneqq D(v_{n-1}, v_{n+1}) \supsetneqq D(v_n, v_{n+1})$. Since Γ is a finite graph, this process must eventually terminate in a vertex w_N such that w_N is a v_{N-1}-expander and $D(v_{N-1}, w_N)$ consists solely of compressible components. This proves Claim 1.

Claim 2. *Let $v, w \in J$ be such that $[v]_J$ is incompressible, w is a v-expander, and $D(v, w) \cap J$ consists solely of compressible components. Then sep$(v, w) \backslash ([v]_J \cup [w]_J)$ contains only compressible components.*

Suppose on the contrary that $x \in \text{sep}(v, w) \backslash ([v]_J \cup [w]_J)$ is such that $[x]_J$ is incompressible. Then there exists an $x^* \in [x]_J$ such that $D(x^*, v) \neq D(x^*, w)$. Then Corollary 3.3 implies that $D(x^*, w) \subseteq D(v, w)$ consists solely of compressible components. Since w is a v-expander, Lemma 3.29 implies that $D(x^*, w) \neq D(v, w)$. The contrapositive of Corollary 3.3 thus implies that $D(v, x^*) = D(v, w)$. But this contradicts the fact that $D(v, w)$ consists solely of compressible components. Therefore no such $[x]_J$ exists. This proves Claim 2.

By Claim 1, there exists $v, w \in J$ such that $[v]_J$ is incompressible, w is a v-expander, and $D(v, w)$ consists solely of compressible components. Claim 2 implies that $[v]_J$ is the only incompressible component in sep(v, w). It follows that any minimal separating set in sep(v, w) has at most one incompressible component. This proves the statement. $\square$

3.6 Computing $\mathrm{Im}(\rho_{\Gamma,J})$

If a separating set $\mathrm{sep}(v,w)$ is minimal, then by Proposition 3.25, we see that $\mathrm{sep}(v,w) = \mathrm{sep}(a,b)$ for all $a,b \in \mathrm{sep}(v,w)$. This makes minimal separating sets particularly nice to work with. The following lemma gives another nice property of minimal separating sets.

Lemma 3.32. *Let $sep(v,w)$ be minimal with respect to inclusion. Then for each $a \in sep(v,w)$ there exists an $a^* \in [a]_J$ such that for all $b \in sep(v,w)\backslash[a]_J$ we have that $D(a^*,b) \cap sep(v,w)\backslash[a]_J = [b]_J$.*

Proof. First, let $a \in [v]_J \cup [w]_J$. Without loss of generality, assume that $a \in [v]_J$. If $\mathrm{sep}(v,w) = [v]_J \cup [w]_J$ then the statement is trivial. If not, let $c \in \mathrm{sep}(v,w)\backslash([v]_J \cup [w]_J)$. Then by Proposition 3.25 combined with minimality, we have that $\mathrm{sep}(v,w) = \mathrm{sep}(c,w)$. Thus we may assume that $a \notin [v]_J \cup [w]_J$.

Let $a^* \in [a]_J$ be such that $D(a^*,v) \neq D(a^*,w)$. Then for any $c \in \mathrm{sep}(v,w)$ such that $c \notin [a]_J \cup [v]_J$, we show that $D(a^*,v) \neq D(a^*,c)$. This implies that $D(a^*,v) \cap \mathrm{sep}(v,w)\backslash[a]_J = [v]_J$.

To begin, Proposition 3.25 implies that $\mathrm{sep}(v,c) \subseteq \mathrm{sep}(v,w)$. Then by minimality $\mathrm{sep}(v,c) = \mathrm{sep}(v,w)$. Hence there exists a $w^* \in [w]_J$ such that $D(w^*,v) \neq D(w^*,c)$. Since $D(a^*,v) \neq D(a^*,w) = D(a^*,w^*)$, Corollary 3.3 implies that $D(a^*,v) \subseteq D(w^*,v)$. But $c \notin D(w^*,v)$. Therefore $c \notin D(a^*,v)$, as claimed.

