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NUMERICAL METHODS FOR OPTIMAL TRAJECTORY PLANNING
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BY

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Abstract

Optimal control theory focuses on finding the inputs that optimize the performance measure of a system subject to differential constraints. Differential game theory focuses on problems involving two separate parties, one of which tries to find the inputs to minimize a performance measure, while the other party tries to find the inputs which maximize the same performance measure. Both parties involved are subject to differential constraints. Both optimal control and differential game problems have a high degree of complexity except for the simplest of problems. This leads to the need for numerical methods to find the solutions to optimal control and differential game problems. In this thesis, we present our original numerical toolbox capable of finding feedback control policies, which solve optimal control and differential game problems by computing the solutions to the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs equations.

Chapter 1. Introduction

1.1 Motivation and Goals

Optimal control theory is a field that deals with finding the inputs that minimize or maximize a performance measure subject to differential constraints. Differential game theory is a field similar to optimal control theory, wherein two competing parties are involved. Specifically, one party tries to minimize some performance measure, while the other party tries to maximize the same performance measure. Analytical solutions to optimal control and differential game problems cannot be found in numerous problems of practical interest and therefore, numerical approaches must be pursued.

Existing numerical methods to numerically solve optimal control and differential game problems were presented in [1] and [2] and the majority of existing methods fall into three categories, namely direct methods, indirect methods, and hybrid methods. Direct methods solve optimal control problems by using optimization methods such as gradient descent method to directly compute the input that minimizes or maximizes the performance measure, and solve differential game problems by finding the inputs which satisfy the saddle point condition. Indirect methods solve optimal control and differential game problems by solving the underlying two point boundary problem given by the theory of calculus of variations approach. Hybrid methods combine both direct methods and indirect methods to solve optimal control and differential game problems. A major drawback of these methods is that the control inputs are parameterized as functions of time. However, it would be ideal to create numerical methods that produce control policies that solve optimal control and differential game problems as functions of the system's state.

In this thesis, we present our original numerical toolbox that solves optimal control and differential game by providing state-feedback control laws in the form of lookup tables. This original toolbox aims to use level set methods to calculate control inputs by solving the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs equations. To our best knowledge, this approach is unprecedented.

This thesis is structured as follows. In Chapter 2, we will discuss both necessary and sufficient conditions to solve optimal control problems, and we will present optimal control problems of practical interest and their solutions. In Chapter 3, we will discuss both necessary and sufficient conditions to solve differential game problems, and we will present differential game problems of practical interest and their solutions. In Chapter 4, we will discuss level set methods which have been applied in our toolbox to solve optimal control and differential game problems, and we will apply our toolbox to solve the problems presented in Chapters 2 and 3. Finally, in Chapter 5, we will draw conclusions and outline future research directions.

Chapter 2. Optimal Control

2.1 Introduction

In this chapter, we discuss necessary and sufficient conditions to solve optimal control problems, and present the solutions to notable problems. The same problems will later be solved in Chapter 4 using our original numerical toolbox.

2.2 Necessary and Sufficient Conditions for Optimality

2.2.1 Problem Statement

First, we must define the set of admissible controls. Given $u : [t_0, t_f] \to \mathbb{U} \subseteq \mathbb{R}^n$ the set of admissible control inputs is defined as

$$\mathcal{U} \triangleq \{u(\cdot) : u(\cdot) \text{ is PWC}[t_0, t_f], u(t) \in \mathbb{U}, t \in [t_0, t_f]\}; \tag{2.1}$$

where $\text{PWC}[t_0, t_f]$ denotes the set of piecewise continuous functions on $[t_0, t_f] \subseteq \mathbb{R}$. Our goal is to find $u(\cdot) \in \mathcal{U}$ that minimizes the performance measure.

$$J[x_0, u(\cdot)] = \psi(t_f, x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t))\ dt, \tag{2.2}$$

where $L : [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable and denotes the *Lagrangian function*, $\psi : [t_0, t_f] \times \mathbb{R}^n \to \mathbb{R}$ denotes the cost at time t_f and endpoint $x(t_f)$, and the state vector $x : [t_0, t_f] \to \mathbb{R}^n$ verifies the differential constraints

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0, \quad x(t_f) = x_f, \quad t \in [t_0, t_f]. \tag{2.3}$$

2.2.2 Calculus of Variations Approach

In this section, we provide necessary conditions to solve the optimal control problem outlined in Section 2.2.1. To this goal, firstly we include the constraints (2.3) in the performance measure (2.2) by means of Lagrange multipliers. Successively, we solve the corresponding unconstrained optimal control problem.

It follows from (2.2) and (2.3) that the optimal control problem in Section 2.2.1 is equivalent to finding $u^*(\cdot) \in \mathcal{U}$ so that the performance measure

$$\begin{aligned} \hat{J}[x_0, u(\cdot)] = {} & \psi(t_f, x(t_f)) + \int_{t_0}^{t_f} \Big[L(t, x(t), u(t)) \\ & + \lambda^{\mathrm{T}}(t)\big[f(t, x(t), u(t)) - \dot{x}(t)\big]\Big]\, dt, \end{aligned} \tag{2.4}$$

verifies the optimality condition

$$\hat{J}[x_0, u^*(\cdot)] = \min_{u(\cdot)\in\mathcal{U}} \hat{J}[x_0, u(\cdot)], \tag{2.5}$$

where $\lambda : [t_0, t_f] \to \mathbb{R}^n$ denote the *Lagrange multipliers*. In the following, we will further characterize the costate vector $\lambda(\cdot)$. Let us define *the Hamiltonian function*

$$\begin{aligned} H(t, x, u, \lambda) & \triangleq L(t, x, u) + \lambda^{\mathrm{T}} f(t,\ x, u), \\ & (t, x, u, \lambda) \in [t_0,\ t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n. \end{aligned} \tag{2.6}$$

Then, integrating equation (2.4) by parts yields

$$\begin{aligned} \hat{J}[x_0, u(\cdot)] = {} & \psi(t_f, x(t_f)) - \lambda^{\mathrm{T}}(t_f)x(t_f) + \lambda^{\mathrm{T}}(t_0)x(t_0) \\ & + \int_{t_0}^{t_f} \Big[H(t, x(t), u(t), \lambda(t)) + \dot{\lambda}^{\mathrm{T}}(t)x(t)\Big] dt. \end{aligned} \tag{2.7}$$

In order to find $u^*(\cdot) \in \mathcal{U}$ that verifies (2.5), we consider first-order variations of the control input and assume that $x_0,\ t_0,\ t_f$ are given. Specifically, given $v(\cdot) \in \mathbb{R}^m$

such that $u(\cdot) + \alpha v(\cdot) \in \mathcal{U},\ \alpha \in [0, \alpha_0)$, the variation in the state is given by

$$\begin{aligned} \dot{x}(t,\alpha) &= f(t,\, x(t,\alpha), u(t) + \alpha v(t)) \\ x(t_0,\alpha) &= x_0,\ x(t_f,\alpha) = x(t_f,0) + \alpha \left.\frac{\partial x(t_f,\alpha)}{\partial \alpha}\right|_{\alpha=0},\ t \in [t_0, t_f], \end{aligned} \tag{2.8}$$

and it follows from Taylor's theorem that

$$x(t,\alpha) = x(t,0) + \alpha \left.\frac{\partial x(t,\alpha)}{\partial \alpha}\right|_{\alpha=0} + O(\alpha). \tag{2.9}$$

Next, we define

$$\delta x(t) \triangleq \left.\frac{\partial x(t,\alpha)}{\partial \alpha}\right|_{\alpha=0}, \qquad t \in [t_0, t_f], \tag{2.10}$$

and note that it follows from (2.3) that

$$\delta \dot{x}(t) = A(t)\delta x(t) + B(t)v(t), \qquad \delta x(t_0) = 0, \qquad t \in [t_0, t_f] \tag{2.11}$$

where

$$A(t) \triangleq \frac{\partial f(t, x(t), u(t))}{\partial x}, \tag{2.12}$$

$$B(t) \triangleq \frac{\partial f(t, x(t), u(t))}{\partial u}, \tag{2.13}$$

and $\delta x(t_f)$ will be defined shortly. In this case, it follows from (2.7) that

$$\begin{aligned} \delta \hat{J}[x_0, u(\cdot), v(\cdot)] = {} & \frac{\partial \psi(t_f, x(t_f))}{\partial x}\delta x(t_f) - \lambda^{\mathrm{T}}(t_f)\delta x(t_f) + \lambda^{\mathrm{T}}(t_0)\delta x(t_0) \\ & + \int_{t_0}^{t_f} \Big[\Big(\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x} + \dot{\lambda}^{\mathrm{T}}(t)\Big)\delta x(t) \\ & \quad + \frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial u} v(t)\Big] dt \end{aligned} \tag{2.14}$$

Since $x(t_0)$ is given, $\delta x(t_0) = 0$ and hence $\lambda(t)$ can be chosen such that

$$\dot{\lambda}^{\mathrm{T}}(t) = -\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x}, \qquad t \in [t_0, t_f]. \tag{2.15}$$

The boundary conditions for (2.15) can be found by setting

$$\left(\frac{\partial \psi(t_f, x(t_f))}{\partial x} - \lambda^{\mathrm{T}}(t_f) \right) \delta x(t_f) = 0. \tag{2.16}$$

Since $\delta x(t_f)$ is free, one can choose

$$\lambda^{\mathrm{T}}(t_f) = \frac{\partial \psi(t_f, x(t_f))}{\partial x}. \tag{2.17}$$

Therefore, it follows from (2.14) that

$$\delta \hat{J}[x_0, u(\cdot), v(\cdot)] = \int_{t_0}^{t_f} \frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial u} v(t) \; dt. \tag{2.18}$$

By the first-order necessary condition for optimality, if $\delta \hat{J}(t, x(t), u(t)) \geq 0$, then $u(\cdot)$ is a local minimizer [7]. Assuming that $u(\cdot) \in \mathring{\mathcal{U}}$, that is that $u(\cdot)$ is in the interior of the admissible set, it must hold that $\delta \hat{J}[x_0, u(\cdot), v(\cdot)] = 0$. This leads to the optimality condition

$$\frac{\partial H(t, x, u, \lambda)}{\partial u} = 0, \qquad (t, x, u, \lambda) \in [t_0, t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n. \tag{2.19}$$

In calculus of variations, the set of equations (2.15) and (2.19) is known as the Euler-Lagrange equations.

In the following, we present optimality conditions for alternative sets of boundary conditions. Specifically, if t_f is free and $x(t_f)$ is given, then

$$\dot{\lambda}^{\mathrm{T}}(t) = -\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x},$$

$$H(t_f, x(t_f), u(t_f), \lambda(t_f)) = 0, \qquad t \in [t_0, t_f], \quad (2.20)$$

with optimality condition

$$\frac{\partial H(t,x,u,\lambda)}{\partial u} \geq 0, \qquad (t,x,u,\lambda) \in [t_0,t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n. \qquad (2.21)$$

In case both t_f and $x(t_f)$ are given, then

$$\dot{\lambda}^{\mathrm{T}}(t) = -\frac{\partial H(t,x(t),u(t),\lambda(t))}{\partial x}, \qquad \lambda^{\mathrm{T}}(t_f) = 0, \qquad t \in [t_0,t_f], \qquad (2.22)$$

with optimality condition

$$\frac{\partial H(t,x,u,\lambda)}{\partial u} = 0, \qquad (t,x,u,\lambda) \in [t_0,t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n. \qquad (2.23)$$

In case both time t_f and $x(t_f)$ are free, then the costate dynamics are

$$\dot{\lambda}^{\mathrm{T}}(t) = -\frac{\partial H(t,x(t),u(t),\lambda(t))}{\partial x}, \qquad \lambda^{\mathrm{T}}(t_f) = 0, \qquad t \in [t_0,t_f], \qquad (2.24)$$

with optimality condition

$$\frac{\partial H(t,x,u,\lambda)}{\partial u} = 0, \qquad (t,x,u,\lambda) \in [t_0,t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n. \qquad (2.25)$$

2.2.3 Hamiltonian Approach

In this section, a different approach to solve the optimal control problem is presented. Let us consider the system with differential constraint

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad t \in [t_0,t_f], \qquad x(t_0) = x_0, \qquad (2.26)$$

where $x(t_f)$ is free. Let us then define the *optimal cost-to-go function* as

$$V(t, x_0) \triangleq \min_{u(\cdot)\in\mathcal{U}} \left[\psi(t_f, x(t_f)) + \int_t^{t_f} L(\tau, x(\tau), u(\tau))d\tau\right]. \tag{2.27}$$

The total derivative of $V(t, x(t))$ is defined as

$$\left.\frac{dV(t, x(t))}{dt}\right|_{u(\cdot)} = \lim_{s\to t} \frac{V(s, x(s)) - V(t, x(t))}{s - t}, \tag{2.28}$$

along the trajectory of

$$\dot{x}(s) = f(s, x(s), u(s)), \qquad x(t) = x, \qquad x(t_f) = x_f, \qquad s \in [t, t_f]. \tag{2.29}$$

Applying the principle of optimality, which is proven in Theorem 11.2 of [9], to (2.27) yields

$$\begin{aligned} V(t, x_0) &= \min_{u(\cdot)\in\mathcal{U}} \left[\int_t^{t_f} L(\tau, x(\tau), u(\tau))d\tau\right] \\ &= \min_{u(\cdot)\in\mathcal{U}} \left[\int_t^{t_1} L(\tau, x(\tau), u(\tau))d\tau \right. \\ &\qquad \left. + \int_{t_1}^{t_f} L(\tau, x(\tau), u(\tau))d\tau\right] \\ &= \min_{u(\cdot)\in\mathcal{U}} \left[\int_t^{t_1} L(\tau, x(\tau), u(\tau))d\tau\right] + V(t_1, x(t_1)). \end{aligned} \tag{2.30}$$

Now, suppose that $u^*(\cdot) \in \mathcal{U}$ minimizes $J(t, x(\cdot), u(\cdot))$ and let $x^*(\cdot)$ be the corresponding trajectory, then it follows from (2.30) that

$$V(t, x_0) = \int_t^{t_1} L(\tau, x^*(\tau), u^*(\tau))d\tau + V(t_1, x(t_1)). \tag{2.31}$$

Therefore, it follows that

$$\frac{V(t_1, x(t_1)) - V(t, x_0)}{t_1 - t} + \frac{1}{t_1 - t}\int_t^{t_1} L(\tau, x^*(\tau), u^*(\tau))d\tau = 0 \tag{2.32}$$

and, letting $t_1 \to t$ yields

$$\left.\frac{dV(t,x(t))}{dt}\right|_{u(\cdot)=u^*(\cdot)} + L(t,x(t),u^*(t)) = 0. \tag{2.33}$$

However, for arbitrary $u(\cdot) \in \mathcal{U}$, it holds that

$$V(t,x_0) \le \int_t^{t_1} L(\tau,x(\tau),u(\tau))d\tau + V(t_1,x(t_1)), \tag{2.34}$$

and it follows from (2.33) that

$$-\frac{\partial V(t,x(t))}{\partial t} = \min_{u\in\mathcal{U}}\left[L(t,x(t),u(t)) + \frac{\partial V(t,x(t))}{\partial x}f(t,x(t),u(t))\right]$$
$$V(t_f,x(t_f)) = \psi(t_f,x(t_f)). \tag{2.35}$$

Equation (2.35) is known as the *Hamilton-Jacobi-Bellman (HJB) equation* and is a necessary condition for optimality. It can be proven that solving

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u\in\mathcal{U}}\left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x}f(t,x,u)\right]$$
$$V(t_f,x(t_f)) = \psi(t_f,x(t_f)). \tag{2.36}$$

is also a sufficient condition for optimality [9].

2.3 Minimum Time to Reach Problems

A common problem of interest in the field of optimal control is the minimum time to reach problem. This problem is useful to create trajectories to reach some endpoint as quickly as possible. In this problem, (2.2) specializes to

$$J[x_0,u(\cdot)] = \int_{t_0}^{t_f} 1\ dt, \tag{2.37}$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0, \quad x(t_f) = x_f, \quad t \in [t_0, t_f]. \tag{2.38}$$

Solving this problem using the calculus of variations approach, it follows from (2.6) that

$$H(t, x(t), u(t), \lambda(t)) = 1 + \lambda^{\mathrm{T}}(t) f(t, x(t), u(t)). \tag{2.39}$$

In this case, it follows from (2.15) that

$$\dot{\lambda}(t) = -\left[\frac{\partial f(t, x(t), u(t))}{\partial x}\right]^{\mathrm{T}} \lambda(t),$$
$$H(t_f, x(t_f), u(t_f), \lambda(t_f)) = 0, \qquad t \in [t_0, t_f] \tag{2.40}$$

and it follows from (2.19) that

$$\lambda^T(t) \frac{\partial f(t, x(t), u(t))}{\partial u} = 0. \tag{2.41}$$

2.3.1 Brachistochrone Problem

A common example of the minimum time problem is known as the brachistochrone problem. Specifically, we are to find $u(\cdot) \in \mathcal{U}$ such that (2.37) is minimized and (2.38) specializes to

$$\dot{x}_1(t) = \sqrt{2gx_2(t)} \cos u(t), \qquad x_1(t_0) = 0, \qquad x_1(t_f) = l, \qquad t \in [t_0, t_f], \tag{2.42}$$
$$\dot{x}_2(t) = \sqrt{2gx_2(t)} \sin u(t) \qquad x_2(t_0) = 0, \qquad x_2(t_f) = x_{2f}, \qquad t \in [t_0, t_f], \tag{2.43}$$

where $g > 0$, $l > 0$, and x_{2f} is free. An interpretation of the problem is the following: find the control such that a point of mass $m > 0$ starting at $x(t_0) = 0$ reaches $s(t_f) = [l \ x_{2f}]^{\mathrm{T}}$ in minimum time under the effect of the force of gravity.

