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PERIODIC SOLUTIONS OF THE DISPERSION-MANAGED NONLINEAR SCHRÖDINGER EQUATION

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Contents

Abstract

We study the existence, uniqueness and stability of solutions to the initial-value problem for the periodic dispersion-managed nonlinear Schröd-inger (DMNLS) equation, an equation that models the propagation of periodic, nonlinear, quasi-monochromatic electromagnetic pulses in a dispersion-managed fiber. The periodic DMNLS equation we derive is the same as the non-periodic DMNLS equation (1.2), except with a subtle difference in the operator $T_{(s)} = T_{D(s)} = e^{-iD(s)\partial_x^2}$. The periodic function $D(s)$ still controls the dispersive properties of the optical fiber.

With respect to the Cauchy problem for the periodic DMNLS equation, under certain assumptions on the variable dispersion, we use a Strichartz estimate (Theorem 3.16) on the family of operators $T_{D(s)}$ to prove global well-posedness for initial data in H^r for non negative integer values of r.

Lastly, we prove results on the existence and stability of ground state solutions by considering the convergence of minimizing sequences for certain variational problems. In the case $\alpha > 0$, the convergence follows from the Rellich-Kondrachov Theorem; in the case $\alpha = 0$, we use a concentration-compactness argument due to Kunze [15], but with significant modifications.

Chapter 1

Introduction

1.1 About the Dispersion-Managed Schrödinger Equation

The one-dimensional nonlinear Schrödinger equation (NLS) with periodically varying dispersion coefficient

$$iu_z + d(z)u_{\tau\tau} + |u|^2 u = 0 \tag{1.1}$$

is an important model equation for pulses in long distance fiber optics communication systems [1]. This equation describes the amplitude of a signal transmitted through an optical fiber cable where the dispersion is varied periodically along the fiber. Here z represents the distance along the fiber, τ represents the time, and $d(z)$ determines the dispersive properties along the fiber which is usually taken to be piecewise constant. The NLS equation and its modified forms also often appear as models for other types of waves, including plasma waves and water waves. In this thesis, we will be focusing on an averaged version of (1.1) called the dispersion-managed nonlinear Schrödinger (DMNLS) equation.

With optic fiber cables that use localized pulses to transmit information, there could be unwanted interaction between individual pulses and thus unwanted spreading of signals. This is due to the effects of dispersion within the fiber. To remedy this situation, in the 1990's, researchers considered constructing a fiber by fusing together fibers with sections of large anomalous dispersion (sections where $d(z)$ is positive) alternating with sections of large normal dispersion (sections where $d(z)$ is negative), in such way that the dispersion has average value near zero [21] and the intensity and clarity of the signals are not compromised. This technique is referred to as dispersion management. It proved to be a success in alleviating the unwanted interactions, and today is a standard technique used in long distance fiber optic transmissions.

Gabitov and Turitsyn [7] put forward a model equation for nonlinear optical waves in a dispersion-managed fiber. To derive it, one assumes that

$$d(z) = \langle d \rangle + \frac{\Delta(z/\epsilon)}{\epsilon},$$

that is, $d(z)$ is periodic with small period $\epsilon << 1$, large in absolute value, of size $O\left(1/\epsilon\right)$, and has mean value $\langle d \rangle$ which is of $O(1)$. Also, $\Delta(z)$ is a function of period 1 with mean value zero; i.e.,

$$\int_0^1 \Delta(z)\, dz = 0.$$

Under these assumptions and taking the limit as ϵ goes to 0, it was shown that one can approximate solutions of (1.1) by solutions of the averaged equation

$$iu_z + \alpha u_{\tau\tau} + \int_0^1 T^{-1}(s)[|T(s)u|^2 T(s)u]\, ds = 0, \tag{1.2}$$

where $\alpha = \langle d \rangle$. This equation is what we refer to as the dispersion managed nonlinear Schrödinger (DMNLS) equation. As a matter of fact, when ϵ is very small, solutions of (1.2) approximate those of (1.1) well (see [22]).

In the above equation, the invertible operator $T(s)$ is defined as follows. For $s \in \mathbb{R}$, define

$$D(s) := \int_0^s \Delta(z)\, dz.$$

Note that $D(s)$ is also periodic with period 1, and controls the dispersive properties of the optical fiber. For $0 \le s \le 1$, define $T(s) : L^2 \mapsto L^2$ by

$$T(s) = e^{-iD(s)\partial_\tau^2}.$$

In other words, if we take the Fourier transform of $T(s)$, we have that

$$\mathcal{F}(T(s)u)[\omega] = e^{-iD(s)\omega^2}\mathcal{F}(u)[\omega].$$

Note that $T(s)$ is a unitary operator on $L^2(\mathbb{R})$, with inverse given by $T^{-1}(s) = e^{iD(s)\partial_\tau^2}$.

In the special case where $D(s) = \alpha s$, $T(s)$ becomes the solution operator for the initial-value problem for the linear Schrödinger equation $iu_s + \alpha u_{xx} = 0$, and will be denoted by $S(\alpha s)$. That is, we define

$$S(\alpha s) = e^{i\alpha s\partial_\tau^2}.$$

Note that here we do not enforce the condition that $D(s)$ be periodic.

For a detailed description of the process of the derivation of (1.2), see [1, 2].

1.2 Previous Results

One of the first questions that is commonly asked about an initial-value problem for a partial differential equation is whether it is well-posed. We say an initial-value problem is well-posed in a function space X if given an arbitrary choice of initial data in X, the problem has a unique solution in X which depends continuously on the initial data. If the unique solution exists for all time, then we say that the initial-value problem is globally well-posed, otherwise, we say that the initial-value problem is locally well-posed.

Another commonly studied question, especially when dealing with nonlinear equations, is whether there exist important special solutions. For equations that model physical phenomena, it is also important to check the stability of such solutions because stable solutions are likely to be observed in reality and some have been shown to play an important role in the development of general solutions to many wave equations. In fact, to prove stability of solutions often requires having a well-posedness result so as to assure existence of nearby solutions for all time. Examples of important solutions to (1.2) are standing wave solutions. These are solitary-wave solutions of the form $u(z,\tau) = e^{i\theta z}\phi(\tau)$ where θ is a constant.

Recently, results on the well-posedness of the initial-value problem and the existence and stability of standing wave solutions have been established for (1.2) in spaces of non-periodic functions. Zharnitsky et al. [23] proved global well-posedness in the standard L^2 based Sobolev space $H^1(\mathbb{R})$, in the case $\alpha \neq 0$, under the assumption that the dispersion profile $\Delta(s)$ is piecewise constant on $[0,1]$. Albert and Kahlil [3] proved a global well-posedness result for the DMNLS equation also in the L^2 based Sobolev space $H^r(\mathbb{R})$ for all $r \geq 0$ when $\alpha \neq 0$, and in $L^2(\mathbb{R})$ when $\alpha = 0$, under the assumption that $D(s)$ is absolutely continuous with a derivative that is piecewise constant and bounded away from zero. We should also note that Hundertmark et al. [10, 12], already showed that the DMNLS equation is globally well-posed in $L^2(\mathbb{R})$ when $\alpha = 0$.

A standard procedure for proving the existence of stable standing wave solutions is to solve a constrained variational problem for a conserved functional for the equation. This method usually involves showing that a maximizing or minimizing sequence for the variational problem has a subsequence that converges (up to symmetries) to a maximizer or respectively a minimizer, which represents the profile of a stable standing wave. If a profile standing wave is a constrained minimizer for a functional which can be interpreted as an energy functional for the equation, then we call the standing wave a ground state.

For the DMNLS equation, Zharnitsky et al. [23] gave a result on the existence of ground state solutions and their stability in $H^1(\mathbb{R})$ in the case $\alpha > 0$. Here stability is interpreted as follows: if we call $\mathcal{S}$ the set of all minimizers (or the set consisting of all the ground state profiles) of the functional $E(v)$ defined by

$$E(v) = \int_{-\infty}^{\infty} \alpha |v_\tau|^2 d\tau - \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} |T_{D(s)} v|^4 \, d\tau \, ds,$$

subject to the constraint

$$P(v) = \int_{-\infty}^{\infty} |v|^2 d\tau = \lambda \quad \text{for each } \lambda < \infty,$$

and we know that an H^1 function, $u_0(\tau)$, is close in norm to a fixed element ϕ in $\mathcal{S}$, then the unique solution $u(z,\tau)$ of (1.2) whose initial condition is u_0 will remain close in norm to some element in $\mathcal{S}$ for all values of z. Kunze [15], by applying an enhanced version of Lions' concentration compactness method [17], showed that for $\alpha = 0$ and $\Delta(s)$ piecewise constant, minimizers of $E(v)$ and thus ground states do exist in $L^2(\mathbb{R})$. Further, arbitrary minimizing sequences have subsequences which, up to symmetries, converge to ground states. This result, along with the well-posedness in $L^2(\mathbb{R})$, implies that ground states are stable.

There are proofs of the existence of minimizers in $H^1(\mathbb{R})$ for $\alpha > 0$ and in

$L^2(\mathbb{R})$ for $\alpha = 0$, under very mild conditions on the dispersion profile. The approach in these proofs avoid the use of Lions' concentration compactness argument or Ekeland's variational principle, and can be extended to show the existence of minimizers for a family of nonlocal and nonlinear variational problems. For more on these proofs, see [5, 10, 12, 13, 16]. Finally, there have been discoveries about the smoothness and exponential decay estimates for solutions of the DMNLS equation both when the average dispersion is zero (see for example [6, 9, 11, 19]), and when the average dispersion is non-negative [9].

1.3 Statement of Main Results

In this thesis, we will establish some results on the well-posedness of the initial value problem and existence and stability of ground state solutions for the periodic DMNLS equation, an averaged version of (1.1) in spaces of periodic functions. Our main results are as follows:

In Chapter 2, in a similar manner as [1], we derive the periodic DMNLS equation from (1.1) under the assumption that $d(z)$ is periodic with period ϵB, and $\Delta(z)$ is periodic of period B. It turns out that the periodic DMNLS equation is given by

$$iu_z + \alpha u_{\tau\tau} + \frac{1}{B}\int_0^B T^{-1}_{D(s)}[|T_{D(s)}u|^2\ T_{D(s)}u]\ ds = 0.$$

We note that this equation is very similar to (1.2), but has a different operator $T_{D(s)}$.

In Chapter 3, we use a slightly altered version of the L^4 Strichartz estimate [4] on the operator $T_{D(s)}$ (Theorem 3.16), and a Banach contraction mapping argument to prove that the periodic DMNLS equation is globally well-posed in $H^r(\mathbb{T})$ for all non negative integer values of r and for all $\alpha \geq 0$ (Theorem

3.20), given the assumption below.

Assumption 1: The function $\Delta(s)$ is piecewise constant. That is, there exist some numbers $s_0, s_1, ..., s_n$ with $0 = s_0 < s_1 < ... < s_{n-1} < s_n = B$, such that for all $j \in 0, ..., n-1$, $\Delta(s) = c_j$ for $s \in [s_j, s_{j+1}]$.

An L^4 Strichartz estimate for the operator $S(t)$ was proved by Bourgain [4]. An interesting fact concerning the well-posedness in $L^2(\mathbb{T})$ is that in the case $\alpha > 0$, solutions $u(z, \tau)$ with initial data $u(0, \tau)$ in $L^2(\mathbb{T})$ are actually in $L^4(\mathbb{T})$ for almost all $z \geq 0$ (Theorem 3.24). To prove this, a Banach contraction mapping argument is also applied. The difference is that Bourgain $X^{s,b}$ estimates are required along with the same Strichartz estimate on the operator $T_{D(s)}$. These Bourgain estimates are similar to those explained in Ginibre [8], Kenig et al. [14], Tao [20], and Bourgain [4].

In Chapter 4, we consider the existence and stability of ground state solutions to (1.2). The periodic DMNLS equation has the following conserved functionals: the energy functional

$$E(v) = \int_0^{2\pi} \alpha |v_\tau|^2 d\tau - \frac{1}{2B} \int_0^B \int_0^{2\pi} |T_{D(s)} v|^4 \, d\tau \, ds,$$

and

$$P(v) = \int_0^{2\pi} |v|^2 d\tau.$$

For $\alpha > 0$, we consider the problem of minimizing $E(v)$ under the constraint that $P(v)$ be held constant, and show that minimizers exist in $H^1(\mathbb{T})$. Hence, ground states exist in $H^1(\mathbb{T})$ and are stable. The proof involves utilizing the Strichartz estimate on $T_{D(s)}$ to show that any minimizing sequence is bounded in $H^1(\mathbb{T})$. Then by the Rellich-Kondrachov theorem for compact manifolds, there exists a subsequence which will converge strongly in $L^2(\mathbb{T})$. Finally, we apply the weak lower semicontinuity of the norm in a Hilbert space to show that a subsequence converges strongly in $H^1(\mathbb{T})$ to a minimizer. A simple proof by contradiction is then used to prove stability.

For $\alpha = 0$ and $D(s) = s$, we provide a sufficient condition for existence and stability of ground states by considering the problem of maximizing $W(v) = -E(v)$ under the constraint that $P(v)$ be held constant. The condition for existence is expressed in terms of the following function. For $w \in L^2(\mathbb{T})$ define

$$A(\hat{w}) = 2\pi \sum_n \sum_{p\neq 0} \sum_{l\neq 0} \frac{i}{2lp} \left[e^{-2lpB} - 1\right] \hat{w}(n)\, \bar{\hat{w}}(n-l)\, \bar{\hat{w}}(n-p)\, \hat{w}(n-p-l).$$

Theorem 1.1. *Let $\lambda > 0$. Suppose there exists a function $w \in L^2$ with $P(w) = \lambda$, such that $A(\hat{w}) - 2\pi B\|\hat{w}\|_{l^4}^4 > 0$. Then there exists at least one maximizer for $W(v)$ in $L^2(\mathbb{T})$. Moreover, given the above condition, every maximizing sequence for J_λ (see Section 4.2) has a subsequence which, after being suitably translated in Fourier space, converges strongly in L^2 to some maximizer.*

The proof of the above theorem is a concentration compactness argument, and it requires a significant modification of the argument used by Kunze [15] in his proof of existence of non-periodic ground states. In essence, the proof shows that if we can find a function w that satisfies the above requirements, then for every maximizing sequence $\{u_j\}$, the sequence of its Fourier transforms $\mathcal{F}(u_j)[n]$ is tight in l^2. This then enables us to use the Rellich-Kondrachov theorem and the weak lower semicontinuity of the norm in a Hilbert space to show that a subsequence, after translations, will converge strongly in L^2 to a maximizer.

Lastly, in Chapter 5, we identify some values of B for which a function w exists that satisfies the requirements in Theorem 1.1, and other values of B for which no such w exists.

1.4 Preliminaries

Notation and Standard Results

The set of all real numbers and the set of all integers will be denoted by $\mathbb{R}$ and $\mathbb{Z}$, respectively. The torus will be denoted by $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

For any measurable function f on $I \subseteq \mathbb{R}$ and any $p \in [1, \infty)$, we define

$$\|f\|_{L^p(I)} = \left(\int_I |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

so that $L^p(I)$ denotes the space of all functions f such that $\|f\|_{L^p(I)}$ is finite. Likewise, if f is defined for $n \in \mathbb{Z}$, we denote l^p as the space of functions f such that

$$\|f\|_{l^p} := \left(\sum_{n \in \mathbb{Z}} |f(n)|^p \right)^{\frac{1}{p}}$$

is finite.

If $f(t, x)$ is a measurable function depending on the variables t and x, and f is defined for $(t, x) \in I \times J \subset \mathbb{R} \times \mathbb{T}$, for $p \in [1, \infty)$ and $q \in [1, \infty)$, we define

$$\|f\|_{L_t^p L_x^q(I \times J)} := \| \|f\|_{L_x^q(J)} \|_{L_t^p(I)} = \left(\int_I \left(\int_J |f(t, x)|^q \, dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}},$$

and $L_t^p L_x^q(I \times J)$ denotes the space of all f for which $\|f\|_{L_t^p L_x^q(I \times J)}$ is finite. If $p = q$, we will express $L_t^p L_x^p(I \times J)$ as $L_{t,x}^p(I \times J)$. In case I and J are all of $\mathbb{R}$ and $\mathbb{T}$ respectively, we exclude the reference to I and J.

We will also use c to denote any positive constant. However, if c is dependent on variable parameters such as ϵ, we record the dependence using the notation c_ϵ.

If $f(x)$ is integrable on $\mathbb{T}$, we define the Fourier transform of f in x by

$$\mathcal{F}_x(f)[n] = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-inx} f(x) \, dx,$$

for $n \in \mathbb{Z}$, and the inversion formula is given by

$$f(x) = \sum_{n\in\mathbb{Z}} \mathcal{F}_x(f)[n]e^{inx}.$$

If $g(t)$ is integrable on $\mathbb{R}$, we define the Fourier transform of g in t by

$$\mathcal{F}_t(g)[\kappa] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\kappa}g(t)\,dt,$$

for $\kappa \in \mathbb{R}$, and its inverse is given by

$$\mathcal{F}_t^{-1}(F)[t] = \int_{\mathbb{R}} e^{it\kappa}F(\kappa)\,d\kappa.$$

Likewise, if $h(t,x)$ is an integrable function on $\mathbb{R}\times\mathbb{T}$, we define the Fourier transform of h in t and x jointly by

$$\mathcal{F}(h)[\kappa,n] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} e^{-it\kappa}\left(\int_{\mathbb{T}} e^{-inx}h(t,x)\,dx\right)dt$$
$$= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-i(nx+t\kappa)}h(t,x)\,dx\,dt,$$

and its inverse is given by

$$\mathcal{F}^{-1}(H)[t,x] = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} e^{i(t\kappa+nx)}H(\kappa,n)\,d\kappa.$$

One can also define the Fourier transform and inverse Fourier transform of any tempered distribution in such a way that it agrees with the above definition. At times, we will express $\mathcal{F}_t(g)[\kappa]$, $\mathcal{F}_x(f)[n]$, and $\mathcal{F}(h)[\kappa,n]$ as $\hat{g}(\kappa)$, $\hat{f}(n)$, and $\hat{h}(\kappa,n)$ respectively.

The convolution of two functions f and g on $\mathbb{R}$ is defined by

$$f * g(\omega) := \int_{\mathbb{R}} f(\omega - z)g(z)dz.$$

The discrete convolution of two functions f and g on $\mathbb{Z}$ is defined by

$$f * g(n) := \sum_{m\in\mathbb{Z}} f(n-m)g(m).$$

For $s > 0$, the inhomogeneous Sobolev space on the torus, $H^s(\mathbb{T})$, is defined to be the space of all measurable functions u on $\mathbb{T}$ for which

$$\|u\|_{H^s(\mathbb{T})} = \|\langle n\rangle^s \hat{u}(n)\|_{l^2} = \left(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\hat{u}(n)|^2\right)^{\frac{1}{2}},$$

is finite. Here, $\langle n\rangle^s = (1+|n|)^s$. Similarly, we define $H^s(\mathbb{R})$ to be the space of all measurable functions u on $\mathbb{R}$ whose norm

$$\|u\|_{H^s(\mathbb{R})} = \|\langle \tau\rangle^s \hat{u}(\tau)\|_{L^2} = \left(\int_{\mathbb{R}} \langle \tau\rangle^{2s} |\hat{u}(\tau)|^2 \, d\tau\right)^{\frac{1}{2}}$$

is finite. It is also useful to note that when s is an integer, $\|u\|_{H^s(\mathbb{R})}$ can be defined as

$$\|u\|_{H^s(\mathbb{R})} = \sum_{i=0}^{s} \|u^{(i)}\|_{L^2}.$$

For $I \subset \mathbb{R}$, we define $C_t^0(I)$ as the space of measurable functions $f(t)$ on I such that

$$\|f\|_{C_t^0(I)} = \sup_{t\in I} |f(t)|$$

is finite.

We define $C_t^0 L_x^2(I \times \mathbb{T})$ as the space of measurable functions $f(t,x)$ on $I \times \mathbb{T}$ such that

$$\|f\|_{C_t^0 L_x^2(I\times\mathbb{T})} = \sup_{t\in I} \|f\|_{L_x^2}$$

is finite. We can also define $C_t^0 H_x^s(I \times \mathbb{T})$ in the same manner.

For $s,\, b \in \mathbb{R},$ the Sobolev space $H^{s,b}(\mathbb{R}\times\mathbb{T})$ is defined as the closure of the set of all Schwartz functions $u(t,x)$ on $\mathbb{R}\times\mathbb{T}$ with respect to the norm

$$\|u\|_{H^{s,b}} := \|\langle n\rangle^s\langle\kappa\rangle^b\mathcal{F}(u)[\kappa,n]\|_{L^2_\kappa l^2_n(\mathbb{R}\times\mathbb{Z})}.$$

Let $h:\mathbb{Z}\to\mathbb{R}$ be a continuous function, and $s,b\in\mathbb{R}$. The Bourgain type space $X^{s,b}_{\kappa=h(n)}(\mathbb{R}\times\mathbb{T})$ is defined to be the closure of the set of all Schwartz functions $u(t,x)$ on $\mathbb{R}\times\mathbb{T}$ with respect to the norm

$$\|u\|_{X^{s,b}_{\kappa=h(n)}(\mathbb{R}\times\mathbb{T})} := \|\langle n\rangle^s\langle\kappa-h(n)\rangle^b\mathcal{F}(u)[\kappa,n]\|_{L^2_\kappa l^2_n(\mathbb{R}\times\mathbb{Z})}.$$

We will also denote the space $X^{s,b}_{\kappa=h(n)}(\mathbb{R}\times\mathbb{T})$ as $X^{s,b}_{\kappa=h(n)}$.

Theorem 1.2. *(Banach algebra property). If $s>\frac{1}{2}$, then there exists a $c_s>0$, such that for every $u,v\in H^s(\mathbb{T})$*

$$\|uv\|_{H^s}\le c_s\|u\|_{H^s}\|v\|_{H^s},$$

thus $uv\in H^s(\mathbb{T})$.

Proof. For $s>\frac{1}{2}$,

$$\begin{aligned}
\|uv\|_{H^s} &= \|\langle n\rangle^s\widehat{uv}(n)\|_{l^2} \\
&= \|\langle n\rangle^s\left(\hat{u}*\hat{v}\right)\|_{l^2} \\
&= \left\|\langle n\rangle^s\sum_{m\in\mathbb{Z}}\hat{u}(n-m)\hat{v}(m))\right\|_{l^2} \\
&= \left\|\sum_{m\in\mathbb{Z}}\langle n\rangle^s\hat{u}(n-m)\hat{v}(m))\right\|_{l^2} \\
&\le c_s\left\|\sum_{m\in\mathbb{Z}}\left(\langle n-m\rangle^s+\langle m\rangle^s\right)\hat{u}(n-m)\hat{v}(m))\right\|_{l^2}
\end{aligned}$$

$$= c_s \left\| \sum_{m\in\mathbb{Z}} \langle n-m\rangle^s \hat{u}(n-m)\hat{v}(m) + \sum_{m\in\mathbb{Z}} \langle m\rangle^s \hat{u}(n-m)\hat{v}(m) \right\|_{l^2}$$

$$\leq c_s \left(\left\| \sum_{m\in\mathbb{Z}} \langle n-m\rangle^s \hat{u}(n-m)\hat{v}(m) \right\|_{l^2} + \left\| \sum_{m\in\mathbb{Z}} \langle m\rangle^s \hat{u}(n-m)\hat{v}(m) \right\|_{l^2} \right)$$

$$= c_s \left(\|(\langle n\rangle^s \hat{u}) * \hat{v}\|_{l^2} + \|(\langle n\rangle^s \hat{v}) * \hat{u}\|_{l^2} \right)$$

$$\leq c_s \left(\|u\|_{H^s} \|\hat{v}\|_{L^1} + \|\hat{u}\|_{L^1} \|v\|_{H^s} \right)$$

$$\leq c_s \left(\|u\|_{H^s} \|v\|_{H^s} + \|u\|_{H^s} \|v\|_{H^s} \right),$$

where we used Young's Convolution Inequality in the last few estimates. □

Chapter 2

Derivation of Periodic DMNLS Equation

Consider the following nondimensionalized NLS equation

$$iu_z + d(z)u_{\tau\tau} + |u|^2 u = 0, \tag{2.1}$$

where the dispersion coefficient function,$d(z)$, is periodic of period ϵB, has a large absolute value of size $O(1/\epsilon)$ with $\epsilon << 1$, and has mean value $\langle d \rangle$ which is $O(1)$. Hence, $d(z) = \langle d \rangle + \frac{\Delta(z/\epsilon)}{\epsilon}$ where $\Delta(z)$ is a periodic function of period B and mean value zero. That is,

$$\int_0^B \Delta(z)\, dz = 0.$$

Let $u = u(Z, \tau, s; \epsilon)$ where $s = \dfrac{z}{\epsilon}$, $Z = z$, and $\tau \in \mathbb{T}$. Then

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\frac{ds}{dz} + \frac{\partial u}{\partial Z}\frac{dZ}{dz} = \frac{1}{\epsilon}\frac{\partial u}{\partial s} + \frac{\partial u}{\partial Z}.$$

Thus (2.1) can be written as

$$i\left(\frac{1}{\epsilon}\frac{\partial u}{\partial s} + \frac{\partial u}{\partial Z}\right) + \left(\langle d \rangle + \frac{\Delta(s)}{\epsilon}\right)\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0$$

which implies that,

$$\frac{1}{\epsilon}\left(i\frac{\partial u}{\partial s}+\Delta(s)\frac{\partial^2 u}{\partial\tau^2}\right)+i\frac{\partial u}{\partial Z}+\langle d\rangle\frac{\partial^2 u}{\partial\tau^2}+|u|^2u=0.$$

If we formally expand u as a series in the small parameter ϵ, writing it as

$$u=u_0+\epsilon u_1+\epsilon^2 u_2+...,\tag{2.2}$$

we get that

$$\frac{1}{\epsilon}\left[i\left(\frac{\partial u_0}{\partial s}+\epsilon\frac{\partial u_1}{\partial s}+\epsilon^2\frac{\partial u_2}{\partial s}+...\right)+\Delta(s)\left(\frac{\partial^2 u_0}{\partial\tau^2}+\epsilon\frac{\partial^2 u_1}{\partial\tau^2}+\epsilon^2\frac{\partial^2 u_2}{\partial\tau^2}+...\right)\right]+$$
$$i\left(\frac{\partial u_0}{\partial Z}+\epsilon\frac{\partial u_1}{\partial Z}+\epsilon^2\frac{\partial u_2}{\partial Z}+...\right)+\langle d\rangle\left(\frac{\partial^2 u_0}{\partial\tau^2}+\epsilon\frac{\partial^2 u_1}{\partial\tau^2}+\epsilon^2\frac{\partial^2 u_2}{\partial\tau^2}+...\right)+$$
$$\left|u_0+\epsilon u_1+\epsilon^2 u_2+...\right|^2\left(u_0+\epsilon u_1+\epsilon^2 u_2+...\right)=0.$$

At leading order $O(1/\epsilon)$, we see that

$$i\frac{\partial u_0}{\partial s}+\Delta(s)\frac{\partial^2 u_0}{\partial\tau^2}=0.\tag{2.3}$$

We can solve the initial-value problem for (2.3) with initial data $u_0(Z,\tau,0)=U(Z,\tau)$ on $\mathbb{T}$ by means of the Fourier transform. Write $u_0=\sum_{n\in\mathbb{Z}}A_n(u_0)e^{in\tau}$, where $A_n(u_0)=\frac{1}{2\pi}\int_{\mathbb{T}}e^{-in\tau}u_0\,d\tau$ represents the Fourier coefficients of u_0. Taking the Fourier transform of (2.3) with respect to τ and evaluating at n, we find that

$$\begin{aligned}\frac{\partial A_n(u_0)}{\partial s}+in^2\Delta(s)A_n(u_0)&=0\\ A_n(u_0(Z,\tau,0))&=A_n(U).\end{aligned}\tag{2.4}$$

Since (2.4) is a linear ODE, we can solve it by using the integrating factor

$I = e^{\int_0^s in^2\Delta(\tilde{s})\, d\tilde{s}}$ to get

$$\frac{\partial(A_n(u_0)e^{in^2 D(s)})}{\partial s} = 0, \text{ where } D(s) = \int_0^s \Delta(\tilde{s})\, d\tilde{s}.$$

Integrating both sides gives:

$$A_n(u_0)e^{in^2 D(s)} = P(Z, n) \text{ for some function } P.$$

Evaluating at $s = 0$ and applying the initial condition in (2.4), we get that

$$A_n(u_0(Z, \tau, 0)) = P(Z, n) = A_n(U).$$

In other words,

$$A_n(u_0) = A_n(U)e^{-in^2 D(s)}. \tag{2.5}$$

Thus,

$$u_0 = \sum_{n\in\mathbb{Z}} A_n(u_0)e^{in\tau} = \sum_{n\in\mathbb{Z}} A_n(U)e^{-in^2 D(s)}e^{in\tau}. \tag{2.6}$$

For future purposes, we will view u_0 as

$$u_0 = T_{D(s)}U = \sum_{n\in\mathbb{Z}} A_n(U)e^{i(n\tau - n^2 D(s))}, \tag{2.7}$$

where the Fourier multiplier operator T_ζ is defined as $T_\zeta = e^{i\zeta\partial_\tau^2}$, so that the Fourier transform of $T_\zeta f$ with respect to τ evaluated at n is given by

$$\mathcal{F}(T_\zeta f)[n] = e^{-in^2\zeta}\mathcal{F}(f)[n].$$

Note that the inverse operator is also defined as $T_\zeta^{-1} = e^{-i\zeta\partial_\tau^2}$, so that

$$\mathcal{F}(T_\zeta^{-1} f)[n] = e^{in^2\zeta}\mathcal{F}(f)[n]. \tag{2.8}$$

Now at order $O(1)$,

$$i\frac{\partial u_1}{\partial s} + \Delta(s)\frac{\partial^2 u_1}{\partial \tau^2} = -i\frac{\partial u_0}{\partial Z} - \langle d\rangle \frac{\partial^2 u_0}{\partial \tau^2} - |u_0|^2 u_0, \tag{2.9}$$

where $u_1 = \sum_{n\in\mathbb{Z}} B_n(u_1)e^{in\tau}$ with $B_n(u_1) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}u_1\, d\tau$.

Again, taking the Fourier transform of (2.9) with respect to τ and evaluating at n, we find that

$$\begin{aligned}\frac{\partial B_n(u_1)}{\partial s} + in^2\Delta(s)B_n(u_1) = -\frac{\partial A_n(u_0)}{\partial Z} - in^2\langle d\rangle A_n(u_0)+ \\ i\mathcal{F}(|u_0|^2u_0)[n].\end{aligned} \tag{2.10}$$

However, (2.7) implies that

$$\mathcal{F}(|u_0|^2u_0)[n] = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\ |u_0|^2 u_0\, d\tau$$

$$= \frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\ |T_{D(s)}U|^2\ T_{D(s)}U\ d\tau.$$

Therefore, (2.10) becomes

$$\begin{aligned}\frac{\partial B_n(u_1)}{\partial s} + in^2\Delta(s)B_n(u_1) = -\frac{\partial A_n(u_0)}{\partial Z} - in^2\langle d\rangle A_n(u_0)+ \\ i\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\ |T_{D(s)}U|^2\ T_{D(s)}U\ d\tau.\end{aligned}$$

This is also an ODE with the same integrating factor as before, so can be solved to get that

$$\frac{\partial(B_n(u_1)e^{in^2D(s)})}{\partial s} = \Bigg[-\frac{\partial A_n(u_0)}{\partial Z} - in^2\langle d\rangle A_n(u_0) + i\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\ |T_{D(s)}U|^2\ T_{D(s)}U\ d\tau\Bigg] e^{in^2D(s)}.$$

Hence,

$$B_n(u_1)e^{in^2D(s)}\Big|_0^s = \int_0^s \Bigg[-\frac{\partial A_n(u_0)}{\partial Z} - in^2\langle d\rangle A_n(u_0) + i\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\ |T_{D(s)}U|^2\ T_{D(s)}U\ d\tau\Bigg] e^{in^2D(s)}\ ds.$$

Since $D(s)$ is periodic in s and Δ has zero mean, $A_n(u_0)$ is also periodic in s. To remove the secular term, we need

$$\int_0^B \Bigg[-\frac{\partial A_n(u_0)}{\partial Z} - in^2\langle d\rangle A_n(u_0) + i\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\ |T_{D(s)}U|^2\ T_{D(s)}U\ d\tau\Bigg] e^{in^2D(s)}\ ds = 0.$$

Using (2.5) we have that

$$\int_0^B \Bigg[-\frac{\partial(A_n(U)e^{-in^2D(s)})}{\partial Z} - in^2\langle d\rangle A_n(U)e^{-in^2D(s)}\Bigg] e^{in^2D(s)}\ ds + \int_0^B \Bigg[i\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}|T_{D(s)}U|^2\ T_{D(s)}U\ d\tau\Bigg] e^{in^2D(s)}\ ds = 0,$$

which implies that

$$\int_0^B -\frac{\partial(A_n(U))}{\partial Z} - in^2\langle d\rangle A_n(U)\ ds + \int_0^B \Bigg[i\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\ |T_{D(s)}U|^2\ T_{D(s)}U\ d\tau\Bigg] e^{in^2D(s)}\ ds = 0,$$

or

$$B\left[-\frac{\partial(A_n(U))}{\partial Z} - in^2\langle d\rangle A_n(U)\right] +$$
$$\int_0^B \left[i\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\;|T_{D(s)}U|^2\;T_{D(s)}U\;d\tau\right] e^{in^2D(s)}\;ds = 0.$$

Multiplying by $\dfrac{-ie^{in\tau}}{B}$, and summing over all n results in

$$i\frac{\partial}{\partial Z}\sum_{n\in\mathbb{Z}} A_n(U)e^{in\tau} - \langle d\rangle \sum_{n\in\mathbb{Z}} n^2 A_n(U)e^{in\tau} +$$
$$\frac{1}{B}\int_0^B \sum_{n\in\mathbb{Z}} \left[\frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\tau}\;|T_{D(s)}U|^2\;T_{D(s)}U\;d\tau\right] e^{i(n^2D(s)+n\tau)}\;ds = 0.$$

To conclude, we use (2.8) together with the fact that $U = \sum_{n\in\mathbb{Z}} A_n(U)e^{in\tau}$ and $U_{\tau\tau} = -\sum_{n\in\mathbb{Z}} n^2 A_n(U)e^{in\tau}$ to obtain the averaged equation

$$iU_Z + \langle d\rangle U_{\tau\tau} + \frac{1}{B}\int_0^B T_{D(s)}^{-1}[|T_{D(s)}U|^2\;T_{D(s)}U]\;ds = 0. \tag{2.11}$$

Equation (2.11) is a periodic version of the DMNLS equation (1.2). The above analysis shows that at least formally, it is necessary that $U(Z,\tau)$ satisfy equation (2.11) in order for the expansion in (2.2) to be valid. One expects that, in the limit as $\epsilon \to 0$, solutions of (2.11) should be good approximations to solutions of (2.1).