Next, let $b \in \mathrm{sep}(v,w)\backslash([a]_J \cup [v]_J)$. Then Proposition 3.25 combined with minimality tells us that $\mathrm{sep}(v,w) = \mathrm{sep}(b,v)$. We showed above that $D(a^*,b) \neq D(a^*,v)$. But this is the only assumption we needed to run the above argument. Therefore $D(a^*,b) \cap \mathrm{sep}(b,v)\backslash[a]_J = [b]_J$.

□

Using Lemma 3.32, we make the following definition.

Definition 3.33. Let $\operatorname{sep}(v, w)$ be minimal with respect to inclusion. For each $x \in \operatorname{sep}(v, w) \backslash ([v]_J \cup [w]_J)$, fix an element $x^* \in [x]_J$ via Lemma 3.32. Then for the given choices for all $[x]_J \subseteq \operatorname{sep}(v, w)$, we define

$$\operatorname{sep}^*(v, w) := \{x^* \mid x \in \operatorname{sep}(v, w)\}.$$

Note that this definition depends on the choice of x^*. However, no matter what choice is made, it contains a unique representative of each component $[x]_J$ of Γ_J that intersects $\operatorname{sep}(v, w)$. Furthermore, Lemma 3.32 says that for each $a^*, b^* \in \operatorname{sep}^*(v, w)$ we have that $D(a^*, b^*) \cap \operatorname{sep}^*(v, w) = \{b^*\}$.

With this definition in hand, we prove the following corollary.

Corollary 3.34. *Let $sep(v, w)$ be minimal with respect to inclusion. Then for all distinct $a^*, b^*, c^* \in sep^*(v, w)$, we have $D(a^*, c^*) = D(b^*, c^*)$.*

Proof. By Lemma 3.32, we have that $c^* \notin D(b^*, a^*)$ and $b^* \notin D(a^*, c^*)$. By a double application of Corollary 3.3, this implies that $D(b^*, c^*) = D(a^*, c^*)$. This completes the proof. $\square$

We now consider a set that is slightly bigger than $\operatorname{sep}(v, w)$. We define this set now.

Definition 3.35. Let $\operatorname{sep}(v, w)$ be minimal with respect to inclusion. Fix a set $\operatorname{sep}^*(v, w)$. Then we define

$$C(v, w) := \{c \in J \mid \exists a^*, b^* \in \operatorname{sep}^*(v, w) \text{ such that } D(a^*, b^*) = D(a^*, c)\}.$$

Using properties of $\mathrm{sep}(v,w)$, we can show that $C(v,w)$ contains a special point. This will allow us to construct a division and apply the TML-decomposition. However, if $C(v,w) = J$, then this application will be trivial. We therefore deal with this case separately.

Note that if $\mathrm{sep}(v,w) = J$ is minimal, then $C(v,w) = J$. This case is easily dealt with. Indeed the minimality of $\mathrm{sep}(v,w)$, combined with Proposition 3.25, implies that $\mathrm{sep}(a,b) = J$ for all $a,b \in J$ such that $[a]_J \neq [b]_J$. Then by Theorem 2.7, we have that $\mathrm{Im}(\rho_{\Gamma,J}) \cong \Gamma_{k_J-1}(2)$. Thus we may safely ignore this case.

Proposition 3.36. *Let sep$(v,w) \subsetneq J$ be minimal. Let sep$(v,w) \neq [v]_J \cup [w]_J$. Then if $C(v,w) = J$, then δ has a splitting point.*

Proof. Pick some $c \notin \mathrm{sep}(v,w)$ and some $s^*, t^* \in \mathrm{sep}^*(v,w)$ such that $D(t^*,s^*) = D(t^*,c)$. Then define $A := D(t^*,s^*)\backslash[s^*]_J$ and let $B := D(t^*,s^*)^{\complement}\backslash[s^*]_J$. We will show that s^* is a splitting point with splitting $A \sqcup [s^*]_J \sqcup B$.