The control input $u(\cdot)$ denotes the angle between the velocity vector of the position and the axis $x_1(\cdot)$. The Hamiltonian function for the brachistochrone problem is given by

$$H(t,x,u,\lambda) = 1 + \lambda^{\mathrm{T}} \begin{bmatrix} \sqrt{2gx_2}\cos u \\ \sqrt{2gx_2}\sin u \end{bmatrix},$$

$$(t,x,u,\lambda) \in [t_0,t_f] \times \mathbb{R}^2 \times [0,2\pi) \times \mathbb{R}^2 \quad (2.44)$$

In this case, it follows from (2.15) and (2.19) the optimality conditions are given by

$$\begin{bmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{[\lambda_1(t)\cos u(t) + \lambda_2(t)\sin u(t)]}{\sqrt{2gx_2(t)}} \end{bmatrix}, \qquad t \in [t_0,t_f]$$

$$\lambda_2(t)\cos u(t) = \lambda_1(t)\sin u(t), \quad (2.45)$$

subject to

$$1 + \lambda_1(t_f)\sqrt{2gx_2(t_f)}\cos u(t_f) + \lambda_2(t_f)\sqrt{2gx_2(t_f)}\sin u(t_f) = 0. \quad (2.46)$$

The solution to (2.45) is

$$\lambda_1(t) = -\frac{\omega}{g}, \qquad t \in [t_0,t_f], \quad (2.47)$$

$$\lambda_2(t) = -\frac{\omega}{g}\cot(\omega t), \quad (2.48)$$

where

$$\omega \triangleq \left(\frac{\pi}{4}\frac{g}{l}\right)^{1/2}. \quad (2.49)$$

Moreover, the optimal control law is given by

$$u(t) = \frac{\pi}{2} - \omega t, \qquad t \in [t_0, t_f], \tag{2.50}$$

and the corresponding optimal trajectory is given by

$$x_1(t) = \frac{2l}{\pi}\left(\omega t - \frac{\sin(2\omega t)}{2}\right), \tag{2.51}$$

$$x_2(t) = \frac{2l}{\pi}\sin^2(\omega t). \tag{2.52}$$

2.4 H_∞ Optimal Control

Consider the linear, time invariant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \qquad x(0) = x_0, \qquad t \in [t_0, t_f], \tag{2.53}$$

$$z(t) = Cx(t) + D_1u(t) + D_2w(t), \tag{2.54}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $E \in \mathbb{R}^{n\times p}$, $D_1 \in \mathbb{R}^{l\times m}$, $D_2 \in \mathbb{R}^{l\times p}$, and $w(\cdot)$ denotes an external disturbance, and consider the performance measure

$$J[x_0, u(\cdot), w(\cdot)] = \frac{1}{2}\int_{t_0}^{t_f}\left(z^{\mathrm{T}}(\tau)z(\tau) - \gamma^2 w^{\mathrm{T}}(\tau)w(\tau)\right)d\tau, \tag{2.55}$$

where $\gamma > 0$, $x(t_0)$ and t_0 are given, and $x(t_f)$ and t_f are free. The goal is to find a control $u^*(\cdot)$ and a disturbance $w^*(\cdot)$ such that

$$J[u^*(t), w(t)] \le J[u^*(t), w^*(t)] \le J[u(t), w^*(t)], \qquad t \in [t_0, t_f]. \tag{2.56}$$

It follows from (2.54) and (2.55) that

$$J[x_0, u(\cdot), w(\cdot)] = \frac{1}{2}\int_{t_0}^{t_f}\Big[[Cx(\tau) + D_1u(\tau) + D_2w(\tau)]^{\mathrm{T}}[Cx(\tau) + D_1u(\tau) + D_2w(\tau)]$$

$$- \gamma^2 w^{\mathrm{T}}(\tau) w(\tau) \Big] \, d\tau, \tag{2.57}$$

which is equivalent to

$$J[u(\cdot), w(\cdot)] = \frac{1}{2} \int_{t_0}^{T} \left(x^{\mathrm{T}}(\tau) C^{\mathrm{T}} C x(\tau) - 2x^{\mathrm{T}}(\tau) [C^{\mathrm{T}} D_1 \; C^{\mathrm{T}} D_2] \begin{bmatrix} u(\tau) \\ w(\tau) \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} u(\tau) \\ w(\tau) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} D_1^{\mathrm{T}} D_1 & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 - \gamma^2 I \end{bmatrix} \begin{bmatrix} u(\tau) \\ w(\tau) \end{bmatrix} \right) d\tau. \tag{2.58}$$

In order to recast this problem in the same formulation as Section 2.2,

$$\begin{aligned} S &\triangleq \begin{bmatrix} C^{\mathrm{T}} D_1 & C^{\mathrm{T}} D_2 \end{bmatrix}, \\ R &\triangleq \begin{bmatrix} D_1^{\mathrm{T}} D_1 & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 - \gamma^2 I \end{bmatrix}, \\ \tilde{B} &\triangleq \begin{bmatrix} B & E \end{bmatrix}, \\ \tilde{u}(t) &\triangleq \begin{bmatrix} u(t) \\ w(t) \end{bmatrix}, \qquad t \in [t_0, t_f] \end{aligned} \tag{2.59}$$

Then, (2.53) is equivalent to

$$\dot{x}(t) = Ax(t) + \tilde{B}\tilde{u}(t), \qquad x(t_0) = x_0, \qquad t \in [t_0, t_f], \tag{2.60}$$

and (2.55) is equivalent to

$$J[x_0, u(\cdot), w(\cdot)] = \frac{1}{2} \int_{t_0}^{t_f} \left[(x^{\mathrm{T}}(\tau) C^{\mathrm{T}} C x(\tau) - 2x^{\mathrm{T}}(\tau) S \tilde{u}(\tau) + \tilde{u}(\tau) R \tilde{u}(\tau) \right] d\tau. \tag{2.61}$$

It follows from (2.6) that the Hamiltonian function is given by

$$H(t,x,\tilde{u},\lambda) = \frac{1}{2}(x^{\mathrm{T}}C^{\mathrm{T}}Cx - 2x^{\mathrm{T}}S\tilde{u} + \tilde{u}R\tilde{u}) + \lambda^{\mathrm{T}}(Ax + \tilde{B}\tilde{u}),$$
$$(t,x,u,\lambda) \in [t_0,t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n, \quad (2.62)$$

and it follows from (2.19) that the necessary condition for the optimal control is given by

$$\frac{\partial H(t,x,\tilde{u},\lambda)}{\partial \tilde{u}} = R\tilde{u} + S^{\mathrm{T}}x + \tilde{B}^{\mathrm{T}}\lambda = 0, \quad (2.63)$$

which implies that

$$\tilde{u}^*(t) = -R^{-1}[S^{\mathrm{T}}x(t) + \tilde{B}\lambda(t)], \qquad t \in [t_0,t_f]. \quad (2.64)$$

Moreover, it follows from (2.15) that

$$\dot{\lambda}(t) = -\frac{\partial H(t,x(t),\tilde{u}(t),\lambda(t))}{\partial x} = -C^{\mathrm{T}}Cx(t) - A^{\mathrm{T}}\lambda(t) - S\tilde{u}(t),$$
$$\lambda(t_f) = 0, \qquad t \in [t_0,t_f], \quad (2.65)$$

and the solution to (2.65) is given by

$$\lambda(t) = P(t)x(t), \qquad t \in [t_0,t_f], \quad (2.66)$$

where $P : [t_0,t_f] \to \mathbb{R}^{n\times n}$ is such that

$$\dot{P}(t)x(t) + P(t)\left[Ax(t) + \tilde{B}\tilde{u}(t)\right] = -C^{\mathrm{T}}Cx(t) - A^{\mathrm{T}}P(t)x(t) - S\tilde{u}(t),$$
$$P(t_f) = 0, \qquad t \in [t_0,t_f]. \quad (2.67)$$

Hence, it follows from (2.64) that

$$-\dot{P}(t) = P(t)A + A^{\mathrm{T}}P(t) + C^{\mathrm{T}}C - [P(t)\tilde{B} + S]R^{-1}[\tilde{B}^{\mathrm{T}}P(t) + S^{\mathrm{T}}],$$
$$P(t_f) = 0, \qquad t \in [t_0, t_f], \quad (2.68)$$

which implies that

$$\begin{bmatrix} u^*(t) \\ w^*(t) \end{bmatrix} = \begin{bmatrix} D_1^{\mathrm{T}}D_1 & D_1^{\mathrm{T}}D_2 \\ D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2 - \gamma^2 I \end{bmatrix}^{-1} \left[\begin{bmatrix} D_1^{\mathrm{T}}C \\ D_2^{\mathrm{T}}C \end{bmatrix} + \begin{bmatrix} B^{\mathrm{T}} \\ E^{\mathrm{T}} \end{bmatrix} P(t) \right] x(t) \qquad (2.69)$$

2.5 Linear-Quadratic Regulator Problem

Another relevant optimal control problem is the linear-quadratic regulator problem. Consider the linear, time invariant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \qquad x(t_f) = x_f, \qquad t \in [t_0, t_f], \qquad (2.70)$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Our goal is to minimize the performance measure

$$J[x_0, u(\cdot)] = \frac{1}{2}x^{\mathrm{T}}(t_f)Q(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} x^{\mathrm{T}}(t)Q(t)x(t) + u^{\mathrm{T}}(t)R(t)u(t)\ dt, \quad (2.71)$$

where $Q(t) = Q^{\mathrm{T}}(t) > I$ and $R(t) = R^{\mathrm{T}}(t) > I$. It follows from (2.36) that

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u(\cdot)\in\mathcal{U}} \left[\frac{1}{2}\left[x^{\mathrm{T}}Q(t)x + u^{\mathrm{T}}R(t)u\right] + \frac{\partial V(t,x)}{\partial x}(Ax + Bu) \right],$$
$$V(t_f, x(t_f)) = \frac{1}{2}x^{\mathrm{T}}(t_f)Q(t_f)x(t_f), \quad (2.72)$$

which implies that

$$u^*(t) = -R^{-1}(t)B^{\mathrm{T}}\frac{\partial V(t,x)}{\partial x}, \qquad t \in [t_0, t_f] \qquad (2.73)$$

Hence, it follows from (2.72) that

$$-\frac{\partial V(t,x)}{\partial t} = \frac{\partial V(t,x)}{\partial x}Ax + \frac{1}{2}x^{\mathrm{T}}Q(t)x - \frac{1}{2}\frac{\partial V(t,x)}{\partial x}BR^{-1}(t)B^{\mathrm{T}}\frac{\partial V(t,x)}{\partial x}, \tag{2.74}$$

which is verified by

$$V(t,x(t)) = \frac{1}{2}x^{\mathrm{T}}P(t)x, \qquad P(t_f) = Q(t_f), \qquad t \in [t_0, t_f]. \tag{2.75}$$

Therefore, it follows from (2.74), that

$$\frac{1}{2}x^{\mathrm{T}}[\dot{P}(t) + P(t)A + A^{\mathrm{T}}P(t) - P(t)BR^{-1}(t)BP(t) + Q(t)]x = 0, \qquad (t,x) \in [t_0, t_f] \times \mathbb{R}^n, \tag{2.76}$$

which implies that

$$u^*(t) = -R^{-1}(t)B^{\mathrm{T}}P(t)x(t) \tag{2.77}$$

and

$$-\dot{P}(t) = P(t)A + A^{\mathrm{T}}P(t) - P(t)BR^{-1}(t)B^{\mathrm{T}}P(t) + Q(t) \qquad P(t_f) = Q(t_f), \qquad t \in [t_0, t_f], \tag{2.78}$$

which is known as the *matrix Riccati equation.* If $t_f \to \infty$ and

$$J[x_0, u(\cdot)] = \frac{1}{2}\int_{t_0}^{\infty} x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)dt, \tag{2.79}$$

then the optimal feedback control policy is given by

$$u^*(t) = -R^{-1}B^{\mathrm{T}}Px(t), \tag{2.80}$$

where P denotes the solution to the *algebraic Riccati equation*

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q. \tag{2.81}$$

Chapter 3. Differential Games

3.1 Introduction

In this chapter, we discuss necessary and sufficient conditions to solve differential game problems, and present the solutions to notable problems. The same problems will later be solved in Chapter 4 using our original numerical methods. Differential games are useful to model situations in which there are opponents that aim to achieve opposite goals.

3.2 Problem Statement

First, we must define the sets of admissible controls. Given $u : [t_0, t_f] \to \mathbb{U} \subseteq \mathbb{R}^{m_1}$ and $v : [t_0, t_f] \to \mathbb{V} \subseteq \mathbb{R}^{m_2}$, the sets of admissible control inputs are defined as

$$\mathcal{U} \triangleq \{u(\cdot) : u(\cdot) \ \in \text{PWC}\ [t_0, t_f], u(t) \in \mathbb{U}, t \in [t_0, t_f]\}, \tag{3.1}$$

$$\mathcal{V} \triangleq \{v(\cdot) : v(\cdot) \ \in \text{PWC}\ [t_0, t_f], v(t) \in \mathbb{V}, t \in [t_0, t_f]\}, \tag{3.2}$$

where PWC $[t_0, t_f]$ denotes the set of piecewise continuous function on $[t_0, t_f] \subseteq \mathbb{R}$. Then, consider the performance measure

$$J[x_0, u(\cdot), v(\cdot)] \triangleq \psi(t_f, x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t), v(t))\ dt, \tag{3.3}$$

where $L : [t_0, t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}$ is continuously differentiable, $\psi : [t_0, t_f] \times \mathbb{R}^n \to \mathbb{R}$ denotes the cost at time t_f and endpoint $x(t_f)$, and the state vector $x : [t_0, t_f] \to \mathbb{R}^n$ verifies the differential constraint

$$\dot{x}(t) = f(t, x(t), u(t), v(t)),$$
$$x(t_0) = x_0, \qquad \phi(t_f, x(t_f)) = 0, \qquad t \in [t_0, t_f], \tag{3.4}$$

where $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$ denotes the terminal constraint. Our goal is to find $(u^*(\cdot), v^*(\cdot)) \in \mathcal{U} \times \mathcal{V}$ such that

$$J[x_0, u^*(\cdot), v^*(\cdot)] = \min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} J[x_0, u(\cdot), v(\cdot)] \tag{3.5}$$

assuming that x_0, t_0, and t_f are given [8].

In this thesis, we assume that

$$\min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} J[x_0, u(\cdot), v(\cdot)] = \max_{v(\cdot)\in\mathcal{V}} \min_{u(\cdot)\in\mathcal{U}} J[x_0, u(\cdot), v(\cdot)]. \tag{3.6}$$

3.3 Calculus of Variations Approach

In this section, we provide necessary conditions to solve the differential game problem outlined in Section 3.2. To this goal, firstly we include the constraints (3.4) in the performance measure (3.3) by means of Lagrange multipliers. Successively, we solve the corresponding unconstrained differential game problem.

It follows from (3.3) and (3.4) that the differential game problem in Section 3.2 is equivalent to finding both $u^*(\cdot) \in \mathcal{U}$ and $v^*(\cdot) \in \mathcal{V}$ so that the performance measure

$$\begin{aligned}\hat{J}[x_0, u(\cdot), v(\cdot)] = \psi(t_f, x(t_f)) &+ \nu^{\mathrm{T}}\phi(t_f, x(t_f)) \\ &+ \int_{t_0}^{t_f} \Big[L(t, x(t), u(t), v(t)) \\ &\qquad + \lambda^{\mathrm{T}}(t)\big[f(t, x(t), u(t), v(t)) - \dot{x}(t)\big]\Big]\, dt,\end{aligned} \tag{3.7}$$

verifies the condition

$$\hat{J}[x_0, u^*(\cdot), v^*(\cdot)] = \min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} \hat{J}[x_0, u(\cdot), v(\cdot)] = \max_{v(\cdot)\in\mathcal{V}} \min_{u(\cdot)\in\mathcal{U}} \hat{J}[x_0, u(\cdot), v(\cdot)], \tag{3.8}$$

where $\nu \in \mathbb{R}^p$ and $\lambda : [t_0, t_f] \to \mathbb{R}^n$ denote the *Lagrange multipliers*. In the following, we will further characterize the *costate vectors* $\lambda(\cdot)$ and ν.