Chapter 3

Well-posedness of the Initial-Value Problem

From now on, the variables Z and τ in the previous section will be denoted by t and x respectively. We begin with the following lemmas that are required for the proof of the well-posedness results in Theorems 3.20 and 3.24.

Lemma 3.1. *For every $u \in L^2(\mathbb{T})$,*

$$\overline{T_{D(s)}u} = T_{D(s)}^{-1}\bar{u}.$$

Proof.

$$\begin{aligned}
\overline{T_{D(s)}u} &= \sum_{n\in\mathbb{Z}} \overline{\hat{u}(n)e^{i(nx-n^2D(s))}} \\
&= \sum_{n\in\mathbb{Z}} \overline{\hat{u}(n)}e^{-i(nx-n^2D(s))} \\
&= \sum_{k\in\mathbb{Z}} \hat{\bar{u}}(k)e^{i(kx+k^2D(s))} \text{ with } k=-n, \\
&= T_{D(s)}^{-1}\bar{u}.
\end{aligned}$$

□

Lemma 3.2. *For $u, v \in L^2(\mathbb{T})$,*

$$\langle u, T_{D(s)}^{-1}v\rangle = \langle T_{D(s)}u, v\rangle.$$

Therefore, $T_{D(s)}$ is a unitary operator on $L^2(\mathbb{T})$.

Proof. By Parseval's identity, we can show that

$$\begin{aligned}
\langle u, T_{D(s)}^{-1} v\rangle &= \int_{\mathbb{T}} u\, \overline{T_{D(s)}^{-1} v}\, dx \\
&= 2\pi \sum_{n\in\mathbb{Z}} \hat{u}\, \overline{\widehat{T_{D(s)}^{-1} v}} \\
&= 2\pi \sum_{n\in\mathbb{Z}} \hat{u}(n)\, \overline{\hat{v}(n) e^{in^2 D(s)}} \\
&= 2\pi \sum_{n\in\mathbb{Z}} \hat{u}(n)\, \overline{\hat{v}(n)} e^{-in^2 D(s)} \\
&= 2\pi \sum_{m\in\mathbb{Z}} \bar{\hat{v}}\, \widehat{T_{D(s)} u} \\
&= \int_{\mathbb{T}} T_{D(s)} u\, \bar{v}\, dx, \\
&= \langle T_{D(s)} u, v\rangle.
\end{aligned}$$

□

Lemma 3.3. *For $r \in \mathbb{R}$ and for $u \in H^r(\mathbb{T})$,*

$$\left\| T_{D(s)} u \right\|_{H^r} = \|u\|_{H^r}.$$

Proof. $\left\| T_{D(s)} u \right\|_{H^r} = \left\| \langle n\rangle^r\, e^{-in^2 D(s)} \hat{u}(n) \right\|_{l^2} = \| \langle n\rangle^r\, \hat{u}(n)\|_{l^2} = \|u\|_{H^r}.$

□

Remark 3.4. *A similar proof can be used to show that $\|S(\alpha t) u\|_{H^r} = \|u\|_{H^r}$ for $u \in H^r(\mathbb{T})$.*

Lemma 3.5. *Suppose $\Delta(s)$ satisfies Assumption 1. Then for $u \in H^r$ with $r \geq 1$,*

$$T_{D(s)} u \in C_s^0 H_x^r([0, B] \times \mathbb{T}).$$

Proof. Observe that

$$\left\|T_{D(s)}u - T_{D(s_0)}u\right\|_{H^r}^2 = \sum_{n\in\mathbb{Z}} \langle n\rangle^{2r} |\hat{u}(n)|^2 \left|e^{-in^2D(s)} - e^{-in^2D(s_0)}\right|^2.$$

Since $D'(s)$ is piecewise constant, $D(s)$ is continuous. The function e^x is also continuous for all values of x, thus $e^{-in^2D(s)}$ is continuous. This implies that $\lim_{s\to s_0} e^{-in^2D(s)} - e^{-in^2D(s_0)} = 0$ for $s \in [0, B]$. In addition, since

$$\left|e^{-in^2D(s)} - e^{-in^2D(s_0)}\right|^2 \le 4$$

for all s, then

$$\left\|T_{D(s)}u - T_{D(s_0)}u\right\|_{H^r}^2 \le 4\|u\|_{H^r}^2 < \infty.$$

So by the Lebesgue Dominated Convergence Theorem,

$$\lim_{s\to s_0} \left\|T_{D(s)}u - T_{D(s_0)}u\right\|_{H^r}^2 = \sum_{n\in\mathbb{Z}} \langle n\rangle^{2r} |\hat{u}(n)|^2 \lim_{s\to s_0} \left|e^{-in^2D(s)} - e^{-in^2D(s_0)}\right|^2$$

$$= 0,$$

concluding the proof of the theorem. □

Let us rewrite the periodic DMNLS equation. We will write (2.11) as

$$iu_t + \alpha u_{xx} + F(u) = 0, \tag{3.1}$$

with $\alpha \in \mathbb{R}$, and

$$F(u) = \frac{1}{B}\int_0^B T_{D(s)}^{-1}[|T_{D(s)}u|^2\, T_{D(s)}u]\, ds.$$

Definition 3.6. *Suppose $u_0 \in L^2(\mathbb{T})$. We say $u(t,x) \in C_t^0L_x^2(I \times \mathbb{T})$ is a strong solution of (3.1) with initial data u_0 if* *(a) for all $t \in I$,*

$$u(t,x) = S(\alpha t)u_0 + i\int_0^t S(\alpha(t-t'))F(u(t',x))\,dt'$$

with $S(\alpha t)$ as previously defined, and

(b) if we fix $t \in I$ and let $u = u(t,\cdot)$, then

$$|T_{D(s)}u|^2\, T_{D(s)}u \in C_s^0 L_x^2([0,B]\times\mathbb{T}),$$

making $F(u)$ well-defined.

3.1 Time Independent Quantities

let $P : L^2 \to \mathbb{R}$ and $E : L^2 \to \mathbb{R}$ be defined as

$$P(u) = \int_{\mathbb{T}} |u|^2\ dx,$$

and

$$E(u) = \int_{\mathbb{T}} \alpha|u_x|^2 dx - \frac{1}{2B}\int_0^B \int_{\mathbb{T}} |T_{D(s)}u|^4\ dx\ ds.$$

Theorem 3.7. *Suppose $u(t,x) \in C_t^0 H_x^r(I\times\mathbb{T})$ is a strong solution to (3.1) for r large enough. Then $P(u(t,x))$ and $E(u(t,x))$ are independent of t.*

Proof. Note that

$$u_t = i\alpha u_{xx} + i\tfrac{1}{B}\int_0^B T_{D(s)}^{-1}[|T_{D(s)}u|^2 T_{D(s)}u]\ ds.$$

For $r \geq 1$, applying the Banach Algebra property for $H^r(\mathbb{T})$ and Lemma 3.5, we get that $u_t \in C_t^0 H_x^{r-2}(I\times\mathbb{T})$. Also, since u is periodic for fixed t, the value of u and its derivatives with respect to x at the end points are the same. Now for u, a strong solution to (3.1), we see that

$$\int_{\mathbb{T}} \bar{u}\left(iu_t + \alpha u_{xx} + F(u)\right)\ dx = 0 \tag{3.2}$$

and

$$\int_{\mathbb{T}} u\left(-i\bar{u}_t + \alpha \bar{u}_{xx} + \overline{F}(u)\right)\ dx = 0. \tag{3.3}$$

Subtract (3.3) from (3.2) to get

$$0 = i\int_{\mathbb{T}} u_t\bar{u} + \bar{u}_t u\ dx + \alpha\int_{\mathbb{T}} \bar{u}u_{xx} - u\bar{u}_{xx}\ dx + \int_{\mathbb{T}} \bar{u}F - u\bar{F}\ dx$$

$$= i\frac{d}{dt}\int_{\mathbb{T}} \frac{1}{2}|u|^2\ dx + \alpha\int_{\mathbb{T}} \bar{u}u_{xx} - u\bar{u}_{xx}\ dx + \int_{\mathbb{T}} \bar{u}F - u\bar{F}\ dx.$$

If $u \in H^r(\mathbb{T})$ for r sufficiently large, then by applying integration by parts, we can address the second integral as follows:

$$\int_{\mathbb{T}} \bar{u}u_{xx} - u\bar{u}_x\ dx \ = \ \bar{u}u_x - u\bar{u}_x\ |_0^{2\pi}\ + \ \int_{\mathbb{T}} \bar{u}_x u_x - u_x\bar{u}_x\ dx \ = \ 0.$$

To calculate the third integral, note that, by Lemma 3.2,

$$\int_{\mathbb{T}} \bar{u}F\ dx = \frac{1}{B}\int_{\mathbb{T}} \bar{u}\int_0^B T_{D(s)}^{-1}[|T_{D(s)}u|^2\ T_{D(s)}u]\ ds\ dx$$

$$= \frac{1}{B}\int_0^B \int_{\mathbb{T}} \bar{u}\ T_{D(s)}^{-1}[|T_{D(s)}u|^2\ T_{D(s)}u]\ dx\ ds$$

$$= \frac{1}{B}\int_0^B \int_{\mathbb{T}} \overline{T_{D(s)}u}|T_{D(s)}u|^2\ T_{D(s)}u\ dx\ ds$$

$$= \frac{1}{B}\int_0^B \int_{\mathbb{T}} |T_{D(s)}u|^4\ dx\ ds.$$

Then by taking the complex conjugate, we get that

$$\int_{\mathbb{T}} u\bar{F}\ dx = \frac{1}{B}\int_0^B \int_{\mathbb{T}} |T_{D(s)}u|^4\ dx\ ds.$$

Thus,

$$\int_{\mathbb{T}} \bar{u}F - u\bar{F}\ dx = 0,$$

and to conclude,

$$\frac{d}{dt}\int_{\mathbb{T}} |u|^2\ \ dx = 0.$$

Hence $P(u) = \int_{\mathbb{T}} |u|^2\ \ dx$ is independent of t.

To show that $E(u)$ is independent of t, by the product rule, we can write that

$$\frac{dE(u)}{dt} = \frac{d}{dt}\left[\int_{\mathbb{T}} \alpha|u_x|^2 dx - \frac{1}{2B}\int_0^B \int_{\mathbb{T}} |T_{D(s)}u|^4\ dx\ ds\right]$$

$$= \alpha\int_{\mathbb{T}} u_x\bar{u}_{xt} + u_{xt}\bar{u}_x\ dx - \frac{1}{B}\int_{\mathbb{T}}\int_0^B T_{D(s)}u\ T_{D(s)}u_t(\overline{T_{D(s)}u})^2\ ds\ dx$$

$$-\frac{1}{B}\int_{\mathbb{T}}\int_0^B (T_{D(s)}u)^2\ \overline{T_{D(s)}u_t\ T_{D(s)}u}\ ds\ dx$$

$$= \int_{\mathbb{T}} u_t\left\{-\alpha\bar{u}_{xx} - \frac{1}{B}\int_0^B \overline{T_{D(s)}^{-1}[|T_{D(s)}u|^2\ T_{D(s)}u]}\ ds\right\} dx + c.c.,$$

where we used integration by parts and Lemma 3.2 in the last step, and *c.c.* represents the complex conjugate. However,

$$-\alpha\bar{u}_{xx} - \int_0^B \overline{T_{D(s)}^{-1}[|T_{D(s)}u|^2\ T_{D(s)}u]}\ ds = -i\bar{u}_t.$$

Therefore,

$$\frac{dE(u)}{dt} = \int_{\mathbb{T}} -iu_t\bar{u}_t + iu_t\bar{u}_t\ dx = 0.$$

$\square$

3.2 Estimates for Linear and Nonlinear Terms

Lemma 3.8. *Suppose* $\alpha \in \mathbb{R}$, $r \in \mathbb{R}$, *and* $T > 0$. *For all* $u_0(x) \in H^r(\mathbb{T})$, *we have*

$$\|S(\alpha t)u_0\|_{C_t^0 H_x^r([0,T]\times\mathbb{T})} = \|u_0\|_{H_x^r}. \tag{3.4}$$

Proof. We have

$$\|S(\alpha t)u_0\|_{C_t^0 H_x^r([0,T]\times\mathbb{T})} = \sup_{t\in[0,T]} \|S(\alpha t)u_0\|_{H_x^r}.$$

Apply Remark 3.4 to get the desired result. $\square$

Lemma 3.9. *Suppose* $\alpha \in \mathbb{R}$, $r \in \mathbb{R}$, *and* $T > 0$. *For all* $F \in C_t^0 H_x^r\left([0,T]\times\mathbb{T}\right)$, *we have*

$$\left\|\int_0^t S(\alpha(t-t'))F(t',x)\,dt'\right\|_{C_t^0 H_x^r([0,T]\times\mathbb{T})} \le T\|F\|_{C_t^0 H_x^r([0,T]\times\mathbb{T})}. \tag{3.5}$$

Proof. We have

$$\begin{aligned}
\left\|\int_0^t S(\alpha(t-t'))F(t',x)\,dt'\right\|_{C_t^0 H_x^r} &= \sup_{t\in[0,T]}\left\|\int_0^t S(\alpha(t-t'))F(t',x)\,dt'\right\|_{H_x^r} \\
&\le \sup_{t\in[0,T]}\int_0^t \|S(\alpha(t-t'))F(t',x)\|_{H_x^r}\,dt' \\
&= \sup_{t\in[0,T]}\int_0^t \|F(t',x)\|_{H_x^r}\,dt' \\
&\le \sup_{t\in[0,T]}\int_0^t \sup_{t'\in[0,T]}\|F(t',x)\|_{H_x^r}\,dt' \\
&= T\sup_{t'\in[0,T]}\|F(t',x)\|_{H_x^r}
\end{aligned}$$

$$= T\,\|F\|_{C_t^0 H_x^r([0,T]\times\mathbb{T})}.$$

□

Corollary 3.10. *Suppose* $T > 0$*. For all* $F \in C_t^0 L_x^2\left([0,T]\times\mathbb{T}\right),$ *we have*

$$\left\|\int_0^t F(t',x)\,dz'\right\|_{C_t^0 L_x^2([0,T]\times\mathbb{T})} \le T\|F\|_{C_t^0 L_x^2([0,T]\times\mathbb{T})}. \tag{3.6}$$

Proof. The proof is similar to Lemma 3.9. □

In the next few lemmas, we let $\eta(t) \in C_0^\infty(\mathbb{R})$ be a function that is supported on $[-2,2]$ and equals 1 on $[-1,1]$, and set $\eta_T(t) = \eta\left(\frac{t}{T}\right)$ for $T > 0$.

Lemma 3.11. *Suppose* $T > 0$ *and* $\alpha \in \mathbb{R}$*. For all* $u_0(x) \in L^2(\mathbb{T})$*, we have*

$$\|\eta_T(t)S(\alpha t)u_0\|_{C_t^0 L_x^2} \le \|u_0\|_{L_x^2}. \tag{3.7}$$

Proof. We have

$$\begin{aligned}
\|\eta_T(t)S(\alpha t)u_0\|_{C_t^0 L_x^2} &= \sup_{t\in\mathbb{R}} \|\eta_T(t)S(\alpha t)u_0\|_{L_x^2} \\
&= \sqrt{2\pi}\sup_{t\in\mathbb{R}} \|\mathcal{F}_x(\eta_T(t)S(\alpha t)u_0)[n]\|_{l_n^2} \\
&= \sqrt{2\pi}\sup_{t\in\mathbb{R}} \eta_T(t)\|\mathcal{F}_x(S(\alpha t)u_0)[n]\|_{l_n^2} \\
&= \sqrt{2\pi}\sup_{t\in\mathbb{R}} \eta_T(t)\|e^{-i\alpha n^2 t}\mathcal{F}_x(u_0)[n]\|_{l_n^2} \\
&\le \sqrt{2\pi}\|\mathcal{F}_x(u_0)\|_{l_n^2} \\
&= \|u_0\|_{L_x^2}.
\end{aligned}$$

$\square$

Definition 3.12. *Let $\mathcal{Y}$ be the space of all functions $f(t,x)$ whose derivatives of all orders with respect to t and x exist, and such that for all m, n, $k \in \mathbb{N}$ there exists $c_{m,k,n}$ such that $|t|^k\,|\partial_t^m \partial_x^n f(t,x)| \leq c_{m,k,n}$ for all $(t,x) \in \mathbb{R}\times\mathbb{T}$.*

Lemma 3.13. *There exists $c > 0$ such that for all $T \in (0,1]$ and all $F \in \mathcal{Y}$*

$$\left\| \eta_T(t) \int_0^t S(\alpha(t-t'))F(t',x)\,dt' \right\|_{C_t^0 L_x^2} \leq c\,T^{\frac{1}{16}} \|F\|_{X^{0,-3/8}_{\kappa=-\alpha n^2}}. \tag{3.8}$$

Proof. Let $\omega(z) \in C_0^\infty$ be a bump function that is supported on $[-3,3]$ and equals 1 on $[-2,2]$. If we define $\omega_T(t) = \omega\left(\frac{t}{T}\right)$, then

$$\left\| \eta_T(t) \textstyle\int_0^t S(\alpha(t-t'))F(t',x)\,dt' \right\|_{C_t^0 L_x^2}$$

$$= \left\| \eta_T(t) \textstyle\int_0^t S(\alpha(t-t'))\;\omega_T(t')F(t',x)\,dt' \right\|_{C_t^0 L_x^2}$$

$$= \sup_{t\in\mathbb{R}} \left\| \eta_T(t) \int_0^t S(\alpha(t-t'))\;\omega_T(t')F(t',x)\,dt' \right\|_{L_x^2}$$

$$= \sqrt{2\pi}\sup_{t\in\mathbb{R}} \left\| \mathcal{F}_x\left(\eta_T(t) \int_0^t S(\alpha(t-t'))H(t',x)\,dt' \right) \right\|_{l_n^2},$$

where $H(t,x) = \omega_T(t)F(t,x)$. Let $\mathcal{H}(\kappa,n) = \mathcal{F}(H)[\kappa,n]$, and observe that

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{-inx} H(t',x)\,dx = \int_{\mathbb{R}} e^{it'\kappa}\mathcal{H}(\kappa,n)\,d\kappa = \mathcal{F}_{t'}^{-1}(\mathcal{H})[t'].$$

Therefore, for any fixed $b > \dfrac{1}{2}$, we have

$$\left\| i\eta_T(t) \int_0^t S(\alpha(t-t'))F(t',x)\,dt' \right\|_{C_t^0 L_x^2}$$

$$= \sqrt{2\pi} \sup_{t\in\mathbb{R}} \left\| \eta_T(t) \int_0^t \mathcal{F}_x(S(\alpha(t-t'))H)[n] \; dt' \right\|_{l_n^2}$$

$$= \sqrt{2\pi} \sup_{t\in\mathbb{R}} \left\| \eta_T(t) \int_0^t e^{-i\alpha n^2(t-t')} \mathcal{F}_x(H)[n] \; dt' \right\|_{l_n^2}$$

$$= \sqrt{2\pi} \sup_{t\in\mathbb{R}} \left\| \eta_T(t) \int_0^t e^{-\alpha n^2(t-t')} \mathcal{F}_{t'}^{-1}(\mathcal{H})[t'] \; dt' \right\|_{l_n^2}$$

$$= \sup_{t\in\mathbb{R}} \left\| \eta_T(t) \int_{\mathbb{R}} \int_0^t e^{-i\alpha n^2(t-t')+it'\kappa} \; \mathcal{H}(\kappa, n) \; dt' \; d\kappa \right\|_{l_n^2}$$

$$= \sqrt{2\pi} \sup_{t\in\mathbb{R}} \left\| \eta_T(t) \int_{\mathbb{R}} e^{-i\alpha n^2 t} \int_0^t e^{it'(\kappa+\alpha n^2)} \mathcal{H}(\kappa, n) \; dt' \; d\kappa \right\|_{l_n^2}$$

$$= \sqrt{2\pi} \sup_{t\in\mathbb{R}} \left\| \eta_T(t) \int_{\mathbb{R}} e^{-i\alpha n^2 t} \left(\int_0^t e^{it'(\kappa+\alpha n^2)} dt' \right) \mathcal{H}(\kappa, n) \; d\kappa \right\|_{l_n^2}$$

$$\leq \sqrt{2\pi} \left\| \int_{\mathbb{R}} \langle \kappa + \alpha n^2 \rangle^{-1} |\mathcal{H}(\kappa, n)| \; d\kappa \right\|_{l_n^2}$$

$$= \sqrt{2\pi} \left\| \int_{\mathbb{R}} \langle \kappa + \alpha n^2 \rangle^{-1+b-b} |\mathcal{H}(\kappa, n)| \; d\kappa \right\|_{l_n^2}$$

$$\leq c_b \; \| \langle \kappa + \alpha n^2 \rangle^{b-1} \mathcal{H}(\kappa, n) \|_{L_\kappa^2 l_n^2}$$

$$= c_b \|H\|_{X^{0,b-1}_{\kappa=-\alpha n^2}},$$

where in the last estimate, the Cauchy-Schwarz inequality was applied. Specifically, if $b = \dfrac{9}{16}$ then

$$\left\| i\eta_T(t) \int_0^t S(\alpha(t-t'))F(t', x) \; dz' \right\|_{C_t^0 L_x^2} \leq c \; \|\omega_T(t) F(t, x)\|_{X^{0,\frac{-7}{16}}_{\kappa=-\alpha n^2}}, \tag{3.9}$$

where c is an absolute constant. Applying Lemma 2.11 from [20] results in

$$\|\omega_T(t) F(t, x)\|_{X^{0,\frac{-7}{16}}_{\kappa=-\alpha n^2}} = \left\| \omega \left(\tfrac{t}{T} \right) F(t, x) \right\|_{X^{0,-\frac{7}{16}}_{\kappa=-\alpha n^2}}$$

$$\leq c\, T^{\frac{1}{16}} \|F\|_{X^{0,-\frac{3}{8}}_{\kappa=-\alpha n^2}},$$

where c depends only on ω. Combine this estimate with (3.9), to conclude the proof of Lemma 3.13. □

The following estimates can also be found in [8] and [14].

Lemma 3.14. *Suppose $b > \frac{1}{2}$. Then, there exists $c > 0$ such that for all $T \in [0, 1]$,*

$$\|\eta_T(t) S(\alpha t) u_0\|_{X^{s,b}_{\kappa=-\alpha n^2}} \leq c\, T^{\frac{1}{2}-b} \|u_0\|_{H^s}. \tag{3.10}$$

Proof. Note that

$$\eta_T(t) S(\alpha t) u_0 = \eta\left(\frac{t}{T}\right) \sum_{n \in \mathbb{Z}} e^{inx - i\alpha n^2 t} \hat{u_0}(n)$$

$$= \int_{\mathbb{R}} e^{i\frac{t}{T}\omega}\ \hat{\eta}(\omega)\ d\omega \sum_{n \in \mathbb{Z}} e^{inx - i\alpha n^2 t} \hat{u_0}(n)$$

$$= \sum_{n \in \mathbb{Z}} e^{inx} \hat{u_0}(n) \int_{\mathbb{R}} e^{it\left(\frac{\omega}{T} - \alpha n^2\right)}\ \hat{\eta}(\omega)\ d\omega$$

$$= \sum_{n \in \mathbb{Z}} e^{inx} \hat{u_0}(n)\ T \int_{\mathbb{R}} e^{it\kappa}\ \hat{\eta}(T(\kappa + \alpha n^2))\ d\kappa$$

$$= T \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} e^{inx} e^{it\kappa} \hat{u_0}(n)\ \hat{\eta}(T(\kappa + \alpha n^2))\ d\kappa,$$

where $\hat{\eta}(\omega)$ represents the Fourier transform of $\eta(t)$ evaluated at ω, and $\kappa = \frac{\omega}{T} - \alpha n^2$. Therefore,

$$\|\eta_T(t) S(\alpha t) u_0\|^2_{X^{s,b}_{\kappa=-\alpha n^2}}$$

$$= T^2 \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} \left(1 + \left|\kappa + \alpha n^2\right|\right)^{2b} |\hat{u_0}(n)|^2\ \left|\hat{\eta}\left(T\left(\kappa + \alpha n^2\right)\right)\right|^2\ d\kappa$$

$$= \sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |\hat{u_0}(n)|^2 \left(T^2 \int_{\mathbb{R}} \left(1+\left|\kappa+\alpha n^2\right|\right)^{2b} \left|\hat{\eta}\left(T\left(\kappa+\alpha n^2\right)\right)\right|^2 \, d\kappa \right)$$

$$= \sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |\hat{u_0}(n)|^2 J_n.$$

Since $b > \frac{1}{2}$, we have that

$$J_n \leq c\Big[T^2 \int_{\mathbb{R}} \left|\hat{\eta}\left(T\left(\kappa+\alpha n^2\right)\right)\right|^2 \, d\kappa + T^2 \int_{\mathbb{R}} \left|\kappa+\alpha n^2\right|^{2b} \left|\hat{\eta}\left(T\left(\kappa+\alpha n^2\right)\right)\right|^2 \, d\kappa \Big]$$

$$= c_b \left[T \int_{\mathbb{R}} |\hat{\eta}(\omega)|^2 \, d\kappa + T^{1-2b} \int_{\mathbb{R}} |\omega|^{2b} |\hat{\eta}(\omega)|^2 \, d\kappa \right]$$

$$\leq c_b \left[T \|\hat{\eta}\|_{L^2}^2 + T^{1-2b} \|\eta\|_{H^b}^2 \right]$$

$$\leq c_b \, T^{1-2b},$$

where $\omega = T(\kappa+\alpha n^2)$, and c only depends on b. Thus

$$\|\eta_T(t) S(\alpha t) u_0\|^2_{X^{s,b}_{\kappa=-\alpha n^2}} \leq c \, T^{1-2b} \sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |\hat{u_0}(n)|^2 .$$

Taking the square root on both sides results in (3.10), and completes the proof of Lemma 3.14. □

Lemma 3.15. *Let* $-\frac{1}{2} < b' \leq 0 \leq b \leq 1+b$. *There exists* $c > 0$ *such that for all* $T \in [0,1]$ *and all* $F \in \mathcal{Y}$,

$$\left\| \eta_T(t) \int_0^t S(\alpha(t-t')) F(t',x) \, dt' \right\|_{X^{s,b}_{\kappa=-\alpha n^2}} \leq c \, T^{1+b'-b} \|F\|_{X^{s,b'}_{\kappa=-\alpha n^2}}. \quad (3.11)$$

Proof. To establish (3.11), we need to first prove the following inequality

$$\left\| \eta_T(t) \int_0^t f(t')\, dt' \right\|_{H^b} \le c\, T^{1+b'-b} \, \|f\|_{H^{b'}} \,. \tag{3.12}$$

We write

$$\eta_T(t) \int_0^t f(t')\, dt' = \eta_T(t) \int_0^t \int_{\mathbb{R}} e^{it'\kappa} \hat{f}(\kappa)\, d\kappa\, dt'$$

$$= \eta_T(t) \int_{\mathbb{R}} \hat{f}(\kappa) \int_0^t e^{it'\kappa}\, dt'\, d\kappa$$

$$= \eta_T(t) \int_{\mathbb{R}} \hat{f}(\kappa) \frac{1}{i\kappa} (e^{it\kappa} - 1)\, d\kappa$$

$$= \eta_T(t) \int_{|\kappa|T \ge 1} (i\kappa)^{-1} (e^{it\kappa} - 1) \hat{f}(\kappa)\, d\kappa + \eta_T(t) \int_{|\kappa|T \le 1} (i\kappa)^{-1} \left(e^{it\kappa} - 1 \right) \hat{f}(\kappa)\, d\kappa$$

$$= \eta_T(t) \int_{|\kappa|T \ge 1} (i\kappa)^{-1} (e^{it\kappa} - 1) \hat{f}(\kappa)\, d\kappa + \eta_T(t) \int_{|\kappa|T \le 1} (i\kappa)^{-1} \left(\sum_{n \ge 0} \frac{(it\kappa)^n}{n!} - 1 \right) \hat{f}(\kappa)\, d\kappa$$

$$= \eta_T(t) \int_{|\kappa|T \ge 1} (i\kappa)^{-1} (e^{it\kappa} - 1) \hat{f}(\kappa)\, d\kappa + \eta_T(t) \int_{|\kappa|T \le 1} (i\kappa)^{-1} \left(\sum_{n \ge 1} \frac{(it\kappa)^n}{n!} \right) \hat{f}(\kappa)\, d\kappa$$

$$= \eta_T(t) \sum_{n \ge 1} \frac{(t)^n}{n!} \int_{|\kappa|T \le 1} (i\kappa)^{n-1} \hat{f}(\kappa)\, d\kappa \; + \eta_T(t) \int_{|\kappa|T \ge 1} (i\kappa)^{-1} e^{it\kappa} \hat{f}(\kappa)\, d\kappa$$

$$- \eta_T(t) \int_{|\kappa|T \ge 1} (i\kappa)^{-1} \hat{f}(\kappa)\, d\kappa$$

$$= I + J - K.$$

Therefore,

$$\left\| \eta_T(t) \int_0^t f(t')\, dt' \right\|_{H^b} \leq \|I\|_{H^b} + \|J\|_{H^b} + \|K\|_{H^b} \,. \tag{3.13}$$

We work on the norms of each of the terms on the right hand side of (3.13). Utilizing the triangle inequality, the fact that $|\kappa| \leq T^{-1}$, and the Cauchy-Schwarz inequality respectively, we have that

$$\begin{aligned}
\|I\|_{H^b} &= \left\| \eta_T(t) \sum_{n\geq 1} \frac{(t)^n}{n!} \int_{|\kappa|T\leq 1} (i\kappa)^{n-1} \hat{f}(\kappa)\, d\kappa \right\|_{H^b} \\
&\leq \sum_{n\geq 1} \frac{1}{n!} \left\| z^n \eta_T(t) \int_{|\kappa|T\leq 1} (i\kappa)^{n-1} \hat{f}(\kappa)\, d\kappa \right\|_{H^b} \\
&= \sum_{n\geq 1} \frac{1}{n!} \left| \int_{|\kappa|T\leq 1} (i\kappa)^{n-1} \hat{f}(\kappa)\, d\kappa \right| \|t^n \eta_T(t)\|_{H^b} \\
&\leq \sum_{n\geq 1} \frac{1}{n!} \|t^n \eta_T(t)\|_{H^b} \int_{|\kappa|T\leq 1} |i\kappa|^{n-1} \left|\hat{f}(\kappa)\right|\, d\kappa \\
&\leq \sum_{n\geq 1} \frac{1}{n!} \|t^n \eta_T(t)\|_{H^b} \int_{|\kappa|T\leq 1} T^{1-n} \langle\kappa\rangle^{b'-b'} \left|\hat{f}(\kappa)\right|\, d\kappa \\
&\leq \sum_{n\geq 1} \frac{1}{n!} \|t^n \eta_T(t)\|_{H^b} T^{1-n} \left\| \langle\kappa\rangle^{b'} \hat{f}(\kappa) \right\|_{L^2_{|\kappa|T\leq 1}} \left\| \langle\kappa\rangle^{-b'} \right\|_{L^2_{|\kappa|T\leq 1}} \\
&\leq \sum_{n\geq 1} \frac{1}{n!} \|t^n \eta_T(t)\|_{H^b} T^{1-n} \|f\|_{H^{b'}} \left\| \langle\kappa\rangle^{-b'} \right\|_{L^2_{|\kappa|T\leq 1}} \,.
\end{aligned} \tag{3.14}$$

Note also that since $|\kappa| \leq T^{-1}$, then $(1+|\kappa|)^{-2b'} \leq (1+|T^{-1}|)^{-2b'} \leq c_{b'}\, T^{2b'}$. So,

$$\left\| \langle\kappa\rangle^{-b'} \right\|_{L^2_{|\kappa|T\leq 1}} = \left(\int_{|\kappa|T\leq 1} \langle\kappa\rangle^{-2b'}\, d\kappa \right)^{\frac{1}{2}}$$

$$\leq c_{b'} \left(\int_{|\kappa| T \leq 1} T^{2b'} \, d\kappa \right)^{\frac{1}{2}}$$

$$\leq c_{b'} \, T^{b'} \left(\int_{-T^{-1}}^{T^{-1}} d\kappa \right)^{\frac{1}{2}}$$

$$\leq c_{b'} \, T^{b'-\frac{1}{2}}. \tag{3.15}$$

In addition,

$$\|t^n \eta_T(t)\|_{H^b} = \left\| T^n \left(\frac{t}{T}\right)^n \eta\left(\frac{t}{T}\right) \right\|_{H^b}$$

$$= T^n \left\| \left(\frac{t}{T}\right)^n \eta\left(\frac{t}{T}\right) \right\|_{H^b}$$

$$= T^n \left\| g_n\left(\frac{t}{T}\right) \right\|_{H^b},$$

where $g_n(t) = t^n \eta(t)$. By the definition of Fourier transform,

$$g_n\left(\frac{t}{T}\right) = \int_{\mathbb{R}} e^{i\left(\frac{t}{T}\right)\omega} \, \mathcal{F}_t(g_n)[\omega] \, d\omega = T \int_{\mathbb{R}} e^{it\kappa} \mathcal{F}_t(g_n)[T\kappa] \, d\kappa.$$

Therefore,

$$\left\| g_n\left(\frac{t}{T}\right) \right\|_{H^b}^2 = T^2 \int_{\mathbb{R}} (1+|\kappa|)^{2b} \, |\mathcal{F}_t(g_n)[T\kappa]|^2 \, d\kappa$$

$$\leq c_b \, T^2 \left(\int_{\mathbb{R}} |\mathcal{F}_t(g_n)[T\kappa]|^2 \, d\kappa + \int_{\mathbb{R}} |\kappa|^{2b} \, |\mathcal{F}_t(g_n)[T\kappa]|^2 \, d\kappa \right)$$

$$= c_b \, T \left(\int_{\mathbb{R}} |\mathcal{F}_t(g_n)[\omega]|^2 \, d\omega + \int_{\mathbb{R}} T^{-2b} \, |\omega|^{2b} \, |\mathcal{F}_t(g_n)[\omega]|^2 \, d\omega \right)$$

$$\leq c_b \, T^{1-2b} \left(\int_{\mathbb{R}} |\mathcal{F}_t(g_n)[\omega]|^2 \, d\omega + \int_{\mathbb{R}} |\omega|^{2b} \, |\mathcal{F}_t(g_n)[\omega]|^2 \, d\omega \right)$$

$$\leq c_b\, T^{1-2b} \left(\|g_n\|_{L^2}^2 + \|g_n\|_{H^b}^2 \right)$$

$$\leq c_b\, T^{1-2b} \left(\|g_n\|_{L^2}^2 + \left[\|g_n\|_{L^2} + \left\| \frac{dg_n}{dt} \right\|_{L^2} \right]^2 \right)$$

$$\leq c_b\, T^{1-2b} \left(\|t^n \eta(t)\|_{L^2}^2 + \left[\|t^n \eta(t)\|_{L^2} + \|t^n \eta'(t)\|_{L^2} + \left\| n t^{n-1} \eta(t) \right\|_{L^2} \right]^2 \right)$$

$$\leq c_b\, 4^n\, T^{1-2b} \left(\|\eta(t)\|_{L^2}^2 + \left[\|\eta(t)\|_{L^2} + \|\eta'(t)\|_{L^2} + n\, \|\eta(t)\|_{L^2} \right]^2 \right)$$

$$\leq c_b\, 4^n\, T^{1-2b} \left(k^2 + [k(1+n) + C]^2 \right)$$

$$\leq c_b\, 4^n\, T^{1-2b} (1+n)^2,$$

where k and C are fixed constants, and we used the fact that η and its derivative are in L^2. Thus,

$$\|t^n \eta T(t)\|_{H^b} = T^n \left\| g_n \left(\frac{t}{T} \right) \right\|_{H^b} \leq c_b\, 2^n\, T^{\frac{1}{2}-b+n} (1+n). \tag{3.16}$$

To conclude, substituting (3.15) and (3.16) into (3.14) results in

$$\|I\|_{H^b} \leq c \sum_{n\geq 1} \frac{2^n (1+n)}{n!}\, T^{\frac{1}{2}-b+n} T^{1-n} T^{b'-\frac{1}{2}}\, \|f\|_{H^{b'}}$$

$$\leq c \sum_{n\geq 1} \frac{2^n (1+n)}{n!}\, T^{1+b'-b}\, \|f\|_{H^{b'}}$$

$$\leq c\, T^{1+b'-b}\, \|f\|_{H^{b'}}\,, \tag{3.17}$$

where c depends only on b and b'.