First, we have that given any $a \in A$ and any $r^* \in \mathrm{sep}^*(v,w)\backslash([s^*]_J \cup [t^*]_J)$, Corollary 3.34 implies that $D(t^*,a) = D(t^*,s^*) = D(r^*,s^*)$. By Lemma 3.32, we have that $D(r^*,s^*) \neq D(r^*,t^*)$. This in turn implies that $a \notin D(r^*,t^*)$. By Corollary 3.34, we have $D(r^*,t^*) = D(s^*,t^*)$. Thus $a \notin D(s^*,t^*)$. Equivalently, $t^* \notin D(s^*,a)$. Then Corollary 3.3 implies that $D(s^*,a) \subseteq D(t^*,a) = D(t^*,s^*)$. Finally, we conclude that $D(s^*,a) \cap J\backslash[s^*]_J \subseteq A$. This also implies that for all $b \in B$ we have $D(s^*,b) \cap J\backslash[s^*]_J \subseteq B$.

Next, note that given any $b \in B$ we have $b \notin D(t^*,s^*)$. By Corollary 3.3, we have that $D(t^*,s^*) \subseteq D(b,s^*)$. This implies that $B^{\complement} \subseteq D(b,s^*)\backslash[b]_J$.

It remains to show that for any $a \in A$ we have $A^{\complement} \subseteq D(a,s^*)\backslash[a]_J$. Note that since $a \in \mathrm{sep}(v,w)^{\complement}$ and $s \in \mathrm{sep}(v,w)$, Proposition 3.24 implies that $\mathrm{sep}(v,w) \subseteq D(a,s^*)$. Thus it suffices to show that $D(a,b) = D(a,s^*)$ for any $b \in B\backslash\mathrm{sep}(v,w)$.

Assume that $B \nsubseteq \text{sep}(v, w)$ and let $b \in B \backslash \text{sep}(v, w)$. By our hypothesis, there exists $x^*, y^* \in \text{sep}^*(v, w)$ such that $D(y^*, x^*) = D(y^*, b)$. By Corollary 3.34, we have $D(y^*, b) = D(y^*, x^*) = D(t^*, x^*) = D(t^*, b) \neq D(t^*, a)$. This shows that $a \notin D(y^*, b)$. Corollary 3.3 then implies that $D(y^*, b) \subseteq D(a, b)$. It follows that $x^* \in D(a, b)$. Proposition 3.24 implies that $\text{sep}(v, w) \subseteq D(a, b)$. Therefore $D(a, b) = D(a, x^*) = D(a, s^*)$, which was to be shown.

□

We now present an algorithm that can compute the isomorphism class of $\text{Im}(\rho_{\Gamma,J})$ for general (Γ, J).

Theorem 3.37. *Given an arbitrary finite graph Γ and vertex set $J \subseteq V(\Gamma)$, the group Im$(\rho_{\Gamma,J})$ can be computed up to isomorphism.*

Proof. We induct on $|J|$.

First, by using repeated applications of Proposition 3.28, we may assume that for all $v \in J$ either $[v]_J = \{v\}$ or $[v]_J$ is incompressible. By Proposition 3.31, there exists some minimal separating set $\text{sep}(v, w)$ such that $[v]_J = \{v\}$. If $\text{sep}(v, w) = J$, then by minimality and Proposition 3.25, we have that $\text{sep}(a, b) = J$ for all distinct $a, b \in J$. Hence $\text{Im}(\rho_{\Gamma,J}) = \Gamma_{k_J - 1}(2)$ by Theorem 2.7. Thus we may assume that $\text{sep}(v, w) \neq J$. Furthermore, by Lemma 3.26, we may assume that $\text{sep}(v, w) \neq [v]_J \cup [w]_J$. We prove a couple of nice facts about $C(v, w)$:

Claim 1. *For any $a^*, b^* \in$ sep$^*(v, w)$ and any $j \in C(v, w)^{\complement}$, we have that $D(a^*, j) = D(b^*, j)$.*

Let $a^*, b^* \in \text{sep}^*(v, w)$ be such that $a^* \neq b^*$ and let $j \in C(v, w)^{\complement}$. Then by definition of $C(v, w)$, we have that $D(a^*, j) \neq D(a^*, b^*)$ and $D(b^*, j) \neq D(b^*, a^*)$.