Let us define

$$\Phi(t_f, x(t_f)) \triangleq \psi(t_f, x(t_f)) + \nu^{\mathrm{T}}\phi(t_f, x(t_f)), \tag{3.9}$$

and the *Hamiltonian function*

$$H(t, x, u, v, \lambda) \triangleq L(t, x, u, v) + \lambda^{\mathrm{T}} f(t, x, u, v)$$
$$(t, x, u, v, \lambda) \in [t_0, t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \times \mathbb{R}^n. \tag{3.10}$$

Then, integrating equation (3.7) by parts yields

$$J[x_0, u(\cdot), v(\cdot)] = \Phi(t_f, x(t_f)) - \lambda^T(t_f)x(t_f) + \lambda^T(t_0)x(t_0)$$
$$+ \int_{t_0}^{t_f} [H(t, x(t), u(t), v(t)) + \dot{\lambda}^T(t)x(t)]\, dt. \tag{3.11}$$

In order to find $(u^*(\cdot), v^*(\cdot)) \in \mathcal{U} \times \mathcal{V}$ that verify (3.8), we consider first-order variations of the control inputs and assume that $x_0,\ t_0,\ t_f$ are given. First let us define $w(\cdot) \triangleq [u^{\mathrm{T}}(\cdot),\ v^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ and $\mathcal{W} = \mathcal{U} \times \mathcal{V}$, then given $r(\cdot) \in \mathbb{R}^{m_1+m_2}$ such that $w(\cdot) + \alpha r(\cdot) \in \mathcal{W}$, $\alpha \in [0, \alpha_0)$, the variation in the state vector is given by

$$\dot{x}(t, \alpha) = f(t, x(t, \alpha), w(t) + \alpha r(t))$$
$$\phi(t_f, x(t_f, \alpha)) = 0, \qquad x(t_0, \alpha) = x_0, \qquad t \in [t_0, t_f]. \tag{3.12}$$

It follows from Taylor's theorem that

$$x(t, \alpha) = x(t, 0) + \alpha \left.\frac{\partial x(t, \alpha)}{\partial \alpha}\right|_{\alpha=0} + O(\alpha). \tag{3.13}$$

Next, we define

$$\delta x(t) \triangleq \left.\frac{\partial x(t,\alpha)}{\partial \alpha}\right|_{\alpha=0}, \qquad t \in [t_0, t_f], \tag{3.14}$$

and note that it follows from (3.4) that

$$\begin{aligned} &\delta \dot{x}(t) = A(t)\delta x(t) + B(t) r(t), \\ &\left.\frac{\partial \phi(t, x(t))}{\partial x}\right|_{t=t_f} \delta x(t_f) = 0, \qquad \delta x(t_0) = 0, \qquad t \in [t_0, t_f], \end{aligned} \tag{3.15}$$

where

$$A(t) \triangleq \frac{\partial f(t, x(t), u(t), v(t))}{\partial x}, \tag{3.16}$$

$$B(t) \triangleq \left[\frac{\partial f(t, x(t), u(t), v(t))}{\partial u}, \ \frac{\partial f(t, x(t), u(t), v(t))}{\partial v}\right]. \tag{3.17}$$

In this case, it follows from (3.11) that

$$\begin{aligned} \delta \hat{J}[x_0, u(\cdot), v(\cdot), r(\cdot)] = {} & \frac{\partial \Phi(t_f, x(t_f))}{\partial x} \delta x(t_f) - \lambda^T(t_f)\delta x(t_f) + \lambda^T(t_0)\delta x(t_0) \\ & + \int_{t_0}^{t_f} \Bigg[\Big(\frac{\partial H(t, x(t), u(t), v(t), \lambda(t))}{\partial x} + \dot{\lambda}^T(t) \Big) \delta x(t) \\ & + \begin{bmatrix} \dfrac{\partial H(t, x(t), u(t), v(t), \lambda(t))}{\partial u} \\ \\ \dfrac{\partial H(t, x(t), u(t), v(t), \lambda(t))}{\partial v} \end{bmatrix}^{\mathrm{T}} r(t) \Bigg] dt. \end{aligned} \tag{3.18}$$

Since $x(t_0)$ is given, $\delta x(t_0) = 0$ and hence $\lambda(t)$ can be chosen such that

$$\dot{\lambda}^{\mathrm{T}}(t) = -\frac{\partial H(x(t), u(t), \lambda(t), t)}{\partial x}, \qquad t \in [t_0, t_f]. \tag{3.19}$$

The boundary conditions for (3.19) can be found by setting

$$\left(\frac{\partial\Phi(t_f, x(t_f))}{\partial x} - \lambda^T(t_f)\right)\delta x(t_f) = 0. \tag{3.20}$$

Since $\delta x(t_f)$ may be non-zero, one can choose

$$\lambda^T(t_f) = \frac{\partial\psi(t_f, x(t_f))}{\partial x} + \nu^{\mathrm{T}}\frac{\partial\phi(t_f, x(t_f))}{\partial x}. \tag{3.21}$$

Therefore, it follows from (3.18) that

$$\delta\hat{J}[x_0, u(\cdot), v(\cdot), r(\cdot)] = \int_{t_0}^{t_f} \begin{bmatrix} \dfrac{\partial H(t, x(t), u(t), v(t), \lambda(t))}{\partial u} \\ \\ \dfrac{\partial H(t, x(t), u(t), v(t), \lambda(t))}{\partial v} \end{bmatrix}^{\mathrm{T}} r(t)dt. \tag{3.22}$$

By the first order necessary condition for optimality, if $\delta\hat{J}[x_0, u(\cdot), v^*(\cdot), r(\cdot)] \geq 0$, then $u(\cdot)$ is a local minimizer and if $\delta\hat{J}[x_0, u^*(\cdot), v(\cdot), r(\cdot)] \leq 0$, then $v(\cdot)$ is a local maximizer. Assuming that $(u(\cdot), v(\cdot)) \in \mathring{\mathcal{U}} \times \mathring{\mathcal{V}}$, that is, the pair $(u(\cdot), v(\cdot))$ is the in the interior of the admissible set, it must hold that $\delta\hat{J}[x_0, u(\cdot), v(\cdot), w_1(\cdot), w_2(\cdot)] = 0$. This leads to the *differential games saddle point condition*

$$\frac{\partial H(t, x, u, v, \lambda)}{\partial u} = 0, \qquad [t, x, u, v, \lambda] \in [t_0, t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \times \mathbb{R}^n, \tag{3.23}$$
$$\frac{\partial H(t, x, u, v, \lambda)}{\partial v} = 0. \tag{3.24}$$

A similar procedure can be used to solve differential games, where in t_f is free, x_0 and t_0 are given, and $x(t_f)$ verifies $\psi(t_f, x(t_f))$. In this case, the first order necessary conditions are

$$\frac{\partial H(t, x, u, v, \lambda)}{\partial u} = 0, \qquad [t, x, u, v, \lambda] \in [t_0, t_f] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \times \mathbb{R}^n, \tag{3.25}$$

$$\frac{\partial H(t,x,u,v,\lambda)}{\partial v} = 0, \tag{3.26}$$

with costate dynamics

$$\begin{aligned} \dot{\lambda}^{\mathrm{T}}(t) &= -\frac{\partial H(t,x(t),u(t),\lambda(t))}{\partial x}, \qquad t \in [t_0, t_f], \\ \lambda^{T}(t_f) &= \frac{\partial \psi(t_f, x(t_f))}{\partial x} + \nu^{\mathrm{T}} \frac{\phi(t_f, x(t_f))}{\partial x}, \\ \frac{\partial \psi(t_f, x(t_f))}{\partial t} + \nu^{\mathrm{T}} \frac{\partial \phi(t_f, x(t_f))}{\partial t} &= -H(t_f, x(t_f), u(t_f), v(t_f), \lambda(t_f)). \end{aligned} \tag{3.27}$$

3.4 Hamiltonian Approach

In this section, an alternative approach to solve the differential game problem is presented. Let us consider the differential constraint

$$\dot{x}(t) = f(t, x(t), u(t), v(t)), \qquad x(t_0) = x_0, \qquad t \in [t_0, t_f], \tag{3.28}$$

where $x(t_f)$ is free. Let us then define the *optimal cost-to-go function* for differential games as

$$V(t, x_0) \triangleq \min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} \left[\psi(t_f, x(t_f)) + \int_{t_0}^{t_f} L(\tau, x(\tau), u(\tau), v(\tau)) d\tau \right], \qquad t \in [t_0, t_f]. \tag{3.29}$$

The total derivative of $V(t, x(t))$ is defined as

$$\left. \frac{dV(t,x(t))}{dt} \right|_{u(\cdot),v(\cdot)} = \lim_{s\to t} \frac{V(s,x(s)) - V(t,x(t))}{s-t}, \tag{3.30}$$

along the trajectory of

$$\dot{x}(s) = f(s, x(s), u(s), v(s)), \qquad x(t) = x, \qquad x(t_f) = x_f, \qquad s \in [t, t_f]. \tag{3.31}$$

Applying the principle of optimality, which is discussed in Section 2.2.3, to (3.29) and assuming that $v^*(\cdot) \in \mathcal{V}$ is a maximizer yields

$$\begin{aligned} V(t, x_0) &= \min_{u(\cdot)\in\mathcal{U}} \left[\int_t^{t_f} L(\tau, x(\tau), u(\tau), v^*(\tau)) d\tau \right] \\ &= \min_{u(\cdot)\in\mathcal{U}} \left[\int_t^{t_1} L(\tau, x(\tau), u(\tau), v^*(\tau)) d\tau \right. \\ &\qquad \left. + \int_{t_1}^{t_f} L(\tau, x(\tau), u(\tau), v^*(\tau)) d\tau \right] \\ &= \min_{u(\cdot)\in\mathcal{U}} \left[\int_t^{t_f} L(\tau, x(\tau), u(\tau), v^*(\tau)) d\tau \right] + V(t_1, x(t_1)). \end{aligned} \tag{3.32}$$

Now, suppose that $u^*(\cdot) \in \mathcal{U}$ minimizes $J(t, x(\cdot), u(\cdot), v^*(\cdot))$ and let $x^*(\cdot)$ denote the corresponding trajectory, then it follows from (3.32) that

$$V(t, x_0) = \int_t^{t_1} L(\tau, x^*(\tau), u^*(\tau), v^*(\tau)) d\tau + V(t_1, x(t_1)), \qquad t \in [t_0, t]. \tag{3.33}$$

Therefore, it follows that

$$\frac{V(t_1, x(t_1)) - V(t, x_0)}{t_1 - t} + \frac{1}{t_1 - t} \int_t^{t_1} L(\tau, x^*(\tau), u^*(\tau), v^*(\tau)) d\tau = 0 \tag{3.34}$$

and, letting $t_1 \to t$ yields

$$\left. \frac{dV(t, x(t))}{dt} \right|_{u^*(\cdot), v^*(\cdot)} + L(t, x(t), u^*(t), v^*(t)) = 0. \tag{3.35}$$

However, for arbitrary $u(\cdot) \in \mathcal{U}$, it holds that

$$V(t, x_0) \leq \int_t^{t_1} L(\tau, x(\tau), u(\tau), v^*(\tau)) d\tau + V(t_1, x(t_1)) \tag{3.36}$$

Let us then repeat the process by applying the principal of optimality to to (3.29) and assuming that $u^*(\cdot)$ is a minimizer yields

$$\begin{aligned} V(t,x_0) &= \max_{v(\cdot)\in\mathcal{V}}\left[\int_t^{t_f} L(\tau,x(\tau),u^*(\tau),v(\tau))d\tau\right] \\ &= \max_{v(\cdot)\in\mathcal{V}}\left[\int_t^{t_1} L(\tau,x(\tau),u^*(\tau),v(\tau))d\tau\right. \\ &\qquad \left.+\int_{t_1}^{t_f} L(\tau,x(\tau),u^*(\tau),v(\tau))d\tau\right] \\ &= \max_{v(\cdot)\in\mathcal{V}}\left[\int_t^{t_f} L(\tau,x(\tau),u^*(\tau),v(\tau))d\tau\right] + V(t_1,x(t_1)). \end{aligned} \tag{3.37}$$

Now, suppose that $v^*(\cdot)\in\mathcal{V}$ maximizes $J(t,x(\cdot),u^*(\cdot),v(\cdot))$ and let $x^*(\cdot)$ denote the corresponding trajectory, then it follows from (3.37) that

$$V(t,x_0) = \int_t^{t_1} L(\tau,x^*(\tau),u^*(\tau),v^*(\tau))d\tau + V(t_1,x(t_1)), \qquad t\in[t_0,t]. \tag{3.38}$$

Therefore it follows that

$$\frac{V(t_1,x(t_1))-V(t,x_0)}{t_1-t} + \frac{1}{t_1-t}\int_t^{t_1} L(\tau,x^*(\tau),u^*(\tau),v^*(\tau))d\tau = 0. \tag{3.39}$$

Letting $t_1\to t$ yields

$$\left.\frac{dV(t,x(t))}{dt}\right|_{u^*(\cdot),v^*(\cdot)} + L(t,x(t),u^*(t),v^*(t)) = 0. \tag{3.40}$$

However for arbitrary $v(\cdot)\in\mathcal{V}$, it holds that

$$V(t,x_0) \geq \int_t^{t_1} L(\tau,x(\tau),u^*(\tau),v(\tau))d\tau + V(t_1,x(t_1)), \tag{3.41}$$

and it follows from (3.35) and (3.40) that

$$\begin{aligned} -\frac{\partial V(t,x(t))}{\partial t} &+ \frac{\partial V(t,x(t))}{\partial x} f(t,x(t),u^*(t),v^*(t)) \\ &+ L(t,x(t),u^*(t),v^*(t)) = 0, \end{aligned} \tag{3.42}$$

which can be rewritten as

$$-\frac{\partial V(t,x(t))}{\partial t} = \min_{u\in\mathcal{U}} \max_{v\in\mathcal{V}} \left[\frac{\partial V(t,x(t))}{\partial x} f(t,x(t),u(t),v(t)) \right.$$
$$\left. + L(t,x(t),u(t),v(t)) \right]$$
$$V(t_f,x(t_f)) = \psi(t_f,x(t_f)). \quad (3.43)$$

Equation (3.43) is known as the *Hamilton-Jacobi-Isaacs (HJI) equation* [8], [9].

3.5 Converse Differential Games

The Hamilton-Jacobi-Issacs equation (3.43) is a first-order partial differential equation, whose analytical solution is impossible to find in many problems of practical interest. To overcome this difficulty, in this section we discuss converse differential games. Consider the performance measure

$$J[x_0,u(\cdot),v(\cdot)] = \int_0^\infty [L_1(x(t)) + L_u(x(t))u(t) + L_v(x(t))v(t)$$
$$+ u^{\mathrm{T}}(t)R_u(x(t))u(t) + v^{\mathrm{T}}(t)R_v(x(t))v(t)]\, dt. \quad (3.44)$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_u : \mathbb{R}^n \to \mathbb{R}^{1\times m_1}$, $L_v : \mathbb{R}^n \to \mathbb{R}^{1\times m_2}$, $R_u : \mathbb{R}^n \to \mathbb{R}^{m_1\times m_1}$, and $R_v : \mathbb{R}^n \to \mathbb{R}^{m_2\times m_2}$ are continuous on $\mathbb{R}^n$ and $R_u(x) = R_u^{\mathrm{T}}(x) > 0$, $x \in \mathbb{R}^n$, $R_v(x) = R_v^{\mathrm{T}}(x) < 0$, and

$$\dot{x}(t) = f(x(t)) + G_u(x(t))u(t) + G_v(x(t))v(t), \qquad x(0) = x_0, \qquad t \geq 0, \quad (3.45)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G_u : \mathbb{R}^n \to \mathbb{R}^{n\times m_1}$, and $G_v : \mathbb{R}^n \to \mathbb{R}^{n\times m_2}$.

Assume that there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, such that

$$0 = \min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} \left(\frac{\partial V(x)}{\partial x} [f(x,u,v) + G_u(x)u(x) + G_v(x)v(x)] + L(x,u,v) \right),$$

$$x \in \mathbb{R}^n \quad (3.46)$$

First order conditions to maximize (3.46) are

$$\frac{\partial V(x)}{\partial x} G_v(x) + L_v(x) + 2v^{\mathrm{T}} R_v(x) = 0, \qquad (x, v) \in \mathbb{R}^n \times \mathcal{V}, \tag{3.47}$$

$$\frac{\partial V(x)}{\partial x} G_u(x) + L_u(x) + 2u^{\mathrm{T}}(t) R_u(x) = 0, \qquad (x, u) \in \mathbb{R}^n \times \mathcal{U}, \tag{3.48}$$

which imply that

$$u^*(x) = -\frac{1}{2} R_u^{-1}(x) \left[\frac{\partial V(x)}{\partial x} G_u(x) + L_u(x) \right]^{\mathrm{T}}, \tag{3.49}$$

$$v^*(x) = -\frac{1}{2} R_v^{-1}(x) \left[\frac{\partial V(x)}{\partial x} G_v(x) + L_v(x) \right]^{\mathrm{T}}. \tag{3.50}$$

Therefore, it follows from (3.29) with $\psi(t, x) = 0, (t, x) \in [0, \infty) \times \mathbb{R}^n$, that

$$V(x_0) = J[x_0, u^*(\cdot), v^*(\cdot)], \quad x_0 \in \mathbb{R}^n. \tag{3.51}$$

3.6 Pursuer Evader Problem

Consider the performance measure

$$J[x_0, u(\cdot), v(\cdot)] = \int_{t_0}^{t_f} 1 \ dt, \tag{3.52}$$

subject to the differential constraints

$$\dot{x}_1(t) = w_1 \sin \theta(t), \qquad x_1(t_0) = x_{10}, \qquad t \in [t_0, t_f], \tag{3.53}$$

$$\dot{x}_2(t) = w_1 \cos \theta(t), \qquad x_2(t_0) = x_{20}, \tag{3.54}$$

$$\dot{x}_3(t) = w_2 \sin v(t), \qquad x_3(t_0) = x_{30}, \tag{3.55}$$

$$\dot{x}_4(t) = w_2 \cos v(t), \qquad x_4(t_0) = x_{40}, \tag{3.56}$$

$$\dot{\theta}(t) = \frac{w_1}{R} u(t) \tag{3.57}$$

where $w_1 > 0$, $w_2 > 0$ denote the velocities of the pursuer and the evader, respectively, $\theta : [t_0, t_f] \to [0, 2\pi)$ denotes the direction of the pursuer's velocity, $v : [t_0, t_f] \to [0, 2\pi)$ denotes the direction of the evader's velocity, $R \in \mathbb{R}^+$ denotes the minimum turn radius, and $u : [t_0, t_f] \to [-1, 1]$ denotes the ratio between the minimum turn radius and the instantaneous turn radius.