For the third term, we have

$$\|K\|_{H^b} = \left\| \eta_T(t) \int_{|\kappa| T \geq 1} (i\kappa)^{-1} \hat{f}(\kappa)\, d\kappa \right\|_{H^b}$$

$$= \left| \int_{|\kappa| T \geq 1} (i\kappa)^{-1} \hat{f}(\kappa) \, d\kappa \right| \|\eta_T(t)\|_{H^b}$$

$$\leq \left(\int_{|\kappa| T \geq 1} |i\kappa|^{-1} |\hat{f}(\kappa)| \, d\kappa \right) \|\eta_T(t)\|_{H^b}$$

$$= \left(\int_{|\kappa| T \geq 1} |i\kappa|^{-1} \langle \kappa \rangle^{b'-b'} |\hat{f}(\kappa)| \, d\kappa \right) \|\eta_T(t)\|_{H^b}$$

$$\leq \left\| |i\kappa|^{-1} \langle \kappa \rangle^{-b'} \right\|_{L^2_{|\kappa| T \geq 1}} \left\| \langle \kappa \rangle^{b'} |\hat{f}(\kappa)| \right\|_{L^2_{|\kappa| T \geq 1}} \|\eta_T(t)\|_{H^b}$$

$$= \left\| |i\kappa|^{-1} \langle \kappa \rangle^{-b'} \right\|_{L^2_{|\kappa| T \geq 1}} \|f\|_{H^{b'}} \|\eta_T(t)\|_{H^b} . \tag{3.18}$$

To obtain an estimate for the H^b norm on the right hand side of (3.18), note that

$$\eta_T(t) = \int_{\mathbb{R}} e^{i \frac{t}{T} \omega} \hat{\eta}(\omega) \, d\omega = T \int_{\mathbb{R}} e^{it\kappa} \hat{\eta}(T\kappa) \, d\kappa.$$

So,

$$\|\eta_T(t)\|_{H^b}^2 = T^2 \int_{\mathbb{R}} \langle \kappa \rangle^{2b} |\hat{\eta}(T\kappa)|^2 \, d\kappa$$

$$= T^2 \int_{\mathbb{R}} \langle \kappa \rangle^{2b} |\hat{\eta}(T\kappa)|^2 \, d\kappa$$

$$\leq c_b \, T^2 \left(\int_{\mathbb{R}} |\hat{\eta}(T\kappa)|^2 \, d\kappa \right) + c \, T^2 \left(\int_{\mathbb{R}} |\kappa|^{2b} |\hat{\eta}(T\kappa)|^2 \, d\kappa \right)$$

$$= c_b \, T \left(\int_{\mathbb{R}} |\hat{\eta}(\kappa)|^2 \, d\kappa \right) + c \, T^{1-2b} \left(\int_{\mathbb{R}} |\kappa|^{2b} |\hat{\eta}(\kappa)|^2 \, d\kappa \right)$$

$$\leq c_b \, T^{1-2b}. \tag{3.19}$$

Also, if $|\kappa| T \geq 1$, then $|\kappa| \geq T^{-1} \geq 1$. So,

$$\left\| |\kappa|^{-1} \langle \kappa \rangle^{-b'} \right\|_{L^2_{|\kappa| T \geq 1}} = \left(\int_{|\kappa| T \geq 1} |i\kappa|^{-2} \langle \kappa \rangle^{-2b'} \, d\kappa \right)^{\frac{1}{2}}$$

$$= \left(\int_{|\kappa| T \geq 1} \frac{(1+\kappa)^{-2b'}}{\kappa^2} \, d\kappa \right)^{\frac{1}{2}}$$

$$\leq c_{b'} \left(\int_{|\kappa| T \geq 1} \frac{(2\kappa)^{-2b'}}{\kappa^2} \, d\kappa \right)^{\frac{1}{2}}$$

$$= c_{b'} \left(\int_{|\kappa| T \geq 1} \kappa^{-2b'-2} \, d\kappa \right)^{\frac{1}{2}}$$

$$= c_{b'} \left(\lim_{k \to \infty} \left[\frac{\kappa^{-2b'-1}}{-2b'-1} \right]_{T^{-1}}^{k} \right)^{\frac{1}{2}}$$

$$\leq c_{b'} \, T^{\frac{1}{2}+b'}. \tag{3.20}$$

Applying estimates (3.19) and (3.20) to (3.18), we have

$$\|K\|_{H^b} \leq c \, T^{1+b'-b} \, \|f\|_{H^{b'}} \, , \tag{3.21}$$

where again c depends only on b and b'.

For the second term, we let $h(t) = \int_{|\kappa| T \geq 1} (i\kappa)^{-1} e^{it\kappa} \hat{f}(\kappa) \, d\kappa$. Then

$$\|J\|_{H^b} = \left\| \langle \kappa \rangle^b \hat{\eta}_T * \hat{h} \right\|_{L^2}$$

$$= \left\| \langle \kappa \rangle^b \int_{\mathbb{R}} \hat{\eta}_T(\kappa - p) \hat{h}(p) \, dp \right\|_{L^2}$$

$$= \left\| \int_{\mathbb{R}} \langle \kappa \rangle^b \hat{\eta}_T(\kappa - p) \hat{h}(p) \, dp \right\|_{L^2}$$

$$= \left\| \int_{\mathbb{R}} \langle \kappa + p - p \rangle^b \hat{\eta}_T(\kappa - p) \hat{h}(p) \, dp \right\|_{L^2}$$

$$\leq \left\| c_b \int_{\mathbb{R}} (1 + |p| + |\kappa - p|)^b \, \hat{\eta}_T(\kappa - p) \hat{h}(p) \, dp \right\|_{L^2}$$

$$\leq \left\| c_b \int_{\mathbb{R}} |\kappa - p|^b \, \hat{\eta}_T(\kappa - p) \hat{h}(p) \, dp \right\|_{L^2} + \left\| c \int_{\mathbb{R}} \langle p \rangle^b \hat{\eta}_T(\kappa - p) \hat{h}(p) \, dp \right\|_{L^2}$$

$$= c_b \left\| \left(|\kappa|^b \, \hat{\eta}_T \right) * \hat{h} \right\|_{L^2} + c \, \left\| \hat{\eta}_T * \left(\langle \kappa \rangle^b \hat{h} \right) \right\|_{L^2}$$

$$\leq c_b \, \left\| |\kappa|^b \, \hat{\eta}_T \right\|_{L^1} \|h\|_{L^2} + c \, \|\hat{\eta}_T\|_{L^1} \|h\|_{H^b} , \tag{3.22}$$

where in the last three estimates, we used Minkowski's inequality and Young's convolution inequality. The norms in (3.22) are estimated as follows:

$$\|\hat{\eta}_T\|_{L^1} = T \int_{\mathbb{R}} |\hat{\eta}(T\kappa)| \, d\kappa = \int_{\mathbb{R}} |\hat{\eta}(\omega)| \, d\omega \leq c, \tag{3.23}$$

$$\left\| |\kappa|^b \, \hat{\eta}_T \right\|_{L^1} = T \int_{\mathbb{R}} |\kappa|^b \, |\hat{\eta}(T\kappa)| \, d\kappa = T^{-b} \int_{\mathbb{R}} |\hat{\eta}(\omega)| \, d\omega \leq c \, T^{-b}, \tag{3.24}$$

and

$$\begin{aligned}
\|h\|_{H^b}^2 &= \left\| \int_{|\kappa| T \geq 1} (i\kappa)^{-1} e^{it\kappa} \hat{f}(\kappa) \, d\kappa \right\|_{H^b}^2 \\
&= \left\| \int_{\mathbb{R}} 1_{|\kappa| T \geq 1} \, (i\kappa)^{-1} e^{it\kappa} \hat{f}(\kappa) \, d\kappa \right\|_{H^b}^2 \\
&= \left\| \int_{\mathbb{R}} e^{it\kappa} g(\kappa) \, d\kappa \right\|_{H^b}^2 \\
&= \left\| \langle \kappa \rangle^b g(\kappa) \right\|_{L^2}^2 \\
&= \left\| \langle \kappa \rangle^{b-b'+b'} 1_{|\kappa| T \geq 1} \, (i\kappa)^{-1} \hat{f}(\kappa) \right\|_{L^2}^2 \\
&= \int_{|\kappa| T \geq 1} \langle \kappa \rangle^{2b-2b'+2b'} |\kappa|^{-2} \left| \hat{f}(\kappa) \right|^2 \, d\kappa \\
&\leq \sup_{|\kappa| T \geq 1} |\kappa|^{-2} \langle \kappa \rangle^{2b-2b'} \int_{|\kappa| T \geq 1} \langle \kappa \rangle^{2b'} \left| \hat{f}(\kappa) \right|^2 \, d\kappa \\
&\leq c \, T^{2-2b+2b'} \, \|f\|_{H^{b'}}^2 ,
\end{aligned}$$

for $b, b' \in \mathbb{R}$, $b \leq 1 + b'$. Taking the square root on both sides of the inequality, we have

$$\|h\|_{H^b} \leq c\, T^{1-b+b'}\, \|f\|_{H^{b'}}\,. \tag{3.25}$$

Specifically, if $b = 0$ then

$$\|h\|_{L^2} \leq c\, T^{1+b'}\, \|f\|_{H^{b'}}\,. \tag{3.26}$$

Lastly, we apply (3.23), (3.24), (3.25), and (3.26) to (3.22), to get

$$\|J\|_{H^b} \leq c\, T^{1+b'-b}\, \|f\|_{H^{b'}}\,. \tag{3.27}$$

The estimation in (3.12) results from substituting (3.17), (3.21), and (3.27) into (3.13). To get (3.11), note that

$$\begin{aligned}
&\left\|\eta_T(t) \int_0^t S(\alpha(t-t'))\, F(t',x)\, dt'\right\|^2_{X^{s,b}_{\kappa=-\alpha n^2}} \\
&\quad= \left\|S(-\alpha t)\, \eta_T(t) \int_0^t S(\alpha(t-t'))\, F(t',x)\, dt'\right\|^2_{H^b_t H^s_x} \\
&\quad= \left\|\eta_T(t) \int_0^t S(-\alpha t')\, F(t',x)\, dt'\right\|^2_{H^b_t H^s_x} \\
&\quad= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \langle n\rangle^{2s} \langle\kappa\rangle^{2b} \left|\mathcal{F}\left(\eta_T(t) \int_0^t S(-\alpha t')\, F(t',x)\, dt'\right)[\kappa, n]\right|^2 d\kappa \\
&\quad= \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} \int_{\mathbb{R}} \langle\kappa\rangle^{2b} \left|\mathcal{F}_t\left(\eta_T(t) \int_0^t H(t',n)\, dt'\right)[\kappa]\right|^2 d\kappa,
\end{aligned}$$

where $H(t,n) = \mathcal{F}_x(S(-\alpha t)\ F)[n]$. For a fixed n, let $H(t,n) = f_n(t)$.

By applying the estimate in (3.12), it follows that

$$\left\|\eta_T(t)\int_0^t S(\alpha(t-t'))F(t',x)\,dt'\right\|^2_{X^{s,b}_{\kappa=-\alpha n^2}}$$

$$= \sum_{n\in\mathbb{Z}}\langle n\rangle^{2s}\int_{\mathbb{R}}\langle\kappa\rangle^{2b}\left|\mathcal{F}_t\left(\eta_T(t)\int_0^t f_n(t')\,dt'\right)[\kappa]\right|^2\,d\kappa$$

$$= \sum_{n\in\mathbb{Z}}\langle n\rangle^{2s}\left\|\eta_T(t)\int_0^t f_n(t')\,dt'\right\|^2_{H^b}$$

$$\leq c\sum_{n\in\mathbb{Z}}\langle n\rangle^{2s}T^{2+2b'-2b}\,\|f_n\|^2_{H^{b'}}$$

$$= c\,T^{2+2b'-2b}\sum_{n\in\mathbb{Z}}\langle n\rangle^{2s}\int_{\mathbb{R}}\langle\kappa\rangle^{2b'}\left|\hat{f}_n(\kappa)\right|^2\,d\kappa$$

$$= c\,T^{2+2b'-2b}\sum_{n\in\mathbb{Z}}\langle n\rangle^{2s}\int_{\mathbb{R}}\langle\kappa\rangle^{2b'}\left|\mathcal{F}_t\left(H(t,n)\right)[\kappa]\right|^2\,d\kappa$$

$$= c\,T^{2+2b'-2b}\sum_{n\in\mathbb{Z}}\langle n\rangle^{2s}\int_{\mathbb{R}}\langle\kappa\rangle^{2b'}\left|\mathcal{F}_t\left(\mathcal{F}_x(S(-\alpha t)F)[n]\right)[\kappa]\right|^2\,d\kappa$$

$$= c\,T^{2+2b'-2b}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\langle n\rangle^{2s}\langle\kappa\rangle^{2b'}\left|\mathcal{F}\left(S(-\alpha t)F\right)[\kappa,n]\right|^2\,d\kappa$$

$$= c\,T^{2+2b'-2b}\,\|S(-\alpha t)F\|^2_{H^{b'}_z H^s_x}$$

$$= c\,T^{2+2b'-2b}\,\|F\|^2_{X^{s,b'}_{\kappa=-\alpha n^2}}\,.$$

Taking the square root of both sides concludes the proof of Lemma 3.15. □

Theorem 3.16. *(Strichartz Estimate). Suppose $D'(s) = \Delta(s)$ satisfies Assumption 1. Then for all $u \in S(\mathbb{T})$ and for all $s \in [0,B]$,*

$$\|T_{D(s)}u\|_{L^4_{s,x}} \leq c\,\|u\|_{L^2_x} \qquad (3.28)$$

Proof. The following proof is similar to that of Proposition 2.1 in [4]. First,

$$\|T_{D(s)}u\|_{L^4_{s,x}} = \left(\int_0^B \|T_{D(s)}u\|^4_{L^4_x}\, ds\right)^{\frac{1}{4}}$$

$$= \left(\sum_{j=0}^{n-1}\int_{s_j}^{s_{j+1}} \|T_{D(s)}u\|^4_{L^4_x}\, ds\right)^{\frac{1}{4}}$$

$$\leq K\sum_{j=0}^{n-1}\left(\int_{s_j}^{s_{j+1}} \|T_{D(s)}u\|^4_{L^4_x}\, ds\right)^{\frac{1}{4}}$$

$$\leq K\sum_{j=0}^{n-1} \|T_{D(s)}u\|_{L^4_{s,x}((s_j,s_{j+1})\times\mathbb{T})}$$

with $s_0, s_1, ..., s_n$ as in Assumption 1. Thus it is enough to prove Theorem 3.16 on an arbitrary finite interval $[a,b]$ instead of on $[0,B]$, and under the assumption that $D'(s) = k$, a constant on $[a,b]$.

With this in mind, we write

$$\|T_{D(s)}u\|^4_{L^4_{s,x}((a,b)\times\mathbb{T})} \leq \|T_{D(s)}u\|^4_{L^4_{s,x}((p,q)\times\mathbb{T})}$$

for some p, q such that for a given k , $[a,b] \subseteq [p,q]$ and $q - p = \dfrac{2M\pi}{k}$, for some $M \in \mathbb{Z}$.

Let $g = T_{D(s)}u$, then by definition of Fourier transform, $g = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{i(nx-n^2D(s))}$ with $\hat{u}(n) = \mathcal{F}_x(u)[n]$. Thus,

$$\|g\|^2_{L^4_{s,x}((p,q)\times\mathbb{T})} = \|g\cdot\bar{g}\|_{L^2_{s,x}((p,q)\times\mathbb{T})}$$

$$= \left\|\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(m)e^{i((n-m)x-(n^2-m^2)D(s))}\right\|_{L^2_{s,x}((p,q)\times\mathbb{T})}$$

$$= \left\|\sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2 + \sum_{n\in\mathbb{Z}}\sum_{m\neq n\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(m)e^{i((n-m)x-(n^2-m^2)D(s))}\right\|_{L^2_{s,x}((p,q)\times\mathbb{T})}$$

$$\leq \left\| \sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2 \right\|_{L^2_{s,x}} + \left\| \sum_{n\in\mathbb{Z}} \sum_{m\neq n\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(m) e^{i((n-m)x-(n^2-m^2)D(s))} \right\|_{L^2_{s,x}}$$

$$\leq \left\| \sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2 \right\|_{L^2_{s,x}} + \left\| \sum_{n\in\mathbb{Z}} \sum_{l\neq 0\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(n-l) e^{ilx} e^{-il(2n-l)D(s)} \right\|_{L^2_{s,x}}, \quad (3.29)$$

where $l = n - m$.

The second term on the right hand side of (3.29) can be dealt with as follows. Let $b_l = \sum_{n\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(n-l)e^{-il(2n-l)D(s)}$. Then

$$\left\| \sum_{n\in\mathbb{Z}} \sum_{l\neq 0\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(n-l) e^{ilx} e^{-il(2n-l)D(s)} \right\|_{L^2_{s,x}} = \left\| \sum_{l\neq 0\in\mathbb{Z}} b_l e^{ilx} \right\|_{L^2_{s,x}}$$

$$= \left(\int_p^q \int_{\mathbb{T}} \left| \sum_{l\neq 0\in\mathbb{Z}} b_l e^{ilx} \right|^2 dx\, ds \right)^{\frac{1}{2}}$$

$$\leq c \left(\int_p^q \sum_{l\neq 0\in\mathbb{Z}} |b_l|^2 \, ds \right)^{\frac{1}{2}}$$

$$\leq c \left(\sum_{l\neq 0\in\mathbb{Z}} \int_p^q b_l \cdot \bar{b}_l \, ds \right)^{\frac{1}{2}}$$

$$= c \left(\sum_{l\neq 0\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(n-l)\hat{u}(r-l)\bar{\hat{u}}(r) \int_p^q e^{-il(2n-2r)D(s)} \, ds \right) \right)^{\frac{1}{2}}.$$

Now we observe that since $q - p = \frac{2M\pi}{k}$, the integral

$$\int_p^q e^{-il(2n-2r)D(s)} \, ds = \int_p^q e^{-il(2n-2r)(ks+d)} \, ds$$

is equal to $q - p$ when $n = r$; and is equal to 0 when $n \neq r$. So,

$$\left\| \sum_{n\in\mathbb{Z}} \sum_{l\neq 0\in\mathbb{Z}} \hat{u}(n)\bar{\hat{u}}(n-l) e^{ilx} e^{-il(2n-l)D(s))} \right\|_{L^2_{s,x}}$$

$$\le c\left(\frac{2M\pi}{k}\sum_{l\neq 0\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\hat{u}(n)\bar{\hat{u}}(n-l)\hat{u}(n-l)\bar{\hat{u}}(n)\right)^{\frac{1}{2}}$$

$$= c\left(\frac{2M\pi}{k}\sum_{n\neq m}|\hat{u}(n)\bar{\hat{u}}(m)|^2\right)^{\frac{1}{2}} \quad \text{since } m = n-l$$

$$\le c\left(\frac{2M\pi}{k}\sum_{n}\sum_{m}|\hat{u}(n)|^2|\hat{u}(m)|^2\right)^{\frac{1}{2}}$$

$$= c\left(\frac{2M\pi}{k}\sum_{n}|\hat{u}(n)|^2\ \sum_{m}|\hat{u}(m)|^2\right)^{\frac{1}{2}},$$

$$\le c\ \|u\|^2_{L^2_x}.$$

Putting the estimate above into (3.29) results in

$$\|g\|^2_{L^4_{s,x}((a,b)\times\mathbb{T})} \le \|g\|^2_{L^4_{s,x}((p,q)\times\mathbb{T})} \le c\ \left(\left\|\sum_{n\in\mathbb{Z}}|\hat{u}(n)|^2\right\|_{L^2_{s,x}((p,q)\times\mathbb{T})} + \|u\|^2_{L^2_x}\right)$$

$$\le c\ \left(\left|\sum_{n\in\mathbb{Z}}|\hat{u}(n)|^2\right| + \|u\|^2_{L^2_x}\right)$$

$$\le c\ \|u\|^2_{L^2_x}.$$

Finally, (3.28) can be obtained from the above estimate. □

3.3 Well-posedness in H^r

Lemma 3.17. *Fix $B \in \mathbb{R}$ and $M > 0$, and suppose r is a non-negative integer. Suppose $u_1, u_2, u_3 \in H^r_x$, and define*

$$R(s) := T^{-1}_{D(s)}[T_{D(s)}u_1 \cdot \overline{T_{D(s)}u_2} \cdot T_{D(s)}u_3]$$

for $s \in [0, B]$. If we let

$$F(u_1,u_2,u_3) := \frac{1}{B}\int_0^B R(s)\,ds,$$

then we have that

$$\|F(u_1,u_2,u_3)\|_{C_t^0H_x^r([0,M]\times\mathbb{T})} \le c\,\|u_1\|_{C_t^0H_x^r([0,M]\times\mathbb{T})}\cdot\|u_2\|_{C_t^0H_x^r([0,M]\times\mathbb{T})} \cdot\|u_3\|_{C_t^0H_x^r([0,M]\times\mathbb{T})}, \tag{3.30}$$

where c *is independent of* u_1,u_2,u_3.

Proof. For $r\in\mathbb{Z}$ and $r>0$,

$$\|F\|_{C_t^0H_x^r([0,M]\times\mathbb{T})} = \frac{1}{B}\sup_{t\in[0,M]}\left\|\int_0^B T_{D(s)}^{-1}[T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3]\,ds\right\|_{H_x^r}$$

$$\le \frac{1}{B}\sup_{t\in[0,M]}\int_0^B \left\|T_{D(s)}^{-1}[T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3]\right\|_{H_x^r}\,ds$$

$$= \frac{1}{B}\sup_{t\in[0,M]}\int_0^B \left\|T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3\right\|_{H_x^r}\,ds$$

$$= \frac{c}{B}\sup_{t\in[0,M]}\int_0^B \|T_{D(s)}u_1\|_{H_x^r}\|\overline{T_{D(s)}u_2}\|_{H_x^r}\|T_{D(s)}u_3\|_{H_x^r}\,ds$$

$$= c\,\|u_1\|_{C_t^0H_x^r([0,M]\times\mathbb{T})}\|u_2\|_{C_t^0H_x^r([0,M]\times\mathbb{T})}\|u_3\|_{C_t^0H_x^r([0,M]\times\mathbb{T})}$$

For $r=0$,

$$\|F\|_{C_t^0L_x^2([0,M]\times\mathbb{T})} = \frac{1}{B}\sup_{t\in[0,M]}\left\|\int_0^B T_{D(s)}^{-1}[T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3]\,ds\right\|_{L_x^2}$$

$$\le \frac{1}{B}\sup_{t\in[0,M]}\int_0^B \left\|T_{D(s)}^{-1}[T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3]\right\|_{L_x^2}\,ds$$

$$= \frac{1}{B}\sup_{t\in[0,M]}\int_0^B \sup_{\|v\|_{L^2}\le 1}\left|\int_{\mathbb{T}} T_{D(s)}^{-1}[T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3]\cdot\bar{v}\,dx\right|\,ds$$

$$= \frac{1}{B}\sup_{t\in[0,M]}\int_0^B \sup_{\|v\|_{L^2}\le 1}\left|\int_{\mathbb{T}} T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3\cdot\overline{T_{D(s)}v}\,dx\right|\,ds$$

$$\leq \frac{1}{B} \sup_{t\in[0,M]} \int_0^B \sup_{\|v\|_{L^2}\leq 1} \int_{\mathbb{T}} |T_{D(s)}u_1 \cdot \overline{T_{D(s)}u_2} \cdot T_{D(s)}u_3 \cdot \overline{T_{D(s)}v}| \; dx \; ds.$$

It follows from Holder's inequality and the Strichartz estimate found in Theorem 3.16 that

$$\|F\|_{C_t^0 L_x^2([0,M]\times\mathbb{T})} \leq \frac{1}{B} \sup_{t\in[0,M]} \sup_{\|v\|_{L^2}\leq 1} \|T_{D(s)}u_1\|_{L^4_{s,x}} \; \|T_{D(s)}u_2\|_{L^4_{s,x}} \; \|T_{D(s)}u_3\|_{L^4_{s,x}} \; \|T_{D(s)}v\|_{L^4_{s,x}}$$

$$\leq \frac{c}{B} \sup_{t\in[0,M]} \sup_{\|v\|_{L^2}\leq 1} \|u_1\|_{L_x^2} \; \|u_2\|_{L_x^2} \; \|u_3\|_{L_x^2} \; \|v\|_{L_x^2}$$

$$\leq \frac{c}{B} \sup_{t\in[0,M]} \|u_1\|_{L_x^2} \; \|u_2\|_{L_x^2} \; \|u_3\|_{L_x^2}$$

$$= c \; \|u_1\|_{C_t^0 L_x^2([0,M]\times\mathbb{T})} \; \|u_2\|_{C_t^0 L_x^2([0,M]\times\mathbb{T})} \; \|u_3\|_{C_t^0 L_x^2([0,M]\times\mathbb{T})} \quad \square$$

Lemma 3.18. *Suppose $s \in [0, B]$ and $M > 0$, and suppose r is a non-negative integer. Let $F(u_1, u_2, u_3)$ be as defined in Lemma 3.17. For $u \in \mathcal{Y}$ define*

$$F(u(t,x)) = F(u) := F(u,u,u) = \frac{1}{B} \int_0^B T^{-1}_{D(s)}[|T_{D(s)}u|^2 \; T_{D(s)}u] \; ds.$$

Then for all $u, v \in \mathcal{Y}$, we have that

$$\|F(u)\|_{C_t^0 H_x^r([0,M]\times\mathbb{T})} \leq c \; \|u\|^3_{C_t^0 H_x^r([0,M]\times\mathbb{T})} \tag{3.31}$$

and

$$\|F(u) - F(v)\|_{C_t^0 H_x^r([0,M]\times\mathbb{T})} \leq c \; \|u - v\|_{C_t^0 H_x^r} (\|u\|_{C_t^0 H_x^r} + \|v\|_{C_t^0 H_x^r})^2, \tag{3.32}$$

where c is independent of u and v.

Proof. The estimate (3.31) follows immediately from (3.30). Observe that

$$F(u) - F(v) = F(u-v,u,u) + F(u-v,v,v) + F(v,u-v,u). \tag{3.33}$$

Thus,

$$\begin{aligned}\|F(u) - F(v)\|_{C_t^0 H_x^r([0,M]\times\mathbb{T})} \le \|F(u-v,u,u)\|_{C_t^0 H_x^r} + \|F(u-v,v,v)\|_{C_t^0 H_x^r} \\ + \|F(v,u-v,u)\|_{C_t^0 H_x^r}\end{aligned}$$

$$\le c\,(\|u-v\|_{C_t^0 H_x^r}\|u\|^2_{C_t^0 H_x^r} + \|u-v\|_{C_t^0 H_x^r}\|v\|^2_{C_t^0 H_x^r} + \|u-v\|_{C_t^0 H_x^r}\|u\|_{C_t^0 H_x^r}\|v\|_{C_t^0 H_x^r})$$

$$\le c\,\|u-v\|_{C_t^0 H_x^r}(\|u\|_{C_t^0 H_x^r} + \|v\|_{C_t^0 H_x^r})^2. \qquad \square$$

Remark 3.19. *It follows from Lemma 3.18 that for $\alpha = 0$, $E(u)$ is a continuous functional on $L^2(\mathbb{T})$. Indeed, we have*

$$|E(u) - E(v)| = |\,\langle F(u), u\rangle - \langle F(v), v\rangle\,|$$

$$\le c(|\,\langle F(u) - F(v), u\rangle\,| + |\,\langle F(v), u-v\rangle\,|)$$

$$\le c(\|F(u) - F(v)\|_{L_x^2}\|u\|_{L_x^2} + \|F(v)\|_{L_x^2}\|u-v\|_{L_x^2})$$

$$\le c(\|u-v\|_{L^2}(\|u\|_{L_x^2} + \|v\|_{L_x^2})^2\|u\|_{L_x^2} + \|v\|^3_{L_x^2}\|u-v\|_{L_x^2}),$$

$$\le c\|u-v\|_{L_x^2}(\|u\|^3_{L_x^2} + \|v\|^3_{L_x^2}).$$

Theorem 3.20. *The periodic DMNLS equation is globally well-posed in $H^r(\mathbb{T})$ for non-negative integer values of r. That is, suppose $\alpha \in \mathbb{R}$. Let $D(s)$ satisfy Assumption 1. If $u_0 \in H^r$, then for every $M > 0$, equation (3.1) has a unique strong solution $u \in C_t^0 H_x^r([0,M]\times\mathbb{T})$ with initial data u_0. The map $u_0 \mapsto u$ is locally Lipschitz from H_x^r to $C_t^0 H_x^r([0,M]\times\mathbb{T})$. With M as defined above, $P(u)$ is independent of t, for $t \in [0,M]$. Finally, the number M can be taken* *arbitrarily large.*

Proof. Let r be an integer. We start by obtaining a local solution via a Banach contraction mapping argument.

