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KLINGEN VECTORS OF LEVEL 2 FOR GSp(4)

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To my Grandparents

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Contents

List of Tables

Abstract

The theory of Siegel modular forms generalizes classical elliptic modular forms which is, in fact, the degree one case. Dimension formulas for spaces of elliptic modular forms have been much studied, however, the situation for Siegel modular forms still needs a lot of work to do. In particular, the dimension formula for the space of Siegel cusp forms of degree 2 with respect to the congruence subgroup Γ is related to the dimensional data of spaces of fixed vectors with respect to the congruence subgroup Γ for the irreducible, admissible representations of $\mathrm{GSp}(4, F)$, where F is a $\mathfrak{p}$-adic field.

In this thesis, we use a variety of methods to determine the dimensions of the spaces of invariant vectors under the Klingen congruence subgroup of level $\mathfrak{p}^2$ for all irreducible, admissible representations of the algebraic group $\mathrm{GSp}(4)$ over F.

Chapter 1

Introduction

1.1 Historical background

The classical theory of newforms is an important topic in holomorphic modular forms, and the local theory of newforms lies at the intersection of representation theory, modular forms theory, and applications to number theory. In 1970, Atkin and Lehner [1] introduced the classical newforms theory for GL(2). In 1973, Casselman [8] used representation theoretic methods to study the local theory of new- and oldforms for representations of GL(2). In 1981, Jacquet, Piatetski-Shapiro and Shalika [11] generalized the local newforms theory for GL(2) to GL(n) for generic representations. In 2005, Schmidt [22] developed a theory of local new- and oldforms for representations of GSp(4) over a $\mathfrak{p}$-adic field with Iwahori-invariant vectors. In 2007, Roberts and Schmidt [15] had a satisfactory local theory of new- and oldforms for representations of GSp(4) with trivial central character, in which they considered the vectors fixed by the paramodular groups $\mathrm{K}(\mathfrak{p}^n)$.

In this thesis, we are going to investigate the vectors fixed by the Klingen congruence subgroup $\mathrm{Kl}(\mathfrak{p}^2)$ as defined in (1.6) of $\mathrm{GSp}(4,F)$ with trivial central character, where F is a $\mathfrak{p}$-adic field.

1.2 Definitions and notations

Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be the ring of integers of F and $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. We fix a generator ϖ of $\mathfrak{p}$. Let q be the cardinality of $\mathfrak{o}/\mathfrak{p}$. We let ν or $|\cdot|$ be the normalized absolute value on F; thus $\nu(\varpi)=q^{-1}$. Throughout this paper we use the Haar measure on F such that $\mathfrak{o}$ has volume 1 and the Haar measure on $F^\times$ defined by $d^*x = dx/|x|$, where dx is our Haar measure on F. The algebraic group GSp(4) is defined as

$$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) \mid {}^tgJg = \lambda(g)J,\, \lambda(g)\in \mathrm{GL}(1)\}, \quad J = \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}. \tag{1.1}$$

We shall sometimes abbreviate GSp(4) by G. The homomorphism

$$\lambda\colon \mathrm{GSp}(4) \longrightarrow \mathrm{GL}(1) \tag{1.2}$$

is called the multiplier homomorphism. Its kernel is the symplectic group Sp(4). There are three different conjugacy classes of parabolic subgroups of GSp(4), represented by, the Borel subgroup B, Siegel parabolic subgroup P and Klingen parabolic subgroup Q. They have the following forms

$$B = \begin{bmatrix} *&*&*&*\\ &*&*&*\\ &&*&*\\ &&&* \end{bmatrix}, \quad P = \begin{bmatrix} *&*&*&*\\ *&*&*&*\\ &&*&*\\ &&*&* \end{bmatrix}, \quad Q = \begin{bmatrix} *&*&*&*\\ &*&*&*\\ &*&*&*\\ &&&* \end{bmatrix}. \tag{1.3}$$

To be more precise, we have

1. Every element of B can be written in the form

$$g = \begin{bmatrix} a &&& \\ & b && \\ && cb^{-1} & \\ &&& ca^{-1} \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 & x & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda & \mu & \kappa \\ & 1 && \mu \\ && 1 & -\lambda \\ &&& 1 \end{bmatrix},$$

where $a, b, c \in F^\times$ and $x, \lambda, \mu, \kappa \in F$. It is not hard to check that $\lambda(g) = c$.

2. Every element of P can be written in the form

$$g = \begin{bmatrix} a & b && \\ c & d && \\ && \lambda a/\Delta & -\lambda b/\Delta \\ && -\lambda c/\Delta & \lambda d/\Delta \end{bmatrix} \begin{bmatrix} 1 && \mu & \kappa \\ & 1 & x & \mu \\ && 1 & \\ &&& 1 \end{bmatrix},$$

where $\Delta = ad - bc \in F^\times, \lambda \in F^\times$, and $\mu, \kappa, x \in F$. We have $\lambda(g) = \lambda$. We will sometimes write

$$A' = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] {}^tA^{-1} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \quad \text{for} \quad A \in \mathrm{GL}(2, F).$$

Using the above notation, a general element in the Levi subgroup of P can be written as

$$M = \left[\begin{smallmatrix} A & \\ & \lambda A' \end{smallmatrix}\right], \quad A \in \mathrm{GL}(2, F),\ \lambda \in F^\times.$$

3. Every element of Q can be written in the form

$$g = \begin{bmatrix} t &&& \\ & a & b & \\ & c & d & \\ &&& \Delta t^{-1} \end{bmatrix} \begin{bmatrix} 1 & \lambda & \mu & \kappa \\ & 1 && \mu \\ && 1 & -\lambda \\ &&& 1 \end{bmatrix},$$

where $t \in F^\times$, and $\lambda, \mu, \kappa \in F$. We have $\lambda(g) = \Delta$.

The Jacobi subgroup G^J of Q and its center Z^J are defined as follows

$$G^J := \begin{bmatrix} 1 & * & * & * \\ & * & * & * \\ & * & * & * \\ &&& 1 \end{bmatrix}, \quad Z^J := \begin{bmatrix} 1 &&& * \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix}. \tag{1.4}$$

The usual 8-element Weyl group W of $\mathrm{GSp}(4,F)$ is generated by the elements

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}. \tag{1.5}$$

We define the Klingen congruence subgroup of level $\mathfrak{p}^n$ as

$$\mathrm{Kl}(\mathfrak{p}^n) := \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}. \tag{1.6}$$

Recall that the paramodular group of level $\mathfrak{p}^n$ is defined as

$$\mathrm{K}(\mathfrak{p}^n) := \{g \in \mathrm{GSp}(4,F) \mid \det(g) \in \mathfrak{o}^\times\} \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}. \tag{1.7}$$

We also define the *middle group* as

$$\mathrm{M}(\mathfrak{p}^2) := \{g \in \mathrm{GSp}(4,F) \mid \det(g) \in \mathfrak{o}^\times\} \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{o} \end{bmatrix}. \tag{1.8}$$

Obviously, we have $\mathrm{K}(\mathfrak{p}^2) \supset \mathrm{M}(\mathfrak{p}^2) \supset \mathrm{Kl}(\mathfrak{p}^2)$. Let (π, V) a smooth representation of $\mathrm{GSp}(4,F)$. Then we have $V^{\mathrm{K}(\mathfrak{p}^2)} \subset V^{\mathrm{M}(\mathfrak{p}^2)} \subset V^{\mathrm{Kl}(\mathfrak{p}^2)}$, where V^Γ means the space of invariant vectors under the subgroup Γ.

Moreover, by (2.7) of [15], we have the Iwahori factorization for $\mathrm{Kl}(\mathfrak{p}^2)$, i.e.,

$$\begin{aligned} \mathrm{Kl}(\mathfrak{p}^2) &= \begin{bmatrix} 1 & & & \\ \mathfrak{p}^2 & 1 & & \\ \mathfrak{p}^2 & & 1 & \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & & & \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ & & & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & & & \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ & & & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & & & \\ \mathfrak{p}^2 & 1 & & \\ \mathfrak{p}^2 & & 1 & \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & 1 \end{bmatrix}. \end{aligned} \tag{1.9}$$

Furthermore, one can show that

$$\mathrm{M}(\mathfrak{p}^2) = \bigsqcup_{v\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & & v\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^2). \tag{1.10}$$

It follows from (1.9) and (1.10) that the Iwahori factorization also holds for the middle group $\mathrm{M}(\mathfrak{p}^2)$, i.e.,

$$\begin{aligned} \mathrm{M}(\mathfrak{p}^2) &= \begin{bmatrix} 1 & & & \\ \mathfrak{p}^2 & 1 & & \\ \mathfrak{p}^2 & & 1 & \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & & & \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ & & & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & & & \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ & & & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & & & \\ \mathfrak{p}^2 & 1 & & \\ \mathfrak{p}^2 & & 1 & \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & 1 \end{bmatrix}. \end{aligned} \tag{1.11}$$

1.3 Representations of $\mathrm{GSp}(4, F)$

1.3.1 Some general representation theory

Let G be a group of td-type as in [7, §1.1], with a countable basis. Let $\mathcal{S}(G)$ be the complex vector space of all locally constant, compactly supported complex valued functions on G.

Definition 1.1. A representation (π, V) of G is a complex vector space V along with a homomorphism π of $\mathrm{GSp}(4, F)$ into the group $\mathrm{Aut}(V)$ of invertible $\mathbb{C}$-linear endomorphisms of V. The representation will be denoted simply by π or by V where convenient.

A representation (π, V) is *smooth* if for every vector $v \in V$, the stabilizer of v in G, given by

$$\mathrm{Stab}_G(v) = \{g \in G \mid \pi(g)v = v\}, \tag{1.12}$$

is open. We say that (π, V) is *admissible* if π is smooth and for any open compact subgroup K of G, the space of invariants in V under K, denoted V^K, is finite dimensional.

Given a smooth representation (π, V) of G, a subspace W of V is said to be stable or invariant under G if for every $w \in W$ and every $g \in G$ we have $\pi(g)w \in W$.

Definition 1.2. A smooth representation (π, V) is *irreducible* if the only G stable subspaces of V are 0 and V. We say that π is reducible if π is not irreducible.

If (π, V) is a smooth representation of G, then an irreducible constituent or irreducible subquotient of π is an irreducible representation of G that is isomorphic to W/W', where $W' \subset W \subset V$ are G stable subspaces of V.

If (π, V) and (π', V') are two (smooth) representations of G, then we denote the space of G intertwining operators from V into V' by $\mathrm{Hom}_G(\pi, \pi')$, i.e., $\mathrm{Hom}_G(\pi, \pi')$ is the space of all linear maps $f\colon V \to V'$ such that

$$f(\pi(g)v) = \pi'(g)f(v) \tag{1.13}$$

for all $v \in V$ and all $g \in G$.

Let (π, V) be a smooth representation of G. Let V^* be the space of linear functionals on V, i.e., $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. An obvious action π^* of G on V^* is given by

$$(\pi^*(g)f)(v) = f(\pi(g^{-1})v) \tag{1.14}$$

for $g \in G, v \in V$ and $f \in V^*$. However, this representation (π^*, V^*) is in general not smooth. Let $V^\vee$ be the subspace of V^* which consists of those linear functionals in V^* whose stabilizers are open in G under the above action. This representation denoted $(\pi^\vee, V^\vee)$ is called the *contragredient representation* of (π, V). And if π admits a central character, then we denote it by ω_π.

Definition 1.3. A character of G is a smooth one–dimensional representation of G, i.e., a continuous homomorphism from G to $\mathbb{C}^\times$.

Let 1_G denote the trivial representation of G, i.e., the trivial character of G.

Definition 1.4. A representation π of G is unitary if there is a *positive definite* G–invariant Hermitian form on the space of π.

One of the most basic ways of constructing representations is by the process of induction. Let H be a closed subgroup of G, and let (σ, W) be a smooth representation of H. Then $\pi = \text{c-Ind}_H^G(\sigma)$ is the representation of G whose space is the vector space of all functions $f\colon G \to W$ such that

1. $f(hg) = \sigma(h)f(g)$ for $h \in H$ and $g \in G$,

2. there exists a compact open subgroup K of G such that $f(gk) = f(g)$ for $k \in K$ and $g \in G$, and there exists a compact set $X \subset G$ such that f vanishes off of HX.

This is called compact induction. The group G acts on this space by right translation, i.e., given $x \in G$ we have

$$(x \cdot f)(g) = f(gx) \quad \text{for all } g \in G. \tag{1.15}$$

Suppose G is unimodular, i.e., every left Haar measure is also a right Haar measure. And assume that M and U are closed subgroups of G such that M normalizes U, $M \cap U = 1, P = MU$ is closed in G, U is unimodular, and $P\backslash G$ is compact. Fix a Haar measure du on U. For $p \in P$ let $\delta_P(p)$ be the positive number such that for all $f \in \mathcal{S}(U)$,

$$\int_U f(p^{-1}up)du = \delta_P(p) \int_U f(u)du.$$

We call $\delta_P : P \to \mathbb{C}^\times$ the modular character of P. Suppose that (σ, W) is a smooth representation of M. Then the normalized induction $\mathrm{Ind}_P^G(\sigma)$ is the representation of G by right translation on the complex vector space of smooth functions f on G with values in σ such that

$$f(mug) = \delta_P(m)^{1/2}\sigma(m)f(g) \tag{1.16}$$

for $m \in M, u \in U$ and $g \in G$.

Proposition 1.5. *Let (σ, W) be a smooth representation of a closed subgroup H of G. Then*

1. *If $H\backslash G$ is compact and σ is admissible, then $\text{c-Ind}_H^G(\sigma) = \mathrm{Ind}_H^G(\delta_H^{-\frac{1}{2}}\sigma)$ is admissible.*

2. *(Frobenius reciprocity) If (π, V) is a smooth representation, then composition with the map $f \mapsto f(1)$ induces an isomorphism of $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma))$ with $\mathrm{Hom}_H(\pi|_H, \sigma)$.*

This proposition can be referred to Lemma 2.26 and Theorem 2.28 of [3].

1.3.2 Generic representations of $\mathrm{GSp}(4, F)$

Let ψ be a fixed non-trivial additive character of F. Fix $c_1, c_2 \in F^\times$, and consider the character ψ_{c_1,c_2} of $U(F)$, the unipotent radical of the Borel subgroup $B(F)$, given by

$$\psi_{c_1,c_2}(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix}) = \psi(c_1x + c_2y). \tag{1.17}$$

Definition 1.6. An irreducible, admissible representation π of $\mathrm{GSp}(4, F)$ is called generic if $\mathrm{Hom}_{U(F)}(\pi, \psi_{c_1,c_2}) \neq 0$.

This definition is independent on the choice of c_1 or c_2. If π is generic, then there exists a Whittaker model for π with respect to ψ_{c_1,c_2}, i.e., π can be realized as a space of functions $W\colon \mathrm{GSp}(4, F) \longrightarrow \mathbb{C}$ that satisfy the transformation property

$$W(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix} g) = \psi(c_1x + c_2y)W(g), \qquad \text{all } g \in \mathrm{GSp}(4, F), \tag{1.18}$$

and $\mathrm{GSp}(4, F)$ acts on this space by right translations. By [18], such a Whittaker model is unique. We denote it by $\mathcal{W}(\pi, \psi_{c_1,c_2})$.

1.3.3 Parabolically induced representations of $\mathrm{GSp}(4, F)$

The irreducible, admissible representations of $\mathrm{GSp}(4, F)$ come in two classes. The first class consists of all those representations that can be obtained as subquotients of parabolically induced representations from one of the parabolic subgroups B, P or Q. These representations have been classified and described in [20] by Sally and Tadić. Furthermore, Roberts and Schmidt reproduce the list, see [15, section 2.2]. The second class consists of all the other representations, which are called supercuspidal. In this section, we will not give explicit descriptions of the supercuspidal representations. However, we are going to give explicit descriptions of the parabolically induced representations as follows.

Borel-induced representations. Let $B = MN$ be the Borel subgroup, where

$$N = \{ \begin{bmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} : \ x, \lambda, \mu, \kappa \in F \}$$

and

$$M = \{ \begin{bmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix} : \ a, b, c \in F^{\times} \}$$

Let χ_1, χ_2 and σ are characters of $F^{\times}$, and consider the character of B given by

$$\begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} \longmapsto \chi_1(a)\chi_2(b)\sigma(c).$$

The representation of $\mathrm{GSp}(4, F)$ obtained by normalized parabolic induction of this character of B is denoted by $\chi_1 \times \chi_2 \rtimes \sigma$. Then the standard model of this representations consists of all locally constant functions $f : \mathrm{GSp}(4, F) \to \mathbb{C}$ with the transformation property

$$f(hg) = \mid a^2 b \mid\mid c \mid^{-\frac{3}{2}} \chi_1(a)\chi_2(b)\sigma(c)f(g), \quad h \in B, g \in G. \tag{1.19}$$

Note here that the modular character of B is given by $\delta_B(h) = |a|^4|b|^2|c|^{-3}$. The group $\mathrm{GSp}(4, F)$ acts on this space by right translations. The central character of $\chi_1 \times \chi_2 \rtimes \sigma$ is $\chi_1\chi_2\sigma^2$. Moreover, group I to VI in the Table 1.1 contain representations supported in B, i.e., these representations are constituents of induced representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$.

Klingen-induced representations. Let $Q = MN$ be the Klingen parabolic subgroup, where

$$N = \{ \begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix} : \ x, y, z \in F \}$$

and

$$M = \{ \begin{bmatrix} \lambda & & \\ & A & \\ & & \lambda^{-1}det(A) \end{bmatrix} : \ A \in \mathrm{GL}(2, F), \ \lambda \in F^{\times} \}$$

Let χ is a character of $F^{\times}$, and let (π, V) be an admissible representation of $\mathrm{GL}(2, F)$ (for systematic reasons, this $\mathrm{GL}(2, F)$ should be consider as the group $\mathrm{GSp}(2, F)$). Then we denote by $\chi \rtimes \pi$ the representation of $\mathrm{GSp}(4, F)$ obtained by normalized parabolic induction from the representation of Q on V given by

$$\begin{bmatrix} \lambda & * & * \\ & A & * \\ & & \lambda^{-1}det(A) \end{bmatrix} \longmapsto \chi(\lambda)\pi(A).$$

Then the standard model of this representations consists of all locally constant functions $f : \mathrm{GSp}(4, F) \to V$ with the transformation property:

$$f(hg) = \mid \lambda^2 \det(A)^{-1} \mid \chi(\lambda)\pi(A)f(g), \quad h \in Q, g \in G, \tag{1.20}$$

We again note that the modular character of Q is given by $\delta_Q(h) = |\lambda|^4 |\det(A)|^{-2}$. If π has central character ω_π, then the central character of $\chi \rtimes \pi$ is $\chi\omega_\pi$. Moreover, groups VII, VIII and IX in the Table 1.1 contain representations supported in Q, i.e., they are constituents of induced representations of the form $\chi \rtimes \pi$.

Siegel-induced representations. Let $P = MN$ be the Siegel parabolic subgroup, where

$$N = \{ \begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} : \ x, y, z \in F \}$$

and

$$M = \{ \begin{bmatrix} A & \\ & \lambda A' \end{bmatrix} : \ A \in \mathrm{GL}(2, F), \ \lambda \in F^{\times} \}, \quad \text{with } A' = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} {}^t A^{-1} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

Let (π, V) be an admissible representation of $\mathrm{GL}(2, F)$, and let σ be a character of $F^\times$. Then we denote by $\pi \rtimes \sigma$ the representation of $\mathrm{GSp}(4,F)$ obtained by normalized parabolic induction from the representation of P on V given by

$$\left[\begin{smallmatrix} A & \\ & \lambda A' \end{smallmatrix}\right] \longmapsto \sigma(\lambda)\pi(A).$$

Then the standard model of this representations consists of all locally constant functions $f : \mathrm{GSp}(4,F) \to V$ with the transformation property:

$$f(hg) = \mid \det(A)\lambda^{-1} \mid^{\frac{3}{2}} \sigma(\lambda)\pi(A)f(g), \quad h \in P, g \in G, \tag{1.21}$$

We denote that the modular character of P is given by $\delta_P(h) = |\det(A)|^3|\lambda|^{-3}$. If π has central character ω_π, then the central character of $\pi \rtimes \sigma$ is $\omega_\pi\sigma^2$. Moreover, groups X and XI in the Table 1.1 contain representations supported in P, i.e., they are constituents of induced representations of the form $\pi \rtimes \sigma$.

The "tempered" column in Table 1.1 gives the conditions on the inducing data under which a representation is tempered. The "L^2" column in Table 1.1 indicates which of the tempered representations are square-integrable. The "generic" column in Table 1.1 indicates the generic representations.

1.4 Main methods

Note that $\mathrm{Kl}(\mathfrak{p}^0) = \mathrm{GSp}(4,\mathfrak{o}) = K$, i.e., the hyperspecial maximal compact open subgroup of $\mathrm{GSp}(4,F)$. We already know the dimensional data for $\mathrm{Kl}(\mathfrak{p})$ and for $\mathrm{Kl}(\mathfrak{p}^0) = K$, see Table 3 in [22]. For $\mathrm{Kl}(\mathfrak{p}^2)$, the situation becomes much more complicated. In order to figure out the dimensional data of the spaces of $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for $\mathrm{GSp}(4,F)$, we use a variety of methods:

Table 1.1: Irreducible non-supercuspidal representations of $\mathrm{GSp}(4,F)$.

	constituents of		representation	tempered	L^2	g
I	$\chi_1 \times \chi_2 \rtimes \sigma$		(irreducible)	χ_i, σ unit.		•
II	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	χ, σ unit.		•
	$(\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2})$	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$			
III	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	π, σ unit.		•
	$(\chi \notin \{1, \nu^{\pm 2}\})$	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$			
IV	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	σ unit.	•	•
		b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$			
		c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$			
		d	$\sigma 1_{\mathrm{GSp}(4)}$			
V	$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	σ unit.	•	•
	$(\xi^2 = 1,\ \xi \neq 1)$	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$			
		c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$			
		d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$			
VI	$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$	a	$\tau(S, \nu^{-1/2}\sigma)$	σ unit.		•
		b	$\tau(T, \nu^{-1/2}\sigma)$	σ unit.		
		c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$			
		d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$			
VII	$\chi \rtimes \pi$		(irreducible)	χ, π unit.		•
VIII	$1_{F^\times} \rtimes \pi$	a	$\tau(S, \pi)$	π unit.		•
		b	$\tau(T, \pi)$	π unit.		
IX	$\nu\xi \rtimes \nu^{-1/2}\pi$	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	π unit.	•	•
	$(\xi \neq 1,\ \xi\pi = \pi)$	b	$L(\nu\xi, \nu^{-1/2}\pi)$			
X	$\pi \rtimes \sigma$		(irreducible)	π, σ unit.		•
XI	$\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	π, σ unit.	•	•
	$(\omega_\pi = 1)$	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$			

- Double coset decompositions. This method works for the full parabolically induced representations of $\mathrm{GSp}(4, F)$. For example, Proposition 3.7 gives the dimensional data for some Iwahori-spherical representations. On the other hand, Proposition 4.5 gives the dimensional data for the non Iwahori-spherical representations of group I.

- Intertwining operators. For some reducible parabolically induced representations which consist of several constituents, we use this method to determine how the fixed vectors distribute among the constituents of an induced representation. For instance, Proposition 3.14 shows the dimensional data for the Iwahori-spherical representations of groups V and VI.

- P_3-theory (see [15, section 2.5]). For some reducible parabolically induced representations, it seems impossible to calculate the intertwining operator directly. In this case, we use the method of P_3-theory instead to obtain the desired information. For example, we use this method to investigate the representations of group VIII as in Section 5.2.

- Hyperspecial parahoric restriction (see [19]) of depth zero representations. For some representations, none of the previous methods apply. In particular, we deal with the Siegel-induced representations and supercuspidal representations in the final Section 6.2. We first prove Lemma 4.2, which puts us into a depth zero situation. Then we consider the hyperspecial parahoric restriction of these kinds of representations to obtain the desired dimensional data. See Theorem 6.6 for more details.

1.5 Main result

In fact, after going through all irreducible admissible representations of $\mathrm{GSp}(4,F)$ by applying the above methods, we also obtain the dimensional data of the spaces of $\mathrm{M}(\mathfrak{p}^2)$-invariant vectors for $\mathrm{GSp}(4,F)$. Thus, we have the following main result.

Theorem 1.7. *The dimensions of the spaces of* $\mathrm{M}(\mathfrak{p}^2)$ *and* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for all irreducible, admissible representations of* $\mathrm{GSp}(4,F)$*, where* F *is a* $\mathfrak{p}$*-adic field, are given in* Table 1.2.

Table 1.2: Dimensions of the spaces of $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors.

		representation	inducing data	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$	χ_1, χ_2, σ unr.	**8**	**11**
		(irreducible)	$a(\sigma)=0, a(\chi_1)=a(\chi_2)=1$	2	3
			$a(\sigma)=1, a(\chi_1)=a(\chi_2)=0$	0	3
			$a(\sigma)=1, a(\chi_1)=1, a(\chi_2)=0$	0	2
			$a(\sigma)=1, a(\chi_1)=0, a(\chi_2)=1$	0	2
			$a(\sigma)=1, a(\chi_i)=1, a(\chi_i\sigma)=0$	2	3
			$a(\sigma)=1, a(\chi_i)=1, a(\chi_i\sigma)=1$	0	1
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	χ, σ unr.	**5**	**7**
			$a(\sigma)=0, a(\chi)=1$	2	3
			$a(\sigma)=1, a(\chi)=0$	0	2
			$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=0$	1	2
			$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=1$	0	1
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	χ, σ unr.	**3**	**4**
			$a(\sigma)=0, a(\chi)=1$	0	0

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Table 1.2 – *Continued from previous page*

		representation	inducing data	M($\mathfrak{p}^2$)	Kl($\mathfrak{p}^2$)
			$a(\sigma)=1, a(\chi)=0$	0	1
			$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=0$	1	1
			$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=1$	0	0
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	χ, σ unr.	**3**	**5**
			$a(\sigma)=1, a(\chi)=0$	0	2
			$a(\sigma)=1, a(\chi)=1$	0	1
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	χ, σ unr.	**5**	**6**
			$a(\sigma)=1, a(\chi)=0$	0	1
			$a(\sigma)=1, a(\chi)=1$	0	1
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	σ unr.	**1**	**2**
			$a(\sigma)=1$	0	1
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	σ unr.	**2**	**3**
			$a(\sigma)=1$	0	1
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	σ unr.	**4**	**5**
			$a(\sigma)=1$	0	1
	d	$\sigma 1_{\mathrm{GSp}(4)}$	σ unr.	**1**	**1**
			$a(\sigma)=1$	0	0
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	ξ, σ unr.	**3**	**5**
			$a(\sigma)=0, a(\xi)=1$	1	2
			$a(\sigma)=1, a(\xi)=0$	0	2
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	1	2
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	1
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	ξ, σ unr.	**2**	**2**

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Table 1.2 – *Continued from previous page*

		representation	inducing data	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
			$a(\sigma)=0, a(\xi)=1$	1	1
			$a(\sigma)=1, a(\xi)=0$	0	0
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	0	0
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	0
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	ξ, σ unr.	**2**	**2**
			$a(\sigma)=0, a(\xi)=1$	0	0
			$a(\sigma)=1, a(\xi)=0$	0	0
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	1	1
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	0
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	ξ, σ unr.	**1**	**2**
			$a(\sigma)=0, a(\xi)=1$	0	0
			$a(\sigma)=1, a(\xi)=0$	0	1
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	0	0
			$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	0
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	σ unr.	**3**	**5**
			$a(\sigma)=1$	0	2
	b	$\tau(T, \nu^{-1/2}\sigma)$	σ unr.	**0**	**0**
			$a(\sigma)=1$	0	0
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	σ unr.	**2**	**2**
			$a(\sigma)=1$	0	0
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	σ unr.	**3**	**4**
			$a(\sigma)=1$	0	1
VII		$\chi \rtimes \pi$	$a(\pi)=2, \chi$ unr.	0	2

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Table 1.2 – *Continued from previous page*

		representation	inducing data	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
		(irreducible)	$a(\pi)=2, a(\chi)=1$	0	1
VIII	a	$\tau(S,\pi)$	$a(\pi)=2$	0	2
	b	$\tau(T,\pi)$	$a(\pi)=2$	0	0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	$a(\pi)=2, \xi$ unr.	0	1
			$a(\pi)=2, a(\xi)=1$	0	1
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$a(\pi)=2, \xi$ unr.	0	1
			$a(\pi)=2, a(\xi)=1$	0	0
X		$\pi\rtimes\sigma$ (irreducible)	$a(\pi)=2, \sigma$ unr.	2	3
			$a(\pi)=2, a(\sigma)=1$	0	1
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$a(\pi)=2, \sigma$ unr.	1	2
			$a(\pi)=2, a(\sigma)=1$	0	1
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$a(\pi)=2, \sigma$ unr.	1	1
			$a(\pi)=2, a(\sigma)=1$	0	0
s.c.		generic	depth zero	0	1
		non-generic		0	0

Chapter 2

Preliminaries

In this chapter, we will start with a brief introduction of a type of Kirillov theory, called P_3-theory.

2.1 P_3-theory

First, the subgroup P_3 of $\mathrm{GL}(3,F)$ is defined as

$$P_3 := \begin{bmatrix} * & * & * \\ * & * & * \\ & & 1 \end{bmatrix}. \tag{2.1}$$

The representation theory of the analogous subgroup P_n of $\mathrm{GL}(n,F)$ plays an important role in the representation theory of $\mathrm{GL}(n,F)$, and there is an extensive theory of P_n smooth representations, see [3]. Let Z be the center of $\mathrm{GSp}(4)$. Then we have

$$Z(F) = \left\{ \begin{bmatrix} z & & & \\ & z & & \\ & & z & \\ & & & z \end{bmatrix}, z \in F^\times \right\} \cong F^\times. \tag{2.2}$$

Recall the center Z^J of the Jacobi group G^J as defined in (1.4), then we have

$$Z^J(F) := \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cong F. \tag{2.3}$$

We write Z for $Z(F)$ and Z^J for $Z^J(F)$ for simplicity, and similarly for other subgroups of $\mathrm{GSp}(4, F)$. The following lemma gives the connection to P_3 representations.

Lemma 2.1 ([15], Lemma 2.5.1)**.** *The group Z^J is a normal subgroup of Q. Moreover, there is a homomorphism $i\colon Q \longrightarrow P_3$ defined by*

$$i(\begin{bmatrix} ad-bc & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -y & x & z \\ & 1 & & x \\ & & 1 & y \\ & & & 1 \end{bmatrix} \begin{bmatrix} u & & & \\ & u & & \\ & & u & \\ & & & u \end{bmatrix}) = \begin{bmatrix} a & b & \\ c & d & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & x \\ & 1 & y \\ & & 1 \end{bmatrix}. \tag{2.4}$$

The kernel of i is $Z^J Z$, so that we get an isomorphism $Q/Z^J Z \cong P_3$.

Let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ with trivial central character. We define

$$V(Z^J) := \langle v - \pi(z)v \mid z \in Z^J, v \in V \rangle. \tag{2.5}$$

This is a $\mathbb{C}$ vector subspace of V. Then Q acts on $V(Z^J)$, so that Q acts on $V_{Z^J} := V/V(Z^J)$. Moreover, we get an action of $Q/Z^J Z$ on V_{Z^J} since Z and Z^J act trivially on V_{Z^J}. By Lemma 2.1, we thus obtain an action of P_3 on V_{Z^J}. Let $p\colon V \longrightarrow V_{Z^J}$ be the projection map. Then we have

$$p(\pi(q)v) = i(q)p(v), \quad \text{for } q \in Q \text{ and } v \in V. \tag{2.6}$$

Thus, we can treat V_{Z^J} as a representation of P_3. In particular, there are three subgroups corresponding to $\mathrm{GL}(0, F) = 1, \mathrm{GL}(1, F) = F^\times$ and $\mathrm{GL}(2, F)$. Then the representations of P_3 can be induced arise from representations of $\mathrm{GL}(0, F) =$

$1, \mathrm{GL}(1,F) = F^\times$ and $\mathrm{GL}(2,F)$.

- For the first group $\left[\begin{smallmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right]$, let Y be the vector space of a fixed smooth representation of $\mathrm{GL}(0,F)$. Define a unitary character Θ of this group by

$$\Theta(\left[\begin{smallmatrix} 1 & u_{12} & * \\ & 1 & u_{23} \\ & & 1 \end{smallmatrix}\right]) = \psi(u_{12} + u_{23})$$

  and let $Y \otimes \Theta$ be the smooth representation of this group defined by $u \cdot y = \Theta(u)y$ for u in this group and $y \in Y$. We consider the smooth representations

$$\tau^{P_3}_{\mathrm{GL}(0)}(Y) = \text{c-Ind}^{P_3}_{\left[\begin{smallmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right]}(Y \otimes \Theta) \cong (\dim Y) \cdot \text{c-Ind}^{P_3}_{\left[\begin{smallmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right]}\Theta.$$

  And the representation

$$\tau^{P_3}_{\mathrm{GL}(0)}(\mathbf{1}) = \text{c-Ind}^{P_3}_{\left[\begin{smallmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right]}\Theta \tag{2.7}$$

  is irreducible.

- For the second group $\left[\begin{smallmatrix} * & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right]$, let (χ, X) be a smooth representation of $\mathrm{GL}(1,F)$. Define a smooth representation $\chi \otimes \Theta$ of this group by letting $\chi \otimes \Theta$ have the same space X as χ and setting

$$(\chi \otimes \Theta)(\left[\begin{smallmatrix} a & * & * \\ & 1 & y \\ & & 1 \end{smallmatrix}\right]) = \psi(y)\chi(a).$$

  We consider the smooth representation

$$\tau^{P_3}_{\mathrm{GL}(1)}(\chi) = \text{c-Ind}^{P_3}_{\left[\begin{smallmatrix} * & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right]}(\chi \otimes \Theta) \tag{2.8}$$

of P_3. If χ is irreducible, i.e., χ is a character, then this representation is irreducible.

- For the last group $\left[\begin{smallmatrix} * & * & * \\ * & * & * \\ & & 1 \end{smallmatrix}\right]$, i.e., P_3, let ρ be a smooth representation of $\mathrm{GL}(2,F)$. Define a smooth representation $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ of P_3 by letting $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ have the same space as ρ and action defined by

$$\tau_{\mathrm{GL}(2)}^{P_3}(\rho)(\left[\begin{smallmatrix} a & b & * \\ c & d & * \\ & & 1 \end{smallmatrix}\right]) = \rho(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]).$$

If ρ is irreducible, then $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ is irreducible.

Theorem 2.2 ([15], Theorem 2.5.3)**.** *Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. The quotient $V_{Z^J} = V/V(Z^J)$ is a smooth representation of Q/Z^JZ, and hence via Lemma 2.1 defines a smooth representation of P_3. As a representation of P_3, V_{Z^J} has finite length. Hence, V_{Z^J} has a finite filtration by P_3 subspaces such that the successive quotients are irreducible and of the form $\tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}), \tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ or $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ for some character χ of $F^\times$, or some irreducible, admissible representation ρ of $\mathrm{GL}(2,F)$. Moreover, the following statements hold:*

i) There exists a chain of P_3 subspaces $0 \subset V_2 \subset V_1 \subset V_0 = V_{Z^J}$ such that:

$$\begin{aligned} V_2 &\cong \tau_{\mathrm{GL}(0)}^{P_3}(V_{U,\psi_{-1,1}}) \cong \dim \mathrm{Hom}_U(V, \psi_{-1,1}) \cdot \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}), \\ V_1/V_2 &\cong \tau_{\mathrm{GL}(1)}^{P_3}(V_{U,\psi_{-1,0}}), \\ V_0/V_1 &\cong \tau_{\mathrm{GL}(2)}^{P_3}(V_{N_Q}). \end{aligned}$$

Here, the complex vector space $V_{U,\psi_{-1,1}}$ defines a smooth representation of $\mathrm{GL}(0,F)$, the vector space $V_{U,\psi_{-1,0}}$ admits a smooth action of $\mathrm{GL}(1,F) \cong$

$F^\times$ induced by the operators

$$\pi(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix}), \qquad a \in F^\times,$$

and V_{N_Q} admits a smooth action of $\mathrm{GL}(2,F)$ induced by the operators

$$\pi(\begin{bmatrix} \det(g) & & \\ & g & \\ & & 1 \end{bmatrix}), \qquad g \in \mathrm{GL}(2,F).$$

ii) The representation π is generic if and only if $V_2 \neq 0$, and if π is generic, then $V_2 \cong \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$.

iii) We have $V_2 = V_{Z^J}$ if and only if π is supercuspidal. If π is supercuspidal and generic, then $V_{Z^J} = V_2 \cong \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$ is non-zero and irreducible. If π is supercuspidal and non-generic, then $V_{Z^J} = V_2 = 0$.

Moreover, Table A.5 and Table A.6 of [15] give the P_3-filtrations V_0/V_1 and V_1/V_2 for non-supercuspidal representations of $\mathrm{GSp}(4,F)$, respectively.

2.2 Twisted Jacquet-type modules

Let N be the unipotent radical of the Siegel parabolic. Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$. Similarly with the definitions of $V(Z^J)$ and V_{Z^J} in the previous section, we let

$$V(N) = \langle \pi(n)v - v \mid v \in V, n \in N \rangle \quad \text{and} \quad V_N = V/V(N) \tag{2.9}$$

be the usual Jacquet module with respect to N. To further define the twisted Jacquet modules as desired, we shall first define a character $\theta = \theta_{a,b,c}$ of N for some $a, b, c \in F$ by

$$\theta\left(\begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) = \psi(ax + by + cz) \tag{2.10}$$

for $x, y, z \in F$. Here, ψ is a fixed non-trivial additive character of F. Then the twisted Jacquet module $V_{N,\theta_{a,b,c}}$ is defined as follows

$$V_{N,\theta_{a,b,c}} = V/V(N, \theta_{a,b,c}), \text{ where } V(N, \theta_{a,b,c}) = \langle \pi(n)v - \theta_{a,b,c}(n)v \mid v \in V, n \in N \rangle.$$

Furthermore, we note that $V_{\text{▖}} := V_{N,\theta_{-1,0,0}}$ is a module for

$$T_{\text{▖}} := \left\{ \begin{bmatrix} a & & & \\ & 1 & & \\ & & 1 & \\ & & & a^{-1} \end{bmatrix} \mid a \in F^\times \right\} \cong F^\times, \tag{2.11}$$

and $V_{\text{▝}} := V_{N,\theta_{0,0,-1}}$ is a module for

$$T_{\text{▝}} := \left\{ \begin{bmatrix} 1 & & & \\ & a & & \\ & & a^{-1} & \\ & & & 1 \end{bmatrix} \mid a \in F^\times \right\} \cong F^\times. \tag{2.12}$$

As a immediate consequence, we have the following lemma.

Lemma 2.3. *The $F^\times$ modules $V_{\text{▖}}$ and $V_{\text{▝}}$ are isomorphic.*

Proof. It is easy to check

$$\pi(s_1)(V\left(\begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix}, \theta_{0,0,-1}\right)) = V\left(\begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix}, \theta_{-1,0,0}\right)$$

and $s_1 T_{\text{▖}} s_1^{-1} = T_{\text{▝}}$. Hence the map $v \mapsto \pi(s_1)v$ induces an isomorphism $V_{\text{▖}} \to V_{\text{▝}}$ that intertwines the actions of $T_{\text{▖}}$ and $T_{\text{▝}}$. $\square$

2.3 Jacobi representations

Let $(\pi^m_{SW}, \mathcal{S}(F))$ be the Schrödinger-Weil representation of the Jacobi group G^J as defined in (1.4) with central character ψ^m, where $\mathcal{S}(F)$ is the space of locally constant, compactly supported functions on F. By [2, section 2.5], the formulas for π^m_{SW} are as follows:

$$\pi^m_{SW}\left(\begin{bmatrix}1&&&\\&a&b&\\&c&d&\\&&&1\end{bmatrix}\right) f = \pi^m_W\left(\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right],1\right) f,$$

where the Weil representation π^m_W of $\widetilde{\mathrm{SL}}(2,F)$ satisfies

$$(\pi^m_W\left(\left[\begin{smallmatrix}1&b\\&1\end{smallmatrix}\right],\varepsilon\right) f)(x) =\varepsilon\psi(mbx^2)f(x), \tag{2.13}$$

$$(\pi^m_W\left(\left[\begin{smallmatrix}a&\\&a^{-1}\end{smallmatrix}\right],\varepsilon\right) f)(x) =\varepsilon\delta_\psi(a)(a,m)|a|^{\frac{1}{2}}f(ax), \tag{2.14}$$

$$\pi^m_W\left(\left[\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right],\varepsilon\right) f =\varepsilon\gamma_\psi(m)\hat{f}. \tag{2.15}$$

Here, the symbol δ_ψ denotes the Weil character as defined in [2, section 5.1], (a,m) is the Hilbert symbol and the notation γ_ψ indicates the Weil constant depending on the character ψ (or ψ^m). The Fourier transform in the last formula is given by

$$\hat{f}(x) = q^{-\frac{\nu(2m)}{2}}\int_F \psi(2mxy)f(y)\,dy.$$

Furthermore,

$$\pi^m_{SW}\left(\begin{bmatrix}1&\lambda&\mu&\kappa\\&1&&\mu\\&&1&-\lambda\\&&&1\end{bmatrix} f\right)(x) = \psi^m(\kappa+(2x+\lambda)\mu)f(x+\lambda). \tag{2.16}$$

Now we take $m = -1$. Let (τ, V) be a smooth representation of the Jacobi group G^J with central character ψ^{-1}. Then by Theorem 2.6.2 of [2] we have

$$\tau \cong \tilde{\tau} \otimes \pi_{SW}^{-1} \tag{2.17}$$

as G^J representations. Here, $(\tilde{\tau}, \widetilde{V})$ is a smooth representation of the metaplectic group $\widetilde{\mathrm{SL}}(2, F)$. And the complete list of irreducible, admissible, genuine representations $\tilde{\tau}$ of $\widetilde{\mathrm{SL}}(2, F)$ have been classified in [2, Section 5.3]. In particular, we would like to put the list as in the following proposition.

Proposition 2.4. *Recall that $m = -1$. The following is a complete list of irreducible, admissible, genuine representations of $\widetilde{\mathrm{SL}}(2, F)$.*

i) Those supercuspidal representations which are not equal to negative Weil representations.

ii) The principal series representations π_χ with $\chi^2 \neq |\ |^{\pm 1}$.

iii) The special representations σ_ξ with $\xi \in F^\times / F^{\times 2}$.

iv) The positive (even) Weil representations $\pi_{\mathrm{W}}^{\xi +}$ with $\xi \in F^\times / F^{\times 2}$.

v) The negative (odd) Weil representations $\pi_{\mathrm{W}}^{\xi -}$ with $\xi \in F^\times / F^{\times 2}$.

The action of G^J on the tensor product $\widetilde{V} \otimes \mathcal{S}(F)$ is given by

$$\tau\left(\begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix}\right)(v \otimes f) = (\tilde{\tau}(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], 1)v) \otimes (\pi_{SW}^{-1}\left(\begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix}\right) f) \tag{2.18}$$

for $v \in \widetilde{V}$ and $f \in \mathcal{S}(F)$. Moreover, Theorem 5.8.3 of [2] gives a complete list of the irreducible, admissible representations of $G^J(F)$ with non-trivial central character ψ^m.

The following lemma is a special case of Proposition 5.7.1 of [2].