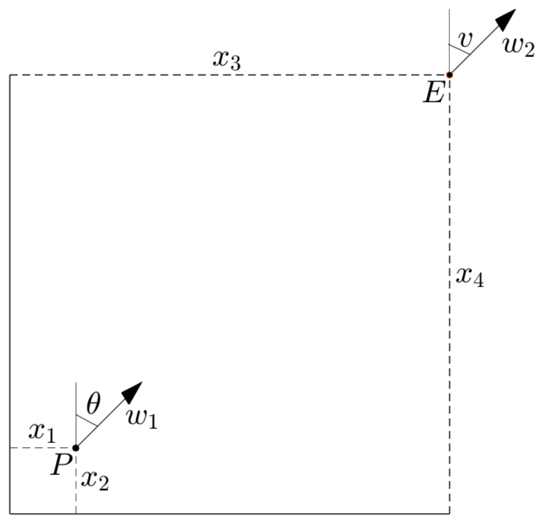

Figure 3.1: Pursuer-Evader Problem

The termination of the game occurs when the pursuer is within a ball of radius $l > 0$ from the evader, that is,

$$\sqrt{[x_3(t_f) - x_1(t_f)]^2 + [x_4(t_f) - x_2(t_f)]^2} \leq l. \tag{3.58}$$

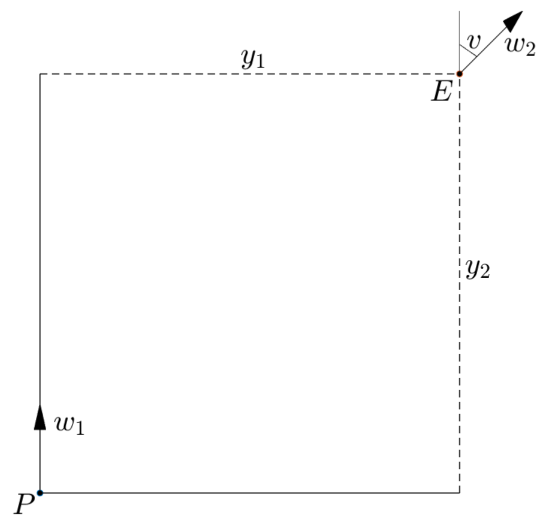

Figure 3.2: Pursuer-Evader Reduced Problem

To solve this problem, we introduce an alternative reference frame so that $y_1(\cdot)$ denotes the distance between the pursuer and the evader perpendicular to the pursuer's velocity vector and $y_2(\cdot)$ denotes the distance between the pursuer and the evader parallel to the pursuer's velocity vector. In this reference frame, the differential constraints (3.53)-(3.57) are equivalent to

$$\dot{y}_1(t) = -\frac{w_1}{R} y_2(t)u(t) + w_2 \sin v(t) \qquad y_1(t_0) = y_{10}, \qquad t \in [t_0, t_f] \tag{3.59}$$

$$\dot{y}_2(t) = -\frac{w_1}{R} y_1(t)u(t) - w_1 + w_2 \cos v(t) \qquad y_2(t_0) = y_{20}, \tag{3.60}$$

and the terminal condition (3.58) is equivalent to

$$\sqrt{y_1^2(t_f) + y_2^2(t_f)} \leq l. \tag{3.61}$$

In this problem, (3.10) specializes to

$$H(t, x, u, v, \lambda) = -\frac{w_1}{R}[x_2\lambda_1 - x_1\lambda_2]u - w_1\lambda_2 + w_2[\lambda_1 \sin v + \lambda_2 \cos v]$$
$$(t, x, u, v, \lambda) \in [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^n. \tag{3.62}$$

Setting

$$A(t) \triangleq y_2(t)\lambda_1(t) - y_1(t)\lambda_2(t), \qquad t \in [t_0, t_f] \tag{3.63}$$

$$\sigma(t) \triangleq \mathrm{sgn}(A(t)), \tag{3.64}$$

$$\rho(t) \triangleq \sqrt{\lambda_1^2(t) + \lambda_2^2(t)}, \tag{3.65}$$

yields

$$\cos v(t) = \frac{\lambda_2(t)}{\rho(t)}, \tag{3.66}$$

$$\sin v(t) = \frac{\lambda_1(t)}{\rho(t)}. \tag{3.67}$$

Then, it follows from (3.23), (3.24), (3.59), (3.60), (3.62)–(3.67), and (2.15) that

$$0 = -\sigma(t)\frac{w_1}{R}A(t) - w_1\lambda_2(t) + w_2\rho(t), \qquad t \in [t_0, t_f] \tag{3.68}$$

$$\dot{y}_1(t) = \sigma(t)\frac{w_1}{R}y_2(t) - w_2\frac{\lambda_1(t)}{\rho(t)}, \qquad y_1(t_0) = y_{10}, \tag{3.69}$$

$$\dot{y}_2(t) = -\sigma(t)\frac{w_1}{R}y_2(t) + w_1 - w_2\frac{\lambda_2(t)}{\rho(t)}, \qquad y_2(t_0) = y_{20}, \tag{3.70}$$

$$\dot{\lambda}_1(t) = \sigma(t)\frac{w_1}{R}\lambda_2(t), \qquad \lambda_1(t_f) = \frac{y_{1f}}{l}, \tag{3.71}$$

$$\dot{\lambda}_2(t) = -\sigma(t)\frac{w_1}{R}\lambda_1(t), \qquad \lambda_2(t_f) = \frac{y_{2f}}{l}. \tag{3.72}$$

Solving this system of equations yields

$$u^*(t) = -\mathrm{sgn}(\theta(t) - v^*(t)), \qquad t \in [t_0, t_f], \tag{3.73}$$

$$\theta(t) = \tan^{-1}\frac{x_1(t)}{x_2(t)}, \tag{3.74}$$

$$\dot{v}^*(t) = \frac{w_1}{R}\mathrm{sgn}(\theta(t) - v^*(t)). \tag{3.75}$$

3.7 Target-Attacker-Defender

A scenario that is similar in nature to the Pursuer-Evader problem is known as the Target-Attacker-Defender problem. In this problem, there are three vehicles. The target is attempting to evade the attacker, while the defender is attempting to prevent the capture of the target even at the cost of itself. Consider the performance measure

$$J[x_0, u(\cdot), v(\cdot)] = \int_{t_0}^{t_f} \dot{R}(t)\ dt, \tag{3.76}$$

where $R : [t_0, t_f] \to \mathbb{R}_+$ denotes the distance between the attacker and the target, subject to the differential constraints

$$\dot{x}_1(t) = V_T \cos(\xi(t) + v_1(t)), \qquad x_1(t_0) = x_{10}, \qquad t \in [t_0, t_f], \tag{3.77}$$

$$\dot{x}_2(t) = V_T \sin(\xi(t) + v_1(t)), \qquad x_2(t_0) = x_{20}, \tag{3.78}$$

$$\dot{x}_3(t) = V_A \cos \hat{\mathcal{X}}(t), \qquad x_3(t_0) = x_{30}, \tag{3.79}$$

$$\dot{x}_4(t) = V_A \sin \hat{\mathcal{X}}(t), \qquad x_4(t_0) = x_{40}, \tag{3.80}$$

$$\dot{x}_5(t) = V_D \cos \hat{\psi}(t), \qquad x_5(t_0) = x_{50}, \tag{3.81}$$

$$\dot{x}_6(t) = V_D \sin \hat{\psi}(t), \qquad x_6(t_0) = x_{60}, \tag{3.82}$$

where

$$\hat{\mathcal{X}}(t) = \xi(t) + \theta(t) - u(t), \tag{3.83}$$

$$\hat{\psi}(t) = v_1(t) + \theta(t) + \xi(t) - \pi. \tag{3.84}$$

$\xi : [t_0, t_f] \to [0, 2\pi)$ denotes the angle between the horizontal axis and the line connecting the attacker and the defender, $\theta : [t_0, t_f] \to [0, 2\pi)$ denotes the angle between the line connecting the attacker and the target and the line connecting the defender and the attacker, $u : [t_0, t_f] \to [0, 2\pi)$ denotes the angle between the attacker's velocity vector and the line connecting the attacker to the defender, $v_2 : [t_0, t_f] \to [0, 2\pi)$ denotes the angle between the line connecting the attacker to the defender and the defender's velocity vector, $v_1 : [t_0, t_f] \to [0, 2\pi)$ denotes the angle between the line connecting the target to the attacker and the target's velocity vector, and $V_T \in \mathbb{R}^+$, $V_A \in \mathbb{R}^+$, and $V_D \in \mathbb{R}^+$ denote the velocities of the target, the attacker, and the defender respectively.

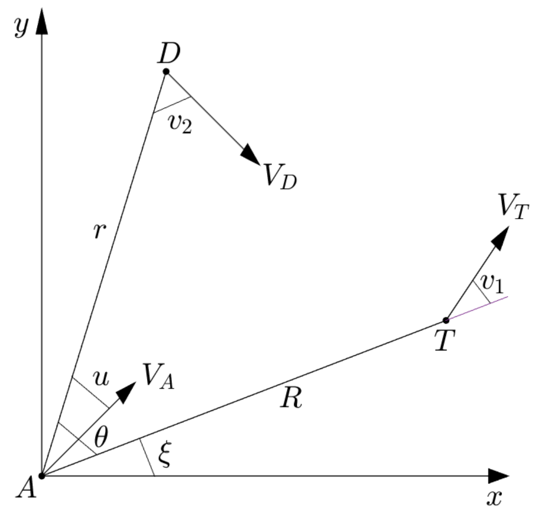

Figure 3.3: Target-Attacker-Defender Scenario

If we assume that $V_A = V_D$, then (3.77)-(3.82) can be reduced to

$$\dot{R}(t) = \alpha \cos v_1(t) - \cos(\theta(t) - u(t)), \qquad R(t_0) = R_0, \qquad t \in [t_0, t_f], \tag{3.85}$$

$$\dot{r}(t) = -\cos u(t) - \cos v_1(t), \qquad r(t_0) = r_0, \tag{3.86}$$

$$\begin{aligned} \dot{\theta}(t) = &- \frac{\alpha}{R(t)} \sin v_1(t) + \frac{1}{R} \sin(\theta(t) - u(t)) - \frac{1}{r(t)} \sin v_2(t) \\ &+ \frac{1}{r(t)} \sin u(t), \qquad \theta(t_0) = \theta_0, \end{aligned} \tag{3.87}$$

where $\alpha = \frac{V_T}{V_A}$, $r : [t_0, t_f] \to \mathbb{R}^+$. The termination of the game happens when $r(t_f) = r_c$, for some $r_c > 0$. The goal of the Target-Attacker-Defender is to find

$$J[x_0, u^*(\cdot), v_1^*(\cdot), v_2^*(\cdot)] = \min_{u(\cdot) \in \mathcal{U}} \max_{[v_1(\cdot)^{\mathrm{T}}, v_2(\cdot)^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{V}} J[x_0, u(\cdot), v_1(\cdot), v_2(\cdot)]. \tag{3.88}$$

In this case, (3.10) specializes to

$$
\begin{aligned}
H(t,x(\cdot),u(\cdot),v_1(\cdot),v_2(\cdot),\lambda(\cdot)) =& \alpha\cos v_1(t) - \cos(\theta(t)-u(t)) \\
&+ \lambda_R(t)[\alpha\cos v_1(t) - \cos(\theta(t)-u(t))] \\
&- \lambda_r(t)[\cos u(t) + \cos v_2(t)] \\
&+ \lambda_\theta(t)\Big[-\frac{\alpha}{R(t)}\sin v_1(t) + \frac{1}{R}\sin(\theta(t) \\
&- u(t)) - \frac{1}{r(t)}\sin v_2(t) + \frac{1}{r(t)}\sin u(t)\Big], \qquad (3.89)
\end{aligned}
$$

and it follows from (3.23) and (3.24) that

$$
u^*(t) = \sin^{-1}\frac{A(t)}{B(t)}, \qquad t\in[t_0,t_f], \tag{3.90}
$$

$$
v_1^*(t) = \sin^{-1}\frac{\lambda_\theta(t)}{r(t)\sqrt{\lambda_r^2(t)+\lambda_\theta^2(t)/r^2(t)}}, \tag{3.91}
$$

$$
v_2^*(t) = \sin^{-1}\frac{\lambda_\theta(t)}{R(t)\sqrt{(1-\lambda_r(t))^2+\lambda_\theta^2(t)/R^2(t)}}, \tag{3.92}
$$

where

$$
A(t) \triangleq [1-\lambda_R(t)]\sin\theta(t) - \frac{\lambda_\theta(t)}{R(t)}\cos\theta(t) + \frac{\lambda_\theta(t)}{r(t)}, \tag{3.93}
$$

$$
\begin{aligned}
B(t) \triangleq \Bigg(&\left[(1-\lambda_R(t))\sin\theta(t) - \frac{\lambda_\theta(t)}{R(t)}\cos\theta(t) + \frac{\lambda_\theta(t)}{r(t)}\right]^2 \\
&+ \left[(1-\lambda_R(t))\cos\theta(t) - \frac{\lambda_\theta(t)}{R(t)}\sin\theta(t) + \lambda_r(t)\right]^2\Bigg)^{1/2}, \qquad (3.94)
\end{aligned}
$$

and it follows from (2.15) that

$$
\dot\lambda_R(t) = \frac{\lambda_\theta(t)}{R^2(t)}[\sin(\theta(t)-u(t)) - \alpha\sin(v_1(t))], \qquad \lambda_R(t_f) = 0, \tag{3.95}
$$

$$
\dot\lambda_r(t) = \frac{\lambda_\theta(t)}{r^2(t)}[\sin(u(t)) - \sin(v_2(t))], \qquad \lambda_r(t_f) = \frac{1}{\alpha^2+2[\alpha+\cos\theta(t_f)]}, \tag{3.96}
$$

$$\dot{\lambda}_\theta(t) = (1 - \lambda_r(t))\sin(\theta(t) - u(t)) - \frac{\lambda_\theta(t)}{R(t)}\cos(\theta(t) - u(t)), \qquad \lambda_\theta(t_f) = 0. \tag{3.97}$$

For more details, see [11].

Chapter 4. A Toolbox to Solve Optimal Control and Differential Game Problems

4.1 Introduction

In this chapter, we will describe some numerical methods known as *level set methods* to solve Hamilton-Jacobi equations. As shown in Chapters 2 and 3, these equations play a key role in the solution of optimal control and differential game problems. Level set methods have been implemented by the MATLAB toolbox [19]. In this work, we modified this toolbox to solve the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs equation, that is, to find numerically both the value function and the state-feedback control laws needed to solve the underlying optimal control and differential game problems. To our best knowledge, this result is unprecedented.

4.2 Level Set Methods

Level set methods are a collection of numerical differentiation and integration schemes to solve partial differential equations that satisfy the *level set equation*

$$\frac{\partial \phi(t,x)}{\partial t} + H\left(t, x, \frac{\partial \phi(t,x)}{\partial x}\right) = 0. \tag{4.1}$$

where $\phi : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ denotes the surface

$$\phi(t,x) = 0, \qquad [t,x] \in [0,\infty) \times \mathbb{R}^n, \tag{4.2}$$

$$H\left(t, x, \frac{\partial \phi(t,x)}{\partial x}\right) = U^{\mathrm{T}}(t,x)\frac{\partial \phi(t,x)}{\partial x}, \qquad [t,x] \in [0,\infty) \times \mathbb{R}^n, \tag{4.3}$$

and $U : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the velocity of the surface $\phi(\cdot,\cdot)$.

In order to find solutions to (4.1), it is necessary to capture spatial derivatives along the surface $\phi(\cdot,\cdot)$. Due to the presence of discontinuities that may arise when propagating the surface $\phi(\cdot,\cdot)$, it is necessary to approximate these derivatives using specialized schemes. The authors in [12] present several schemes to overcome the presence of discontinuities, which are summarized in the following.

4.2.1 Differencing Schemes

Three common schemes used to approximate spatial derivatives are known as *central differencing*, *forward differencing*, and *backward differencing*. For simplicity of explanation, in the following we consider the case for $n = 1$, for arbitrary n, see [12]. The central differencing approximation is given by

$$\frac{\partial\phi(t_j, x_i)}{\partial x} \approx \frac{\phi(t_j, x_{i+1}) - \phi(t_j, x_{i-1})}{2\Delta x}, \qquad j = 0, \ldots, T, \qquad i = 1, \ldots, N, \quad (4.4)$$

the forward differencing approximation is given by

$$\frac{\partial\phi(t_j, x_i)}{\partial x}^{+} \approx \frac{\phi(t_j, x_{i+1}) - \phi(t_j, x_i)}{\Delta x}, \quad (4.5)$$

and the backward differencing approximation is given by

$$\frac{\partial\phi(t_j, x_i)}{\partial x}^{-} \approx \frac{\phi(t_j, x_i) - \phi(t_j, x_{i-1})}{\Delta x}, \quad (4.6)$$

where t_j denotes a sample of the uniformly discretized time interval $[0, T]$, x_i denotes a sample of the uniformly discretized domain $D \subseteq \mathbb{R}^n$, and $\Delta x \triangleq x_{i+1} - x_i$; in this thesis, we assume that Δx is constant for $i = 1, \ldots, N$ and that the values of $\phi(t_j, x_i)$ are available for $i < 1$ and $i > N$ if need.

When using these schemes to approximate partial differential equations, a method known as *upwind differencing* is commonly used to determine which approximation to use. Upwind differencing gives us a systematic approach to determine what approximation we should use by evaluating the current direction of the surface's velocity. According to upwind differencing, if $U(t_j, x_i) < 0,\ j = 0, \ldots, T,\ i = 1, \ldots, N$, then the forward differencing approximation should be used, while if $U(t_j, x_i) > 0$, then the backward differencing approximation should be used.

Although these methods are efficient, forward and backward differencing yield only first-order accuracy, while centeral differencing yields second-order accuracy for smooth regions of the surface $\phi(\cdot, \cdot)$. In most cases, it is desirable to use higher-order accuracy schemes even though they are more computationally expensive.