We denote the closed ball of radius a centered at the origin in $C_t^0 H_x^r([0,M] \times \mathbb{T})$ by

$$\Lambda_{M,a} = \{u \in C_t^0 H_x^r([0,M] \times \mathbb{T}) : \|u\|_{C_t^0 H_x^r} \leq a\}.$$

Here we will show that if $K > 0$, then for every $a \in [2K, \infty)$ there exists $M > 0$ such that if $u_0 \in H^r$ satisfies $\|u_0\|_{H^r} \leq K$, and $M' \in (0, M]$, then there is a unique strong solution to (3.1) in $\Lambda_{M',a}$ with initial data u_0.

Fix $u_0 \in H^r$ such that $\|u_0\|_{H^r} \leq K$. For each $M > 0$, we define $Q : C_t^0 H_x^r([0,M] \times \mathbb{T}) \to C_t^0 H_x^r([0,M] \times \mathbb{T})$ by setting, for $t \in [0, M]$ and $u \in C_t^0 H_x^r([0,M] \times \mathbb{T})$,

$$Q(u)(t) = S(\alpha t)u_0 + i \int_0^t S(\alpha(t-t'))F(u(t',x))\, dt', \tag{3.34}$$

with $F(u)$ as defined in Lemma 3.18. We have shown in Lemmas 3.8, 3.9, and 3.18 that

$$\begin{aligned} \|Q(u)\|_{C_t^0 H_x^r([0,M]\times\mathbb{T})} &\leq \|u_0\|_{H_x^r} + M\|F\|_{C_t^0 H_x^r} \\ &\leq \|u_0\|_{H_x^r} + cM\, \|u\|^3_{C_t^0 H_x^r}. \end{aligned} \tag{3.35}$$

Furthermore, if $u, v \in C_t^0 L_x^2([0,M] \times \mathbb{T})$, then Lemma 3.9 and (3.32) imply that

$$Q(u) - Q(v) = i \int_0^t S(\alpha(t-t'))[F(u(t',x)) - F(v(t',x))]\, dt'$$

satisfies

$$\|Q(u)-Q(v)\|_{C_t^0H_x^r([0,M]\times\mathbb{T})} \leq \ cM\ \|u-v\|_{C_t^0H_x^r}(\|u\|_{C_t^0H_x^r}+\|v\|_{C_t^0H_x^r})^2. \tag{3.36}$$

Now suppose $a \geq 2K$, choose $M = \frac{1}{(8ca^2)}$, and suppose $0 < M' \leq M$. For all $u, v \in \Lambda_{M',a}$ we have that

$$\|Q(u)\|_{C_t^0H_x^r([0,M']\times\mathbb{T})} \leq \frac{a}{2}+\frac{a}{2} = a$$

and

$$\|Q(u)-Q(v)\|_{C_t^0H_x^r([0,M']\times\mathbb{T})} \leq \frac{1}{2}\|u-v\|_{C_t^0H_x^r([0,M']\times\mathbb{T})}.$$

Therefore $Q(u)$ defines a contraction from the closed ball $\Lambda_{M',a}$ to itself, and so by the Banach Contraction Mapping Theorem, $Q(u)$ has a unique fixed point $u \in \Lambda_{M',a}$. This fixed point is a strong solution to (3.1) with initial data u_0. Note also that every strong solution with initial data u_0 is also a fixed point of Q, so there exists a unique strong solution in $u \in \Lambda_{M',a}$ with initial data u_0.

We next prove continuity of the fixed point with respect to the initial data. Let $K > 0$, $u_0, v_0 \in H^r$ with $\|u_0\|_{H^r} \leq K$, and $\|v_0\|_{H^r} \leq K$. For $a = 2K$, and M defined as above, let u and v be unique solutions in $\Lambda_{M,a}$ with initial data u_0 and v_0 respectively. Then we will show that

$$\|u-v\|_{C_t^0H_x^r([0,M]\times\mathbb{T})} \leq c\|u_0-v_0\|_{H^r}. \tag{3.37}$$

Suppose Q is defined on $C_t^0H_x^r([0,M]\times\mathbb{T})$ as in (3.34). Then we know that

$$\|Q(u)-Q(v)\|_{C_t^0H_x^r([0,M]\times\mathbb{T})} \leq \frac{1}{2}\|u-v\|_{C_t^0H_x^r([0,M]\times\mathbb{T})}. \tag{3.38}$$

Define $\Psi : C_t^0 H_x^r([0,M]\times\mathbb{T}) \times L_x^2 \to C_t^0 H_x^r([0,M]\times\mathbb{T})$ by

$$\Psi(u,w) = S(\alpha t)w + i\int_0^t S(\alpha(t-t'))F(u)\,dt'.$$

Then $\Psi(u,u_0) = u$ and $\Psi(v,v_0) = v$. So for every $t \in [0,M]$,

$$\|u-v\|_{C_t^0 H_x^r} = \|\Psi(u,u_0) - \Psi(v,u_0) + \Psi(v,u_0) - \Psi(v,v_0)\|_{C_t^0 H_x^r}$$

$$\le \|\Psi(u,u_0) - \Psi(v,u_0)\|_{C_t^0 H_x^r} + \|\Psi(v,u_0) - \Psi(v,v_0)\|_{C_t^0 H_x^r}$$

$$= \|Q(u) - Q(v)\|_{C_t^0 H_x^r} + \|u_0 - v_0\|_{H^r}$$

$$\le \frac{1}{2}\|u-v\|_{C_t^0 H_x^r} + \|u_0 - v_0\|_{H^r}.$$

This gives the desired continuity with respect to the initial data.

Next, we will show that if $u_0 \in L^2$ and $M > 0$, then there can not be two different strong solutions of (3.1) with initial data u_0.

Suppose u and v are two strong solutions in $C_t^0 L_x^2([0,M]\times\mathbb{T})$ with the same initial data u_0, and let

$$T = \sup\{t \in [0,M] : u(t') = v(t') \text{ for all } t' \in [0,t]\}.$$

Continuity in t implies that $u(T) = v(T)$. Suppose $T < M$; then for every $\epsilon \in (0, M-T)$, there exists $t \in [T, T+\epsilon]$ such that $u(t) \neq v(t)$. Define $u_1 = u(T) = v(T)$. If $a_1 = 2\|u_1\|_{L^2}$, then there exists $M_1 > 0$ such that for every $\epsilon \in (0, M_1]$, (3.1) has a unique solution in Λ_{ϵ,a_1} with initial data u_1. Choose ϵ such that $\epsilon < \min(M_1, M-T)$ and $a_1 \ge \max(\|v(t) - u_1\|_{L_x^2}, \|u(t) - u_1\|_{L_x^2})$ for all $t \in [T, T+\epsilon]$. Then the functions $\tilde{u}(t) := u(t-T)$ and $\tilde{v}(t) := v(t-T)$ are two distinct strong solutions in Λ_{ϵ,a_1} with initial data u_1, which is a contradiction. This implies that we must have $T = M$, and therefore $u = v$

in $C_t^0 L_x^2\,([0,M]\times\mathbb{T})$.

For a given $u_0 \in H^r$, let

$$M(u_0,r) = \sup\{M>0 : \exists \text{ a strong solution of (3.1) in } C_t^0 H_x^r$$

$$\text{with initial data } u_0\}.$$

Note that by the Banach contraction mapping theorem, $M(u_0,r)>0$; and since every strong solution in $C_t^0 H_x^r\,([0,M]\times\mathbb{T})$ is also a strong solution in $C_t^0 L_x^2\,([0,M]\times\mathbb{T})$, we have that if two solutions with the same initial data are defined on different time intervals, then they must agree on the smaller of the two intervals. Therefore, there is a well-defined function $u(t,x)$ for $t\in[0,M(u_0,r))$ such that for every $M\in[0,M(u_0,r))$, u is the unique strong solution of (3.1) in $C_t^0 H_x^r\,([0,M]\times\mathbb{T})$ with initial data u_0. Moreover, if $M(u_0,r)<\infty$, then $\lim_{t\to M(u_0,r)}\|u(t)\|_{H^r}=\infty$. Otherwise, we obtain a contradiction by choosing $u(M)$ as the initial data with M sufficiently close to $M(u_0,r)$, and then use the fixed point argument above to extend the solution u to an interval $[0,M+\epsilon)$, with $M+\epsilon>M(u_0,r)$.

We next establish persistence of regularity: if a solution $u(t,x)\in L^2$ has initial data that is in H^r for some positive integer r, then u will remain in H^r for as long as it is in L^2.

Suppose to begin with that $u_0\in H^1$. Then $M(u_0,1)=M(u_0,0)$. We will show that if $0<M<M(u_0,0)$ and u is a strong solution in $C_t^0 H_x^1\,([0,M]\times\mathbb{T})$ with initial data u_0, then $\|u(t)\|_{H_x^1}$ remains bounded for $z\in[0,M]$. To see this, first observe that

$$\|u(t)\|_{H_x^1} \le \|S(\alpha t)u_0\|_{H_x^1} + \int_0^t \|S(\alpha(t-t'))F(u(t'))\|_{H_x^1}\,dt'$$

$$= \|u_0\|_{H^1} + \int_0^t \|F(t')\|_{H^1_x}\, dz'$$

$$\leq \|u_0\|_{H^1} + c\int_0^t \|F(t')\|_{L^2_x} + \|F_x(t')\|_{L^2_x}\, dt'.$$

From (3.31), we have that $\|F(t)\|_{L^2_x} \leq c\, \|u(t)\|^3_{L^2_x} \leq c\, \|u(t)\|^2_{L^2_x}\|u(t)\|_{H^1_x}$. Also,

$$\|F_x(t)\|_{L^2} = \frac{1}{B}\left\|\int_0^B \frac{d}{dx}T^{-1}_{D(s)}[T_{D(s)}u(t)\cdot\overline{T_{D(s)}u(t)}\cdot T_{D(s)}u(t)]\, ds\right\|_{L^2_x}$$

$$= \frac{1}{B}\left\|\int_0^B T^{-1}_{D(s)}\left[\frac{d}{dx}\Big(T_{D(s)}u(t)\cdot\overline{T_{D(s)}u(t)}\cdot T_{D(s)}u(t)\Big)\right]\, ds\right\|_{L^2_x}$$

$$\leq \frac{1}{B}\left\|\int_0^B T^{-1}_{D(s)}[T_{D(s)}u(t)\cdot\overline{T_{D(s)}u_x(t)}\cdot T_{D(s)}u(t)]\, ds\right\|_{L^2_x} +$$

$$\frac{1}{B}\left\|\int_0^B T^{-1}_{D(s)}[2\, T_{D(s)}u_x(t)\cdot\overline{T_{D(s)}u(t)}\cdot T_{D(s)}u(t)]\, ds\right\|_{L^2_x}$$

$$\leq c\, \|u(t)\|^2_{L^2_x}\|u_x(t)\|_{L^2_x} \leq c\, \|u(t)\|^2_{L^2_x}\|u(t)\|_{H^1_x}$$

where we have used the product rule and Minkowski's inequality and (3.30). Therefore,

$$\|u(t)\|_{H^1_x} \leq \|u_0\|_{H^1} + c\int_0^t \|u(t')\|^2_{L^2_x}\|u(t')\|_{H^1_x}\, dt'$$

$$\leq \|u_0\|_{H^1} + cR\int_0^t \|u(t')\|_{H^1_x}\, dt'$$

where $R = \|u\|^2_{C^0_t L^2_x([0,M]\times\mathbb{T})} < \infty$. Then from Gronwall's inequality it follows that, for all $t \in [0, M]$,

$$\|u(t)\|_{H^1_x} \leq \|u_0\|_{H^1}\, e^{cRt} \leq \|u_0\|_{H^1}\, e^{cRM} < \infty.$$

Next, suppose $u_0 \in H^2$. Then $M(u_0, 2) = M(u_0, 0)$. Again, for all $t \in$

$[0, M]$,

$$\|u(t)\|_{H^2_x} \leq \|S(\alpha t)u_0\|_{H^2_x} + \int_0^t \|S(\alpha(t-t'))F(u(t'))\|_{H^2_x}\, dt'$$

$$= \|u_0\|_{H^2} + \int_0^t \|F(t')\|_{H^2_x}\, dt'$$

$$\leq \|u_0\|_{H^2} + c\int_0^t \|F(t')\|_{L^2_x} + \|F_x(t')\|_{L^2_x} + \|F_{xx}(t')\|_{L^2_x}\, dz'.$$

$$\leq \|u_0\|_{H^2} + c\int_0^t \|u(t')\|^2_{L^2_x}\|u(t')\|_{H^1_x} + \|u(t')\|^2_{L^2_x}\|u(t')\|_{H^1_x} + \|F_{xx}(t')\|_{L^2_x}\, dz'.$$

$$\|F_{xx}(t)\|_{L^2_x} = \frac{1}{B}\left\|\int_0^B \frac{d^2}{dx^2}\, T^{-1}_{D(s)}[T_{D(s)}u(t)\cdot \overline{T_{D(s)}u(t)}\cdot T_{D(s)}u(t)]\, ds\right\|_{L^2_x}$$

$$\leq \frac{1}{B}\left\|\int_0^B T^{-1}_{D(s)}\left[\frac{d}{dx}\left(T_{D(s)}u(t)\cdot \overline{T_{D(s)}u_x(t)}\cdot T_{D(s)}u(t)\right)\right]\, ds\right\|_{L^2_x} +$$

$$\frac{1}{B}\left\|\int_0^B T^{-1}_{D(s)}\left[2\,\frac{d}{dx}\left(T_{D(s)}u_x(t)\cdot \overline{T_{D(s)}u(t)}\cdot T_{D(s)}u(t)\right)\right]\, ds\right\|_{L^2_x}$$

$$\leq c\,\|u(t)\|_{L^2_x}\|u_x(t)\|^2_{L^2_x} + c\,\|u(t)\|^2_{L^2_x}\|u_{xx}(t)\|_{L^2_x}$$

$$\leq c\,(\|u(t)\|_{L^2_x}\|u(t)\|^2_{H^1_x} + \|u(t)\|^2_{L^2_x}\|u(t)\|_{H^2_x})$$

$$\leq c\,\|u(t)\|^2_{L^2_x}\|u(t)\|_{H^2_x},$$

where we have used the product rule and Minkowski's inequality, (3.30) and the fact that $\|u(t)\|_{H^1_x}$ is bounded for $t \in [0, M]$. Therefore,

$$\|u(t)\|_{H^2} \leq \|u_0\|_{H^2} + c\int_0^t \|u(t')\|^2_{L^2_x}\|u(t')\|_{H^2_x}\, dt'$$

$$\leq \|u_0\|_{H^2} + cR\int_0^t \|u(t')\|_{H^2_x}\, dt'.$$

Then from Gronwall's inequality it follows that, for all $t \in [0, M]$,

$$\|u(t)\|_{H^2_x} \leq \|u_0\|_{H^2}\, e^{cRt} \leq \|u_0\|_{H^2}\, e^{cRM} < \infty.$$

By using the same strategy as above, we can show that if $u_0 \in H^r$, then $M(u_0, r) = M(u_0, 0)$ for every positive integer value of r.

We will next show that if $u_0 \in H^r$, then $M(u_0, 0) = \infty$, and $P(u(t))$ and $E(u(t))$ are independent of t for $t \geq 0$. For this proof, we let $M < M(u_0, 0)$, so that a strong solution u with initial value u_0 exists in $C_t^0 L_x^2\left([0, M] \times \mathbb{T}\right)$. Note that since $M(u_0, r) = M(u_0, 0)$, u is also a strong solution in $C_t^0 H_x^r\left([0, M] \times \mathbb{T}\right)$. We can then conclude the independence of $P(u(t))$ and $E(u(t))$ by applying Theorem 3.7. We showed that $P(u(t))$ and $E(u(t))$ are constant for $t \in [0, M]$, for all $M < M(u_0, 0)$. Therefore, $\|u(t)\|_{L_x^2}$ is constant for $0 \leq t < M(u_0, 0)$ and we can conclude that $M(u_0, 0) = \infty$.

Next, we show that if $u_0 \in L^2$ then $M(u_0, 0) = \infty$, and $P(u(t))$ is independent of t for $t \geq 0$. Moreover, if $\alpha = 0$, then $E(u(t))$ is also independent of t for $t \geq 0$.

To prove this, first choose $K > 0$ such that $\|u_0\|_{L^2} < K$. As we have shown above, there exists $M_K > 0$ so that whenever u_0, $v_0 \in L^2$ with $\|u_0\|_{L^2} \leq K$ and $\|v_0\|_{L^2} \leq K$, there exist corresponding strong solutions u and v in $C_t^0 L_x^2\left([0, M_K] \times \mathbb{T}\right)$ satisfying

$$\|u - v\|_{C_t^0 L_x^2([0, M_K] \times \mathbb{T})} \leq c \|u_0 - v_0\|_{L^2}. \tag{3.39}$$

Since H^2 is dense in L^2, we can let ϕ_n be a sequence of functions in H^2 such that $\|\phi_n\|_{L^2} \leq K$ for all n, and $\phi_n \to u_0$ in L^2. As shown above, for each n there exists a strong solution v_n in $C_t^0 L_x^2\left([0, M_K] \times \mathbb{T}\right)$ with $P(v_n(t)) = P(\phi_n)$ and $E(v_n(t)) = E(\phi_n)$ for all $t \in [0, M_K]$. From (3.39), we have that $v_n \to u$ in $C_t^0 L_x^2\left([0, M_K] \times \mathbb{T}\right)$. Hence for all $t \in [0, M_K]$ we have that $P(u(t)) = \lim_{n\to\infty} P(v_n(t)) = \lim_{n\to\infty} P(\phi_n) = P(u_0)$; and if $\alpha = 0$, since $E(u)$ is continuous (see Remark 3.19), then $E(u(t)) = \lim_{n\to\infty} E(v_n(t)) = \lim_{n\to\infty} E(\phi_n) =$

$E(u_0)$.

Now, since we have that $\|u(M_K)\|_{L^2} = \|u_0\|_{L^2} < K$, we can repeat the argument with $u(M_K)$ as initial data, to obtain a strong solution $u \in C_t^0 L_x^2\left([0, 2M_K] \times \mathbb{T}\right)$ with $P(u(t))$ constant for $t \in [0, 2M_K]$. Iterating this argument gives that $M(u_0, 0) = \infty$ and $P(u(t))$ is constant for all $z \geq 0$; moreover, $E(u(t))$ is constant for $t \geq 0$, if $\alpha = 0$.

We now show that the map from initial data to strong solutions in H^r is locally Lipschitz: for every $K > 0$ and $M > 0$, there exists $C > 0$ such that if $u_0, v_0 \in H^r$ with $\|u_0\|_{H^r} \leq K$ and $\|v_0\|_{H^r} \leq K$, and u and v are strong solutions in $C_t^0 H_x^r\left([0, M] \times \mathbb{T}\right)$ with initial data u_0 and v_0, then

$$\|u - v\|_{C_t^0 H_x^r([0,M]\times\mathbb{T})} \leq \|u_0 - v_0\|_{H^r}.$$

Suppose $\|u_0\|_{H^r} \leq K$ and $\|v_0\|_{H^r} \leq K$, let M be given, and let u and v be the corresponding strong solutions in $C_t^0 H_x^r\left([0, M] \times \mathbb{T}\right)$. From above we have that $\|u(t)\|_{L_x^2} \leq K$ and $\|v(t)\|_{L_t^2} \leq K$ for all t. Define $R = \max(\|u\|_{C_t^0 H_x^r([0,M]\times\mathbb{T})}, \|v\|_{C_t^0 H_x^r([0,M]\times\mathbb{T})})$. Then for all $t \in [0, M]$, we have that

$$\|u(t) - v(t)\|_{H_x^r} =$$

$$\left\| S(\alpha t)(u_0 - v_0) + i\int_0^t S(\alpha(t - t'))[F(u(t')) - F(v(t'))]\, dt' \right\|_{H_x^r}$$

$$\begin{aligned} &\leq \|u_0 - v_0\|_{H^r} + \int_0^t \|[F(u(t')) - F(v(t'))]\|_{H_x^r}\, dt' \\ &\leq \|u_0 - v_0\|_{H^r} + CKR\int_0^t \|u(t') - v(t')\|_{H_x^r}\, dt'. \end{aligned}$$

So from Gronwall's inequality it follows that

$$\|u(t) - v(t)\|_{H_x^r} \leq e^{CKRt}\|u_0 - v_0\|_{H^r} \leq e^{CKRM}\|u_0 - v_0\|_{H^r}$$

for all $t \in [0, M]$.

$\square$

3.4 Well-posedness in $L^2 \cap L^4$

In this section we prove a result that shows that the periodic DMNLS equation is locally well-posed in $L^2 \cap L^4([0, M] \times \mathbb{T})$. This result can also be interpreted as establishing a smoothing property for DMNLS: for every choice of initial data in $L^2(\mathbb{T})$, although this data may not be in $L^4(\mathbb{T})$, the unique solution $u(t, x)$ found in Theorem 3.20 is in $L^4_{t,x}([0, M] \times \mathbb{T})$, and therefore $u(\cdot, x)$ is in $L^4(\mathbb{T})$ for almost every $t \in [0, M]$.

The proof of this result requires the following lemmas.

Lemma 3.21. *Suppose $\alpha > 0$. Then there exists $C > 0$ such that for all $u_0(x) \in L^2(\mathbb{T})$, $T \in (0, 1]$, and $G \in \mathcal{Y}$, the function $Q(t, x)$ defined by*

$$Q(t, x) := \eta_1(t) S(\alpha t) u_0 + i \eta_T(t) \int_0^t S(\alpha(t - t')) G(t', x) \, dt'$$

satisfies

$$\|Q\|_{C_t^0 L_x^2} + \|Q\|_{L^4_{t,x}} \le C \left(\|u_0\|_{L^2} + T^{\frac{1}{16}} \|G\|_{L^{\frac{4}{3}}_{t,x}} \right).$$

Proof. From the triangle inequality, Lemma 3.11 and Lemma 3.13, we have that

$$\begin{aligned} \|Q\|_{C_t^0 L_x^2} &\le \|\eta_1(t) S(\alpha t) u_0\|_{C_t^0 L_x^2} + \left\| \eta_T(t) \int_0^t S(\alpha(t - t')) G(t', x) \, dt' \right\|_{C_t^0 L_x^2} \\ &\le c \left(\|u_0\|_{L^2} + T^{\frac{1}{16}} \|G\|_{X^{0, -\frac{3}{8}}_{\kappa = -\alpha n^2}} \right). \end{aligned} \tag{3.40}$$

Also, applying another periodic Strichartz estimate (see [4, 20]) results in

$$\|Q\|_{L^4_{t,x}} \le c\,\|Q\|_{X^{0,\frac{3}{8}}_{\kappa=-\alpha n^2}} \le c\,\|Q\|_{X^{0,\frac{9}{16}}_{\kappa=-\alpha n^2}}. \tag{3.41}$$

Hence by the triangle inequality, Lemma 3.14 and Lemma 3.15 with $b' = -\frac{3}{8}$, we have that

$$\begin{aligned}\|Q\|_{X^{0,\frac{9}{16}}_{\kappa=-\alpha n^2}} &\le \|\eta_1(t)S(\alpha t)u_0\|_{X^{0,\frac{9}{16}}_{\kappa=-\alpha n^2}} + \left\|\eta_T(t)\int_0^t S(\alpha(t-t'))G(t',x)\,dt'\right\|_{X^{0,\frac{9}{16}}_{\kappa=-\alpha n^2}} \\ &\le c\left(\|u_0\|_{L^2} + T^{\frac{1}{16}}\,\|G\|_{X^{0,-\frac{3}{8}}_{\kappa=-\alpha n^2}}\right). \end{aligned} \tag{3.42}$$

Note also that by duality and the Bourgain estimate in (3.41),

$$\begin{aligned}\|G\|_{X^{0,-\frac{3}{8}}_{\kappa=-\alpha n^2}} &= \sup_{v\in X^{0,\frac{3}{8}}_{\kappa=-\alpha n^2},\|v\|\le 1} \int G\bar{v}\,dx\,dt \\ &\le \sup_{v\in X^{0,\frac{3}{8}}_{\kappa=-\alpha n^2},\|v\|\le 1} \|v\|_{L^4_{t,x}}\|G\|_{L^{\frac{4}{3}}_{t,x}} \\ &\le \sup_{v\in X^{0,\frac{3}{8}}_{\kappa=-\alpha n^2},\|v\|\le 1} \|v\|_{X^{0,\frac{3}{8}}_{\kappa=-\alpha n^2}}\|G\|_{L^{\frac{4}{3}}_{t,x}} \\ &\le \|G\|_{L^{\frac{4}{3}}_{t,x}}. \end{aligned} \tag{3.43}$$

Now Lemma 3.21 follows if we apply (3.43) to the sum of (3.40) and (3.42). □

Lemma 3.22. *There exists $c > 0$ such that for $u_1, u_2, u_3 \in C^\infty$ we have*

$$\|F(u_1,u_2,u_3)\|_{L^{\frac{4}{3}}_x} \le c\,\|u_1\|_{L^4_x}\|u_2\|_{L^4_x}\|u_3\|_{L^4_x}, \tag{3.44}$$

where $F(u_1,u_2,u_3)$ is as defined in Lemma 3.17.

Proof. The proof is similar to that of Lemma 3.17.

$$\|F(u_1,u_2,u_3)\|_{L_x^{\frac{4}{3}}} = \sup_{\|v\|_{L_x^4}\le 1}\left|\int_{\mathbb{T}} F(u_1,u_2,u_3)\bar{v}\, dx\right|$$

$$= \frac{1}{B}\sup_{\|v\|_{L_x^4}\le 1}\left|\int_{\mathbb{T}}\left(\int_0^B T_{D(s)}^{-1}[T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3]\, ds\right)\cdot\bar{v}\, dx\right|$$

$$= \frac{1}{B}\sup_{\|v\|_{L_x^4}\le 1}\left|\int_0^B\int_{\mathbb{T}} T_{D(s)}^{-1}[T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3]\cdot\bar{v}\, dx\, ds\right|$$

$$= \frac{1}{B}\sup_{\|v\|_{L_x^4}\le 1}\left|\int_0^B\int_{\mathbb{T}} T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3\cdot\overline{T_{D(s)}v}\, dx\, ds\right|$$

$$\le \frac{1}{B}\sup_{\|v\|_{L_x^4}\le 1}\int_0^B\int_{\mathbb{T}} |T_{D(s)}u_1\cdot\overline{T_{D(s)}u_2}\cdot T_{D(s)}u_3\cdot\overline{T_{D(s)}v}|\, dx\, ds.$$

It follows from Holder's inequality and the Strichartz estimate found in Theorem 3.16 that

$$\|F\|_{L_x^{\frac{4}{3}}} \le c\sup_{\|v\|_{L_x^4}\le 1}\|T_{D(s)}u_1\|_{L^4_{s,x}}\|T_{D(s)}u_2\|_{L^4_{s,x}}\|T_{D(s)}u_3\|_{L^4_{s,x}}\|T_{D(s)}v\|_{L^4_{s,x}}$$

$$\le c\sup_{\|v\|_{L_x^4}\le 1}\|u_1\|_{L_x^2}\|u_2\|_{L_x^2}\|u_3\|_{L_x^2}\|v\|_{L_x^2}$$

$$\le c\sup_{\|v\|_{L_x^4}\le 1}\|u_1\|_{L_x^4}\|u_2\|_{L_x^4}\|u_3\|_{L_x^4}\|v\|_{L_x^4}$$

$$\le c\,\|u_1\|_{L_x^4}\|u_2\|_{L_x^4}\|u_3\|_{L_x^4}. \qquad \square$$

Lemma 3.23. *Let $F(u_1,u_2,u_3)$ be as defined in Lemma 3.17, and $u\in C^\infty$. Define*

$$F(u(t,x)) = F(u) := F(u,u,u) = \frac{1}{B}\int_0^B T_{D(s)}^{-1}[|T_{D(s)}u|^2 T_{D(s)}u]\, ds.$$

Then for all $u, v\in C^\infty$, we have

$$\|F(u)\|_{L_{t,x}^{\frac{4}{3}}} \le c\,\|u\|^3_{L^4_{t,x}} \tag{3.45}$$

and

$$\|F(u)-F(v)\|_{L^{\frac{4}{3}}_{t,x}} \le c\,\|u-v\|_{L^4_{t,x}}(\|u\|_{L^4_{t,x}}+\|v\|_{L^4_{t,x}})^2. \tag{3.46}$$

Proof. The estimate (3.45) follows immediately from (3.44) and integrating with respect to t. Again utilizing (3.33), we have

$$\begin{aligned}\|F(u)-F(v)\|_{L^{\frac{4}{3}}_{t,x}} &\le \|F(u-v,u,u)\|_{L^{\frac{4}{3}}_{t,x}} + \|F(u-v,v,v)\|_{L^{\frac{4}{3}}_{t,x}} \\ &\quad + \|F(v,u-v,u)\|_{L^{\frac{4}{3}}_{t,x}}\end{aligned}$$

$$\le c\left(\int_{\mathbb{R}} (\|u-v\|_{L^4_x}\|u\|^2_{L^4_x})^{\frac{4}{3}}\,dt\right)^{\frac{3}{4}} + c\left(\int_{\mathbb{R}} (\|u-v\|_{L^4_x}\|v\|^2_{L^4_x})^{\frac{4}{3}}\,dt\right)^{\frac{3}{4}}$$

$$+c\left(\int_{\mathbb{R}} (\|u-v\|_{L^4_x}\|u\|_{L^4_x}\|v\|_{L^4_x})^{\frac{4}{3}}\,dt\right)^{\frac{3}{4}}.$$

Apply Holder's inequality to each of the sums to get

$$\begin{aligned}\|F(u)-F(v)\|_{L^{\frac{4}{3}}_{t,x}} &\le c\left\|\|u-v\|^{\frac{4}{3}}_{L^4_x}\right\|^{\frac{3}{4}}_{L^3_t}\cdot\left\|\|u\|^{\frac{8}{3}}_{L^4_x}\right\|^{\frac{3}{4}}_{L^{\frac{3}{2}}_t} \\ &\quad +c\left\|\|u-v\|^{\frac{4}{3}}_{L^4_x}\right\|^{\frac{3}{4}}_{L^3_t}\cdot\left\|\|v\|^{\frac{8}{3}}_{L^4_x}\right\|^{\frac{3}{4}}_{L^{\frac{3}{2}}_t} \\ &\quad +c\left\|\|u-v\|^{\frac{4}{3}}_{L^4_x}\right\|^{\frac{3}{4}}_{L^3_t}\cdot\left\|\|u\|^{\frac{4}{3}}_{L^4_x}\right\|^{\frac{3}{4}}_{L^3_t}\cdot\left\|\|v\|^{\frac{4}{3}}_{L^4_x}\right\|^{\frac{3}{4}}_{L^3_t}\end{aligned}$$

$$= c\,(\|u-v\|_{L^4_{t,x}}\|u\|^2_{L^4_{t,x}} + \|u-v\|_{L^4_{t,x}}\|v\|^2_{L^4_{t,x}} + \|u-v\|_{L^4_{t,x}}\cdot\|u\|_{L^4_{t,x}}\|v\|_{L^4_{t,x}})$$

$$\le c\,\|u-v\|_{L^4_{t,x}}(\|u\|_{L^4_{t,x}}+\|v\|_{L^4_{t,x}})^2.$$

□

Theorem 3.24. *Suppose $\alpha\in\mathbb{R}\backslash\{0\}$ and $D(s)$ satisfies Assumption 1. If $u_0\in L^2$, then for every $M>0$, the unique strong solution $u\in C^0_tL^2_x([0,M]\times\mathbb{T})$ found in Theorem 3.20 with initial data u_0, exists in $L^4_{t,x}([0,M]\times\mathbb{T})$.*

Proof. We denote the closed ball of radius a centered at the origin in $C^0_tL^2_x\cap$

$L^4_{t,x}([0,M]\times\mathbb{T})$ by

$$\Lambda_{M,a} = \{u \in C^0_t L^2_x \cap L^4_{t,x}([0,M]\times\mathbb{T})\colon \|\!|u|\!\| \le a\},$$

where $\|\!|u|\!\| = \|u\|_{C^0_t L^2_x} + \|u\|_{L^4_{t,x}}$. Fix $u_0 \in L^2$ such that $\|u\|_{L^2} \le K$. For each $M > 0$, we define $Q : C^0_t L^2_x \cap L^4_{t,x}([0,M]\times\mathbb{T}) \to C^0_t L^2_x \cap L^4_{t,x}([0,M]\times\mathbb{T})$ by setting, for $t \in [0,M]$ and $u \in C^0_t L^2_x \cap L^4_{t,x}([0,M]\times\mathbb{T})$,

$$Q(u)(t) := \eta_1(t) S(\alpha t) u_0 + i\eta_M(t) \int_0^t S(\alpha(t-t'))F(u(t',x))\,dt',$$

with η as previously defined, and F defined in Lemma 3.17. Then by Lemmas 3.21 and 3.23, we have that

$$\|\!|Q(u)|\!\| \le c\left(\|u_0\|_{L^2} + M^{\frac{1}{16}}\|F\|_{L^{\frac{4}{3}}_{t,x}}\right)$$

$$\le c\left(\|u_0\|_{L^2} + M^{\frac{1}{16}}\|u\|^3_{L^4_{t,x}}\right).$$

In addition, if $u, v \in C^0_t L^2_x \cap L^4_{t,x}([0,M]\times\mathbb{T})$, then Lemma 3.21 and Lemma 3.23 imply that

$$\|\!|Q(u) - Q(v)|\!\| \le c\, M^{\frac{1}{16}} \|u-v\|_{L^4_{t,x}} (\|u\|_{L^4_{t,x}} + \|v\|_{L^4_{t,x}})^2 \tag{3.47}$$

Now suppose $a \ge K$, and choose $M = \left(\frac{1}{8ca^2}\right)^{16}$ so that for all $u, v \in \Lambda_{M,a}$,

$$\|\!|Q(u)|\!\| < \frac{a}{2} + \frac{a}{2} = a,$$

and

$$\|\!|Q(u) - Q(v)|\!\| < \frac{1}{2}\|u-v\|_{L^4_{t,x}}.$$

Therefore $Q(u)$ defines a contraction from the closed ball $\Lambda_{M,a}$ to itself, and so by the Banach Contraction Mapping Theorem, $Q(u)$ has a unique fixed point $u \in \Lambda_{M,a}$. This fixed point is also a strong solution of (3.1) with initial data u_0, and is in $L^4_{t,x}([0,M] \times \mathbb{T})$.