Lemma 2.5. *As above, let $V = \widetilde{V} \otimes \mathcal{S}(F)$ be a smooth representation of G^J for which the center acts via the character ψ^{-1}. Let $\ell : \widetilde{V} \to \mathbb{C}$ be a linear functional with the property*

$$\ell(\widetilde{\tau}(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right], 1)v) = \ell(v) \qquad \textit{for all } x \in F,$$

so that ℓ can be identified with a linear functional $\overline{\ell}$ on $\widetilde{V}_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]}$. Then the map

$$\ell^J : \widetilde{V} \otimes \mathcal{S}(F) \longrightarrow \mathbb{C}, \qquad \ell^J(v \otimes f) = \ell(v) f(0)$$

satisfies

$$\ell^J(\tau\left(\left[\begin{smallmatrix} 1 & & \mu & \\ & 1 & x & \mu \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]\right)(v \otimes f)) = \ell^J(v \otimes f). \tag{2.19}$$

and can therefore be identified with a linear functional $\overline{\ell}^J$ on $V_{\left[\begin{smallmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}$. The map that sends $\overline{\ell}$ to $\overline{\ell}^J$ defines an isomorphism of $\mathbb{C}$ vector spaces

$$\mathrm{Hom}(\widetilde{V}_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]}, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}(V_{\left[\begin{smallmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}, \mathbb{C}).$$

Assume that there exists a character χ of $F^\times$ such that $\overline{\ell}$ satisfies

$$\overline{\ell}(\tilde{\tau}(\left[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right], \varepsilon)v) = \varepsilon \delta_\psi(a) \chi(a) |a| \overline{\ell}(v) \quad \textit{for all } v \in \widetilde{V}_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]} \tag{2.20}$$

and $a \in F^\times$. Then

$$\overline{\ell}^J\left(\left[\begin{smallmatrix} 1 & & & \\ & a & & \\ & & a^{-1} & \\ & & & 1 \end{smallmatrix}\right] v\right) = \chi(a) |a|^{\frac{3}{2}} \overline{\ell}^J(v) \qquad \textit{for all } v \in V_{\left[\begin{smallmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]} \tag{2.21}$$

and $a \in F^\times$.

Proof. The (2.19) follows from (2.13) and (2.16). It is clear that our map is injective. Let L be in $\mathrm{Hom}(V_{\left[\begin{smallmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}, \mathbb{C})$, and also denote by L the composition of L with the quotient map

$$V \longrightarrow V_{\left[\begin{smallmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}. \tag{2.22}$$

We now follow the argument from page 128-9 of[2]. Fix $v \in \widetilde{V}$. Define L_v : $\mathcal{S}(F) \to \mathbb{C}$ by $L_v(f) = L(v \otimes f)$. We have

$$\begin{aligned} L_v(\pi_{SW}\left(\begin{bmatrix} 1 & & \mu & \\ & 1 & & \mu \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) f) =& L(v \otimes \pi_{SW}\left(\begin{bmatrix} 1 & & \mu & \\ & 1 & & \mu \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) f) \\ =& L(\begin{bmatrix} 1 & & \mu & \\ & 1 & & \mu \\ & & 1 & \\ & & & 1 \end{bmatrix}(v \otimes f)) \\ =& L(v \otimes f) = L_v(f). \end{aligned}$$

It follows that L_v is an element of $\mathrm{Hom}_{\left[\begin{smallmatrix} 1 & & * & \\ & 1 & & * \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}((\pi_{SW}, \mathcal{S}(F)), \mathbb{C})$. By iii) of Lemma 5.1.1 of [16], this space is one-dimensional and spanned by the map defined by $f \mapsto f(0)$. Therefore, there exists $\ell(v) \in \mathbb{C}$ such that $L_v(f) = \ell(v)f(0)$ for $f \in \mathcal{S}(F)$; moreover, it is clear that $\ell(v)$ is the unique element of $\mathbb{C}$ with this property. We have proven that for every $v \in V$, there exists a unique $\ell(v) \in \mathbb{C}$ such that

$$L(v \otimes f) = \ell(v)f(0)$$

for all $f \in \mathcal{S}(F)$. Calculations using this formula imply that $\ell : \widetilde{V} \longrightarrow \mathbb{C}$ defined by $v \mapsto \ell(v)$ is a linear map, and that in fact

$$\ell(\tilde{\tau}([\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}], 1)v) = \ell(v)$$

for all $x \in F$ and $v \in V$. Therefore, $\overline{\ell}$ is in $\mathrm{Hom}(\widetilde{V}_{[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}]}, \mathbb{C})$. Since $\overline{\ell}$ maps to L, our map is surjective are claimed.

The statement (2.21) follows from a straightforward calculation using (2.14).

□

2.4 Hyperspecial parahoric restriction

Let $K = \mathrm{GSp}(4, \mathfrak{o})$ be the hyperspecial parahoric subgroup of $G = \mathrm{GSp}(4, F)$. Let $\Gamma(\mathfrak{p})$ be the principal congruence subgroup of level $\mathfrak{p}$ defined as

$$\Gamma(\mathfrak{p}) := \left[\begin{smallmatrix} 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right].$$

Definition 2.6. A smooth representation (π, V) of $\mathrm{GSp}(4, F)$ is a *depth zero* representation if the space of $\Gamma(\mathfrak{p})$-invariant vectors is non-zero.

For more information about the depth of an irreducible admissible representation of an algebraic group over a $\mathfrak{p}$-adic field, see [13, section 5].

With a similar process of the construction of supercuspidal representation for $\mathrm{GL}(2, F)$ from a cuspidal representation $\mathrm{GL}(2, \mathfrak{o}/\mathfrak{p})$ in [5, section 4.8], we can obtain a supercuspidal representation of $\mathrm{GSp}(4, F)$ from a compact induction with respect to the subgroup ZK, where Z is the center of $\mathrm{GSp}(4, F)$.

Lemma 2.7. *Let* $\pi \cong \text{c-Ind}_{ZK}^{G}(\tau)$ *be a depth zero supercuspidal irreducible admissible representation of* G*, where* τ *is an extension to* ZK *of an irreducible* *cuspidal admissible representation σ of $K/\Gamma(\mathfrak{p}) = \mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p})$. The hyperspeical parahoric restriction $r_K(\pi)$ is irreducible and isomorphic to σ.*

Proof. This is a special case of Lemma 2.18 of [19]. □

Therefore, this lemma give us an effective approach to studying the depth zero supercuspidal irreducible admissible representations of $\mathrm{GSp}(4, F)$ by investigating the cuspidal irreducible admissible representations of finite group $\mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p})$.

Furthermore, by Theorem 2.19 of [19] we have the result that the hyperspecial parahoric restriction commutes with parabolic induction. For example, let π be a Siegel-induced representation of $\mathrm{GSp}(4, F)$, i.e., π is of type X, XIa or XIb. Then π is a subquotient of an induced representation of the form $\tau \rtimes \sigma$, where τ is a supercuspidal representation of $\mathrm{GL}(2, F)$, and σ is a character of $\mathrm{GL}(1, F) = F^\times$. Then we have

$$r_K(\tau \rtimes \sigma) \cong r_{\mathrm{GL}(2,\mathfrak{o})}(\tau) \rtimes r_{\mathrm{GL}(1,\mathfrak{o})}(\sigma).$$

Evidently, we can get the similar conclusions for Borel- and Klingen-induced representations.

Chapter 3

Iwahori-spherical representations

An admissible representation (π, V) of $G(F) = \mathrm{GSp}(4, F)$ is called ***Iwahori-spherical*** if it has non-zero I-invariant vectors, where I is the Iwahori subgroup defined as follows

$$I := \mathrm{GSp}(4, \mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{3.1}$$

Note that the Iwahori-spherical representations are exactly the constituents of the Borel-induced representations with an unramified character of $B(F)$. This fact follows from a basic result of Borel and Casselman; see Lemma 4.7 of [4] and Theorem 3.3.3. of [9]. For more information about the Iwahori-spherical representations of $\mathrm{GSp}(4, F)$; see [22, section 1.3].

In this chapter, we determine the dimensions of the spaces of $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for all Iwahori-spherical representations of $\mathrm{GSp}(4, F)$. Main methods are double coset decompositions and intertwining operators.

3.1 Double coset representatives

Recall the definition of the *middle subgroup*

$$\mathrm{M}(\mathfrak{p}^2) := \{g \in \mathrm{GSp}(4,F) \mid \det(g) \in \mathfrak{o}^\times\} \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{o} \end{bmatrix}. \tag{3.2}$$

We also define the element ι as

$$\iota := \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix}. \tag{3.3}$$

Then we conjugate the middle subgroup $\mathrm{M}(\mathfrak{p}^2)$ by this element. In particular,

$$\mathrm{M}(\mathfrak{p}^2)^\iota := \iota \cdot \mathrm{M}(\mathfrak{p}^2) \cdot \iota^{-1} = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{3.4}$$

Let $K = \mathrm{GSp}(4,\mathfrak{o})$. By the Iwasawa decomposition we have

$$\mathrm{GSp}(4,F) = B(F)K, \tag{3.5}$$

where $B(F)$ is the Borel subgroup. Then we consider the double cosets

$$B(\mathfrak{o})\backslash \mathrm{GSp}(4,\mathfrak{o})/\mathrm{M}(\mathfrak{p}^2)^\iota, \tag{3.6}$$

where $B(\mathfrak{o})$ is the Iwahori subgroup I defined as in (3.1). Furthermore, we have the following isomorphism

$$B(\mathfrak{o})\backslash \mathrm{GSp}(4,\mathfrak{o})/\mathrm{M}(\mathfrak{p}^2)^\iota \xrightarrow{\simeq} B(\mathfrak{o}/\mathfrak{p})\backslash \mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p})/H, \tag{3.7}$$

where $H = \mathrm{M}(\mathfrak{p}^2)^\iota/\Gamma(\mathfrak{p})$. Thus we further have the following isomorphism

$$B(F)\backslash \mathrm{GSp}(F)/\mathrm{M}(\mathfrak{p}^2)^\iota \simeq B(\mathfrak{o}/\mathfrak{p})\backslash \mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p})/H. \tag{3.8}$$

Using the Bruhat decomposition for the finite group $\mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p})$, we obtain the following proposition.

Proposition 3.1. *A complete and minimal set of representatives for the double cosets $B(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)^\iota$ is given by the following 8 elements.*

$$\mathbf{I}_4, \quad s_1, \quad s_1\begin{bmatrix} 1&1&&\\ &1&&\\ &&1&-1\\ &&&1 \end{bmatrix}, \quad s_2s_1,$$
$$s_2s_1\begin{bmatrix} 1&1&&\\ &1&&\\ &&1&-1\\ &&&1 \end{bmatrix}, \quad s_1s_2s_1, \quad s_1s_2s_1\begin{bmatrix} 1&1&&\\ &1&&\\ &&1&-1\\ &&&1 \end{bmatrix}, \quad s_1s_2s_1\begin{bmatrix} 1&&1&\\ &1&&1\\ &&1&\\ &&&1 \end{bmatrix}.$$

Proof. First, by the Bruhat decomposition $G = \bigsqcup_{w\in W} BwB$. It follows that

$$B\backslash G = \bigsqcup_{w\in W} w\begin{bmatrix} 1&*&*&*\\ &1&*&*\\ &&1&*\\ &&&1 \end{bmatrix}. \tag{3.9}$$

By some straightforward calculation, we have

$$\begin{aligned} B(\mathfrak{o}/\mathfrak{p})\backslash \mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p}) = {}& \mathbf{I}_4 \cup s_1\begin{bmatrix} 1&*&&\\ &1&&\\ &&1&*\\ &&&1 \end{bmatrix} \cup s_2\begin{bmatrix} 1&&&\\ &1&*&\\ &&1&\\ &&&1 \end{bmatrix} \cup s_1s_2\begin{bmatrix} 1&&*&\\ &1&*&*\\ &&1&\\ &&&1 \end{bmatrix} \\ & \cup s_2s_1\begin{bmatrix} 1&*&&*\\ &1&&\\ &&1&*\\ &&&1 \end{bmatrix} \cup s_1s_2s_1\begin{bmatrix} 1&*&*&*\\ &1&&*\\ &&1&*\\ &&&1 \end{bmatrix} \cup s_2s_1s_2\begin{bmatrix} 1&&*&*\\ &1&*&*\\ &&1&\\ &&&1 \end{bmatrix} \cup s_1s_2s_1s_2\begin{bmatrix} 1&*&*&*\\ &1&*&*\\ &&1&*\\ &&&1 \end{bmatrix}. \end{aligned}$$

Furthermore, for $x, y, z \in \mathfrak{o}/\mathfrak{p}$ we have

$$B(\mathfrak{o}/\mathfrak{p})\backslash \mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p})/H = \mathbf{I}_4 \cup s_1\begin{bmatrix} 1&x&&\\ &1&&\\ &&1&-x\\ &&&1 \end{bmatrix} \cup s_2s_1\begin{bmatrix} 1&x&&z\\ &1&&\\ &&1&-x\\ &&&1 \end{bmatrix} \cup s_1s_2s_1\begin{bmatrix} 1&x&y&z\\ &1&&y\\ &&1&-x\\ &&&1 \end{bmatrix}. \tag{3.10}$$

Here, we omit the details since they are elementary. Furthermore, for most cases,

$x, y, z \in \mathfrak{o}/\mathfrak{p}$ above can be reduced to 0 or 1. In fact, it easily follows from straightforward calculations that the right hand side of (3.10) can be reduced to the following 16 elements.

$$\mathbf{I}_4, \quad s_1, \quad s_1 B_1, \quad s_2 s_1, \quad s_2 s_1 B_2, \quad s_2 s_1 B_1, \quad s_2 s_1 C_1,$$
$$s_1 s_2 s_1, \quad s_1 s_2 s_1 B_2, \quad s_1 s_2 s_1 B_1, \quad s_1 s_2 s_1 C_1,$$
$$s_1 s_2 s_1 B_3, \quad s_1 s_2 s_1 C_2, \quad s_1 s_2 s_1 C_3, \quad s_1 s_2 s_1 D_1, \quad s_1 s_2 s_1 D_2.$$

Here, we let

$$B_1 = \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

and

$$C_1 = \begin{bmatrix} 1 & 1 & & 1 \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 & 1 & \\ & 1 & & 1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & & 1 & 1 \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

and

$$D_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 1 & 1 & w \\ & 1 & & 1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \quad w \neq 0, 1.$$

Next, we are going to find all the equivalent classes in $B(\mathfrak{o}/\mathfrak{p})\backslash \mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p})/H$. In fact, we have the following equivalent relations.

1. The element $s_1 s_2 s_1 B_2$ is equivalent to $\mathbf{I}_4$. In particular,

$$\begin{bmatrix} 1 & & & -1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \mathbf{I}_4 = s_1 s_2 s_1 B_2 \cdot \begin{bmatrix} -1 & & & \\ & 1 & 1 & \\ & & 1 & \\ 1 & & & -1 \end{bmatrix}. \tag{3.11}$$

2. The element $s_2 s_1 B_2$ is equivalent to s_1. In particular,

$$\begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot s_1 = s_2 s_1 B_2 \cdot \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & -1 \end{bmatrix}. \tag{3.12}$$

3. The element $s_1 s_2 s_1 C_3$ is equivalent to $s_1 B_1$. In particular,

$$\begin{bmatrix} -1 & 1 & -1 & \\ & 1 & & 1 \\ & & 1 & 1 \\ & & & -1 \end{bmatrix} \cdot s_1 B_1 = s_1 s_2 s_1 C_3 \cdot \begin{bmatrix} -1 & & & \\ & 1 & 1 & \\ & & 1 & \\ 1 & & & -1 \end{bmatrix}. \tag{3.13}$$

4. The element $s_2 s_1 C_1$ is equivalent to $s_2 s_1 B_1$. In particular,

$$\begin{bmatrix} 1 & -1 & & -1 \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \cdot s_2 s_1 B_1 = s_2 s_1 C_1 \cdot \begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{3.14}$$

5. The element $s_1 s_2 s_1 C_1$ is equivalent to $s_2 s_1 B_1$. In particular,

$$\begin{bmatrix} 1 & 1 & 1 & \\ & -1 & & -1 \\ & & -1 & 1 \\ & & & 1 \end{bmatrix} \cdot s_2 s_1 B_1 = s_1 s_2 s_1 C_1 \cdot \begin{bmatrix} 1 & & & \\ & 0 & -1 & \\ & 1 & 1 & \\ -1 & & & 1 \end{bmatrix}. \tag{3.15}$$

6. The element $s_1 s_2 s_1 D_1$ is equivalent to $s_2 s_1 B_1$. In particular,

$$\begin{bmatrix} 1 & 1 & 1 & \\ & -1 & 1 & -2 \\ & & -1 & 1 \\ & & & 1 \end{bmatrix} \cdot s_2 s_1 B_1 = s_1 s_2 s_1 D_1 \cdot \begin{bmatrix} 1 & & & \\ & -1 & -2 & \\ & 1 & 1 & \\ -1 & & & 1 \end{bmatrix}. \tag{3.16}$$

7. The element $s_1 s_2 s_1 D_2$ is equivalent to $s_2 s_1 B_1$. In particular,

$$\begin{bmatrix} \frac{1}{w} & 1 & \frac{1}{w} & \\ & -w & \frac{1}{w} & -w-1 \\ & & -\frac{1}{w} & 1 \\ & & & w \end{bmatrix} \cdot s_2 s_1 B_1 = s_1 s_2 s_1 D_2 \cdot \begin{bmatrix} 1 & & & \\ & -\frac{1}{w} & -w-1 & \\ & \frac{1}{w} & 1 & \\ -\frac{1}{w} & & & 1 \end{bmatrix}. \tag{3.17}$$

8. The element $s_1 s_2 s_1 C_2$ is equivalent to $s_1 s_2 s_1 B_1$. In particular,

$$\begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot s_1 s_2 s_1 B_1 = s_1 s_2 s_1 C_2 \cdot \begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{3.18}$$

□

Corollary 3.2. *Let $G(F) = \mathrm{GSp}(4, F)$ and P, Q be the two parabolic subgroups* *as defined in* (1.3).

1. *For the double cosets* $P(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)^\iota$*, a complete and minimal set of representatives is given by the following* 3 *elements.*

$$\mathbf{I}_4, \quad s_2s_1, \quad s_2s_1\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

2. *For the double cosets* $Q(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)^\iota$*, a complete and minimal set of representatives is given by the following* 5 *elements.*

$$\mathbf{I}_4, \quad s_1, \quad s_1\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \quad s_1s_2s_1, \quad s_1s_2s_1\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

Proof. The assertions easily follow from the Proposition 3.1. □

Proposition 3.3. *Let* $G(F) = \mathrm{GSp}(4, F)$ *and* B, P, Q *be three parabolic subgroups as defined in* (1.3).

i) *For the double cosets* $B(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)$*, a complete and minimal set of representatives is given by the following 8 elements.*

$$\mathbf{I}_4, \quad s_1, \quad \begin{bmatrix} 1 & & & \\ \varpi & 1 & & \\ & & 1 & \\ & & -\varpi & 1 \end{bmatrix}, \quad s_2s_1,$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}, \quad s_1s_2s_1, \quad s_1\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}, \quad s_2s_1\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}.$$

ii) *For the double cosets* $P(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)$*, a complete and minimal set of representatives is given by the following 3 elements.*

$$\mathbf{I}_4, \quad s_2s_1, \quad \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}.$$

iii) For the double cosets $Q(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)$, a complete and minimal set of representatives is given by the following 5 elements.

$$\mathbf{I}_4, \quad s_1, \quad \begin{bmatrix} 1 &&& \\ \varpi & 1 && \\ && 1 & \\ && -\varpi & 1 \end{bmatrix}, \quad s_1 s_2 s_1, \quad s_1 \begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi && 1 & \\ & \varpi && 1 \end{bmatrix}.$$

Before giving the proof of this proposition, we need the following lemma.

Proposition 3.4. *With ι defined as in (3.3), we have*

$$\iota^{-1} s_1 \iota = \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi && \\ && \varpi^{-1} & \\ &&& \varpi \end{bmatrix} \cdot s_1, \qquad \iota^{-1} s_2 \iota = s_2, \tag{3.19}$$

$$\iota^{-1} \begin{bmatrix} 1 & 1 && \\ & 1 && \\ && 1 & -1 \\ &&& 1 \end{bmatrix} \iota = \begin{bmatrix} 1 & \varpi^{-1} && \\ & 1 && \\ && 1 & -\varpi^{-1} \\ &&& 1 \end{bmatrix}, \qquad \iota^{-1} \begin{bmatrix} 1 && 1 & \\ & 1 && 1 \\ && 1 & \\ &&& 1 \end{bmatrix} \iota = \begin{bmatrix} 1 && \varpi^{-1} & \\ & 1 && \varpi^{-1} \\ && 1 & \\ &&& 1 \end{bmatrix}. \tag{3.20}$$

Proof. The assertion easily follows from the straightforward calculations. □

Proof of Proposition 3.3. We observe that

$$\mathrm{M}(\mathfrak{p}^2)^\iota := \begin{bmatrix} \varpi &&& \\ & 1 && \\ && 1 & \\ &&& \varpi^{-1} \end{bmatrix} \mathrm{M}(\mathfrak{p}^2) \begin{bmatrix} \varpi^{-1} &&& \\ & 1 && \\ && 1 & \\ &&& \varpi \end{bmatrix} = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{3.21}$$

By the Iwasawa decomposition $\mathrm{GSp}(4,F) = B(F)K$ with $K = \mathrm{GSp}(4,\mathfrak{o})$, we have

$$B(F)\backslash\mathrm{GSp}(4,F)/\mathrm{M}(\mathfrak{p}^2)^\iota \simeq B(\mathfrak{o})\backslash\mathrm{GSp}(4,\mathfrak{o})/\mathrm{M}(\mathfrak{p}^2)^\iota \simeq B(\mathfrak{o}/\mathfrak{p})\backslash\mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p})/H, \tag{3.22}$$

where $H = \mathrm{M}(\mathfrak{p}^2)^\iota/\Gamma(\mathfrak{p})$. The second isomorphism follows from taking the quotient by the principal congruence subgroup $\Gamma(\mathfrak{p})$. Using the Bruhat decomposition for the finite group $\mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p})$, straightforward calculations show that the double coset space on the right of (3.22) has eight elements. Conjugation back by ι, we obtain the asserted representatives given in part i). Parts ii) and iii) follow easily from part i). □

As an immediate consequence of this result, we obtain the dimension of the space $V^{\mathrm{M}(\mathfrak{p}^2)}$ of $\mathrm{M}(\mathfrak{p}^2)$-invariant vectors in certain induced representations. More precisely,

i) Let χ_1, χ_2 and σ be unramified characters of $F^\times$, and let V be the standard space of the Borel induced representation $\chi_1 \times \chi_2 \rtimes \sigma$. Then Proposition 3.3 i) implies that $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 8$.

ii) Similarly, let χ and σ be unramified characters of $F^\times$, and let V be the standard space of the Siegel induced representation $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$. Then Proposition 3.3 ii) implies that $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 3$.

iii) Finally, let χ and σ be as in ii), and let V be the standard space of the Klingen induced representation $\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$. Then Proposition 3.3 iii) implies that $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 5$.

We use a similar process as in the previous section. First, we conjugate $\mathrm{Kl}(\mathfrak{p}^2)$ by ι and we have

$$\mathrm{Kl}(\mathfrak{p}^2)^\iota := \iota \cdot \mathrm{Kl}(\mathfrak{p}^2) \cdot \iota^{-1} = \mathrm{GSp}(4, \mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{3.23}$$

Proposition 3.5. *With $G(F)$ and B, P, Q as in Proposition 3.3, we have*

i) For the double cosets $B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)^\iota$, a complete and minimal set of *representatives is given by the following* 11 *elements.*

$$\mathbf{I}_4, \quad s_1, \quad s_1 \begin{bmatrix}1&1&&\\&1&&\\&&1&-1\\&&&1\end{bmatrix}, \quad s_2s_1,$$

$$s_2s_1 \begin{bmatrix}1&1&&\\&1&&\\&&1&-1\\&&&1\end{bmatrix}, \quad s_2s_1 \begin{bmatrix}1&&&\varpi\\&1&&\\&&1&\\&&&1\end{bmatrix}, \quad s_1s_2s_1, \quad s_1s_2s_1 \begin{bmatrix}1&1&&\\&1&&\\&&1&-1\\&&&1\end{bmatrix},$$

$$s_1s_2s_1 \begin{bmatrix}1&&1&\\&1&&1\\&&1&\\&&&1\end{bmatrix}, \quad s_1s_2s_1 \begin{bmatrix}1&&&\varpi\\&1&&\\&&1&\\&&&1\end{bmatrix}, \quad s_1s_2s_1 \begin{bmatrix}1&&1&\\&1&&1\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\varpi\\&1&&\\&&1&\\&&&1\end{bmatrix}.$$

ii) For the double cosets $P(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)^\iota$*, a complete and minimal set of representatives is given by the following* 4 *elements.*

$$\mathbf{I}_4, \quad s_2s_1, \quad s_2s_1 \begin{bmatrix}1&1&&\\&1&&\\&&1&-1\\&&&1\end{bmatrix}, \quad s_2s_1 \begin{bmatrix}1&&&\varpi\\&1&&\\&&1&\\&&&1\end{bmatrix}.$$

iii) For the double cosets $Q(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)^\iota$*, a complete and minimal set of representatives is given by the following* 6 *elements.*

$$\mathbf{I}_4, \quad s_1, \quad s_1 \begin{bmatrix}1&1&&\\&1&&\\&&1&-1\\&&&1\end{bmatrix}, \quad s_1s_2s_1, \quad s_1s_2s_1 \begin{bmatrix}1&1&&\\&1&&\\&&1&-1\\&&&1\end{bmatrix}, \quad s_1s_2s_1 \begin{bmatrix}1&&&\varpi\\&1&&\\&&1&\\&&&1\end{bmatrix}.$$

Proof. Let r_i be the representatives of $B(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)^\iota$ in Proposition 3.1 and we define the matrix F as

$$F := \begin{bmatrix}1&&&\varpi\\&1&&\\&&1&\\&&&1\end{bmatrix} \in \mathrm{M}(\mathfrak{p}^2)^\iota. \tag{3.24}$$

It follows from (3.4) and (3.23) that

$$G(F) = \bigsqcup_{i=1}^{8} B(F) r_i \mathrm{M}(\mathfrak{p}^2)^\iota = \bigcup_{\epsilon\in\{0,1\}} \bigcup_{i=1}^{8} B(F) r_i \begin{bmatrix}1&&&\epsilon\varpi\\&1&&\\&&1&\\&&&1\end{bmatrix} \mathrm{Kl}(\mathfrak{p}^2)^\iota. \tag{3.25}$$

It is easy to show that r_i is not equivalent to r_jF if $i \neq j$ in the double cosets $B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)^\iota$ with F defined as in (3.24). Next, we need to check whether r_i is equivalent to r_iF for any $i \in \{1, 2, \cdots, 8\}$. With a similar method as in the proof of Proposition 3.1, we have

1. For $r_1 = \mathbf{I}_4$, we have $\begin{bmatrix} 1&&&\varpi\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix} \cdot r_1 = r_1F \cdot \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix}$.
2. For $r_2 = s_1$, we have $\begin{bmatrix} 1&&&\\ &1&\varpi&\\ &&1&\\ &&&1 \end{bmatrix} \cdot r_2 = r_2F \cdot \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix}$.
3. For $r_3 = s_1 \begin{bmatrix} 1&1&&\\ &1&&\\ &&1&-1\\ &&&1 \end{bmatrix}$, we have $\begin{bmatrix} 1&&&\\ &1&\varpi&\\ &&1&\\ &&&1 \end{bmatrix} \cdot r_3 = r_3F \cdot \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix}$.
4. For $r_5 = s_2s_1 \begin{bmatrix} 1&1&&\\ &1&&\\ &&1&-1\\ &&&1 \end{bmatrix}$, we have $\begin{bmatrix} 1&&&\varpi\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix} \cdot r_5 = r_5F \cdot \begin{bmatrix} 1&&-\varpi&\\ &1&\varpi&-\varpi\\ &&1&\\ &&&1 \end{bmatrix}$.
5. For $r_7 = s_1s_2s_1 \begin{bmatrix} 1&1&&\\ &1&&\\ &&1&-1\\ &&&1 \end{bmatrix}$, we have $\begin{bmatrix} 1&&&\\ &1&\varpi&\\ &&1&\\ &&&1 \end{bmatrix} \cdot r_7 = r_7F \cdot \begin{bmatrix} 1&&-\varpi&\\ &1&\varpi&-\varpi\\ &&1&\\ &&&1 \end{bmatrix}$.

The assertions of parts ii) and iii) easily follow from part i). □

Proposition 3.6. *With $G(F)$ and B, P, Q as in Proposition 3.3, we have*

i) For the double cosets $B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$, a complete and minimal set of representatives is given by the following 11 elements.

$$\mathbf{I}_4,\quad s_1,\quad \begin{bmatrix} 1&&&\\ \varpi&1&&\\ &&1&\\ &&-\varpi&1 \end{bmatrix},\quad s_2s_1,\quad \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix},\quad s_1s_2s_1,\quad s_1\begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix},$$
$$s_2s_1\begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix},\quad s_1\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix},\quad \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix},\quad s_1\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix}.$$

ii) For the double cosets $P(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$, a complete and minimal set of representatives is given by the following 4 elements.

$$\mathbf{I}_4,\quad s_2s_1,\quad \begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix},\quad \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix}.$$

iii) *For the double cosets $Q(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$, a complete and minimal set of representatives is given by the following 6 elements.*

$$\mathbf{I}_4, \quad s_1, \quad \begin{bmatrix} 1&&&\\ \varpi&1&&\\ &&1&\\ &&-\varpi&1 \end{bmatrix}, \quad s_1s_2s_1, \quad s_1\begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix}, \quad \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix}.$$

Proof. The proof is analogous to that of Proposition 3.3. In fact, we let r_i be the representatives of $B(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)$ in Proposition 3.3, then

$$G(F) = \bigsqcup_{i=1}^{8} B(F)r_i\mathrm{M}(\mathfrak{p}^2) = \bigcup_{\epsilon\in\{0,1\}}\bigcup_{i=1}^{8} B(F)r_i\begin{bmatrix} 1&&&\epsilon\varpi^{-1}\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix}\mathrm{Kl}(\mathfrak{p}^2). \tag{3.26}$$

After checking for equalities among the above double cosets, we get a complete and minimal set of representatives for the double cosets $B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$ which has order 11. Then we go through a similar process as in the proof of Proposition 3.3 to obtain the desired assertions. □

Again, as an immediate consequence of this result, we obtain the dimension of the space $V^{\mathrm{Kl}(\mathfrak{p}^2)}$ of $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors in certain induced representations. More precisely,

i) Let χ_1, χ_2 and σ be unramified characters of $F^\times$, and let V be the standard space of the Borel induced representation $\chi_1 \times \chi_2 \rtimes \sigma$. Then Proposition 3.6 i) implies that $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = 11$.

ii) Similarly, let χ and σ be unramified characters of $F^\times$, and let V be the standard space of the Siegel induced representation $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$. Then Proposition 3.6 ii) implies that $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = 4$.

iii) Finally, let χ and σ be as in ii), and let V be the standard space of the Klingen induced representation $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$. Then Proposition 3.6 iii)

Table 3.1: Dimensions of the spaces of $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for Iwahori-spherical representations of groups I to IV.

		constituent of	representation	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$	(irreducible)	8	11
II	a	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	5	7
	b	$(\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2})$	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	3	4
III	a	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	3	5
	b	$(\chi \notin \{1, \nu^{\pm 2}\})$	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	5	6
IV	a	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	1	2
	b		$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	3
	c		$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	4	5
	d		$\sigma 1_{\mathrm{GSp}(4)}$	1	1

implies that $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = 6$.

3.2 Dimensions of the spaces of fixed vectors for groups I to IV

Proposition 3.7. *Table 3.1 shows the dimensions of the spaces of* $\mathrm{M}(\mathfrak{p}^2)$ *and* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for the irreducible, admissible representations* (π, V) *of groups* I *to* IV*. Here,* χ_1, χ_2, χ *and* σ *are unramified characters of* $F^\times$*.*

Proof. The results of groups I to III follow from Proposition 3.3 and Proposition 3.6. As for group IV, we consider the representation $\sigma\mathbf{1}_{\mathrm{GSp}(4)}$ of type IVd, which always has dimension 1. This holds for both the $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$ case. Moreover, by (2.9) in [15, section 2.2] we have the following relations

$$\dim(\mathrm{IVb})^\Gamma + \dim(\mathrm{IVd})^\Gamma = \dim(\mathrm{IIb})^\Gamma, \qquad \dim(\mathrm{IVc})^\Gamma + \dim(\mathrm{IVd})^\Gamma = \dim(\mathrm{IIIb})^\Gamma.$$

Here, Γ means the subgroup $\mathrm{M}(\mathfrak{p}^2)$ or $\mathrm{Kl}(\mathfrak{p}^2)$. Thus we can get the dimensions of $V^{\mathrm{M}(\mathfrak{p}^2)}$ and $V^{\mathrm{Kl}(\mathfrak{p}^2)}$ for the Iwahori-spherical representations of group IV. $\square$

3.3 Intertwining operators for groups V and VI

The symbols $F, \mathfrak{o}, \mathfrak{p}, q, \varpi$ have the usual meaning. Consider the degenerate principal series representation

$$\xi\nu^s \mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi^{-1}\nu^{-s}, \qquad \xi \text{ an unramified character of } F^\times, s \in \mathbb{C}. \tag{3.27}$$

Let $V_{\xi,s}$ be the standard model for this induced representation, consisting of smooth functions $f\colon G(F) \to \mathbb{C}$ with the transformation property

$$f\left(\left[\begin{smallmatrix} A & * \\ & uA' \end{smallmatrix}\right] g\right) = \xi(u^{-1}\det(A))|u^{-1}\det(A)|^{s+3/2} f(g). \tag{3.28}$$

By a similar discussion as in [23, section 2], we study an intertwining operator $\mathcal{A}(s)\colon V_{\xi,s} \to V_{\xi^{-1},-s}$, given by

$$(\mathcal{A}(s)f)(g) = \int\limits_N f(s_2s_1s_2ng)\, dn, \tag{3.29}$$

where N is the unipotent radical of the Siegel parabolic P. By Proposition 2.1 of [23], the intertwining operator as in (3.29) is well-defined. Let $V_{\xi,s}^{\mathrm{Kl}(\mathfrak{p}^2)}$ denote the subspace of $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors. By Proposition 3.6 ii), any $f \in V_{\xi,s}^{\mathrm{Kl}(\mathfrak{p}^2)}$ is determined by the four numbers

$$\alpha := f(\mathbf{I}_4), \quad \beta := f(s_2s_1), \quad \gamma := f(s_2s_1y_1s_1), \quad \delta := f(y_2). \tag{3.30}$$

Here, y_1 and y_2 are defined as follows

$$y_1 := \begin{bmatrix} 1 & \varpi & & \\ & 1 & & \\ & & 1 & -\varpi \\ & & & 1 \end{bmatrix}, \quad y_2 := \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix}. \tag{3.31}$$

In particular, $\dim_{\mathbb{C}}(V_{\xi,s}^{\mathrm{Kl}(\mathfrak{p}^2)}) = 4$. We need to compute $\mathcal{A}(s)f$ for such a function. Since $\mathcal{A}(s)f$ is again $\mathrm{Kl}(\mathfrak{p}^2)$-invariant, we only have to compute $(\mathcal{A}(s)f)(w)$ for $w \in \{\mathbf{I}_4, s_2s_1, s_2s_1y_1s_1, y_2\}$. We are going to use the following lemma in the calculation for $(\mathcal{A}(s)f)(w)$.

Proposition 3.8. *Let R_i be the representatives of $B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$ in Proposition 3.6 i). Then those 11 elements $\{R_i \mid 1 \le i \le 11\}$ are the same with the following 11 elements in $B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$.*

$$\mathbf{I}_4, \quad s_1, \quad s_1y_1s_1, \quad s_2s_1, \quad s_2s_1y_1s_1, \quad s_1s_2s_1, \tag{3.32}$$

$$s_1s_2s_1y_1s_1, \quad s_1y_1s_1s_2s_1, \quad s_1y_2, \quad y_2, \quad s_1B_1y_2. \tag{3.33}$$

Proof. The first 10 elements easily follows from the straightforward computation. As for the last element $s_1B_1y_2$ in (3.32), the matrix identity

$$s_2s_1y_1s_1y_2 = B_1y_2k', \quad \text{with} \quad k' = \begin{bmatrix} 1 & & -1 & \\ & 0 & 1 & \\ & -1 & -\varpi & 1 \\ & & & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \tag{3.34}$$

implies that $s_2s_1y_1s_1y_2$ and y_2 define the same element of $B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$. Since the matrix k' is exactly belong to $\mathrm{Kl}(\mathfrak{p}^2)$. More precisely,

$$k' = \begin{bmatrix} 1 & & -1 & \\ & 0 & 1 & \\ & -1 & -\varpi & 1 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & -\varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -1 & \\ & 1 & & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Thus, we have $s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}$ and $s_1B_1y_2$ define the same element of

$B(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$. $\square$

3.3.1 First integration

Lemma 3.9. *Let* $f \in V_{\xi,s}^{\mathrm{Kl}(\mathfrak{p}^2)}$ *and* $\alpha, \beta, \gamma, \delta$ *as in* (3.30). *Let* $\{\omega_i\}_{i=1}^{11}$ *be the* 11 *elements as in* (3.32). *Then, for any unramified character* ξ*,*

$$\int_F f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \omega_i\right) d\kappa$$

$$= \begin{cases} q^{-2}\alpha + \frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta + q^{-1}(1-q^{-1})\delta, & \omega_1 = \mathbf{I}_4, \\ (1 + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}})\alpha, & \omega_2 = s_1, \\ q^{-2}\alpha + \frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta + q^{-1}(1-q^{-1})\delta, & \omega_3 = s_1 y_1 s_1, \\ (1 + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}})\beta, & \omega_4 = s_2 s_1, \\ q^{-2}\gamma + \frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta + q^{-1}(1-q^{-1})\delta, & \omega_5 = s_2 s_1 y_1 s_1, \\ \beta + \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\alpha + (1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\delta, & \omega_6 = s_1 s_2 s_1, \\ \gamma + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\alpha, & \omega_7 = s_1 s_2 s_1 y_1 s_1, \\ (1 + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}})\beta, & \omega_8 = s_1 y_1 s_1 s_2 s_1, \\ (1 + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}})\delta, & \omega_9 = s_1 y_2, \\ q^{-2}\alpha + \frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta + q^{-1}(1-q^{-1})\delta, & \omega_{10} = y_2, \\ (1 + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}})\delta, & \omega_{11} = s_1 B_1 y_2. \end{cases}$$

Proof. The proof is analogous to that of Lemma 2.2 of [23]. Similarly, the main matrix identity that we use is

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}=\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}\underbrace{\begin{bmatrix}&&&1\\&1&&\\&&1&\\-1&&&\end{bmatrix}}_{=s_1s_2s_1}\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}. \tag{3.35}$$

1. For $\omega_1=\mathbf{I}_4$, we shall separate into three parts to calculate.

   (a) $$\int_{\mathfrak{p}^2} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix})\,d\kappa=\int_{\mathfrak{p}^2} f(\mathbf{I}_4)\,d\kappa=q^{-2}f(\mathbf{I}_4)=q^{-2}\alpha.$$

   (b) $$\begin{aligned}&\int_{F\backslash\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix})\,d\kappa\\ =&\int_{F\backslash\mathfrak{p}} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix})\,d\kappa\\ =&\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_2s_1)=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta.\end{aligned}$$

   (c) $$\begin{aligned}&\int_{\mathfrak{p}\backslash\mathfrak{p}^2} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix})\,d\kappa=q^{-1}\int_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa\varpi&&&1\end{bmatrix})\,d\kappa\\ =&q^{-1}(1-q^{-1})f(y_2)=q^{-1}(1-q^{-1})\delta.\end{aligned}$$

2. For $\omega_2=s_1$, we shall separate into two parts to calculate.

   (a) $$\int_{\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1)\,d\kappa=\int_{\mathfrak{p}} f(s_1\begin{bmatrix}1&&&\\&1&&\\&\kappa&1&\\&&&1\end{bmatrix})\,d\kappa=q^{-1}f(\mathbf{I}_4)=q^{-1}\alpha.$$

   (b) $$\begin{aligned}&\int_{F\backslash\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1)\,d\kappa\\ =&\int_{F\backslash\mathfrak{p}} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1)\,d\kappa\end{aligned}$$

$$=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(\mathbf{I}_4)=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\alpha.$$

3. $\omega_3 = s_1y_1s_1$. This is exactly the same with the case $\omega_1 = \mathbf{I}_4$. In fact,

$$\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1y_1s_1)\,d\kappa=\int\limits_F f(\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\\kappa&&-\varpi&1\end{bmatrix})\,d\kappa=\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}\cdot\mathbf{I}_4)\,d\kappa.$$

4. For $\omega_4 = s_2s_1$, we shall separate into two parts to calculate.

   (a) $\int\limits_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_2s_1)\,d\kappa = 1\cdot f(s_2s_1)=\beta.$

   (b)

$$\begin{aligned}&\int\limits_{F\setminus\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_2s_1)\,d\kappa\\=&\int\limits_{F\setminus\mathfrak{o}} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}s_2s_1)\,d\kappa\\=&\frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_2s_1)=\frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta.\end{aligned}$$

5. For $\omega_5 = s_2s_1y_1s_1$, we shall separate into three parts to calculate.

   (a) $\int\limits_{\mathfrak{p}^2} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_2s_1y_1s_1)\,d\kappa = q^{-2}f(s_2s_1y_1s_1)=q^{-2}\gamma.$

   (b)

$$\begin{aligned}&\int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_2s_1y_1s_1)\,d\kappa\\=&\int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}s_2s_1y_1s_1)\,d\kappa\\=&\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_1s_2s_1s_2s_1y_1s_1\begin{bmatrix}1&&-\kappa^{-1}\varpi&\kappa^{-1}\\&1&\kappa^{-1}\varpi^2&-\kappa^{-1}\varpi\\&&1&\\&&&1\end{bmatrix})\end{aligned}$$

$$=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_2s_1s_2s_1)$$

$$=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_2s_1)=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta.$$

(c)

$$\int\limits_{\mathfrak{p}\backslash\mathfrak{p}^2} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_2s_1y_1s_1)\,d\kappa = q^{-1}\int\limits_{\mathfrak{o}^\times} f(s_2s_1y_1s_1\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa\varpi&&&1\end{bmatrix})\,d\kappa$$

$$=q^{-1}\int\limits_{\mathfrak{o}^\times} f(s_2\begin{bmatrix}1&&&\\&1&&\\&&\kappa&\\&&&\kappa\end{bmatrix}s_1y_1s_1y_2)\,d\kappa = q^{-1}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&\kappa&&\\&&1&\\&&&\kappa\end{bmatrix}B_1y_2k')\,d\kappa$$

$$=q^{-1}(1-q^{-1})f(y_2)=q^{-1}(1-q^{-1})\delta.$$

6. For $\omega_6=s_1s_2s_1$, we shall separate into three parts to calculate.

(a) $\int\limits_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1)\,d\kappa = \int\limits_{\mathfrak{o}} f(s_1s_2s_1\begin{bmatrix}1&&&-\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}\,d\kappa = f(s_2s_1)=\beta.$

(b)

$$\int\limits_{F\backslash\mathfrak{p}^{-1}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1)\,d\kappa$$

$$=\int\limits_{F\backslash\mathfrak{p}^{-1}} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1)\,d\kappa$$

$$=\frac{(1-q^{-1})\xi^2(\varpi)(q^{-s-\frac{1}{2}})^2}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(\mathbf{I}_4)=\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\alpha.$$

(c)

$$\int\limits_{\mathfrak{p}^{-1}\backslash\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1)\,d\kappa = q^2\int\limits_{\varpi\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa^{-1}&&&1\end{bmatrix}s_1s_2s_1)\,d\kappa$$

$$=q^2\int\limits_{\varpi\mathfrak{o}^\times} f(\begin{bmatrix}-\kappa&&&\\&1&&\\&&1&\\&&&-\kappa^{-1}\end{bmatrix}\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1)\,d\kappa$$

$$=q^{-s+\frac{1}{2}}\xi(\varpi)\int\limits_{\varpi\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix})\,d\kappa = q^{-s-\frac{1}{2}}\xi(\varpi)\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa\varpi&&&1\end{bmatrix})\,d\kappa$$
$$=q^{-s-\frac{1}{2}}(1-q^{-1})\xi(\varpi)f(y_2) = q^{-s-\frac{1}{2}}(1-q^{-1})\xi(\varpi)\delta.$$