4.2.2 Hamilton-Jacobi Essentially Non-Oscillatory Scheme

An alternative, more accurate, scheme to approximate the spatial derivatives of (4.1) is the Hamilton-Jacobi essentially non-oscillatory scheme. Specifically, for $n = 1$ the *zero-th divided difference* of $\phi(t_j, x_i)$ is defined as

$$D^0\phi(t_j, x_i) \triangleq \phi(t_j, x_i), \qquad j = 0, \ldots, T, \qquad i = 1, \ldots, N, \tag{4.7}$$

the *first divided difference* of $\phi(t_j, x_i)$ is defined as

$$D^1\phi(t_j, x_{i+1/2}) \triangleq \frac{D^0\phi(t_j, x_{i+1}) - D^0\phi(t_j, x_i)}{\Delta x}, \tag{4.8}$$

the *second divided difference* of $\phi(t_j, x_i)$ is defined as

$$D^2\phi(t_j, x_i) \triangleq \frac{D^1\phi(t_j, x_{i+1/2}) - D^1\phi(t_j, x_{i-1/2})}{2\Delta x}, \tag{4.9}$$

and the *third divided difference* of $\phi(t_j, x_i)$ is defined as

$$D^3\phi(t_j, x_{i+1/2}) \triangleq \frac{D^2\phi(t_j, x_{i+1}) - D^2\phi(t_j, x_i)}{3\Delta x}. \tag{4.10}$$

We then approximate $\phi(t,x)$ with the polynomial interpolation such that

$$\phi(t,x) \approx Q_0(t,x,x_i) + Q_1(t,x,x_i) + Q_2(t,x,x_i) + Q_3(t,x,x_i), \tag{4.11}$$

where, $Q_0 : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$Q_0(t,x,x_i) = D^0\phi(t_j,x_i)(x-x_i)^0, \qquad j=1,\dots,T, \qquad i=1,\dots,N \tag{4.12}$$

$$\frac{\partial Q_0(t_j,x,x_i)}{\partial x} = 0, \tag{4.13}$$

and $Q_k : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ k=1,2,3$, are defined in the following. It follows from (4.11) that

$$\frac{\partial \phi(t_j,x_i)}{\partial x} \approx \frac{\partial Q_1(t_j,x,x_i)}{\partial x} + \frac{\partial Q_2(t_j,x,x_i)}{\partial x} + \frac{\partial Q_3(t_j,x,x_i)}{\partial x}. \tag{4.14}$$

Next, we apply upwind differencing to determine whether $\frac{\partial \phi(t_j,x_i)}{\partial x},\ j=1,\dots,T,\ i=1,\dots,N$, must be approximated using forward or backward differencing. In case of backward differencing, it holds that

$$Q_1(t_j,x,x_i) = D^1\phi(t_j,x_{i-1/2})(x-x_i),$$
$$j=1,\dots,T, \qquad i=1,\dots,N. \tag{4.15}$$

$$\frac{\partial Q_1(t_j,x,x_i)}{\partial x} = D^1\phi(t_j,x_{i-1/2}) \tag{4.16}$$

The terms $Q_2(t_j,x,x_i)$ and $\frac{\partial Q_2(t_j,x,x_i)}{\partial x}$ are computed as follows. If $|D^2\phi(t_j,x_{i-1})| \leq |D^2\phi(t_j,x_i)|,\ j=1,\dots,T,\ i=1,\dots,N$, then

$$Q_2(t_j,x,x_i) = D^2\phi(t_j,x_{i-1})(x-x_{i-1})(x-x_i) \tag{4.17}$$

$$\frac{\partial Q_2(t_j,x,x_i)}{\partial x} = D^2\phi(t_j,x_{i-1})\Delta x, \tag{4.18}$$

otherwise

$$Q_2(t_j, x, x_i) = D^2\phi(t_j, x_i)(x - x_i)(x - x_{i+1}) \tag{4.19}$$

$$\frac{\partial Q_2(t_j, x, x_i)}{\partial x} = D^2\phi(t_j, x_i)\Delta x. \tag{4.20}$$

Similarly, it holds for $Q_3(t_j, x, x_i)$ and $\frac{\partial Q_3(t_j,x,x_i)}{\partial x}$ that if $|D^2\phi(t_j, x_{i-1})| \leq |D^2\phi(t_j, x_i)|,\ j = 1, \ldots, T,\ i = 1, \ldots, N$, then $Q_3(\cdot, \cdot, \cdot)$ and $\frac{\partial Q_3(\cdot,\cdot,\cdot)}{\partial x}$ are given by (4.21) and (4.22), respectively. Otherwise, if $|D^2\phi(t_j, x_{i-1})| \geq |D^2\phi(t_j, x_i)|,\ j = 1, \ldots, T,\ i = 1, \ldots, N$, then $Q_3(\cdot, \cdot, \cdot)$ and $\frac{\partial Q_3(\cdot,\cdot,\cdot)}{\partial x}$ are given by (4.23) and (4.24), respectively.

$$Q_3(t_j,x,x_i)=\begin{cases} D^3\phi(t_j,x_{i-3/2})(x-x_{i-2})(x-x_{i-1})(x-x_i), & |D^3\phi(t_j,x_{i-3/2})|\le|D^3\phi(t_j,x_{i-1/2})|,\\ D^3\phi(t_j,x_{i-3/2})(x-x_{i-1})(x-x_i)(x-x_{i+1}), & |D^3\phi(t_j,x_{i-3/2})|\ge|D^3\phi(t_j,x_{i-1/2})|, \end{cases} \tag{4.21}$$

$$\frac{\partial Q_3(t_j,x,x_i)}{\partial x}=\begin{cases} 2D^3\phi(t_j,x_{i-3/2})\Delta x^2, & |D^3\phi(t_j,x_{i-3/2})|\le|D^3\phi(t_j,x_{i-1/2})|,\\ 2D^3\phi(t_j,x_{i-1/2})\Delta x^2, & |D^3\phi(t_j,x_{i-3/2})|\ge|D^3\phi(t_j,x_{i-1/2})|, \end{cases} \tag{4.22}$$

$$Q_3(t_j,x,x_i)=\begin{cases} D^3\phi(t_j,x_{i-1/2})(x-x_{i-1})(x-x_i)(x-x_{i+1}), & |D^3\phi(t_j,x_{i-1/2})|\le|D^3\phi(t_j,x_{i+1/2})|,\\ D^3\phi(t_j,x_{i+1/2})(x-x_i)(x-x_{i+1})(x-x_{i+2}), & |D^3\phi(t_j,x_{i-1/2})|\ge|D^3\phi(t_j,x_{i+1/2})|, \end{cases} \tag{4.23}$$

$$\frac{\partial Q_3(t_j,x,x_i)}{\partial x}=\begin{cases} 2D^3\phi(t_j,x_{i-1/2})\Delta x^2, & |D^3\phi(t_j,x_{i-1/2})|\le|D^3\phi(t_j,x_{i+1/2})|,\\ 2D^3\phi(t_j,x_{i+1/2})\Delta x^2, & |D^3\phi(t_j,x_{i-1/2})|\ge|D^3\phi(t_j,x_{i+1/2})|. \end{cases} \tag{4.24}$$

In case of forward differencing, it holds that

$$Q_1(t_j, x, x_i) = D^1\phi(t_j, x_{i+1/2})(x - x_i),$$

$$j = 1, \ldots, T, \qquad i = 1, \ldots, N. \tag{4.25}$$

$$\frac{\partial Q_1(t_j, x, x_i)}{\partial x} = D^1\phi(t_j, x_{i+1/2}) \tag{4.26}$$

The terms $Q_2(t_j, x, x_i)$ and $\frac{\partial Q_2(t_j,x,x_i)}{\partial x}$ are computed as follows. If $|D^2\phi(t_j, x_i)| \leq |D^2\phi(t_j, x_{i+1})|,\ j = 1, \ldots, T,\ i = 1, \ldots, N$, then

$$Q_2(t_j, x, x_i) = D^2\phi(t_j, x_i)(x - x_i)(x - x_{i+1}) \tag{4.27}$$

$$\frac{\partial Q_2(t_j, x, x_i)}{\partial x} = D^2\phi(t_j, x_i)\Delta x, \tag{4.28}$$

otherwise

$$Q_2(t_j, x, x_i) = D^2\phi(t_j, x_{i+1})(x - x_{i+1})(x - x_{i+2}) \tag{4.29}$$

$$\frac{\partial Q_2(t_j, x, x_i)}{\partial x} = D^2\phi(t_j, x_{i+1})\Delta x. \tag{4.30}$$

Similarly, it holds for $Q_3(t_j, x, x_i)$ and $\frac{\partial Q_3(t_j,x,x_i)}{\partial x}$ that if
$|D^2\phi(t_j, x_i)| \leq |D^2\phi(t_j, x_{i+1})|,\ j = 1, \ldots, T,\ i = 1, \ldots, N$, then $Q_3(\cdot,\cdot,\cdot)$ and $\frac{\partial Q_3(\cdot,\cdot,\cdot)}{\partial x}$ are given by (4.31) and (4.32), respectively. Otherwise, if $|D^2\phi(t_j, x_i)| \geq |D^2\phi(t_j, x_{i+1})|,\ j = 1, \ldots, T,\ i = 1, \ldots, N$, then $Q_3(\cdot,\cdot,\cdot)$ and $\frac{\partial Q_3(\cdot,\cdot,\cdot)}{\partial x}$ are given by (4.33) and (4.34), respectively.

$$Q_3(t_j, x, x_i) = \begin{cases} D^3\phi(t_j, x_{i-1/2})(x - x_{i-1})(x - x_i)(x - x_{i+1}), & |D^3\phi(t_j, x_{i-1/2})| \leq |D^3\phi(t_j, x_{i+1/2})|, \\ D^3\phi(t_j, x_{i-1/2})(x - x_{i-1})(x - x_{i+1})(x - x_{i+2}), & |D^3\phi(t_j, x_{i-1/2})| \geq |D^3\phi(t_j, x_{i+1/2})|, \end{cases} \tag{4.31}$$

$$\frac{\partial Q_3(t_j, x_i)}{\partial x} = \begin{cases} 2D^3\phi(t_j, x_{i-1/2})\Delta x^2, & |D^3\phi(t_j, x_{i-1/2})| \leq |D^3\phi(t_j, x_{i+1/2})|, \\ 2D^3\phi(t_j, x_{i+1/2})\Delta x^2, & |D^3\phi(t_j, x_{i-1/2})| \geq |D^3\phi(t_j, x_{i+1/2})|, \end{cases} \tag{4.32}$$

$$Q_3(t_j, x, x_i) = \begin{cases} D^3\phi(t_j, x_{i+1/2})(x - x_i)(x - x_{i+1})(x - x_{i+2}), & |D^3\phi(t_j, x_{i+1/2})| \leq |D^3\phi(t_j, x_{i+3/2})|, \\ D^3\phi(t_j, x_{i+3/2})(x - x_{i+1})(x - x_{i+2})(x - x_{i+3}), & |D^3\phi(t_j, x_{i-1/2})| \geq |D^3\phi(t_j, x_{i+3/2})|, \end{cases} \tag{4.33}$$

$$\frac{\partial Q_3(t_j, x_i)}{\partial x} = \begin{cases} 2D^3\phi(t_j, x_{i+1/2})\Delta x^2, & |D^3\phi(t_j, x_{i+1/2})| \leq |D^3\phi(t_j, x_{i+3/2})|, \\ 2D^3\phi(t_j, x_{i+3/2})\Delta x^2, & |D^3\phi(t_j, x_{i+1/2})| \geq |D^3\phi(t_j, x_{i+3/2})|. \end{cases} \tag{4.34}$$

Once we have determined $\frac{\partial Q_1(t_j,x,x_i)}{\partial x}$, $\frac{\partial Q_2(t_j,x,x_i)}{\partial x}$, and $\frac{\partial Q_3(t_j,x,x_i)}{\partial x}$ (4.14) yields third order accuracy.

4.2.3 Hamilton-Jacobi Weighted Essentially Non-Oscillatory Scheme

The *Hamilton-Jacobi weighted essentially non-oscillatory* scheme is a method that provides the optimal weighting for the convex combination of the possible approximations given by the Hamilton-Jacobi essentially non-oscillatory scheme for a more accurate approximation. Specifically, assuming that $n = 1$, let us first consider the backward differencing problem. In this case, define

$$v_1(t_j, x_i) \triangleq \frac{\partial\phi(t_j, x_{i-2})}{\partial x}^{-}, \qquad j = 1, \ldots, T, \qquad i = 1, \ldots, N, \tag{4.35}$$

$$v_2(t_j, x_i) \triangleq \frac{\partial\phi(t_j, x_{i-1})}{\partial x}^{-}, \tag{4.36}$$

$$v_3(t_j, x_i) \triangleq \frac{\partial\phi(t_j, x_i)}{\partial x_i}^{-}, \tag{4.37}$$

$$v_4(t_j, x_i) \triangleq \frac{\partial\phi(t_j, x_{i+1})}{\partial x}^{-}, \tag{4.38}$$

$$v_5(t_j, x_i) \triangleq \frac{\partial\phi(t_j, x_{i+2})}{\partial x}^{-}. \tag{4.39}$$

According to the Hamilton-Jacobi essentially non-oscillatory scheme, there are three approximations of $\frac{\partial\phi(t_n,x_i)}{\partial x}^{-}$, namely

$$\frac{\partial\phi(t_j, x_i)}{\partial x}^{-}_{1} = \frac{v_1(t_j, x_i)}{3} - \frac{7v_2(t_j, x_i)}{6} + \frac{11v_3(t_j, x_i)}{6}, \tag{4.40}$$

$$\frac{\partial\phi(t_j, x_i)}{\partial x}^{-}_{2} = -\frac{v_2(t_j, x_i)}{6} - \frac{5v_3(t_j, x_i)}{6} + \frac{v_4(t_j, x_i)}{6}, \tag{4.41}$$

and

$$\frac{\partial\phi(t_j, x_i)}{\partial x}^{-}_{3} = \frac{v_3(t_j, x_i)}{3} + \frac{5v_4(t_j, x_i)}{6} - \frac{v_5(t_j, x_i)}{6}. \tag{4.42}$$

In case of forward differencing, define

$$v_1(t_j,x_i) \triangleq \frac{\partial\phi(t_n,x_{i+2})}{\partial x}^+, \qquad j=1,\ldots,T, \qquad i=1,\ldots,N, \tag{4.43}$$

$$v_2(t_j,x_i) \triangleq \frac{\partial\phi(t_n,x_{i+1})}{\partial x}^+, \tag{4.44}$$

$$v_3(t_j,x_i) \triangleq \frac{\partial\phi(t_n,x_i)}{\partial x}^+, \tag{4.45}$$

$$v_4(t_j,x_i) \triangleq \frac{\partial\phi(t_n,x_{i-1})}{\partial x}^+, \tag{4.46}$$

$$v_5(t_j,x_i) \triangleq \frac{\partial\phi(t_n,x_{i-2})}{\partial x}^+. \tag{4.47}$$

According to the Hamilton-Jacobi essentially non-oscillatory scheme, there are three approximations of $\frac{\partial\phi(t_n,x_i)}{\partial x}^+$, namely

$$\frac{\partial\phi(t_j,x_i)}{\partial x}^+_1 = \frac{v_1(t_j,x_i)}{3} - \frac{7v_2(t_j,x_i)}{6} + \frac{11v_3(t_j,x_i)}{6}, \tag{4.48}$$

$$\frac{\partial\phi(t_j,x_i)}{\partial x}^+_2 = -\frac{v_2(t_j,x_i)}{6} - \frac{5v_3(t_j,x_i)}{6} + \frac{v_4(t_j,x_i)}{6}, \tag{4.49}$$

and

$$\frac{\partial\phi(t_j,x_i)}{\partial x}^+_3 = \frac{v_3(t_j,x_i)}{3} + \frac{5v_4(t_j,x_i)}{6} - \frac{v_5(t_j,x_i)}{6}. \tag{4.50}$$

In backward differencing, $\frac{\partial\phi(t_n,x_i)}{\partial x}^-$, can also be expressed as a convex combination of (4.40)-(4.42), that is,

$$\frac{\partial\phi(t_j,x_i)}{\partial x}^- = \omega_1\frac{\partial\phi(t_j,x_i)}{\partial x}^-_1 + \omega_2\frac{\partial\phi(t_j,x_i)}{\partial x}^-_2 + \omega_3\frac{\partial\phi(t_j,x_i)}{\partial x}^-_3, \tag{4.51}$$

where $0 \le \omega_k \le 1,\ k=1,2,3$, and $\omega_1+\omega_2+\omega_3=1$. Simlarly, in forward differencing, $\frac{\partial\phi(t_j,x_i)}{\partial x}^+$ can be approximated through a convex combination of (4.48)-(4.50), that is,

$$\left.\frac{\partial\phi(t_n,x_i)}{\partial x}\right.^{+} = \omega_1 \left.\frac{\partial\phi(t_n,x_i)}{\partial x}\right.^{+}_{1} + \omega_2 \left.\frac{\partial\phi(t_n,x_i)}{\partial x}\right.^{+}_{2} + \omega_3 \left.\frac{\partial\phi(t_n,x_i)}{\partial x}\right.^{+}_{3}. \tag{4.52}$$

The weights $\omega_k,\ k = 1, 2, 3$ can be chosen according to the scheme presented in [15], which is shown in the following. First, we define

$$\begin{aligned} S_1(t_j,x_i) \triangleq\ & \frac{13}{12}[v_1(t_j,x_i) - 2v_2(t_j,x_i) + v_3(t_j,x_i)]^2 \\ & + \frac{1}{4}[v_1(t_j,x_i) - 4v_2(t_j,x_i) + 3v_3(t_j,x_i)]^2, \\ & \qquad j = 1,\ldots,T, \qquad i = 1,\ldots,N, \end{aligned} \tag{4.53}$$

$$\begin{aligned} S_2(t_j,x_i) \triangleq\ & \frac{13}{12}[v_1(t_j,x_i) - 2v_2(t_j,x_i) + v_4(t_j,x_i)]^2 \\ & + \frac{1}{4}[v_2(t_j,x_i) - v_4(t_j,x_i)]^2, \end{aligned} \tag{4.54}$$

$$\begin{aligned} S_3(t_j,x_i) \triangleq\ & \frac{13}{12}[v_3(t_j,x_i) - 2v_4(t_j,x_i) + v_5(t_j,x_i)]^2 \\ & + \frac{1}{4}[3v_3(t_j,x_i) - 4v_4(t_j,x_i) + v_5(t_j,x_i)]^2. \end{aligned} \tag{4.55}$$

Next, define

$$\alpha_1(t_j,x_i) \triangleq \frac{0.1}{[S_1(t_j,x_i) + \epsilon(t_j,x_i)]^2}, \qquad j = 1,\ldots,T, \qquad i = 1,\ldots,N, \tag{4.56}$$

$$\alpha_2(t_j,x_i) \triangleq \frac{0.6}{[S_2(t_j,x_i) + \epsilon(t_j,x_i)]^2}, \tag{4.57}$$

$$\alpha_3(t_j,x_i) \triangleq \frac{0.3}{[S_3(t_j,x_i) + \epsilon(t_j,x_i)]^2}, \tag{4.58}$$

where

$$\epsilon(t_j,x_i) \triangleq 10^{-6}\max[v_1(t_j,x_i)^2, v_2(t_j,x_i)^2, v_3(t_j,x_i)^2, v_4(t_j,x_i)^2, v_5(t_j,x_i)^2]. \tag{4.59}$$

Finally, the weighting coefficients of (4.51) and (4.52) can be chosen as

$$\omega_1(t_j, x_i) = \frac{\alpha_1(t_j, x_i)}{\alpha_1(t_j, x_i) + \alpha_2(t_j, x_i) + \alpha_3(t_j, x_i)}, \quad j = 1, \ldots, T, \quad i = 1, \ldots, N, \tag{4.60}$$

$$\omega_2(t_j, x_i) = \frac{\alpha_2(t_j, x_i)}{\alpha_1(t_j, x_i) + \alpha_2(t_j, x_i) + \alpha_3(t_j, x_i)}, \tag{4.61}$$

$$\omega_3(t_j, x_i) = \frac{\alpha_3(t_j, x_i)}{\alpha_1(t_j, x_i) + \alpha_2(t_j, x_i) + \alpha_3(t_j, x_i)}. \tag{4.62}$$

Whenever $\phi(\cdot, \cdot)$ is smooth the optimal weights are given by $\omega_1 = 0.1$, $\omega_2 = 0.6$, and $\omega_3 = 0.3$ and the Hamilton-Jacobi weighted essentially non-oscillatory scheme yields fifth order accurate approximations. If $\phi(\cdot, \cdot)$ is not smooth, then there is no guarantee on higher accuracy of the Hamilton-Jacobi weighted essentially non-oscillatory scheme and may yield the same accuracy as the Hamilton-Jacobi essentially non-oscillatory scheme.