Therefore by a double application of Corollary 3.3, we have that $D(a^*, j) = D(b^*, j)$. If $a^* = b^*$ then $D(a^*, j) = D(b^*, j)$. This proves the claim.

Claim 2. *For any $u^* \in sep^*(v, w)$ and any $j \in C(v, w)^{\complement}$, we have that $D(u^*, j) \cap C(v, w) \backslash [u^*]_J = \varnothing$.*

Let $u^* \in \text{sep}^*(v, w)$, $j \in C(v, w)^{\complement}$. Let $c \in C(v, w)$ such that $[u^*]_J \neq [c]_J$. By definition of $C(v, w)$, there exists $x^*, y^* \in \text{sep}^*(v, w)$ such that $D(x^*, c) = D(x^*, y^*)$. By Claim 1, $D(u^*, j) = D(x^*, j)$. But by definition of $C(v, w)$, we have that $D(x^*, j) \neq D(x^*, y^*) = D(x^*, c)$. Therefore $c \notin D(u^*, j)$. This proves the claim.

Claim 3. *For any $j \in C(v, w)^{\complement}$, we have that $D(j, v) \supseteq C(v, w)$*

Let $j \in C(v, w)^{\complement}$. Given any $c \in C(v, w)$, there exists $x^*, y^* \in \text{sep}^*(v, w)$ such that $D(x^*, c) = D(x^*, y^*)$. By Proposition 3.24, we have that $D(j, v) \supseteq \text{sep}(v, w)$. This implies that $D(j, v) = D(j, y^*)$. By definition of $C(v, w)$, we have $D(x^*, y^*) \neq D(x^*, j)$. Therefore by Corollary 3.3, we have $D(x^*, c) = D(x^*, y^*) \subseteq D(j, y^*) = D(j, v)$. Thus $c \in D(j, v)$. This proves the claim.

If $C(v, w) = J$, then since $\text{sep}(v, w) \neq J$ and $\text{sep}(v, w) \neq [v]_J \cup [w]_J$, we are done by Proposition 3.36 combined with Proposition 3.22. If not, then we may write $J = \bigsqcup_{i=1}^{m} A_i$ where $A_1 = C(v, w)$ and for all $1 < i \leqslant m$, we have $A_i = [a_i]_J$ for some $a_i \in J$. We show that this is a non-trivial division of J. Hence we are done by the TML decomposition (Theorem 1.2).

It is trivial to verify that for any $a \in J$, we have that a is a special point in $[a]_J$. Hence we need only show that A_1 contains a special point. We show that v is a special point in A_1. We assumed earlier that $[v]_J = \{v\}$. Hence by default $v \in \text{sep}^*(v, w)$. Then Claim 2 implies that for all $j \in J$ either $D(v, j) \cap J \subseteq A_1$

or $D(v, j) \cap J \subseteq A_1^{\complement}$, and Claim 3 implies that for all $j \in A_1^{\complement}$ we have that $A_1 \subseteq D(j, v)$. Therefore v is a special point in A_1.

$\square$

Chapter 4

Additional Results

4.1 Star groups

In this section we show that every element in a certain family of subgroups of $\Gamma_n(2)$ can be written as $\mathrm{Im}(\rho_{\Gamma,J})$ for some (Γ, J). We will use the following notation from the proof of Theorem 2.7. Given $1 \leqslant i, j \leqslant n$, $i \neq j$, let $E_{i,j}$ denote the matrix identical to the identity matrix except in the (i, j)-entry, which equals 2 and let F_i denote the matrix identical to the identity matrix except in the (i, i)-entry, which equals -1.