7. For $\omega_7 = s_1s_2s_1y_1s_1$, we shall separate into two parts to calculate.

   (a) $\int\limits_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} s_1s_2s_1y_1s_1)\,d\kappa = f(s_2s_1y_1s_1) = \gamma.$

   (b)

$$\int\limits_{F\setminus\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} s_1s_2s_1y_1s_1)\,d\kappa$$
$$=\int\limits_{\nu(\kappa)\le -1} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix} s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix} s_1s_2s_1y_1s_1)\,d\kappa$$
$$=\frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(\mathbf{I}_4) = \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\alpha.$$

8. For $\omega_8 = s_1y_1s_1s_2s_1$, we shall separate into two parts to calculate.

   (a) $\int\limits_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} s_1y_1s_1s_2s_1)\,d\kappa = f(s_1y_1s_1s_2s_1) = f(s_2s_1) = \beta.$

   (b)

$$\int\limits_{F\setminus\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} s_1y_1s_1s_2s_1)\,d\kappa$$
$$=\int\limits_{F\setminus\mathfrak{o}} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix} s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix} s_1y_1s_1s_2s_1)\,d\kappa$$
$$=\frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(s_2y_1s_1s_2s_1) = \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta.$$

9. For $\omega_9 = s_1y_2$, we shall separate into two parts to calculate.

(a) $\displaystyle\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_1 y_2)\, d\kappa = \int_{\mathfrak{o}} f(s_1 y_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \kappa & 1 & \\ & & & 1 \end{bmatrix})\, d\kappa = f(y_2) = \delta.$

(b)

$$\begin{aligned}
&\int_{F\backslash\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_1 y_2)\, d\kappa \\
=&\int_{F\backslash\mathfrak{o}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 y_2)\, d\kappa \\
=&\frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(y_2) = \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\delta.
\end{aligned}$$

10. For $\omega_{10} = y_2$, we shall separate into two parts to calculate.

(a)

$$\begin{aligned}
&\int_{\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_2)\, d\kappa = q^{-1}\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix})\, d\kappa \\
=&q^{-1}\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (\kappa+1)\varpi & & & 1 \end{bmatrix})\, d\kappa = q^{-1}\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\varpi & & & 1 \end{bmatrix})\, d\kappa \\
=&q^{-1}\left[\int_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\varpi & & & 1 \end{bmatrix})\, d\kappa + \int_{\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\varpi & & & 1 \end{bmatrix})\, d\kappa\right] \\
=&q^{-1}\left[(1-q^{-1})\delta + q^{-1}\alpha\right] = q^{-1}(1-q^{-1})\delta + q^{-2}\alpha.
\end{aligned}$$

(b)

$$\begin{aligned}
&\int_{F\backslash\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_2)\, d\kappa \\
=&\int_{F\backslash\mathfrak{p}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} y_2)\, d\kappa
\end{aligned}$$

$$=\int_{F\backslash\mathfrak{o}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix}\begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1s_2s_1y_2 \begin{bmatrix} 1+\kappa^{-1}\varpi & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ -\kappa^{-1}\varpi^2 & & & 1-\kappa^{-1}\varpi \end{bmatrix})\,d\kappa$$

$$=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_1s_2s_1y_2)=\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta.$$

11. For $\omega_{11}=s_1B_1y_2$, we shall separate into two parts to calculate.

(a) $\displaystyle\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_1B_1y_2)\,d\kappa=\int_{\mathfrak{o}} f(s_1B_1y_2\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \kappa & 1 & \\ & & & 1 \end{bmatrix})\,d\kappa=f(y_2)=\delta.$

(b)

$$\int_{F\backslash\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_1B_1y_2)\,d\kappa$$

$$=\int_{F\backslash\mathfrak{o}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix}\begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1s_2s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1B_1y_2)\,d\kappa$$

$$=\frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_1s_2B_1y_2)=\frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\delta.$$

□

3.3.2 Double integration

Lemma 3.10. *With f and $\omega_i, 1\le i\le 11$ as in Lemma 3.9, we define*

$$\mathcal{B}(\omega_i):=\int_{F^2} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix}\omega_i\right)d\mu\,d\kappa, \tag{3.36}$$

then for any unramified character ξ, we have $\mathcal{B}(\omega_i)$

$$= \begin{cases} q^{-4}\alpha + \frac{q^{-1}(1-q^{-1})(1+q+\xi(\varpi)q^{-s+\frac{1}{2}})}{1-\xi^2(\varpi)q^{-2s}}\beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta, & i=1, \\ \left(q^{-2} + \frac{q^{-1}(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}(\xi(\varpi)q^{-s+\frac{1}{2}}+1)}{1-\xi^2(\varpi)q^{-2s}}\right)\alpha + \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\beta & \\ \quad +q^{-1}(1-q^{-1})\gamma + \frac{(1-q^{-1})^2\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi^2(\varpi)q^{-2s}}\delta, & i=2, \\ q^{-4}\alpha + \frac{q^{-1}(1-q^{-1})(1+q+\xi(\varpi)q^{-s+\frac{1}{2}})}{1-\xi^2(\varpi)q^{-2s}}\beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta, & i=3, \\ \frac{(1-q^{-1})\xi^4(\varpi)q^{-4s-2}}{1-\xi^2(\varpi)q^{-2s}}\alpha + \left(1 + \frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}}+\xi^2(\varpi)q^{-2s})}{1-\xi^2(\varpi)q^{-2s}}\right)\beta & \\ \quad +(1-q^{-1})\xi^2(\varpi)q^{-2s-2}\gamma + \frac{(1-q^{-1})^2\xi^2(\varpi)q^{-2s-1}}{1-\xi^2(\varpi)q^{-2s}}\delta, & i=4, \\ q^{-4}\alpha + \frac{q^{-1}(1-q^{-1})(1+q+\xi(\varpi)q^{-s+\frac{1}{2}})}{1-\xi^2(\varpi)q^{-2s}}\beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta, & i=5, \\ \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s-1}\left(\xi^2(\varpi)q^{-2s+1}+\xi(\varpi)q^{-s+\frac{1}{2}}+1\right)}{1-\xi^2(\varpi)q^{-2s}}\alpha + \beta & \\ \quad +(1-q^{-1})\xi^2(\varpi)q^{-2s}\gamma + (1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\delta, & i=6, \\ \left(q^{-2} + \frac{q^{-1}(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}(\xi(\varpi)q^{-s+\frac{1}{2}}+1)}{1-\xi^2(\varpi)q^{-2s}}\right)\alpha + \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\beta & \\ \quad +q^{-1}(1-q^{-1})\gamma + \frac{(1-q^{-1})^2\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi^2(\varpi)q^{-2s}}\delta, & i=7, \\ \frac{(1-q^{-1})\xi^4(\varpi)q^{-4s-2}}{1-\xi^2(\varpi)q^{-2s}}\alpha + \left(1 + \frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}}+\xi^2(\varpi)q^{-2s})}{1-\xi^2(\varpi)q^{-2s}}\right)\beta & \\ \quad +(1-q^{-1})\xi^2(\varpi)q^{-2s-2}\gamma + \frac{(1-q^{-1})^2\xi^2(\varpi)q^{-2s-1}}{1-\xi^2(\varpi)q^{-2s}}\delta, & i=8, \\ \frac{q^{-1}(1-q^{-1})\xi^3(\varpi)q^{-3s-\frac{1}{2}}}{1-\xi^2(\varpi)q^{-2s}}\alpha + \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\beta + (1-q^{-1})\xi(\varpi)q^{-s-\frac{3}{2}}\gamma & \\ \quad + \left(q^{-1} + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\left(\xi(\varpi)q^{-s-\frac{1}{2}}-q^{-1}+1\right)}{1-\xi^2(\varpi)q^{-2s}}\right)\delta, & i=9, \\ q^{-4}\alpha + \frac{q^{-1}(1-q^{-1})(1+q+\xi(\varpi)q^{-s+\frac{1}{2}})}{1-\xi^2(\varpi)q^{-2s}}\beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta, & i=10, \\ \frac{q^{-1}(1-q^{-1})\xi^3(\varpi)q^{-3s-\frac{1}{2}}}{1-\xi^2(\varpi)q^{-2s}}\alpha + \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\beta + (1-q^{-1})\xi(\varpi)q^{-s-\frac{3}{2}}\gamma & \\ \quad + \left(q^{-1} + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\left(\xi(\varpi)q^{-s-\frac{1}{2}}-q^{-1}+1\right)}{1-\xi^2(\varpi)q^{-2s}}\right)\delta, & i=11. \end{cases}$$

Proof. The proof is analogous to that of Lemma 2.3 of [23]. Similarly, the main matrix identity that we use is

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} = \begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} \underbrace{\begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}}_{=s_2 s_1 s_2} \begin{bmatrix} 1 & & \mu^{-1} & \\ & 1 & & \mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \qquad (3.37)$$

1. For $\omega_1 = \mathbf{I}_4$, we have

   (a) $$\int\limits_F \int\limits_{\mathfrak{p}^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} \cdot \mathbf{I}_4)\ d\mu\ d\kappa = q^{-2} \int\limits_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \cdot \mathbf{I}_4)\ d\kappa.$$

   (b)

$$\begin{aligned} & \int\limits_F \int\limits_{F \setminus \mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} \cdot \mathbf{I}_4)\, d\mu\, d\kappa \\ = & \int\limits_F \int\limits_{F \setminus \mathfrak{p}} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} s_2 s_1 s_2 \begin{bmatrix} 1 & & \mu^{-1} & \\ & 1 & & \mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \mathbf{I}_4)\, d\mu\, d\kappa \\ = & \frac{1 - q^{-1}}{1 - \xi^2(\varpi) q^{-2s}} \int\limits_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1)\, d\mu\, d\kappa. \end{aligned}$$

   (c)

$$\begin{aligned} & \int\limits_F \int\limits_{\mathfrak{p} \setminus \mathfrak{p}^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} \cdot \mathbf{I}_4)\, d\mu\, d\kappa \\ = & \int\limits_F \left(q^{-1} \cdot \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu\varpi & & 1 & \\ & \mu\varpi & & 1 \end{bmatrix})\, d\mu \right)\ d\kappa \\ = & q^{-1} \int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -\mu & \\ & & & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -\kappa\mu^{-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\varpi & & 1 & \\ & -\varpi & & 1 \end{bmatrix})\, d\mu\, d\kappa \\ = & q^{-1} \int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -\kappa & & & 1 \end{bmatrix} s_2 s_1 y_1 s_1 s_2^{-1})\, d\mu\, d\kappa \\ = & q^{-1}(1 - q^{-1}) \int\limits_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 y_1 s_1)\, d\kappa. \end{aligned}$$

2. For $\omega_2 = s_1$, we have

(a) $$\int\limits_F \int\limits_{\mathfrak{p}^2} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa & \mu && 1 \end{bmatrix} s_1)\, d\mu\, d\kappa = q^{-2} \int\limits_F f(\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa &&& 1 \end{bmatrix} s_1)\, d\kappa.$$

(b)

$$\begin{aligned}
&\int\limits_F \int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa & \mu && 1 \end{bmatrix} s_1)\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix} -\mu^{-1} && -1 & \\ \mu^{-2}\kappa & -\mu^{-1} && -1 \\ && -\mu & \\ && -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa\mu^{-2} &&& 1 \end{bmatrix} s_2 s_1 s_2 \begin{bmatrix} 1 && \mu^{-1} & \\ & 1 && \mu^{-1} \\ && 1 & \\ &&& 1 \end{bmatrix} s_1)\, d\mu\, d\kappa \\
=&\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}} \int\limits_F f(\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa &&& 1 \end{bmatrix} s_1 s_2 s_1)\, d\kappa.
\end{aligned}$$

(c)

$$\begin{aligned}
&\int\limits_F \int\limits_{\mathfrak{p}\setminus\mathfrak{p}^2} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa & \mu && 1 \end{bmatrix} s_1)\, d\mu\, d\kappa = \int\limits_F \int\limits_{\varpi\mathfrak{o}^\times} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ & \mu && 1 \end{bmatrix} s_1)\, d\mu\, d\kappa \\
=&q^{-1} \int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && -\mu & \\ &&& -\mu \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ -\varpi && 1 & \\ & -\varpi && 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && -\mu^{-1} & \\ &&& -\mu^{-1} \end{bmatrix} s_1)\, d\mu\, d\kappa \\
=&q^{-1} \int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa &&& 1 \end{bmatrix} s_2 s_1 y_1 s_1 s_2^{-1} s_1)\, d\mu\, d\kappa \\
=&q^{-1}(1-q^{-1}) \int\limits_F f(\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa &&& 1 \end{bmatrix} s_1 s_2 s_1 y_1 s_1)\, d\kappa.
\end{aligned}$$

3. For $\omega_3 = s_1 y_1 s_1$, this is exactly the case $\omega_1 = \mathbf{I}_4$. In fact,

$$\begin{aligned}
&\int\limits_F \int\limits_F f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa & \mu && 1 \end{bmatrix} s_1 y_1 s_1)\, d\mu\, d\kappa = \int\limits_F \int\limits_F f(\begin{bmatrix} 1 &&& \\ \varpi & 1 && \\ && 1 & \\ && -\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa+2\mu\varpi & \mu && 1 \end{bmatrix})\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_F f(\begin{bmatrix} 1 &&& \\ \varpi & 1 && \\ && 1 & \\ && -\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa & \mu && 1 \end{bmatrix})\, d\mu\, d\kappa = \int\limits_F \int\limits_F f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa & \mu && 1 \end{bmatrix} \cdot \mathbf{I}_4)\, d\mu\, d\kappa.
\end{aligned}$$

4. For $\omega_4 = s_2 s_1$, we have

(a) $$\int\limits_F \int\limits_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1)\, d\mu\, d\kappa = \int\limits_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1)\, d\kappa.$$

(b)

$$\begin{aligned}
&\int\limits_F \int\limits_{\nu(\mu)\le -2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1)\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_{F\backslash \mathfrak{p}^{-1}} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} s_2 s_1 s_2 \begin{bmatrix} 1 & & \mu^{-1} & \\ & 1 & & \mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1)\, d\mu\, d\kappa \\
=&\frac{(1-q^{-1})(\xi^2(\varpi)q^{-2s})^2}{1-\xi^2(\varpi)q^{-2s}} \int\limits_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix})\, d\kappa.
\end{aligned}$$

(c)

$$\begin{aligned}
&\int\limits_F \int\limits_{\nu(\mu)=-1} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1)\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_{\varpi^{-1}\mathfrak{o}^\times} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} s_2 s_1 s_2 \begin{bmatrix} 1 & & \mu^{-1} & \\ & 1 & & \mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1)\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_{\varpi^{-1}\mathfrak{o}^\times} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} s_2 \begin{bmatrix} 1 & & & \\ \mu^{-1} & 1 & & \\ & & 1 & \\ & & -\mu^{-1} & 1 \end{bmatrix})\, d\mu\, d\kappa \\
=&\xi^2(\varpi)q^{-2s} \int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 \begin{bmatrix} 1 & & & \\ \mu\varpi & 1 & & \\ & & 1 & \\ & & -\mu\varpi & 1 \end{bmatrix})\, d\mu\, d\kappa \\
=&\xi^2(\varpi)q^{-2s}(1-q^{-1}) \int\limits_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 y_1 s_1)\, d\kappa.
\end{aligned}$$

5. For $\omega_5 = s_2 s_1 y_1 s_1$, it is the same as the first case $\omega_1 = \mathbf{I}_4$. In fact,

$$\int\limits_F \int\limits_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1 y_1 s_1)\, d\mu\, d\kappa$$

$$=\int\limits_F\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}s_2\begin{bmatrix}\frac{1}{\varpi}&1&&\\&&&\\&&\frac{1}{-\varpi}&1\end{bmatrix})\,d\mu\,d\kappa$$

$$=\int\limits_F\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}\begin{bmatrix}1&&&\\\mu-\varpi&1&&\\&&1&\\&\mu-\varpi&&1\end{bmatrix})\,d\mu\,d\kappa$$

$$=\int\limits_F\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}\cdot\mathbf{I}_4)\,d\mu\,d\kappa.$$

6. For $\omega_6 = s_1s_2s_1$, we have

(a) $\int\limits_F\int\limits_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1s_2s_1)\,d\mu\,d\kappa = 1\cdot\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1)\,d\kappa.$

(b)

$$\int\limits_F\int\limits_{\nu(\mu)\le -2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1s_2s_1)\,d\mu\,d\kappa$$

$$=\int\limits_F\int\limits_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix}-\mu^{-1}&&-1&\\\mu^{-2}\kappa&-\mu^{-1}&&-1\\&&-\mu&\\&&-\kappa&-\mu\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa\mu^{-2}&&&1\end{bmatrix}s_2s_1s_2\begin{bmatrix}1&\mu^{-1}&&\\&1&&\mu^{-1}\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1)\,d\mu\,d\kappa$$

$$=\frac{(1-q^{-1})(\xi^2(\varpi)q^{-2s})^2}{1-\xi^2(\varpi)q^{-2s}}\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1)\,d\kappa.$$

(c)

$$\int\limits_F\int\limits_{\nu(\mu)=-1} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1s_2s_1)\,d\mu\,d\kappa$$

$$=\int\limits_F\int\limits_{\varpi^{-1}\mathfrak{o}^\times} f(\begin{bmatrix}-\mu^{-1}&&-1&\\\mu^{-2}\kappa&-\mu^{-1}&&-1\\&&-\mu&\\&&-\kappa&-\mu\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa\mu^{-2}&&&1\end{bmatrix}s_2s_1s_2\begin{bmatrix}1&\mu^{-1}&&\\&1&&\mu^{-1}\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1)\,d\mu\,d\kappa$$

$$=\xi^2(\varpi)q^{-2s}\int\limits_F\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2\begin{bmatrix}\frac{1}{\mu\varpi}&1&&\\&&&\\&&\frac{1}{-\mu\varpi}&1\end{bmatrix})\,d\mu\,d\kappa$$

$$=\xi^2(\varpi)q^{-2s}\int\limits_F\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1y_1s_1)\,d\mu\,d\kappa$$

$$=\xi^2(\varpi)q^{-2s}(1-q^{-1})\int_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1y_1s_1)\,d\kappa.$$

7. For $\omega_7=s_1s_2s_1y_1s_1$, this is exactly same with the case $\omega_2=s_1$. In fact,

$$\begin{aligned}&\int_F\int_F f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1s_2s_1y_1s_1)\,d\mu\,d\kappa\\=&\int_F\int_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\-\varpi&&1&\\&-\varpi&&1\end{bmatrix}s_1s_2)\,d\mu\,d\kappa\\=&\int_F\int_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\mu-\varpi&&1&\\&\mu-\varpi&&1\end{bmatrix}s_1)\,d\mu\,d\kappa=\int_F\int_F f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1)\,d\mu\,d\kappa.\end{aligned}$$

8. For $\omega_8=s_1y_1s_1s_2s_1$, this is exactly the case $\omega_4=s_2s_1$. In fact,

$$\begin{aligned}&\int_F\int_F f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1y_1s_1s_2s_1)\,d\mu\,d\kappa=\int_F\int_F f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\&&-\varpi&1\end{bmatrix}s_2s_1)\,d\mu\,d\kappa\\=&\int_F\int_F f(\begin{bmatrix}1&&&\\\varpi&1&&\\&&1&\\&&-\varpi&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa+2\mu\varpi&\mu&&1\end{bmatrix}s_2s_1)\,d\mu\,d\kappa=\int_F\int_F f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_2s_1)\,d\mu\,d\kappa.\end{aligned}$$

9. For $\omega_9=s_1y_2$, we have

(a) $$\int_F\int_{\mathfrak{p}^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1y_2)\,d\mu\,d\kappa=q^{-2}\int_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1y_2)\,d\kappa.$$

(b)

$$\begin{aligned}&\int_F\int_{\mathfrak{p}\backslash\mathfrak{p}^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1y_2)\,d\mu\,d\kappa\\=&q^{-1}\int_F\int_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1\begin{bmatrix}1&&&\\&1&&\\\varpi&&1&\\&\varpi&&1\end{bmatrix}y_2)\,d\mu\,d\kappa\\=&q^{-1}\int_F\int_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1y_1s_1y_2)\,d\mu\,d\kappa\end{aligned}$$

$$=q^{-1}(1-q^{-1})\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1B_1y_2)\,d\kappa.$$

(c)

$$\int\limits_F\int\limits_{F\setminus\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1y_2)\,d\mu\,d\kappa$$

$$=\int\limits_F\int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix}-\mu^{-1}&&-1&\\\mu^{-2}\kappa&-\mu^{-1}&&-1\\&&-\mu&\\&&-\kappa&-\mu\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa\mu^{-2}&&&1\end{bmatrix}s_2s_1s_2\begin{bmatrix}1&\mu^{-1}&&\\&1&&\\&&1&\mu^{-1}\\&&&1\end{bmatrix}s_1y_2)\,d\mu\,d\kappa$$

$$=\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}}\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1y_2)\,d\kappa.$$

Define $I=\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_1s_2s_1y_2)\,d\kappa.$

i. $I_1=\int\limits_{\mathfrak{o}} f(s_1s_2s_1y_2\begin{bmatrix}1-\kappa\varpi&&&-\kappa\\&1&&\\&&1&\\\kappa\varpi^2&&&1+\kappa\varpi\end{bmatrix})\,d\kappa=f(s_2s_1)=\beta.$

ii.

$$I_2=\int\limits_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1y_2)\,d\kappa$$

$$=\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s-1}}{1-\xi(\varpi)q^{-s-\frac12}}\delta.$$

iii.

$$I_3=\int\limits_{\nu(\kappa)=-1} f(\begin{bmatrix}-\kappa^{-1}&&&\\&1&&\\&&1&\\&&&-\kappa\end{bmatrix}\begin{bmatrix}1&&&\kappa\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1\begin{bmatrix}1&&&\kappa^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2s_1y_2)\,d\kappa$$

$$=\xi(\varpi)q^{-s-\frac12}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi(1-\kappa)&&&1\end{bmatrix})\,d\kappa.$$

- if $1-\kappa \in \mathfrak{p}, i.e. \kappa \in 1+\mathfrak{p}$, then $(1-\kappa)\mathfrak{p} \in \mathfrak{p}^2$, therefore

$$\xi(\varpi)q^{-s-\frac{1}{2}} \int\limits_{1+\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi(1-\kappa) & & & 1 \end{bmatrix})\, d\kappa = q^{-1}\xi(\varpi)q^{-s-\frac{1}{2}}\alpha.$$

- if $1-\kappa \notin \mathfrak{p}, i.e. 1-\kappa \in \mathfrak{o}^\times$. Furthermore, $\kappa \in \mathfrak{o}^\times \backslash 1+\mathfrak{p}$.

$$\xi(\varpi)q^{-s-\frac{1}{2}} \int\limits_{\mathfrak{o}^\times \backslash 1+\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi(1-\kappa) & & & 1 \end{bmatrix})\, d\kappa = (1-2q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\delta.$$

In conclusion, we have

$$\int\limits_F \int\limits_{F\backslash \mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1 y_2)\, d\mu\, d\kappa = \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \cdot$$

$$\cdot \left[\beta + \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\delta + q^{-1}\xi(\varpi)q^{-s-\frac{1}{2}}\alpha + (1-2q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\delta\right].$$

(d) In this case, we will use the following matrix identity

$$s_2 s_1 s_2 y_1 s_1 y_2 = \begin{bmatrix} 1 & -\varpi & & -\varpi \\ & 1 & & \\ & & 1 & \varpi \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 s_2. \tag{3.38}$$

$$\begin{aligned}
&\int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1 y_2)\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} s_2 s_1 s_2 \begin{bmatrix} 1 & & \mu^{-1} & \\ & 1 & & \mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 y_2)\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 s_1 y_2 \begin{bmatrix} 1 & & & \\ & \mu^{-1} & & \\ & & \mu & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \varpi & 1 & & \\ & & 1 & \\ & & -\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \mu & & \\ & & \mu^{-1} & \\ & & & 1 \end{bmatrix})\, d\mu\, d\kappa \\
=&\int\limits_F \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 s_1 y_2 s_1 y_1 s_1)\, d\mu\, d\kappa
\end{aligned}$$

$$=(1-q^{-1})\int\limits_{F} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1 y_2)\, d\mu\, d\kappa$$

$$=(1-q^{-1})\int\limits_{F} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -\varpi & & -\varpi \\ & 1 & & \\ & & 1 & \varpi \\ & & & 1 \end{bmatrix} s_1 s_2 s_1)\, d\mu\, d\kappa.$$

Furthermore, we define

$$J := \int\limits_{F} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} Y s_1 s_2 s_1)\, d\kappa, \quad Y := \begin{bmatrix} 1 & -\varpi & & -\varpi \\ & 1 & & \\ & & 1 & \varpi \\ & & & 1 \end{bmatrix}. \tag{3.39}$$

i. $J_1 = \int\limits_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} Y s_1 s_2 s_1)\, d\kappa = f(s_2 s_1) = \beta.$

ii. In this case, we shall use the following matrix identity

$$\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 Y s_1 s_2 s_1 = y_2 \begin{bmatrix} -1 & 1 & & \\ & 1 & & \\ & -\varpi & 1 & 1 \\ & & & -1 \end{bmatrix}. \tag{3.40}$$

$$\begin{aligned} J_2 &= \int\limits_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} Y s_1 s_2 s_1)\, d\kappa \\ &= \int\limits_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} Y s_1 s_2 s_1)\, d\kappa \\ &= \frac{(1-q^{-1})\xi^2(\varpi) q^{-2s-1}}{1-\xi(\varpi) q^{-s-\frac{1}{2}}} f(s_1 s_2 s_1 Y s_1 s_2 s_1) = \frac{(1-q^{-1})\xi^2(\varpi) q^{-2s-1}}{1-\xi(\varpi) q^{-s-\frac{1}{2}}} \delta. \end{aligned}$$

iii.

$$\begin{aligned} J_3 &= \int\limits_{\nu(\kappa)=-1} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} Y s_1 s_2 s_1)\, d\kappa \\ &= \xi(\varpi) q^{-s-\frac{1}{2}} \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\varpi & & & 1 \end{bmatrix} B_1^{-1} y_2 Y_1)\, d\kappa \end{aligned}$$

$$
\begin{aligned}
&=\xi(\kappa)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix}\begin{bmatrix} 1&-1&&\\ &1&&\\ &&1&1\\ &&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \kappa\varpi&&&1 \end{bmatrix})\,d\kappa\\
&=\xi(\kappa)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1&-1&&\\ &1&&\\ \kappa\varpi&&1&1\\ (\kappa+1)\varpi&-\varpi&&1 \end{bmatrix})\,d\kappa\\
&=\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1&-1&&\\ &1&&\\ -\varpi&&1&1\\ &-\varpi&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ (\kappa+1)\varpi&&&1 \end{bmatrix})\,d\kappa.
\end{aligned}
$$

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\begin{aligned}
&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{-1+\mathfrak{p}} f(\begin{bmatrix} 1&-1&&\\ &1&&\\ -\varpi&&1&1\\ &-\varpi&&1 \end{bmatrix})\,d\kappa\\
=&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{-1+\mathfrak{p}} f(s_2s_1y_1s_1)\,d\kappa=\xi(\varpi)q^{-s-\frac{3}{2}}\gamma.
\end{aligned}
$$

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\begin{aligned}
&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times\backslash -1+\mathfrak{p}} f(\begin{bmatrix} 1&-1&&\\ &1&&\\ \kappa\varpi&&1&1\\ (\kappa+1)\varpi&-\varpi&&1 \end{bmatrix})d\,\kappa\\
=&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times\backslash -1+\mathfrak{p}} f(\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ (\kappa+1)\varpi&&&1 \end{bmatrix}\begin{bmatrix} 1&-1&&\\ &1&&\\ \kappa\varpi&&1&1\\ &\kappa\varpi&&1 \end{bmatrix})d\kappa\\
=&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times\backslash -1+\mathfrak{p}} f(\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ (\kappa+1)\varpi&&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ &1&&\\ \kappa\varpi&&1&\\ &\kappa\varpi&&1 \end{bmatrix}\begin{bmatrix} 1&-1&&\\ &1&&\\ &\kappa\varpi&1&1\\ &&&1 \end{bmatrix})d\kappa\\
=&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times\backslash -1+\mathfrak{p}} f(\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ (\kappa+1)\varpi&&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ &1&&\\ \kappa\varpi&&1&\\ &\kappa\varpi&&1 \end{bmatrix})d\kappa\\
=&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times\backslash -1+\mathfrak{p}} f(\begin{bmatrix} 1&&&\\ &1&&\\ \varpi&&1&\\ &\varpi&&1 \end{bmatrix}\begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ \varpi&&&1 \end{bmatrix})d\kappa\\
=&\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times\backslash -1+\mathfrak{p}} f(s_2s_1y_1s_1y_2)d\kappa\\
=&(1-2q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}f(y_2)=(1-2q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\delta.
\end{aligned}
$$

In conclusion, we have

$$\int_F \int_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1 y_2)\, d\mu\, d\kappa = (1-q^{-1})\cdot$$
$$\cdot \left[\beta + \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\delta + q^{-1}\xi(\varpi)q^{-s-\frac{1}{2}}\gamma + (1-2q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\delta\right].$$

10. For $\omega_{10} = y_2$, it turns out to be the same as the first case $\omega_1 = \mathbf{I}_4$. In fact,

$$\begin{aligned} &\int_F \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_2)\, d\mu\, d\kappa \\ =& \int_F \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa+\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix})\, d\mu\, d\kappa \\ =& \int_F \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} \cdot \mathbf{I}_4)\, d\mu\, d\kappa. \end{aligned}$$

11. For $\omega_{11} = s_1 B_1 y_2$, it is the same as the case $\omega_9 = s_1 y_2$. In fact,

$$\int_F \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1 B_1 y_2)\, d\mu\, d\kappa = \int_F \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} s_1 y_2)\, d\mu\, d\kappa$$
$$= \int_F \int_F f(\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa+2\mu & \mu & & 1 \end{bmatrix} s_1 y_2)\, d\mu\, d\kappa = \int_F \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1 y_2)\, d\mu\, d\kappa.$$

□

3.3.3 Triple integration

Let

$$\tilde{\omega}_1 = \mathbf{I}_4, \quad \tilde{\omega}_2 = s_2 s_1, \quad \tilde{\omega}_3 = s_2 s_1 y_1 s_1, \quad \tilde{\omega}_4 = y_2. \tag{3.41}$$

Lemma 3.11. *With f and ξ as in the previous lemmas, we have*

$$(\mathcal{A}(s)f)(\tilde{\omega}_i) = \int\limits_{F^3} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1 s_2 \tilde{\omega}_i\right) dx\, d\mu\, d\kappa =$$

$$\begin{cases} \mathcal{B}(\omega_4) + \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s+1}}{1-\xi(\varpi)q^{-s+\frac{1}{2}}} \cdot \mathcal{B}(\omega_2) + (1-q^{-1})\xi(\varpi)q^{-s+\frac{1}{2}} \cdot \mathcal{B}(\omega_9) & i=1 \\ \left(1 + \frac{(1-q^{-1})\xi(\varpi)q^{-s+\frac{1}{2}}}{1-\xi(\varpi)q^{-s+\frac{1}{2}}}\right) \cdot \mathcal{B}(\omega_1) & i=2 \\ \frac{\xi^2(\varpi)q^{-2s-1}(1-\xi(\varpi)q^{-s-\frac{1}{2}})}{1-\xi(\varpi)q^{-s+\frac{1}{2}}}\alpha + \frac{1-\xi(\varpi)q^{-s-\frac{1}{2}}}{1-\xi(\varpi)q^{-s+\frac{1}{2}}}\beta + \frac{(1-q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}(1-\xi(\varpi)q^{-s-\frac{1}{2}})}{1-\xi(\varpi)q^{-s+\frac{1}{2}}}\delta & \\ + \frac{(1-\xi(\varpi)q^{-s-\frac{1}{2}})\xi^2(\varpi)q^{-2s-1}\left(\xi^2(\varpi)q^{-2s}+(q-1)\xi(\varpi)q^{-s-\frac{1}{2}}+q-q^{-1}-1\right)}{(1-\xi^2(\varpi)q^{-2s})(1-\xi(\varpi)q^{-s+\frac{1}{2}})}\gamma & i=3 \\ \mathcal{B}(\omega_4) + \xi(\varpi)q^{-s-\frac{1}{2}} \cdot \mathcal{B}(\omega_2) + \frac{\xi^2(\varpi)q^{-2s}+(1-2q^{-1})\xi(\varpi)q^{-s+\frac{1}{2}}}{1-\xi(\varpi)q^{-s+\frac{1}{2}}} \cdot \mathcal{B}(\omega_9) & i=4 \end{cases}$$

Proof. The proof is analogous to that of Lemma 2.3 of [23]. Similarly, the main matrix identity that we use is

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -\mu x^{-1} & -x^{-1} & -1 & \\ & & -x & \\ & & -\mu & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\mu x^{-1} & & 1 & \\ \kappa-\mu^2 x^{-1} & -\mu x^{-1} & & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}}_{=s_2} \begin{bmatrix} 1 & & & \\ & 1 & x^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{3.42}$$

1. For $\omega_1 = \mathbf{I}_4$, we have

   (a) $\displaystyle\int\limits_{F^2}\int\limits_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1 s_2 \cdot \mathbf{I}_4)\, dx\, d\mu\, d\kappa = \int\limits_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1)\, d\mu\, d\kappa.$

   (b)

$$\int\limits_{F^2}\int\limits_{\nu(x)\leq -2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2 s_1 s_2 \cdot \mathbf{I}_4)\, dx\, d\mu\, d\kappa$$

$$= \int\limits_{F^2}\int\limits_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix} 1 & & & \\ -\mu x^{-1} & -x^{-1} & -1 & \\ & & -x & \\ & & -\mu & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\mu x^{-1} & & 1 & \\ \kappa-\mu^2 x^{-1} & -\mu x^{-1} & & 1 \end{bmatrix} s_2 \begin{bmatrix} 1 & & & \\ & 1 & x^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1)\, dx\, d\mu\, d\kappa$$

$$= \frac{(1-q^{-1})\xi^2(\varpi)q^{1-2s}}{1-\xi(\varpi)q^{\frac{1}{2}-s}} \int\limits_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1)\, d\mu\, d\kappa.$$

(c)

$$\int_{F^2}\int_{\nu(x)=-1} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2s_1s_2 \cdot \mathbf{I}_4)\, dx\, d\mu\, d\kappa$$

$$= \int_{F^2}\int_{\varpi^{-1}\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ -\mu x^{-1} & -x^{-1} & -1 & \\ & & -x & \\ & & -\mu & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\mu x^{-1} & & 1 & \\ \kappa-\mu^2x^{-1} & -\mu x^{-1} & & 1 \end{bmatrix} s_2 \begin{bmatrix} 1 & & & \\ & 1 & x^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2s_1)\, dx\, d\mu\, d\kappa$$

$$= \xi(\varpi)q^{\frac{1}{2}-s} \int_{F^2}\int_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x\varpi & & & 1 \end{bmatrix})\, dx\, d\mu\, d\kappa$$

$$= \xi(\varpi)q^{\frac{1}{2}-s} \int_{F^2}\int_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_1y_2)\, dx\, d\mu\, d\kappa.$$

2. For $\omega_2 = s_2s_1$, we have

(a) $\displaystyle\int_{F^2}\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2s_1s_2s_2s_1)\, dx\, d\mu\, d\kappa = \int_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix})\, d\mu\, d\kappa.$

(b)

$$\int_{F^2}\int_{F\setminus\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2s_1s_2s_2s_1)\, dx\, d\mu\, d\kappa$$

$$= \int_{F^2}\int_{F\setminus\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ -\mu x^{-1} & -x^{-1} & -1 & \\ & & -x & \\ & & -\mu & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\mu x^{-1} & & 1 & \\ \kappa-\mu^2x^{-1} & -\mu x^{-1} & & 1 \end{bmatrix} s_2 \begin{bmatrix} 1 & & & \\ & 1 & x^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dx\, d\mu\, d\kappa$$

$$= \frac{(1-q^{-1})\xi(\varpi)q^{\frac{1}{2}-s}}{1-\xi(\varpi)q^{\frac{1}{2}-s}} \int_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix})\, d\mu\, d\kappa.$$

3. For $\omega_3 = s_2s_1y_1s_1$, we have

(a) $\displaystyle\int_{F^2}\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & x & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2y_1s_1)\, dx\, d\mu\, d\kappa = \int_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2y_1s_1)\, d\mu\, d\kappa.$

To calculate $\displaystyle\int_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} s_2y_1s_1)\, d\mu\, d\kappa$, we have

i. $\displaystyle\int_F\int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} s_2y_1s_1) d\mu d\kappa = \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2y_1s_1) d\kappa.$

- $\displaystyle\int_{\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 y_1 s_1)\, d\kappa = \int_{\mathfrak{p}} f(s_2 y_1 s_1)\, d\kappa = q^{-1}\beta.$
-

$$
\begin{aligned}
&\int_{F \backslash \mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 y_1 s_1)\, d\kappa \\
=&\int_{F \backslash \mathfrak{p}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 y_1 s_1)\, d\kappa \\
=&\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(s_2 s_1) = \frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\beta.
\end{aligned}
$$

ii.

$$
\begin{aligned}
&\int_{F} \int_{F \backslash \mathfrak{p}^{-1}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} s_2 y_1 s_1)\, d\mu\, d\kappa \\
=&\int_{F} \int_{F \backslash \mathfrak{p}^{-1}} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} \\
& \qquad s_2 s_1 s_2 s_2 y_1 s_1 \begin{bmatrix} 1 & & & \\ -\mu^{-1} & 1 & & \\ & & 1 & \\ & & \mu^{-1} & 1 \end{bmatrix})\, d\mu\, d\kappa \\
=&\frac{(1-q^{-1})(\xi^2(\varpi)q^{-2s})^2}{1-\xi^2(\varpi)q^{-2s}} \int_{F} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 y_1 s_1)\, d\kappa.
\end{aligned}
$$

iii.

$$
\begin{aligned}
&\int_{F} \int_{\nu(\mu)=-1} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} s_2 y_1 s_1)\, d\mu\, d\kappa \\
=&\xi^2(\varpi)q^{-2s} \int_{F} \int_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 y_1 s_1 \begin{bmatrix} 1 & & & \\ \mu\varpi & 1 & & \\ & & 1 & \\ & & -\mu\varpi & 1 \end{bmatrix})\, d\mu\, d\kappa \\
=&\xi^2(\varpi)q^{-2s} \int_{F} \int_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 \begin{bmatrix} 1 & (\mu+1)\varpi & & \\ & 1 & & \\ & & 1 & -(\mu+1)\varpi \\ & & & 1 \end{bmatrix} s_1)\, d\mu\, d\kappa.
\end{aligned}
$$

- $$\xi^2(\varpi)q^{-2s}\int\limits_F\int\limits_{-1+\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}\cdot \mathbf{I}_4)\,d\mu\,d\kappa$$
$$=q^{-1}\xi^2(\varpi)q^{-2s}\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix})\,d\kappa.$$

- $$\xi^2(\varpi)q^{-2s}\int\limits_F\int\limits_{\mathfrak{o}^\times\backslash-1+\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_2s_1\begin{bmatrix}1&(\mu+1)\varpi&&\\&1&&\\&&1&-(\mu+1)\varpi\\&&&1\end{bmatrix}s_1)\,d\mu\,d\kappa$$
$$=(1-2q^{-1})\xi^2(\varpi)q^{-2s}\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}s_2s_1y_1s_1)\,d\kappa.$$

(b)

$$\int\limits_{F^2}\int\limits_{F\backslash\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&x&1&\\\kappa&\mu&&1\end{bmatrix}s_2y_1s_1)\,dx\,d\mu\,d\kappa$$
$$=\int\limits_{F^2}\int\limits_{F\backslash\mathfrak{o}} f(\begin{bmatrix}1&&&\\-\mu x^{-1}&-x^{-1}&-1&\\&&-x&\\&&-\mu&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\-\mu x^{-1}&&1&\\\kappa-\mu^2x^{-1}&-\mu x^{-1}&&1\end{bmatrix}$$
$$s_2\begin{bmatrix}1&&&\\&1&x^{-1}&\\&&1&\\&&&1\end{bmatrix}s_2y_1s_1)\,dx\,d\mu\,d\kappa$$
$$=\int\limits_{F^2}\int\limits_{F\backslash\mathfrak{o}} f(\begin{bmatrix}1&&&\\-\mu x^{-1}&-x^{-1}&-1&\\&&-x&\\&&-\mu&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\-\mu x^{-1}&&1&\\\kappa-\mu^2x^{-1}&-\mu x^{-1}&&1\end{bmatrix}y_1s_1\begin{bmatrix}1&&&\\&1&&\\&&1&\\-x^{-1}&&&1\end{bmatrix})\,dx\,d\mu\,d\kappa.$$

i.

$$\int\limits_{F^2}\int\limits_{\nu(x)\le-2} f(\begin{bmatrix}1&&&\\-\mu x^{-1}&-x^{-1}&-1&\\&&-x&\\&&-\mu&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\-\mu x^{-1}&&1&\\\kappa-\mu^2x^{-1}&-\mu x^{-1}&&1\end{bmatrix}y_1s_1)\,dx\,d\mu\,d\kappa$$
$$=\frac{(1-q^{-1})(\xi(\varpi)q^{-s+\frac12})^2}{1-\xi(\varpi)q^{-s+\frac12}}\int\limits_{F^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}y_1s_1)\,d\mu\,d\kappa.$$

Let

$$K = \int_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1)\, d\mu\, d\kappa = \int_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ 2\mu\varpi & \mu & & 1 \end{bmatrix})\, d\mu\, d\kappa,$$

then we have

•

$$K_1 = \int_F \int_{\mathfrak{p}^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1)\, d\mu\, d\kappa = q^{-2} \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1)\, d\kappa$$
$$= q^{-2} \int_F f(y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix})\, d\kappa.$$

– $$q^{-2} \int_{\mathfrak{p}} f(y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix})\, d\kappa = q^{-2} \int_{\mathfrak{p}} f(\mathbf{I}_4) d\kappa = q^{-3}\alpha.$$

–

$$q^{-2} \int_{F\backslash\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1)\, d\kappa$$
$$= q^{-2} \int_{F\backslash\mathfrak{p}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} y_1 s_1)\, d\kappa$$
$$= q^{-2} \frac{1 - q^{-1}}{1 - \xi(\varpi) q^{-s-\frac{1}{2}}} f(s_2 s_1 y_1 s_1) = q^{-2} \frac{1 - q^{-1}}{1 - \xi(\varpi) q^{-s-\frac{1}{2}}} \gamma.$$

In conclusion, we have

$$K_1 = q^{-3}\alpha + q^{-2} \frac{1 - q^{-1}}{1 - \xi(\varpi) q^{-s-\frac{1}{2}}} \gamma. \tag{3.43}$$

•

$$
\begin{aligned}
K_2 =& \int\limits_{F} \int\limits_{\nu(\mu)=1} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ 2\mu\varpi & \mu & & 1 \end{bmatrix}) \, d\mu \, d\kappa \\
=& \int\limits_{F} \int\limits_{\nu(\mu)=1} f(y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \mu & 1 & & \\ & & 1 & \\ & \mu & & 1 \end{bmatrix}) \, d\mu \, d\kappa \\
=& q^{-1} \int\limits_{F} \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ (\kappa+\mu)\varpi & \kappa & 1 & \\ \kappa\varpi^2 & (\kappa+\mu)\varpi & & 1 \end{bmatrix}) \, d\mu \, d\kappa \\
=& q^{-1} \int\limits_{F} \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa-\mu & 1 & \\ (\kappa-\mu)\varpi^2 & \kappa\varpi & & 1 \end{bmatrix}) \, d\mu \, d\kappa \\
=& q^{-1} \int\limits_{F} \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -\mu\varpi^2 & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -\mu & 1 & \\ & & & 1 \end{bmatrix}) \, d\mu \, d\kappa \\
=& q^{-1} \int\limits_{F} \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix}) \, d\mu \, d\kappa \\
=& q^{-1}(1-q^{-1}) \int\limits_{F} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix}) \, d\kappa.
\end{aligned}
$$