Let $K > 0$, $u_0, v_0 \in L^2$ with $\|u_0\|_{L^2} \leq K$, and $\|v_0\|_{L^2} \leq K$. For $a = 2K$, and M defined as above, let u and v be the unique solutions in $\Lambda_{M,a}$ with initial data u_0 and v_0 respectively. Then we will show that

$$|||u - v||| \leq c\|u_0 - v_0\|_{L^2}. \tag{3.48}$$

Define $\Psi : C^0_t L^2_x \cap L^4_{t,x}\,([0,M] \times \mathbb{T}) \times L^2_x \to C^0_t L^2_x \cap L^4_{t,x}\,([0,M] \times \mathbb{T})$ by

$$\Psi(u,w) = \eta_1(t)S(\alpha t)u_0 + i\eta_M(t)\int_0^t S(\alpha(t-t'))F(u(t',x))\,dt'.$$

Then $\Psi(u, u_0) = u$ and $\Psi(v, v_0) = v$. So for every $t \in [0, M]$,

$$|||u - v||| = |||\Psi(u,u_0) - \Psi(v,u_0) + \Psi(v,u_0) - \Psi(v,v_0)|||$$

$$\leq |||\Psi(u,u_0) - \Psi(v,u_0)||| + |||\Psi(v,u_0) - \Psi(v,v_0)|||$$

$$= |||Q(u) - Q(v)||| + \|u_0 - v_0\|_{L^2}$$

$$\leq \frac{1}{2}\|u - v\|_{L^4_{t,x}} + \|u_0 - v_0\|_{L^2}$$

$$\leq \frac{1}{2}|||u - v||| + \|u_0 - v_0\|_{L^2},$$

which in turn implies (3.48).

□

Chapter 4

Existence and Stability of Ground State Solutions

Solitary wave solutions of (3.1) are solutions of the form

$$u(t,x) = e^{i\theta t}\phi(x)$$

where $\theta \in \mathbb{R}$ and $\phi \in L^2(\mathbb{T})$. Note that if we substitute $u = e^{i\theta t}\phi(x)$ into (3.1), we see that $u(t,x)$ is a standing wave solution if and only if $E'(v) = \theta P'(v)$. Here $E'(v)$ and $P'(v)$ denote the Frechet derivatives of the already introduced functionals $E(v)$ and $P(v)$ respectively. Therefore, we have (see [18]) that ϕ is a critical point of the variational problem of minimizing $E(v)$ subject to $P(v) = \lambda$. Hence, $u(t,x)$ is a standing wave solution if and only if ϕ is a critical point of $E(v)$ subject to the constraint $P(v) = \lambda$, and $u(t,x)$ is a ground state solution if and only if ϕ is a global minimizer for $E(v)$ subject to this constraint. The aim of this chapter is to show that stable ground states exist by finding global minimizers of this constrained variational problem.

4.1 Existence of Minimizers in H^1

In this section, for $\alpha \neq 0$, we find a solution to the problem of minimizing the functional

$$E(v) = \frac{\alpha}{B}\int_0^B \int_{\mathbb{T}} |v_x|^2 \, dx \, ds - \frac{1}{2B}\int_0^B \int_{\mathbb{T}} |T_{D(s)}v|^4 \, dx \, ds \tag{4.1}$$

under the constraints $v \in H^1(\mathbb{T})$ and $P(v) = \int_{\mathbb{T}} |v|^2 \, dx = \lambda$. Let I_λ be defined as follows:

$$I_\lambda = \inf\{E(v) : v \in H^1 \text{ and } P(v) = \lambda\}.$$

Theorem 4.1. *Suppose that $\alpha > 0$ and that $\Delta'(s)$ satisfies Assumption 1. Then for every $\lambda > 0$, there exists at least one minimizer for I_λ in $H^1(\mathbb{T})$. Moreover, every minimizing sequence for I_λ has a subsequence which converges strongly in H^1 to some minimizer u.*

Proof. We will prove Theorem 4.1 for $B = 1$. The proof for general values of B follows similar steps as the one below. Fix $\lambda > 0$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of $E(v)$ subject to $P(v) = \lambda$. That is, we have that $u_n \in H^1(\mathbb{T})$ and $P(u_n) = \lambda$ for all $n \in \mathbb{N}$, and $E(u_n) \to I_\lambda$ as $n \to \infty$.

Since $\{E(u_n)\}$ is a convergent sequence, then it is bounded. Therefore,

$$\|u_n\|_{H^1}^2 = \|u_n\|_{L^2}^2 + \|u_{n_x}\|_{L^2}^2$$

$$= \lambda + \int_{\mathbb{T}} |u_{n_x}|^2 \, dx$$

$$= \lambda + \int_0^1 \int_{\mathbb{T}} |u_{n_x}|^2 \, dx \, ds$$

$$= \lambda + \int_0^1 \int_{\mathbb{T}} |u_{n_x}|^2 \, dx \, ds - \frac{1}{2\alpha}\int_0^1 \int_{\mathbb{T}} |T_{D(s)}u_n|^4 \, dx \, ds$$

$$+\frac{1}{2\alpha}\int_0^1\int_{\mathbb{T}}|T_{D(s)}u_n|^4\,dx\,ds$$

$$= \frac{1}{\alpha}\left(\alpha\lambda + E(u_n) + \frac{1}{2}\int_0^1\int_{\mathbb{T}}|T_{D(s)}u_n|^4\,dx\,ds\right)$$

$$\le \frac{1}{|\alpha|}\left(|\alpha|\lambda + |E(u_n)| + \frac{1}{2}\|T_{D(s)}u_n\|^4_{L^4_{s,x}}\right)$$

$$\le \frac{1}{|\alpha|}\left(|\alpha|\lambda + |E(u_n)| + C\|u_n\|^4_{L^2}\right),$$

by the Strichartz estimate in (3.28). Also, because $||u_n||^4_{L^2} = P(u_n) = \lambda$ for all n, it follows that $\{u_n\}$ is bounded in $H^1(\mathbb{T})$. This in turn implies that $\{u_n\}$ has a weakly convergent subsequence in $H^1(\mathbb{T})$, which we continue to denote by $\{u_n\}$. We will also define $u \in H^1(\mathbb{T})$ to be the weak limit of this subsequence. By the Rellich-Kondrachov Theorem for compact manifolds, $H^1(\mathbb{T})$ is compactly embedded in $L^2(\mathbb{T})$. Hence we can conclude that $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, that converges strongly to u in $L^2(\mathbb{T})$.

As a result of the weak convergence of $\{u_n\}$ to u in $H^1(\mathbb{T})$, by the weak lower semicontinuity of the norm in a Hilbert space, we have that $\|u\|_{H^1(\mathbb{T})} \le \liminf_{n\to\infty}\|u_n\|_{H^1(\mathbb{T})}$. Again, by the $L^4_{s,x}$ Strichartz estimate in (3.28), we know that $\|T_{D(s)}(u_n - u)\|_{L^4_{s,x}} \le c\|u_n - u\|_{L^2}$. So, $T_{D(s)}u_n \to T_{D(s)}u$ in $L^4_{s,x}$, and $\|T_{D(s)}u_n\|_{L^4_{s,x}} \to \|T_{D(s)}u\|_{L^4_{s,x}}$. Thus,

$$\lim_{n\to\infty}\|u_n\|^2_{H^1(\mathbb{T})} = \frac{1}{\alpha}\left(I_\lambda + \alpha\lambda + \frac{1}{2}\|T_{D(s)}u\|_{L^4_{s,x}}\right).$$

Convergence of $\{u_n\}$ to u in L^2 implies that $\lim_{n\to\infty} P(u_n) = P(u)$, which in turn means that $P(u) = \lambda$. Hence, by the definition of I_λ, $E(u) \ge I_\lambda$. Note also that

$$I_\lambda = \lim_{n\to\infty} E(u_n)$$

$$= \lim_{n\to\infty} \alpha \int_0^1 \int_{\mathbb{T}} |u^\lambda_{n_x}|^2 \, dx \, ds - \frac{1}{2} \int_{\mathbb{T}} \int_0^1 |T_{D(s)} u_n|^4 \, ds \, dx$$

$$= \lim_{n\to\infty} \Big(\alpha \int_{\mathbb{T}} |u^\lambda_{n_x}|^2 \, dx - \frac{1}{2} \int_{\mathbb{T}} \int_0^1 |T_{D(s)} u_n|^4 \, ds \, dx$$

$$+ \alpha \int_{\mathbb{T}} |u_n|^2 \, dx - \alpha \int_{\mathbb{T}} |u_n|^2 \, dx \Big)$$

$$= \lim_{n\to\infty} \left(\alpha \|u_n\|^2_{H^1(\mathbb{T})} - \frac{1}{2} \int_{\mathbb{T}} \int_0^1 |T_{D(s)} u_n|^4 \, ds \, dx - \alpha \int_{\mathbb{T}} |u_n|^2 \, dx \right)$$

$$= \lim_{n\to\infty} \alpha \|u_n\|^2_{H^1(\mathbb{T})} - \frac{1}{2} \int_{\mathbb{T}} \int_0^1 |T_{D(s)} u|^4 \, ds \, dx - \alpha\lambda.$$

By the lower semicontinuity of the norm it follows that

$$I_\lambda \geq \alpha \|u\|^2_{H^1(\mathbb{T})} - \frac{1}{2} \int_{\mathbb{T}} \int_0^1 |T_{D(s)} u|^4 \, ds \, dx - \alpha\lambda$$

$$= \alpha \|u\|^2_{H^1(\mathbb{T})} - \frac{1}{2} \int_{\mathbb{T}} \int_0^1 |T_{D(s)} u|^4 \, ds \, dx - \alpha \int_{\mathbb{T}} |u|^2 \, dx$$

$$= E(u).$$

So, $E(u) = I_\lambda$. Since $\{u_n\}$ is a minimizing sequence, then

$$\lim_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{T})} = \lim_{n\to\infty} \frac{1}{\alpha} \left(\alpha\lambda + E(u_n) + \frac{1}{2} \int_0^1 \int_{\mathbb{T}} |T_{D(s)} u_n|^4 \, dx \, ds \right)$$

$$= \frac{1}{\alpha} \left(I_\lambda + \alpha\lambda + \frac{1}{2} \|T_{D(s)} u\|_{L^4_{s,x}} \right)$$

$$= \frac{1}{\alpha} \left(E(u) + \alpha\lambda + \frac{1}{2} \|T_{D(s)} u\|_{L^4_{s,x}} \right),$$

$$= \|u\|^2_{H^1(\mathbb{T})}.$$

Again, as a result of the weak convergence of $\{u_n\}$ to u in $H^1(\mathbb{T})$ and $\lim_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{T})} = \|u\|^2_{H^1(\mathbb{T})}$, we have that $\{u_n\}$ converges strongly to u in $H^1(\mathbb{T})$. Therefore $\{u_n\}$ and $\{u_{n_x}\}$ converge in $L^2(\mathbb{T})$ to u and u_x respectively. Equation (3.28) implies that $\|T_{D(s)}(u_n - u)\|_{L^4_{s,x}} \leq c\|u_n - u\|_{L^2}$. So, $\{T_{D(s)}u_n\}$ converges strongly to $T_{D(s)}u$ in $L^4_{s,x}$, which in turn implies that $\|T_{D(s)}u_n\|_{L^4_{s,x}}$ converges to $\|T_{D(s)}u\|_{L^4_{s,x}}$. With this in mind, we find that the limit of $E(u_n)$ is

$$\begin{aligned}
\lim_{n\to\infty} E(u_n) &= \lim_{n\to\infty} \alpha\|u_{n_x}\|^2_2 - \frac{1}{2}\lim_{n\to\infty} \|T_{D(s)}u_n\|^4_{L^4_{s,x}} \\
&= \alpha\|u_x\|^2_2 - \frac{1}{2}\|T_{D(s)}u\|^4_{L^4_{s,x}} \\
&= E(u).
\end{aligned}$$

This shows that u is a constrained minimizer of $E(v)$. $\square$

4.2 Existence of Minimizers (Sufficient Condition)

In the case $\alpha = 0$, we cannot apply the argument of the preceding subsection to minimize $E(u)$. This is because, if we take a minimizing sequence $\{u_j\} \subset H^1$, there is no H^1 bound on $\{u_j\}$ that enables us to apply the Rellich-Kondrachov Theorem to show that a subsequence converges strongly in L^2 to a minimizer. However, it is still possible to establish strong convergence in L^2, if we can decompose the minimizing sequence into two parts; a regular low-frequency part which is bounded in H^1, to which we can apply the Rellich-Kondrachov Theorem, and a small high-frequency part in L^2, to which we can apply weak lower semicontinuity of the norm in a Hilbert space. Kunze [15] explained that in the non-periodic case, the minimizing sequence can be separated into the two parts if the corresponding sequence of the Fourier transforms has a subsequence that is tight in the sense of measures. A variant of the method of concentration compactness was then applied to show that a subsequence of the Fourier transforms is indeed tight.

It turns out that this idea is useful in the periodic case as well, but requires significant modifications. In this section, we explain how to adopt Kunze's method in the periodic case to provide a proof of the existence of minimizers in L^2.

We note that the problem of minimizing $E(v)$ subject to $P(v) = \lambda$ is the same as maximizing $W(v)$ subject to $P(v) = \lambda$, where

$$W(v) = -E(v) = \int_0^B \int_{\mathbb{T}} |T(s)v|^4 \, dx \, ds \qquad B > 0 \tag{4.2}$$

under the constraint $v \in L^2(\mathbb{T})$, $P(v) = \int_{\mathbb{T}} |u|^2 \, dx = \lambda$, and the operator $T(s)$ is the Fourier multiplier operator $T_{D(s)} = e^{iD(s)\partial_x^2}$ with $D(s) = s$. In this section and onward, the variable B here represents half the period of $\Delta(s)$.

Note that if we define $D(s)$ to be

$$D(s) = \begin{cases} s & 0 \le s \le B \\ 2B - s & B \le s \le 2B \quad B > 0, \end{cases}$$

then we have that

$$\|T_{D(s)}v\|^4_{L^4_{s,x}([0,2B]\times\mathbb{T})} = 2\|T(s)v\|^4_{L^4_{s,x}([0,B]\times\mathbb{T})}.$$

Therefore, it is sufficient to consider maximizing $\|T(s)v\|^4_{L^4_{s,x}([0,B]\times\mathbb{T})}$ subject to $P(v) = \lambda$. Define

$$J_\lambda = \sup\{W(u) : u \in L^2 \text{ and } P(u) = \lambda\}. \tag{4.3}$$

We will repeat the statement provided in Theorem 1.1.

Theorem 4.2. *Let $\lambda > 0$. Suppose there exists a function $w \in L^2$ with $P(w) = \lambda$, such that $A(\hat{w}) - 2\pi B\|\hat{w}\|^4_{l^4} > 0$. Then there exists at least one maximizer for J_λ, in $L^2(\mathbb{T})$. Moreover, given the above condition, every maximizing sequence for J_λ has a subsequence which, after being suitably translated in Fourier space, converges strongly in L^2 to some maximizer. Here, we define*

$$A(\hat{w}) = 2\pi \sum_n \sum_{p\neq 0} \sum_{l\neq 0} \frac{i}{2lp}\left[e^{-2ilpB} - 1\right] \hat{w}(n)\,\bar{\hat{w}}(n-l)\,\hat{w}(n-p-l)\,\bar{\hat{w}}(n-p).$$

To construct the proof of the above theorem, it is necessary to introduce some supplementary results and estimates.

4.2.1 The Concentration Compactness Method

The concentration compactness method is commonly used to prove stability of wave solutions of a differential equation by ensuring the compactness of minimizing or maximizing sequences. The concentration compactness principle says that for any L^2 bounded sequence of functions, there is a subsequence that satisfies exactly one of the three possibilities: the sequence is tight, or it is vanishing (tends to zero uniformly on balls of fixed radius), or it splits into two other functions with separated supports and fixed masses. We will state a specific case of the principle below along with a proof.

Lemma 4.3. *Suppose* $p > 0$, *and let* $\{a_j\} \subset l^2$ *be a sequence of sequences such that* $\|a_j\|_{l^2}^2 = p$ *for* $j \in \mathbb{N}$. *Then, there is a subsequence of* $\{a_j\}$, *still denoted by* $\{a_j\}$, *for which one of the following three statements is true.*

1) Concentration / Tightness:

There exists a sequence of integers $m_1, m_2, m_3, ...$ *such that for every* $\epsilon > 0$, *there exists* $r = r(\epsilon)$ *with the property that*

$$\sum_{m_j - r}^{m_j + r} |a_j(n)|^2 \, dx \geq p - \epsilon, \qquad \textit{for all } j \in \mathbb{N}.$$

2) Vanishing:

For every $r > 0$,

$$\lim_{j \to \infty} \sup_{m \in \mathbb{Z}} \sum_{m-r}^{m+r} |a_j(n)|^2 = 0.$$

3) Dichotomy / Splitting:

There is a number $\alpha \in (0, p)$ *and sequences* $\{b_j\}$ *and* $\{c_j\}$ *in* l^2 *such that* $d(supp(b_j), supp(c_j)) \to \infty$, $\|a_j - b_j - c_j\|_{l^2}^2 \to 0$, $\|b_j\|_{l^2}^2 \to \alpha$, *and*

$\|c_j\|_{l^2}^2 \to p - \alpha.$

Specifically, for every $\delta \in (0, \alpha)$, *there exists* $j_0 = j_0(\delta)$ *and two integers* $r_1^* = r_1^*(\delta)$, $r_2^* = r_2^*(\delta)$ *such that*

$$\alpha - \delta < \sup_{m \in \mathbb{Z}} \sum_{m-r_2^*}^{m+r_2^*} |a_j(n)|^2 < \alpha + \delta \quad \textit{for all } j \geq j_0.$$

Also, for each $j \in \mathbb{N}$, *we may select* $m_j \in \mathbb{Z}$ *satisfying*

$$\alpha - \delta < \sum_{m_j - r_1^*}^{m_j + r_1^*} |a_j(n)|^2 < \alpha + \delta \quad \textit{for all } j \geq j_0.$$

Moreover, $lim_{\delta \to 0}\, r_2^*(\delta) - r_1^*(\delta) = \infty$.

Proof. It is sufficient to prove the lemma for a sequence $\{a_j\} \subset l^2$ such that $\|a_j\|_{l^2}^2 = 1$ for all $j \in \mathbb{N}$. Define, for each $j \in \mathbb{N}$ and $r \in \mathbb{N}$,

$$M_j(r) = \sup_{m \in \mathbb{Z}} \sum_{m-r}^{m+r} |a_j(n)|^2.$$

Then, M_j is such that $0 \leq M_j \leq 1$, and $M_j(r)$ as a function of r is non-decreasing. An elementary argument shows that any uniformly bounded sequence of non-decreasing functions has a subsequence converging pointwise to a non-negative and non-decreasing limit function. Hence, $\{M_j\}$ has such a subsequence which will be denoted again by $\{M_j\}$. We also continue to denote by $\{a_j\}$ the subsequence of $\{a_j\}$ corresponding to this subsequence of $\{M_j\}$. Let M be the non-decreasing function to which M_j converges. If $\alpha := \lim_{r \to \infty} M(r)$, then there are three possibilities: $\alpha = 1$, $\alpha = 0$, and $0 < \alpha < 1$.

Case 1: $\alpha = 0$.

Let $\epsilon > 0$. For each $r \in \mathbb{N}$,

$$|M_j(r)| = |M_j(r) - M(r) + M(r)| \leq |M_j(r) - M(r)| + |M(r)|.$$

Since $M(r)$ is non-negative and non-decreasing with limit 0, then $|M(r)| = 0$

for all $r \in \mathbb{N}$. Note that by the definition of pointwise limit, there exists N such that for $j \geq N$, $|M_j(r) - M(r)| < \epsilon$. Therefore, $|M_j(r)| < \epsilon$ for $j \geq N$, which implies that $\{a_j\}$ is vanishing

Case 2: $\alpha = 1$.

In this case, there exists r_0 such that for all $r \geq r_0$, $|M(r) - 1| < \frac{1}{2}$. Specifically, $|M(r_0) - 1| < \frac{1}{2}$, and $M(r_0) > \frac{1}{2}$. For any $j \in \mathbb{Z}$, choose $m_j \in \mathbb{Z}$ such that

$$M_j(r_0) \leq \sum_{m_j - r_0}^{m_j + r_0} |a_j(n)|^2 + \frac{1}{j}. \tag{4.4}$$

Now, for $0 < \epsilon < \frac{1}{3}$, there exists r_ϵ such that $|M(r_\epsilon) - 1| < \epsilon$. That is, $M(r_\epsilon) > 1 - \epsilon > \frac{2}{3}$. Again, for any $j \in \mathbb{Z}$, choose $m_j^\epsilon \in \mathbb{Z}$ such that

$$M_j(r_\epsilon) \leq \sum_{m_j^\epsilon - r_\epsilon}^{m_j^\epsilon + r_\epsilon} |a_j(n)|^2 + \frac{1}{j}. \tag{4.5}$$

Equations (4.4) and (4.5) imply that

$$\sum_{m_j^\epsilon - r_\epsilon}^{m_j^\epsilon + r_\epsilon} |a_j(n)|^2 + \sum_{m_j - r_0}^{m_j + r_0} |a_j(n)|^2 \geq M_j(r_\epsilon) + M_j(r_0) - \frac{2}{j}.$$

Therefore, for sufficiently large j, we have

$$\sum_{m_j^\epsilon - r_\epsilon}^{m_j^\epsilon + r_\epsilon} |a_j(n)|^2 + \sum_{m_j - r_0}^{m_j + r_0} |a_j(n)|^2 \geq M_j(r_\epsilon) + M_j(r_0) > \frac{1}{2} + \frac{2}{3} - \frac{2}{j} > 1 = \|a_j\|_{l^2}^2.$$

Thus, for large j, we have that

$$\{n \in [m_j^\epsilon - r_\epsilon \, , \, m_j^\epsilon + r_\epsilon]\} \cap \{n \in [m_j - r_0 \, , \, m_j + r_0]\} \neq \emptyset.$$

That is,

$$[m_j^\epsilon - r_\epsilon \, , \, m_j^\epsilon + r_\epsilon] \subset [m_j - (2r_\epsilon + r_0) \, , \, m_j + (2r_\epsilon + r_0)].$$

Therefore,

$$\sum_{m_j-(2r_\epsilon+r_0)}^{m_j+(2r_\epsilon+r_0)} |a_j(n)|^2 \geq 1 - \epsilon.$$

Now if we choose $r \geq (2r_\epsilon + r_0)$, we get that the inequality in *1)* is satisfied for sufficiently large j. This proves that $\{a_j\}$ is tight.

Case 3: $0 < \alpha < 1$.

Let $\delta \in (0, \alpha)$. We know that for $\delta > 0$, there exists r^* such that $\alpha - \delta < M(r) \leq \alpha$ for $r \geq r^*$. Choose r_1^*, $r_2^* > r^*$ with $r_2^* - r_1^* \geq 6\delta^{-1}$. Since $M_j(r_1^*) \to M(r_1^*)$ and $M_j(r_2^*) \to M(r_2^*)$, there exists j_0 such that for $j \geq j_0$, $\alpha - \delta < M_j(r_2^*) < \alpha + \delta$ and $\alpha - \delta < M_j(r_1^*) < \alpha + \delta$. Hence, for $j \geq j_0$,

$$\alpha - \delta < \sup_{m \in \mathbb{Z}} \sum_{m-r_2^*}^{m+r_2^*} |a_j(n)|^2 < \alpha + \delta,$$

and for each $j \geq j_0$, we can choose m_j such that

$$\alpha - \delta < \sum_{m_j-r_1^*}^{m_j+r_1^*} |a_j(n)|^2 < \alpha + \delta.$$

For m_j chosen, define b_j satisfying $b_j(n) = a_j(n)$ for $|n - m_j| \leq r_1^*$ and $b_j(n) = 0$ for $|n - m_j| \geq r_1^* + 2\delta^{-1}$. Also define c_j satisfying $c_j(n) = a_j(n)$ for $|n - m_j| \geq r_2^*$ and $c_j(n) = 0$ for $|n - m_j| \leq r_2^* - 2\delta^{-1}$. Now,

$$\|a_j - (b_j + c_j)\|_{l^2}^2 \leq \sum_{m_j+r_1^*}^{m_j+r_2^*} |a_j(n)|^2 + \sum_{m_j-r_2^*}^{m_j-r_1^*} |a_j(n)|^2$$

$$= \sum_{m_j - r_2^*}^{m_j + r_2^*} |a_j(n)|^2 - \sum_{m_j - r_1^*}^{m_j + r_1^*} |a_j(n)|^2$$

$$\leq \alpha + \delta - (\alpha - \delta)$$

$$= 2\delta.$$

Therefore,

$$\|a_j - b_j - c_j\|_{l^2}^2 \to 0.$$

Similarly,

$$\left| \|b_j\|_{l^2}^2 - \alpha \right| \leq \left| \sum_{m_j - r_1^* - 2\delta^{-1}}^{m_j + r_1^* + 2\delta^{-1}} |a_j(n)|^2 - \alpha \right|$$

$$\leq \left| \sum_{m_j - r_1}^{m_j + r_1} |a_j(n)|^2 - \alpha \right| + \left| \sum_{r_1^* \leq |n - m_j| \leq r_2^*} |a_j(n)|^2 \right|$$

$$\leq \delta + 2\delta = 3\delta,$$

which in turn implies that $\|b_j\|_{l^2}^2 \to \alpha$. Lastly,

$$\left| \|c_j\|_{l^2}^2 - (1 - \alpha) \right| = \left| \|c_j\|_{l^2}^2 - (\|a_j\|_{l^2}^2 - \alpha) \right|$$

$$= \left| \|c_j\|_{l^2}^2 - \|a_j\|_{l^2}^2 + \|b_j\|_{l^2}^2 + \alpha - \|b_j\|_{l^2}^2 \right|$$

$$\leq \left| \|c_j\|_{l^2}^2 - \|a_j\|_{l^2}^2 + \|b_j\|_{l^2}^2 \right| + \left| \alpha - \|b_j\|_{l^2}^2 \right|$$

$$\leq \Bigg| \sum_{|n - m_j| \geq r_2^*} |a_j(n)|^2 - \sum_{n \in \mathbb{Z}} |a_j(n)|^2 + \sum_{|n - m_j| \leq r_1^*} |a_j(n)|^2 +$$

$$+ \sum_{r_1^* \leq |n - m_j| \leq r_1^* + 2\delta^{-1}} |b_j(n)|^2 + \sum_{r_2^* - 2\delta^{-1} \leq |n - m_j| \leq r_2^*} |c_j(n)|^2 \Bigg| + 3\delta$$

$$= \Bigg| \sum_{r_1^* \leq |n - m_j| \leq r_1^* + 2\delta^{-1}} |b_j(n)|^2 \; - \sum_{r_1^* \leq |n - m_j| \leq r_2^*} |a_j(n)|^2 +$$

$$+ \sum_{r_2^*-2\delta^{-1}\le|n-m_j|\le r_2^*} |c_j(n)|^2 \Bigg| + 3\delta$$

$$\le \left| \sum_{r_1^*\le|n-m_j|\le r_2^*} |a_j(n)|^2 \right| + \left| \sum_{r_1^*\le|n-m_j|\le r_1^*+2\delta^{-1}} |b_j(n)|^2 \right| +$$

$$+ \left| \sum_{r_2^*-2\delta^{-1}\le|n-m_j|\le r_2^*} |c_j(n)|^2 \right| + 3\delta$$

$$\le 3 \left| \sum_{r_1^*\le|n-m_j|\le r_2^*} |a_j(n)|^2 \right| + 3\delta$$

$$\le 6\delta + 3\delta = 9\delta.$$

This shows that, $\|c_j\|_{l^2}^2 \to 1-\alpha$. □

4.2.2 Other Estimates

Lemma 4.4. *Suppose* $u_1, u_2, u_3,$ *and* u_4 *are in* $L^2(\mathbb{T})$*. Then*

$$\left| \int_0^B \int_{\mathbb{T}} T(s)u_1\, \overline{T(s)u_2}\, T(s)u_3\, \overline{T(s)u_4}\, dx\, ds \right| \le c\|u_1\|_{L^2}\|u_2\|_{L^2}\|u_3\|_{L^2}\|u_4\|_{L^2}. \tag{4.6}$$

Proof. $\left| \int_0^B \int_{\mathbb{T}} T(s)u_1\, \overline{T(s)u_2}\, T(s)u_3\, \overline{T(s)u_4}\, dx\, ds \right|$

$$\le \int_0^B \int_{\mathbb{T}} \left| T(s)u_1\, \overline{T(s)u_2}\, T(s)u_3\, \overline{T(s)u_4} \right|\, dx\, ds$$

$$\le \|T(s)u_1\|_{L^4_{s,x}}\ \|\overline{T(s)u_2}\|_{L^4_{s,x}}\ \|T(s)u_3\|_{L^4_{s,x}}\ \|\overline{T(s)u_4}\|_{L^4_{s,x}}$$
$$\le c\|u_1\|_{L^2}\ \|u_2\|_{L^2}\ \|u_3\|_{L^2}\ \|u_4\|_{L^2},$$

where we applied Holder's inequality and the Strichartz estimate in Theorem 3.16. □

Lemma 4.5. *For J_λ defined as above in (4.3), we have $J_\lambda = \lambda^2 J_1$.*

Proof. For u with $P(u) = \lambda$, let $v = \lambda^{-\frac{1}{2}}u$. Then, $\|v\|_{L^2}^2 = \|\lambda^{-\frac{1}{2}}u\|_{L^2}^2 = \lambda^{-1}\|u\|_{L^2}^2 = 1$. Also,

$$\begin{aligned}
W(u) &= \int_0^B \int_{\mathbb{T}} |T(s)u|^4 \, dx \, ds \\
&= \int_0^B \int_{\mathbb{T}} |T(s)(\lambda^{\frac{1}{2}}v)|^4 \, dx \, ds \\
&= \int_0^B \int_{\mathbb{T}} \lambda^{\frac{4}{2}} |T(s)v|^4 \, dx \, ds \\
&= \lambda^2 \|T(s)v\|_{L^4}^4 \\
&= \lambda^2 W(v).
\end{aligned}$$

Taking the supremum of this identity over all $u \in L^2$ with $P(u) = \lambda$ concludes the proof. $\square$

Lemma 4.6. *For the Fourier multiplier operator $T(s) = e^{is\partial_x^2}$, we have that*

$$\|T(s)u\|_{L^4_{s,x}([0,B]\times\mathbb{T})}^4 = 4\pi B\|\hat{u}\|_{l^2}^4 - 2\pi B\|\hat{u}\|_{l^4}^4 + A(\hat{u}),$$

where $\hat{u}(n) = \mathcal{F}_x(u)[n]$, and

$$A(\hat{u}) = 2\pi \sum_n \sum_{p\neq 0} \sum_{l\neq 0} \frac{i}{2lp} \left[e^{-2ilpB} - 1\right] \hat{u}(n)\, \bar{\hat{u}}(n-l)\, \hat{u}(n-p-l)\, \bar{\hat{u}}(n-p).$$

Proof. We will begin the proof in a similar fashion as Theorem 3.16. Let $g = T(s)u$; then by the definition of $T(s)$, $g = \sum_n \hat{u}(n)e^{i(nx-n^2s)}$. Thus,

$$\|T(s)u\|_{L^4_{s,x}([0,B]\times\mathbb{T})}^4 = \|g\|_{L^4_{s,x}([0,B]\times\mathbb{T})}^4 = \|g\cdot\bar{g}\|_{L^2_{s,x}([0,B]\times\mathbb{T})}^2$$

$$= \left\| \sum_n \sum_m \hat{u}(n)\, \bar{\hat{u}}(m)\, e^{i((n-m)x-(n^2-m^2)s)} \right\|_{L^2_{s,x}}^2$$

$$=\left\| \sum_n \sum_l \hat{u}(n)\, \bar{\hat{u}}(n-l)\, e^{ilx}\, e^{-il(2n-l)s} \right\|^2_{L^2_{s,x}},$$

where $l = n - m$. Let $b_l = \sum_n \hat{u}(n)\, \bar{\hat{u}}(n-l)\, e^{-il(2n-l)s}$, then

$$\begin{aligned}
\left\| \sum_n \sum_l \hat{u}(n)\, \bar{\hat{u}}(n-l)\, e^{ilx} e^{-il(2n-l)s} \right\|^2_{L^2_{s,x}} &= \left\| \sum_l b_l\, e^{ilx} \right\|^2_{L^2_{s,x}} \\
&= \int_0^B \int_{\mathbb{T}} \left| \sum_l b_l\, e^{ilx} \right|^2 dx\, ds \\
&= 2\pi \int_0^B \sum_l |b_l|^2\, ds \\
&= 2\pi \sum_l \int_0^B b_l \cdot \bar{b}_l\, ds \\
&= 2\pi \sum_l \sum_n \sum_r \hat{u}(n)\, \bar{\hat{u}}(n-l)\, \hat{u}(r-l)\, \bar{\hat{u}}(r) \int_0^B e^{-il(2n-2r)s}\, ds.
\end{aligned}$$

Now by direct integration,

$$\int_0^B e^{-il(2n-2r)s}\, ds = \begin{cases} \frac{i}{2l(n-r)} \left[e^{-2ilB(n-r)} - 1 \right] & n \neq r \text{ and } l \neq 0 \\ B & n = r \text{ or } l = 0. \end{cases}$$

Therefore,

$$\left\| \sum_n \sum_l \hat{u}(n)\, \bar{\hat{u}}(n-l)\, e^{ilx}\, e^{-il(2n-l)s} \right\|^2_{L^2_{s,x}} =$$

$$2\pi \sum_l \sum_n \sum_r \hat{u}(n)\, \bar{\hat{u}}(n-l) ... \begin{cases} \frac{i}{2l(n-r)} \left[e^{-2ilB(n-r)} - 1 \right] & n \neq r \text{ and } l \neq 0 \\ B & n = r \text{ or } l = 0. \end{cases} \tag{4.7}$$

We split the sum in (4.7) into four parts.