4.3 Numerical Estimation of the Hamiltonian Function

To solve (4.1) using level set methods, it is necessary to use specialized schemes to evaluate the Hamiltonian function so that the temporal integration of $\phi(\cdot, \cdot)$ is stable. To present these schemes, let us consider the case where $n = 2$, for arbitrary n see [12]. Let $x_{i,k} = [x_{1,i},\ x_{2,k}]^{\mathrm{T}} \in \mathbb{R}^2$, then define the numerical Hamiltonian function

$$\begin{aligned}
&\hat{H}\left(t_j, x_{i,k}, \frac{\partial \phi(t_n, x_{i,k})}{\partial x_1}^-, \frac{\partial \phi(t_n, x_{i,k})}{\partial x_1}^+, \frac{\partial \phi(t_n, x_{i,k})}{\partial x_2}^-, \frac{\partial \phi(t_n, x_{i,k})}{\partial x_2}^+\right) \\
&\triangleq H\left(t_j, x_{i,k}, \frac{1}{2}\left(\frac{\partial \phi(t_n, x_{i,k})}{\partial x_1}^- + \frac{\partial \phi(t_n, x_{i,k})}{\partial x_1}^+\right), \frac{1}{2}\left(\frac{\partial \phi(t_n, x_{i,k})}{\partial x_2}^- + \frac{\partial \phi(t_n, x_{i,k})}{\partial x_2}^+\right)\right) \\
&\quad - \alpha^{x_1}\frac{1}{2}\left(\frac{\partial \phi(t_n, x_{i,k})}{\partial x_1}^- - \frac{\partial \phi(t_n, x_{i,k})}{\partial x_1}^+\right) - \alpha^{x_2}\frac{1}{2}\left(\frac{\partial \phi(t_n, x_{i,k})}{\partial x_2}^- - \frac{\partial \phi(t_n, x_{i,k})}{\partial x_2}^+\right), \\
&\qquad j = 0, \ldots, T, \quad i = 1, \ldots, N, \quad k = 1, \ldots, N.
\end{aligned} \tag{4.63}$$

Let

$$H_{\frac{\partial\phi(t,x)}{\partial x}} = \frac{\partial}{\partial z}H(t,x,z)\bigg|_{z=\frac{\partial\phi(t,x)}{\partial x}}, \tag{4.64}$$

and

$$\alpha^{x_1} \triangleq \max_{\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\in I^{x_1},\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\in I^{x_2}} \left| H_{\frac{\partial\phi(t,x)}{\partial x_1}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_n,x_{i,k})}{\partial x_2}\right)\right|, \tag{4.65}$$

$$\alpha^{x_2} \triangleq \max_{\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\in I^{x_1},\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\in I^{x_2}} \left| H_{\frac{\partial\phi(t,x)}{\partial x_2}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_n,x_{i,k})}{\partial x_2}\right)\right|. \tag{4.66}$$

The first scheme given is the *Lax-Friedrichs* scheme as presented in [16]. According to this scheme, define the intervals

$$I^{x_1} \triangleq \left[\min_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\right), \max_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\right)\right],$$

$$j=0,\ldots,T, \qquad i=1,\ldots,N, \qquad k=1,\ldots,N \tag{4.67}$$

$$I^{x_2} \triangleq \left[\min_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right), \max_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right)\right], \tag{4.68}$$

evaluated over $i = 1, ..., N$, and $k = 1, ..., N$, we then evaluate α^{x_1} and α^{x_1} on the intervals I^{x_1} and I^{x_1}.

The next scheme given is the *Stencil Lax-Friedrichs* scheme. According to this scheme to find α^{x_1} and α^{x_2} define the intervals

$$I^{x_1} \triangleq \left[\min_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\right), \max_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\right)\right], \tag{4.69}$$

$$I^{x_2} \triangleq \left[\min_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right), \max_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right)\right], \tag{4.70}$$

evaluated over $i-3, ..., i+3$ and $k-3, ..., k+3$, we then evaluate α^{x_1} and α^{x_2} on the intervals I^{x_1} and I^{x_2}. This improves the local accuracy of the numerical Hamiltonian function since the accuracy decays for larger values of α^{x_1} and α^{x_2}.

The third scheme given is the *Local Lax-Friedrichs* scheme as presented in [17]. In this scheme, α^{x_1} and α^{x_2} are computed separately. Specifically, α^{x_1} is defined as in (4.65) and evaluated over the intervals,

$$I^{x_1} \triangleq \left[\frac{\partial\phi(t_j, x_{i,k})}{\partial x_1}^-, \frac{\partial\phi(t_j, x_{i,k})}{\partial x_1}^+\right], \tag{4.71}$$

$$I^{x_2} \triangleq \left[\min_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j, x_{i,k})}{\partial x_2}\right), \max_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j, x_{i,k})}{\partial x_2}\right)\right]. \tag{4.72}$$

Similarly, α^{x_2} is defined as in (4.66) and evaluated over the intervals,

$$I^{x_1} \triangleq \left[\min_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j, x_{i,k})}{\partial x_1}\right), \max_{x_{i,k}\in\mathbb{R}^2}\left(\frac{\partial\phi(t_j, x_{i,k})}{\partial x_1}\right)\right], \tag{4.73}$$

$$I^{x_2} \triangleq \left[\frac{\partial\phi(t_j, x_{i,k})}{\partial x_2}^-, \frac{\partial\phi(t_j, x_{i,k})}{\partial x_2}^+\right]. \tag{4.74}$$

The fourth scheme given is the *Local Local Lax-Friedrichs* scheme as presented in [18]. In this scheme,

$$I^{x_1} \triangleq \left[\frac{\partial\phi(t_n, x_{i,k})}{\partial x_1}^-, \frac{\partial\phi(t_n, x_{i,k})}{\partial x_1}^+\right], \tag{4.75}$$

$$I^{x_2} \triangleq \left[\frac{\partial\phi(t_n, x_{i,k})}{\partial x_2}^-, \frac{\partial\phi(t_n, x_{i,k})}{\partial x_2}^+\right], \tag{4.76}$$

and α^{x_1} and α^{x_2} are evaluated on the intervals I^{x_1} and I^{x_2}. The last scheme considered in this thesis is the *Roe-Fix* scheme. For this scheme, the numerical Hamiltonian is defined as

$$\begin{aligned}
&\hat{H}\left(t_j, x_{i,k}, \frac{\partial\phi(t_n, x_{i,k})}{\partial x_1}^-, \frac{\partial\phi(t_n, x_{i,k})}{\partial x_1}^+, \frac{\partial\phi(t_n, x_{i,k})}{\partial x_2}^-, \frac{\partial\phi(t_n, x_{i,k})}{\partial x_2}^+\right) \\
&\triangleq H\left(t_j, x_{i,k}, \frac{\partial\phi(t_j, x_{i,k})}{\partial x_1}^*, \frac{\partial\phi(t_j, x_{i,k})}{\partial x_2}^*\right) - \alpha^{x_1}\frac{1}{2}\left(\frac{\partial\phi(t_j, x_{i,k})}{\partial x_1}^- - \frac{\partial\phi(t_j, x_{i,k})}{\partial x_1}^+\right) \\
&\quad - \alpha^{x_2}\frac{1}{2}\left(\frac{\partial\phi(t_j, x_{i,k})}{\partial x_2}^- - \frac{\partial\phi(t_j x_{i,k})}{\partial x_2}^+\right), \\
&\qquad j = 0, \ldots, T, \qquad i = 1, \ldots, N, \qquad k = 1, \ldots, N.
\end{aligned} \tag{4.77}$$

In the case that $H_{\frac{\partial\phi(t,x)}{\partial x_1}}(\cdot,\cdot,\cdot)$ and $H_{\frac{\partial\phi(t,x)}{\partial x_2}}(\cdot,\cdot,\cdot)$ have constants signs over the intervals I^{x_1} and I^{x_2}, then

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^* = \begin{cases} \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^-, & H_{\frac{\partial\phi(t,x)}{\partial x_1}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) > 0, \\ \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^+, & H_{\frac{\partial\phi(t,x)}{\partial x_1}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) < 0, \end{cases} \tag{4.78}$$

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^* = \begin{cases} \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^-, & H_{\frac{\partial\phi(t,x)}{\partial x_2}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) > 0, \\ \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^+, & H_{\frac{\partial\phi(t,x)}{\partial x_2}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) < 0, \end{cases} \tag{4.79}$$

$$\alpha^{x_1} = 0, \tag{4.80}$$

$$\alpha^{x_2} = 0. \tag{4.81}$$

In the case that the sign of $H_{\frac{\partial\phi(t,x)}{\partial x_1}}(\cdot,\cdot,\cdot)$ changes, but the sign of $H_{\frac{\partial\phi(t,x)}{\partial x_2}}(\cdot,\cdot,\cdot)$ remains constant over the intervals I^{x_1} and I^{x_2}, then

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^* = \frac{1}{2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^+ + \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^-\right), \tag{4.82}$$

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^* = \begin{cases} \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^-, & H_{\frac{\partial\phi(t,x)}{\partial x_2}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) > 0, \\ \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^+, & H_{\frac{\partial\phi(t,x)}{\partial x_2}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) < 0, \end{cases} \tag{4.83}$$

$$\alpha^{x_1} = \max_{\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\in I^{x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\in I^{x_2}} \left| H_{\frac{\partial\phi(t,x)}{\partial x_1}}\left(t_j, x_i, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_n,x_{i,k})}{\partial x_2}\right)\right|, \tag{4.84}$$

$$\alpha^{x_2} = 0. \tag{4.85}$$

In the case that the sign of $H_{\frac{\partial\phi(t,x)}{\partial x_1}}(\cdot,\cdot,\cdot)$ remains constant, but the sign of $H_{\frac{\partial\phi(t,x)}{\partial x_2}}(\cdot,\cdot,\cdot)$ changes over the intervals I^{x_1} and I^{x_2}, then

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^* = \begin{cases} \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^-, & H_{\frac{\partial\phi(t,x)}{\partial x_1}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) > 0, \\ \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^+, & H_{\frac{\partial\phi(t,x)}{\partial x_1}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\right) < 0, \end{cases} \tag{4.86}$$

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^* = \frac{1}{2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^+ + \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^-\right), \tag{4.87}$$

$$\alpha^{x_1} = 0, \tag{4.88}$$

$$\alpha^{x_2} = \max_{\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\in I^{x_1},\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\in I^{x_2}} \left| H_{\frac{\partial\phi(t,x)}{\partial x_2}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_n,x_{i,k})}{\partial x_2}\right)\right|. \tag{4.89}$$

In the case that the signs of both $H_{\frac{\partial\phi(t,x;y)}{\partial x}(\cdot,\cdot,\cdot)}$ and $H_{\frac{\partial\phi(t,x;y)}{\partial y}}(\cdot,\cdot,\cdot)$ change over the intervals I^{x_1} and I^{x_2}, then

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^* = \frac{1}{2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^+ + \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}^-\right), \tag{4.90}$$

$$\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^* = \frac{1}{2}\left(\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^+ + \frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}^-\right), \tag{4.91}$$

$$\alpha^{x_1} = \max_{\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\in I^{x_1},\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\in I^{x_2}} \left| H_{\frac{\partial\phi(t,x)}{\partial x_1}}\left(t_j, x_i, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_n,x_{i,k})}{\partial x_2}\right)\right|, \tag{4.92}$$

$$\alpha^{x_2} = \max_{\frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}\in I^{x_1},\frac{\partial\phi(t_j,x_{i,k})}{\partial x_2}\in I^{x_2}} \left| H_{\frac{\partial\phi(t,x)}{\partial x_2}}\left(t_j, x_{i,k}, \frac{\partial\phi(t_j,x_{i,k})}{\partial x_1}, \frac{\partial\phi(t_n,x_{i,k})}{\partial x_2}\right)\right|. \tag{4.93}$$

4.4 Transformation of Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs Equations

If the time derivative does not appear explicitly in the Hamilton-Jacobi-Bellman or the Hamilton-Jacobi-Isaacs equations, then it is necessary to reduce these equations to the same form as (4.1). To this goal, we follow the method described in [20]. First, recall that the optimal cost to go is defined as

$$V(x_0) \triangleq \min_{u(\cdot)\in\mathcal{U}} \int_0^\infty L(x(\tau), u(\tau))d\tau, \tag{4.94}$$

where

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0, \qquad t \in [0, \infty), \tag{4.95}$$

and $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and $L : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$. In this section, we assume that $L(x,u) \neq 0, (x,u) \in \mathbb{R}^n \times \mathbb{U}$. The stationary Hamilton-Jacobi-Bellman equation is given by

$$\begin{aligned} \tilde{H}\left(x, \frac{\partial V(x)}{\partial x}\right) &= -L(x,u), \qquad (x,u) \in (\mathbb{R}^n \backslash \mathcal{T}) \times \mathbb{U}, \\ V(x) = 0, &\qquad x \in \partial T. \end{aligned} \tag{4.96}$$

where $\mathcal{T} \subset \mathbb{R}^n$ is closed, and

$$\tilde{H}\left(x, \frac{\partial V(x)}{\partial x}\right) \triangleq \frac{\partial V(x)}{\partial x} f(x,u). \tag{4.97}$$

Now, assume that $L(x(t), u(t)) \neq 0,\ (x,u) \in \mathbb{R}^n \times \mathbb{U}$, and let $\phi : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ so that

$$\frac{\partial V(x)}{\partial x} = \left(\frac{\partial \phi(t,x)}{\partial t}\right)^{-1} \frac{\partial \phi(t,x)}{\partial x}, \tag{4.98}$$

where $V(x) = \{t \ : \ \phi(t,x) = 0,\ x \in \mathbb{R}^n\}$. It follows from (4.96) that

$$\left(\frac{\partial \phi(t,x)}{\partial t}\right)^{-1} \frac{\partial \phi(t,x)}{\partial x} f(x,u) = L(x,u), \tag{4.99}$$

which, upon rearranging, yields

$$\frac{\partial \phi(x,t)}{\partial t} - \frac{\partial \phi(x,t)}{\partial x} \cdot \frac{f(x,u)}{L(x,u)} = 0. \tag{4.100}$$

The same process applies to the Hamilton-Jacobi-Isaacs equation, where

$$V(x) \triangleq \min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} \int_0^\infty L(x(\tau), u(\tau))d\tau, \tag{4.101}$$

and

$$\dot{x}(t) = f(x(t), u(t), v(t)), \qquad x(0) = x_0, \qquad t \in [0, \infty), \tag{4.102}$$

where $f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}^n$ and $L : \mathbb{R}^n \times \mathbb{U} \times \mathbb{V} \to \mathbb{R}$. The stationary Hamilton-Jacobi-Isaacs equation is given by

$$\begin{aligned} \tilde{H}\left(x, \frac{\partial V(x)}{\partial x}\right) &= -L(x, u, v), \qquad (x, u, v) \in (\mathbb{R}^n \backslash \mathcal{T}) \times \mathbb{U} \times \mathbb{V}, \\ V(x) &= 0, \qquad x \in \partial T \end{aligned} \tag{4.103}$$

where

$$\tilde{H}\left(x, \frac{\partial V(x)}{\partial x}\right) \triangleq \frac{\partial V(x)}{\partial x} f(x, u, v). \tag{4.104}$$

Now, assume that $(x, u, v) \in \mathbb{R}^n \times \mathbb{U} \times \mathbb{V}$, and let $\phi : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ so that

$$\frac{\partial V(x)}{\partial x} = \left(\frac{\partial \phi(t, x)}{\partial t}\right)^{-1} \frac{\partial \phi(t, x)}{\partial x}, \tag{4.105}$$

where $V(x) = \{t \ : \ \phi(t, x) = 0, \ x \in \mathbb{R}^n\}$. It follows from (4.103) that

$$\left(\frac{\partial \phi(t, x)}{\partial t}\right)^{-1} \frac{\partial \phi(t, x)}{\partial x} f(x, u, v) = L(x, u, v), \tag{4.106}$$

which, upon rearranging, yields

$$\frac{\partial \phi(t, x)}{\partial t} - \frac{\partial \phi(t, x)}{\partial x} \cdot \frac{f(x, u, v)}{L(x, u, v)} = 0. \tag{4.107}$$

Both (4.100) and (4.107) are Hamilton-Jacobi PDEs in the same form as (4.1) with $U(t,x) = -\frac{f(x,u)}{L(x,u)}$ for the transformed Hamilton-Jacobi-Bellman equation or $U(t,x) = -\frac{f(x,u,v)}{L(x,u,v)}$ for the transformed Hamilton-Jacobi-Isaacs equation, which allows us to solve for $V(\cdot)$ using level set methods. In both cases, it is assumed that the inputs are known beforehand. This requires us to modify the toolbox [19] to solve for $u(\cdot)$, $v(\cdot)$, and $V(\cdot)$ simultaneously.

4.5 Illustrative Numerical Examples

In this section, we will present the capability of our toolbox to solve optimal control and differential game problems. We will present our solutions to the problems that were outlined in Chapters 2 and 3 and discuss the effectiveness of our toolbox.

Specifically, upon solving the Hamilton-Jacobi-Bellman equation, using the numerical integration schemes discussed in this chapter, our toolbox iteratively searches over a discretized set of inputs to find $u(\cdot)$ that minimizes $V(x_i)$. Similarly, upon solving the Hamilton-Jacobi-Isaacs equation, our toolbox iteratively searches over a discretized set of inputs to find $u(\cdot)$ and $v(\cdot)$ such that $V(x_i)$ verifies (4.101).

4.5.1 Brachistochrone Problem

The first numerical example is the Brachistochrone problem presented in Section 2.3.1. This initial problem shows us that the toolbox is able to approximate the solution for the optimal control policy of a trajectory. The numerical control policy is able to approximate the analytical control policy as can be seen in Figure 4.1. For the control policy, the mean error is 8.06% and the standard deviation is 3.26 degrees.