– $q^{-1}(1-q^{-1}) \int\limits_{\mathfrak{p}} f(y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix}) \, d\kappa = q^{-2}(1-q^{-1})\alpha.$

–

$$
\begin{aligned}
& q^{-1}(1-q^{-1}) \int\limits_{F \backslash \mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1) \, d\kappa \\
=& q^{-1}(1-q^{-1}) \int\limits_{F \backslash \mathfrak{p}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
& \qquad\qquad s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} y_1 s_1) \, d\kappa \\
=& q^{-1}(1-q^{-1}) \frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(s_2 s_1 y_1 s_1) \\
=& q^{-1}(1-q^{-1}) \frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} \gamma.
\end{aligned}
$$

In conclusion, we have

$$K_2 = q^{-2}(1-q^{-1})\alpha + q^{-1}(1-q^{-1})\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\gamma. \quad (3.44)$$

-

$$\begin{aligned} K_3 =& \int_F \int_{\nu(\mu)\le 0} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1)\, d\mu\, d\kappa \\ =& \int_F \int_{F\backslash\mathfrak{p}} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix} \\ & \qquad s_2 s_1 s_2 \begin{bmatrix} 1 & \mu^{-1} & & \\ & 1 & & \mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} y_1 s_1)\, d\mu\, d\kappa \\ =& \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}} \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1)\, d\kappa. \end{aligned}$$

–

$$\begin{aligned} & \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}} \int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1)\, d\kappa \\ =& \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}} f(s_2 s_1) = \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}} \beta. \end{aligned}$$

–

$$\begin{aligned} & \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}} \int_{\nu(\kappa)\le -2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1)\, d\kappa \\ =& \frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}} \int_{\nu(\kappa)\le -2} \\ & f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1)\, d\kappa \end{aligned}$$

$$=\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}})^2}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}f(s_2s_1y_1s_1)$$
$$=\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}})^2}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\gamma.$$

–

$$\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\int\limits_{\nu(\kappa)=-1} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2s_1s_2y_1s_1)\,d\mu\,d\kappa$$
$$=\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times} f(s_2s_1y_1s_1\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\varpi & & & 1 \end{bmatrix})\,d\kappa$$
$$=\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times} f(s_2s_1y_1s_1y_2)\,d\kappa$$
$$=\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times} f(y_2)d\kappa$$
$$=\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\xi(\varpi)q^{-s-\frac{1}{2}}(1-q^{-1})\delta.$$

In conclusion, we have

$$K_3=\frac{1-q^{-1}}{1-\xi^2(\varpi)q^{-2s}}\cdot \cdot\left[\beta+\frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}})^2}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\gamma+\xi(\varpi)q^{-s-\frac{1}{2}}(1-q^{-1})\delta\right]. \tag{3.45}$$

ii.

$$\int\limits_{F^2}\int\limits_{\nu(x)=-1} f(\begin{bmatrix} 1 & & & \\ -\mu x^{-1} & -x^{-1} & -1 & \\ & & -x & \\ & & -\mu & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ -\mu x^{-1} & & 1 & \\ \kappa-\mu^2x^{-1} & -\mu x^{-1} & & 1 \end{bmatrix} s_2s_2y_1s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -x^{-1} & & & 1 \end{bmatrix})\,dx\,d\mu\,d\kappa$$

$$=(1-q^{-1})\xi(\varpi)q^{-s+\frac{1}{2}}\int\limits_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1 y_2)\, d\mu\, d\kappa.$$

Let

$$L=\int\limits_{F^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1 y_2)\, d\mu\, d\kappa, \tag{3.46}$$

then we have

A.

$$\begin{aligned} L_1 =& \int\limits_{F}\int\limits_{\mathfrak{p}^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1 y_2)\, d\mu\, d\kappa \\ =& q^{-2}\int\limits_{F} f(y_1 s_1 y_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi & \kappa & 1 & \\ \kappa\varpi^2 & \kappa\varpi & & 1 \end{bmatrix})\, d\kappa. \end{aligned}$$

- $q^{-2}\int\limits_{\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1 y_2)\, d\kappa = q^{-2}\int\limits_{\mathfrak{p}} f(y_1 s_1 y_2)\, d\kappa = q^{-3}\delta.$
-

$$\begin{aligned} & q^{-2}\int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix}\begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} y_1 s_1 y_2)\, d\kappa \\ =& q^{-2}\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(s_2 s_1 y_1 s_1 y_2) = q^{-2}\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\delta. \end{aligned}$$

In conclusion, we have

$$L_1 = q^{-2}\left[q^{-1}+\frac{1-q^{-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\right]\delta. \tag{3.47}$$

B.

$$L_2 = \int\limits_{F}\int\limits_{\mathfrak{p}\setminus\mathfrak{p}^2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1 y_2)\, d\mu\, d\kappa$$

$$=q^{-1}\int\limits_F\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} y_1s_1y_2 \begin{bmatrix}1&&&\\\mu\varpi&1&&\\&\mu\varpi&1&\\&&&1\end{bmatrix})\,d\mu\,d\kappa$$

$$=q^{-1}\int\limits_F\int\limits_{\mathfrak{o}^\times} f(y_1s_1 \begin{bmatrix}1&&&\\&1&&\\(\kappa+\mu)\varpi&\kappa&1&\\\kappa\varpi^2&(\kappa+\mu)\varpi&&1\end{bmatrix} y_2)\,d\mu\,d\kappa$$

$$=q^{-1}\int\limits_F\int\limits_{\mathfrak{o}^\times} f(y_1s_1 \begin{bmatrix}1&&&\\&1&&\\\kappa\varpi&\kappa-\mu&1&\\(\kappa-\mu)\varpi^2&\kappa\varpi&&1\end{bmatrix} y_2)\,d\mu\,d\kappa$$

$$=q^{-1}\int\limits_F\int\limits_{\mathfrak{o}^\times} f(y_1s_1 \begin{bmatrix}1&&&\\&1&&\\\kappa\varpi&\kappa&1&\\\kappa\varpi^2&\kappa\varpi&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&-\mu&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\-\mu\varpi^2&&&1\end{bmatrix} y_2)\,d\mu\,d\kappa$$

$$=q^{-1}(1-q^{-1})\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} y_1s_1y_2)\,d\kappa.$$

C. In order to calculate

$$L_3=\int\limits_F\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix} y_1s_1y_2)\,d\mu\,d\kappa, \tag{3.48}$$

we are going to change the order. In particular, we have

$$\begin{aligned}L_3=&\int\limits_F\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix} y_1s_1y_2)\,d\mu\,d\kappa\\=&\int\limits_{\mathfrak{o}^\times}\int\limits_F f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} y_1s_1y_2)\,d\kappa\,d\mu.\end{aligned}$$

•

$$\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix} y_1s_1y_2)\,d\kappa\,d\mu$$

$$=q^{-1}\int\limits_{\mathfrak{o}^\times} f(y_1s_1y_2\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\2\mu\varpi&\mu&&1\end{bmatrix})\,d\mu$$

$$=q^{-1}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\(2\mu+1)\varpi&&&1\end{bmatrix})\,d\mu$$

$$=q^{-1}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (2\mu+1)\varpi & & & 1 \end{bmatrix})\, d\mu.$$

– If $2\mu+1 \in \mathfrak{p}$, then by the matrix identity

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} s_2 s_1 B_1 s_1 = s_2 s_1 \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ & -1 & & -1 \\ & & -1 & -1 \\ & & & 1 \end{bmatrix}$$

we have

$$q^{-1} \int\limits_{2\mu+1\in\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{bmatrix})\, d\mu = q^{-1} \int\limits_{2\mu+1\in\mathfrak{p}} f(s_2 s_1 B_1 s_1)\, d\mu$$

$$=q^{-1} f(s_2 s_1) \int\limits_{2\mu+1\in\mathfrak{p}} d\mu = q^{-1}\beta \int\limits_{2\mu+1\in\mathfrak{p}} d\mu.$$

– If $2\mu+1 \notin \mathfrak{p}$, then by the matrix identity

$$\begin{bmatrix} 1 & & 1 & \\ -\varpi & 1 & & 1 \\ & & 1 & \\ & & \varpi & 1 \end{bmatrix} s_2 s_1 B_1 s_1 y_2 = s_2 s_1 \begin{bmatrix} 1 & & & \\ & -1 & & \\ & -\varpi & -1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}$$

we have

$$q^{-1} \int\limits_{\mathfrak{o}^\times \setminus \{2\mu+1\in\mathfrak{p}\}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (2\mu+1)\varpi & & & 1 \end{bmatrix})\, d\mu$$

$$=q^{-1} \int\limits_{\mathfrak{o}^\times \setminus \{2\mu+1\in\mathfrak{p}\}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix})\, d\mu$$

$$=q^{-1} \int\limits_{\mathfrak{o}^\times \setminus \{2\mu+1\in\mathfrak{p}\}} f(s_2 s_1 B_1 s_1 y_2)\, d\mu$$

$$=q^{-1} f(s_2 s_1) \int\limits_{\mathfrak{o}^\times \setminus \{2\mu+1\in\mathfrak{p}\}} d\mu = q^{-1}\beta \int\limits_{\mathfrak{o}^\times \setminus \{2\mu+1\in\mathfrak{p}\}} d\mu.$$

In conclusion, we have

$$\begin{aligned}
&\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}y_1s_1y_2)\,d\kappa\,d\mu\\
=&q^{-1}\beta(\int\limits_{2\mu+1\in\mathfrak{p}}+\int\limits_{\mathfrak{o}^\times\setminus\{2\mu+1\in\mathfrak{p}\}})\,d\mu=(1-q^{-1})q^{-1}\beta.
\end{aligned}$$

•

$$\begin{aligned}
&\int\limits_{\mathfrak{o}^\times}\int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}y_1s_1y_2)\,d\kappa\,d\mu\\
=&\int\limits_{\mathfrak{o}^\times}\int\limits_{F\setminus\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\kappa&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}y_1s_1y_2)\,d\kappa\,d\mu\\
=&\int\limits_{\mathfrak{o}^\times}\int\limits_{F\setminus\mathfrak{p}} f(s_2s_1y_1s_1y_2\begin{bmatrix}1&&&\\\mu\kappa^{-1}&1&\kappa^{-1}&\\&&1&\\-\mu^2\kappa^{-1}&&-\mu\kappa^{-1}&1\end{bmatrix})\,d\kappa\,d\mu\\
=&\int\limits_{\mathfrak{o}^\times}\int\limits_{F\setminus\mathfrak{p}} f(y_2k'\begin{bmatrix}1&&&\\\mu\kappa^{-1}&1&\kappa^{-1}&\\&&1&\\-\mu^2\kappa^{-1}&&-\mu\kappa^{-1}&1\end{bmatrix})\,d\kappa\,d\mu.
\end{aligned}$$

–

$$\begin{aligned}
&\int\limits_{\mathfrak{o}^\times}\int\limits_{\nu(\kappa)\le-2} f(f(y_2k'\begin{bmatrix}1&&&\\\mu\kappa^{-1}&1&\kappa^{-1}&\\&&1&\\-\mu^2\kappa^{-1}&&-\mu\kappa^{-1}&1\end{bmatrix})\,d\kappa\,d\mu\\
=&\frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}})^2}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\int\limits_{\mathfrak{o}^\times}f(y_2)d\mu\\
=&\frac{(1-q^{-1})^2\xi^2(\varpi)q^{-2s-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}}\delta.
\end{aligned}$$

–

$$\int\limits_{\mathfrak{o}^\times}\int\limits_{\nu(\kappa)=-1} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}y_1s_1y_2)\,d\kappa\,d\mu$$

$$
\begin{aligned}
&= \int\limits_{\mathfrak{o}^\times} \int\limits_{\nu(\kappa)=-1} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} y_1 s_1 y_2\right) d\kappa\, d\mu \\
&= \int\limits_{\mathfrak{o}^\times} \int\limits_{\nu(\kappa)=-1} f\left(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 \right. \\
&\qquad\qquad\qquad\qquad \left. \begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} y_1 s_1 y_2\right) d\kappa\, d\mu \\
&= \xi(\varpi) q^{-s-\frac{1}{2}} \\
&\quad \cdot \int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}^\times} f\left(s_2 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} y_1 s_1 y_2 \begin{bmatrix} 1 & & & \\ \mu\kappa\varpi & 1 & & \\ & & 1 & \\ -\mu^2\kappa\varpi & & -\mu\kappa\varpi & 1 \end{bmatrix}\right) d\kappa\, d\mu \\
&= \xi(\varpi) q^{-s-\frac{1}{2}} \int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}^\times} f\left(s_2 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} y_1 s_1 y_2 \begin{bmatrix} 1 & & & \\ \kappa\varpi & 1 & & \\ & & 1 & \\ -\mu\kappa\varpi & & -\kappa\varpi & 1 \end{bmatrix}\right) d\kappa\, d\mu.
\end{aligned}
$$

By the matrix identity

$$
\begin{aligned}
&s_2 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} y_1 s_1 y_2 \begin{bmatrix} 1 & & & \\ \kappa\varpi & 1 & & \\ & & 1 & \\ -\mu\kappa\varpi & & -\kappa\varpi & 1 \end{bmatrix} \\
=&s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} s_1 y_1 s_1 y_2 \begin{bmatrix} 1 & & & \\ \kappa\varpi & 1 & & \\ & & 1 & \\ -\mu\kappa\varpi & & -\kappa\varpi & 1 \end{bmatrix} \\
=&s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \varpi & 1 & & \\ & & 1 & \\ & & -\varpi & 1 \end{bmatrix} y_2 \begin{bmatrix} 1 & & & \\ \kappa\varpi & 1 & & \\ & & 1 & \\ -\mu\kappa\varpi & & -\kappa\varpi & 1 \end{bmatrix} \\
=&s_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ (\kappa+1)\varpi & 1 & & \\ & & 1 & \\ (1-\mu\kappa)\varpi & & -(\kappa+1)\varpi & 1 \end{bmatrix} = s_2 \begin{bmatrix} 1 & & & \\ (\kappa+1)\varpi & 1 & & \\ \mu & & 1 & \\ (1+\mu)\varpi & \mu & -(\kappa+1)\varpi & 1 \end{bmatrix} \\
=&\begin{bmatrix} 1 & & & \\ \mu & 1 & & \\ -(\kappa+1)\varpi & & 1 & \\ (1+\mu)\varpi & -(\kappa+1)\varpi & -\mu & 1 \end{bmatrix} s_2 \\
=&\begin{bmatrix} 1 & & & \\ \mu & 1 & & \\ & & 1 & \\ & & -\mu & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -(\kappa+1)\varpi & & 1 & \\ (1-\mu\kappa)\varpi & -(\kappa+1)\varpi & & 1 \end{bmatrix} s_2,
\end{aligned}
$$

then the integral

$$
\int\limits_{\mathfrak{o}^\times} \int\limits_{\nu(\kappa)=-1} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1 y_2\right) d\kappa\, d\mu
$$

$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} f(s_2s_1\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\&\mu&&1\end{bmatrix}y_1s_1y_2\begin{bmatrix}1&&&\\\kappa\varpi&1&&\\&&1&\\-\mu\kappa\varpi&&-\kappa\varpi&1\end{bmatrix})\,d\kappa\,d\mu$$

$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\-(\kappa+1)\varpi&1&&\\&&1&\\(1-\mu\kappa)\varpi&-(\kappa+1)\varpi&&1\end{bmatrix})\,d\kappa\,d\mu.$$

① If $1+\kappa\in\mathfrak{p}$, then we have

$$\xi(\varpi)q^{-s-\frac{1}{2}}\int_{\mathfrak{o}^\times}\int_{1+\kappa\in\mathfrak{p}} f(\begin{bmatrix}1&&&\\-(\kappa+1)\varpi&1&&\\&&1&\\(1-\mu\kappa)\varpi&-(\kappa+1)\varpi&&1\end{bmatrix})\,d\kappa\,d\mu$$

$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int_{\mathfrak{o}^\times}\int_{1+\kappa\in\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\(1-\mu\kappa)\varpi&&&1\end{bmatrix})\,d\kappa\,d\mu$$

$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int_{\mathfrak{o}^\times}\int_{1+\kappa\in\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\(1+\mu)\varpi&&&1\end{bmatrix})\,d\kappa\,d\mu.$$

a') If $1+\mu\in\mathfrak{p}$, then we have

$$\xi(\varpi)q^{-s-\frac{1}{2}}\int_{1+\mu\in\mathfrak{p}}\int_{1+\kappa\in\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\(1+\mu)\varpi&&&1\end{bmatrix})\,d\kappa\,d\mu$$

$$=\xi(\varpi)q^{-s-\frac{1}{2}}\alpha\int_{1+\mu\in\mathfrak{p}}\int_{1+\kappa\in\mathfrak{p}} d\kappa\,d\mu=q^{-2}\xi(\varpi)q^{-s-\frac{1}{2}}\alpha.$$

b') If $1+\mu\notin\mathfrak{p}$, then we have

$$\xi(\varpi)q^{-s-\frac{1}{2}}\int_{1-\mu\notin\mathfrak{p}}\int_{1+\kappa\in\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\(1-\mu)\varpi&&&1\end{bmatrix})\,d\kappa\,d\mu$$

$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int_{1-\mu\notin\mathfrak{p}}\int_{1+\kappa\in\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\&&1&\\\varpi&&&1\end{bmatrix})\,d\kappa\,d\mu$$

$$=q^{-1}(1-2q^{-1})\xi(\varpi)q^{-s-\frac{1}{2}}\delta.$$

② If $1+\kappa \notin \mathfrak{p}$:

$$\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times}\int\limits_{1+\kappa\notin\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ -(\kappa+1)\varpi & & 1 & \\ (1-\mu\kappa)\varpi & -(\kappa+1)\varpi & & 1 \end{bmatrix})\,d\kappa\,d\mu$$
$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times}\int\limits_{1+\kappa\notin\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (1-\mu\kappa)\varpi & & & 1 \end{bmatrix})\,d\kappa\,d\mu$$
$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{\mathfrak{o}^\times}\int\limits_{1+\kappa\notin\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (1+\mu)\varpi & & & 1 \end{bmatrix})\,d\kappa\,d\mu.$$

$c')$ If $1+\mu \in \mathfrak{p}$, then we have

$$\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{1+\mu\in\mathfrak{p}}\int\limits_{1+\kappa\notin\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix})\,d\kappa\,d\mu$$
$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{1+\mu\in\mathfrak{p}}\int\limits_{1+\kappa\notin\mathfrak{p}} f(s_2s_1y_1s_1)\,d\kappa\,d\mu$$
$$=\xi(\varpi)q^{-s-\frac{1}{2}}q^{-1}(1-2q^{-1})\gamma.$$

$d')$ If $1+\mu \notin \mathfrak{p}$, then we have

$$\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{1-\mu\notin\mathfrak{p}}\int\limits_{1+\kappa\notin\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix})\,d\kappa\,d\mu$$
$$=\xi(\varpi)q^{-s-\frac{1}{2}}\int\limits_{1-\mu\notin\mathfrak{p}}\int\limits_{1+\kappa\notin\mathfrak{p}} f(s_2s_1y_1s_1y_2)\,d\kappa\,d\mu$$
$$=\xi(\varpi)q^{-s-\frac{1}{2}}(1-2q^{-1})^2\delta.$$

–

$$\int\limits_{\mathfrak{o}^\times}\int\limits_{\nu(\kappa)=0} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix}y_1s_1y_2)\,d\kappa\,d\mu$$

$$
\begin{aligned}
&= \int\limits_{\mathfrak{o}^\times} \int\limits_{\nu(\kappa)=0} f(y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi+\mu & \kappa & 1 & \\ (\kappa\varpi+2\mu+1)\varpi & \kappa\varpi+\mu & & 1 \end{bmatrix}) \, d\kappa \, d\mu \\
&= \int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}^\times} f(y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \kappa\varpi+\mu & & 1 & \\ (\kappa\varpi+2\mu+1)\varpi & \kappa\varpi+\mu & & 1 \end{bmatrix}) \, d\kappa \, d\mu \\
&= \int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}^\times} f(y_1 s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ (-\kappa\varpi+2\mu+1)\varpi & \mu & & 1 \end{bmatrix}) \, d\kappa \, d\mu \\
&= \int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (2\mu+1)\varpi & & & 1 \end{bmatrix}) \, d\kappa \, d\mu \\
&= (1-q^{-1}) \int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (2\mu+1)\varpi & & & 1 \end{bmatrix}) \, d\mu \\
&= (1-q^{-1})^2 \beta.
\end{aligned}
$$

For the last equality, it follows from the previous calculation for $\int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1 y_2) \, d\kappa \, d\mu$. Furthermore, we can combine them as

$$
\begin{aligned}
&\int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1 y_2) \, d\kappa \, d\mu \\
=& \Big(\int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{p}} + \int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}^\times} \Big) f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ & \mu & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} y_1 s_1 y_2) \, d\kappa \, d\mu \\
=& q^{-1}(1-q^{-1})\beta + (1-q^{-1})^2\beta = (1-q^{-1})\beta.
\end{aligned}
$$

D.

$$
\begin{aligned}
L_4 &= \int\limits_{F} \int\limits_{F \setminus \mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} y_1 s_1 y_2) \, d\mu \, d\kappa \\
&= \int\limits_{F} \int\limits_{F \setminus \mathfrak{o}} f(\begin{bmatrix} -\mu^{-1} & & -1 & \\ \mu^{-2}\kappa & -\mu^{-1} & & -1 \\ & & -\mu & \\ & & -\kappa & -\mu \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa\mu^{-2} & & & 1 \end{bmatrix}
\end{aligned}
$$

$$
s_2 s_1 s_2 \begin{bmatrix} 1 & & \mu^{-1} & \\ & 1 & & \mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} y_1 s_1 y_2)\, d\mu\, d\kappa
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \int_F f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1 y_2)\, d\kappa.
$$

•

$$
\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \int_{\mathfrak{o}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1 y_2)\, d\kappa
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \int_{\mathfrak{o}} f(s_2 s_1 s_2 y_1 s_1 y_2 \begin{bmatrix} 1-\kappa\varpi & & \kappa\varpi & -\kappa \\ \kappa\varpi^2 & 1 & -\kappa\varpi^2 & \kappa\varpi \\ & & 1 & \\ \kappa\varpi^2 & & -\kappa\varpi^2 & 1+\kappa\varpi \end{bmatrix})\, d\kappa
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} f(s_1 s_2 y_1 s_1 y_2) = \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \beta.
$$

•

$$
\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \int_{\nu(\kappa)\leq -2} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1 y_2)\, d\kappa
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \int_{\nu(\kappa)\leq -2} f(\begin{bmatrix} -\kappa^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa \end{bmatrix} \begin{bmatrix} 1 & & & \kappa \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1
$$

$$
\begin{bmatrix} 1 & & & \kappa^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1 y_2)\, d\kappa
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}})^2}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(s_2 s_1 y_1 s_1 y_2)
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \frac{(1-q^{-1})(\xi(\varpi)q^{-s-\frac{1}{2}})^2}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} f(y_2)
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \frac{(1-q^{-1})\xi^2(\varpi)q^{-2s-1}}{1-\xi(\varpi)q^{-s-\frac{1}{2}}} \delta.
$$

•

$$
\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}} \int_{\nu(\kappa)=-1} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \kappa & & & 1 \end{bmatrix} s_2 s_1 s_2 y_1 s_1 y_2)\, d\kappa
$$

$$=\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}}\int_{\nu(\kappa)=-1} f(\begin{bmatrix} -\kappa^{-1} &&& \\ & 1 && \\ && 1 & \\ &&& -\kappa \end{bmatrix}\begin{bmatrix} 1 &&& \kappa \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix} s_1s_2s_1$$

$$\begin{bmatrix} 1 &&& \kappa^{-1} \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix} s_2s_1s_2y_1s_1y_2)\,d\kappa$$

$$=\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}}\xi(\varpi)q^{-s-\frac{1}{2}}\int_{\mathfrak{o}^\times} f(s_2s_1y_1s_1y_2\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \kappa\varpi &&& 1 \end{bmatrix})\,d\kappa$$

$$=\frac{(1-q^{-1})\xi^2(\varpi)q^{-2s}}{1-\xi^2(\varpi)q^{-2s}}\xi(\varpi)q^{-s-\frac{1}{2}}\int_{\mathfrak{o}^\times} f(s_2s_1y_1s_1\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ (\kappa+1)\varpi &&& 1 \end{bmatrix})\,d\kappa.$$

– If $\kappa+1\in\mathfrak{p}$, i.e., $\kappa\in -1+\mathfrak{p}$, then the integral becomes

$$\int_{-1+\mathfrak{p}} f(s_2s_1y_1s_1\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ (\kappa+1)\varpi &&& 1 \end{bmatrix})\,d\kappa=\int_{-1+\mathfrak{p}} f(s_2s_1y_1s_1)\,d\kappa=\gamma.$$

– If $\kappa+1\notin\mathfrak{p}$, i.e., $\kappa+1\in\mathfrak{o}^\times$, then the integral becomes

$$\int_{\mathfrak{o}^\times\setminus -1+\mathfrak{p}} f(s_2s_1y_1s_1\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ (\kappa+1)\varpi &&& 1 \end{bmatrix})\,d\kappa$$

$$=\int_{\mathfrak{o}^\times\setminus -1+\mathfrak{p}} f(s_2s_1y_1s_1\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \varpi &&& 1 \end{bmatrix})\,d\kappa$$

$$=\int_{\mathfrak{o}^\times\setminus -1+\mathfrak{p}} f(y_2)d\kappa=(1-2q^{-1})\delta.$$

4. For $\omega_4=y_2$, we have

(a)

$$\int_{F^2}\int_{\mathfrak{o}} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \mu & x & 1 & \\ \kappa & \mu && 1 \end{bmatrix} s_2s_1s_2y_2)\,dx\,d\mu\,d\kappa$$

$$=\int_{F^2}\int_{\mathfrak{o}} f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \mu && 1 & \\ \kappa & \mu && 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 & -\varpi & \\ && 1 & \\ &&& 1 \end{bmatrix} s_2s_1)\,dx\,d\mu\,d\kappa$$

$$
= \int_{F^2} \int_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&-\varpi&\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}1&&&\\\mu\varpi&1&&\\\mu&&1&\\\kappa&\mu&-\mu\varpi&1\end{bmatrix} s_2 s_1)\, dx\, d\mu\, d\kappa
$$

$$
= \int_{F^2} \int_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\\mu\varpi&1&&\\&&1&\\&&-\mu\varpi&1\end{bmatrix} \begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa+\mu^2\varpi&\mu&&1\end{bmatrix} s_2 s_1)\, dx\, d\mu\, d\kappa
$$

$$
= \int_{F^2} \int_{\mathfrak{o}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix} s_2 s_1)\, dx\, d\mu\, d\kappa = \int_{F^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix} s_2 s_1)\, d\mu\, d\kappa.
$$

(b)

$$
\int_{F^2} \int_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&x&1&\\\kappa&\mu&&1\end{bmatrix} s_2 s_1 s_2 y_2)\, dx\, d\mu\, d\kappa
$$

$$
= \int_{F^2} \int_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix}1&&&\\-\mu x^{-1}&-x^{-1}&-1&\\&&-x&\\&&-\mu&1\end{bmatrix} \begin{bmatrix}1&&&\\&1&&\\-\mu x^{-1}&&1&\\\kappa-\mu^2 x^{-1}&-\mu x^{-1}&&1\end{bmatrix}
$$

$$
s_2 \begin{bmatrix}1&&&\\&1&x^{-1}&\\&&1&\\&&&1\end{bmatrix} s_2 s_1 y_2)\, dx\, d\mu\, d\kappa
$$

$$
= \xi(x)^{-1}|x|^{-s-\frac{1}{2}} \int_{F^2} \int_{F\setminus\mathfrak{p}^{-1}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix} s_2 \begin{bmatrix}1&&&\\&1&x^{-1}&\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}1&&&\\&1&-\varpi&\\&&1&\\&&&1\end{bmatrix} s_2 s_1)\, dx\, d\mu\, d\kappa
$$

$$
= \frac{(1-q^{-1})\xi^2(\varpi) q^{-2s+1}}{1-q^{-s+\frac{1}{2}}} \int_{F^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix} s_1 y_2)\, dx\, d\mu\, d\kappa.
$$

(c)

$$
\int_{F^2} \int_{\varpi^{-1}\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\\mu&x&1&\\\kappa&\mu&&1\end{bmatrix} s_2 s_1 s_2 y_2)\, dx\, d\mu\, d\kappa
$$

$$
= \int_{F^2} \int_{\varpi^{-1}\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\-\mu x^{-1}&-x^{-1}&-1&\\&&-x&\\&&-\mu&1\end{bmatrix} \begin{bmatrix}1&&&\\&1&&\\-\mu x^{-1}&&1&\\\kappa-\mu^2 x^{-1}&-\mu x^{-1}&&1\end{bmatrix}
$$

$$
s_2 \begin{bmatrix}1&&&\\&1&x^{-1}&\\&&1&\\&&&1\end{bmatrix} s_2 s_1 y_2)\, dx\, d\mu\, d\kappa
$$

$$
= \int_{F^2} \int_{\varpi^{-1}\mathfrak{o}^\times} \xi(x)^{-1}|x|^{-s-\frac{1}{2}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix} s_2 \begin{bmatrix}1&&&\\&1&x^{-1}&\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}1&&&\\&1&-\varpi&\\&&1&\\&&&1\end{bmatrix} s_2 s_1)\, dx\, d\mu\, d\kappa
$$

$$=\xi(\varpi)q^{-s+\frac{1}{2}}\int\limits_{F^2}\int\limits_{\mathfrak{o}^\times} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&\varpi(1-x)&1&\\&&&1\end{bmatrix}s_1)\,dx\,d\mu\,d\kappa.$$

i. If $1-x\in\mathfrak{p}$, i.e., $x\in 1+\mathfrak{p}$, then we have

$$\begin{aligned}
&\xi(\varpi)q^{-s+\frac{1}{2}}\int\limits_{F^2}\int\limits_{1+\mathfrak{p}} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1)\,dx\,d\mu\,d\kappa\\
=&\xi(\varpi)q^{\frac{1}{2}-s}\int\limits_{F^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1)\,d\mu\,d\kappa\cdot\int\limits_{1+\mathfrak{p}}dx\\
=&\xi(\varpi)q^{-\frac{1}{2}-s}\int\limits_{F^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1)\,d\mu\,d\kappa.
\end{aligned}$$

ii. If $x\in\mathfrak{o}^\times\backslash 1+\mathfrak{p}$, i.e., $1-x\in\mathfrak{o}^\times$, then we have

$$\begin{aligned}
&\int\limits_{F^2}\int\limits_{\mathfrak{o}^\times\backslash 1+\mathfrak{p}}\xi(\varpi)|\varpi|^{s-\frac{1}{2}}f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&\varpi(1-x)&1&\\&&&1\end{bmatrix}s_1)\,dx\,d\mu\,d\kappa\\
=&\int\limits_{F^2}\int\limits_{\mathfrak{o}^\times\backslash 1+\mathfrak{p}}\xi(\varpi)|\varpi|^{s-\frac{1}{2}}f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&\varpi&1&\\&&&1\end{bmatrix}s_1)\,dx\,d\mu\,d\kappa\\
=&\int\limits_{F^2}\int\limits_{\mathfrak{o}^\times\backslash 1+\mathfrak{p}}\xi(\varpi)|\varpi|^{s-\frac{1}{2}}f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1y_2)\,dx\,d\mu\,d\kappa\\
=&\xi(\varpi)q^{-s+\frac{1}{2}}\int\limits_{F^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1y_2)\,d\mu\,d\kappa\cdot\int\limits_{\mathfrak{o}^\times\backslash 1+\mathfrak{p}}dx\\
=&(1-2q^{-1})\xi(\varpi)q^{-s+\frac{1}{2}}\int\limits_{F^2} f(\begin{bmatrix}1&&&\\&1&&\\\mu&&1&\\\kappa&\mu&&1\end{bmatrix}s_1y_2)\,d\mu\,d\kappa.
\end{aligned}$$

□

3.3.4 Dimensions of the spaces of fixed vectors for groups V and VI

Proposition 3.12. *Let $V_{\xi,s}$ be the standard space of the induced representation $\xi\nu^s 1_{\mathrm{GL}(2)} \rtimes \xi^{-1}\nu^{-s}$.*

i) Let ξ be the trivial character of $F^\times$. Taking $s = -\frac{1}{2}$, we obtain an intertwining operator

$$\mathcal{A}(-1/2)\colon V_{1,-1/2} \to V_{1,1/2}. \tag{3.49}$$

The restriction of this operator to the four-dimensional space of $\mathrm{Kl}(\mathfrak{p}^2)$-fixed vectors is zero.

ii) Let ξ be the non-trivial unramified quadratic character of $F^\times$. Taking $s = \frac{1}{2}$, we obtain an intertwining operator

$$\mathcal{A}(1/2)\colon V_{\xi,1/2} \to V_{\xi,-1/2}. \tag{3.50}$$

With respect to a suitable basis, the restriction of this operator to the four-dimensional space of $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors has matrix

$$\begin{bmatrix} \frac{q^{-2}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & \frac{q^{-1}(1-q^{-2})}{2} & -\frac{q^{-1}(1-q^{-2})}{2} \\ \frac{q^{-4}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & \frac{q^{-3}(1-q^{-2})}{2} & \frac{q^{-2}(1-q^{-2})}{2} \\ \frac{q^{-2}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & \frac{q^{-1}(1-q^{-2})}{2} & -\frac{q^{-1}(1-q^{-2})}{2} \\ -\frac{q^{-3}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & -\frac{q^{-2}(1-q^{-2})}{2} & q^{-2}(1+q^{-1}) \end{bmatrix}. \tag{3.51}$$

In particular, this matrix has rank 2.

Proof. Let

$$A := \xi(\varpi)q^{-s-\frac{1}{2}}, \quad B := \xi^2(\varpi)q^{-2s}, \quad C := \xi(\varpi)q^{-s+\frac{1}{2}}, \tag{3.52}$$

then we have $C = qA, B = AC = qA^2$.

From the calculation for the double integration in section 3.3.2, it is enough to just show the following cases. More precisely, we need to calculate

1.

$$\begin{aligned}\mathcal{B}(\omega_1) = &\int\limits_{F^2} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix}\omega_1\right)\, d\mu\, d\kappa \\ =&q^{-4}\alpha + \frac{q^{-1}(1-q^{-1})(1+q+C)}{1-B}\beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta.\end{aligned}$$

2.

$$\begin{aligned}\mathcal{B}(\omega_2) = &\int\limits_{F^2} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix}\omega_2\right)\, d\mu\, d\kappa \\ =&(q^{-2} + \frac{q^{-1}(1-q^{-1})A(C+1)}{1-B})\alpha + \frac{1-q^{-1}}{1-B}\beta + q^{-1}(1-q^{-1})\gamma + \frac{(1-q^{-1})^2A}{1-B}\delta.\end{aligned}$$

3.

$$\begin{aligned}\mathcal{B}(\omega_4) = &\int\limits_{F^2} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix}\omega_4\right)\, d\mu\, d\kappa \\ =&\frac{q^{-2}(1-q^{-1})B^2}{1-B}\alpha + (1 + \frac{(1-q^{-1})(A+B)}{1-B})\beta \\ &+ q^{-2}(1-q^{-1})B\gamma + \frac{q^{-1}(1-q^{-1})^2B}{1-B}\delta.\end{aligned}$$

4.

$$\begin{aligned}\mathcal{B}(\omega_6) = &\int\limits_{F^2} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix}\omega_6\right)\, d\mu\, d\kappa \\ =&\frac{(1-q^{-1})A^2(C^2+C+1)}{1-B}\alpha + \beta + (1-q^{-1})B\gamma + (1-q^{-1})A\delta.\end{aligned}$$

5.

$$
\begin{aligned}
\mathcal{B}(\omega_9) = &\int_{F^2} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & & 1 \end{bmatrix} \omega_9\right) d\mu\, d\kappa \\
=&\frac{q^{-1}(1-q^{-1})AB}{1-B}\alpha + \frac{1-q^{-1}}{1-B}\beta + q^{-1}(1-q^{-1})A\gamma \\
&+ (q^{-1} + \frac{(1-q^{-1})A(A-q^{-1}+1)}{1-B})\delta.
\end{aligned}
$$

i) Since $\xi(\varpi) = 1, s = -\frac{1}{2}$, then we have

$$
A = \xi(\varpi)q^{-s-\frac{1}{2}} = 1, \quad B = \xi^2(\varpi)q^{-2s} = q, \quad C = \xi(\varpi)q^{-s+\frac{1}{2}} = q.
$$

(a) $\mathcal{B}(\omega_1) = q^{-4}\alpha - q^{-2}(1+2q)\beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta$.

(b) $\mathcal{B}(\omega_2) = -q^{-1}\alpha - q^{-1}\beta + q^{-1}(1-q^{-1})\gamma - q^{-1}(1-q^{-1})\delta$.

(c) $\mathcal{B}(\omega_4) = -q^{-1}\alpha - q^{-1}\beta + q^{-1}(1-q^{-1})\gamma - q^{-1}(1-q^{-1})\delta$.

(d) $\mathcal{B}(\omega_6) = -q^{-1}(q^2+q+1)\alpha + \beta + q(1-q^{-1})\gamma + (1-q^{-1})\delta$.

(e) $\mathcal{B}(\omega_9) = -q^{-1}\alpha - q^{-1}\beta + q^{-1}(1-q^{-1})\gamma - q^{-1}(1-q^{-1})\delta$.

From above result, we can see that $\mathcal{B}(\omega_2) = \mathcal{B}(\omega_4) = \mathcal{B}(\omega_9)$.

With $(\mathcal{A}(s)f)(\tilde{\omega}_i), 1 \le i \le 4$ as defined in Lemma 3.11, then we have

(a) $(\mathcal{A}(s)f)(\tilde{\omega}_1) = 0$.

(b) $(\mathcal{A}(s)f)(\tilde{\omega}_2) = 0$.

(c) $(\mathcal{A}(s)f)(\tilde{\omega}_3) = 0 \cdot \alpha - 0 \cdot \beta - 0 \cdot \gamma + 0 \cdot \delta = 0$.

(d) $(\mathcal{A}(s)f)(\tilde{\omega}_4) = 0$.

In conclusion, we have $\mathcal{A}(-\frac{1}{2}) = 0$.

ii) Since $\xi(\varpi) = -1, s = \frac{1}{2}$, then

$$A = \xi(\varpi)q^{-s-\frac{1}{2}} = -q^{-1}, B = \xi^2(\varpi)q^{-2s} = q^{-1},\ C = \xi(\varpi)q^{-s+\frac{1}{2}} = -1.$$

(a) $\mathcal{B}(\omega_1) = q^{-4}\alpha + \beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta$.

(b) $\mathcal{B}(\omega_2) = q^{-2}\alpha + \beta + q^{-1}(1-q^{-1})\gamma - q^{-1}(1-q^{-1})\delta$.

(c) $\mathcal{B}(\omega_4) = q^{-4}\alpha + \beta + q^{-3}(1-q^{-1})\gamma + q^{-2}(1-q^{-1})\delta$.

(d) $\mathcal{B}(\omega_6) = q^{-2}\alpha + \beta + q^{-1}(1-q^{-1})\gamma - q^{-1}(1-q^{-1})\delta$.

(e) $\mathcal{B}(\omega_9) = -q^3\alpha + \beta - q^{-2}(1-q^{-1})\gamma + 2q^{-2}\delta$.

Similarly, from above result, we can see that $\mathcal{B}(\omega_1) = \mathcal{B}(\omega_4), \mathcal{B}(\omega_2) = \mathcal{B}(\omega_6)$. Therefore, we have

(a) $(\mathcal{A}(s)f)(\tilde{\omega}_1) = \frac{q^{-2}(1+q^{-1})}{2}\alpha + \frac{1+q^{-1}}{2}\beta + \frac{q^{-1}(1-q^{-2})}{2}\gamma - \frac{q^{-1}(1-q^{-2})}{2}\delta$.

(b) $(\mathcal{A}(s)f)(\tilde{\omega}_2) = \frac{q^{-4}(1+q^{-1})}{2}\alpha + \frac{1+q^{-1}}{2}\beta + \frac{q^{-3}(1-q^{-2})}{2}\gamma + \frac{q^{-2}(1-q^{-2})}{2}\delta$.

(c) $(\mathcal{A}(s)f)(\tilde{\omega}_3) = \frac{q^{-2}(1+q^{-1})}{2}\alpha + \frac{1+q^{-1}}{2}\beta + \frac{q^{-1}(1-q^{-2})}{2}\gamma - \frac{q^{-1}(1-q^{-2})}{2}\delta$.

(d) $(\mathcal{A}(s)f)(\tilde{\omega}_4) = -\frac{q^{-3}(1+q^{-1})}{2}\alpha + \frac{1+q^{-1}}{2}\beta - \frac{q^{-2}(1-q^{-2})}{2}\gamma + q^{-2}(1+q^{-1})\delta$.

In conclusion, we have

$$\mathcal{A}(\frac{1}{2}) = \begin{bmatrix} \frac{q^{-2}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & \frac{q^{-1}(1-q^{-2})}{2} & -\frac{q^{-1}(1-q^{-2})}{2} \\ \frac{q^{-4}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & \frac{q^{-3}(1-q^{-2})}{2} & \frac{q^{-2}(1-q^{-2})}{2} \\ \frac{q^{-2}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & \frac{q^{-1}(1-q^{-2})}{2} & -\frac{q^{-1}(1-q^{-2})}{2} \\ -\frac{q^{-3}(1+q^{-1})}{2} & \frac{1+q^{-1}}{2} & -\frac{q^{-2}(1-q^{-2})}{2} & q^{-2}(1+q^{-1}) \end{bmatrix}. \tag{3.53}$$

With a similar process as in Proposition 2.5 of [23] to obtain the assertions.

□

Corollary 3.13. *With $V_{\xi,s}$, $\mathcal{A}(-1/2)$ and $\mathcal{A}(1/2)$ as in Proposition 3.12, we have* *i) The restriction of the operator* $\mathcal{A}(-1/2)$ *to the three-dimensional space of* $\mathrm{M}(\mathfrak{p}^2)$*-invariant vectors is zero.*

ii) With respect to a suitable basis, the restriction of the operator $\mathcal{A}(1/2)$ *to the three-dimensional space of* $\mathrm{M}(\mathfrak{p}^2)$*-invariant vectors has matrix*

$$\begin{bmatrix} \frac{q^{-2}(1+q^{-1})}{2} & \frac{1+q}{2} & \frac{q^{-1}(1-q^{-2})}{2} \\ \frac{q^{-4}(1+q^{-1})}{2} & \frac{q^{-2}(1+q)}{2} & \frac{q^{-3}(1-q^{-2})}{2} \\ \frac{q^{-2}(1+q^{-1})}{2} & \frac{1+q}{2} & \frac{q^{-1}(1-q^{-2})}{2} \end{bmatrix}. \tag{3.54}$$

In particular, this matrix has rank 1*.*

Proof. The matrix identity

$$y_2 = \begin{bmatrix} -\varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\varpi \end{bmatrix} \begin{bmatrix} 1 & & & \varpi \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 \cdot s_2 s_1 \cdot \begin{bmatrix} 1 & & & \varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{3.55}$$

implies that y_2 and s_2s_1 define the same element of $P(F)\backslash G(F)/\mathrm{M}(\mathfrak{p}^2)$. Then the assertions follow easily from Proposition 3.12. □

Proposition 3.14. *Table 3.2 shows the dimensions of the spaces of* $\mathrm{M}(\mathfrak{p}^2)$ *and* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for the irreducible, admissible representations* (π, V) *of groups* V *and* VI*. Here,* ξ *and* σ *are unramified characters of* $F^\times$*.*

Proof. By (2.11) in [15, section 2.2], it follows that the dimension of the space of Γ-invariant vectors of type VIb is equal to the rank of the matrix $\mathcal{A}(-1/2)$ as in (3.49), where Γ is the subgroup $\mathrm{M}(\mathfrak{p}^2)$ or $\mathrm{Kl}(\mathfrak{p}^2)$. Moreover, we have the relations

$$\dim(\mathrm{VIb})^\Gamma + \dim(\mathrm{VId})^\Gamma = \dim(\mathrm{IIb})^\Gamma, \qquad \dim(\mathrm{VIc})^\Gamma + \dim(\mathrm{VId})^\Gamma = \dim(\mathrm{IIIb})^\Gamma.$$

Similarly by (2.10) in [15, section 2.2], the dimension of the space of Γ-invariant vectors of type Vd is equal to the rank of the matrix $\mathcal{A}(1/2)$ as in (3.50). Again

Table 3.2: Dimensions of the spaces of $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for Iwahori-spherical representations of groups V to VI.

		constituent of	representation	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
V	a	$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	3	5
	b	$(\xi^2 = 1,\ \xi \neq 1)$	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	2	2
	c		$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	2	2
	d		$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	1	2
VI	a	$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$	$\tau(S, \nu^{-1/2}\sigma)$	3	5
	b		$\tau(T, \nu^{-1/2}\sigma)$	0	0
	c		$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	2	2
	d		$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	3	4

we have the relation $\dim(\mathrm{Vb})^\Gamma + \dim(\mathrm{Vd})^\Gamma = \dim(\mathrm{IIb})^\Gamma$. Moreover, we consider the representation of type Vb as the representation of type Vc twisted by some character σ. It follows that $\dim(\mathrm{Vb})^\Gamma = \dim(\mathrm{Vc})^\Gamma$. Then by Proposition 3.12 and Corollary 3.13, we obtain the desired dimensional data as in (3.2). □

Chapter 4

Non Iwahori-spherical: Borel-induced representations

In this chapter, we obtain the desired dimensional data for the Borel-induced representations which are non Iwahori-spherical representations.