First we sum over all values of l, n, r such that $l \neq 0$ and $n = r$. This gives

$$2\pi B \sum_{l\neq 0} \sum_{n} \hat{u}(n)\, \bar{\hat{u}}(n-l)\, \hat{u}(n-l)\, \bar{\hat{u}}(n)$$

$$= 2\pi B \sum_{n} \sum_{m\neq n} \hat{u}(n)\, \bar{\hat{u}}(m)\, \hat{u}(m)\, \bar{\hat{u}}(n)$$

$$= 2\pi B \left(\sum_{n} \sum_{m} \hat{u}(n)\, \bar{\hat{u}}(m)\, \hat{u}(m)\, \bar{\hat{u}}(n) - \sum_{n} \hat{u}(n)\, \bar{\hat{u}}(n)\, \hat{u}(n)\, \bar{\hat{u}}(n) \right)$$

$$= 2\pi B \left(\|\hat{u}\|_{l^2}^4 - \|\hat{u}\|_{l^4}^4 \right).$$

Next we sum over all values of $l,\ n,\ r$ such that $l = 0$ and $n \neq r$. This gives

$$2\pi B \sum_{n} \sum_{r\neq n} \hat{u}(n)\, \bar{\hat{u}}(r)\, \hat{u}(r)\, \bar{\hat{u}}(n)$$

$$= 2\pi B \left(\sum_{n} \sum_{r} \hat{u}(n)\, \bar{\hat{u}}(r)\, \hat{u}(r)\, \bar{\hat{u}}(n) - \sum_{n} \hat{u}(n)\, \bar{\hat{u}}(n)\, \hat{u}(n)\, \bar{\hat{u}}(n) \right)$$

$$= 2\pi B \left(\|\hat{u}\|_{l^2}^4 - \|\hat{u}\|_{l^4}^4 \right).$$

Next we sum over all values of $l,\ n,\ r$ such that $l = 0$ and $n = r$. This gives

$$2\pi B \sum_{n} \hat{u}(n)\, \bar{\hat{u}}(n)\, \hat{u}(n)\, \bar{\hat{u}}(n)$$

$$= 2\pi B \|\hat{u}\|_{l^4}^4.$$

Finally, we sum over all values of $l,\ n,\ r$ such that $l \neq 0$ and $n \neq r$. This gives

$$2\pi \sum_{l\neq 0} \sum_{n} \sum_{r\neq n} \frac{i}{2l(n-r)} \left[e^{-2ilB(n-r)} - 1 \right]\, \hat{u}(n)\, \bar{\hat{u}}(n-l)\, \hat{u}(r-l)\, \bar{\hat{u}}(r)$$

$$= 2\pi \sum_{n} \sum_{p\neq 0} \sum_{l\neq 0} \frac{i}{2lp} \left[e^{-2ilpB} - 1 \right] \hat{u}(n)\, \bar{\hat{u}}(n-l)\, \hat{u}(n-p-l)\, \bar{\hat{u}}(n-p)$$

$$= A(\hat{u}).$$

Adding these four parts, we obtain from (4.7) that

$$\begin{aligned}\|T(s)u\|^4_{L^4_{s,x}([0,B]\times\mathbb{T})} &= 4\pi B\left(\|\hat{u}\|^4_{l^2} - \|\hat{u}\|^4_{l^4}\right) + 2\pi B\|\hat{u}\|^4_{l^4} + A(\hat{u}) \\ &= 4\pi B\|\hat{u}\|^4_{l^2} - 2\pi B\|\hat{u}\|^4_{l^4} + A(\hat{u}).\end{aligned} \tag{4.8}$$

□

The next two lemmas are modeled on Lemmas 2.9 and 2.11 in Kunze [15].

Lemma 4.7. *Assume $u, v, w, h \in L^2$ are such that $u = v + w + h$. Then*

$$\begin{aligned}\big|\|T(s)u\|^4_{L^4_{s,x}} - \|T(s)v\|^4_{L^4_{s,x}} - \|T(s)w\|^4_{L^4_{s,x}}\big| \le c(1 + \|u\|^3_{L^2} + \|v\|^3_{L^2} + \\ \|w\|^3_{L^2})\,\|h\|_{L^2} + (|\Lambda_1(v,w)| + ... + |\Lambda_7(v,w)|),\end{aligned} \tag{4.9}$$

where the remainder terms Λ_1 through Λ_7 are given by

$$\Lambda_1(v,w) = 4\int_0^B \int_{\mathbb{T}} |T(s)v|^2 \; |T(s)w|^2 \, dx \, ds$$

$$\Lambda_2(v,w) = 2\int_0^B \int_{\mathbb{T}} |T(s)v|^2 \; \overline{T(s)v} \; T(s)w \, dx \, ds$$

$$\Lambda_3(v,w) = \int_0^B \int_{\mathbb{T}} \overline{T(s)v}^2 \; (T(s)w)^2 \, dx \, ds$$

$$\Lambda_4(v,w) = 2\int_0^B \int_{\mathbb{T}} (T(s)w)^2 \; \overline{T(s)v} \; \overline{T(s)w} \, dx \, ds$$

$$\Lambda_5(v,w) = 2\int_0^B \int_{\mathbb{T}} (T(s)v)^2 \; \overline{T(s)v} \; \overline{T(s)w} \, dx \, ds$$

$$\Lambda_6(v,w) = \int_0^B \int_{\mathbb{T}} (T(s)v)^2 \; \overline{T(s)w}^2 \, dx \, ds$$

$$\Lambda_7(v,w) = 2\int_0^B \int_{\mathbb{T}} |T(s)w|^2 \; \overline{T(s)w} \; T(s)v \, dx \, ds.$$

Proof. It is clear that

$$\|T(s)u\|^4_{L^4_{s,x}} - \|T(s)v\|^4_{L^4_{s,x}} - \|T(s)w\|^4_{L^4_{s,x}}$$

$$= \int_0^B \int_{\mathbb{T}} |T(s)v + T(s)w + T(s)h|^4 - |T(s)v|^4 - |T(s)w|^4 \, dx \, ds.$$

Expand $|T(s)v + T(s)w + T(s)h|^4$ and divide into a principal part whose terms have at least one $T(s)h$ or $\overline{T(s)h}$ as a factor, and a remainder whose terms are the expressions $\Lambda_1(v,w)$, ... , $\Lambda_7(v,w)$ defined above. After taking the absolute value on both sides of the equality, apply the Triangle Inequality to the right hand side. The terms in the principal part can be written either in the form $\int_0^B \int_{\mathbb{T}} T(s)f_1 \; \overline{T(s)f_2} \; T(s)f_3 \; \overline{T(s)h} \; dx \; ds$, or in the form $\int_0^B \int_{\mathbb{T}} \overline{T(s)f_1} \; T(s)f_2 \; \overline{T(s)f_3} \; T(s)h \, dx \, ds$, with $f_1, f_2, f_3 \in \{u, v, w, h\}$. Apply Lemma 4.4 and Young's inequality to these terms to get the desired result. □

Lemma 4.8. *Suppose $u, v \in L^2$ are such that, for some $n_0 \in \mathbb{Z}$, $\delta > 0$, and $r_1^*, r_2^* \in \mathbb{Z}$ with $r_1^* - r_2^* \geq 6\delta^{-1}$, we have $\hat{v}(n) = 0$ for $|n - n_0| \geq r_1^* + 2\delta^{-1}$, and $\hat{w}(n) = 0$ for $|n - n_0| \leq r_2^* - 2\delta^{-1}$. Then*

$$\left| \int_{\mathbb{T}} F(v,v,w)\bar{w} \, dx \right| \leq (2\pi B + c\delta^{\frac{1}{2}}) \|\hat{v}\|^2_{l^2} \, \|\hat{w}\|^2_{l^2}, \tag{4.10}$$

$$\left| \int_{\mathbb{T}} F(v,v,w)\bar{v} \, dx \right| \leq c\|\hat{v}\|^3_{l^2} \, \|\hat{w}\|_{l^2} \, \delta^{\frac{1}{2}}, \tag{4.11}$$

$$\left| \int_{\mathbb{T}} F(w,v,w)\bar{v} \, dx \right| \leq c\|\hat{v}\|^2_{l^2} \, \|\hat{w}\|^2_{l^2} \, \delta^{\frac{1}{2}}, \tag{4.12}$$

$$\left| \int_{\mathbb{T}} F(w,w,w)\bar{v} \, dx \right| \leq c\|\hat{v}\|_{l^2} \, \|\hat{w}\|^3_{l^2} \, \delta^{\frac{1}{2}}, \tag{4.13}$$

$$\left| \int_{\mathbb{T}} F(v,v,v)\bar{w} \, dx \right| \leq c\|\hat{v}\|^3_{l^2} \, \|\hat{w}\|_{l^2} \, \delta^{\frac{1}{2}}, \tag{4.14}$$

$$\left| \int_{\mathbb{T}} F(v,w,v)\bar{w} \, dx \right| \leq c\|\hat{v}\|^2_{l^2} \, \|\hat{w}\|^2_{l^2} \, \delta^{\frac{1}{2}}, \tag{4.15}$$

$$\left| \int_{\mathbb{T}} F(v,w,w)\bar{w} \, dx \right| \leq c\|\hat{v}\|_{l^2} \, \|\hat{w}\|^3_{l^2} \, \delta^{\frac{1}{2}}, \tag{4.16}$$

with F as defined in Lemma 3.17, and $c>0$ independent of n_0, δ, r_1^*, and r_2^*.

Proof. According to Parseval's Theorem,

$$\int_{\mathbb{T}} F(u_1,u_2,u_3)\,\overline{u_4}\,dx = 2\pi\sum_n \widehat{F}(u_1,u_2,u_3)[n]\,\overline{\widehat{u_4}}[n]$$

$$= 2\pi\sum_n \overline{\widehat{u_4}}(n)\int_0^B e^{in^2 s}\left(\widehat{T(s)u_1} * \widehat{\overline{T(s)u_2}} * \widehat{T(s)u_3}\right)ds$$

$$= 2\pi\sum_n\sum_{n_1}\sum_{n_2}\overline{\widehat{u_4}}(n)\int_0^B e^{in^2 s}\ \widehat{T(s)u_1}(n-n_1-n_2)\,\widehat{\overline{T(s)u_2}}(n_1)\,\widehat{T(s)u_3}(n_2)\,ds$$

$$= 2\pi\sum_n\sum_{n_1}\sum_{n_2}\left(\int_0^B e^{-i2s(n-n_2)(n_1+n_2)}\,ds\right)\widehat{u_1}(n-n_1-n_2)\,\widehat{\overline{u_2}}(n_1)\,\widehat{u_3}(n_2)\,\overline{\widehat{u_4}}(n)$$

$$= 2\pi\sum_{n_3}\sum_{n_1}\sum_{n_2}\left(\int_0^B e^{-i2s(n_1+n_3)(n_1+n_2)}\,ds\right)\widehat{u_1}(n_3)\,\widehat{\overline{u_2}}(n_1)\,\widehat{u_3}(n_2)\,\overline{\widehat{u_4}}(n_1+n_2+n_3),$$

where $n_3 = n-n_1-n_2$. Therefore,

$$\left|\int_{\mathbb{T}} F(u_1,u_2,u_3)\bar{v}_4\,dx\right| \le 2\pi\sum_{n_3}\sum_{n_1}\sum_{n_2}\left|\int_0^B e^{-is\alpha(n_1,n_2,n_3)}\,ds\right|\,\left|\widehat{u_1}(n_3)\right|\left|\widehat{\overline{u_2}}(n_1)\right|$$
$$\left|\widehat{u_3}(n_2)\right|\,\left|\overline{\widehat{u_4}}(n_1+n_2+n_3)\right|, \tag{4.17}$$

where $\alpha(n_1,n_2,n_3) = 2(n_1+n_3)(n_1+n_2)$. We will be estimating each of the integrals by splitting these sums into partial sums. With respect to equation (4.10), let $u_1=u_2=v$ and $u_3=u_4=w$. Then (4.17) can be written as,

$$2\pi\sum_{n_3}\sum_{n_1}\sum_{n_2}\left|\int_0^B e^{-is\alpha(n_1,n_2,n_3)}\,ds\right|\,\left|\widehat{v}(n_3)\right|\left|\widehat{\overline{v}}(n_1)\right|\left|\widehat{w}(n_2)\right|\left|\overline{\widehat{w}}(n_1+n_2+n_3)\right| \tag{4.18}$$

$$\le (I)+(II)+(III)+(IV),$$

where (I) is the sum over all n_1, n_2, n_3 such that $|n_1+n_2|=0$, (II) is the sum over all n_1, n_2, n_3 such that $|n_1+n_3|=0$, (III) is the sum over all n_1, n_2, n_3 such that $|n_1+n_2| \geq |n_1+n_3| \geq 1$, and (IV) is the sum over all n_1, n_2, n_3 such that $|n_1+n_3| \geq |n_1+n_2| \geq 1$.

To estimate (I), we write

$$
\begin{aligned}
(I) &= 2\pi B \sum_{n_3} \sum_{n_2} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(-n_2)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{w}}(n_3)\right| \\
&= 2\pi B \sum_{n_3} \sum_{n_2} \left|\widehat{v}(n_3)\right| \left|\overline{\widehat{v}}(n_2)\right| \left|\widehat{w}(n_2)\right| \left|\overline{\widehat{w}}(n_3)\right| \\
&= 0,
\end{aligned}
\tag{4.19}
$$

because the assumption on the supports of v and w implies that either $\hat{v}=0$ or $\hat{w}=0$.

To estimate (II), we write

$$
\begin{aligned}
(II) &= 2\pi B \sum_{n_3} \sum_{n_2} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(-n_3)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{w}}(n_2)\right| \\
&= 2\pi B \sum_{n_3} \sum_{n_2} \left|\widehat{v}(n_3)\right| \left|\overline{\widehat{v}}(n_3)\right| \left|\widehat{w}(n_2)\right| \left|\overline{\widehat{w}}(n_2)\right| \\
&= 2\pi B \|\widehat{v}\|_{l^2}^2 \, \|\widehat{w}\|_{l^2}^2 .
\end{aligned}
\tag{4.20}
$$

To obtain estimates for (III) and (IV), we note that by direct integration we have

$$
(4.18) = 2\pi B \sum_{n_3} \sum_{n_1} \sum_{n_2} \left| \frac{1-e^{iB\alpha(n_1,n_2,n_3)}}{B\alpha(n_1,n_2,n_3)} \right| \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right|
$$

$$\left|\overline{\widehat{w}}(n_1+n_2+n_3)\right|$$

$$\leq 2\pi c \sum_{n_3}\sum_{n_1}\sum_{n_2} \frac{1}{1+|n_1+n_2||n_1+n_3|} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\overline{\widehat{w}}(n_1+n_2+n_3)\right|. \tag{4.21}$$

In order that $\widehat{v}(n_3)\,\widehat{\overline{v}}(n_1)\,\widehat{w}(n_2)\,\overline{\widehat{w}}(n_1n_2+n_3)$ be non-zero, we must have that

$$|n_2-n_0| \geq r_2^* - 2\delta^{-1},\ |n_3-n_0| \leq r_1^* + 2\delta^{-1},\ |n_1+n_0| \leq r_1^* + 2\delta^{-1},$$

$$\text{and } |n_1+n_2+n_3-n_0| \geq r_2^* - 2\delta^{-1}.$$

To estimate (III), we first observe that if $|n_1+n_2| \geq |n_1+n_3| \geq 1$, then

$$\begin{aligned} 1+|n_1+n_2|\,|n_1+n_3| &\geq |n_1+n_2|\,|n_1+n_3| \\ &\geq |n_1+n_2|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &= |(n_1+n_2+n_3-n_0)-(n_3-n_0)|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &\geq (|n_1+n_2+n_3-n_0|-|n_3-n_0|)^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &\geq (r_2^*-2\delta^{-1}-r_1^*-2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &\geq (2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}}. \end{aligned}$$

Let $K(n) = \chi_{|n|\geq 1}\,|n|^{-\frac{3}{2}},\ K_* = K(n)\,[(|\hat{w}(-.)| * |\overline{\widehat{w}}|)(n)]$. Then by equation (4.21), we have that

$$(III) \leq 2\pi c \sum_{n_3}\sum_{n_1}\sum_{n_2} \left[\frac{\chi_{|n_1+n_2|\geq|n_1+n_3|\geq 1}}{1+|n_1+n_2||n_1+n_3|}\right] \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\overline{\widehat{w}}(n_1+n_2+n_3)\right|$$

$$\leq 2\pi c \delta^{\frac{1}{2}} \sum_{n_3}\sum_{n_1}\sum_{n_2} \chi_{|n_1+n_3|\geq 1}\ |n_1+n_3|^{-\frac{3}{2}} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\overline{\widehat{w}}(n_1+n_2+n_3)\right|$$

$$= 2\pi c\delta^{\frac{1}{2}} \sum_{n_3}\sum_{n_1} K(n_1+n_3)\left[\left(\left|\widehat{w}(-.)\right| * \left|\overline{\widehat{w}}\right|\right)(n_1+n_3)\right]\left|\widehat{v}(n_3)\right|\left|\widehat{\overline{v}}(n_1)\right|$$

$$= 2\pi c\delta^{\frac{1}{2}} \sum_{n_3}\sum_{n_1} K_*(n_1+n_3)\left|\widehat{v}(n_3)\right|\left|\widehat{\overline{v}}(n_1)\right|$$

$$= 2\pi c\delta^{\frac{1}{2}} \sum_{n_1} \left|\widehat{\overline{v}}(-n_1)\right| \left(K_* * \left|\widehat{v}(-.)\right|\right)(n_1)$$

$$\leq 2\pi c\delta^{\frac{1}{2}} \left\|\left|\widehat{\overline{v}}\right|\right\|_{l^2} \left\|K_* * \left|\widehat{v}(-.)\right|\right\|_{l^2}$$

$$\leq 2\pi c\delta^{\frac{1}{2}} \left\|\widehat{\overline{v}}\right\|_{l^2} \left\|K_*\right\|_{l^1} \left\|\left|\widehat{v}(-.)\right|\right\|_{l^2}$$

$$= 2\pi c\delta^{\frac{1}{2}} \left\|\widehat{v}\right\|_{l^2} \left\|K_*\right\|_{l^1} \left\|\widehat{v}\right\|_{l^2},$$

where the Cauchy-Schwarz inequality and Young's inequality were applied to the last few estimates. However,

$$\begin{aligned}\|K_*\|_{l^1} &= \left\|K(n)(|\hat{w}(-.)| * |\overline{\widehat{w}}|)\right\|_{l^1}\\ &\leq \|K\|_{l^1}\left\||\hat{w}(-.)| * |\overline{\widehat{w}}|\right\|_{l^\infty}\\ &\leq \|K\|_{l^1}\|\hat{w}\|_{l^2}\|\hat{w}\|_{l^2}\\ &\leq c\,\|\hat{w}\|_{l^2}\|\hat{w}\|_{l^2}.\end{aligned}$$

Note $\|K\|_{l^1} = \sum_n \chi_{|n|\geq 1}\, |n|^{-\frac{3}{2}} = 2\sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty$. Therefore,

$$(III) \leq c\delta^{\frac{1}{2}} \|\hat{v}\|_{l^2}^2 \|\hat{w}\|_{l^2}^2. \tag{4.22}$$

To estimate (IV), we first observe that if $|n_1+n_3| \geq |n_1+n_2| \geq 1$, then

$$\begin{aligned}1 + |n_1+n_2|\,|n_1+n_3| &\geq |n_1+n_2|^2\\ &= |(n_1+n_2+n_3-n_0)-(n_3-n_0)|^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}\\ &\geq (r_2^*-2\delta^{-1}-r_1^*-2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}\\ &\geq (2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}.\end{aligned}$$

Then we obtain

$$(IV) \leq 2\pi c \sum_{n_3} \sum_{n_1} \sum_{n_2} \left[\frac{\chi_{|n_1+n_3| \geq |n_1+n_2| \geq 1}}{1 + |n_1 + n_2||n_1 + n_3|} \right] \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{w}}(n_1 + n_2 + n_3)\right|$$

$$\leq c\delta^{\frac{1}{2}} \left\|\hat{w}\right\|_{l^2}^2 \left\|\hat{v}\right\|_{l^2}^2 , \tag{4.23}$$

by an argument similar to that used to obtain (4.22).

To summarize, (4.19), (4.20), (4.22), and (4.23) show that

$$\left| \int_{\mathbb{T}} F(v, v, w)\bar{w} \; dx \right| \leq (2\pi B + c\delta^{\frac{1}{2}}) \|\hat{v}\|_{l^2}^2 \; \|\hat{w}\|_{l^2}^2. \tag{4.24}$$

Thus we have proved (4.10).

For (4.11), let $u_1 = u_2 = u_4 = v$ and $u_3 = w$. Then equation (4.17) can be written as

$$2\pi \sum_{n_3} \sum_{n_1} \sum_{n_2} \left| \int_0^B e^{-i2s(n_1+n_3)(n_1+n_2)} \; ds \right| \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_1+n_2+n_3)\right| \tag{4.25}$$

$$\leq (I) + (II) + (III) + (IV),$$

where (I) to (IV) are as previously explained.

For (I), we have

$$(I) = 2\pi B \sum_{n_3} \sum_{n_2} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(-n_2)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_3)\right|$$

$$= 2\pi B \sum_{n_3} \sum_{n_2} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_2)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_3)\right|$$

$$= 0, \tag{4.26}$$

based on the assumption on the supports of v and w.

Also, for (II), we have

$$(II) = 2\pi B \sum_{n_3} \sum_{n_2} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(-n_3)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_2)\right|$$

$$= 0, \tag{4.27}$$

applying the same reasoning as in (I).

Again, to obtain estimates (III) and (IV), we note that by direct integration we have

$$(4.25) = 2\pi B \sum_{n_3} \sum_{n_1} \sum_{n_2} \left| \frac{1 - e^{iB\alpha(n_1,n_2,n_3)}}{B\alpha(n_1,n_2,n_3)} \right| \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_1+n_2+n_3)\right|$$

$$\leq 2\pi c \sum_{n_3} \sum_{n_1} \sum_{n_2} \frac{1}{1 + |n_1+n_2||n_1+n_3|} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_1+n_2+n_3)\right|. \tag{4.28}$$

In order for $\widehat{v}(n_3)\, \widehat{\overline{v}}(n_1)\, \widehat{w}(n_2)\, \widehat{\overline{v}}(n_1+n_2+n_3)$ to be non-zero, we must have

$$|n_2 - n_0| \geq r_2^* - 2\delta^{-1},\ |n_3 - n_0| \leq r_1^* + 2\delta^{-1},\ |n_1 + n_0| \leq r_1^* + 2\delta^{-1},$$

$$\text{and } |n_1 + n_2 + n_3 - n_0| \leq r_1^* + 2\delta^{-1}.$$

To estimate (III), we observe that if $|n_1 + n_2| \geq |n_1 + n_3| \geq 1$, then

$$\begin{aligned} 1 + |n_1+n_2|\,|n_1+n_3| &\geq |n_1+n_2|\,|n_1+n_3| \\ &\geq |n_1+n_2|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &= |(n_2-n_0) - (-n_1-n_0)|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \end{aligned}$$

$$\geq (|n_2 - n_0| - |n_1 + n_0|)^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}}$$
$$\geq (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}}$$
$$\geq (2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}}.$$

Let $K(n) = \chi_{|n|\geq 1} \, |n|^{-\frac{3}{2}}, \; K_* = K(n) \, [(|\hat{w}(-.)| * |\overline{\hat{v}}|)(n)]$. Then by equation (4.28), we have that

$$(III) \leq 2\pi c \sum_{n_3} \sum_{n_1} \sum_{n_2} \left[\frac{\chi_{|n_1+n_2|\geq|n_1+n_3|\geq 1}}{1 + |n_1 + n_2||n_1 + n_3|} \right] \; \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_1 + n_2 + n_3)\right|$$

$$\leq 2\pi c \delta^{\frac{1}{2}} \sum_{n_3} \sum_{n_1} \sum_{n_2} \chi_{|n_1+n_3|\geq 1} \; |n_1 + n_3|^{-\frac{3}{2}} \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \left|\widehat{\overline{v}}(n_1 + n_2 + n_3)\right|$$

$$= 2\pi c \delta^{\frac{1}{2}} \sum_{n_3} \sum_{n_1} K(n_1 + n_3) \left[\left(\left|\widehat{w}(-.)\right| * \left|\widehat{\overline{v}}\right| \right) (n_1 + n_3) \right] \; \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right|$$

$$= 2\pi c \delta^{\frac{1}{2}} \sum_{n_3} \sum_{n_1} K_*(n_1 + n_3) \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right|$$

$$= 2\pi c \delta^{\frac{1}{2}} \sum_{n_1} \left|\widehat{\overline{v}}(-n_1)\right| (K_* * \left|\widehat{v}(-.)\right|)(n_1)$$

$$\leq 2\pi c \delta^{\frac{1}{2}} \left\| \left|\widehat{\overline{v}}\right| \right\|_{l^2} \; \left\| K_* * \left|\widehat{v}(-.)\right| \right\|_{l^2}$$

$$\leq 2\pi c \delta^{\frac{1}{2}} \left\| \widehat{\overline{v}} \right\|_{l^2} \; \|K_*\|_{l^1} \; \|\widehat{v}(-.)\|_{l^2}$$

$$= 2\pi c \delta^{\frac{1}{2}} \, \|\widehat{v}\|_{l^2} \; \|K_*\|_{l^1} \; \|\widehat{v}\|_{l^2} \,,$$

$$\leq c \delta^{\frac{1}{2}} \, \|\hat{w}\|_{l^2} \, \|\hat{v}\|_{l^2}^3 \,. \tag{4.29}$$

To estimate (IV), we observe that if $|n_1 + n_3| \geq |n_1 + n_2| \geq 1$, then $1 + |n_1 + n_2| \, |n_1 + n_3| \geq |n_1 + n_3|^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}}$

$$
\begin{aligned}
&= |(n_0 - n_2) - (n_0 - n_1 - n_2 - n_3)|^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}} \\
&\geq (|n_0 - n_2| - |n_0 - n_1 - n_2 - n_3|)^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}} \\
&\geq (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}} \\
&\geq (2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
(IV) = 2\pi c \sum_{n_3} \sum_{n_1} \sum_{n_2} \left[\frac{\chi_{|n_1+n_3| \geq |n_1+n_2| \geq 1}}{1 + |n_1+n_2||n_1+n_3|} \right] \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \left|\widehat{w}(n_2)\right| \\
\left|\widehat{\overline{v}}(n_1+n_2+n_3)\right| \\
\leq c\delta^{\frac{1}{2}} \, \|\hat{w}\|_{l^2} \, \|\hat{v}\|_{l^2}^3 \, , \qquad (4.30)
\end{aligned}
$$

by an argument similar to that used in (4.29).

To summarize, (4.26), (4.27), (4.29), and (4.30) show that

$$
\left| \int_{\mathbb{T}} F(v, v, w) \bar{v} \, dx \right| \leq c \|\hat{v}\|_{l^2}^3 \, \|\hat{w}\|_{l^2} \, \delta^{\frac{1}{2}}. \qquad (4.31)
$$

We note that the proof of the estimates in equations (4.12) to (4.16) follow the same procedure as the proof of the estimate in equation (4.11). In fact, the estimates for the sums analogous to (I) and (II) in equations (4.12) to (4.16) are the same as that of equation (4.11). Since the estimates for the sums analogous to (III) and (IV) also follow the same strategy as that of (4.11), we only need to check that the estimates $1+|n_1+n_2|\,|n_1+n_3| \geq (2\delta^{-1})^{\frac{1}{2}} \, |n_1+n_3|^{\frac{3}{2}}$ and $1 + |n_1+n_2|\,|n_1+n_3| \geq (2\delta^{-1})^{\frac{1}{2}} \, |n_1+n_2|^{\frac{3}{2}}$ are satisfied, and to choose the appropriate function K_*.

Concerning (4.12), let $u_1 = u_3 = w$ and $u_2 = u_4 = v$. In order for $\widehat{w}(n_3)\, \widehat{\overline{v}}(n_1)\, \widehat{w}(n_2)\, \overline{\widehat{v}}(n_1+n_2+n_3)$ to be non-zero, we must have that

$$|n_2 - n_0| \geq r_2^* - 2\delta^{-1},\ |n_3 - n_0| \geq r_2^* - 2\delta^{-1},\ |n_1 + n_0| \leq r_1^* + 2\delta^{-1},$$

$$\text{and } |n_1 + n_2 + n_3 - n_0| \leq r_1^* + 2\delta^{-1}.$$

To estimate (III), we observe that if $|n_1+n_2| \geq |n_1+n_3| \geq 1$, then

$$\begin{aligned} 1 + |n_1+n_2|\,|n_1+n_3| &\geq |n_1+n_2|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &= |(n_2-n_0) - (-n_1-n_0)|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &\geq (|n_2-n_0| - |n_1+n_0|)^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &\geq (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\ &\geq (2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}}. \end{aligned}$$

To estimate (IV), we observe that if $|n_1+n_3| \geq |n_1+n_2| \geq 1$, then

$$\begin{aligned} 1 + |n_1+n_2|\,|n_1+n_3| &\geq |n_1+n_2|^{\frac{3}{2}}\,|n_1+n_3|^{\frac{1}{2}} \\ &= |(n_3-n_0) - (-n_1-n_0)|^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}} \\ &\geq (|n_3-n_0| - |n_1+n_0|)^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}} \\ &\geq (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}} \\ &\geq (2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}. \end{aligned}$$

Let $K_* = K(n)\,[(|\hat{w}(-.)| * |\overline{\hat{v}}|)(n)]$. We then proceed as with the proof of (4.11) to verify (4.12).

Regarding (4.13), let $u_1 = u_2 = u_3 = w$, and $u_4 = v$. The term, $\widehat{w}(n_3)\, \widehat{\overline{w}}(n_1)\, \widehat{w}(n_2)\, \overline{\widehat{v}}(n_1+n_2+n_3)$ is non-zero if

$$|n_2 - n_0| \geq r_2^* - 2\delta^{-1},\ |n_3 - n_0| \geq r_2^* - 2\delta^{-1},\ |n_1 + n_0| \geq r_2^* - 2\delta^{-1},$$

$$\text{and } |n_1 + n_2 + n_3 - n_0| \le r_1^* + 2\delta^{-1}.$$

To estimate (III), we observe that $|n_1+n_2| \ge |n_1+n_3| \ge 1$ implies that

$$\begin{aligned}
1 + |n_1 + n_2|\,|n_1 + n_3| &\ge |n_1 + n_3|^{\frac{1}{2}}\,|n_1 + n_3|^{\frac{3}{2}} \\
&= |(n_0 - n_2) - (-n_1 - n_2 - n_3 + n_0)|^{\frac{1}{2}}\,|n_1 + n_3|^{\frac{3}{2}} \\
&\ge (|n_0 - n_2| - |n_0 - (n_1 + n_2 + n_3)|)^{\frac{1}{2}}\,|n_1 + n_3|^{\frac{3}{2}} \\
&\ge (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}}\,|n_1 + n_3|^{\frac{3}{2}} \\
&\ge (2\delta^{-1})^{\frac{1}{2}}\,|n_1 + n_3|^{\frac{3}{2}}.
\end{aligned}$$

To estimate (IV), we observe that if $|n_1 + n_3| \ge |n_1 + n_2| \ge 1$, then

$$\begin{aligned}
1 + |n_1 + n_2|\,|n_1 + n_3| &\ge |n_1 + n_2|^{\frac{3}{2}}\,|n_1 + n_2|^{\frac{1}{2}} \\
&= |(n_0 - n_3) - (-n_1 - n_2 - n_3 + n_0)|^{\frac{1}{2}}\,|n_1 + n_2|^{\frac{3}{2}} \\
&\ge (|n_0 - n_3| - |n_0 - (n_1 + n_2 + n_3)|)^{\frac{1}{2}}\,|n_1 + n_2|^{\frac{3}{2}} \\
&\ge (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}}\,|n_1 + n_2|^{\frac{3}{2}} \\
&\ge (2\delta^{-1})^{\frac{1}{2}}\,|n_1 + n_2|^{\frac{3}{2}}.
\end{aligned}$$

Let $K_* = K(n)\,[(|\hat{w}(-.)| * |\overline{\hat{v}}|)(n)]$. Again, we proceed as before to verify (4.13).