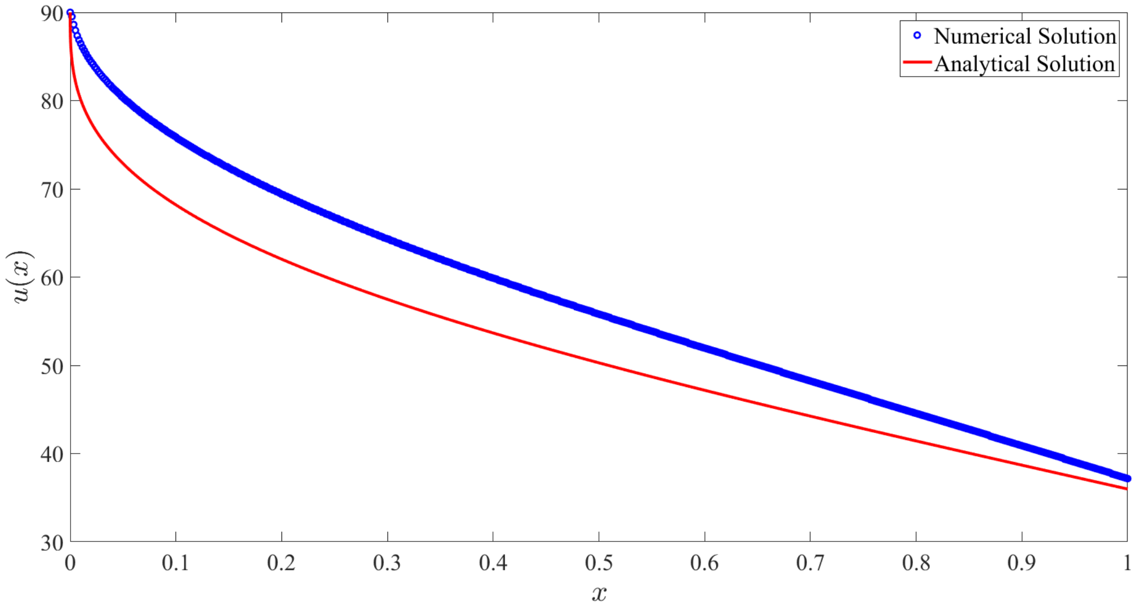

Figure 4.1: Brachistochrone Problem

4.5.2 Linear Quadratic Regulator

The second problem of interest is the time-invariant Linear Quadratic Regulator presented in Section 2.5. The toolbox is able to accurately approximate the optimal policy as can be seen in Figure 4.2. In this example, the average error between the analytical control policy and the control policy found using our numerical methods is 8.0% and the standard deviation is 2.56.

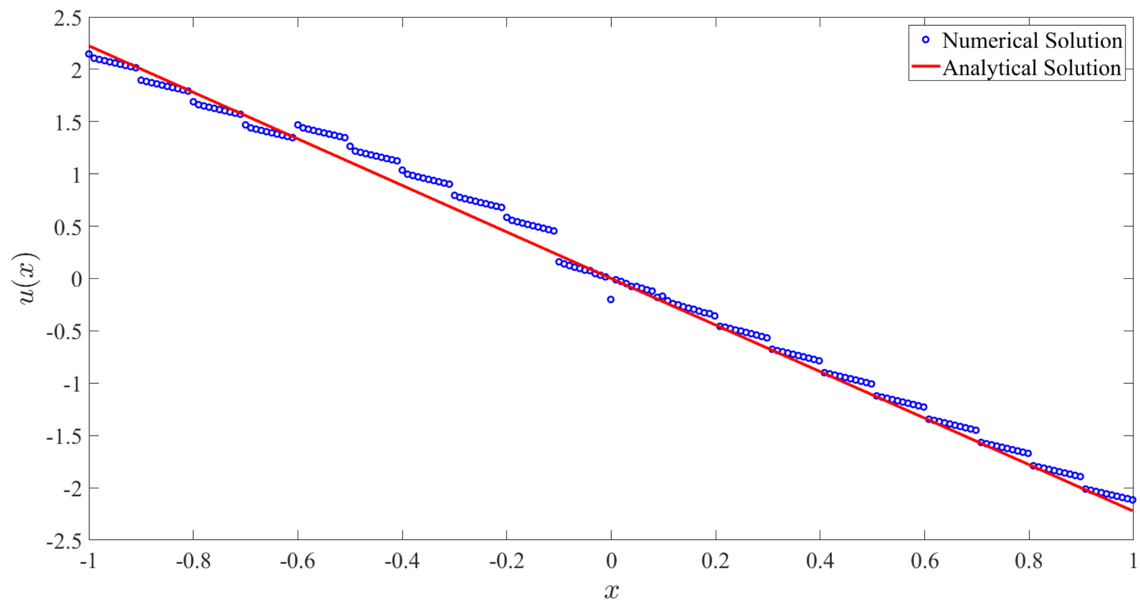

Figure 4.2: Linear Quadratic Regulator Control

In addition, the toolbox was also able to accurately approximate the optimal cost-to-go as can be seen in Figure 4.2. In this example, the average error between the analytical cost and the cost found using our numerical methods is 2.0% and the standard deviation is 0.3.

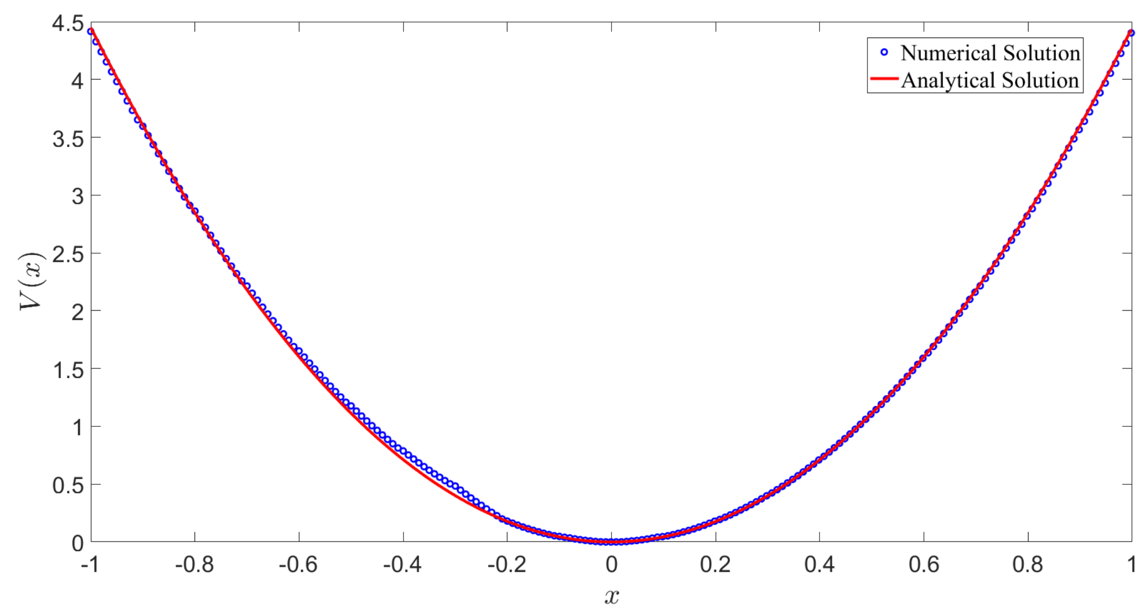

Figure 4.3: Linear Quadratic Regulator Cost-to-Go

4.5.3 H_∞ Optimal Control

The next problem of interest is the H_∞ optimal control problem discussed in Section 2.4. This control problem is equivalent to a differential game problem for which the analytical solution is known. As can be seen in Figure 4.4 the control policy found by the toolbox is an accurate approximation of the optimal control policy. In this example, the average error between the analytical control policy and the control policy found using our numerical methods is 7.38% and the standard deviation is 0.09.

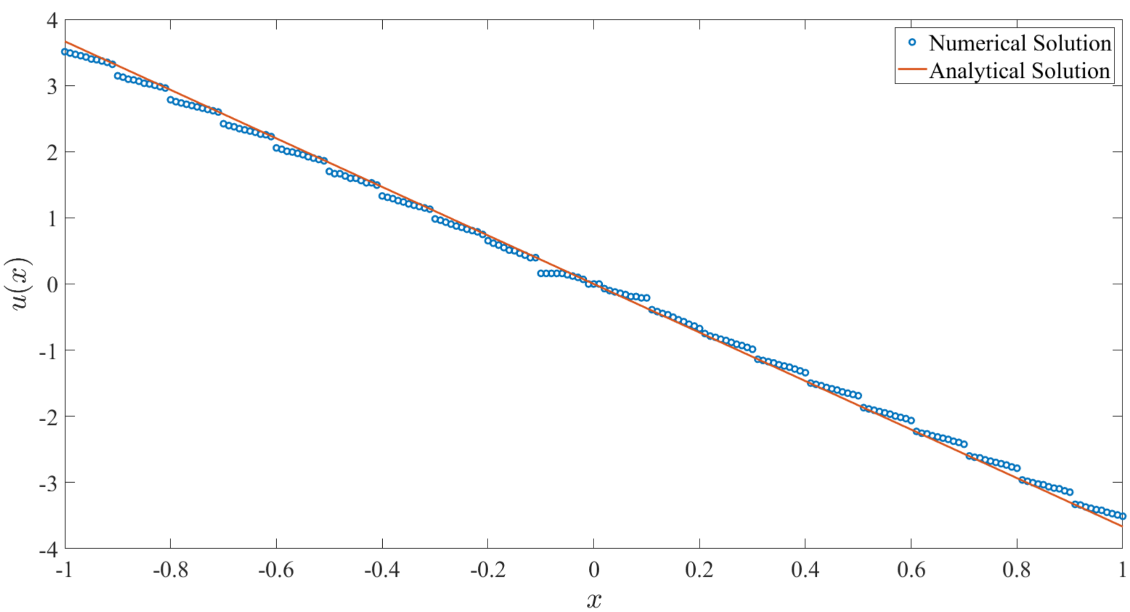

Figure 4.4: H_∞ Control

As can be seen in Figure 4.4, the noise found by the toolbox is a good approximation for the worst-case noise. In this example, the average error between the analytical disturbance and the disturbance found using our numerical methods was 15.51% and the standard deviation is 0.04.

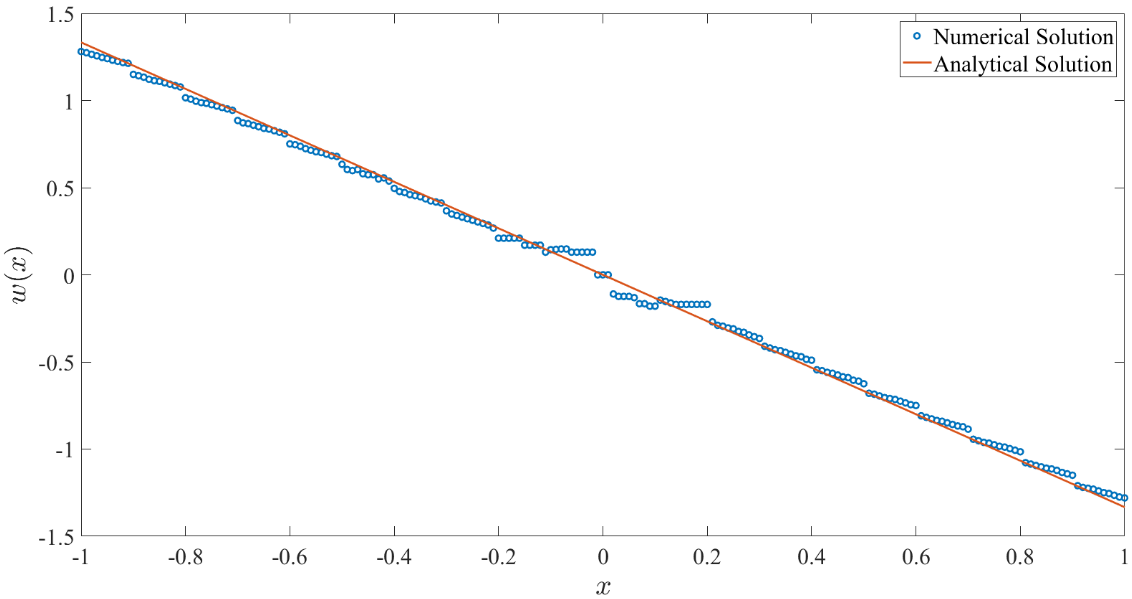

Figure 4.5: H_∞ Noise

As can be seen in Figure 4.6, the value of the game captured by our toolbox is an accurate approximation of the analytical value of the game. In this example, the average error between the analytical cost and the cost found using our numerical methods is 0.35% and the standard deviation is 0.004.

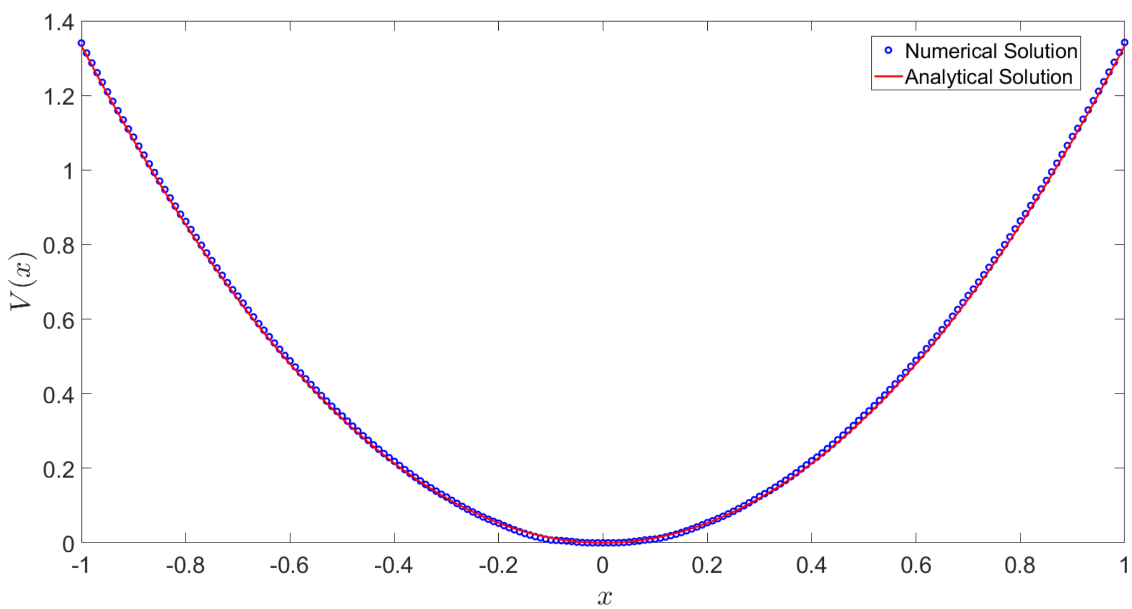

Figure 4.6: H_∞ Cost-to-Go

4.5.4 Pursuer Evader Problem

The next problem that we considered to validate our toolbox is the pursuer evader differential game problem discussed in Section 3.6. This differential game is guaranteed to have a solution only if the pursuer's speed is greater than the evader's speed. Therefore for our simulation we assumed that the speed of pursuer is 2 m/s and its minimum turn radius is 1 m, while speed of the evader is 1 m/s.

To test the validity of the solution found by our numerical solver we simulated three different scenarios. In all three cases the pursuer was able to capture the evader which shows that our toolbox was able to find a feedback solution for the pursuer to capture the evader. In the first simulation, the pursuer started at (0,0) and the evader started at (0.6, 0.6), and as can be seen in Figure 4.7 the pursuer is able to capture the evader.

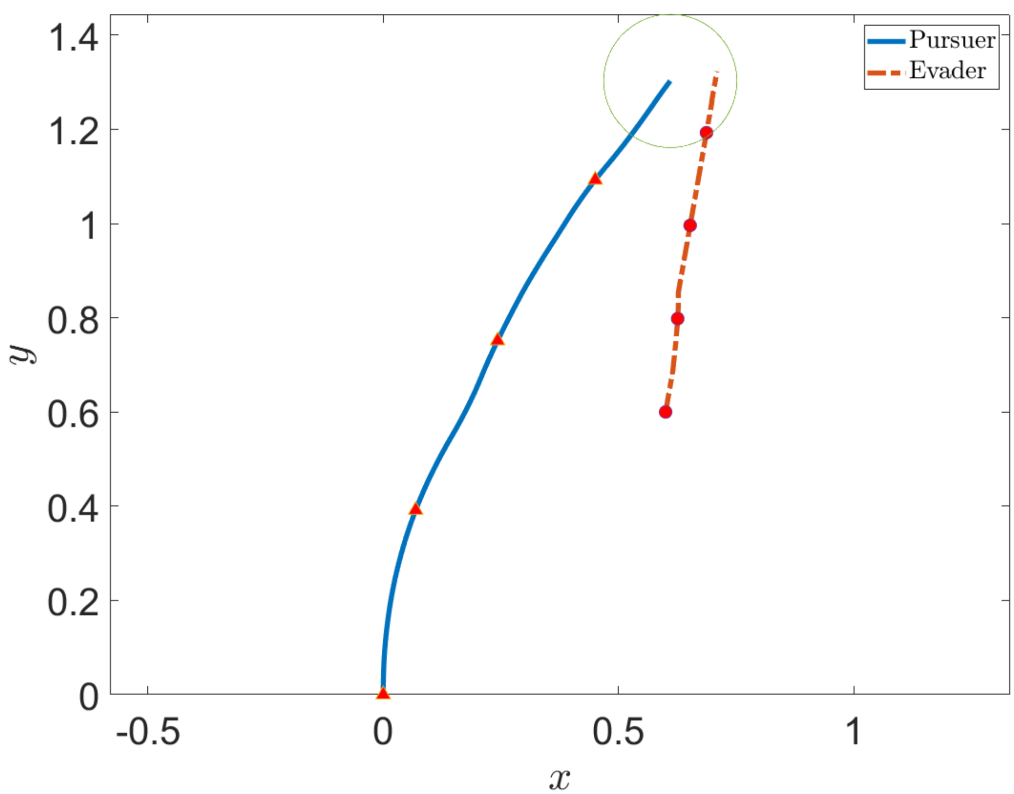

Figure 4.7: Pursuer Evader Problem with Initial Conditions x = 0.6, y = 0.6

In the second simulation the pursuer started at (0,0) and the evader started at (0.6, 0.8), as can be seen in Figure 4.8 the pursuer was still able to capture the evader.