4.1 Depth zero representations

4.1.1 $\mathrm{Kl}(\mathfrak{p}^n)$-vectors and $\mathrm{Kl}_1(\mathfrak{p}^n)$-vectors

Let

$$\mathrm{Kl}(\mathfrak{p}^n) = \mathrm{GSp}(4, \mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} \tag{4.1}$$

and

$$\mathrm{Kl}_1(\mathfrak{p}^n) = \mathrm{GSp}(4, \mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{n-1} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^{n-1} & \mathfrak{o} \end{bmatrix}. \tag{4.2}$$

Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$. Let $n \geq 2$ be an integer. We define a linear map $\alpha : V^{\mathrm{Kl}(\mathfrak{p}^n)} \longrightarrow V^{\mathrm{Kl}_1(\mathfrak{p}^n)}$ by

$$\alpha(v) = \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\right) v. \tag{4.3}$$

We further define a linear map $\beta : V^{\mathrm{Kl}_1(\mathfrak{p}^n)} \longrightarrow V^{\mathrm{Kl}(\mathfrak{p}^n)}$ by

$$\beta(v) = \sum_{g \in \mathrm{GL}(2,\mathfrak{o})/\Gamma_0(\mathfrak{p})} \pi\left(\begin{bmatrix} 1 && \\ & g & \\ && \det(g) \end{bmatrix}\right) v = \sum_{y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ & y & 1 & \\ &&& 1 \end{bmatrix}\right) v + \pi(s_2)v. \tag{4.4}$$

Lemma 4.1. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4,F)$. *Let n be an integer such that $n \geq 2$. Let α and βbe the maps defines above.*

i) α is injective.

ii) Suppose $v \in V^{\mathrm{Kl}_1(\mathfrak{p}^n)}$ is such that $\beta(v) = 0$. Then v is invariant under

$$\begin{bmatrix} 1 &&& \\ & 1 && \\ \mathfrak{p}^{n-1} && 1 & \\ & \mathfrak{p}^{n-1} && 1 \end{bmatrix} \tag{4.5}$$

Proof. i) We calculate, for $v \in V^{\mathrm{Kl}(\mathfrak{p}^n)}$,

$$\begin{aligned}
&\beta(\alpha(v)) \\
&= \sum_{y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ & y & 1 & \\ &&& 1 \end{bmatrix}\right) \alpha(v) + \pi(s_2)\alpha(v) \\
&= \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ & y & 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\right) v + \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(s_2 \begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\right) v \\
&= \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ yx\varpi^{n-1} & y & 1 & \\ yx^2\varpi^{2n-2} & yx\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&\qquad + \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ x\varpi^{n-1} && 1 & \\ & x\varpi^{n-1} && 1 \end{bmatrix} s_2\right) v
\end{aligned}$$

$$
\begin{aligned}
&= qv + \sum_{x \in (\mathfrak{o}/\mathfrak{p})^\times} \sum_{y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ yx\varpi^{n-1} && 1 & \\ & yx\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&\qquad + \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ x\varpi^{n-1} && 1 & \\ & x\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&= qv + \sum_{x \in (\mathfrak{o}/\mathfrak{p})^\times} \sum_{y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ y\varpi^{n-1} && 1 & \\ & y\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&\qquad + \sum_{y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ y\varpi^{n-1} && 1 & \\ & y\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&= qv + \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ y\varpi^{n-1} && 1 & \\ & y\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&= qv + \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ y\varpi^{n-1} && 1 & \\ & y\varpi^{n-1} && 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\right) v \\
&= qv + \sum_{y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ y\varpi^{n-1} && 1 & \\ & y\varpi^{n-1} && 1 \end{bmatrix}\right) \alpha(v).
\end{aligned}
$$

Hence, if $\alpha(v) = 0$, then $v = 0$.

ii) We calculate, for $v \in V^{\mathrm{Kl}_1(\mathfrak{p}^n)}$,

$$
\begin{aligned}
&\alpha(\beta(v)) \\
&= \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\right) \beta(v) \\
&= \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 && \\ & y & 1 & \\ &&& 1 \end{bmatrix}\right) v + \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix} s_2\right) v \\
&= \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ -yx\varpi^{n-1} & y & 1 & \\ yx^2\varpi^{2n-2} & -yx\varpi^{n-1} && 1 \end{bmatrix}\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix}\right) v \\
&\qquad + \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix} s_2\right) v \\
&= \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ & 1 && \\ -yx\varpi^{n-1} & y & 1 & \\ & -yx\varpi^{n-1} && 1 \end{bmatrix}\right) v + \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix} s_2\right) v
\end{aligned}
$$

$$
\begin{aligned}
=&qv + \sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ -yx\varpi^{n-1} & 1 && \\ & y & 1 & \\ & -yx\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&\qquad\qquad + \sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ && -x\varpi^{n-1} & 1 \end{bmatrix} s_2\right) v \\
=&qv + \sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ & y & 1 & \\ & x\varpi^{n-1} && 1 \end{bmatrix}\right) v + \sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi\left(s_2\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ & x\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
=&qv + \sum_{x,y\in\mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ & y & 1 & \\ & x\varpi^{n-1} && 1 \end{bmatrix}\right) v - \sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi\left(\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ & x\varpi^{n-1} && 1 \end{bmatrix}\right) v \\
&\quad + \sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi\left(s_2\begin{bmatrix} 1 &&& \\ x\varpi^{n-1} & 1 && \\ && 1 & \\ & x\varpi^{n-1} && 1 \end{bmatrix}\right) v
\end{aligned}
$$

The three sums are all invariant under the group (4.5). Hence, if $\beta(v) = 0$, then v is also invariant under the group (4.5). □

4.1.2 The case $n = 2$

Now consider

$$
\mathrm{Kl}(\mathfrak{p}^2) = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{o} \end{bmatrix} \text{ and } \mathrm{Kl}_1(\mathfrak{p}^2) = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \qquad (4.6)
$$

We also define

$$
\mathrm{Kl}_{11}(\mathfrak{p}^2) = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \qquad (4.7)
$$

By Lemma 4.1, we have an injective map $\alpha : V^{\mathrm{Kl}(\mathfrak{p}^2)} \to V^{\mathrm{Kl}_1(\mathfrak{p}^2)}$, and a map $\beta : V^{\mathrm{Kl}_1(\mathfrak{p}^2)} \to V^{\mathrm{Kl}(\mathfrak{p}^2)}$ whose kernel is contained in $V^{\mathrm{Kl}_{11}(\mathfrak{p}^2)}$. Observe that

$$
\mathrm{Kl}_1(\mathfrak{p}^2)^\omega := \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} \mathrm{Kl}_1(\mathfrak{p}^2) \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& 1 \end{bmatrix} = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix} \qquad (4.8)
$$

and

$$\mathrm{Kl}_{11}(\mathfrak{p}^2)^{\omega} := \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} \mathrm{Kl}_{11}(\mathfrak{p}^2) \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& 1 \end{bmatrix} = \mathrm{GSp}(4, \mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} \end{bmatrix} \qquad (4.9)$$

both contain $\Gamma(\mathfrak{p})$.

Lemma 4.2. *Let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$. Suppose that $V^{\mathrm{Kl}(\mathfrak{p}^2)} \neq 0$. Then π is a depth zero representation, i.e., $V^{\Gamma(\mathfrak{p})} \neq 0$.*

Proof. Suppose that $V^{\mathrm{Kl}(\mathfrak{p}^2)} \neq 0$. By Lemma 4.1 i), then also $V^{\mathrm{Kl}_1(\mathfrak{p}^2)} \neq 0$. Hence there exist non-zero vectors invariant under the group $\mathrm{Kl}_1(\mathfrak{p}^2)^{\omega}$ defined in (4.8). In particular, there exist non-zero vectors invariant under $\Gamma(\mathfrak{p})$. □

As an immediate consequence, we have

- If π is a depth zero supercuspidal irreducible admissible representation of $\mathrm{GSp}(4, F)$, we can study π by restricting it to a irreducible cuspidal admissible representation of the finite group $\mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p})$; see Lemma 2.7.

- If π is a parabolically (Borel-, Klingen- and Siegel-) induced representation of $\mathrm{GSp}(4, F)$. Then by Theorem 5.2 of [14] and the property of depth zero of π, we have the following conclusions.

  i) The depth zero character χ of $F^{\times}$ has the conductor at most 1, i.e., $a(\chi) \leq 1$.

  ii) The depth zero supercuspidal representation τ of $\mathrm{GL}(2, F)$ has the conductor exactly 2, i.e., $a(\pi) = 2$.

4.2 Group I

Recall the Iwahori factorization for $\mathrm{Kl}(\mathfrak{p}^2)$, i.e.,

$$\begin{aligned}\mathrm{Kl}(\mathfrak{p}^2) &= \begin{bmatrix} 1 &&& \\ \mathfrak{p}^2 & 1 && \\ \mathfrak{p}^2 && 1 & \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times &&& \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ &&& \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 && \mathfrak{o} \\ && 1 & \mathfrak{o} \\ &&& 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 && \mathfrak{o} \\ && 1 & \mathfrak{o} \\ &&& 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times &&& \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ &&& \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 &&& \\ \mathfrak{p}^2 & 1 && \\ \mathfrak{p}^2 && 1 & \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & 1 \end{bmatrix}. \end{aligned} \tag{4.10}$$

For this reason, we let

$$A = \begin{bmatrix} 1 &&& \\ u & 1 && \\ v && 1 & \\ w & v & -u & 1 \end{bmatrix}, \quad M = \begin{bmatrix} t &&& \\ & a & b & \\ & c & d & \\ &&& \frac{\Delta}{t} \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & x & y & z \\ & 1 && y \\ && 1 & -x \\ &&& 1 \end{bmatrix}, \tag{4.11}$$

where $\Delta = ad - bc, u, v, w \in \mathfrak{p}^2, x, y, z \in \mathfrak{o}, t \in \mathfrak{o}^\times, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o})$. Consider the full Borel-induced representation $\chi_1 \times \chi_2 \rtimes \sigma$, where χ_1, χ_2 and σ are characters of $F^\times$. The standard space V consists of smooth functions $f\colon B \to \mathbb{C}^\times$ with the transformation property

$$f(\begin{bmatrix} \tilde{a} & * & * & * \\ & \tilde{b} & * & * \\ && \tilde{c}\tilde{b}^{-1} & * \\ &&& \tilde{c}\tilde{a}^{-1} \end{bmatrix} h) = |\tilde{a}^2\tilde{b}||\tilde{c}|^{-3/2}\chi_1(\tilde{a})\chi_2(\tilde{b})\sigma(\tilde{c})f(h), \quad \tilde{a}, \tilde{b}, \tilde{c} \in F^\times \tag{4.12}$$

Let

$$\chi(\tilde{g}) = |\tilde{a}^2\tilde{b}||\tilde{c}|^{-3/2}\chi_1(\tilde{a})\chi_2(\tilde{b})\sigma(\tilde{c}), \tag{4.13}$$

and we take

$$\tilde{g} = \begin{bmatrix} \tilde{a} & * & * & * \\ & \tilde{b} & * & * \\ && \tilde{c}\tilde{b}^{-1} & * \\ &&& \tilde{c}\tilde{a}^{-1} \end{bmatrix} \in B(F), h \in \mathrm{GSp}(4, F). \tag{4.14}$$

Suppose $f \in V^{\mathrm{Kl}(\mathfrak{p}^2)}$, then f is determined on the set of representatives for the double cosets $B(F)\backslash\mathrm{GSp}(4, F)/\mathrm{Kl}(\mathfrak{p}^2)$ as in Proposition 3.6 i). That is to say, f is determined by $f(r_1), f(r_2), \ldots, f(r_{11})$. Consider $B(F)r_i\mathrm{Kl}(\mathfrak{p}^2), i \in \{1, 2, \ldots, 11\}$

and assume $f(r_i) \neq 0$. Then for $\tilde{g} \in B(F), h \in \mathrm{Kl}(\mathfrak{p}^2)$, we need

$$f(\tilde{g} r_i h) = \chi(\tilde{g}) f(r_i), \quad i \in \{1, 2, \dots, 11\}. \tag{4.15}$$

To prove above equation (4.15) is well defined, it is equivalent to show that if for some $\tilde{g}' \in B(F), h' \in \mathrm{Kl}(\mathfrak{p}^2)$, we have

$$\tilde{g} r_i h = \tilde{g}' r_i h'. \tag{4.16}$$

Then, it follows that

$$\tilde{g}'^{-1} \tilde{g} = r_i h' h^{-1} r_i^{-1} \in B(F) \cap r_i \mathrm{Kl}(\mathfrak{p}^2) r_i^{-1}. \tag{4.17}$$

Since $f \in V^{\mathrm{Kl}(\mathfrak{p}^2)}$ and $h, h' \in \mathrm{Kl}(\mathfrak{p}^2)$, it follows from (4.16) that

$$\chi(\tilde{g}) f(r_i) = \chi(\tilde{g}') f(r_i). \tag{4.18}$$

Thus, we obtain

$$\chi(\tilde{g}'^{-1} \tilde{g}) f(r_i) = f(r_i), \quad f(r_i) \neq 0. \tag{4.19}$$

By the equation (4.17), the well-definedness of (4.15) is equivalent to the following condition for the character χ as defined in (4.12).

$$\chi \text{ must be trivial on } B(F) \cap r_i \cdot \mathrm{Kl}(\mathfrak{p}^2) \cdot r_i^{-1}, \quad i \in \{1, 2, \dots, 11\}. \tag{4.20}$$

Because we are concerning the non Iwahori-spherical representations, the character χ should not be *unramified*, i.e.,

$$\chi_1, \chi_2 \text{ and } \sigma \text{ cannot be unramified at the same time.} \tag{4.21}$$

In addition, all the representations of $\mathrm{GSp}(4, F)$ are with the trivial central character. That is to say, $\chi_1\chi_2\sigma^2 = 1$. Furthermore, with the discussion in previous section 4.1.2, we have

$$a(\chi_1) \leq 1, \quad a(\chi_2) \leq 1, \quad a(\sigma) \leq 1. \tag{4.22}$$

i) For $r_1 = \mathbf{I}_4$, to ensure that $r_1 \mathrm{Kl}(\mathfrak{p}^2) r_1^{-1} \in B(F)$, we need

$$u = v = w = c = 0.$$

And the matrix will be

$$\begin{bmatrix} t & & & \\ & a & & \\ & & d & \\ & & & \frac{ad}{t} \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(t)\chi_2(a)\sigma(ad) = 1, \quad \forall t, a, d \in \mathfrak{o}^\times. \tag{4.23}$$

It is easy to see that we need the characters χ_1, χ_2 and σ are all *unramified.* This will not happen since we are considering the non Iwahori-spherical representations; see (4.21).

ii) For $r_2 = s_1$, to ensure that $r_2 \mathrm{Kl}(\mathfrak{p}^2) r_2^{-1} \in B(F)$, we need

$$x = v = c = w = 0.$$

And the matrix will be

$$\begin{bmatrix} a &&& \\ & t && \\ && \frac{ad}{t} & \\ &&& d \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ && 1 & * \\ &&& 1 \end{bmatrix}.$$

By the similar reason for the case of $r_1 = \mathbf{I}_4$, we need the condition that χ_1, χ_2 and σ are all *unramified* and this is impossible; see (4.21).

iii) For $r_3 = s_2 s_1$, to ensure that $r_3 \mathrm{Kl}(\mathfrak{p}^2) r_3^{-1} \in B(F)$, we need

$$x = c = z = v = 0.$$

And the matrix will be

$$\begin{bmatrix} a &&& \\ & \frac{ad}{t} && \\ && t & \\ &&& d \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ && 1 & * \\ &&& 1 \end{bmatrix}.$$

By the similar reason for the case of $r_1 = \mathbf{I}_4$, we need the condition that χ_1, χ_2 and σ are all *unramified* and this is impossible; see (4.21).

iv) For $r_4 = s_1 s_2 s_1$, to ensure that $r_4 \mathrm{Kl}(\mathfrak{p}^2) r_4^{-1} \in B(F)$, we need

$$x = y = z = c = 0$$

And the matrix will be

$$\begin{bmatrix} \frac{ad}{t} &&& \\ & a && \\ && d & \\ &&& t \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ && 1 & * \\ &&& 1 \end{bmatrix}.$$

By the similar reason for the case of $r_1 = \mathbf{I}_4$, we need the condition that χ_1, χ_2 and σ are all *unramified* and this is impossible; see (4.21).

In fact, r_1, r_2, r_3 and r_4 are the elements of Weyl group W of $\mathrm{GSp}(4, F)$ which is 8-element group generated by s_1 and s_2; see (1.5). From above discussion, all of these four cases cannot support non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vectors for non Iwahori-spherical representations.

5. For $r_5 = \begin{bmatrix} 1 & & & \\ \varpi & 1 & & \\ & & 1 & \\ & & -\varpi & 1 \end{bmatrix}$, to ensure that $r_5 \mathrm{Kl}(\mathfrak{p}^2) r_5^{-1} \in B(F)$, we need

$$c = w = v = 0, t = \frac{a\varpi}{(\varpi + u)(1 - x\varpi)} \in \mathfrak{o}^{\times}.$$

And the matrix will be

$$\begin{bmatrix} \frac{a\varpi}{\varpi+u} & & & \\ & \frac{a}{1-x\varpi} & & \\ & & d(1-x\varpi) & \\ & & & d(1+\frac{u}{\varpi}) \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(a)\chi_2(a)\sigma(ad) = (\chi_1\chi_2)(a)\sigma(ad) = \sigma(a^{-1}d) = 1, \quad \forall a, d \in \mathfrak{o}^{\times}. \quad (4.24)$$

The last equality is because of the trivial central character ($\chi_1\chi_2\sigma^2 = 1$). Then σ has to be *unramified*. It further implies that $a(\chi_1\chi_2) = 0$. Since we are studying the non Iwahori–spherical representations, then we have

$$a(\chi_1) = a(\chi_2) = 1, \quad a(\sigma) = 0. \quad (4.25)$$

6. For $r_6 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}$, to ensure that $r_6 \mathrm{Kl}(\mathfrak{p}^2) r_6^{-1} \in B(F)$, we need

i) $t = \frac{\varpi d}{(1-\pi y)(\pi + v)} = \frac{d}{(1-\pi y)(1+v\varpi^{-1})} \in \mathfrak{o}^{\times}$.

ii) $b = \frac{du}{\pi+v} \in \mathfrak{p}$.

iii) $w = -u\varpi \in \mathfrak{p}^3$.

iv) $c = \frac{d\varpi(x(-2+\varpi y)+\varpi z)}{(-1+\varpi y)^2} \in \mathfrak{p}$.

And the matrix will be

$$\begin{bmatrix} \frac{d\varpi}{\varpi+v} & & & \\ & a(1-\varpi y)+\frac{d\varpi u((-2+\varpi y)x+\varpi z)}{(\varpi+v)(-1+\varpi y)} & & \\ & & \frac{d}{1-\varpi y} & \\ & & & a(1+\frac{v}{\varpi})-\frac{du((-2+\varpi y)x+\varpi z)}{(-1+\varpi y)^2} \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(d)\chi_2(a)\sigma(ad) = (\chi_1\sigma)(d)(\chi_2\sigma)(a) = 1, \quad \forall a, d \in \mathfrak{o}^\times. \tag{4.26}$$

By the trivial central character ($\chi_1\chi_2\sigma^2 = 1$), we have

$$(\chi_1\sigma)(a^{-1}d) = (\chi_2\sigma)(ad^{-1}) = 1, \quad \forall a, d \in \mathfrak{o}^\times. \tag{4.27}$$

It implies that

$$a(\chi_i\sigma) = 0, \quad i \in \{1, 2\}. \tag{4.28}$$

Again, since we are studying the non Iwahori–spherical representations, then we have

$$a(\chi_1) = a(\chi_2) = a(\sigma) = 1, a(\chi_i\sigma) = 0, \quad i \in \{1, 2\}. \tag{4.29}$$

7. For $r_7 = s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}$, to ensure that $r_7\mathrm{Kl}(\mathfrak{p}^2)r_7^{-1} \in B(F)$, we need

   i) $x = \varpi z \in \mathfrak{p}$.

ii) $c = -\frac{d\varpi^2 z}{1-\varpi y} \in \mathfrak{p}^2$.

iii) $t = \frac{d}{(1-\varpi y)(1+\varpi^{-1}v)} \in \mathfrak{o}^\times$.

iv) $b = \frac{d(2\varpi u+uv+w)}{(\varpi+v)^2} \in \mathfrak{o}$.

And the matrix will be

$$\begin{bmatrix} a(1-y\varpi)+\frac{d\varpi^2(2\varpi u+uv+w)z}{(\varpi+v)^2} & & & \\ & \frac{d\varpi}{\varpi+v} & & \\ & & a(1+\frac{v}{\varpi})-\frac{d\varpi(2\varpi u+uv+w)z}{(\varpi+v)(-1+\varpi y)} & \\ & & & \frac{d}{1-\varpi y} \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(a)\chi_2(d)\sigma(ad) = (\chi_1\sigma)(a)(\chi_2\sigma)(d) = 1, \quad \forall a, d \in \mathfrak{o}^\times. \tag{4.30}$$

By the trivial central character ($\chi_1\chi_2\sigma^2 = 1$), we have

$$(\chi_1\sigma)(ad^{-1}) = (\chi_2\sigma)(a^{-1}d) = 1, \quad \forall a, d \in \mathfrak{o}^\times. \tag{4.31}$$

It implies that

$$a(\chi_i\sigma) = 0, \quad i \in \{1, 2\}. \tag{4.32}$$

Again, since we are studying the non Iwahori–spherical representations, then we have

$$a(\chi_1) = a(\chi_2) = a(\sigma) = 1, a(\chi_i\sigma) = 0, \quad i \in \{1, 2\}. \tag{4.33}$$

8. For $r_8 = s_2 s_1 \begin{bmatrix} 1 & & & \\ \varpi & 1 & & \\ & \varpi & 1 & \\ & & & 1 \end{bmatrix}$, to ensure that $r_8 \mathrm{Kl}(\mathfrak{p}^2) r_8^{-1} \in B(F)$, we need

$$z = x = c = 0, t = \frac{d\varpi}{(\varpi+v)(1-\varpi y)} \in \mathfrak{o}^\times.$$

And the matrix will be

$$\begin{bmatrix} a(1-\varpi y) & & & \\ & a(1+\frac{v}{\varpi}) & & \\ & & \frac{d\varpi}{\varpi+v} & \\ & & & \frac{d}{1-\varpi y} \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(a)\chi_2(a)\sigma(ad) = (\chi_1\chi_2)(a)\sigma(ad) = \sigma(a^{-1}d) = 1, \quad \forall a, d \in \mathfrak{o}^\times. \tag{4.34}$$

The last equality is because of the trivial central character ($\chi_1\chi_2\sigma^2 = 1$). Then σ has to be *unramified*. It further implies that $a(\chi_1\chi_2) = 0$. Since we are studying the non Iwahori–spherical representations, then we have

$$a(\chi_1) = a(\chi_2) = 1, \quad a(\sigma) = 0. \tag{4.35}$$

9. For $r_9 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix}$, to ensure that $r_9 \mathrm{Kl}(\mathfrak{p}^2) r_9^{-1} \in B(F)$, we need

   i) $x = \frac{cu-av}{t(\varpi+w)} \in \mathfrak{p}$.

   ii) $y = \frac{du-bv}{t(\varpi+w)} \in \mathfrak{p}$.

   iii) $c = \frac{av^2}{\varpi+w+uv} \in \mathfrak{p}^3$.

   iv) $z = \frac{bc-ad}{t^2(\varpi+w)} + \frac{1}{\varpi} = \frac{\frac{bc-ad}{t^2}+(1+\varpi^{-1}w)}{\varpi+w} \in \mathfrak{o}$.

   And the matrix will be

$$\begin{bmatrix} \frac{a\varpi(-bv^2+d(\varpi+uv+w))}{t(\varpi+w)(\varpi+uv+w)} & & & \\ & \frac{a(\varpi+w)}{\varpi+uv+w} & & \\ & & d+\frac{v(du-bv)}{\varpi+w} & \\ & & & t(1+\frac{w}{\varpi}) \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(\frac{ad}{t})\chi_2(a)\sigma(ad) = 1. \tag{4.36}$$

By the above condition iv), we have

$$\frac{bc - ad}{t^2} + (1 + \varpi^{-1}w) \in \mathfrak{p}. \tag{4.37}$$

Since $c \in \mathfrak{p}^3, w \in \mathfrak{p}^2$, it follows that

$$-\frac{ad}{t^2} + 1 \in \mathfrak{p} \Longleftrightarrow ad \in t^2 + \mathfrak{p}. \tag{4.38}$$

Therefore, by (4.36) and the trivial central character, we have

$$\chi_1(t)\chi_2(a)\sigma(t^2) = \chi_2(at^{-1}) = 1, \quad \forall a, t \in \mathfrak{o}^\times. \tag{4.39}$$

Thus, we have the conclusion that χ_2 is *unramified*, i.e., $a(\chi_2) = 0$. Again, since we are studying the non Iwahori–spherical representations, then we have two possibilities

(a) $a(\sigma) = 1, a(\chi_1) = 0, a(\chi_2) = 0.$

(b) $a(\sigma) = 1, a(\chi_1) = 1, a(\chi_2) = 0.$

10. For $r_{10} = s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix}$, to ensure that $r_{10}\mathrm{Kl}(\mathfrak{p}^2)r_{10}^{-1} \in B(F)$, we need

$$x = c = v = 0, a = \frac{t^2(\varpi + w)(1 - \varpi z)}{d\varpi} \in \mathfrak{o}^\times.$$

And the matrix will be

$$\begin{bmatrix} \frac{t^2(\varpi+w)(1-\varpi z)}{d\varpi} & & & \\ & t(1-\varpi z) & & \\ & & t(1+\frac{w}{\varpi}) & \\ & & & d \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(\frac{t^2}{d})\chi_2(t)\sigma(t^2) = \chi_1(td^{-1}) = 1, \quad \forall d, t \in \mathfrak{o}^\times. \tag{4.40}$$

Thus, we have the conclusion that χ_1 is *unramified*, i.e., $a(\chi_1) = 0$. Again, since we are studying the non Iwahori–spherical representations, then we have two possibilities

(a) $a(\sigma) = 1, a(\chi_1) = 0, a(\chi_2) = 0$.

(b) $a(\sigma) = 1, a(\chi_1) = 0, a(\chi_2) = 1$.

11. For $r_{11} = s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}$, to ensure that $r_{11}\mathrm{Kl}(\mathfrak{p}^2)r_{11}^{-1} \in B(F)$, we need

   i) $x = z = c = 0$.

   ii) $d = \frac{t(\varpi+v)(1-\varpi y)}{\varpi} \in \mathfrak{o}^\times$.

   iii) $t = \frac{(a+b\varpi)(\varpi+v)}{(\varpi+2\varpi u+uv+w)(1-\varpi y)} \in \mathfrak{o}^\times$.

   And the matrix will be

$$\begin{bmatrix} a(1-\varpi y) & & & \\ & \frac{(a+b\varpi)(\varpi+v)}{\varpi+2\varpi u+uv+w} & & \\ & & a(1+\frac{v}{\varpi}) & \\ & & & \frac{(a+b\varpi)(\varpi+v)^2}{\varpi(\varpi+2\varpi u+uv+w)(1-\varpi y)} \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

Table 4.1: Inducing data for non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vectors of Borel-induced cases.

$B(F)r_i\mathrm{Kl}(\mathfrak{p}^2)$	Inducing data
r_1	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_2	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_3	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_4	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_5	$a(\sigma) = 0$
r_6	$a(\chi_1\sigma) = a(\chi_2\sigma) = 0$
r_7	$a(\chi_1\sigma) = a(\chi_2\sigma) = 0$
r_8	$a(\sigma) = 0$
r_9	$a(\chi_2) = 0$
r_{10}	$a(\chi_1) = 0$
r_{11}	$a(\chi_1\chi_2\sigma^2) = 1$

It follows from the condition (4.20) and (4.22) that

$$\chi_1(a)\chi_2(a+b\varpi)\sigma(a^2(1+a^{-1}b\varpi)) = (\chi_1\chi_2\sigma^2)(a) = 1, \quad \forall a \in \mathfrak{o}^\times. \tag{4.41}$$

It is always true by the fact of trivial central character. Thus, there is no the other restriction for the characters χ_1, χ_2 and σ. Again, however, they cannot be *unramified* at the same time since we are focusing on the non Iwahori-spherical representations.

Proposition 4.3. *Let $i \in \{1, 2, \ldots, 11\}$. The inducing data for the double cosets $B(F)r_i\mathrm{Kl}(\mathfrak{p}^2)$ to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector is given in Table 4.1. We assume the central character is trivial, i.e., $\chi_1\chi_2\sigma^2 = 1$. Moreover, if we are*

Table 4.2: Inducing data for $\mathrm{Kl}(\mathfrak{p}^2)$ of non Iwahori-spherical Borel-induced case.

$B(F)r_i\mathrm{Kl}(\mathfrak{p}^2)$	Inducing data
r_1	**NA**
r_2	**NA**
r_3	**NA**
r_4	**NA**
r_5	$a(\sigma)=0, a(\chi_1)=a(\chi_2)=1$
r_6	$a(\sigma)=a(\chi_1)=a(\chi_2)=1, a(\chi_j\sigma)=0, j\in\{1,2\}$
r_7	$a(\sigma)=a(\chi_1)=a(\chi_2)=1, a(\chi_j\sigma)=0, j\in\{1,2\}$
r_8	$a(\sigma)=0, a(\chi_1)=a(\chi_2)=1$
r_9	$a(\sigma)=1, a(\chi_1)\in\{0,1\}, a(\chi_2)=0$
r_{10}	$a(\sigma)=1, a(\chi_1)=0, a(\chi_2)\in\{0,1\}$
r_{11}	$\{a(\sigma)\in\{0,1\}, a(\chi_j)\in\{0,1\}\}\setminus\{a(\chi_j)=a(\sigma)=0\}, j\in\{1,2\}$

just considering non Iwahori-spherical representations, then above table shall be as Table 4.2.

Proof. The conclusions follows from the above discussion. □

Corollary 4.4. *Let $i\in\{1,2,\ldots,8\}$. The inducing data for the double cosets $B(F)r_i\mathrm{M}(\mathfrak{p}^2)$ to support a non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector is given in Table 4.3. We assume the central character is trivial, i.e., $\chi_1\chi_2\sigma^2=1$. Moreover, if we are just considering non Iwahori-spherical representations, then above table shall be as Table 4.4.*

Proof. The proof is analogue to that of Proposition 4.3. □

Proposition 4.5. *Table 4.5 gives the dimensions of spaces of $\mathrm{M}(\mathfrak{p}^2)$- and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for an admissible full Borel-induced representation $\chi_1\times\chi_2\rtimes\sigma$ which is non Iwahori-spherical with trivial central character.*

Proof. It easily follows from Proposition 4.3 and Corollary 4.4. □

Table 4.3: Inducing data for non-zero $\mathrm{M}(\mathfrak{p}^2)$ vectors of Borel-induced cases.

$B(F)r_i\mathrm{M}(\mathfrak{p}^2)$	Inducing data
r_1	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_2	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_3	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_4	$a(\chi_1) = a(\chi_2) = a(\sigma) = 0$
r_5	$a(\sigma) = 0$
r_6	$a(\chi_1\sigma) = a(\chi_2\sigma) = 0$
r_7	$a(\chi_1\sigma) = a(\chi_2\sigma) = 0$
r_8	$a(\sigma) = 0$

Table 4.4: Inducing data for $\mathrm{M}(\mathfrak{p}^2)$ of non Iwahori-spherical Borel-induced case.

$B(F)r_i\mathrm{M}(\mathfrak{p}^2)$	Inducing data
r_1	**NA**
r_2	**NA**
r_3	**NA**
r_4	**NA**
r_5	$a(\sigma) = 0, a(\chi_1) = a(\chi_2) = 1$
r_6	$a(\sigma) = a(\chi_1) = a(\chi_2) = 1, a(\chi_j\sigma) = 0, j \in \{1, 2\}$
r_7	$a(\sigma) = a(\chi_1) = a(\chi_2) = 1, a(\chi_j\sigma) = 0, j \in \{1, 2\}$
r_8	$a(\sigma) = 0, a(\chi_1) = a(\chi_2) = 1$

Table 4.5: Dimensions of the spaces of $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for non Iwahori-spherical representations of group I.

inducing data				$\dim V^\Gamma$	
$a(\sigma)$	$a(\chi_1)$	$a(\chi_2)$	$a(\chi_i\sigma)$	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
0	1	1		2	3
1	0	0		0	3
	1	0		0	2
	0	1		0	2
	1	1	0	2	3
			1	0	1

4.3 Groups II to VI

Since Proposition 4.5 already gives the dimensional data for group I in which the Borel-induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible, we are going to investigate the representations of group II to VI in this section. In fact, the analogue dimensional data follows from Proposition 4.5 for most cases, see Table 4.6.

Table 4.6: Dimensions of the spaces of $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for non Iwahori-spherical representations of groups II to VI.

representation		inducing data	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
II	a	$a(\sigma)=0, a(\chi)=1$	2	3
		$a(\sigma)=1, a(\chi)=0$	0	2
		$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=0$	1	2
		$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=1$	0	1
	b	$a(\sigma)=0, a(\chi)=1$	0	0
		$a(\sigma)=1, a(\chi)=0$	0	1
		$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=0$	1	1

Continued on next page

Table 4.6 – *Continued from previous page*

representation		inducing data	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
		$a(\sigma)=1, a(\chi)=1, a(\chi\sigma)=1$	0	0
III	a	$a(\sigma)=1, a(\chi)=0$	0	2
		$a(\sigma)=1, a(\chi)=1$	0	1
	b	$a(\sigma)=1, a(\chi)=0$	0	1
		$a(\sigma)=1, a(\chi)=1$	0	1
IV	a	$a(\sigma)=1$	0	1
	b	$a(\sigma)=1$	0	1
	c	$a(\sigma)=1$	0	1
	d	$a(\sigma)=1$	0	0
V	a	$a(\sigma)=0, a(\xi)=1$	1	2
		$a(\sigma)=1, a(\xi)=0$	0	2
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	1	2
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	1
	b	$a(\sigma)=0, a(\xi)=1$	1	1
		$a(\sigma)=1, a(\xi)=0$	0	0
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	0	0
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	0
	c	$a(\sigma)=0, a(\xi)=1$	0	0
		$a(\sigma)=1, a(\xi)=0$	0	0
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	1	1
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	0
	d	$a(\sigma)=0, a(\xi)=1$	0	0
		$a(\sigma)=1, a(\xi)=0$	0	1

Continued on next page

Table 4.6 – *Continued from previous page*

representation		inducing data	M($\mathfrak{p}^2$)	Kl($\mathfrak{p}^2$)
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=0$	0	0
		$a(\sigma)=1, a(\xi)=1, a(\xi\sigma)=1$	0	0
VI	a	$a(\sigma)=1$	0	2
	b	$a(\sigma)=1$	0	0
	c	$a(\sigma)=1$	0	0
	d	$a(\sigma)=1$	0	1

The dimensional data for groups II and III easily follows from Proposition 4.5 since types IIb and IIIb are related to the double cosets $P(F)\backslash G(F)/\Gamma$ and $Q(F)\backslash G(F)/\Gamma$, respectively. Here, Γ is represented as the congruence subgroup $\mathrm{Kl}(\mathfrak{p}^2)$ or $\mathrm{M}(\mathfrak{p}^2)$. As for the dimensional data for group IV, in particular, we consider the representation of type IVd which is the trivial representation of GSp(4) twisted by the character σ. Furthermore, we have the following claim.

Claim 4.6. There is no non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector for the type of IVd which is non Iwahori-spherical. Similarly, in the non Iwahori-spherical case, there is no non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector for the type of IVd either.

Proof of Claim 4.6. Since we are concerning the non Iwahori-spherical representations, then we have $a(\sigma)=1$ for the type of IVd. Suppose that there is such a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector v. In particular, we can take $g = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & x & \\ & & & x \end{bmatrix} \in \mathrm{Kl}(\mathfrak{p}^2)$, for any $x \in \mathfrak{o}^\times$ and we have

$$(\sigma \mathbf{1}_{\mathrm{GSp}(4)})(g).v = \sigma(x)v = v, \quad \forall x \in \mathfrak{o}^\times. \tag{4.42}$$

This implies that $a(\sigma)=0$, i.e., σ is *unramified*. However, this contradicts the condition of $a(\sigma) = 1$. With the similar discussion, the assertion also holds for the case of $\mathrm{M}(\mathfrak{p}^2)$-invariant vector. $\square$

Then the dimensional data for the other types IVa, IVb and IVc easily follows from the relations

$$\dim(\mathrm{IVb})^{\Gamma} + \dim(\mathrm{IVd})^{\Gamma} = \dim(\mathrm{IIb})^{\Gamma}, \qquad \dim(\mathrm{IVc})^{\Gamma} + \dim(\mathrm{IVd})^{\Gamma} = \dim(\mathrm{IIIb})^{\Gamma}.$$

Next, we are going to figure out the dimensional data for groups V and VI. Recall the relations in the proof of Proposition 3.14,

$$\dim(\mathrm{VIb})^{\Gamma} + \dim(\mathrm{VId})^{\Gamma} = \dim(\mathrm{IIb})^{\Gamma}, \quad \dim(\mathrm{VIc})^{\Gamma} + \dim(\mathrm{VId})^{\Gamma} = \dim(\mathrm{IIIb})^{\Gamma}.$$
$$\dim(\mathrm{Vb})^{\Gamma} + \dim(\mathrm{Vd})^{\Gamma} = \dim(\mathrm{IIb})^{\Gamma}, \quad \dim(\mathrm{Vb})^{\Gamma} = \dim(\mathrm{Vc})^{\Gamma}.$$

Then we can immediately obtain the dimensional data for $\mathrm{M}(\mathfrak{p}^2)$-invariant vectors by combing above relations and the data for $\mathrm{K}(\mathfrak{p}^2)$-invariant vectors; see Table A.12 of [15]. Now we consider the $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors, for group VI, it is enough to consider the σ ramified case, i.e., $a(\sigma) = 1$. By (3.2), we know that $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = 0$ for the type of VIb in the Iwahori-spherical case. Then, it is easy to see that $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = 0$ for the type of VIb still holds in the non Iwahori-spherical case; the reason is that if a double coset does not support an invariant function if $a(\sigma) = 0$, then it also does not support an invariant function if $a(\sigma) = 1$. Once the dimensions for VIb are known, the dimensions for the other representations in group VI follow as before.

However, we shall need more tools to completely obtain the desired dimensional data, which is summarized in Table 1.2, for group V. First, the following lemma is very useful to determine the dimensional data for $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for $\mathrm{GSp}(4,F)$.

Lemma 4.7. *Let (π, V) be an admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let $v \in V$ be a vector invariant under the group* $\begin{bmatrix} \mathfrak{o}^\times & & & \mathfrak{o} \\ & \mathfrak{o}^\times & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.
Then

$$P(v) = 0 \quad \Longleftrightarrow \quad v \text{ is invariant under } \begin{bmatrix} 1 & & & \mathfrak{p}^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{4.43}$$

Here, P is the projection $V \to V_{Z^J,\psi^{-1}}$.

Proof. See [17]. □

The following theorem plays an important role in determining the dimensions of spaces of $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for some cases.

Theorem 4.8. *Let (π, V) be an irreducible, admissible, non-generic representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let π^J be any irreducible subquotient of $V_{Z^J,\psi^{-1}}$. Write*

$$\pi^J \cong \tilde{\tau} \otimes \pi_{SW}^{-1} \tag{4.44}$$

with an irreducible, admissible representation $\tilde{\tau}$ of $\widetilde{\mathrm{SL}}(2,F)$. *If $\tilde{\tau}$ is non-spherical and $V^{\mathrm{M}(\mathfrak{p}^2)} = 0$, then $V^{\mathrm{Kl}(\mathfrak{p}^2)} = 0$.*

Proof. Suppose that v is a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector in (π, V); we will obtain a contradiction. We consider the projection $P\colon V \longrightarrow V_{Z^J,\psi^{-1}}$. It easily follows from $v \in V$ that $P(v)$ is invariant under $G^J(\mathfrak{o})$. That is to say, $P(v)$ is a spherical vector in the Jacobi representation $V_{Z^J,\psi^{-1}}$. One can show that such a representation must be of the form $\tilde{\tau} \otimes \pi_{SW}^{-1}$ with $\tilde{\tau}$ either a spherical principal series representation or an unramified even Weil representation.

Moreover, it follows from Theorem 7.1.4 of [16] that the G^J-module $V_{Z^J,\psi^{-1}}$

has finite length. In particular, we consider a composition series

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V_{Z^J, \psi^{-1}}, \tag{4.45}$$

where V_i/V_{i-1} is an irreducible G^J-module τ_i for $i \in \{1, \ldots, n\}$. As in (2.17), we may write

$$V_i/V_{i-1} \cong \widetilde{V}_i \otimes \mathcal{S}(F), \tag{4.46}$$

where $\widetilde{V}_i$ is the space of an irreducible, admissible representation $\tilde{\tau}_i$ of the metaplectic group $\widetilde{\mathrm{SL}}(2, F)$, and $\mathcal{S}(F)$ is the space of the Schrödinger-Weil representation π_{SW}^{-1}.

We assume $P(v) \neq 0$, then $P(v)$ defines a non-zero vector u in V_i/V_{i-1} for some i, which is $G^J(\mathfrak{o})$-invariant. Hence V_i/V_{i-1} is a spherical G^J-representation. Then we can conclude that $\widetilde{\tau}_i$ is a spherical principal series representation or an unramified even Weil representation $\widetilde{\tau}_i = \pi_W^{m+}$ for some m. However, we assume that $\tilde{\tau}$ is non-spherical which is not of this kind. Thus, we get a contradiction, proving that $P(v) = 0$. In addition, it follows from Lemma 4.7 that v is invariant under $\begin{bmatrix} 1 & & \mathfrak{p}^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$. Thus, v is also a $\mathrm{M}(\mathfrak{p}^2)$-invariant vector which contradicts the assumption of $V^{\mathrm{M}(\mathfrak{p}^2)} = 0$. □

Now we come back to determining the dimensions of the spaces $V^{\mathrm{M}(\mathfrak{p}^2)}$ and $V^{\mathrm{Kl}(\mathfrak{p}^2)}$ for group V. The diemsnion of the space $V^{\mathrm{M}(\mathfrak{p}^2)}$ easily follow from those of the spaces of $\mathrm{K}(\mathfrak{p}^2)$-invariant vectors; see Table A.12 of [15]. As for the space $V^{\mathrm{Kl}(\mathfrak{p}^2)}$, in fact, the dimensional data for most of the cases easily follows from that of $V^{\mathrm{M}(\mathfrak{p}^2)}$. And only the case of $a(\sigma) = 1, a(\xi) = 0$ needs more work to do. Furthermore, we note that $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 0$ in this case.