Definition 4.1. Let $I \subseteq \{1, 2, \ldots, n\}^2$ be a subset with the following properties. For all $i \in \{1, 2, \ldots, n\}$ we have $(i, i) \in I$, and for each $(i, j), (j, k) \in I$ we have $(i, k) \in I$. Then we call I a star set of width n. Given any star set of width n, the star group associated to I is the group

$$G_I := \{A \in \Gamma_n(2) \mid A_{i,j} = 0 \text{ for all } (i, j) \notin I\}.$$

It is not immediately clear that G_I is a group, in particular that it is closed under inverses. This will follow from Proposition 4.3 in which we compute a generating set for G_I. First, however, we need the following lemma.

Lemma 4.2. *Let I be a star set of width n, and let $S_i := \{(i,j) \mid 1 \leqslant j \leqslant n, (i,j) \in I\}$. Assume that $|S_m|$ is maximal. If $(i,m) \in I$, then $(m,i) \in I$*

Proof. Assume that $(i,m) \in I$. It follows that for all $1 \leqslant j \leqslant n$ such that $(m,j) \in I$ we have that $(i,j) \in I$. Hence $|S_i| \geqslant |S_m|$. But $|S_m|$ is maximal. Hence $|S_i| = |S_m|$. But since $(m,j) \in I \implies (i,j) \in I$, it follows that $i \in \{j \mid (i,j) \in I\} = \{j \mid (m,j) \in I\}$. Therefore $(m,i) \in I$. □

We now compute a generating set for each G_I.

Proposition 4.3. *Let I be a star set. Then G_I is the subgroup of $\Gamma_n(2)$ generated by $\{F_i \mid 1 \leqslant i \leqslant n\} \cup \{E_{i,j} \mid (i,j) \in I, i \neq j\}$*

Proof. Let H_I be the group generated by $\{F_i \mid 1 \leqslant i \leqslant n\} \cup \{E_{i,j} \mid (i,j) \notin I, i \neq j\}$. It is clear that $H_I \leqslant G_I$. Thus it suffices to show that $G_I \leqslant H_I$. We proceed by induction on the width of I. The base case is trivial.

Assume for induction that for every star set I of width less than n, we have $G_I = H_I$. Let I be a subgroup of width n, and let $A \in G_I$. To show that $G_I \leqslant H_I$, we will take an arbitrary element of G_I and multiply it by elements of H_I to reduce it to the identity matrix.

Since $A \in \Gamma_n(2)$, we must have $A_{1,1} \equiv_2 1$. Assume that $|A_{1,1}| > 1$. We show that we can multiply A by elements of H_I to obtain a matrix $A' \in G_I$ such that $|A'_{1,1}| < |A_{1,1}|$. By induction, this will allow us to assume that $|A_{1,1}| = 1$. If $A_{1,1}$ is negative, then $(F_1 A)_{1,1}$ is positive. Since $F_1 \in H_I$, we may assume that $A_{1,1}$ is positive. If $A_{i,1}$ is a multiple of $A_{1,1}$ for all i then A is not invertible. Therefore there exists an i such that $A_{i,1}$ is not a multiple of $A_{1,1}$. Multiplying by F_i on the left if necessary, we may assume that $A_{i,1}$ is positive.

Assume that $A_{i,1} > A_{1,1}$. Then $(i,1) \in I$, so $E_{i,1} \in H_I$. By the division algorithm, there exists $q, r \in \mathbb{Z}$ such that $A_{i,1} = qA_{1,1} + r$ and $0 \leqslant r < A_{1,1}$. If $q = 2k$ for some $k \in \mathbb{Z}$, then $(E_{i,1}^{-k}A)_{i,1} = r < A_{1,1}$ and $(E_{i,1}^{-k}A)_{j,1} = A_{j,1}$ for all $j \neq i$. If $q = 2k - 1$ for some $k \in Z$, then $-A_{1,1} < r - A_{1,1} = A_{i,1} - 2kA_{1,1} < 0$. Therefore $0 < (F_iE_{i,1}^{-k}A)_{i,1} < A_{1,1}$ and $(F_iE_{i,1}^{-k}A)_{j,1} = A_{j,1}$ for all $j \neq i$. We may therefore assume that $0 < A_{i,1} < A_{1,1}$.