For (4.14), let $u_1 = u_2 = u_3 = v$ and $u_4 = w$. Again, $\widehat{v}(n_3)\,\overline{\widehat{v}}(n_1)\,\widehat{v}(n_2)\,\overline{\widehat{w}}(n_1 + n_2 + n_3)$ is non-zero, if

$$|n_2 - n_0| \le r_1^* + 2\delta^{-1},\ |n_3 - n_0| \le r_1^* + 2\delta^{-1},\ |n_1 + n_0| \le r_1^* + 2\delta^{-1},$$

$$\text{and } |n_1 + n_2 + n_3 - n_0| \ge r_2^* - 2\delta^{-1}.$$

For (III), we observe that $|n_1 + n_2| \ge |n_1 + n_3| \ge 1$ implies that

$$\begin{aligned}
1 + |n_1 + n_2|\,|n_1 + n_3| &\ge |n_1 + n_2|^{\frac{1}{2}}|n_1 + n_3|^{\frac{3}{2}} \\
&= |(n_1 + n_2 + n_3 - n_0) - (n_3 - n_0)|^{\frac{1}{2}}\,|n_1 + n_3|^{\frac{3}{2}} \\
&\ge (|n_1 + n_2 + n_3 - n_0| - |n_3 - n_0|)^{\frac{1}{2}}\,|n_1 + n_3|^{\frac{3}{2}}
\end{aligned}$$

$$\geq (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}}$$

$$\geq (2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}}.$$

To estimate (IV), we observe that if $|n_1 + n_3| \geq |n_1 + n_2| \geq 1$, then

$$\begin{aligned} 1 + |n_1 + n_2| \, |n_1 + n_3| &\geq |n_1 + n_2|^{\frac{3}{2}} \, |n_1 + n_3|^{\frac{1}{2}} \\ &= |(n_1 + n_2 + n_3 - n_0) - (n_2 - n_0)|^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}} \\ &\geq ((|n_1 + n_2 + n_3 - n_0| - |n_2 - n_0|)^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}} \\ &\geq (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}} \\ &\geq (2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}}. \end{aligned}$$

Let $K_* = K(n) \, [(|\hat{v}(-.)| * |\overline{\hat{w}}|)(n)]$, and proceed with the proof as before to verify (4.14).

For (4.15), let $u_1 = u_3 = v$ and $u_2 = u_4 = w$.

The term $\widehat{v}(n_3) \, \widehat{\overline{w}}(n_1) \, \widehat{v}(n_2) \, \overline{\widehat{w}}(n_1 + n_2 + n_3)$ is non-zero if

$$|n_2 - n_0| \leq r_1^* + 2\delta^{-1}, \; |n_3 - n_0| \leq r_1^* + 2\delta^{-1}, \; |n_1 + n_0| \geq r_2^* - 2\delta^{-1},$$

$$\text{and } |n_1 + n_2 + n_3 - n_0| \geq r_2^* - 2\delta^{-1}.$$

To estimate (III), we observe that $|n_1 + n_2| \geq |n_1 + n_3| \geq 1$ implies that

$$\begin{aligned} 1 + |n_1 + n_2| \, |n_1 + n_3| &\geq |n_1 + n_2|^{\frac{1}{2}} |n_1 + n_3|^{\frac{3}{2}} \\ &= |(n_1 + n_2 + n_3 - n_0) - (n_3 - n_0)|^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}} \\ &\geq (|n_1 + n_2 + n_3 - n_0| - |n_3 - n_0|)^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}} \\ &\geq (r_2^* - 2\delta^{-1} - r_1^* - 2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}} \\ &\geq (2\delta^{-1})^{\frac{1}{2}} \, |n_1 + n_3|^{\frac{3}{2}}. \end{aligned}$$

To estimate (IV), we observe that if $|n_1 + n_3| \geq |n_1 + n_2| \geq 1$, then

$$\begin{aligned} 1 + |n_1 + n_2| \, |n_1 + n_3| &\geq |n_1 + n_2|^{\frac{3}{2}} \, |n_1 + n_3|^{\frac{1}{2}} \\ &= |(n_1 + n_2 + n_3 - n_0) - (n_2 - n_0)|^{\frac{1}{2}} \, |n_1 + n_2|^{\frac{3}{2}} \end{aligned}$$

$$\geq (|n_1+n_2+n_3-n_0|-|n_2-n_0|)^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}$$
$$\geq (r_2^*-2\delta^{-1}-r_1^*-2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}$$
$$\geq (2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}.$$

Let $K_* = K(n)\,[(|\hat{v}(-.)| * |\overline{\widehat{w}}|)(n)]$, and proceed as before to verify (4.15).

Finally, to prove (4.16), let $u_1 = v$ and $u_2 = u_3 = u_4 = w$. In order for $\widehat{v}(n_3)\ \widehat{\overline{w}}(n_1)\ \widehat{w}(n_2)\ \overline{\widehat{w}}(n_1+n_2+n_3)$ to be non-zero, we must have that

$$|n_2-n_0| \geq r_2^*-2\delta^{-1},\ |n_3-n_0| \leq r_1^*+2\delta^{-1},\ |n_1+n_0| \geq r_2^*-2\delta^{-1},$$

$$\text{and } |n_1+n_2+n_3-n_0| \geq r_2^*-2\delta^{-1}.$$

Thus, to estimate (III), we observe that $|n_1+n_2| \geq |n_1+n_3| \geq 1$ implies that

$$\begin{aligned}
1+|n_1+n_2|\,|n_1+n_3| &\geq |n_1+n_3|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\
&= |(n_1+n_0)-(-n_3+n_0)|^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\
&\geq (|n_1+n_0|-|n_3-n_0|)^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\
&\geq (r_2^*-2\delta^{-1}-r_1^*-2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}} \\
&\geq (2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_3|^{\frac{3}{2}}.
\end{aligned}$$

To estimate (IV), we observe that if $|n_1+n_3| \geq |n_1+n_2| \geq 1$, then

$$\begin{aligned}
1+|n_1+n_2|\,|n_1+n_3| &\geq |n_1+n_2|^{\frac{3}{2}}\,|n_1+n_2|^{\frac{1}{2}} \\
&= |(n_1+n_2+n_3-n_0)-(n_3-n_0)|^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}} \\
&\geq (|n_1+n_2+n_3-n_0|-|n_3-n_0|)^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}} \\
&\geq (r_2^*-2\delta^{-1}-r_1^*-2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}} \\
&\geq (2\delta^{-1})^{\frac{1}{2}}\,|n_1+n_2|^{\frac{3}{2}}.
\end{aligned}$$

Let $K_* = K(n)\,[(|\hat{w}(-.)| * |\overline{\widehat{w}}|)(n)]$, and proceed as before to verify (4.16). □

Lemma 4.9. *If there exists a function w such that $P(w) = \lambda$ and $A(\hat{w}) -$*

$2\pi B\|\hat{w}\|_{l^4}^4 > 0$, *then* $J_\lambda > \dfrac{B\lambda^2}{\pi}$.

Proof. From Lemma 4.6,

$$\|T(s)w\|_{L^4_{s,x}([0,B]\times\mathbb{T})}^4 = 4\pi B\|\hat{w}\|_{l^2}^4 - 2\pi B\|\hat{w}\|_{l^4}^4 + A(\hat{w})$$

$$= \frac{B\lambda^2}{\pi} - 2\pi B\|\hat{w}\|_{l^4}^4 + A(\hat{w}) > \frac{B\lambda^2}{\pi}.$$

Hence by the definition of J_λ,

$$J_\lambda \geq \|T(s)w\|_{L^4_{s,x}([0,B]\times\mathbb{T})}^4 > \frac{B\lambda^2}{\pi}.$$

□

Lemma 4.10. *If a sequence* $\{u_j\} \subset L^2$ *is such that* $\|u_j\|_{L^2}^2 = \lambda$, *and* $\{\widehat{u_j}\}$ *vanishes in the sense defined in Lemma 4.3, then as* $j \to \infty$,

$$\|\widehat{u_j}\|_{l^4}^4 \to 0. \tag{4.32}$$

Proof. Suppose $\{\widehat{u_j}\}$ vanishes. Then for each fixed $r > 0$ and for $\epsilon > 0$, there exists an $N = N(r,\epsilon) \in \mathbb{N}$ such that

$$\sup_{m\in\mathbb{Z}} \sum_{m-r}^{m+r} |\widehat{u_j}(n)|^2 < \frac{\epsilon}{2},$$

for $j \geq N$. Thus, we have that $|\widehat{u_j}(n)|^2 < \frac{\epsilon}{2}$ for $j \geq N$, which in turn implies that $\|\widehat{u_j}\|_{l^\infty} < \epsilon$ for $j \geq N$. So, $\|\widehat{u_j}\|_{l^\infty} \to 0$. To get (4.32), we observe that by interpolation, $\|\widehat{u_j}\|_{l^4}^4 \leq \|\widehat{u_j}\|_{l^2}^2 \|\widehat{u_j}\|_{l^\infty}^2$.

□

Lemma 4.11. *If a sequence* $\{u_j\} \subset L^2$ *is such that* $\|u_j\|_{L^2}^2 = \lambda$ *and* $\{\widehat{u_j}\}$

vanishes in the sense of Lemma 4.3, then

$$D(\widehat{u_j}) \to 0. \tag{4.33}$$

Proof. We have

$$|D(\widehat{u_j})| = \Bigg| 2\pi B \sum_{l\neq 0}\sum_{n}\sum_{r\neq n} \frac{i}{2lB(n-r)} \left[e^{-2ilB(n-r)} - 1\right] \widehat{u_j}(n)$$

$$\bar{\widehat{u_j}}(n-l)\, \widehat{u_j}(r-l)\, \bar{\widehat{u_j}}(r) \Bigg|$$

$$\leq 2\pi c \sum_{l\neq 0}\sum_{n}\sum_{r\neq n} \frac{1}{1+|l||n-r|} |\widehat{u_j}(n)|\, |\bar{\widehat{u_j}}(n-l)|\, |\widehat{u_j}(r-l)|\, |\bar{\widehat{u_j}}(r)|. \tag{4.34}$$

Since $l \neq 0$ and $n \neq r$, we have that $|l| \geq 1$ and $|n-r| \geq 1$. Suppose $1 \leq |l| \leq \beta$ and $1 \leq |n-r| \leq \beta$ for some β, then

$$(4.34) = 2\pi c \sum_{l}\sum_{n}\sum_{r} \frac{\chi_{1\leq|l|\leq\beta,\, 1\leq|n-r|\leq\beta}}{1+|l||n-r|} |\widehat{u_j}(n)|\, |\bar{\widehat{u_j}}(n-l)|\, |\widehat{u_j}(r-l)|\, |\bar{\widehat{u_j}}(r)|$$

$$= 2\pi c \sum_{n}\sum_{l} \chi_{1\leq|l|\leq\beta}\, |\widehat{u_j}(n)|\, |\bar{\widehat{u_j}}(n-l)| \sum_{n-\beta}^{n+\beta} |\widehat{u_j}(r-l)|\, |\bar{\widehat{u_j}}(r)|$$

$$\leq 2\pi c\, \|\widehat{u_j}\|_{l^2} \left(\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u_j}}(r)|^2\right)^{\frac{1}{2}} \sum_{n} |\widehat{u_j}(n)| \sum_{l} \chi_{|l|\leq\beta}\, |\bar{\widehat{u_j}}(n-l)|$$

$$\leq 2\pi c\, \|\widehat{u_j}\|_{l^2} \left(\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u_j}}(r)|^2\right)^{\frac{1}{2}} \sum_{n} |\widehat{u_j}(n)|\, [(\chi_{[-\beta,\beta]} * |\bar{\widehat{u_j}}|)(n)]$$

$$\leq 2\pi c\, \|\widehat{u_j}\|_{l^2}^2\, \|\chi_{[-\beta,\beta]} * |\bar{\widehat{u_j}}|\|_{l^2} \left(\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u_j}}(r)|^2\right)^{\frac{1}{2}}$$

$$\leq 2\pi c\, \|\widehat{u_j}\|_{l^2}^3\, \|\chi_{[-\beta,\beta]}\|_{l^1} \left(\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u_j}}(r)|^2\right)^{\frac{1}{2}}$$

$$= 2\pi c\beta \left\| \widehat{u_j} \right\|_{l^2}^3 \left(\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u}}_j(r)|^2 \right)^{\frac{1}{2}}. \tag{4.35}$$

Now, suppose $|l| > \beta$ and $|n-r| \geq |l|$, then $1 + |l|\,|n-r| \geq |l|\,|n-r| \geq |l|^{\frac{3}{2}}\,|n-r|^{\frac{1}{2}} \geq \beta^{\frac{1}{2}}\,|l|^{\frac{3}{2}}$. Let $K(n) = \chi_{|n|\geq 1}\,|n|^{-\frac{3}{2}}$, and $K_* = K(n)\,[(|\widehat{u_j}| * |\overline{\widehat{u_j}}(-.)|)(n)]$. Thus,

$$(4.34) = 2\pi c \sum_l \sum_n \sum_r \frac{\chi_{|n-r|\geq|l|>\beta}}{1+|l||n-r|} |\widehat{u_j}(n)|\,|\bar{\widehat{u}}_j(n-l)|\,|\widehat{u_j}(r-l)|\,|\bar{\widehat{u}}_j(r)|$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \sum_l \sum_n \sum_r \chi_{|l|\geq 1}\,|l|^{-\frac{3}{2}}\,|\widehat{u_j}(n)|\,|\bar{\widehat{u}}_j(n-l)|\,|\widehat{u_j}(r-l)|\,|\bar{\widehat{u}}_j(r)|$$

$$= 2\pi c\beta^{-\frac{1}{2}} \sum_n |\widehat{u_j}(n)| \sum_l \chi_{|l|\geq 1}\,|l|^{-\frac{3}{2}}\,|\bar{\widehat{u}}_j(n-l)|\,[(|\widehat{u_j}| * |\overline{\widehat{u_j}}(-.)|)(l)]$$

$$= 2\pi c\beta^{-\frac{1}{2}} \sum_n |\widehat{u_j}(n)| \sum_l K_*(l)\,|\bar{\widehat{u}}_j(n-l)|$$

$$= 2\pi c\beta^{-\frac{1}{2}} \sum_n |\widehat{u_j}(n)|\,(K_* * |\bar{\widehat{u}}_j|)(n)$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \left\| |\overline{\widehat{u_j}}| \right\|_{l^2} \left\| K_* * |\widehat{u_j}| \right\|_{l^2}$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \left\| \widehat{u_j} \right\|_{L^2} \left\| K_* \right\|_{l^1} \left\| \widehat{u_j} \right\|_{l^2}$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \left\| \widehat{u_j} \right\|_{l^2}^4, \tag{4.36}$$

by Cauchy Schwarz inequality, Young's inequality, and using the estimate for K_* found in the proof of (4.10). Suppose $|l| \geq |n-r|$, then $1 + |l|\,|n-r| \geq |l|\,|n-r| \geq \beta^{\frac{1}{2}}\,|n-r|^{\frac{3}{2}}$. This means that

$$(4.34) = 2\pi c \sum_l \sum_n \sum_r \frac{\chi_{|l|\geq|n-r|}}{1+|l||n-r|} |\widehat{u_j}(n)|\,|\bar{\widehat{u}}_j(n-l)|\,|\widehat{u_j}(r-l)|\,|\bar{\widehat{u}}_j(r)|$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \sum_l \sum_n \sum_r \chi_{|n-r|\geq 1}|n-r|^{-\frac{3}{2}}|\widehat{u_j}(n)||\bar{\widehat{u}}_j(n-l)||\widehat{u_j}(r-l)||\bar{\widehat{u}}_j(r)|$$

$$= 2\pi c\beta^{-\frac{1}{2}} \sum_n \sum_p \chi_{|p|\geq 1} \; |p|^{-\frac{3}{2}} \, |\widehat{u_j}(n)| \, |\bar{\widehat{u_j}}(n-p)| \sum_l \; |\bar{\widehat{u_j}}(n-l)| \, |\widehat{u_j}(n-p-l)|$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \, \|\widehat{u_j}\|_{l^2}^2 \sum_n |\widehat{u_j}(n)| \sum_p \chi_{|p|\geq 1} \; |p|^{-\frac{3}{2}} \, |\bar{\widehat{u_j}}(n-p)|$$

$$= 2\pi c\beta^{-\frac{1}{2}} \, \|\widehat{u_j}\|_{l^2}^2 \sum_n |\widehat{u_j}(n)| \, [(K * |\bar{\widehat{u_j}}|)(n)]$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \, \|\widehat{u_j}\|_{l^2}^3 \, \left\| K * |\overline{\widehat{u_j}}| \right\|_{l^2}$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \, \|\widehat{u_j}\|_{l^2}^3 \; \|K\|_{l^1} \; \|\widehat{u_j}\|_{l^2}$$

$$\leq 2\pi c\beta^{-\frac{1}{2}} \, \|\widehat{u_j}\|_{l^2}^4 \,, \tag{4.37}$$

where $p = n-r$. The cases where $|n-r| > \beta$ and either $|l| \geq |n-r|$ or $|n-r| \geq |l|$, can be dealt with by applying the same strategies as those used to find (4.36) and (4.37). To conclude, (4.35), (4.36), and (4.37) imply that,

$$|D(\widehat{u_j})| \;\leq\; c\beta^{-\frac{1}{2}} \; \|\widehat{u_j}\|_2^4 + c\beta \, \|\widehat{u_j}\|_2^3 \left(\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u_j}}(r)|^2 \right)^{\frac{1}{2}} . \tag{4.38}$$

If $\{\widehat{u_j}\}$ vanishes, then for each fixed β and for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $j \geq N$,

$$\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u_j}}(r)|^2 < \epsilon^6.$$

In particular, for $\beta = \epsilon^{-2}$, there exists N such that for $j \geq N$,

$$|D(\widehat{u_j})| \leq c\beta^{-\frac{1}{2}} \; \|\widehat{u_j}\|_2^4 \; + \; c\beta \, \|\widehat{u_j}\|_2^3 \left(\sum_{n-\beta}^{n+\beta} |\bar{\widehat{u_j}}(r)|^2 \right)^{\frac{1}{2}}$$

$$= c(\epsilon^{-2})^{-\frac{1}{2}} \; \|\widehat{u_j}\|_2^4 + c\epsilon^{-2} \, \|\widehat{u_j}\|_2^3 \left(\sum_{n-\epsilon^{-2}}^{n+\epsilon^{-2}} |\bar{\widehat{u_j}}(r)|^2 \right)^{\frac{1}{2}}$$

$$\leq c\epsilon \; \|\widehat{u_j}\|_2^4 + c\epsilon^{-2} \, (\epsilon^6)^{\frac{1}{2}} \, \|\widehat{u_j}\|_2^3.$$

$$\leq c\epsilon$$

This shows that $\lim_{j\to\infty} D(\widehat{u_j}) = 0$, and concludes the proof of the lemma. $\square$

4.2.3 Proof of Theorem 1.1

Fix $\lambda > 0$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a maximizing sequence of $W(u)$ subject to $P(u) = \lambda$. Suppose there exists $w \in L^2(\mathbb{T})$ such that $P(w) = \lambda$ and $D(\widehat{w}) - 2\pi B\|\widehat{w}\|_{l^4}^4 > 0$.

It is clear that $\{u_j\}$ is tight as it is defined on a compact space. What is left is to determine which of the three cases in Lemma 4.3 the sequence of Fourier transforms $\{\hat{u}_j\}$ satisfies.

If $\{\hat{u}_j\}$ vanishes, then Lemmas 4.10 and 4.11 imply that $\left\|\widehat{u_j^\lambda}\right\|_{l^4}^4 \to 0,$ and $D\left(\widehat{u_j^\lambda}\right) \to 0$. Also, since w satisfies $D(\widehat{w}) - 2\pi B\|\widehat{w}\|_{l^4}^4 > 0$, by Lemma 4.9, $J_\lambda > \frac{B\lambda^2}{\pi}$. Taking the limit as $j \to \infty$ on both sides of Lemma 4.6 results in

$$\lim_{j\to\infty} \|T(s)u_j\|_{L^4_{s,x}}^4 = \lim_{j\to\infty} 4\pi B \|\widehat{u}_j\|_{l^2}^4 = 4\pi B \frac{\|u_j\|_{L^2}^4}{4\pi^2} = \frac{B\lambda^2}{\pi}.$$

However, since $(u_j)_{j\in\mathbb{N}}$ is a maximizing sequence, we have that

$$\lim_{j\to\infty} \|T(s)u_j\|_{L^4_{s,x}}^4 = \lim_{j\to\infty} W(u_j) = J_\lambda.$$

This implies that $J_\lambda = \frac{B\lambda^2}{\pi}$, thus contradicting the fact that $J_\lambda > \frac{B\lambda^2}{\pi}$. Hence, $(\widehat{u}_j)$ can not vanish.

If the sequence of Fourier transforms exhibits dichotomy, then as in case

3) of Lemma 4.3,

$$\alpha \in \left(0, \frac{\lambda}{2\pi}\right), \ a_j = \widehat{u}_j, \ b_j = \widehat{v}_j, \ \text{and } c_j = \widehat{w}_j.$$

Thus, for every $\delta \in (0, \alpha)$, there exists a j_0 such that for $j \geq j_0$, the following occur:

i) With m_j, r_1^* and r_2^* chosen as in Lemma 4.3, $\widehat{v}_j(n) = 0$ for $|n - m_j| \geq r_1^* + 2\delta^{-1}$, and $\widehat{w}_j(n) = 0$ for $|n - m_j| \leq r_2^* - 2\delta^{-1}$. Also, $\|v_j\|_{L^2}^2 \leq \lambda$ and $\|w_j\|_{L^2}^2 \leq \lambda$.

ii) $\|\widehat{u}_j - \widehat{v}_j - \widehat{w}_j\|_{l^2}^2 \leq 2\delta$, $\|\widehat{v}_j - \alpha\|_{l^2}^2 \leq 3\delta$, and $\|\widehat{w}_j - (\frac{\lambda}{2\pi} - \alpha)\|_{l^2}^2 \leq 9\delta$.

Let $\widehat{h}_j = \widehat{u}_j - \widehat{v}_j - \widehat{w}_j$, so that $h_j = u_j - v_j - w_j$, and $\|h_j\|_{L^2}^2 \leq c\delta$ for $j \geq j_0$.

Thus, utilizing Lemma 4.7, we have that

$$\begin{aligned} |W(u_j) - W(v_j) - W(w_j)| &\leq c\left(1 + \|u_j\|_{L^2}^3 + \|v_j\|_{L^2}^3 + \|w_j\|_{L^2}^3\right)\|h_j\|_{L^2} \\ &\quad + (|\Lambda_1(v_j, w_j)| + ... + |\Lambda_7(v_j, w_j)|) \\ &\leq c\delta^{\frac{1}{2}} + |\Lambda_1(v_j, w_j)| + ... + |\Lambda_7(v_j, w_j)|. \end{aligned}$$

Due to Lemma 3.2, we have that

$$\begin{aligned} \int_{\mathbb{T}} F(u_1, u_2, u_3)\bar{u}_4 \, dx &= \int_{\mathbb{T}} \int_0^B T^{-1}(s)[T(s)u_1 \, \overline{T(s)u_2} \, T(s)u_3] \, \bar{u}_4 \, ds \, dx \\ &= \int_0^B \int_{\mathbb{T}} T(s)u_1 \, \overline{T(s)u_2} \, T(s)u_3 \, \overline{T(s)u_4} \, dx \, ds. \end{aligned}$$

Thus, by Lemma 4.8,

$$\begin{aligned} |W(u_j) - W(v_j) - W(w_j)| &\leq c\delta^{\frac{1}{2}} + 4(2\pi B + c\delta^{\frac{1}{2}})\|\widehat{v}_j\|_{l^2}^2 \, \|\widehat{w}_j\|_{l^2}^2 \\ &\quad + 2c\|\widehat{v}_j\|_{l^2}^3 \, \|\widehat{w}_j\|_{l^2} \, \delta^{\frac{1}{2}} + ... + 2c\|\widehat{v}_j\|_{l^2} \, \|\widehat{w}_j\|_{l^2}^3 \, \delta^{\frac{1}{2}} \end{aligned}$$

$$\leq c\delta^{\frac{1}{2}} + 8\pi B \|\widehat{v}_j\|_{l^2}^2 \, \|\widehat{w}_j\|_{l^2}^2, \qquad \text{for } j \geq j_0.$$

Hence, by Lemma 4.5,

$$\|T(s)u_j\|_{L^4_{s,x}}^4 \leq \|T(s)v_j\|_{L^4_{s,x}}^4 + \|T(s)w_j\|_{L^4_{s,x}}^4 + c\delta^{\frac{1}{2}} + 8\pi B \, \|\widehat{v}_j\|_{l^2}^2 \, \|\widehat{w}_j\|_{l^2}^2$$

$$\leq J_{\|v_j\|_{L^2}^2} + J_{\|w_j\|_{L^2}^2} + c\delta^{\frac{1}{2}} + 8\pi B \, \|\widehat{v}_j\|_{l^2}^2 \, \|\widehat{w}_j\|_{l^2}^2$$

$$= \|v_j\|_{L^2}^4 \, J_1 + \|w_j\|_{L^2}^4 \, J_1 + c\delta^{\frac{1}{2}} + 8\pi B \, \|\widehat{v}_j\|_{l^2}^2 \, \|\widehat{w}_j\|_{l^2}^2$$

$$= 4\pi^2(\|\widehat{v}_j\|_{l^2}^4 \, J_1 + \|\widehat{w}_j\|_{l^2}^4 \, J_1) + c\delta^{\frac{1}{2}} + 8\pi B \, \|\widehat{v}_j\|_{l^2}^2 \, \|\widehat{w}_j\|_{l^2}^2, \quad \text{for } j \geq j_0.$$

Taking the limit as $j \to \infty$ and $\delta \to 0$ on both sides of the inequality results in

$$J_\lambda \leq 4\pi^2 \left(\alpha^2 + \left(\frac{\lambda}{2\pi} - \alpha \right)^2 \right) J_1 + c\delta^{\frac{1}{2}} + 8\pi B\alpha \left(\frac{\lambda}{2\pi} - \alpha \right).$$

On the other hand, by Lemma 4.5, we have that

$$J_\lambda = \lambda^2 J_1 = 4\pi^2 \left(\frac{\lambda}{2\pi} \right)^2 J_1 = 4\pi^2 \left(\alpha + \frac{\lambda}{2\pi} - \alpha \right)^2 J_1.$$

Hence,

$$4\pi^2 \left(\alpha + \frac{\lambda}{2\pi} - \alpha \right)^2 J_1 \leq 4\pi^2 \left(\alpha^2 + \left(\frac{\lambda}{2\pi} - \alpha \right)^2 \right) J_1 + c\delta^{\frac{1}{2}} + 8\pi B\alpha \left(\frac{\lambda}{2\pi} - \alpha \right).$$

It follows that $8\pi^2 \alpha \left(\frac{\lambda}{2\pi} - \alpha \right) J_1 \leq 8\pi B\alpha \left(\frac{\lambda}{2\pi} - \alpha \right)$, which implies that $J_\lambda \leq \frac{B\lambda^2}{\pi}$. This contradicts the fact that $J_\lambda > \frac{B\lambda^2}{\pi}$, and we conclude that $\{\widehat{u}_j\}$ can not exhibit dichotomy.

The final case is that in which the sequence of Fourier transforms is tight. Then there exists a sequence of integers $m_1, m_2, m_3, \ldots$ such that for each $\epsilon > 0$, there exists an integer $r = r(\epsilon) > 0$ with the property that

$$\sum_{m_j-r}^{m_j+r} |\widehat{u}_j|^2 \geq \frac{\lambda}{2\pi} - \epsilon \qquad \text{for } j \in \mathbb{N}.$$

Let $v_j = e^{-im_j x}u_j$; then $\|v_j\|_{L^2} = \sqrt{\lambda}$, $\|v_j\|_{L^4} = \|u_j\|_{L^4}$, and $\widehat{v}_j(n) = \widehat{u}_j(n + m_j)$. Also, using Lemma 4.6, we have that $\|T(s)v_j\|^4_{L^4_{s,x}} = \|T(s)u_j\|^4_{L^4_{s,x}}$. Hence, v_j is also a maximizing sequence. Note that

$$\sum_{m_j-r}^{m_j+r} |\widehat{u}_j|^2 = \sum_{-r}^{r} |\widehat{u}_j(n + m_j)|^2 = \sum_{-r}^{r} |\widehat{v}_j|^2.$$

So, without loss of generality, we can assume $m_j = 0$ for all $j \in \mathbb{N}$. This implies that for each $\epsilon > 0$, there exists an integer $r > 0$ with the property that

$$\sum_{-r}^{r} |\widehat{u}_j|^2 \geq \frac{\lambda}{2\pi} - \epsilon \qquad \text{for } j \in \mathbb{N}.$$

Define $\mu \in C_0^\infty$ such that $\mu(n) = 1$ for $|n| \leq 1$, and $\mu(n) = 0$ for $|n| \geq 2$. Define $\mu_r(n) = \mu(\frac{n}{r})$, where $r > 0$. For each $k \in \mathbb{N}$, let $\epsilon = \frac{1}{k}$ and choose $r(\epsilon) = r(\frac{1}{k}) := r_k$. Let $\mu_k = \mu_{r_k}$, and define $v_{j,k} := (\mu_k \, \widehat{u}_j)\check{}$ and $w_{j,k} := ((1 - \mu_k)\, \widehat{u}_j)\check{}$. Then, $u_j = v_{j,k} + w_{j,k}$ and

$$\|v_{j,k}\|^2_{H^1} = \sum_n (1 + |n|)^2 \; |\mu_k \, \widehat{u}_j|^2$$

$$\leq c \left(\sum_n |\mu_k \, \widehat{u}_j|^2 + \sum_n |n|^2 \, |\mu_k \, \widehat{u}_j|^2 \right)$$

$$\leq c \left(\|\widehat{u}_j\|^2_{l^2} + \sum_{-2r_k}^{2r_k} |n|^2 \, |\mu_k \, \widehat{u}_j|^2 \right)$$

$$\leq c \left(\|\widehat{u}_j\|^2_{l^2} + \||n|^2\|_{l^\infty} \, \||\mu_k \widehat{u}_j|^2\|_{l^1} \right)$$

$$\leq c \, (1 + 4r_k^2) \, \|\widehat{u}_j\|^2_{l^2}$$

$$= c\,(1+4r_k^2). \tag{4.39}$$

In addition,

$$\begin{aligned}
\|w_{j,k}\|_{L^2}^2 &= 2\pi \sum_{|n|\geq r_k} |(1-\mu_k)\widehat{u}_j|^2 \\
&\leq 2\pi \sum_{|n|\geq r_k} |\widehat{u}_j|^2 \\
&= 2\pi \left[\sum_n |\widehat{u}_j|^2 - \sum_{|n|\leq r_k} |\widehat{u}_j|^2 \right] \\
&= 2\pi \left[\frac{\lambda}{2\pi} - \sum_{|n|\leq r_k} |\widehat{u}_j|^2 \right] \\
&\leq 2\pi \left[\frac{\lambda}{2\pi} - \left(\frac{\lambda}{2\pi} - \frac{1}{k} \right) \right] \\
&= \frac{2\pi}{k}.
\end{aligned} \tag{4.40}$$

Now, $\|u_j\|_{L^2} = \sqrt{\lambda}$ implies that there exists a subsequence, still denoted by $\{u_j\}$, that converges weakly to some function $u \in L^2$, with $\|u\|_{L^2} \leq \sqrt{\lambda}$. Fix $k \in \mathbb{N}$. Equation (4.39) implies that $\{v_{j,k}\}_{j\in\mathbb{N}}$ is bounded in $H^1(\mathbb{T})$, and (4.40) implies that $\{w_{j,k}\}_{j\in\mathbb{N}}$ is small in $L^2(\mathbb{T})$. Thus, there exists a subsequence $(j') \subset \mathbb{N}$, and functions $v_k \in H^1(\mathbb{T})$ and $w_k \in L^2(\mathbb{T})$, such that, $\{v_{j',k}\}$ converges weakly to v_k in $H^1(\mathbb{T})$ and $\{w_{j',k}\}$ converges weakly to w_k in $L^2(\mathbb{T})$, with $u = v_k + w_k$. By the weak lower semicontinuity of the norm in a Hilbert space,

$$\|w_k\|_{L^2} \leq \liminf_{j'\to\infty} \|w_{j',k}\|_{L^2} \leq \sqrt{\frac{2\pi}{k}}.$$

By the Rellich-Kondrachov Theorem for compact manifolds, we can conclude that there is a subsequence of $\{v_{j',k}\}$, still denoted by $\{v_{j',k}\}$, that converges strongly to $v_k \in L^2(\mathbb{T})$. Now,

$$
\begin{aligned}
\|u\|_{L^2(\mathbb{T})} &= \|v_k + w_k\|_{L^2(\mathbb{T})} \\
&\geq \|v_k\|_{L^2(\mathbb{T})} - \|w_k\|_{L^2(\mathbb{T})} \\
&\geq \lim_{j'\to\infty} \|v_{j',k}\|_{L^2(\mathbb{T})} - \sqrt{\frac{2\pi}{k}} \\
&= \lim_{j'\to\infty} \|u_{j'} - w_{j',k}\|_{L^2(\mathbb{T})} - \sqrt{\frac{2\pi}{k}} \\
&\geq \limsup_{j'\to\infty} \|u_{j'}\|_{L^2(\mathbb{T})} - \|w_{j',k}\|_{L^2(\mathbb{T})} - \sqrt{\frac{2\pi}{k}} \\
&\geq \limsup_{j'\to\infty} \|u_{j'}\|_{L^2(\mathbb{T})} - 2\sqrt{\frac{2\pi}{k}} \\
&= \sqrt{\lambda} - 2\sqrt{\frac{2\pi}{k}}.
\end{aligned}
$$

Taking the limit as $k \to \infty$, it follows that $\|u\|_{L^2(\mathbb{T})} \geq \sqrt{\lambda}$. To conclude, since $\{u_j\}$ converges weakly to u, and $\|u\|_{L^2(\mathbb{T})} = \sqrt{\lambda} = \lim_{j\to\infty} \|u_j\|_{L^2(\mathbb{T})}$, then $\{u_j\}$ converges strongly to u in $L^2(\mathbb{T})$. Finally, $W(u) : L^2 \to \mathbb{R}$ is continuous (see Remark 3.19), so we can conclude that u is a maximizer of $W(u)$.