Let (π, V) be the representation of type Vb. By Table A.5 and Table A.6 of

[15], we have

$$V_{Z^J} = \tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{\frac{1}{2}}\xi\sigma \times \nu^{-\frac{1}{2}}\sigma)) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^2\sigma). \tag{4.47}$$

For more details about P_3-theory, see [15, section 2.5]. First, we have the following more general lemma for the second part of (4.47).

Lemma 4.9. *Let χ be a character of $F^\times$. For $a \in F$, consider the character*

$$\theta_{a,0}(\left[\begin{smallmatrix} 1 & x & y \\ & 1 & \\ & & 1 \end{smallmatrix}\right]) = \psi(ax).$$

If $a \neq 0$, then

$$\big(\tau^{P_3}_{\mathrm{GL}(1)}(\chi)\big)_{\left[\begin{smallmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{smallmatrix}\right],\theta_{a,0}} = 0.$$

Proof. To any $f \in \tau^{P_3}_{\mathrm{GL}(1)}(\chi)$, associate the function $\tilde{f} : F \times F^\times \to \mathbb{C}$ defined by

$$\tilde{f}(u, b) = f(\left[\begin{smallmatrix} 1 & & \\ u & b & \\ & & 1 \end{smallmatrix}\right]).$$

As in the proof of Lemma 2.5.5 of [15], one sees that $\tilde{f} \in \mathcal{S}(F \times F^\times)$. It is easy to see that the map $f \mapsto \tilde{f}$ from $\tau^{P_3}_{\mathrm{GL}(1)}(\chi)$ to $\mathcal{S}(F \times F^\times)$ thus defined is surjective. It is also injective since the set $\left[\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ * & * \end{smallmatrix}\right]$ is dense in $\mathrm{GL}(2, F)$. Thus we have an isomorphism

$$\tau^{P_3}_{\mathrm{GL}(1)}(\chi) \cong \mathcal{S}(F \times F^\times).$$

Transferring the action of $\left[\begin{smallmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{smallmatrix}\right]$, we see that

$$(\left[\begin{smallmatrix} 1 & & y \\ & 1 & \\ & & 1 \end{smallmatrix}\right]\tilde{f})(u, b) = \psi(yu)\tilde{f}(u, b).$$

We calculate the asserted Jacquet module in stages, by first calculating

$$\left(\tau_{\mathrm{GL}(1)}^{P_3}(\chi)\right)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]} \cong \mathcal{S}(F \times F^\times)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]}. \tag{4.48}$$

Consider the map $\mathcal{S}(F \times F^\times) \to \mathcal{S}(F^\times)$ which maps $\tilde{f}$ to the function $b \mapsto \tilde{f}(0,b)$. Its kernel is $\mathcal{S}(F^\times \times F^\times)$, and we have an exact sequence

$$0 \longrightarrow \mathcal{S}(F^\times \times F^\times) \longrightarrow \mathcal{S}(F \times F^\times) \longrightarrow \mathcal{S}(F^\times) \longrightarrow 0.$$

The sequence

$$0 \longrightarrow \mathcal{S}(F^\times \times F^\times)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]} \longrightarrow \mathcal{S}(F \times F^\times)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]} \longrightarrow \mathcal{S}(F^\times)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]} \longrightarrow 0 \tag{4.49}$$

is also exact. It is easy to see that if $\tilde{f} \in \mathcal{S}(F^\times \times F^\times)$, then

$$\int\limits_{\mathfrak{p}^{-n}} \left[\begin{smallmatrix}1&&\\&1&y\\&&1\end{smallmatrix}\right] \tilde{f}\, dy = 0$$

for large enough n. Hence the module on the left in (4.49) is zero, so that we get an isomorphism

$$\mathcal{S}(F \times F^\times)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]} \cong \mathcal{S}(F^\times)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]}. \tag{4.50}$$

Evidently, the group $\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]$ acts trivially on $\mathcal{S}(F^\times)$, so that

$$\mathcal{S}(F^\times)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]} \cong \mathcal{S}(F^\times). \tag{4.51}$$

Combining (4.48), (4.50) and (4.51), we see that

$$\big(\tau_{\mathrm{GL}(1)}^{P_3}(\chi)\big)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right]} \cong \mathcal{S}(F^\times). \tag{4.52}$$

Hence

$$\big(\tau_{\mathrm{GL}(1)}^{P_3}(\chi)\big)_{\left[\begin{smallmatrix}1&*&*\\&1&\\&&1\end{smallmatrix}\right],\theta_{a,0}} \cong \mathcal{S}(F^\times)_{\left[\begin{smallmatrix}1&*&\\&1&\\&&1\end{smallmatrix}\right],\psi^a}. \tag{4.53}$$

By the definitions involved, the map that induces the isomorphism (4.52) is $f \mapsto \hat{f}$, where

$$\hat{f}(b) = f(\left[\begin{smallmatrix}1&&\\&b&\\&&1\end{smallmatrix}\right]), \qquad b \in F^\times.$$

It is thus easy to transfer the action of the group $\left[\begin{smallmatrix}1&*&\\&1&\\&&1\end{smallmatrix}\right]$: It acts trivially on $\mathcal{S}(F^\times)$. It follows that

$$\mathcal{S}(F^\times)_{\left[\begin{smallmatrix}1&*&\\&1&\\&&1\end{smallmatrix}\right],\psi^a} = 0.$$

This proves the lemma. □

Let (ρ, W) be an irreducible, admissible principal series representation of $\mathrm{GL}(2,F)$. where $\rho = \nu^{\frac{1}{2}}\xi\sigma \times \nu^{-\frac{1}{2}}\sigma$ and ξ is a non-trivial quadratic character of $F^\times$. As for the first part of (4.47), this representation has the same space W as ρ, and the action of Q is given by

$$\left[\begin{smallmatrix}ad-bc&*&*&*\\&a&b&*\\&c&d&*\\&&&1\end{smallmatrix}\right]\left[\begin{smallmatrix}u&&&\\&u&&\\&&u&\\&&&u\end{smallmatrix}\right] w = \omega_\pi(u)(\nu^{\frac{3}{2}}\xi\sigma \times \nu^{\frac{1}{2}}\sigma)(\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right])w. \tag{4.54}$$

The central character is $\omega_\pi = \xi^2\sigma^2 = \sigma^2$, since $\xi^2 = 1, \xi \neq 1$.

Lemma 4.10. *Let (π, V) be a representation of type* Vb. *The $F^\times$ module $V_{\boxplus}$ is one-dimensional and isomorphic to the character $\nu^2\xi$.*

Proof. By Lemma 2.5,

$$V_{\boxplus} = (V_{Z^J})_{N^0,\theta_{-1,0}} = W_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right],\psi^{-1}}.$$

The space on the right is one-dimensional by the existence and uniqueness of Whittaker models for GL(2). It follows from (4.54) that

$$\begin{bmatrix} a &&& \\ & 1 && \\ && 1 & \\ &&& a^{-1} \end{bmatrix} w = \omega_\pi(a^{-1})(\nu^{\frac{3}{2}}\xi\sigma \times \nu^{\frac{1}{2}}\sigma)(\left[\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right])w = |a|^2\xi(a)$$

for all $w \in V_{Z^J}$. Hence $T_{\boxplus} \cong F^\times$ acts on $V_{\boxplus}$ via the character $\nu^2\xi$. □

Proposition 4.11. *As above, let (π, V) be the representation of type* Vb. *Then there is only one non-supercuspidal $\widetilde{\tau}_i$ is isomorphic to $\tilde{\sigma}^m$, where m is such that $(m, \cdot) = \xi$.*

Proof. First, it follows from Theorem 7.1.4 of [16] that the G^J-module $V_{Z^J,\psi^{-1}}$ has finite length. In particular, we consider a composition series

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V_{Z^J,\psi^{-1}}, \tag{4.55}$$

where V_i/V_{i-1} is an irreducible G^J-module τ_i for $i \in \{1, \ldots, n\}$. Then we get

$$0 = (V_0)_{N^0} \subset (V_1)_{N^0} \subset \ldots \subset (V_n)_{N^0} = V_{\boxplus}. \tag{4.56}$$

By Lemma 2.3 and Lemma 4.10, we have $\dim V_{\boxplus} = 1$. It follows that $(V_i/V_{i-1})_{N^0}$ is one-dimensional for exactly one i, and zero for all the other i. By Lemma 2.5,

$$\dim((\widetilde{V}_i)_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]}) = \dim((V_i/V_{i-1})_{N^0}).$$

Hence the $\dim((\widetilde{V}_i)_{\left[\begin{smallmatrix}1 & * \\ & 1\end{smallmatrix}\right]})$ are zero except for one i, where the dimension is one. It follows that all but one of the $\tilde{\tau}_i$ are zero, and the remaining one, say $\tilde{\tau}_{i_0}$, has a one-dimensional Jacquet module. One can show that one-dimensional Jacquet modules occur precisely for special representations $\tilde{\sigma}^m$ and for even Weil representations π_W^{m+}. More precisely, as an $\tilde{A}$-module, $\tilde{\sigma}^m$ has Jacquet module

$$([\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}], \varepsilon) \longmapsto \varepsilon\delta_\psi(a)(m,a)\nu(a)^{\frac{3}{2}},$$

and π_W^{m+} has Jacquet module

$$([\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}], \varepsilon) \longmapsto \varepsilon\delta_\psi(a)(m,a)\nu(a)^{\frac{1}{2}}.$$

By Lemma 4.10, $T_{\boxplus} \cong F^\times$ acts on $V_{\boxplus}$ via the character $\nu^2\xi$. Using (2.21), it follows that $\tilde{\tau}_{i_0} \cong \tilde{\sigma}^m$, where m is such that $(m,\cdot) = \xi$. □

Proposition 4.12. *If $a(\sigma) = 1, a(\xi) = 0$, then the dimensions of spaces of* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for types of* V(a, b, c, d) *are* (2, 0, 0, 1)*.*

Proof. It follows from Proposition 4.11 and Theorem 4.8 that

$$\dim(\mathrm{Vb})^{\mathrm{Kl}(\mathfrak{p}^2)} = \dim(\mathrm{Vc})^{\mathrm{Kl}(\mathfrak{p}^2)} = 0. \tag{4.57}$$

Recalling the relation $\dim(\mathrm{Vb})^\Gamma + \dim(\mathrm{Vd})^\Gamma = \dim(\mathrm{IIb})^\Gamma$, we can conclude that the dimensions for V(a, b, c, d) are (2, 0, 0, 1). □

Chapter 5

Non Iwahori-spherical: Klingen-induced representations

In this chapter, we obtain the desired dimensional data for the Klingen-induced representations which are non Iwahori-spherical representations.

5.1 Group VII

Recall that

$$A = \begin{bmatrix} 1 & & & \\ u & 1 & & \\ v & & 1 & \\ w & v & -u & 1 \end{bmatrix}, \quad M = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \frac{\Delta}{t} \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \tag{5.1}$$

where $\Delta = ad - bc, u, v, w \in \mathfrak{p}^2, x, y, z \in \mathfrak{o}, t \in \mathfrak{o}^\times, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o})$. Consider the full Klingen-induced representation $\chi \rtimes \pi$, where π is a supercuspidal representation of $\mathrm{GL}(2, F)$ and χ is a character of $F^\times$. The standard space V of this representation consists of smooth functions $f\colon Q \to V_\pi$ with the transformation property

$$f(\begin{bmatrix} a & * & & * \\ & g & & * \\ & & & a^{-1}\det(g) \end{bmatrix} h) = |a^2 \det(g)^{-1}|\chi(a)\pi(g)f(h), \quad g \in \mathrm{GL}_2(F), a \in F^\times. \tag{5.2}$$

Let

$$\tilde{\pi}(\tilde{g}) := |a^2 \det(g)^{-1}|\chi(a)\pi(g), \tag{5.3}$$

and we take

$$\tilde{g} = \begin{bmatrix} a & * & & * \\ & g & & * \\ & & & a^{-1}\det(g) \end{bmatrix} \in Q(F), h \in \mathrm{GSp}(4, F). \tag{5.4}$$

Suppose $f \in V^{\mathrm{Kl}(\mathfrak{p}^2)}$, then f is determined on the set of representatives for the double cosets $Q(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$ as in Proposition 3.6 iii). That is to say, f is determined by $f(R_1), f(R_2), \ldots, f(R_6)$. Consider $Q(F)R_j\mathrm{Kl}(\mathfrak{p}^2), j = \{1, \ldots, 6\}$ and assume $f(R_j) \neq 0$. Then, for $\tilde{g} \in Q(F), h \in \mathrm{Kl}(\mathfrak{p}^2)$, we need

$$f(\tilde{g}R_j h) = \tilde{\pi}(\tilde{g})f(R_j), \quad j = \{1, \ldots, 6\}. \tag{5.5}$$

By a similar discussion as in Section 4.2, to check the well-definedness of (5.5), we need the following for the representation $\tilde{\pi}$ as defined in (5.2).

$$\tilde{\pi} \text{ must be trivial on } Q(F) \cap R_j\mathrm{Kl}(\mathfrak{p}^2)R_j^{-1}, \quad j \in \{1, 2, \ldots, 6\}. \tag{5.6}$$

It follows from Lemma 4.2 that the full Klingen-induced representation $\chi \rtimes \pi$ has depth zero. It implies that

$$\chi \text{ and } \pi \text{ both are depth zero.} \tag{5.7}$$

Then by the discussion in Section 4.1.2, we have the conductor condition

$$a(\pi) = 2, \quad a(\chi) \leq 1. \tag{5.8}$$

In particular, the condition of $a(\pi) = 2$ implies that there is a non-zero newform $v_0 \in V_\pi$ such that v_0 is invariant under

$$\Gamma_1(\mathfrak{p}^2) := \left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o}^\times \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{smallmatrix}\right]. \tag{5.9}$$

For each of double coset $Q(F)R_j\mathrm{Kl}(\mathfrak{p}^2), j = \{1, \ldots, 6\}$, let $\tilde{v}_j := f(R_j)$. By the above discussion, to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vector, i.e., $\tilde{v}_j \neq 0$, Up to conjugation, we have

$$\pi(\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o}^\times \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{smallmatrix}\right])\tilde{v}_j = \tilde{v}_j. \tag{5.10}$$

In fact, the conductor condition (5.8) follows from the following proposition.

Proposition 5.1. *If $G = G_1 \times G_2$ with $\mathfrak{p}$-adic groups G_1 and G_2, and if π is the representation $\pi_1 \otimes \pi_2$ of G, where π_i is an irreducible, admissible representation of G_i for $i = 1, 2$, then the depth of the representation π has the property.*

$$\rho(\pi) = \max\{\rho(\pi_1), \rho(\pi_2)\}. \tag{5.11}$$

Proposition 5.2 (Proposition 3.4, [24])**.** *Let π be a supercuspidal representation of $\mathrm{GL}(2, K)$, and let χ be a quasi–character of K^* with conductor $a(\chi)$. Then $\mathrm{cond}(\pi \otimes \chi) \leq \max(\mathrm{cond}\pi, 2a(\chi))$, with equality if π is minimal or $\mathrm{cond}(\pi) \neq 2a(\chi)$. The conductor of ω_π is at most $(1/2)\mathrm{cond}(\pi)$.*

Recall the central character of $\chi \rtimes \pi$ is $\chi\omega_\pi$, where ω_π is the central character of π. Since we are working with representations of $\mathrm{GSp}(4, F)$ with trivial central character, we assume $\chi\omega_\pi = 1$. In particular, we have $a(\chi) = a(\omega_\pi)$. Then it follows from Proposition 5.2 that

$$a(\chi) = a(\omega_\pi) \leq \frac{1}{2}a(\pi) = 1. \tag{5.12}$$

Remark 5.3. If χ is unramified, then so is ω_π. In this case, the condition of $a(\pi) = 2$ implies that the non-zero newform $v_0 \in V_\pi^{\Gamma_1(\mathfrak{p}^2)}$ is even $\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o}^\times \end{smallmatrix}\right]$-invariant, i.e., $v_0 \in V_\pi^{\Gamma_0(\mathfrak{p}^2)}$. In fact, for any $x \in \mathfrak{o}^\times$ we have

$$\pi(\left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right])v_0 = \omega_\pi(x)v_0 = v_0.$$

We can further assume that $\omega_\pi = 1$ by some appropriate twisting. That is to say, we can assume that π has trivial central character if the central character ω_π is unramified.

If χ is ramified, i.e., $a(\chi) = 1$, then so is ω_π. In this case, the condition of $a(\pi) = 2$ implies that the non-zero newform $v_0 \in V_\pi^{\Gamma_1(\mathfrak{p}^2)}$ is even $\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{smallmatrix}\right]$-invariant. By a similar discussion with above case, we just need to show that v_0 is invariant under $\left[\begin{smallmatrix} 1 & \\ & 1+\mathfrak{p} \end{smallmatrix}\right]$. In particular, we take any diagonal element $\left[\begin{smallmatrix} 1 & \\ & d \end{smallmatrix}\right] \in \left[\begin{smallmatrix} 1 & \\ & 1+\mathfrak{p} \end{smallmatrix}\right]$, then we have

$$\pi(\left[\begin{smallmatrix} 1 & \\ & d \end{smallmatrix}\right])v_0 = \omega_\pi(d)\left[\begin{smallmatrix} 1/d & \\ & 1 \end{smallmatrix}\right]v_0 = \omega_\pi(d)v_0 = v_0, \quad \forall d \in 1 + \mathfrak{p}.$$

The last equality is because that $a(\omega_\pi) = a(\chi) = 1$.

In summary, we have the following proposition.

Proposition 5.4. *Let (π, V) be a supercuspidal representation of $\mathrm{GL}(2, F)$ with depth zero. And we assume $v_0 \in V$ is a nonzero newform, i.e., v_0 is invariant* *under* $\Gamma_1(\mathfrak{p}^2)$ *as defined in* (5.9). *Then we have*

$$\pi\left(\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}\right) v_0 = v_0. \tag{5.13}$$

Proof. Since π a depth zero supercuspidal representation of $\mathrm{GL}(2,F)$, then we have

$$\pi\left(\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix}\right) v_0 = v_0.$$

That is to say, there is a non-zero newform v_0 which is $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix}$-invariant. To complete proof, we need to show such v_0 is also $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}$-invariant. In fact, by Proposition 5.2, we have $a(\omega_\pi) \le 1$. Here, ω_π is the central character of π. It implies that for any element $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}$, we have

$$\pi(d \cdot \begin{bmatrix} a/d & b/d \\ c/d & 1 \end{bmatrix})v_0 = \omega_\pi(d)v_0 = v_0.$$

The second equality is because that $a(\omega_\pi) \le 1$ and $d \in 1+\mathfrak{p}$. That is to say, v_0 is also $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}$-invariant. □

All the elements of $\mathrm{Kl}(\mathfrak{p}^2)$ can be written as AMY with A, M, Y as in (5.1). That is to say, any element h of $\mathrm{Kl}(\mathfrak{p}^2)$ has the form of

$$h = \begin{bmatrix} 1 & & & \\ u & 1 & & \\ v & & 1 & \\ w & v & -u & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \frac{\Delta}{t} \end{bmatrix} \begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \tag{5.14}$$

where $\Delta = ad - bc, u, v, w \in \mathfrak{p}^2, x, y, z \in \mathfrak{o}, t \in \mathfrak{o}^\times, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathfrak{o})$.

i) For $R_1 = \mathbf{I}_4$, to ensure that $R_1 \mathrm{Kl}(\mathfrak{p}^2) R_1^{-1} \in Q(F)$, we need

$$u = v = w = 0.$$

And the matrix will be

$$\begin{bmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \frac{ad-bc}{t} \end{bmatrix}.$$

It follows from (5.6) that

$$\chi(t)\pi(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])\tilde{v}_1 = \tilde{v}_1, \quad \forall t \in \mathfrak{o}^\times, \forall \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o}). \tag{5.15}$$

It implies that χ and π both are *unramified*. It is impossible since it contradicts (5.7) or (5.8).

ii) For $R_2 = s_1$, to ensure that $R_2\mathrm{Kl}(\mathfrak{p}^2)R_2^{-1} \in Q(F)$, we need

$$x = c = v = 0.$$

And the matrix will be

$$\begin{bmatrix} a & * & * & * \\ & t & tz & * \\ & tw & \frac{ad}{t}+twz & * \\ & & & d \end{bmatrix}.$$

It follows from (5.6) that

$$\chi(a)\pi(\begin{bmatrix} t & tz \\ tw & \frac{ad}{t}+twz \end{bmatrix})\tilde{v}_2 = \chi(a)\omega_\pi(\frac{ad}{t})\pi(\begin{bmatrix} \frac{t^2}{ad} & \frac{t^2z}{ad} \\ \frac{t^2w}{ad} & 1+\frac{t^2wz}{ad} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \tag{5.16}$$

Since we have trivial central character, i.e., $\chi\omega_\pi = 1$, then we have

$$\chi(\frac{t}{d})\pi(\begin{bmatrix} \frac{t^2}{ad} & \frac{t^2z}{ad} \\ \frac{t^2w}{ad} & 1+\frac{t^2wz}{ad} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \tag{5.17}$$

We observe that

$$\begin{bmatrix} \frac{t^2}{ad} & \frac{t^2z}{ad} \\ \frac{t^2w}{ad} & 1+\frac{t^2wz}{ad} \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix}.$$

And it is easy to see that any element in $\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{smallmatrix}\right]$ can have the form of

$$\left[\begin{smallmatrix} \frac{t^2}{ad} & \frac{t^2z}{ad} \\ \frac{t^2w}{ad} & 1+\frac{t^2wz}{ad} \end{smallmatrix}\right].$$

If follows from $a(\pi) = 2$ that the equation (5.17) becomes to

$$\chi(\frac{t}{d})\tilde{v}_2 = \tilde{v}_2, \qquad \forall t, d \in \mathfrak{o}^\times. \tag{5.18}$$

It implies that χ is *unramified*. Hence also ω_π since $\chi\omega_\pi = 1$. In conclusion, the double cosets $Q(F)R_2\mathrm{Kl}(\mathfrak{p}^2)$ does support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_2$ and the conductor condition is

$$a(\pi) = 2, \quad a(\chi) = 0. \tag{5.19}$$

In addition, by Remark 5.3 this non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vector $\tilde{v}_2$ is $\Gamma_0(\mathfrak{p}^2)$-invariant.

iii) For $R_3 = s_1s_2s_1$, to ensure that $R_3\mathrm{Kl}(\mathfrak{p}^2)R_3^{-1} \in Q(F)$, we need

$$x = y = z = 0.$$

And the matrix will be

$$\left[\begin{smallmatrix} \frac{ad-bc}{t} & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & t \end{smallmatrix}\right].$$

It follows from (5.6) that

$$\chi(\frac{ad-bc}{t})\pi([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}])\tilde{v}_3 = \tilde{v}_3, \quad \forall t \in \mathfrak{o}^\times, \forall [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathrm{GL}(2, \mathfrak{o}). \tag{5.20}$$

It implies that χ and π both are *unramified*. It is impossible since it contradicts (5.7) or (5.8).

iv) For $R_4 = \begin{bmatrix} 1 & & & \\ \varpi & 1 & & \\ & & 1 & \\ & & -\varpi & 1 \end{bmatrix}$, to ensure that $R_4 \mathrm{Kl}(\mathfrak{p}^2) R_4^{-1} \in Q(F)$, we need

(a) $v = -\frac{w}{\varpi} \in \mathfrak{p}^2 \Rightarrow w \in \mathfrak{p}^3$.

(b) $a = \frac{c\varpi(\varpi+u)}{w} \in \mathfrak{o}^\times \Rightarrow c \in \mathfrak{p} \backslash \{0\}$.

(c) $t = -\frac{c\varpi^2}{w(-1+\varpi x)} \in \mathfrak{o}^\times$.

And the matrix will be

$$\begin{bmatrix} \frac{c\varpi^2}{w} & * & * & * \\ & -\frac{c\varpi(\varpi+u)}{w(-1+\varpi x)} & \frac{\varpi^2 c(u+\varpi)((\varpi x-2)y-\varpi z)-bw(\varpi x-1)^2}{w(\varpi x-1)} & * \\ & \frac{c}{1-\varpi x} & \frac{\varpi c((\varpi x-2)y-\varpi z)-d(\varpi x-1)^2}{\varpi x-1} & * \\ & & & \frac{\varpi d(u+\varpi)-bw}{\varpi^2} \end{bmatrix}.$$

It follows from (5.6) that

$$\chi(\frac{c\varpi^2}{w})\pi(\begin{bmatrix} -\frac{c\varpi(\varpi+u)}{w(-1+\varpi x)} & \frac{\varpi^2 c(u+\varpi)((\varpi x-2)y-\varpi z)-bw(\varpi x-1)^2}{w(\varpi x-1)} \\ \frac{c}{1-\varpi x} & \frac{\varpi c((\varpi x-2)y-\varpi z)-d(\varpi x-1)^2}{\varpi x-1} \end{bmatrix})\tilde{v}_4 = \tilde{v}_4. \tag{5.21}$$

In particular, we have $\frac{c\varpi^2}{w} \in \mathfrak{o}^\times$. And we denote that

$$g_0 := -\frac{1}{d} \cdot \begin{bmatrix} -\frac{c\varpi(\varpi+u)}{w(-1+\varpi x)} & \frac{\varpi^2 c(u+\varpi)((\varpi x-2)y-\varpi z)-bw(\varpi x-1)^2}{w(\varpi x-1)} \\ \frac{c}{1-\varpi x} & \frac{\varpi c((\varpi x-2)y-\varpi z)-d(\varpi x-1)^2}{\varpi x-1} \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}\backslash\{0\} & 1+\mathfrak{p} \end{bmatrix}.$$

Then the equation (5.21) becomes

$$\chi(\frac{c\varpi^2}{w})\omega_\pi(d)\pi(g_0)\tilde{v}_4 = \chi(\frac{c\varpi^2}{dw})\pi(g_0)\tilde{v}_4 = \tilde{v}_4, \forall d \in \mathfrak{o}^\times. \tag{5.22}$$

This implies that χ is *unramified* and hence also ω_π. In order to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_4$ we need

$$\pi(g)\tilde{v}_4 = \tilde{v}_4, \quad \forall g \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{bmatrix}. \tag{5.23}$$

Thus, we obtain that $a(\pi) = 1$ which is a contradiction since $a(\pi) = 2$.

v) For $R_5 = s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}$, to ensure that $R_5 \mathrm{Kl}(\mathfrak{p}^2) R_5^{-1} \in Q(F)$, we need

(a) $x = z\varpi \Rightarrow x \in \mathfrak{p}$.

(b) $c = \frac{\varpi^2 dz}{\varpi y - 1} \Rightarrow c \in \mathfrak{p}^2$.

(c) $t = \frac{\varpi d}{(v+\varpi)(1-\varpi y)} \Rightarrow d \in \mathfrak{o}^\times$.

And the matrix will be

$$\begin{bmatrix} a-\varpi ay+\varpi^2 bz & * & * & * \\ & \frac{d\varpi}{\varpi+v} & \frac{\varpi dz}{(v+\varpi)(1-\varpi y)} & * \\ & -\frac{\varpi\left(b(v+\varpi)^2-d(uv+2\varpi u+w)\right)}{v+\varpi} & \frac{a(v+\varpi)^2(\varpi y-1)-\varpi^2 dz(uv+2\varpi u+w)}{\varpi(v+\varpi)(\varpi y-1)} & * \\ & & & \frac{d}{1-\varpi y} \end{bmatrix}.$$

It follows from (5.6) that

$$\chi(a)\pi(\begin{bmatrix} & \frac{d\varpi}{\varpi+v} & \frac{\varpi dz}{(v+\varpi)(1-\varpi y)} & \\ & -\frac{\varpi\left(b(v+\varpi)^2-d(uv+2\varpi u+w)\right)}{v+\varpi} & \frac{a(v+\varpi)^2(\varpi y-1)-\varpi^2 dz(uv+2\varpi u+w)}{\varpi(v+\varpi)(\varpi y-1)} & \end{bmatrix})\tilde{v}_5 = \tilde{v}_5 \qquad (5.24)$$

since $1 - \varpi y + \varpi^2 bza^{-1} \in 1 + \mathfrak{p}$. Again, we denote that

$$g_0 := \frac{\varpi}{a(\varpi+v)} \cdot \begin{bmatrix} \frac{d\varpi}{\varpi+v} & \frac{\varpi dz}{(v+\varpi)(1-\varpi y)} \\ -\frac{\varpi\left(b(v+\varpi)^2-d(uv+2\varpi u+w)\right)}{v+\varpi} & \frac{a(v+\varpi)^2(\varpi y-1)-\varpi^2 dz(uv+2\varpi u+w)}{\varpi(v+\varpi)(\varpi y-1)} \end{bmatrix},$$

and we observe that

$$g_0 \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}.$$

It follows from (5.24) and (5.12) that

$$\chi(a)\omega(\frac{a(\varpi+v)}{\varpi})\pi(g_0)\tilde{v}_5 = (\chi\omega_\pi)(a)\pi(g_0)\tilde{v}_5 = \pi(g_0)\tilde{v}_5 = \tilde{v}_5. \qquad (5.25)$$

Thus, in order to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_5$, we need

$$\pi\left(\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}\right) v_5 = v_5. \tag{5.26}$$

It follows from the Remark 5.3 and Proposition 5.4 that (5.26) is possible. In conclusion, the double cosets $Q(F)R_5\mathrm{Kl}(\mathfrak{p}^2)$ indeed support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_5$ and the conductor condition is

$$a(\pi) = 2, \quad a(\chi) \le 1. \tag{5.27}$$

In addition, this non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vector $\tilde{v}_5$ is invariant under $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}$.

vi) For $R_6 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix}$, to ensure that $R_6\mathrm{Kl}(\mathfrak{p}^2)R_6^{-1} \in Q(F)$, we need

(a) $x = \frac{cu-av}{t(w+\varpi)} \Rightarrow x \in \mathfrak{p}$.

(b) $y = \frac{du-bv}{t(w+\varpi)} \Rightarrow y \in \mathfrak{p}$.

(c) $z = \frac{bc-ad}{t^2(w+\varpi)} + \frac{1}{\varpi} \Rightarrow ad - bc \in t^2 + \mathfrak{p}$.

And the matrix will be

$$\begin{bmatrix} \frac{\varpi(ad-bc)}{t(w+\varpi)} & * & * & * \\ & a+\frac{u(cu-av)}{w+\varpi} & b+\frac{u(du-bv)}{w+\varpi} & * \\ & c+\frac{v(cu-av)}{w+\varpi} & d+\frac{v(du-bv)}{w+\varpi} & * \\ & & & \frac{t(w+\varpi)}{\varpi} \end{bmatrix}.$$

It follows from (5.6) that

$$\chi(\frac{ad-bc}{t})\pi(\begin{bmatrix} a+\frac{u(cu-av)}{w+\varpi} & b+\frac{u(du-bv)}{w+\varpi} \\ c+\frac{v(cu-av)}{w+\varpi} & d+\frac{v(du-bv)}{w+\varpi} \end{bmatrix})\tilde{v}_6 = \tilde{v}_6, \tag{5.28}$$

since $\frac{\varpi}{w+\varpi} \in 1+\mathfrak{p}$. We also observe that

$$\begin{bmatrix} a+\frac{u(cu-av)}{w+\varpi} & b+\frac{u(du-bv)}{w+\varpi} \\ c+\frac{v(cu-av)}{w+\varpi} & d+\frac{v(du-bv)}{w+\varpi} \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}).$$

This implies that χ and π are both *unramified* which is impossible.

In conclusion, we have the following proposition.

Proposition 5.5. *The following table gives the dimensions of spaces of* $\mathrm{M}(\mathfrak{p}^2)$*- and* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for an admissible full Klingen-induced representation* $\chi \rtimes \pi$ *which is non Iwahori-spherical with trivial central character.*

inducing data		$\dim V^\Gamma$	
$a(\pi)$	$a(\chi)$	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
2	0	0	2
	1	0	1

Proof. The dimensions of the spaces of $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors easily follows from above discussion. The proof for the dimensional data for $\mathrm{M}(\mathfrak{p}^2)$-invariant vectors is analogue to that for the $\mathrm{Kl}(\mathfrak{p}^2)$ case. In particular, it sufficient to check R_2 and R_5 cases. Note that the only difference from $\mathrm{Kl}(\mathfrak{p}^2)$ case is that $z \in \mathfrak{p}^{-1}$, see (1.11). Here, we use the same notation $\tilde{v}_j, j=\{2,5\}$ with the $\mathrm{Kl}(\mathfrak{p}^2)$ case.

1. For $R_2 = s_1$, to ensure that $R_2\mathrm{M}(\mathfrak{p}^2)R_2^{-1} \in Q(F)$, we need

$$x = c = v = 0.$$

   And the matrix will be
$$\begin{bmatrix} a & * & * & * \\ & t & tz & * \\ & tw & \frac{ad}{t}+twz & * \\ & & & d \end{bmatrix}.$$

It follows from (5.6) that

$$\chi(a)\pi(\begin{bmatrix} t & tz \\ tw & \frac{ad}{t}+twz \end{bmatrix})\tilde{v}_2 = \chi(a)\omega_\pi(\frac{ad}{t})\pi(\begin{bmatrix} \frac{t^2}{ad} & \frac{t^2 z}{ad} \\ \frac{t^2 w}{ad} & 1+\frac{t^2 wz}{ad} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \qquad (5.29)$$

Again since we have trivial central character, i.e., $\chi\omega_\pi = 1$, then we have

$$\chi(\frac{t}{d})\pi(\begin{bmatrix} \frac{t^2}{ad} & \frac{t^2 z}{ad} \\ \frac{t^2 w}{ad} & 1+\frac{t^2 wz}{ad} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \qquad (5.30)$$

It is easy to that χ is *unramified.* In fact, we can take some special relations

$$ad = t^2, \quad z = w = 0. \qquad (5.31)$$

Hence also $a(\omega_\pi) = 0$. Thus, in order to support a non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_2$, we need

$$\pi(\begin{bmatrix} \frac{t^2}{ad} & \frac{t^2}{ad}z \\ \frac{t^2}{ad}w & 1+\frac{t^2}{ad}wz \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \qquad (5.32)$$

However, we observe that

$$\begin{bmatrix} \frac{t^2}{ad} & \frac{t^2}{ad}z \\ \frac{t^2}{ad}w & 1+\frac{t^2}{ad}wz \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix} \sim \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{bmatrix}.$$

It follows that $a(\pi) = 1$ which is impossible since $a(\pi) = 2$; see (5.8).

2. For $R_5 = s_1 \begin{bmatrix} 1 & & & \\ \varpi & 1 & & \\ & & 1 & \\ & \varpi & & 1 \end{bmatrix}$, to ensure that $R_5\mathrm{M}(\mathfrak{p}^2)R_5^{-1} \in Q(F)$, we need

   (a) $x = z\varpi \Rightarrow x \in \mathfrak{o}$.

   (b) $c = \frac{\varpi^2 dz}{\varpi y - 1} \Rightarrow c \in \mathfrak{p}$.

   (c) $t = \frac{\varpi d}{(v+\varpi)(1-\varpi y)} \Rightarrow d \in \mathfrak{o}^\times$.

And the matrix will be

$$\begin{bmatrix} a-\varpi ay+\varpi^2 bz & * & * & * \\ & \frac{d\varpi}{\varpi+v} & \frac{\varpi dz}{(v+\varpi)(1-\varpi y)} & * \\ & -\frac{\varpi\left(b(v+\varpi)^2-d(uv+2\varpi u+w)\right)}{v+\varpi} & \frac{a(v+\varpi)^2(\varpi y-1)-\varpi^2 dz(uv+2\varpi u+w)}{\varpi(v+\varpi)(\varpi y-1)} & * \\ & & & \frac{d}{1-\varpi y} \end{bmatrix}.$$

It follows from (5.6) that

$$\chi(a)\pi(\begin{bmatrix} \frac{d\varpi}{\varpi+v} & \frac{\varpi dz}{(v+\varpi)(1-\varpi y)} \\ -\frac{\varpi\left(b(v+\varpi)^2-d(uv+2\varpi u+w)\right)}{v+\varpi} & \frac{a(v+\varpi)^2(\varpi y-1)-\varpi^2 dz(uv+2\varpi u+w)}{\varpi(v+\varpi)(\varpi y-1)} \end{bmatrix})\tilde{v}_5 = \tilde{v}_5 \quad (5.33)$$

since $1-\varpi y+\varpi^2 bza^{-1} \in 1+\mathfrak{p}$. Again, we denote that

$$g_0 := \frac{\varpi}{a(\varpi+v)} \cdot \begin{bmatrix} \frac{d\varpi}{\varpi+v} & \frac{\varpi dz}{(v+\varpi)(1-\varpi y)} \\ -\frac{\varpi\left(b(v+\varpi)^2-d(uv+2\varpi u+w)\right)}{v+\varpi} & \frac{a(v+\varpi)^2(\varpi y-1)-\varpi^2 dz(uv+2\varpi u+w)}{\varpi(v+\varpi)(\varpi y-1)} \end{bmatrix},$$

and we observe that

$$g_0 \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}.$$

Similarly, it follows from (5.24) and (5.12) that

$$\chi(a)\omega(\frac{a(\varpi+v)}{\varpi})\pi(g_0)\tilde{v}_5 = (\chi\omega_\pi)(a)\pi(g_0)\tilde{v}_5 = \pi(g_0)\tilde{v}_5 = \tilde{v}_5. \quad (5.34)$$

Thus, in order to support a non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector, we need

$$\pi\left(\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}\right)\tilde{v}_5 = \tilde{v}_5. \quad (5.35)$$

It is impossible since $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix} \sim \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{bmatrix}$. This implies that $a(\pi)=1$ which again contradicts the fact of $a(\pi)=2$, see (5.8) .

In conclusion, there is no non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector in any Klingen-induced representation $\chi \rtimes \pi$. □

5.2 Group VIII

Let (ρ, W) be an irreducible, admissible supercuspidal representation of $\mathrm{GL}(2, F)$. Then $1_{F^\times} \rtimes \rho$ is a unitary representation of $\mathrm{GSp}(4, F)$. It decomposes as a direct sum

$$1_{F^\times} \rtimes \rho = \tau(S, \rho) \oplus \tau(T, \rho),$$

where $\tau(S, \rho)$ is of type VIIIa and $\tau(T, \rho)$ is of type VIIIb. Let (π, V) be the representation of type VIIIb. By the tables in [15], we have

$$V_{Z^J} = \tau_{\mathrm{GL}(2)}^{P_3}(\nu\rho),$$

where this representation has the same space W as ρ, and the action of Q is given by

$$\begin{bmatrix} ad-bc & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & 1 \end{bmatrix} \begin{bmatrix} u & & & \\ & u & & \\ & & u & \\ & & & u \end{bmatrix} w = \omega_\pi(u)(\nu\rho)(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])w. \tag{5.36}$$

(Note that the tables list the P_3-module, which ignores the action of the center.) The central character is $\omega_\pi = \omega_\rho$.

Lemma 5.6. *The $F^\times$ module $V_{\boxplus}$ is one-dimensional and isomorphic to the character ν^2.*

Proof. With a similar discussion with in the proof of Lemma 4.10, the space $V_{\boxplus}$ is one-dimensional. It follows from (5.36) that

$$\begin{bmatrix} a & & & \\ & 1 & & \\ & & 1 & \\ & & & a^{-1} \end{bmatrix} w = \nu^2(a)w$$

for all $w \in V_{Z^J}$. Hence $T_{\boxplus} \cong F^\times$ acts on $V_{\boxplus}$ via the character ν^2. □

By Theorem 7.1.4 of [16], the G^J-module $V_{Z^J,\psi^{-1}}$ has finite length. Consider a composition series

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V_{Z^J,\psi^{-1}}, \tag{5.37}$$

where V_i/V_{i-1} is an irreducible G^J-module τ_i for $i \in \{1, \ldots, n\}$. As in (2.17), we may write

$$V_i/V_{i-1} \cong \widetilde{V}_i \otimes \mathcal{S}(F), \tag{5.38}$$

where $\widetilde{V}_i$ is the space of an irreducible, admissible representation $\tilde{\tau}_i$ of the metaplectic group $\widetilde{\mathrm{SL}}(2, F)$, and $\mathcal{S}(F)$ is the space of the Schrödinger-Weil representation π_{SW}^{-1}.

Proposition 5.7. *As above, let (π, V) be the representation of type* VIIIb*. Then all but one of the $\tilde{\tau}_i$ are supercuspidal, and the remaining one is isomorphic to the special representation $\tilde{\sigma}^1$.*

Proof. From (5.37) we get

$$0 = (V_0)_{\left[\begin{smallmatrix}1&&*&\\&1&*&*\\&&1&\\&&&1\end{smallmatrix}\right]} \subset (V_1)_{\left[\begin{smallmatrix}1&&*&\\&1&*&*\\&&1&\\&&&1\end{smallmatrix}\right]} \subset \ldots \subset (V_n)_{\left[\begin{smallmatrix}1&&*&\\&1&*&*\\&&1&\\&&&1\end{smallmatrix}\right]} = V_{\boxplus}. \tag{5.39}$$

By Lemma 2.3 and Lemma 5.6, we have $\dim V_{\boxplus} = 1$. It follows that

$$(V_i/V_{i-1})_{\left[\begin{smallmatrix}1&&*&\\&1&*&*\\&&1&\\&&&1\end{smallmatrix}\right]}$$

is one-dimensional for exactly one i, and zero for all the other i. By Lemma 2.5,

$$\dim((\widetilde{V}_i)_{\left[\begin{smallmatrix}1&*\\&1\end{smallmatrix}\right]}) = \dim((V_i/V_{i-1})_{\left[\begin{smallmatrix}1&&*&\\&1&*&*\\&&1&\\&&&1\end{smallmatrix}\right]}).$$

Hence the $\dim((\widetilde{V}_i)_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]})$ are zero except for one i, where the dimension is one. It follows that all but one of the $\tilde{\tau}_i$ are supercuspidal, and the remaining one, say $\tilde{\tau}_{i_0}$, has a one-dimensional Jacquet module. One can show that one-dimensional Jacquet modules occur precisely for special representations $\tilde{\sigma}^m$ and for even Weil representations π_{W}^{m+}. More precisely, as an $\tilde{A}$-module, $\tilde{\sigma}^m$ has Jacquet module

$$\left(\left[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right], \varepsilon\right) \longmapsto \varepsilon\delta_\psi(a)(m,a)\nu(a)^{\frac{3}{2}},$$

and π_{W}^{m+} has Jacquet module

$$\left(\left[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right], \varepsilon\right) \longmapsto \varepsilon\delta_\psi(a)(m,a)\nu(a)^{\frac{1}{2}}.$$

By Lemma 5.6, $T_{\boxplus} \cong F^\times$ acts on $V_{\boxplus}$ via the character ν^2. Using (2.21), it follows that $\tilde{\tau}_{i_0} \cong \tilde{\sigma}^1$. $\square$

Theorem 5.8. *There is no non-zero* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vector for type* VIIIb.