Let S_i be as in Lemma 4.2. Up to reindexing, we may assume that $|S_1|$ is maximal. Since $(i,1) \in I$, Lemma 4.2 implies that $(1,i) \in I$. Therefore $E_{1,i} \in H_I$. Since $0 < A_{i,1} < A_{1,1}$, it follows that $|A_{1,1} - 2A_{i,1}| < A_{1,1}$. Hence $|E_{1,i}^{-1}A|_{1,1} < A_{1,1}$ and $(E_{1,i}^{-1}A)_{j,1} = A_{j,1}$ for all $j \neq 1$. Thus we can reduce to the case where $|A_{1,1}| = 1$.

If $A_{1,1} = -1$, then $F_1 \in H_I$ and $(F_1A)_{1,1} = 1$. Thus we may assume that $A_{1,1} = 1$. For all $i \neq 1$ we have that $A_{i,1}$ is even. Assume that $A_{i,1} = 2k \neq 0$. Then $(i,1) \in I$, hence $E_{i,1} \in G_I$. It follows that $E_{i,1}^{-k}A \in G_I$. But $(E_{i,1}^{-k}A)_{i,1} = 0$ and $(E_{i,1}^{-k})_{j,1} = A_{j,1}$ for all $j \neq i$. Repeating this process for each i such that $A_{i,1} \neq 0$, we reduce to a matrix of the form $\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$.

For each $(1,i) \in I$, we have that $E_{1,i} \in H$. If $A_{1,i} = 2k \neq 0$, then $(AE_{1,i}^{-k})_{1,i} = 0$ and $(AE_{1,i}^{-k})_{1,j} = A_{1,j}$ for all $j \neq i$. Repeating this process for each i such that $A_{1,i} \neq 0$, we reduce to a matrix of the form $\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$.

Let $I' := \{(i,j) \in I \mid i \neq 1, j \neq 1\}$. Then I' is a star set of rank $n-1$. By induction, $B \in H_{I'}$. But there is a natural embedding

$$\iota : H_{I'} \to H_I, \quad B \mapsto \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}.$$

Therefore $A \in H_I$, which completes the proof. $\square$

Now that we have a generating set for the star groups, we will show that every star group is the image of some representation $\rho_{\Gamma,J}$. To that end, we first give a condition on J which results in $\mathrm{Im}(\rho_{\Gamma,J})$ being a star group.

Theorem 4.4. *Let $\{v_1, \ldots, v_n\}$ be a set of representatives of the components of Γ_J. Assume that $D(v_n, v_i) \cap \{v_1, \ldots, v_n\} = \{v_i\}$ for all $v_i \in \{v_1, \ldots, v_{n-1}\}$. Let $I := \{(i,j) \mid 1 \leqslant i,j \leqslant n-1, D(v_i, v_j) \cap \{v_1, \ldots, v_n\} = \{v_j\}\}$. Then I is a star set of width $n-1$ and $Im(\rho_{\Gamma,J}) \cong G_I$.*

Proof. First assume that $D(v_i, v_j) \cap \{v_1, \ldots, v_n\} = \{v_j\}$ and $D(v_j, v_k) \cap \{v_1, \ldots, v_n\} = \{v_k\}$. Then since $v_j \notin D(v_i, v_k)$, Corollary 3.3 implies that $D(v_i, v_k) \subseteq D(v_j, v_k)$, so that $D(v_i, v_k) \cap \{v_1, \ldots, v_n\} = \{v_k\}$. This shows that I is a star set.