4.3 Stability

Let $\mathcal{S}_\lambda$ be the set of all maximizers for J_λ; that is,

$$\mathcal{S}_\lambda = \{\phi \in L^2 \ : \ W(\phi) = J_\lambda \ \text{and} \ P(\phi) = \lambda\}.$$

Theorem 4.12. *The set $\mathcal{S}_\lambda$ is nonempty and is a stable set for the initial* *value problem for (3.1) in* $L^2(\mathbb{T})$*. That is, for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that if* $u_0 \in L^2$*, and*

$$\inf_{\phi \in \mathcal{S}_\lambda} \|u_0 - \phi\|_{L^2} \leq \delta,$$

then the solution $u(t,x)$ *of (3.1) with* $u(0,x) = u_0(x)$ *satisfies*

$$\inf_{\phi \in \mathcal{S}_\lambda} \|u_n(t,\cdot) - \phi\|_{L^2} < \epsilon$$

for all $t > 0$.

Proof. Suppose for contradiction that $\mathcal{S}_\lambda$ is not stable. Then there exists $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$, we can find $u_{0n} \in L^2$ such that

$$\inf_{\phi \in \mathcal{S}_\lambda} \|u_{0n} - \phi\|_{L^2} \leq \frac{1}{n},$$

and some $t_n > 0$ such that the solution $u_n(t,x)$, of (3.1) with initial data u_{0n} satisfies

$$\inf_{\phi \in \mathcal{S}_\lambda} \|u_n(t_n,\cdot) - \phi\|_{L^2} \geq \epsilon_0.$$

Let $\epsilon > 0$ be given. We know that since $\inf_{\phi \in \mathcal{S}_\lambda} \|u_{0n} - \phi\|_{L^2} \leq \frac{1}{n}$, then for each n there exists $\phi_n \in \mathcal{S}_\lambda$ such that $\|u_{0n} - \phi_n\|_{L^2} \leq \frac{2}{n}$.

Also, there exists N such that for $n \geq N$, $\frac{1}{n} < \frac{\epsilon}{2}$. Thus for $n \geq N$, we have

$$\begin{aligned}
|\|u_{0n}\|_{L^2} - \sqrt{\lambda}| &= |\|u_{0n}\|_{L^2} - \|\phi_n\|_{L^2}| \\
&\leq \|u_{0n} - \phi_n\|_{L^2} \\
&\leq \frac{2}{n} \\
&< \epsilon.
\end{aligned}$$

This shows that $P(u_{0n})$ converges to λ in L^2.

Moreover, let M be such that $\|u_{0n}\|_{L^2} \leq M$. Then for all $\phi \in \mathcal{S}_\lambda$,

$$|W(u_{0n}) - W(\phi)| \leq c\|u_{0n} - \phi\|_{L^2}(\|u_{0n}\|_{L^2} + \|\phi\|_{L^2})^3.$$

Therefore, for $\epsilon > 0$, there exists $\tilde{N}$ such that for $n \geq \tilde{N}$, $\frac{1}{n} < \frac{\epsilon}{2c(M+\sqrt{\lambda})^3}$. So for $n \geq \tilde{N}$,

$$|W(u_{0n}) - J_\lambda| = |W(u_{0n}) - W(\phi_n)|$$

$$\leq c\|u_{0n} - \phi_n\|_{L^2}(\|u_{0n}\|_{L^2} + \|\phi_n\|_{L^2})^3$$

$$\leq c\frac{2}{n}(M + \sqrt{\lambda})^3$$

$$< \epsilon.$$

This shows that $W(u_{0n})$ converges to J_λ.

Now, define $\{\alpha_n\}$ such that $P(\alpha_n\, u_{0n}) = \lambda$ for all $n \in \mathbb{N}$. Then $\alpha_n \to 1$. Define $v_n = \alpha_n\, u_n(\cdot, t)$. Since $P(u_n)$ and $W(u_n)$ are independent of time, then $P(v_n) = P(\alpha_n\, u_n) = \alpha_n^2\, P(u_n) = \alpha_n^2\, P(u_{0n}) = P(\alpha_n\, u_{0n}) = \lambda$, and

$$\lim_{n\to\infty} W(v_n) = \lim_{n\to\infty} W(\alpha_n\, u_n) = \lim_{n\to\infty} \alpha_n^4\, W(u_n(t,\cdot)) = \lim_{n\to\infty} \alpha_n^4\, W(u_{n_0}) = J_\lambda.$$

This implies that $\{v_n\}$ is a maximizing sequence. From Theorem 1.1, it follows that there exists a maximizer $\psi \in \mathcal{S}_\lambda$ such that a subsequence of $e^{-im_n x}\, v_n$ satisfies $\|e^{-im_n x}\, v_n - \psi\|_2 < \frac{\epsilon_0}{2}$. On the other hand,

$$\epsilon_0 \leq \|u_n(t_n, \cdot) - e^{im_n x}\, \psi\|_{L^2}$$

$$= \left\| \frac{1}{\beta_n} v_n(t_n, \cdot)) - e^{im_n x}\, \psi \right\|_{L^2}$$

$$= \left\| \frac{1}{\beta_n} v_n(t_n,\cdot) - v_n(t_n,\cdot) + v_n(t_n,\cdot) - e^{im_n x}\,\psi \right\|_{L^2}$$

$$= \left\| \frac{1}{\beta_n} v_n(t_n,\cdot) - v_n(t_n,\cdot) \right\|_{L^2} + \|v_n(t_n,\cdot) - e^{im_n x}\,\psi\|_{L^2}$$

$$\leq \left| \frac{1}{\beta_n} - 1 \right| \|v_n(t_n,\cdot)\|_{L^2} + |e^{im_n x}| \|e^{-im_n x}\, v_n(t_n,\cdot) - \psi\|_{L^2}.$$

Taking the limit as $n \to \infty$, we obtain that $\epsilon_0 \leq \frac{\epsilon_0}{2}$, a contradiction.

□

Remark 4.13. *We note that a similar approach can be used to prove stability of the set of all minimizers in* H^1.

Chapter 5

Intervals of Existence of Maximizers in L^2

The goal of this section is to prove results on the existence or nonexistence of functions w satisfying the sufficient condition for stability in Theorem 1.1, for various values of B. We show in Section 5.1 that such functions w do not exist for $B = N\pi,\ N \in \mathbb{N},$ but do exist for $B = \frac{\pi}{4}$ and for $B \in (0, B_0)$ where B_0 is about 0.6958. We show in Section 5.2 that such functions exist for $B = \frac{\pi}{2}$ and for $B \in [B_0, B_1]$ where B_1 is about 1.39.

5.1 Functions w with Real-Valued Fourier Transform

Lemma 5.1. *If $B = N\pi$ for $N \in \mathbb{N}$, then there does not exist any maximizing function for J_λ in $L^2(\mathbb{T})$.*

Proof. Let $B = N\pi$ for $N \in \mathbb{Z}$. If $u \in L^2(\mathbb{T})$ is such that $P(u) = \lambda$, then Lemma 4.6 implies that

$$\|T(s)u\|^4_{L^4_{s,x}([0,N\pi]\times\mathbb{T})} = 4N\pi^2\|\hat{u}\|^4_{l^2} - 2N\pi^2\|\hat{u}\|^4_{l^4} \le N\lambda^2,$$

since $A(\hat{u}) = 0$ when $B = N\pi$. According to the definition of J_λ, we have that

$J_\lambda \leq N\lambda^2$. However, if we define $u_j \in L^2$ by

$$\widehat{u}_j(n) = \begin{cases} \sqrt{\frac{\lambda}{2\pi(2j-1)}} & |n| < j \\ 0 & \text{otherwise,} \end{cases}$$

then $\|\widehat{u}_j\|_{l^2}^2 = \frac{\lambda}{2\pi}$ and, as $j \to \infty$, $\|\widehat{u}_j\|_{l^\infty} \to 0$; hence $\|\widehat{u}_j\|_{l^4} \to 0$ and $\|T(s)u_j\|^4_{L^4_{s,x}([0,N\pi]\times\mathbb{T})} \to N\lambda^2$. This implies that $J_\lambda \geq N\lambda^2$, and therefore that $J_\lambda = N\lambda^2$.

Now suppose u is a maximizer for J_λ in $L^2(\mathbb{T})$. Then we have that

$$\|T(s)u\|^4_{L^4_{s,x}([0,N\pi]\times\mathbb{T})} = N\lambda^2.$$

Lemma 4.6 implies that $\|\widehat{u}\|^4_{l^4} = 0$; thus $u = 0$. This contradicts the fact that $P(u) = \lambda > 0$, and we conclude that there does not exist a maximizer for J_λ in $L^2(\mathbb{T})$. □

From now on, we will write $A(\hat{u})$ as

$$A(\hat{u}) = 2\pi B \sum_{p\neq 0} \sum_{l \neq 0} b_{p,l}\, a_{p,l}, \tag{5.1}$$

where

$$b_{p,l} = \frac{i}{2lpB}\left[e^{-2ilpB} - 1\right]$$
$$= \frac{1}{2lpB}\left[\sin(2lpB) + i(\cos(2lpB) - 1)\right] \tag{5.2}$$

and

$$a_{p,l} = \sum_n \hat{u}(n)\, \bar{\hat{u}}(n-l)\, \hat{u}(n-p-l)\, \bar{\hat{u}}(n-p). \tag{5.3}$$

Lemma 5.2. *For all $p, l \in \mathbb{Z}$, we have*

$$a_{p,l} = a_{-p,-l},$$

$$a_{-p,l} = \overline{a_{p,l}} = a_{p,-l},$$

and

$$a_{p,l} = a_{l,p}.$$

Proof. The equation $a_{p,l} = a_{l,p}$ follows immediately from the definition of $a_{p,l}$.

Also, we have

$$a_{-p,-l} = \sum_n \hat{u}(n)\ \bar{\hat{u}}(n+l)\ \hat{u}(n+p+l)\ \bar{\hat{u}}(n+p)$$

$$= \sum_{\tilde{n}} \hat{u}(\tilde{n}-p-l)\ \bar{\hat{u}}(\tilde{n}-p)\ \hat{u}(\tilde{n})\ \bar{\hat{u}}(\tilde{n}-l)$$

$$= a_{p,l},$$

where in the second sum we changed the index from n to $\tilde{n} = n+p+l$. Finally,

$$a_{-p,l} = \sum_n \hat{u}(n)\ \bar{\hat{u}}(n-l)\ \hat{u}(n+p-l)\ \bar{\hat{u}}(n+p)$$

$$= \sum_{\tilde{n}} \hat{u}(\tilde{n}-p)\ \bar{\hat{u}}(\tilde{n}-p-l)\ \hat{u}(\tilde{n}-l)\ \bar{\hat{u}}(\tilde{n})$$

$$= \overline{a_{p,l}},$$

where we made the change of index from n to $\tilde{n} = n+p$. A similar calculation works for $a_{p,-l}$.

□

For a function u with real-valued Fourier transform $\hat{u}$, we see that $a_{p,l}$ is even as a function of p and l, and $b_{p,l}$ is the sum of an even function and an odd function of p and l (but not both). Hence, if $\hat{u}$ is real-valued, we can rewrite $A(\hat{u})$ as

$$A(\hat{u}) = 2\pi B \sum_{p\neq 0} \left[\sum_{l\neq 0} \frac{1}{2lpB} \; [\sin(2lpB) + i(\cos(2lpB) - 1)] \; a_{p,l} \right]$$

$$= 2\pi B \sum_{p\neq 0} \sum_{l\neq 0} \frac{\sin(2lpB)}{2lpB} \; a_{p,l}$$

$$= 2\pi B \cdot 4 \sum_{p=1}^{\infty} \sum_{l=1}^{\infty} \frac{\sin(2lpB)}{2lpB} \; a_{p,l}. \tag{5.4}$$

From equation (5.4), we have that

$$A(\hat{w}) - 2\pi B \, \|\hat{w}\|_{l^4}^4 = 8\pi B \sum_{p=1}^{\infty} \sum_{l=1}^{\infty} \frac{\sin(2lpB)}{2lpB} \; a_{p,l} - 2\pi B \, \|\hat{w}\|_{l^4}^4 \, . \tag{5.5}$$

Fix $B > 0$, and let $\hat{w}$ be defined by

$$\hat{w}(n) = \begin{cases} 1 & n = 0 \\ r & n = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $r \in \mathbb{R}$. From (5.3) we see that if $p \geq 1$ and $l \geq 1$, then for $a_{p,l} \neq 0$ to hold, we must have that $p < 2$, $l < 2$, $|p - l| \leq 2$, and $|p + l| \leq 2$; and therefore we must have $p = l = 1$. Hence

$$A(\hat{w}) - 2\pi B \, \|\hat{w}\|_{l^4}^4 = 8\pi B \left(\frac{\sin 2B}{2B} \right) a_{1,1} - 2\pi B \, \|\hat{w}\|_{l^4}^4$$

$$= 8\pi B \left(\frac{\sin 2B}{2B} \right) \sum_{n} \; \hat{w}(n) \; \bar{\hat{w}}(n-1) \; \hat{w}(n-2) \; \bar{\hat{w}}(n-1) - 2\pi B \, \|\hat{w}\|_{l^4}^4 \, .$$

If we consider all the possible values for n such that $\hat{w}(n) \; \bar{\hat{w}}(n-1) \; \hat{w}(n-2) \; \bar{\hat{w}}(n-1)$ is non-zero, we see that this can only happen for $n = 1$. Therefore,

$$A(\hat{w}) - 2\pi B \, \|\hat{w}\|_{l^4}^4 = 8\pi B \left(\frac{\sin 2B}{2B} \right) \; \hat{w}(1) \; \hat{w}^2(0) \; \hat{w}(-1) - 2\pi B \, \|\hat{w}\|_{l^4}^4$$

$$= 8\pi B r^2 \left(\frac{\sin 2B}{2B}\right) - 2\pi B(1 + 2r^4).$$

Note that $A(\hat{w}) - 2\pi B \|\hat{w}\|_{l^4}^4 > 0$ if

$$4r^2 \left(\frac{\sin 2B}{2B}\right) > (1 + 2r^4),$$

that is, if

$$\left(\frac{\sin 2B}{2B}\right) > \frac{1 + 2r^4}{4r^2}. \tag{5.6}$$

Since the function $f(r) = \frac{1+2r^4}{4r^2}$ has a minimum value of $\frac{\sqrt{2}}{2}$ at $r = 2^{-\frac{1}{4}}$, then (5.6) will be satisfied at some $r > 0$, as long as

$$\left(\frac{\sin 2B}{2B}\right) > \frac{\sqrt{2}}{2}. \tag{5.7}$$

We see that (5.7) is true for all values of B in the interval $(0, B_0)$, where B_0 is about 0.6958.

To obtain a function w_λ satisfying satisfying the condition of Theorem 1.1, we now set

$$\hat{w}_\lambda(n) = \sqrt{\frac{\lambda}{2\pi}} \frac{\hat{w}(n)}{\|\hat{w}\|_{l^2}},$$

so that

$$\|w_\lambda\|_{L^2}^2 = 2\pi \|\hat{w}_\lambda\|_{l^2}^2 = 2\pi \left(\sqrt{\frac{\lambda}{2\pi}}\right)^2 \frac{\|\hat{w}\|_{l^2}^2}{\|\hat{w}\|_{l^2}^2} = \lambda,$$

and

$$A(\hat{w}_\lambda) - 2\pi B \|\hat{w}_\lambda\|_{l^4}^4 = \frac{\lambda^2}{4\pi^2 \|\hat{w}\|_{l^2}^4} [A(\hat{w}) - 2\pi B \|\hat{w}\|_{l^4}^4] > 0$$

for all B in the interval $(0, B_0)$.

More generally, we can define the function w by setting

$$\hat{w}(n) = \begin{cases} 1 & n = 0 \\ r & n = \pm 1 \\ s & n = \pm 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $r, s \in \mathbb{R}$. From equation (5.3), we see that in order to have $a_{p,l} \neq 0$, we must have that $p < 4$, $l < 4$, $|p - l| \leq 4$, and $|p + l| \leq 4$. In this case, the possibilities for (p, l) are $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$, and $(3,1)$. This implies that

$$A(\hat{w}) - 2\pi B \, \|\hat{w}\|_{l^4}^4 = 8\pi B \left[\left(\frac{\sin 2B}{2B} \right) a_{1,1} + \left(\frac{\sin 4B}{4B} \right) a_{1,2} + \left(\frac{\sin 6B}{6B} \right) a_{1,3} \right.$$

$$\left. + \left(\frac{\sin 4B}{4B} \right) a_{2,1} + \left(\frac{\sin 6B}{6B} \right) a_{3,1} + \left(\frac{\sin 8B}{8B} \right) a_{2,2} \right] - 2\pi B \, \|\hat{w}\|_{l^4}^4 .$$

By Lemma 5.2, it suffices to calculate $a_{p,l}$ for $(p, l) = (1,1)$, $(1,2)$, $(1,3)$, and $(2,2)$. For $a_{1,1}$, the term $\hat{w}(n) \, \bar{\hat{w}}(n-1) \, \hat{w}(n-2) \, \bar{\hat{w}}(n-1)$ is non-zero when $n = 0, 1$ and 2. Evaluating at each of these values and taking the sum results in $a_{1,1} = 2sr^2 + r^2$. For $a_{1,2}$, the only values of n that make $\hat{w}(n) \, \bar{\hat{w}}(n-1) \, \hat{w}(n-3) \, \bar{\hat{w}}(n-2)$ non-zero are $n = 1$ and 2. Evaluating at each of these values and taking the sum results in $a_{1,2} = 2sr^2$. For $a_{2,2}$, the only value of n that makes $\hat{w}(n) \, \bar{\hat{w}}(n-2) \, \hat{w}(n-4) \, \bar{\hat{w}}(n-2)$ non-zero is $n = 2$. Evaluating at this value results in $a_{2,2} = s^2$. Lastly, for $a_{1,3}$, the only value of n that makes $\hat{w}(n) \, \bar{\hat{w}}(n-1) \, \hat{w}(n-4) \, \bar{\hat{w}}(n-3)$ non-zero is $n = 2$. Evaluating at this value results in $a_{1,3} = s^2 r^2$.

For $B = \frac{\pi}{4},$ we have that

$$A(\hat{w}) - \frac{\pi^2}{2}\,\|\hat{w}\|_{l^4}^4 = 8\pi\left[\left(\frac{\sin\frac{\pi}{2}}{2}\right)r^2(2s+1) + \left(\frac{\sin\pi}{4}\right)2sr^2 + \left(\frac{\sin\frac{3\pi}{2}}{6}\right)s^2r^2\right.$$

$$\left. + \left(\frac{\sin\pi}{4}\right)2sr^2 + \left(\frac{\sin\frac{3\pi}{2}}{6}\right)s^2r^2 + \left(\frac{\sin 2\pi}{8}\right)s^2\right] - \frac{\pi^2}{2}(2s^4 + 2r^4 + 1)$$

$$= 8\pi\left[\frac{1}{2}\,r^2(2s+1) + \frac{-1}{3}\,s^2r^2\right] - \frac{\pi^2}{2}(2s^4 + 2r^4 + 1). \tag{5.8}$$

Therefore, for $B = \frac{\pi}{4},$ $A(\hat{w}) - \frac{\pi^2}{2}\,\|\hat{w}\|_{l^4}^4$ has a positive maximum when $r = 1.16426,$ and $s = 0.754222.$

Again, to obtain a function w_λ satisfying the condition of Theorem 1.1, one can set

$$\hat{w}_\lambda(n) = \sqrt{\frac{\lambda}{2\pi}}\,\frac{\hat{w}(n)}{\|\hat{w}\|_{l^2}}.$$

5.2 Functions w with Complex-Valued Fourier Transform

Let $B = \frac{\pi}{2}$, and define the function $\hat{w}$ by

$$\hat{w}(n) = \begin{cases} 1 & n = 0 \\ r & n = \pm 1,\ \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

where now, unlike in Section 5.1, we allow r to take a complex value. For this value of B, equation (5.2) gives

$$b_{p,l} = \frac{1}{lp\pi}\left[\sin(lp\pi) + i\cos(lp\pi) - i\right],$$

$$= \begin{cases} 0 & \text{if either } l \text{ or } p \text{ is even} \\ \frac{-2i}{lp\pi} & \text{otherwise.} \end{cases}$$

Thus

$$A(\hat{w}) - 2\pi B \left\| \hat{w} \right\|_{l^4}^4 = \pi^2 \sum_{p\text{-odd}} \sum_{l\text{-odd}} \frac{-2i}{lp\pi} \, a_{p,l} - \pi^2 \left\| \hat{w} \right\|_{l^4}^4 . \tag{5.9}$$

In order to have $a_{p,l} \neq 0$, we must have that $|p| < 4$, $|l| < 4$, $|p - l| \leq 4$ and $|p + l| \leq 4$. In this case, we see that the possibilities for (p, l) are $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$, $(1, 3)$, $(-1, 3)$, $(1, -3)$, $(-1, -3)$, $(3, 1)$, $(3, 1)$, $(3, -1)$, and $(-3, -1)$. Hence,

$$A(\hat{w}) - 2\pi B \left\| \hat{w} \right\|_{l^4}^4 = -2i\pi \Big[a_{1,1} - a_{-1,1} - a_{1,-1} + a_{-1,-1} + \frac{1}{3} a_{1,3} - \frac{1}{3} a_{-1,3}$$

$$- \frac{1}{3} a_{1,-3} + \frac{1}{3} a_{-1,-3} + \frac{1}{3} a_{3,1} - \frac{1}{3} a_{-3,1} - \frac{1}{3} a_{3,-1} + \frac{1}{3} a_{-3,-1} \Big] - \pi^2 \left\| \hat{w} \right\|_{l^4}^4$$

In addition, Lemma 5.2 shows that we need only calculate $a_{1,1}$ and $a_{1,3}$. For $a_{1,1}$, the only values of n that make $\hat{w}(n)\ \bar{\hat{w}}(n-1)\ \hat{w}(n-2)\ \bar{\hat{w}}(n-1)$ non-zero are $n = 0$, 1, and 2. Hence, $a_{1,1} = 2r\bar{r}^2 + r^2$. Apply a similar strategy to each of the possibilities for $a_{1,3}$ to get $a_{1,3} = r^2 \bar{r}^2$. With this in mind, we get from (5.9) that

$$A(\hat{w}) - 2\pi B \left\| \hat{w} \right\|_{l^4}^4 = -2i\pi \left[(2r\bar{r}^2 + r^2) - (2\bar{r}r^2 + \bar{r}^2) - (2\bar{r}r^2 + \bar{r}^2) + \right.$$

$$(2r\bar{r}^2 + r^2) + \frac{1}{3} r^2 \bar{r}^2 - \frac{1}{3} \bar{r}^2 r^2 - \frac{1}{3} \bar{r}^2 r^2 + \frac{1}{3} r^2 \bar{r}^2 + \frac{1}{3} r^2 \bar{r}^2 - \frac{1}{3} \bar{r}^2 r^2 -$$

$$\left. \frac{1}{3} \bar{r}^2 r^2 + \frac{1}{3} r^2 \bar{r}^2 \right] - \pi^2 (1 + 4|r|^4)$$

$$= -2i\pi \left[2(2r\bar{r}^2 + r^2) - 2(2\bar{r}r^2 + \bar{r}^2) \right] - \pi^2 (1 + 4|r|^4)$$

$$= -2i\pi\left[2(2\bar{r}|r|^2 + r^2) - 2(2r|r|^2 + \bar{r}^2)\right] - \pi^2(1 + 4|r|^4)$$

$$= -4i\pi\left[2|r|^2(\bar{r} - r) + r^2 - \bar{r}^2\right] - \pi^2(1 + 4|r|^4).$$

If we let $r = x + iy$, then

$$A(\hat{w}) - 2\pi B\,\|\hat{w}\|_{l^4}^4 = -4i\pi\left[-4iy(x^2 + y^2) + 4ixy\right] - \pi^2(1 + 4(x^2 + y^2)^2)$$

$$= \pi\left[-16x^2y + 16xy - 16y^3\right] - \pi^2(1 + 4(x^2 + y^2)^2). \qquad (5.10)$$

One sees that the right hand side of (5.10) has a positive maximum at $x = -0.527097$, and $y = -0.947191$. Therefore, for $B = \frac{\pi}{2}$, there exists a function w satisfying the conditions of Theorem 1.1.

Define the function $\hat{w}$, by

$$\hat{w}(n) = \begin{cases} 1 & n = 0 \\ r & n = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

where again we allow r to take a complex value. For this value of B, equations (5.1) and (5.2) give

$$A(\hat{w}) = 2\pi B \sum_{p\neq 0}\sum_{l\neq 0} b_{p,l}\; a_{p,l}$$

$$= \pi \sum_{p\neq 0}\sum_{l\neq 0} \frac{i}{lp}[e^{-2ilpB} - 1]\; a_{p,l}.$$

For $a_{p,l} \neq 0$, we have that $|p| \leq 2$, $|l| \leq 2$, $|p - l| \leq 2$ and $|p + l| \leq 2$. In this case, we see that the possibilities for (p, l) are $(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1,-1)$. Hence,

$$A(\hat{w}) = i\pi\left[(e^{-2iB}-1)(a_{1,1}+a_{-1,-1}) - (e^{2iB}-1)(a_{-1,1}+a_{1,-1})\right].$$

In addition, Lemma 5.2 shows that we need only calculate $a_{1,1}$. For $a_{1,1}$, the only value of n that makes $\hat{w}(n)\ \bar{\hat{w}}(n-1)\ \hat{w}(n-2)\ \bar{\hat{w}}(n-1)$ non-zero is $n=1$. This results in $a_{1,1}=r^2$. With this in mind,

$$A(\hat{w}) = 2i\pi\left[(e^{-2iB}-1)r^2 - (e^{2iB}-1)\bar{r}^2\right]. \tag{5.11}$$

If we let $r = x+iy$, then

$$\begin{aligned} A(\hat{w}) &= 2i\pi\left[\cos(2B)(r^2-\bar{r}^2) - i\sin(2B)(r^2+\bar{r}^2) - (r^2-\bar{r}^2)\right] \\ &= 2i\pi\left[4xyi\cos(2B) - 2i(x^2-y^2)\sin(2B) - 4xyi\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} A(\hat{w}) - 2\pi B\|\hat{w}\|_{l^4}^4 = 2\pi\big[&-4xy\cos(2B) + 2(x^2-y^2)\sin(2B) + 4xy - \\ &B(1+2(x^2+y^2)^2)\big]. \end{aligned} \tag{5.12}$$

For $x = 0.65$ and $y = 0.53$, we have that $A(\hat{w}) - 2\pi B\|\hat{w}\|_{l^4}^4 > 0$ in the interval (B_0, B_1) where $B_0 = 0.67$ and $B_1 = 1.39$.

To summarize, by Theorem 1.1, we have shown that maximizers for J_λ, and thus stable ground state solutions, exist for the values of B in the interval $(0, 1.39)$, and for $B = \frac{\pi}{2}$. However, maximizers do not exist for $N\pi$ when $N \in \mathbb{N}$.

5.3 Open questions

The known values of B for which maximizers exist in L^2 suggest that there could be a possibility to extend the existence all the way to $B = \pi$, by considering larger classes of functions w.

It is not yet known whether the periodic DMNLS equation is globally (or locally) well posed in H^s for small fractional values of $s > 0$, or for $s < 0$. It would seem that the first hurdle is in finding a suitable Strichartz type estimate that will help with the Banach contraction mapping argument.

Also, since ground state solutions correspond to optimizers of the energy functional $E(u)$, and we have only considered minimizers of $E(u)$, it could be possible to find ground state solutions that correspond to maximizers of the energy functional.

Lastly, a more challenging question will be to identify precisely which functions are in the set of minimizers $\mathcal{S}$, and use this information to determine whether ground states are orbitally stable.

Bibliography

[1] M. Ablowitz, *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons*, Cambridge, 2011.

[2] M. J. Ablowitz and G. Biondini, Multiscale pulse dynamics in communication systems with strong dispersion management, Opt. Lett 23 (1998), 1668-1670.

[3] J. Albert and E. Kahlil, On the well-posedness of the Cauchy problem for some nonlocal nonlinear Schrdinger equations, Nonlinearity 30 (2017), 2308-2333.

[4] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations Part 1: Schrödinger equations, Geometric and Functional Analysis 3 (1993) 107-156.

[5] M.-R. Choi, D. Hundertmark, and Y.-R. Lee, Thresholds for existence of dispersion management solitons for general nonlinearities. SIAM J. Math. Anal 49 (2017), 1519-1569.

[6] M.B. Erdoğan, D. Hundertmark, and Y.-R. Lee, Exponential decay of dispersion managed solitons for vanishing average dispersion, Math. Res. Lett. 18 (2011), 1124.

[7] I.R. Gabitov and S. K. Turitsyn, Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation, Optics Letters 21 (1996) 327-329.

[8] J. Ginibre, Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace, Séminaire N. Bourbaki (1994-1995) 163-187.

[9] W.R. Green and D. Hundertmark, Exponential decay of dispersion-managed solitons for general dispersion profiles, Lett. Math. Phys. 106 (2016) 221-249.

[10] D. Hundertmark, P. Kunstmann, and R. Schnaubelt, Stability of dispersion managed solitions for vanishing average dispersion, Archiv der Mathematik 104 (2015), 283-288.

[11] D. Hundertmark and Y.-R. Lee, Decay estimates and smoothness for solutions of the dispersion managed non-linear Schrödinger equation, Commun. Math. Phys. 286 (2009), 851873.

[12] D. Hundertmark and Y.-R. Lee, On non-local variational problems with lack of compactness related to non-linear optics, J. Nonlinear Sci. 22 (2012), 138.

[13] D. Hundertmark, Y.-R. Lee, T. Reid, and V. Zharnitsky, Solitary waves in nonlocal NLS with dispersion averaged saturated nonlinearities, J.Diff. Eq. 265 (2018), 3311-3338.

[14] C. E. Kenig, G. Ponce, and L. Vega, The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, Duke Mathematical J. 71 (1993) 1-21 .

[15] M. Kunze, On a variational problem with lack of compactness related to Strichartz inequality, Calculus of Variations 19 (2004) 307-336.

[16] M. Kunze, J. Moeser, and V. Zharnitsky, Ground states for the higher-order dispersion managed NLS equation in the absence of average dispersion, J. Diff. Eq. 209 (2005), 77 100.

[17] P.-L. Lions, The concentration-compactness principle in the calculus of varations. The locally compact case, Part 1 and 2, Annales de l'I.H.P. Analyse non linaire 1 (1984), no. 2 and no. 4, 109-145 and 223-283.

[18] D. Luenberger, *Optimization by Vector Space Methods*, Wiley, 1969.

[19] M. Stanislavova, Regularity of ground state solutions of DMNLS equations, J. Diff. Eq. 210 (2005), 87 105.

[20] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Regional Conference Series in Mathematics, 2006.

[21] S.K. Turitsyn, B. Bale, and M. P. Fedoruk, Dispersion-managed solitons in fibre systems and lasers, Phys. Reports 521 (2012), 135203.

[22] V. Zharnitsky et al., Stabilizing effects of dispersion management, Physica D152-153 (2001) 794-817.

[23] V. Zharnitsky et al., Ground states of dispersion managed nonlinear Schrödinger equation, Phys. Rev. E 62 (2000) 7358-7364.