Proof. Suppose that v is a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector in type VIIIb; we will obtain a contradiction. Consider the projection $P\colon V \longrightarrow V_{Z^J,\psi^{-1}}$. Then $P(v)$ is invariant under $G^J(\mathfrak{o})$. By Theorem 7.1.4 of [16], the G^J-module $V_{Z^J,\psi^{-1}}$ has finite length. In particular, we consider a composition series as in (5.37). Assume $P(v) \neq 0$, then $P(v)$ defines a non-zero vector u in V_i/V_{i-1} for some i, which is $G^J(\mathfrak{o})$-invariant. Hence V_i/V_{i-1} is a spherical G^J-representation. Then we can conclude that $\widetilde{\pi}_i$ is a spherical principal series representation or $\widetilde{\pi}_i = \pi_{\mathrm{W}}^{m+}$ for some m. But, by Proposition 5.7 we can see that the $\widetilde{\tau}_i$ is not of this kind. Thus, we get a contradiction, proving that $P(v) = 0$. Moreover, it follows from [17] that v is invariant under $\begin{bmatrix} 1 & & & \mathfrak{p}^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$. Thus, v is also a $\mathrm{M}(\mathfrak{p}^2)$-invariant vector in type VIIIb, which contradicts Proposition 5.5. $\square$

5.3 Fourier-Jacobi quotient for the type of IXb

Again, let (ρ, W) be an irreducible, admissible supercuspidal representation of $\mathrm{GL}(2,F)$. Assume that there exists a non-trivial quadratic character ξ of $F^\times$ such that $\xi\rho \cong \rho$. Then there is an exact sequence

$$1 \longrightarrow \delta(\nu\xi, \nu^{-\frac{1}{2}}\rho) \longrightarrow \nu\xi \rtimes \nu^{-\frac{1}{2}}\rho \longrightarrow L(\nu\xi, \nu^{-\frac{1}{2}}\rho) \longrightarrow 1.$$

The representation $\delta(\nu\xi, \nu^{-\frac{1}{2}}\rho)$ is of type IXa, and $L(\nu\xi, \nu^{-\frac{1}{2}}\rho)$ is of type IXb. Let (π, V) be the representation of type IXb. Similarly, by Table A.5 and Table A.6 of [15], we have

$$V_{Z^J} = \tau_{\mathrm{GL}(2)}^{P_3}(\nu^{\frac{1}{2}}\rho),$$

where this representation has the same space W as ρ, and the action of Q is given by

$$\begin{bmatrix} ad-bc & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & 1 \end{bmatrix} \begin{bmatrix} u & & & \\ & u & & \\ & & u & \\ & & & u \end{bmatrix} w = \omega_\pi(u)(\nu^{\frac{1}{2}}\rho)(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])w. \tag{5.40}$$

The central character is $\omega_\pi = \xi\omega_\rho$.

Lemma 5.9. *The $F^\times$ module $V_{\boxplus}$ is one-dimensional and isomorphic to the character $\nu\xi$.*

Proof. The proof is analogous to that of Lemma 5.6. □

Proposition 5.10. *Let (π, V) be the representation of type* IXb. *Then all but one of the $\widetilde{\tau}_i$ are supercuspidal, and the remaining one is isomorphic to π_{W}^{m+}, where m is such that $(m, \cdot) = \xi$.*

Proof. The proof is analogous to that of Proposition 5.7. □

Proposition 5.11. *Let (π, V) be the representation of type* IXb*.*

i) If $a(\xi)=1$*, then* $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)}=0$.

ii) If ξ *is unramified,* i.e., $a(\xi)=0$*, then* $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)}\le 1$.

Proof. The proof of part i) is analogous to that of Theorem 5.8. For part ii), the result follows easily from Proposition 5.10. □

5.4 Intertwining operator for $\mathrm{Kl}_2(\mathfrak{p}^2)$

To determine the precise number of $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)}$ in Proposition 5.11 ii), we would like to consider a new subgroup $\mathrm{Kl}_2(\mathfrak{p}^2)$ defined as follows

$$\mathrm{Kl}_2(\mathfrak{p}^2) := \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{5.41}$$

Clearly, the space of $\mathrm{Kl}_2(\mathfrak{p}^2)$-invariant vectors is a subspace of $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors.

Let π be a supercuspidal representation of $\mathrm{GL}(2,F)$ with the central character $\omega_\pi=\xi$. Here, ξ is the unique non-trivial unramified quadratic character of $F^\times$, characterized by $\xi(\varpi)=-1$. Consider the family of induced representations

$$\nu^s\xi \rtimes \nu^{-s/2}\pi, \qquad \xi\neq 1, \xi\pi=\pi, s\in\mathbb{C}. \tag{5.42}$$

Let $V_{\xi,\pi,s}$ be the standard model for $\nu^s\xi\rtimes\nu^{-s/2}\pi$, i.e., $V_{\xi,\pi,s}$ consists of locally constant functions $f\colon \mathrm{GSp}(4,F)\to V_\pi$ that transform as

$$f\left(\begin{bmatrix} a & * & & * \\ & g & & * \\ & & & a^{-1}\det(g) \end{bmatrix} h\right) = |a|^{2+s}|\det(g)|^{-s/2-1}\xi(a)\pi(g)f(h), \quad a\in F^\times, g\in \mathrm{GL}(2,F). \tag{5.43}$$

Proposition 5.12. *A complete and minimal system of representatives for the double cosets $Q(F)\backslash G(F)/\mathrm{Kl}_2(\mathfrak{p}^2)$ is given by the following 4 elements.*

$$\mathbf{I}_4, \quad s_1, \quad s_1s_2s_1, \quad \begin{bmatrix} 1&&&\\&1&&\\&&1&\\ \varpi&&&1\end{bmatrix}. \tag{5.44}$$

If $a(\pi)=2$ and χ is unramified, then $Q(F)s_1\mathrm{Kl}_2(\mathfrak{p}^2)$ supports a unique $\mathrm{Kl}_2(\mathfrak{p}^2)$-invariant vector. Otherwise, none of the double cosets as above supports a non-zero $\mathrm{Kl}_2(\mathfrak{p}^2)$-invariant vector. Moreover, we say $f(s_1)=v_0$, where v_0 is the newform in π and invariant under $\mathrm{GL}(2,\mathfrak{o})\cap\left[\begin{smallmatrix}\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^2&\mathfrak{o}\end{smallmatrix}\right]$.

Proof. The proof is analogous to that of Proposition 5.5. □

Let $V_{\xi,\pi,s}^{\mathrm{Kl}_2(\mathfrak{p}^2)}$ denote the subspace of $\mathrm{Kl}_2(\mathfrak{p}^2)$-invariant vectors. By Proposition 5.12, any $f\in V_{\xi,\pi,s}^{\mathrm{Kl}_2(\mathfrak{p}^2)}$ is determined by $f(s_1)=v_0$. It follows that $\dim(V_{\xi,\pi,s}^{\mathrm{Kl}_2(\mathfrak{p}^2)})=1$. We shall determine, for $s=1$, the dimensions of the spaces of $\mathrm{Kl}_2(\mathfrak{p}^2)$-invariant vectors for types IXa and IXb. Consider the intertwining operator

$$\mathcal{A}(s)\colon V_{\xi,\pi,s}\longrightarrow V_{\xi,\widehat{\pi},-s}$$

defined by

$$(\mathcal{A}(s)f)(g)=\int\limits_N f\left(s_1s_2s_1ng\right)\,dn, \tag{5.45}$$

where $\widehat{\pi}$ denotes the representation contragredient to π. Here N is the unipotent radical of Klingen parabolic Q. By a similar discussion in Section 3.3, we only need to compute

$$(\mathcal{A}(s)f)(s_1)=\int\limits_{F^3} f\left(\begin{bmatrix}1&&&\\ \lambda&1&&\\ \mu&&1&\\ \kappa&\mu&-\lambda&1\end{bmatrix}s_1\right)\,d\mu\,d\kappa\,d\lambda. \tag{5.46}$$

To compute this $(\mathcal{A}(s)f)(s_1)$, we separate it into two parts. More precisely,

$$\begin{aligned}&(\mathcal{A}(s)f)(s_1)\\ =&\int_{F^2}\int_{\mathfrak{p}} f\left(\begin{bmatrix}1&&&\\ \lambda&1&&\\ \mu&&1&\\ \kappa&\mu&-\lambda&1\end{bmatrix}s_1\right)d\mu\,d\kappa\,d\lambda+\int_{F^2}\int_{F\backslash\mathfrak{p}} f\left(\begin{bmatrix}1&&&\\ \lambda&1&&\\ \mu&&1&\\ \kappa&\mu&-\lambda&1\end{bmatrix}s_1\right)d\mu\,d\kappa\,d\lambda.\end{aligned}\tag{5.47}$$

We write the first (resp. second) integral as $\mathcal{I}^s_{\mathfrak{p}}$ (resp. $\mathcal{I}^s_{F\backslash\mathfrak{p}}$). We further divide the second integral $\mathcal{I}^s_{F\backslash\mathfrak{p}}$ into three parts in terms of the variable κ. In particular,

$$\mathcal{I}^s_{F\backslash\mathfrak{p}}=\int_F\left\{\int_{\mathfrak{o}}+\int_{\varpi^{-1}\mathfrak{o}^\times}+\int_{F\backslash\mathfrak{p}^{-1}}\right\}\int_{F\backslash\mathfrak{p}} f\left(\begin{bmatrix}1&&&\\ \lambda&1&&\\ \mu&&1&\\ \kappa&\mu&-\lambda&1\end{bmatrix}s_1\right)d\mu\,d\kappa\,d\lambda.\tag{5.48}$$

Again, for convenience, these three integrals are denoted by $\mathcal{J}^s_{\mathfrak{o}}, \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}$ and $\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}$, respectively.

5.4.1 A useful lemma

Before we prove a lemma which is very useful for calculation $\mathcal{A}(s)$, we need the following theorem.

Theorem 5.13 ([21], Theorem 3.2.2)**.** *For each infinite-dimensional, irreducible, admissible representation π of $\mathrm{PGL}(2,F)$, the Atkin-Lehner eigenvalue equals $\varepsilon(\frac{1}{2},\pi)$.*

Remark 5.14. In our case, the supercuspidal representation π of $\mathrm{GL}(2,F)$ does not have trivial central character, i.e., $\omega_\pi\neq 1$. In fact, ω_π is an unramified, non-trivial quadratic character of $F^\times$, i.e., $\omega_\pi(\varpi)=-1$. Thus, our newform v_0 is not just invariant under the group $\begin{bmatrix}\mathfrak{o}^\times&\mathfrak{o}\\ \mathfrak{p}^2&1+\mathfrak{p}^2\end{bmatrix}$ but also invariant under $\begin{bmatrix}\mathfrak{o}^\times&\mathfrak{o}\\ \mathfrak{p}^2&\mathfrak{o}^\times\end{bmatrix}$. It is easy to check since ω_π is unramified. In particular, we just need to check v_0 is $\begin{bmatrix}\mathfrak{o}^\times&\\ &\mathfrak{o}^\times\end{bmatrix}$-invariant.

1. Take any $x \in \mathfrak{o}^\times$, we have $\pi(\left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right])v_0 = \omega_\pi(x)v_0 = v_0$.

2. Take any $a \in \mathfrak{o}^\times$, we have $\pi(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])v_0 = v_0$.

Claim 5.15. For our π with $a(\omega_\pi) = 0, \omega_\pi$ is a non-trivial quadratic character, we still have

$$\pi(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right])v_0 = \varepsilon(\frac{1}{2}, \pi)v_0.$$

Proof of Claim 5.15. For $\pi' = \alpha\pi$ with trivial central character, i.e., the above theorem works for π'. That is to say

$$\pi'(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right])v_0 = \varepsilon(\frac{1}{2}, \pi')v_0. \tag{5.49}$$

In fact, we can take such α unramified character and $\alpha^2 = \xi$. Then we put back $\pi' = \alpha\pi$, then we have

$$\begin{aligned}
(\alpha\pi)(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right])v_0 =& \varepsilon(\frac{1}{2}, \alpha\pi)v_0 = \alpha(\varpi)^{a(\pi)-2a(\psi)}\varepsilon(\frac{1}{2}, \pi)v_0 \\
\alpha(\det \left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right])\pi(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right])v_0 =& \alpha(\varpi)^2\varepsilon(\frac{1}{2}, \pi)v_0 \\
\alpha(-\varpi^2)\pi(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right])v_0 =& \alpha(\varpi)^2\varepsilon(\frac{1}{2}, \pi)v_0 \\
\pi(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right])v_0 =& \varepsilon(\frac{1}{2}, \pi)v_0
\end{aligned}$$

□

As a consequence, we have

$$\begin{aligned}
\pi\left(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]\right)v_0 =& \pi\left(\left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]\right)v_0 = \pi\left(\left[\begin{smallmatrix} 1 & \\ & \varpi^{-2} \end{smallmatrix}\right]\right)\pi\left(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right]\right)v_0 \\
=& \varepsilon\pi\left(\left[\begin{smallmatrix} 1 & \\ & \varpi^{-2} \end{smallmatrix}\right]\right)v_0 = \varepsilon\omega_\pi(\varpi^{-2})\pi\left(\left[\begin{smallmatrix} \varpi^2 & \\ & 1 \end{smallmatrix}\right]\right)v_0 \\
=& \varepsilon\xi(\varpi^{-2})\pi\left(\left[\begin{smallmatrix} \varpi^2 & \\ & 1 \end{smallmatrix}\right]\right)v_0 = \varepsilon(-1)^2\pi\left(\left[\begin{smallmatrix} \varpi^2 & \\ & 1 \end{smallmatrix}\right]\right)v_0
\end{aligned}$$

$$=\varepsilon\pi\left(\left[\begin{smallmatrix} \varpi^2 & \\ & 1 \end{smallmatrix}\right]\right) v_0.$$

Here, $\varepsilon = \varepsilon(\frac{1}{2}, \pi) \in \{\pm 1\}$. For more information about ε-factors, see [21].

Lemma 5.16. *Let (π, V) be the supercuspidal representation of* $\mathrm{GL}(2, F)$ *with conductor $a(\pi) = 2$. Let v_0 be the newform in π, characterized by being invariant under* $\mathrm{GL}(2, \mathfrak{o}) \cap \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} \end{smallmatrix}\right]$. *Then*

i) $\displaystyle\int\limits_{\mathfrak{p}^{-m}} \pi(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]) v_0 \, dx = 0, \quad m \in \mathbb{Z}$ *and* $m \geq 1$.

ii) $\displaystyle\int\limits_{\mathfrak{p}^{n}} \pi(\left[\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right]) v_0 \, dy = 0, \quad n \in \mathbb{Z}$ *and* $n \leq 1$.

Proof. We prove i), the argument for ii) being very similar. Define

$$w_m(s) := \int\limits_{\mathfrak{p}^{-m}} \pi(\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]) v_0 \, dx, \quad m \geq 1. \tag{5.50}$$

It is easy to see that $w_1(s)$ is invariant under $\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & \mathfrak{o}^\times \end{smallmatrix}\right]$. In fact, we have

- $\pi(\left[\begin{smallmatrix} \mathfrak{o}^\times & \\ & \mathfrak{o}^\times \end{smallmatrix}\right]) w_1(s) = w_1(s)$.
- $\pi(\left[\begin{smallmatrix} 1 & \mathfrak{p}^{-1} \\ & 1 \end{smallmatrix}\right]) w_1(s) = w_1(s)$.
- $\pi(\left[\begin{smallmatrix} 1 & \\ \mathfrak{p}^2 & 1 \end{smallmatrix}\right]) w_1(s) = w_1(s)$.

Now consider

$$v_1(s) := \pi(\left[\begin{smallmatrix} 1 & \\ & \varpi^{-1} \end{smallmatrix}\right]) w_1(s).$$

It easily follows that $v_1(s)$ is invariant under $\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{smallmatrix}\right]$. However, this implies $v_1(s) = 0$, since the conductor of π is 2. It follows that $w_1(s) = 0$. For $m \geq 2$, $w_m(s) = 0$ follows from Theorem 1 of [12, III, §1]. □

Now we are going to calculate $(\mathcal{A}(s)f)(s_1)$ as defined in (5.46) by using the identities in the proofs of Lemma 3.2, Lemma 3.3 and Lemma 3.4 of [23]. However, the matrix identity (26) in the proof of Lemma 3.4 of [23] needs a slight refinement. In particular, we show the refined statement as follows.

$$
\begin{aligned}
&\int_{F^3} f\left(\begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & -\lambda & 1 \end{bmatrix} s_1 s_2 s_1 w\right) d\mu\, d\kappa\, d\lambda \\
=&\int_{F^3} f\left(\begin{bmatrix} 1 & & & \\ -\lambda & 1 & & \\ -\mu & & 1 & \\ -\kappa & -\mu & \lambda & 1 \end{bmatrix} s_1 s_2 s_1 w\right) d\mu\, d\kappa\, d\lambda \\
=&\int_{F^3} f\left(\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} -1 & & & \\ \lambda & -1 & & \\ \mu & & -1 & \\ \kappa & \mu & -\lambda & -1 \end{bmatrix} s_1 s_2 s_1 w\right) d\mu\, d\kappa\, d\lambda \\
=&\int_{F^3} f\left(\begin{bmatrix} -1 & & & \\ \lambda & -1 & & \\ \mu & & -1 & \\ \kappa & \mu & -\lambda & -1 \end{bmatrix} s_1 s_2 s_1 w\right) d\mu\, d\kappa\, d\lambda
\end{aligned}
$$

Then the matrix identity (26) in the proof of Lemma 3.4 of [23] is as follows

$$
\begin{bmatrix} -1 & & & \\ \lambda & -1 & & \\ \mu & & -1 & \\ \kappa & \mu & -\lambda & -1 \end{bmatrix} = \begin{bmatrix} -\mu^{-1} & & 1 & \\ & -\mu^{-1} & -\lambda-\kappa\mu^{-1} & 1 \\ & & -\mu & \\ & & & -\mu \end{bmatrix}\begin{bmatrix} 1 & & & \\ -\kappa\mu^{-1} & 1 & & \\ & & 1 & \\ -\mu^{-2}\kappa+\mu^{-1}\lambda & & \kappa\mu^{-1} & 1 \end{bmatrix}
$$
$$
\underbrace{\begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}}_{s_2 s_1 s_2}\begin{bmatrix} 1 & & -\mu^{-1} & \\ & 1 & & -\mu^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}.
$$

5.4.2 $\mathcal{I}_{\mathfrak{p}}^{s}$

First, we have the following proposition for $\mathcal{I}_{\mathfrak{p}}^{s}$.

Proposition 5.17. *With v_0 as in Proposition 5.12, we have*

$$
\mathcal{I}_{\mathfrak{p}}^{s} = \frac{q^{-1}(1-q^{-2-2s})}{1-q^{-2s-1}} v_0. \tag{5.51}
$$

Before we prove this proposition, we need the following claim.

Claim 5.18. Let n be a positive integer, then we have

$$\int_{\varpi^{-n}\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa = \begin{cases} 0 & \nu(\lambda^2) < n-1, \\ -q^{n-1} v_0 & \nu(\lambda^2) = n-1, \\ q^{n-1}(q-1) v_0 & \nu(\lambda^2) \geq n. \end{cases} \tag{5.52}$$

Proof of Claim 5.18. Since $\varpi^{-n}\mathfrak{o}^\times = \mathfrak{p}^{-n} \backslash \mathfrak{p}^{-n+1}$, then we have

$$\int_{\varpi^{-n}\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa = \int_{\mathfrak{p}^{-n}} \pi\left(\left[\begin{smallmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa - \int_{\mathfrak{p}^{-n+1}} \pi\left(\left[\begin{smallmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa. \tag{5.53}$$

By Lemma 5.16 i), for any $n \in \mathbb{Z}_{>0}$ we have

$$\int_{\mathfrak{p}^{-n}} \pi\left(\left[\begin{smallmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa = \begin{cases} 0 & \nu(\lambda^2) < n, \\ q^n v_0 & \nu(\lambda^2) \geq n. \end{cases} \tag{5.54}$$

Thus, it follows from the straightforward calculation that

$$\int_{\varpi^{-n}\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa = \begin{cases} 0 & \nu(\lambda^2) < n-1, \\ -q^{n-1} v_0 & \nu(\lambda^2) = n-1, \\ q^{n-1}(q-1) v_0 & \nu(\lambda^2) \geq n. \end{cases}$$

□

Proof of Proposition 5.17. Recall the integral $\mathcal{I}^s_{\mathfrak{p}}$ as defined in (5.47), i.e.,

$$\mathcal{I}^s_{\mathfrak{p}} = \int\limits_{F^2} \int\limits_{\mathfrak{p}} f\left(\left[\begin{smallmatrix} 1 &&& \\ \lambda & 1 && \\ \mu && 1 & \\ \kappa & \mu & -\lambda & 1 \end{smallmatrix}\right] s_1\right) d\mu \, d\kappa \, d\lambda = q^{-1} \int\limits_{F^2} f\left(\left[\begin{smallmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ \kappa && -\lambda & 1 \end{smallmatrix}\right] s_1\right) d\kappa \, d\lambda$$

$$=q^{-1}\int\limits_F \left\{ \int\limits_{\mathfrak{o}} + \int\limits_{F\setminus\mathfrak{o}} \right\} f\left(\begin{bmatrix} 1&&&\\ \lambda&1&&\\ &&1&\\ \kappa&&-\lambda&1 \end{bmatrix} s_1 \right) d\kappa\, d\lambda.$$

1.

$$\begin{aligned} I_{\mathfrak{o}} =& q^{-1}\int\limits_F \int\limits_{\mathfrak{o}} f\left(\begin{bmatrix} 1&&&\\ \lambda&1&&\\ &&1&\\ \kappa&&-\lambda&1 \end{bmatrix} s_1 \right) d\kappa\, d\lambda = q^{-1}\int\limits_F f\left(\begin{bmatrix} 1&&&\\ \lambda&1&&\\ &&1&\\ &&-\lambda&1 \end{bmatrix} s_1 \right) d\lambda \\ =& q^{-1}\int\limits_{\mathfrak{o}} f\left(\begin{bmatrix} 1&&&\\ \lambda&1&&\\ &&1&\\ &&-\lambda&1 \end{bmatrix} s_1 \right) d\lambda \\ &+ q^{-1}\int\limits_{F\setminus\mathfrak{o}} f\left(\begin{bmatrix} -\lambda^{-1}&&&\\ &-\lambda&&\\ &&\lambda^{-1}&\\ &&&\lambda \end{bmatrix} \begin{bmatrix} 1&\lambda&&\\ &1&&\\ &&1&-\lambda\\ &&&1 \end{bmatrix} \begin{bmatrix} &1&&\\ -1&&&\\ &&&1\\ &&-1& \end{bmatrix} \begin{bmatrix} 1&\lambda^{-1}&&\\ &1&&\\ &&1&-\lambda^{-1}\\ &&&1 \end{bmatrix} s_1 \right) d\lambda \\ =& q^{-1}v_0. \end{aligned}$$

2.

$$\begin{aligned} I_{F\setminus\mathfrak{o}} =& q^{-1}\int\limits_F \int\limits_{F\setminus\mathfrak{o}} f\left(\begin{bmatrix} 1&&&\\ \lambda&1&&\\ &&1&\\ \kappa&&-\lambda&1 \end{bmatrix} s_1 \right) d\kappa\, d\lambda \\ =& q^{-1}\int\limits_F \int\limits_{F\setminus\mathfrak{o}} f\left(\begin{bmatrix} \kappa^{-1}&&\kappa^{-1}\lambda&1\\ &1&\kappa^{-1}\lambda^2&\lambda\\ &&1&\\ &&&\kappa \end{bmatrix} \begin{bmatrix} 1&&&\\ \kappa^{-1}\lambda&1&&\\ &&1&\\ &&-\kappa^{-1}\lambda&1 \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} -1&&&-\kappa^{-1}\\ &1&&\\ &&1&\\ &&&-1 \end{bmatrix} s_1 \right) d\kappa\, d\lambda \\ =& q^{-1}\int\limits_F \int\limits_{F\setminus\mathfrak{o}} |\kappa||\kappa^{-1}|^{2+s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1&\kappa^{-1}\lambda^2\\ &1 \end{bmatrix} \right) f\left(\begin{bmatrix} 1&&&\\ \kappa^{-1}\lambda&1&&\\ &&1&\\ &&-\kappa^{-1}\lambda&1 \end{bmatrix} s_1 \right) d^*\kappa\, d\lambda \\ =& q^{-1}\left\{ \int\limits_{\mathfrak{o}} + \int\limits_{F\setminus\mathfrak{o}} \right\} \int\limits_{F\setminus\mathfrak{o}} |\kappa^{-1}|^{s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1&\kappa\lambda^2\\ &1 \end{bmatrix} \right) f\left(\begin{bmatrix} 1&&&\\ \lambda&1&&\\ &&1&\\ &&-\lambda&1 \end{bmatrix} s_1 \right) d^*\kappa\, d\lambda \end{aligned}$$

(a) Let $\imath^{\mathfrak{o}}_{F\setminus\mathfrak{o}}$ be the first part of above integral. Then we have

$$\imath^{\mathfrak{o}}_{F\setminus\mathfrak{o}}$$

$$
\begin{aligned}
=&q^{-1}\int\limits_{\mathfrak{o}}\int\limits_{F\backslash\mathfrak{o}} |\kappa^{-1}|^s\xi(\kappa^{-1})\pi\left(\left[\begin{smallmatrix}1 & \kappa\lambda^2\\ & 1\end{smallmatrix}\right]\right) f\left(\left[\begin{smallmatrix}1 & & & \\ \lambda & 1 & & \\ & & 1 & \\ & & -\lambda & 1\end{smallmatrix}\right] s_1\right) d^*\kappa\, d\lambda \\
=&q^{-1}\int\limits_{\mathfrak{o}}\int\limits_{F\backslash\mathfrak{o}} |\kappa^{-1}|^s\xi(\kappa^{-1})\pi\left(\left[\begin{smallmatrix}1 & \kappa\lambda^2\\ & 1\end{smallmatrix}\right]\right) v_0\, d^*\kappa\, d\lambda \\
=&q^{-1}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\int\limits_{\varpi^m\mathfrak{o}^\times}\int\limits_{\varpi^{-n}\mathfrak{o}^\times} |\lambda||\kappa^{-1}|^{s+1}\xi(\kappa^{-1})\pi\left(\left[\begin{smallmatrix}1 & \kappa\lambda^2\\ & 1\end{smallmatrix}\right]\right) v_0\, d\kappa\, d^*\lambda \\
=&q^{-1}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} q^{-m}q^{-n(s+1)}\xi(\varpi^n)\int\limits_{\varpi^m\mathfrak{o}^\times}\int\limits_{\varpi^{-n}\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix}1 & \kappa\lambda^2\\ & 1\end{smallmatrix}\right]\right) v_0\, d\kappa\, d^*\lambda \\
=&\begin{cases} 0 & 2m < n-1, \\ q^{-1}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{-m-n(s+1)}\int\limits_{\varpi^m\mathfrak{o}^\times} q^{n-1}v_0\, d^*\lambda & 2m = n-1, \\ q^{-1}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{n}q^{-m-n(s+1)}\int\limits_{\varpi^m\mathfrak{o}^\times} q^{n-1}(q-1)v_0\, d^*\lambda & 2m \geq n, \end{cases} \\
=&\sum_{m=0}^{\infty}\sum_{\substack{n=1\\ n\leq 2m}}^{\infty}(-1)^n q^{-m-ns-1}(1-q^{-1})^2 v_0 \\
&-\sum_{m=0}^{\infty}\sum_{\substack{n=1\\ n=2m+1}}^{\infty}(-1)^n q^{-m-ns-1}(1-q^{-1})q^{-1}v_0 \\
=&\sum_{m=1}^{\infty}(1-q^{-1})^2q^{-m-1}\sum_{\substack{n=1\\ n\leq 2m}}^{\infty}(-q^{-s})^n v_0 \\
&-\sum_{m=0}^{\infty}(-1)^{2m+1}q^{-m-(2m+1)s-1}(1-q^{-1})q^{-1}v_0 \\
=&-\sum_{m=1}^{\infty}\frac{q^{-1-s}(1-q^{-1})^2}{1+q^{-s}}(1-q^{-(2m)s})q^{-m}v_0+\sum_{m=0}^{\infty}(1-q^{-1})q^{-s-2}q^{-(2s+1)m}v_0 \\
=&-\frac{q^{-1-s}(1-q^{-1})^2}{1+q^{-s}}\left(\sum_{m=1}^{\infty}q^{-m}-\sum_{m=1}^{\infty}q^{-(2m)s-m}\right)v_0 \\
&+(1-q^{-1})q^{-s-2}\frac{1}{1-q^{-2s-1}}v_0 \\
=&-\frac{q^{-1-s}(1-q^{-1})^2}{1+q^{-s}}\left(\frac{q^{-1}}{1-q^{-1}}-\frac{q^{-2s-1}}{1-q^{-2s-1}}\right)v_0 \\
&+(1-q^{-1})q^{-s-2}\frac{1}{1-q^{-2s-1}}v_0
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{q^{-1-s}(1-q^{-1})^2}{1+q^{-s}}\left(\frac{q^{-1}(1-q^{-2s})}{(1-q^{-1})(1-q^{-2s-1})}\right)v_0 \\
&\quad + (1-q^{-1})q^{-s-2}\frac{1}{1-q^{-2s-1}}v_0 \\
&= -\frac{q^{-2-s}(1-q^{-1})(1-q^{-s})}{1-q^{-2s-1}}v_0 + \frac{(1-q^{-1})q^{-s-2}}{1-q^{-2s-1}}v_0 \\
&= \frac{q^{-2-s}(1-q^{-1})(1-1+q^{-s})}{1-q^{-2s-1}}v_0 = \frac{q^{-2-2s}(1-q^{-1})}{1-q^{-2s-1}}v_0
\end{aligned}
$$

(b) Similarly, let $\imath^{\mathfrak{o}}_{F\backslash\mathfrak{o}}$ be the second part of above integral. Then we have

$$
\begin{aligned}
\imath^{F\backslash\mathfrak{o}}_{F\backslash\mathfrak{o}} &= q^{-1}\int\limits_{F\backslash\mathfrak{o}}\int\limits_{F\backslash\mathfrak{o}} |\kappa^{-1}|^s\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1 & \kappa\lambda^2 \\ & 1\end{bmatrix}\right) f\left(\begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ & & 1 & \\ & & -\lambda & 1\end{bmatrix} s_1\right) d^*\kappa\, d\lambda \\
&= q^{-1}\int\limits_{F\backslash\mathfrak{o}}\int\limits_{F\backslash\mathfrak{o}} |\kappa^{-1}|^s\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1 & \kappa\lambda^2 \\ & 1\end{bmatrix}\right) \\
&\quad f\left(\begin{bmatrix} -\lambda^{-1} & & & \\ & -\lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda\end{bmatrix}\begin{bmatrix} 1 & \lambda & & \\ & 1 & & \\ & & 1 & -\lambda \\ & & & 1\end{bmatrix}\begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}\begin{bmatrix} 1 & \lambda^{-1} & & \\ & 1 & & \\ & & 1 & -\lambda^{-1} \\ & & & 1\end{bmatrix} s_1\right) d^*\kappa\, d\lambda \\
&= 0
\end{aligned}
$$

In conclusion, we have

$$
\mathcal{I}^s_{\mathfrak{p}} = q^{-1}v_0 + \frac{q^{-2-2s}(1-q^{-1})}{1-q^{-2s-1}}v_0 = \frac{q^{-1}(1-q^{-2-2s})}{1-q^{-2s-1}}v_0. \tag{5.55}
$$

□

5.4.3 $\mathcal{I}^s_{F\backslash\mathfrak{p}}$

Next, we are going to calculate $\mathcal{I}^s_{F\backslash\mathfrak{p}}$ as defined in (5.46). In particular, we have

$$
\mathcal{I}^s_{F\backslash\mathfrak{p}} = \int\limits_{F^2}\int\limits_{F\backslash\mathfrak{p}} f\left(\begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ \mu & & 1 & \\ \kappa & \mu & -\lambda & 1\end{bmatrix} s_1\right) d\mu\, d\kappa\, d\lambda
$$

$$
\begin{aligned}
&= \int\limits_{F^2} \int\limits_{F\backslash\mathfrak{p}} f\left(\begin{bmatrix} -1 &&& \\ \lambda & -1 && \\ \mu && -1 & \\ \kappa & \mu & -\lambda & -1 \end{bmatrix} s_1 \right) d\mu\, d\kappa\, d\lambda \\
&= \int\limits_{F^2} \int\limits_{F\backslash\mathfrak{p}} f\Bigg(\begin{bmatrix} -\mu^{-1} && 1 & \\ & -\mu^{-1} & -\lambda-\kappa\mu^{-1} & 1 \\ && -\mu & \\ &&& -\mu \end{bmatrix} \begin{bmatrix} 1 &&& \\ -\kappa\mu^{-1} & 1 && \\ && 1 & \\ -\mu^{-2}\kappa+\mu^{-1}\lambda && \kappa\mu^{-1} & 1 \end{bmatrix} \\
&\qquad\qquad s_2 s_1 s_2 \begin{bmatrix} 1 && -\mu^{-1} & \\ & 1 && -\mu^{-1} \\ && 1 & \\ &&& 1 \end{bmatrix} s_1 \Bigg) d\mu\, d\kappa\, d\lambda \\
&= \int\limits_{F^2} \int\limits_{F\backslash\mathfrak{p}} f\left(\begin{bmatrix} -\mu^{-1} && 1 & \\ & -\mu^{-1} & -\lambda-\kappa\mu^{-1} & 1 \\ && -\mu & \\ &&& -\mu \end{bmatrix} \begin{bmatrix} 1 &&& \\ -\kappa\mu^{-1} & 1 && \\ && 1 & \\ -\mu^{-2}\kappa+\mu^{-1}\lambda && \kappa\mu^{-1} & 1 \end{bmatrix} s_2 s_1 s_2 s_1 \right) d\mu\, d\kappa\, d\lambda \\
&= \int\limits_{F^2} \int\limits_{F\backslash\mathfrak{p}} |\mu| f\left(\begin{bmatrix} -\mu^{-1} && 1 & \\ & -\mu^{-1} & -\lambda-\kappa\mu^{-1} & 1 \\ && -\mu & \\ &&& -\mu \end{bmatrix} \begin{bmatrix} 1 &&& \\ -\kappa\mu^{-1} & 1 && \\ && 1 & \\ -\mu^{-2}\kappa+\mu^{-1}\lambda && \kappa\mu^{-1} & 1 \end{bmatrix} s_1 s_2 s_1 \right) d^*\mu\, d\kappa\, d\lambda \\
&= \int\limits_{F^2} \int\limits_{F\backslash\mathfrak{p}} |\mu^{-1}|^{1+s} \xi(\mu^{-1}) \pi\left(\begin{bmatrix} -\mu^{-1} & -\lambda-\kappa\mu^{-1} \\ & -\mu \end{bmatrix} \right) \\
&\qquad\qquad f\left(\begin{bmatrix} 1 &&& \\ -\kappa\mu^{-1} & 1 && \\ && 1 & \\ -\mu^{-2}\kappa+\mu^{-1}\lambda && \kappa\mu^{-1} & 1 \end{bmatrix} s_1 s_2 s_1 \right) d^*\mu\, d\kappa\, d\lambda \\
&= \int\limits_{F^2} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s} \xi(\mu^{-1}) \pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix} \right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ \kappa && -\lambda & 1 \end{bmatrix} s_1 s_2 s_1 \right) d^*\mu\, d\kappa\, d\lambda \\
&= \int\limits_{F} \int\limits_{\mathfrak{o}} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s} \xi(\mu^{-1}) \pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix} \right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ \kappa && -\lambda & 1 \end{bmatrix} s_1 s_2 s_1 \right) d^*\mu\, d\kappa\, d\lambda \\
&+ \int\limits_{F} \int\limits_{\varpi^{-1}\mathfrak{o}^\times} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s} \xi(\mu^{-1}) \pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix} \right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ \kappa && -\lambda & 1 \end{bmatrix} s_1 s_2 s_1 \right) d^*\mu\, d\kappa\, d\lambda \\
&+ \int\limits_{F} \int\limits_{F\backslash\mathfrak{p}^{-1}} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s} \xi(\mu^{-1}) \pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix} \right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ \kappa && -\lambda & 1 \end{bmatrix} s_1 s_2 s_1 \right) d^*\mu\, d\kappa\, d\lambda
\end{aligned}
$$

First part $\mathcal{J}_{\mathfrak{o}}^s$

$$
\mathcal{J}_{\mathfrak{o}}^s = \int\limits_{F} \int\limits_{\mathfrak{o}} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s} \xi(\mu^{-1}) \pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix} \right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ && -\lambda & 1 \end{bmatrix} s_1 s_2 s_1 \right) d^*\mu\, d\kappa\, d\lambda
$$

$$
= \left\{ \int\limits_{\mathfrak{o}} + \int\limits_{F\setminus\mathfrak{o}} \right\} \int\limits_{\mathfrak{o}} \int\limits_{F\setminus\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right)
$$

$$
f\left(\begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ & & 1 & \\ & & -\lambda & 1 \end{bmatrix} s_1 s_2 s_1\right) d^*\mu \, d\kappa \, d\lambda
$$

$$
= \int\limits_{F\setminus\mathfrak{o}} \int\limits_{\mathfrak{o}} \int\limits_{F\setminus\mathfrak{p}} |\lambda||\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right) f\left(\begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ & & 1 & \\ & & -\lambda & 1 \end{bmatrix} s_1 s_2 s_1\right) d^*\mu \, d\kappa \, d^*\lambda
$$

$$
= \int\limits_{F\setminus\mathfrak{o}} \int\limits_{\mathfrak{o}} \int\limits_{F\setminus\mathfrak{p}} |\lambda||\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right)
$$

$$
f\left(\begin{bmatrix} -\lambda^{-1} & & & \\ & -\lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda \end{bmatrix} \begin{bmatrix} 1 & \lambda & & \\ & 1 & & \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix} \begin{bmatrix} 1 & \lambda^{-1} & & \\ & 1 & & \\ & & 1 & -\lambda^{-1} \\ & & & 1 \end{bmatrix} s_1 s_2 s_1\right) d^*\mu \, d\kappa \, d^*\lambda
$$

$$
= \int\limits_{F\setminus\mathfrak{o}} \int\limits_{\mathfrak{o}} \int\limits_{F\setminus\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right) |\lambda|^{-1-s}\xi(\lambda^{-1})
$$

$$
\pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) \pi\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right) v_0 \, d^*\mu \, d\kappa \, d^*\lambda
$$

$$
= \int\limits_{F\setminus\mathfrak{o}} \int\limits_{\mathfrak{o}} \int\limits_{F\setminus\mathfrak{p}} |\mu|^{1-s}|\lambda|^{-1-s}\xi(\mu\lambda)
$$

$$
\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right) \pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) \pi\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right) v_0 \, d^*\mu \, d\kappa \, d^*\lambda
$$

$$
= \int\limits_{F\setminus\mathfrak{o}} \int\limits_{\mathfrak{o}} \int\limits_{F\setminus\mathfrak{p}} |\mu|^{1-s}|\lambda|^{-1-s}\xi(\mu\lambda)
$$

$$
\pi\left(\begin{bmatrix} \mu^{-1} & \kappa\mu \\ & \mu \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & 2\mu\lambda \\ & 1 \end{bmatrix}\right) \pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) \pi\left(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) v_0 \, d^*\mu \, d\kappa \, d^*\lambda
$$

$$
= \int\limits_{F\setminus\mathfrak{o}} \int\limits_{F\setminus\mathfrak{p}} \int\limits_{\mathfrak{o}} |\mu|^{1-s}|\lambda|^{-1-s}\xi(\mu\lambda)
$$

$$
\pi\left(\begin{bmatrix} \mu^{-1} & \\ & \mu \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & 2\mu\lambda \\ & 1 \end{bmatrix}\right) \pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) \pi\left(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & \\ \kappa\mu^2\lambda^{-2} & 1 \end{bmatrix}\right) v_0 \, d\kappa \, d^*\mu \, d^*\lambda
$$

$$
= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \int\limits_{\varpi^{-i}\mathfrak{o}^\times} \int\limits_{\varpi^{-j}\mathfrak{o}^\times} |\mu|^{1-s}|\lambda|^{-1-s}\xi(\mu\lambda)\pi\left(\begin{bmatrix} \mu^{-1} & \\ & \mu \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & 2\mu\lambda \\ & 1 \end{bmatrix}\right)
$$

$$
\pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) \pi\left(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) \int\limits_{\mathfrak{o}} \pi\left(\begin{bmatrix} 1 & \\ \kappa\mu^2\lambda^{-2} & 1 \end{bmatrix}\right) v_0 \, d\kappa \, d^*\mu \, d^*\lambda
$$

$$
= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \int\limits_{\mathfrak{o}^\times} \int\limits_{\mathfrak{o}^\times} q^{j(1-s)+i(-1-s)}(-1)^{i+j}\pi\left(\begin{bmatrix} \varpi^{j}\mu^{-1} & \\ & \varpi^{-j}\mu \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & 2\varpi^{-i-j}\mu\lambda \\ & 1 \end{bmatrix}\right)
$$

$$\pi\left(\begin{bmatrix}\varpi^{-i}\lambda & \\ & \varpi^{i}\lambda^{-1}\end{bmatrix}\right)\pi\left(\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\right)\underbrace{\int_{\mathfrak{o}}\pi\left(\begin{bmatrix}1 & \\ \varpi^{2i-2j}\kappa\mu^2\lambda^{-2} & 1\end{bmatrix}\right)v_0\,d\kappa}_{\begin{cases}0 & \text{if } 2i-2j\le 1, i.\,e., i-j\le 0\\ v_0 & \text{if } 2i-2j\ge 2, i.\,e., i-j\ge 1\end{cases}}\,d^*\mu\,d^*\lambda$$

$$=\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}(-1)^{i+j}q^{j(1-s)+i(-1-s)}$$
$$\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}\varpi^{j-i}\mu^{-1}\lambda & \\ & \varpi^{-j+i}\mu\lambda^{-1}\end{bmatrix}\right)\pi\left(\begin{bmatrix}1 & 2\varpi^{i-j}\mu\lambda^{-1}\\ & 1\end{bmatrix}\right)\pi\left(\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\right)v_0\,d^*\mu\,d^*\lambda$$

$$=\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}(-1)^{i+j}(1-q^{-1})q^{j(1-s)+i(-1-s)}$$
$$\pi\left(\begin{bmatrix}\varpi^{j-i}\mu^{-1} & \\ & \varpi^{i-j}\mu\end{bmatrix}\right)\pi\left(\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\right)\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}1 & \\ 2\varpi^{i-j}\mu & 1\end{bmatrix}\right)v_0\,d^*\mu.$$