Fix the following basis of $I_J(H_1(\hat{X}; \mathbb{Q}))$.

$$\hat{v}_n - \hat{v}_1, \hat{v}_n - \hat{v}_2, \ldots, \hat{v}_n - \hat{v}_{n-1}.$$

By Corollary 3.16, $\{\hat{\sigma}_{D,v} \mid v_n \notin D\}$ is a generating set. Fix some $\hat{\sigma}_{D(v_i,v_j),v_i}$ such that $v_n \notin D(v_i, v_j)$. Then by Corollary 3.3, we have that $D(v_i, v_j) \cap \{v_1, \ldots, v_n\} \subseteq D(v_n, v_j) \cap \{v_1, \ldots, v_n\} = \{v_j\}$ so $D(v_i, v_j) \cap \{v_1, \ldots, v_n\} = \{v_j\}$. If $i = n$, then by direct calculation $\hat{\sigma}_{D(v_i,v_j),v_i} = F_j$. If $i \neq n$, then by direct calculation $\hat{\sigma}_{D(v_i,v_j),v_i} = F_j E_{i,j}$. Therefore by Proposition 4.3

$$\mathrm{Im}(\rho_{\Gamma,J}) \cong \langle \{F_i \mid 1 \leqslant i \leqslant n-1\} \cup \{E_{i,j} \mid 1 \leqslant i,j \leqslant n-1, (i,j) \in I\} \rangle \cong G_I.$$

□

Now that we have a theorem that recognizes when the image of a given representation is a star group, we can show that all such groups can be written as $\mathrm{Im}(\rho_{\Gamma,J})$ for some (Γ, J).

Theorem 4.5. *Let I be a star set of width n. Then there exists a graph Γ and a vertex set J such that Im$(\rho_{\Gamma,J}) \cong G_I$.*

Proof. Let $J = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ and let $W := \{w_{i,j} \mid 1 \leqslant i < j \leqslant n+1\}$. Let

$$V(\Gamma) := J \sqcup W$$

and let

$$E(\Gamma) := \{(v_i, w_{j,k}) \mid (i,j) \in I \text{ or } (i,k) \in I \text{ or } i = n+1\}$$

By construction, Γ is a bipartite graph, and J consists of pairwise non-adjacent vertices. Since $(v_{n+1}, w_{i,j}) \in E(\Gamma)$ for all $1 \leqslant i < j \leqslant n+1$, we have that $D(v_{n+1}, v_i) \cap J = \{v_i\}$ for all $1 \leqslant i \leqslant n+1$.

Assume that $(i,j) \in I$. It follows that $j \neq n+1$. Let $(v_j, w_{b,c}) \in E(\Gamma)$. Then either $(j,b) \in I$ or $(j,c) \in I$. Since I is a star set, it follows that either $(i,b) \in I$ or $(i,c) \in I$. Hence $(v_i, w_{b,c}) \in E(\Gamma)$. This shows that $\mathrm{lk}(v_j) \subseteq \mathrm{lk}(v_i)$, which implies that $D(v_i, v_j) \cap J = \{v_j\}$.

Now let $1 \leqslant i,j \leqslant n$ be such that $D(v_i, v_j) \cap J = \{v_j\}$. Then since $(v_j, w_{j,n+1}, v_{n+1})$ is a path in Γ, we must have that $w_{j,n+1} \in \mathrm{lk}(v_i)$. This implies that $(i,j) \in I$.

We have shown that $D(v_{n+1}, v_i) \cap J = \{v_i\}$ for all $1 \leqslant i \leqslant n$ and that $I = \{(i,j) \mid 1 \leqslant i,j \leqslant n, D(v_i, v_j) \cap J = \{v_j\}\}$. Therefore by Theorem 4.4, we have that $\mathrm{Im}(\rho_{\Gamma,J}) \cong G_I$. □

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DEDICATION

to

My wife

Lauren Elizabeth Fehr, and

My daughter

Marie Rhetta Morgan

For

Loving me so well