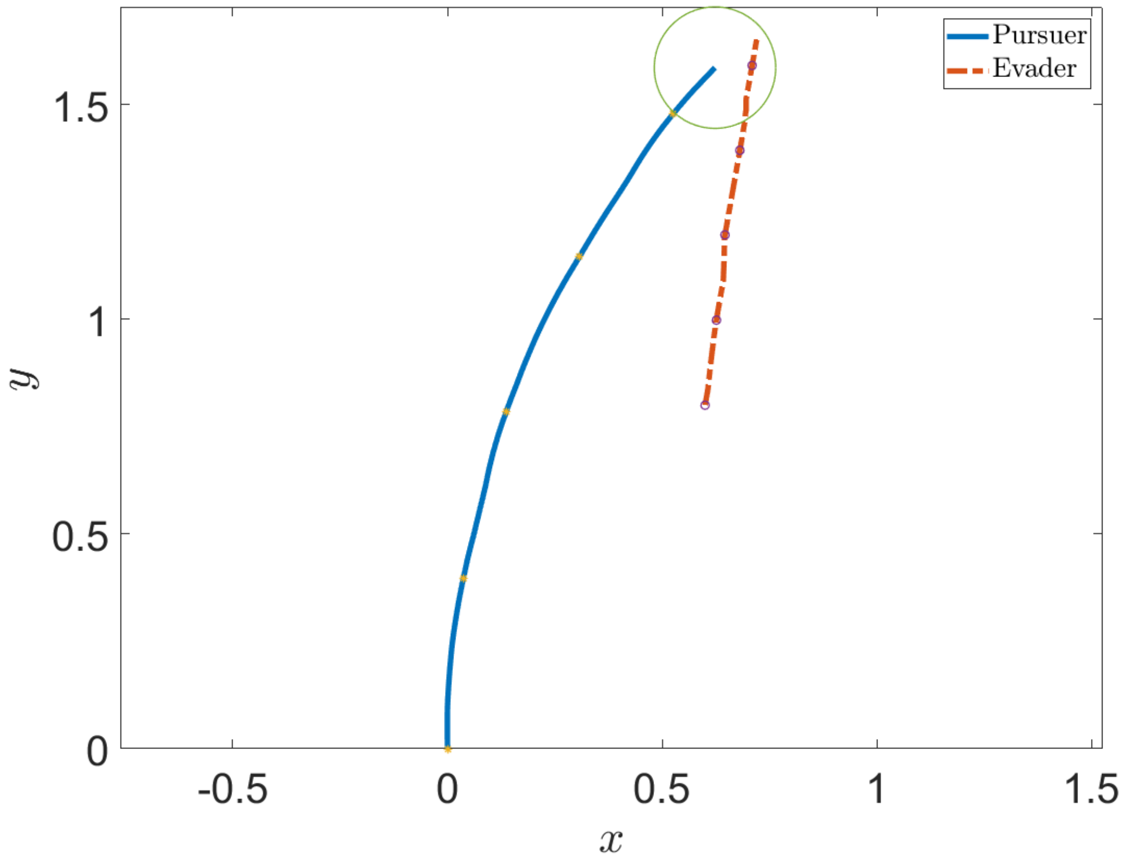

Figure 4.8: Pursuer Evader Problem with Initial Conditions x = 0.6, y = 0.8

In the final simulation the pursuer started at (0,0) and the evader started at (0.8, 0.8), as can be seen in Figure 4.9 the pursuer was still able to capture the evader. From these three simulations, we verify that, as expected, the further away the evader starts the longer it takes to be captured. This result also verifies the validity of our toolbox since we were able to show successful capture of the evader in every situation which is what the theory tells us will happen.

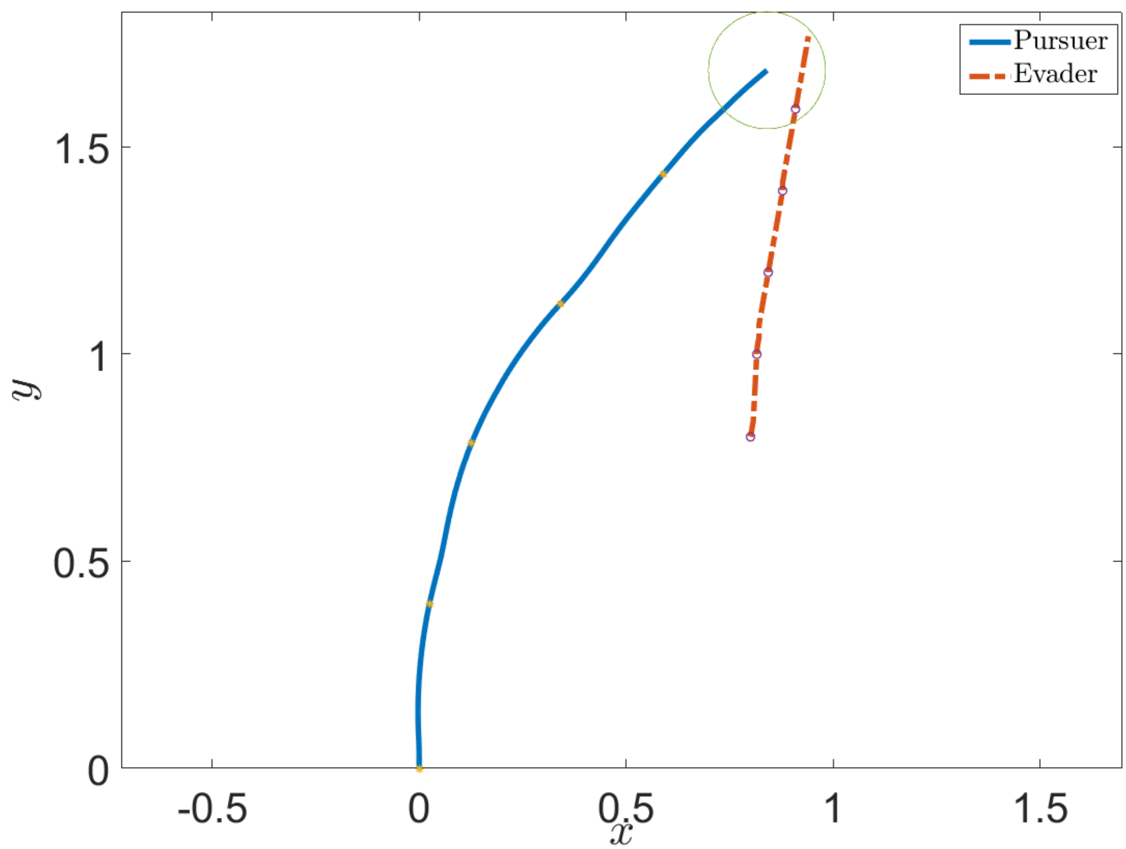

Figure 4.9: Pursuer Evader Problem with Initial Conditions x = 0.8, y = 0.8

4.5.5 Target-Attacker-Defender Problem

The final problem that we solved was the target attacker defender differential game problem discussed in Section 3.7. In this problem, we assume that the ratio of the target's velocity to the attacker's velocity is 0.4 while the ratio of the defender's velocity to the attacker's velocity is 1.

To test the solution found by our toolbox we simulated two different scenarios using the control policy found. In the first scenario considered in this thesis, the attacker starts at (-2,0), the defender starts at (-1,3), and the target starts at (-0.5, 2). Even though the attacker had a larger capture radius, the defender is still able to intercept the attacker as shown in Figure 4.10.

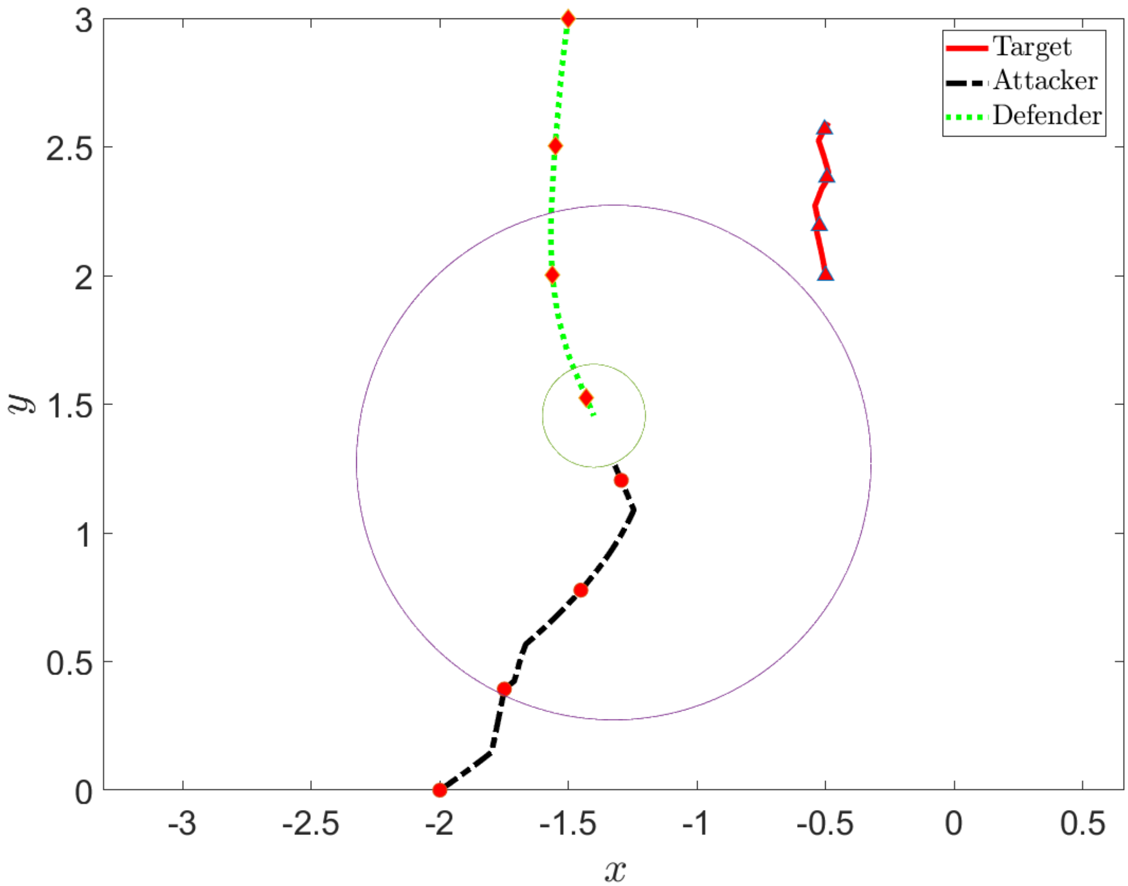

Figure 4.10: Target Attacker Defender Scenario 1

In the second scenario, the attacker starts at (0,0), the defender starts at (0.3,0.4), and the target starts at (0.4, 0.3). As can be seen in Figure 4.11, if the defender and the attacker have equal capture radius, then the defender is able to intercept the attacker again.

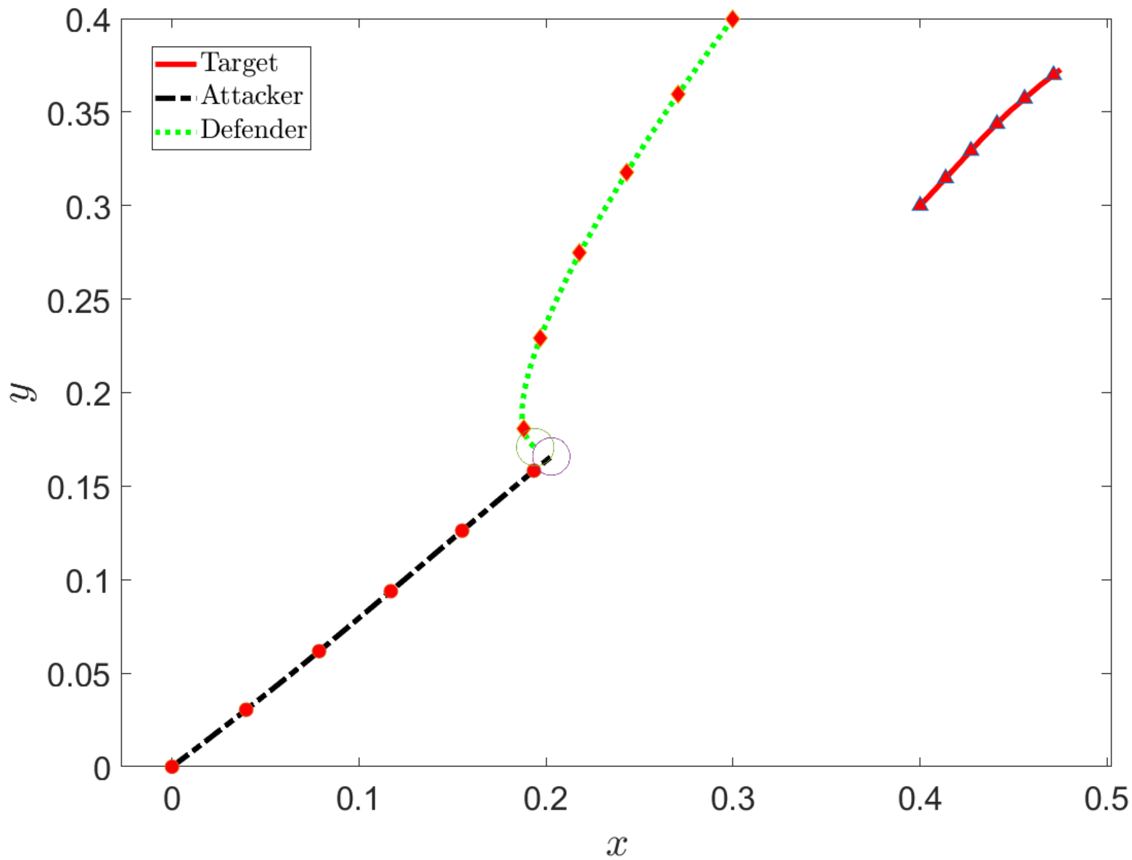

Figure 4.11: Target Attacker Defender Scenario 2

4.6 Conclusion

In this chapter we presented numerical integration schemes needed to solve first order partial differential equations. These schemes have been applied to iteratively solve the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs equations, while searching for optimal control policies over some discretized domain. The validity of our toolbox, which leverages on [19] to solve the Hamilton-Jacobi equation for a given control policy, has been verified by solving optimal control and differential game problems, whose analytical solutions have been presented in Chapters 2 and 3.

Chapter 5. Conclusion

In this thesis, we presented the necessary and sufficient conditions to solve optimal control and differential game problems. We then presented the level set methods which are capable of solving Hamilton-Jacobi PDEs and showed how they were applied in our original toolbox to solve the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs equation. We then demonstrated the capability of our toolbox by presenting the numerical solutions to the problems presented in Chapters 2 and 3. Our toolbox contributes to the existing material by providing a method which can find the inputs that solve the optimal control and differential game problems as well as characterize the cost function through the application of level set methods. To our knowledge, this is unprecented.

Bibliography

[1] R. Anil, *A Survey of Numerical Methods for Optimal Control*, Advances in the Astronautical Sciences. 135., 2010

[2] P. Johnson, *Numerical Solution Methods for Differential Game Problems*, 2007

[3] A. Schweidtmann, A. Mitsos, *Deterministic Global Optimization with Artificial Neural Networks Embedded*, Journal of Optimization Theory and Applications, Volume 180, No. 3, 2019

[4] C. Edwards, D. Penney, D. Calvis, *Differential Equations and Boundary Value Problems*, Pearson, 5 ed., 2015

[5] A. Bryson, Y. Ho, *Applied Optimal Control*, Blaisdell, 1969

[6] U. Boscain, B. Piccoli, *An Introduction to Optimal Control*

[7] D. Liberzon, *Calculus of Variations and Optimal Control Theory*, Princeton, 2011

[8] R. Isaacs, *Differential Games*, New York: Wiley, 1965

[9] D. Bernstein, P. Tsiotras, *A Course in Classical Optimal Control*, 2011

[10] E. Lavretsky, K. Wise, *Robust and Adaptive Control*, Springer, 2013

[11] E. Garcia, D. Casbeer, M. Pachter, *Cooperative Strategies for Optimal Aircraft Defense from an Attacking Missile*, Journal of Guidance, Control, and Dynamics, vol. 38, no. 8, pp.1510-1520, 2015

[12] S. Osher, R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*, Springer, 2003

[13] A. Harten, B. Engquist, S. Osher, S. Chakravarthy, *Uniformly High-Order Accurate Essentially Non-Oscillatory Schemes III*, J. Comput. Phys. 71, pg. 231-303, 1987

[14] X.-D. Liu, S. Osher, T. Chan, *Weighted Essentially Non-Oscillatory Schemes*, J. Comput. Phys. 126, pg. 202-212, 1996

[15] G.-S. Jiang, C.-W. Shu, *Weighted ENO Schemes for Hamilton Jacobi Equations*, SIAM J. Sci. Comput. 21, 2126-2143, 2000

[16] M. Crandall, P.-L. Lions, *Two Approximations of Solutions of Hamilton-Jacobi Equations*, Math. Comput. 43, 1-19, 1984

[17] C.W. Shu, S. Osher, *Efficient Implementation of Essentially Non-Oscillatory Shock Capturing Schemes II*, J. Comput. Phys. 83, 32-78, 1989

[18] S. Osher, C.W. Shu, *High Order Essentially Non-Oscillatory Schemes for Hamilton-Jacobi Equations*, SIAM J. Numer. Anal. 28, 902-921, 1991

[19] I. Mitchell, *A Toolbox of Level Set Methods (Version 1.1) UBC CS TR-2007-11*, 2007

[20] S. Osher, *A Level Set Formulation for the Solution of the Dirichlet Problem for Hamilton-Jacobi Equations*, SIAM J. Numer. Anal. 24, 1145-1152, 1993