Second part $\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}$

$$\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}=\int_F\int_{\varpi^{-1}\mathfrak{o}^\times}\int_{F\backslash\mathfrak{p}}|\kappa||\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix}-\mu^{-1} & -\kappa\mu+2\lambda\\ & -\mu\end{bmatrix}\right)$$
$$f\left(\begin{bmatrix}\kappa^{-1} & & \kappa^{-1}\lambda & 1\\ & 1 & \kappa^{-1}\lambda^2 & \lambda\\ & & 1 & \\ & & & \kappa\end{bmatrix}\begin{bmatrix}1 & & & \\ \kappa^{-1}\lambda & 1 & & \\ & & 1 & \\ & & -\kappa^{-1}\lambda & 1\end{bmatrix}\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1\\ -\kappa^{-1} & & & \end{bmatrix}\right)d^*\mu\,d^*\kappa\,d\lambda$$
$$=\int_F\int_{\varpi^{-1}\mathfrak{o}^\times}\int_{F\backslash\mathfrak{p}}|\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix}-\mu^{-1} & -\kappa\mu+2\lambda\\ & -\mu\end{bmatrix}\right)|\kappa|^{-1-s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix}1 & \kappa^{-1}\lambda^2\\ & 1\end{bmatrix}\right)$$
$$f\left(\begin{bmatrix}1 & & & \\ \kappa^{-1}\lambda & 1 & & \\ & & 1 & \\ & & -\kappa^{-1}\lambda & 1\end{bmatrix}\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1\\ -\kappa^{-1} & & & \end{bmatrix}\right)d^*\mu\,d^*\kappa\,d\lambda$$
$$=\int_F\int_{\varpi^{-1}\mathfrak{o}^\times}\int_{F\backslash\mathfrak{p}}|\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix}-\mu^{-1} & -\kappa\mu+2\lambda\kappa\\ & -\mu\end{bmatrix}\right)|\kappa|^{-s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix}1 & \kappa\lambda^2\\ & 1\end{bmatrix}\right)$$
$$f\left(\begin{bmatrix}1 & & & \\ \lambda & 1 & & \\ & & 1 & \\ & & -\lambda & 1\end{bmatrix}\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1\\ -\kappa^{-1} & & & \end{bmatrix}\right)d^*\mu\,d^*\kappa\,d\lambda$$
$$=-q^{-s}\left\{\int_{\mathfrak{p}}+\int_{F\backslash\mathfrak{p}}\right\}\int_{\mathfrak{o}^\times}\int_{F\backslash\mathfrak{p}}|\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix}-\mu^{-1} & \varpi^{-1}\kappa\mu-2\varpi^{-1}\lambda\kappa\\ & -\mu\end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa\lambda^2 \\ & 1 \end{bmatrix}\right)\pi\left(\begin{bmatrix} \kappa & \\ & 1 \end{bmatrix}\right) f\left(\begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ & & 1 & \\ & & -\lambda & 1 \end{bmatrix} y_2\right) d^*\mu\, d^*\kappa\, d\lambda$$

$$= -\, q^{-s}\int_{F\setminus\mathfrak{p}}\int_{\mathfrak{o}^\times}\int_{F\setminus\mathfrak{p}} |\lambda||\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & \varpi^{-1}\kappa\mu-2\varpi^{-1}\lambda\kappa \\ & -\mu \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa\lambda^2 \\ & 1 \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} \kappa & \\ & 1 \end{bmatrix}\right) f\left(\begin{bmatrix} -\lambda^{-1} & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & -\lambda \end{bmatrix}\begin{bmatrix} 1 & -\lambda & & \\ & 1 & & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix} s_1 \begin{bmatrix} 1 & \lambda^{-1} & & \\ & 1 & & \\ & & 1 & -\lambda^{-1} \\ & & & 1 \end{bmatrix} y_2\right) d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= -\, q^{-s}\int_{F\setminus\mathfrak{p}}\int_{\mathfrak{o}^\times}\int_{F\setminus\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & \varpi^{-1}\kappa\mu-2\varpi^{-1}\lambda\kappa \\ & -\mu \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa\lambda^2 \\ & 1 \end{bmatrix}\right)$$

$$|\lambda|^{-1-s}\xi(\lambda^{-1})\pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= -\, q^{-s}\int_{F\setminus\mathfrak{p}}\int_{\mathfrak{o}^\times}\int_{F\setminus\mathfrak{p}} |\mu|^{1-s}|\lambda|^{-1-s}\xi(\mu\lambda)\pi\left(\begin{bmatrix} \mu^{-1} & \varpi^{-1}\kappa(2\lambda-\mu) \\ & \mu \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa \\ & 1 \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= -\, q^{-s}\int_{F\setminus\mathfrak{p}}\int_{\mathfrak{o}^\times}\int_{F\setminus\mathfrak{p}} |\mu|^{1-s}|\lambda|^{-1-s}\xi(\mu\lambda)\pi\left(\begin{bmatrix} \mu^{-1}\lambda & \\ & \mu\lambda^{-1} \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa(1-2\mu\lambda^{-1}+\mu^2\lambda^{-2}) \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= -\, q^{-s}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} q^{j(1-s)+i(-1-s)}(-1)^{i+j}\pi\left(\begin{bmatrix} \varpi^{j-i}\mu^{-1}\lambda & \\ & \varpi^{i-j}\mu\lambda^{-1} \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa(1-\varpi^{(i-j)}\mu\lambda^{-1})^2 \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= -\, q^{-s}(1-q^{-1})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} q^{j(1-s)+i(-1-s)}(-1)^{i+j}\pi\left(\begin{bmatrix} (\varpi^{i-j}\mu)^{-1} & \\ & \varpi^{i-j}\mu \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa(1-\varpi^{i-j}\mu)^2 \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa$$

$$= -\, q^{-s}(1-q^{-1})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} q^{j(1-s)+i(-1-s)}(-1)^{i+j}\pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa(\mu^{-1}-\varpi^{(i-j)})^2 \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \kappa^{-1}\mu^2\varpi & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa$$

$$= -\, q^{-s}(1-q^{-1})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} q^{j(1-s)+i(-1-s)}(-1)^{i+j}\pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa(1-\varpi^{(i-j)}\mu)^2 \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa$$

$$= - q^{-s}(1-q^{-1}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^{j(1-s)+i(-1-s)}(-1)^{i+j}$$

$$\int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa(1-\varpi^{(i-j)}\mu)^2 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}\right) v_0 \, d^*\mu \, d^*\kappa$$

$$= - q^{-s}(1-q^{-1}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^{j(1-s)+i(-1-s)}(-1)^{i+j}$$

$$\int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\varpi^{(i-j)}\mu)^2 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \kappa\varpi & 1 \end{bmatrix}\right) v_0 \, d^*\mu \, d^*\kappa.$$

1. If $i > j$, we can take a substitution $\kappa \mapsto \kappa(1-\varpi^{(i-j)}\mu)^2$, then we have

$$\mathcal{J}^{s,i>j}_{\varpi^{-1}\mathfrak{o}^\times} = - q^{-s}(1-q^{-1}) \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} q^{j(1-s)+i(-1-s)}(-1)^{i+j} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\right)$$

$$\pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \kappa\varpi(1-\varpi^{(i-j)}\mu)^2 & 1 \end{bmatrix}\right) v_0 \, d^*\mu \, d^*\kappa$$

$$= - q^{-s}(1-q^{-1}) \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} q^{j(1-s)+i(-1-s)}(-1)^{i+j}$$

$$\int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\right) \pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \kappa\varpi & 1 \end{bmatrix}\right) v_0 \, d^*\mu \, d^*\kappa$$

$$= - q^{-s}(1-q^{-1})^2 \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} q^{j(1-s)+i(-1-s)}(-1)^{i+j}$$

$$\int_{\mathfrak{o}^\times} \pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix} \begin{bmatrix} 1 & -\varpi^{-1}\kappa^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \kappa\varpi & 1 \end{bmatrix}\right) v_0 \, d^*\kappa.$$

2. If $i < j$, then we let $I_{i,j}$ be the integral

$$\int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \pi(\begin{bmatrix} 1 & -\varpi^{-1}\kappa(1-\varpi^{(i-j)}\mu)^2 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{bmatrix}) v_0 \, d^*\mu \, d^*\kappa.$$

Then we have

$$
\begin{aligned}
I_{i,j} &= \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}\kappa\mu^2(\varpi^{(j-i)}\mu^{-1}-1)^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{smallmatrix}\right])v\, d^*\mu\, d^*\kappa \\
&= \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}\kappa\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa^{-1}\varpi(\varpi^{(j-i)}\mu^{-1}-1)^2 & 1 \end{smallmatrix}\right])v\, d^*\mu\, d^*\kappa \\
&= \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}\kappa\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{smallmatrix}\right])v\, d^*\mu\, d^*\kappa.
\end{aligned}
$$

For fixed μ, consider the inner integral

$$
\begin{aligned}
I_{i,j,\mu} &= \int\limits_{\mathfrak{o}^\times} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}\kappa\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa^{-1}\varpi & 1 \end{smallmatrix}\right])v\, d^*\kappa \\
&= \sum_{x\in\mathfrak{o}^\times/(1+\mathfrak{p})} \int\limits_{1+\mathfrak{p}} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}xy\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ (xy)^{-1}\varpi & 1 \end{smallmatrix}\right])v\, d^*y \\
&= \sum_{x\in\mathfrak{o}^\times/(1+\mathfrak{p})} \int\limits_{1+\mathfrak{p}} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}xy\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\, d^*y.
\end{aligned}
$$

For fixed $x \in \mathfrak{o}^\times$, consider

$$
\begin{aligned}
I_{i,j,\mu,x} &= \int\limits_{1+\mathfrak{p}} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}xy\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\, d^*y \\
&= \int\limits_{1+\mathfrak{p}} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}xy\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\, dy \\
&= \int\limits_{\mathfrak{p}} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}x(1+y)\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\, dy \\
&= \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}x\mu^2 \\ & 1 \end{smallmatrix}\right]) \int\limits_{\mathfrak{p}} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}xy\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\, dy \\
&= q^{-1}\pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}x\mu^2 \\ & 1 \end{smallmatrix}\right]) \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)}xy\mu^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\, dy
\end{aligned}
$$

$$= q^{-1}\pi(\left[\begin{smallmatrix} 1 & -\varpi^{2(i-j)-1}x\mu^2 \\ & 1 \end{smallmatrix}\right]) \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & \varpi^{2(i-j)}y \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy.$$

Lemma 5.19. *Let $k \geq 1$ be an integer and $x \in \mathfrak{o}^\times$. Then*

$$\int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & \varpi^{-2k}y \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy = 0.$$

Proof. The vector

$$v_1 := \pi(\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v$$

is invariant under $\left[\begin{smallmatrix} 1 & \mathfrak{o} \\ & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1+\mathfrak{p} & \\ & 1+\mathfrak{p} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & \\ \mathfrak{p}^2 & 1 \end{smallmatrix}\right]$. We claim that

$$v_2 := \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & \varpi^{-2k}y \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy$$

is invariant under $\left[\begin{smallmatrix} 1 & \mathfrak{p}^{-2k} \\ & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1+\mathfrak{p} & \\ & 1+\mathfrak{p} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & \\ \mathfrak{p}^{2k+1} & 1 \end{smallmatrix}\right]$. The first two assertions are obvious. To see the third one, let $x \in \mathfrak{p}^{2k+1}$. We calculate

$$\begin{aligned}
\pi(\left[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right])v_2 &= \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \varpi^{-2k}y \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy \\
&= \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} (1+xy\varpi^{-2k})^{-1} & \varpi^{-2k}y \\ & 1+xy\varpi^{-2k} \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ (1+xy\varpi^{-2k})^{-1}x & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy \\
&= \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} (1+xy\varpi^{-2k})^{-1} & \varpi^{-2k}y \\ & 1+xy\varpi^{-2k} \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy \\
&= \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & \varpi^{-2k}y(1+xy\varpi^{-2k})^{-1} \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} (1+xy\varpi^{-2k})^{-1} & \\ & 1+xy\varpi^{-2k} \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy \\
&= \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & \varpi^{-2k}y(1+xy\varpi^{-2k})^{-1} \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dy.
\end{aligned}$$

Let $z = y/(1 + xy\varpi^{-2k})$. Then $y = z/(1 - xz\varpi^{-2k})$. Hence the map $y \mapsto z$ is a bijection of $\mathfrak{o}$ onto itself, and in fact is a legitimate change of variables.

Hence

$$\pi(\left[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right])v_2 = \int\limits_{\mathfrak{o}} \pi(\left[\begin{smallmatrix} 1 & \varpi^{-2k}z \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x^{-1}\varpi & 1 \end{smallmatrix}\right])v\,dz = v_2.$$

This proves our claim about the invariance properties of v_2.

Now consider $v_3 := \pi(\left[\begin{smallmatrix} \varpi^k & \\ & \varpi^{-k} \end{smallmatrix}\right])v_2$. It is invariant under $\left[\begin{smallmatrix} 1+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right]$. The group generated by these three types of matrices contains the principal congruence subgroup $\Gamma(\mathfrak{p})$ (use the Iwahori decomposition for $\Gamma(\mathfrak{p})$). Hence v_3 lies in the space V_0 of $\Gamma(\mathfrak{p})$-invariant vectors.

Now recall that π is a depth zero supercuspidal of $\mathrm{GL}(2,F)$. This implies that

$$\pi = \text{c-Ind}_{ZK}^{\mathrm{GL}(2,F)}(\rho),$$

where ρ is an irreducible, cuspidal representation of $\mathrm{GL}(2,\mathbb{F}_q)$, inflated to a representation of $K = \mathrm{GL}(2,\mathfrak{o})$, and then further extended to a representation of ZK by making the center Z act trivially. In such a situation it is known that V_0 is the same with the space of ρ. It follows that v_3, considered as a vector in the representation ρ of the finite group $\mathrm{GL}(2,\mathbb{F}_q)$, is invariant under $\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]$. But this implies $v_3 = 0$, since ρ is *cuspidal*; see page 410 of [5]. Since $v_3 = 0$, then also $v_2 = 0$. □

It follows from this lemma that $I_{i,j,\mu,x} = 0$. Hence also $I_{i,j,\mu} = 0$ and $I_{i,j} = 0$.

3. If $i = j$, then we have

$$\mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times} = -q^{-s}(1-q^{-1})\sum_{i=0}^{\infty} q^{-2si} \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}\right]\right) v\, d^*\mu\, d^*\kappa$$

$$= -\frac{q^{-s}(1-q^{-1})}{1-q^{-2s}} \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} \pi\left(\begin{bmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2 \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \kappa\varpi & 1 \end{bmatrix}\right) v \, d^*\mu \, d^*\kappa$$

Third part $\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}$

$$\begin{aligned}
&\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}} \\
=&\int\limits_{F}\int\limits_{F\backslash\mathfrak{p}^{-1}}\int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ \kappa && -\lambda & 1 \end{bmatrix} s_1s_2s_1\right) d^*\mu \, d\kappa \, d\lambda \\
=&\int\limits_{F}\int\limits_{F\backslash\mathfrak{p}^{-1}}\int\limits_{F\backslash\mathfrak{p}} |\kappa||\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right) \\
&f\left(\begin{bmatrix} \kappa^{-1} && \kappa^{-1}\lambda & 1 \\ & 1 & \kappa^{-1}\lambda^2 & \lambda \\ && 1 & \\ &&& \kappa \end{bmatrix}\begin{bmatrix} 1 &&& \\ \kappa^{-1}\lambda & 1 && \\ && 1 & \\ && -\kappa^{-1}\lambda & 1 \end{bmatrix} s_1s_2s_1 \begin{bmatrix} -1 &&& -\kappa^{-1} \\ & 1 && \\ && 1 & \\ &&& -1 \end{bmatrix} s_1s_2s_1\right) d^*\mu \, d^*\kappa \, d\lambda \\
=&\int\limits_{F}\int\limits_{F\backslash\mathfrak{p}^{-1}}\int\limits_{F\backslash\mathfrak{p}} |\kappa||\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda \\ & -\mu \end{bmatrix}\right) \\
&\qquad f\left(\begin{bmatrix} \kappa^{-1} && \kappa^{-1}\lambda & 1 \\ & 1 & \kappa^{-1}\lambda^2 & \lambda \\ && 1 & \\ &&& \kappa \end{bmatrix}\begin{bmatrix} 1 &&& \\ \kappa^{-1}\lambda & 1 && \\ && 1 & \\ && -\kappa^{-1}\lambda & 1 \end{bmatrix}\right) d^*\mu \, d^*\kappa \, d\lambda \\
=&\left\{\int\limits_{\mathfrak{p}} + \int\limits_{F\backslash\mathfrak{p}}\right\}\int\limits_{F\backslash\mathfrak{p}^{-1}}\int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda\kappa \\ & -\mu \end{bmatrix}\right) \\
&\qquad |\kappa|^{-s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{bmatrix}\right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ && -\lambda & 1 \end{bmatrix}\right) d^*\mu \, d^*\kappa \, d\lambda \\
=&0 + \int\limits_{F\backslash\mathfrak{p}}\int\limits_{F\backslash\mathfrak{p}^{-1}}\int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda\kappa \\ & -\mu \end{bmatrix}\right) \\
&\qquad |\kappa|^{-s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{bmatrix}\right) f\left(\begin{bmatrix} 1 &&& \\ \lambda & 1 && \\ && 1 & \\ && -\lambda & 1 \end{bmatrix}\right) d^*\mu \, d^*\kappa \, d\lambda \\
=&\int\limits_{F\backslash\mathfrak{p}}\int\limits_{F\backslash\mathfrak{p}^{-1}}\int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} -\mu^{-1} & -\kappa\mu+2\lambda\kappa \\ & -\mu \end{bmatrix}\right) \\
&\quad |\kappa|^{-s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{bmatrix}\right) |\lambda|^{-1-s}\xi(\lambda^{-1})\pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) v_0 \, d^*\mu \, d^*\kappa \, d^*\lambda \\
=&\int\limits_{F\backslash\mathfrak{p}}\int\limits_{F\backslash\mathfrak{p}^{-1}}\int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} \mu^{-1} & \kappa\mu \\ & \mu \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & -2\mu\lambda\kappa \\ & 1 \end{bmatrix}\right)
\end{aligned}$$

$$|\kappa|^{-s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{bmatrix}\right)|\lambda|^{-1-s}\xi(\lambda^{-1})\pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= \int\limits_{F\backslash\mathfrak{p}} \int\limits_{F\backslash\mathfrak{p}^{-1}} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}\xi(\mu^{-1})\pi\left(\begin{bmatrix} \mu^{-1} & \\ & \mu \end{bmatrix}\begin{bmatrix} 1 & \kappa\mu^2 \\ & 1 \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & -2\mu\lambda\kappa \\ & 1 \end{bmatrix}\right)$$

$$|\kappa|^{-s}\xi(\kappa^{-1})\pi\left(\begin{bmatrix} 1 & \kappa\lambda^2 \\ & 1 \end{bmatrix}\right)|\lambda|^{-1-s}\xi(\lambda^{-1})\pi\left(\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= \int\limits_{F\backslash\mathfrak{p}} \int\limits_{F\backslash\mathfrak{p}^{-1}} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}|\kappa|^{-s}|\lambda|^{-1-s}\xi(\mu\kappa\lambda)$$

$$\pi\left(\begin{bmatrix} \mu^{-1} & \\ & \mu \end{bmatrix}\begin{bmatrix} 1 & -2\lambda\mu\kappa \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \kappa(\mu^2+\lambda^2) \\ & 1 \end{bmatrix}\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= \int\limits_{F\backslash\mathfrak{p}} \int\limits_{F\backslash\mathfrak{p}^{-1}} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}|\kappa|^{-s}|\lambda|^{-1-s}\xi(\mu\kappa\lambda)$$

$$\pi\left(\begin{bmatrix} \mu^{-1} & \\ & \mu \end{bmatrix}\begin{bmatrix} 1 & \kappa(\mu^2-2\lambda\mu+\lambda^2) \\ & 1 \end{bmatrix}\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= \int\limits_{F\backslash\mathfrak{p}} \int\limits_{F\backslash\mathfrak{p}^{-1}} \int\limits_{F\backslash\mathfrak{p}} |\mu|^{1-s}|\kappa|^{-s}|\lambda|^{-1-s}\xi(\mu\kappa\lambda)$$

$$\pi\left(\begin{bmatrix} \mu^{-1}\lambda & \\ & \mu\lambda^{-1} \end{bmatrix}\begin{bmatrix} 1 & \kappa(\mu^2\lambda^{-2}-2\lambda^{-1}\mu+1) \\ & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= \sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty} \int\limits_{\varpi^{-i}\mathfrak{o}^\times} \int\limits_{\varpi^{-k}\mathfrak{o}^\times} \int\limits_{\varpi^{-j}\mathfrak{o}^\times} |\mu|^{1-s}|\kappa|^{-s}|\lambda|^{-1-s}\xi(\mu\kappa\lambda)$$

$$\pi\left(\begin{bmatrix} \mu^{-1}\lambda & \\ & \mu\lambda^{-1} \end{bmatrix}\begin{bmatrix} 1 & \kappa(1-\mu\lambda^{-1})^2 \\ & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= \sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty} \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} (-1)^{i+k+j} q^{i(-1-s)+k(-s)+j(1-s)}$$

$$\pi\left(\begin{bmatrix} \varpi^{j-i}\mu^{-1}\lambda & \\ & \varpi^{i-j}\mu\lambda^{-1} \end{bmatrix}\begin{bmatrix} 1 & \varpi^{-k}\kappa(1-\varpi^{(i-j)}\mu\lambda^{-1})^2 \\ & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa\, d^*\lambda$$

$$= \sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty} \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} (1-q^{-1})(-1)^{i+k+j} q^{i(-1-s)+k(-s)+j(1-s)}$$

$$\pi\left(\begin{bmatrix} \varpi^{j-i}\mu^{-1} & \\ & \varpi^{i-j}\mu \end{bmatrix}\begin{bmatrix} 1 & \varpi^{-k}\kappa(1-\varpi^{(i-j)}\mu)^2 \\ & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa$$

$$= (1-q^{-1})\sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\sum_{j=0}^{\infty} \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} (-1)^{i+k+j} q^{i(-1-s)+k(-s)+j(1-s)}$$

$$\pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\begin{bmatrix} 1 & \varpi^{-k}\kappa(1-\varpi^{(i-j)}\mu)^2 \\ & 1 \end{bmatrix}\right) v_0\, d^*\mu\, d^*\kappa$$

1. If $i > j$, i.e., $1 - \varpi^{(i-j)}\mu \in \mathfrak{o}^\times$, then we can make the substitution

$$\kappa \mapsto \kappa(1 - \varpi^{(i-j)}\mu)^{-2}.$$

That is,

$$\begin{aligned}\frac{1}{(1-q^{-1})} \cdot \mathcal{J}^{s,i>j}_{F\backslash\mathfrak{p}^{-1}} &= \sum_{i=1}^{\infty}\sum_{k=2}^{\infty}\sum_{j=0}^{i-1} \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} (-1)^{i+k+j} q^{i(-1-s)+k(-s)+j(1-s)} \\ &\qquad \pi\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \varpi^{-k}\kappa \\ & 1\end{smallmatrix}\right]\right) v_0 \, d^*\mu \, d^*\kappa \\ &= \sum_{i=1}^{\infty}\sum_{k=2}^{\infty}\sum_{j=0}^{i-1} \int_{\mathfrak{o}^\times} (-1)^{i+k+j} q^{i(-1-s)+k(-s)+j(1-s)} \\ &\qquad \pi\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\right) \int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}\kappa \\ & 1\end{smallmatrix}\right]\right) v_0 \, d^*\kappa \, d^*\mu\end{aligned}$$

Now look at the inner integration. Since $k \geq 2$ we have

$$\int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}\kappa \\ & 1\end{smallmatrix}\right]\right) v \, d^*\kappa = \left\{\int_{\mathfrak{o}} - \int_{\mathfrak{p}}\right\} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}\kappa \\ & 1\end{smallmatrix}\right]\right) v \, d^*\kappa = 0 - 0 = 0.$$

2. Similarly, if $i < j$, then

$$1 - \varpi^{(i-j)}\mu = -\varpi^{(i-j)}\mu(1 - \varpi^{(j-i)}\mu^{-1}), \tag{5.56}$$

where $(1 - \varpi^{(j-i)}\mu^{-1}) \in \mathfrak{o}^\times$. Again, we make the substitution

$$\kappa \mapsto \kappa\mu^{-2}(1 - \varpi^{(j-i)}\mu^{-1})^{-2},$$

and get

$$\begin{aligned}\mathcal{J}_{F\backslash\mathfrak{p}^{-1}}^{s,i<j} =&(1-q^{-1})\sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\sum_{j=i+1}^{\infty}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times}(-1)^{i+k+j}q^{i(-1-s)+k(-s)+j(1-s)}\\ &\pi\left(\begin{bmatrix}\varpi^{j-i} & \\ & \varpi^{i-j}\end{bmatrix}\begin{bmatrix}1 & \varpi^{-(k+2(j-i))}\kappa\\ & 1\end{bmatrix}\right)v_0\,d^*\mu\,d^*\kappa\\ =&(1-q^{-1})\sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\sum_{j=i+1}^{\infty}\int_{\mathfrak{o}^\times}(-1)^{i+k+j}q^{i(-1-s)+k(-s)+j(1-s)}\\ &\pi\left(\begin{bmatrix}\varpi^{j-i} & \\ & \varpi^{i-j}\end{bmatrix}\right)\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}1 & \varpi^{-(k+2(j-i))}\kappa\\ & 1\end{bmatrix}\right)v_0\,d^*\kappa\,d^*\mu.\end{aligned}$$

Similar to above, the inner integration gives zero.

3. That is, only the $i=j$ case remains. Our integration becomes

$$\begin{aligned}\mathcal{J}_{F\backslash\mathfrak{p}^{-1}}^{s,i=j} &= (1-q^{-1})\sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times}(-1)^{k}q^{i(-2s)+k(-s)}\pi\left(\begin{bmatrix}1 & \varpi^{-k}\kappa(1-\mu)^2\\ & 1\end{bmatrix}\right)v_0\,d^*\mu\,d^*\kappa\\ &=\frac{1-q^{-1}}{1-q^{-2s}}\sum_{k=2}^{\infty}\int_{\mathfrak{o}^\times}(-1)^{k}q^{-ks}\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}1 & \varpi^{-k}\kappa(1-\mu)^2\\ & 1\end{bmatrix}\right)v_0\,d^*\kappa\,d^*\mu.\end{aligned}$$

(a) If $1-\mu\in\mathfrak{o}^\times$, with the same reason as in the previous cases, we have

$$\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}1 & \varpi^{-k}\kappa(1-\mu)^2\\ & 1\end{bmatrix}\right)v_0\,d^*\kappa = 0,\quad \forall k\geq 2. \tag{5.57}$$

(b) If $1-\mu\in\mathfrak{p}$, i.e., $\mu\in 1+\mathfrak{p}$, then we have

$$\begin{aligned}\mathcal{J}_{F\backslash\mathfrak{p}^{-1}}^{s,i=j} =&\frac{1-q^{-1}}{1-q^{-2s}}\sum_{k=2}^{\infty}\int_{1+\mathfrak{p}}(-1)^{k}q^{-ks}\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}1 & \varpi^{-k}\kappa(1-\mu)^2\\ & 1\end{bmatrix}\right)v_0\,d^*\kappa\,d^*\mu\\ =&\frac{1-q^{-1}}{1-q^{-2s}}\sum_{k=2}^{\infty}(-1)^{k}q^{-ks}\int_{1+\mathfrak{p}}\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}1 & \varpi^{-k}(1-\mu)^2\kappa\\ & 1\end{bmatrix}\right)v_0\,d^*\kappa\,d^*\mu.\end{aligned}$$

i. If $\nu(\varpi^{-k}(1-\mu)^2) \leq -2$, i.e., $k \geq 2 + 2\nu(1-\mu)$, the inner integral is

$$\begin{aligned}
\int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\kappa &= \int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa \\
&= \int_{\mathfrak{o}} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa - \int_{\mathfrak{p}} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa \\
&= 0 - 0 = 0.
\end{aligned}$$

ii. If $\nu(\varpi^{-k}(1-\mu)^2) = -1$, i.e., $k = 1 + 2\nu(1-\mu)$, the inner integral is

$$\begin{aligned}
\int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\kappa &= \int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa \\
&= \int_{\mathfrak{o}} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa - \int_{\mathfrak{p}} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d\kappa \\
&= 0 - q^{-1} v_0 = -q^{-1} v_0.
\end{aligned}$$

iii. If $\nu(\varpi^{-k}(1-\mu)^2) \geq 0$, i.e., $2 \leq k \leq 2\nu(1-\mu)$, the inner integral is

$$\int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\kappa = (1 - q^{-1}) v_0.$$

Let $\nu(1-\mu) = m \geq 1$. It follows from the above discussions that

$$\mathcal{J}^{s,i=j}_{F\backslash\mathfrak{p}^{-1}} = \frac{1-q^{-1}}{1-q^{-2s}} \sum_{k=2}^{\infty} (-1)^k q^{-ks} \int_{1+\mathfrak{p}} \int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\kappa \, d^*\mu$$

$$
\begin{aligned}
&= \frac{1-q^{-1}}{1-q^{-2s}} \sum_{m=1}^{\infty} \sum_{k=2}^{\infty} (-1)^k q^{-ks} \int\limits_{\nu(1-\mu)=m} \int\limits_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\kappa \, d^*\mu \\
&= \frac{1-q^{-1}}{1-q^{-2s}} \sum_{m=1}^{\infty} \sum_{k=2}^{2m} (-1)^k q^{-ks} \int\limits_{\nu(1-\mu)=m} \int\limits_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-k}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\kappa \, d^*\mu \\
&- \frac{1-q^{-1}}{1-q^{-2s}} \sum_{m=1}^{\infty} q^{-(1+2m)s} \int\limits_{\nu(1-\mu)=m} \int\limits_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & \varpi^{-(1+2m)}(1-\mu)^2\kappa \\ & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\kappa \, d^*\mu \\
&= \frac{1-q^{-1}}{1-q^{-2s}} \sum_{m=1}^{\infty} \sum_{k=2}^{2m} (-1)^k q^{-ks} \int\limits_{\nu(1-\mu)=m} (1-q^{-1}) v_0 \, d^*\mu \\
&\qquad - \frac{1-q^{-1}}{1-q^{-2s}} \sum_{m=1}^{\infty} q^{-(1+2m)s} \int\limits_{\nu(1-\mu)=m} (-q^{-1}) v_0 \, d^*\mu \\
&= \frac{(1-q^{-1})^2}{1-q^{-2s}} \sum_{m=1}^{\infty} \sum_{k=2}^{2m} (-1)^k q^{-ks} \int\limits_{\nu(1-\mu)=m} v_0 \, d^*\mu \\
&\qquad + \frac{q^{-1}(1-q^{-1})}{1-q^{-2s}} \sum_{m=1}^{\infty} q^{-(1+2m)s} \int\limits_{\nu(1-\mu)=m} v_0 \, d^*\mu \\
&= \frac{(1-q^{-1})^2(1-q^{-1})}{1-q^{-2s}} \sum_{m=1}^{\infty} q^{-m} \frac{q^{-2s}(1+q^{-(2m-1)s})}{1+q^{-s}} v_0 \\
&\qquad + \frac{q^{-1}(1-q^{-1})(1-q^{-1})}{1-q^{-2s}} \sum_{m=1}^{\infty} q^{-m} q^{-(1+2m)s} v_0 \\
&= \frac{(1-q^{-1})^3 q^{-2s}}{(1-q^{-2s})(1+q^{-s})} \sum_{m=1}^{\infty} \left(q^{-m} + q^s q^{-(2s+1)m}\right) v_0 \\
&\qquad + \frac{(1-q^{-1})^2 q^{-s-1}}{1-q^{-2s}} \sum_{m=1}^{\infty} q^{-(2s+1)m} v_0 \\
&= \frac{(1-q^{-1})^3 q^{-2s}}{(1-q^{-2s})(1+q^{-s})} \left(\frac{q^{-1}}{1-q^{-1}} + q^s \frac{q^{-2s-1}}{1-q^{-2s-1}}\right) v_0 \\
&\qquad + \frac{(1-q^{-1})^2 q^{-s-1}}{1-q^{-2s}} \frac{q^{-2s-1}}{1-q^{-2s-1}} v_0 \\
&= \frac{(1-q^{-1})^3 q^{-2s}}{(1-q^{-2s})(1+q^{-s})} \left(\frac{q^{-1}(1-q^{-2s-1}+q^{-s}-q^{-s-1})}{(1-q^{-1})(1-q^{-2s-1})}\right) v_0 \\
&\qquad + \frac{(1-q^{-1})^2 q^{-s-1}}{1-q^{-2s}} \frac{q^{-2s-1}}{1-q^{-2s-1}} v_0
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\frac{(1-q^{-1})^2q^{-2s-1}(1-q^{-s-1})}{(1-q^{-2s})(1-q^{-2s-1})}+\frac{(1-q^{-1})^2q^{-3s-2}}{(1-q^{-2s})(1-q^{-2s-1})}\right)v_0\\
&=\frac{(1-q^{-1})^2q^{-2s-1}}{(1-q^{-s})(1+q^{-s})(1-q^{-2s-1})}v_0.
\end{aligned}
$$

From above discussion, we can summary the results as in following proposition.

Proposition 5.20. *With $\mathcal{I}^s_{\mathfrak{p}}, \mathcal{I}^s_{F\backslash\mathfrak{p}}$ as defined in (5.47), then we have*

1. $\mathcal{I}^s_{\mathfrak{p}} = \dfrac{q^{-1}(1-q^{-2-2s})}{1-q^{-2s-1}}v_0;$
2. $\mathcal{I}^s_{F\backslash\mathfrak{p}} = \mathcal{J}^s_{\mathfrak{o}} + \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times} + \mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}$, *where* $\mathcal{J}^s_{\mathfrak{o}}, \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}$ *and* $\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}$ *as in* (5.48).

   (a)

$$
\mathcal{J}^s_{\mathfrak{o}} = \sum_{i=1}^{\infty}\sum_{j=0}^{i-1}(-1)^{i+j}(1-q^{-1})q^{j(1-s)+i(-1-s)}\pi\left(\left[\begin{smallmatrix}\varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\left[\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}\right]\right)
\int\limits_{\mathfrak{o}^\times}\pi\left(\left[\begin{smallmatrix}1 & \\ 2\varpi^{i-j}\mu & 1\end{smallmatrix}\right]\right)v_0\,d^*\mu
$$

   (b) $\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times} = \mathcal{J}^{s,i>j}_{\varpi^{-1}\mathfrak{o}^\times} + \mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times}$

      i.

$$
\mathcal{J}^{s,i>j}_{\varpi^{-1}\mathfrak{o}^\times} = -q^{-s}(1-q^{-1})^2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}q^{j(1-s)+i(-1-s)}(-1)^{i+j}
\int\limits_{\mathfrak{o}^\times}\pi\left(\left[\begin{smallmatrix}\varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & -\varpi^{-1}\kappa^{-1}\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right)v_0\,d^*\kappa
$$

      ii.

$$
\mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times} = -\frac{q^{-s}(1-q^{-1})}{1-q^{-2s}}\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times}\pi\left(\left[\begin{smallmatrix}1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right)v_0\,d^*\mu\,d^*\kappa
$$

(c) $\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}} = \dfrac{(1-q^{-1})^2 q^{-2s-1}}{(1-q^{-s})(1+q^{-s})(1-q^{-2s-1})} v_0.$

5.5 Whittaker model of intertwining operator evaluated at 1_2

From Proposition 5.20, we can see that the result for $\mathcal{I}^s_{F\backslash\mathfrak{p}}$ is not so nice and cannot be simplified completely. Next we shall consider the Whittaker model of the newform v_0. In particular, we define

$$\pi' := \alpha\pi, \quad \alpha \text{ unramified character and } \alpha^2 = \xi. \tag{5.58}$$

Then it follows that π' has trivial central character. Furthermore, it follows from Theorem 2.3.2 i) of [21] that $a(\xi) = 1$. Thus by Theorem 3.2.2 of [21], we have

$$\pi\left(\left[\begin{smallmatrix} & 1 \\ \varpi^2 & \end{smallmatrix}\right]\right) v_0 = \varepsilon(\frac{1}{2}, \pi) v_0, \tag{5.59}$$

which follows from straightforward computations with $\pi' = \alpha\pi$. Here, $\varepsilon(\frac{1}{2}, \pi)$ is the ε-factor attached to π. The additive character ψ has the conductor $\mathfrak{o}$, i.e., $\psi\mid_{\mathfrak{o}} \equiv 1$ but $\psi\mid_{\mathfrak{p}^{-1}} \not\equiv 1$. By Corollary 3.5 of [6], we may assume that $v_0(1_2) = 1$. Moreover, we will have the following fact:

Fact 5.21. There is a isomorphism

$$W(\pi,\psi) \xrightarrow{\sim} W(\alpha\pi,\psi)$$
$$W(g) \longmapsto (g \mapsto \alpha(\det(g)))W(g)$$

This is, in fact, a statement at the end of [10, page 36]. That is to say,

$$v_\alpha(g) = \alpha(\det(g)) \cdot v(g) \tag{5.60}$$

Thus, by above discussion, we are going to evaluate $\mathcal{I}^s_{F\backslash\mathfrak{p}}$ at 1_2. And we recall that

$$\mathcal{I}^s_{F\backslash\mathfrak{p}} = \mathcal{J}^s_{\mathfrak{o}} + \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times} + \mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}, \tag{5.61}$$

see (5.48) for the precise details. Let i, j be two non-negative integers. With π, v_0 as in Proposition 5.12 and $s \in \mathbb{C}$, we set

$$\mathcal{K}_{i,j} = \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} \pi(\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\varpi^{(i-j)}\mu)^2 \\ & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}\right]) v_0 \, d^*\mu \, d^*\kappa. \tag{5.62}$$

By Proposition 5.20, we can easily have

$$\mathcal{J}^s_{\mathfrak{o}} = \sum_{i=1}^{\infty}\sum_{j=0}^{i-1}(1-q^{-1})(-1)^{i+j}q^{j(1-s)+i(-1-s)} \tag{5.63}$$

$$\int_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{smallmatrix}\right]\left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ 2\varpi^{i-j}\mu & 1 \end{smallmatrix}\right]\right) v_0 \, d^*\mu, \tag{5.64}$$

$$\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times} = -\, q^{-s}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-q^{-1})(-1)^{i+j}q^{j(1-s)+i(-1-s)}\pi\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{smallmatrix}\right]\right)\mathcal{K}_{i,j}, \tag{5.65}$$

$$\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}} = \frac{(1-q^{-1})^2 q^{-2s-1}}{(1-q^{-s})(1+q^{-s})(1-q^{-2s-1})} v_0. \tag{5.66}$$

Recall that for the integral $\mathcal{K}_{i,j}$ as defined in (5.62), we have already showed that

$$\mathcal{K}_{i,j} = 0 \text{ if } i < j.$$

Now we evaluate each of $\mathcal{J}^s_{\mathfrak{o}}, \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}$ and $\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}$ at 1_2, respectively.

5.5.1 $\mathcal{J}_{\mathfrak{o}}^s(1_2)$

Proposition 5.22.

$$\mathcal{J}_{\mathfrak{o}}^s(1_2) = \frac{(1-q^{-1})q^{-s-1}}{1-q^{-2s}}\varepsilon(\frac{1}{2},\pi)\begin{cases} -q^{-1} & \text{if } q \text{ is odd,} \\ 1-q^{-1} & \text{if } q \text{ is even.} \end{cases}. \tag{5.67}$$

Proof. i) Let

$$\left(\mathcal{J}_{\mathfrak{o}}^s(i,j)\right)(1_2) := \int_{\mathfrak{o}^\times} \pi\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \\ 2\varpi^{i-j}\mu & 1\end{bmatrix}\right) v_0(1_2)\, d^*\mu \tag{5.68}$$

Then we have

$$\begin{aligned} \left(\mathcal{J}_{\mathfrak{o}}^s(i,j)\right)(1_2) &= \int_{\mathfrak{o}^\times} v_0\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \\ 2\varpi^{i-j}\mu & 1\end{bmatrix}\right)\, d^*\mu \\ &= \int_{\mathfrak{o}^\times} v_0\left(\begin{bmatrix} 1 & 2\varpi^{-(i-j)}\mu \\ & 1\end{bmatrix}\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right)\, d^*\mu \\ &= \int_{\mathfrak{o}^\times} \psi(2\varpi^{-(i-j)}\mu)\, d^*\mu \cdot v_0\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) \end{aligned}$$

For the coefficient part of above, we have the following claim.

Claim 5.23.

$$\int_{\mathfrak{o}^\times} \psi(2\varpi^{-(i-j)}\mu)\, d^*\mu = \begin{cases} 0 & \text{if } q \text{ is odd and } i-j \geq 2. \\ -q^{-1} & \text{if } q \text{ is odd and } i-j = 1. \\ 0 & \text{if } q \text{ is even and } i-j \geq 2+\nu(2). \\ -q^{-1} & \text{if } q \text{ is even and } i-j = 1+\nu(2). \\ 1-q^{-1} & \text{if } q \text{ is even and } i-j \leq \nu(2). \end{cases} \tag{5.69}$$

Proof of Claim 5.23. First, we have

$$\int_{\mathfrak{o}^\times} \psi(2\varpi^{-(i-j)}\mu)\, d^*\mu = \left\{ \int_{\mathfrak{o}} - \int_{\mathfrak{p}} \right\} \psi(2\varpi^{-(i-j)}\mu)\, d^*\mu.$$

Since ψ has the conductor $\mathfrak{o}$, then (5.69) easily follows from straightforward calculations. $\square$

In fact, we can investigate $v_0\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right)$, and because of the fact that v_0 is a newform, we have v_0 is invariant under $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o}^\times \end{bmatrix}$. If $b \in \mathfrak{p}^2$, then we have

$$\begin{aligned} v_0\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) =& v_0\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \\ b & 1 \end{bmatrix}\right) \\ =& v_0\left(\begin{bmatrix} 1 & b\varpi^{2(j-i)} \\ & 1 \end{bmatrix}\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) \\ =& \psi(b\varpi^{2(j-i)}) v_0\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) \end{aligned}$$

This means that

1. If $i - j \geq 2$, we have $v_0\left(\begin{bmatrix} \varpi^{j-i} & \\ & \varpi^{i-j} \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) = 0$;
2. If $i - j = 1, i.e., j = i - 1$, the coefficient part will be
    (a) If q is odd, the coefficient $\mathcal{C}_q$ will be $-q^{-1}$;
    (b) If q is even, the coefficient $\mathcal{C}_q$ will be $1 - q^{-1}$.

Consequently, only the case of $j = i - 1$ survives, so that

$$\begin{aligned} \mathcal{J}_{\mathfrak{o}}^s(1_2) =& -(1-q^{-1})q^{s-1}\sum_{i=1}^{\infty}(q^{-2s})^i \mathcal{C}_q v_0\left(\begin{bmatrix} \varpi^{-1} & \\ & \varpi \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) \\ =& -(1-q^{-1})q^{s-1}\sum_{i=1}^{\infty}(q^{-2s})^i \mathcal{C}_q \pi\left(\begin{bmatrix} \varpi^{-1} & \\ & \varpi \end{bmatrix}\right)\pi\left(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right) v_0(1_2) \end{aligned}$$

$$= -(1-q^{-1})q^{s-1}\sum_{i=1}^{\infty}(q^{-2s})^i\mathcal{C}_q\varepsilon(\frac{1}{2},\pi)\pi\left(\begin{bmatrix}\varpi^{-1} & \\ & \varpi\end{bmatrix}\right)\pi\left(\begin{bmatrix}\varpi^{2} & \\ & 1\end{bmatrix}\right)v_0\,(1_2)$$
$$= -(1-q^{-1})q^{s-1}\sum_{i=1}^{\infty}(q^{-2s})^i\mathcal{C}_q\varepsilon(\frac{1}{2},\pi)\omega_\pi(\varpi)v_0\,(1_2)$$
$$=\frac{(1-q^{-1})q^{-s-1}}{1-q^{-2s}}\mathcal{C}_q\varepsilon(\frac{1}{2},\pi).$$

□

5.5.2 $\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}(1_2)$

Proposition 5.24.

$$\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = \varepsilon(1/2,\pi)\frac{q^{-s}(1-q^{-1})}{1-q^{-2s}}\begin{cases} q^{-2}-\epsilon q^{-1} & \textit{if } q \textit{ is odd,} \\ q^{-2} & \textit{if } q \textit{ is even,}\end{cases} \tag{5.70}$$

where $\epsilon = -(\varpi,-1)\varepsilon(1/2,\pi)\in\{\pm1\}$. *Here,* $(\varpi,\cdot)$ *is the Hilbert symbol.*

Proof. ii) Since we already know that

$$\mathcal{K}_{i,j} = 0 \text{ if } i<j, \text{ i.e., } \mathcal{J}^{s,i<j}_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = 0$$

It follows that we only need to calculate $\mathcal{J}^{s,i\geq j}_{\varpi^{-1}\mathfrak{o}^\times}(1_2)$.

Assume that $i>j$, it follows from Proposition 5.20 that

$$\mathcal{J}^{s,i>j}_{\varpi^{-1}\mathfrak{o}^\times} = -q^{-s}(1-q^{-1})^2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}q^{j(1-s)+i(-1-s)}(-1)^{i+j}$$
$$\int_{\mathfrak{o}^\times}\pi\left(\begin{bmatrix}\varpi^{j-i} & \\ & \varpi^{i-j}\end{bmatrix}\begin{bmatrix}1 & -\varpi^{-1}\kappa^{-1} \\ & 1\end{bmatrix}\begin{bmatrix}1 & \\ \kappa\varpi & 1\end{bmatrix}\right)v_0\,d^*\kappa$$

Let $y \in \mathfrak{p}$. Evaluating the integral at 1_2, we have

$$\begin{aligned}
&\int\limits_{\mathfrak{o}^\times} v_0\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1} \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right)\, d^*\kappa \\
=&\int\limits_{\mathfrak{o}^\times} \psi(-\varpi^{-2(i-j)-1}\kappa^{-1}) v_0\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right)\, d^*\kappa \\
=&\int\limits_{\mathfrak{o}^\times} \psi(-\varpi^{-2(i-j)-1}\kappa^{-1}(1+y)) v_0\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right)\, d^*\kappa \\
=&\int\limits_{\mathfrak{o}^\times} \psi(-\varpi^{-2(i-j)-1}\kappa^{-1}y)\varphi(-\varpi^{-2(i-j)-1}\kappa^{-1}) v_0\left(\left[\begin{smallmatrix} \varpi^{j-i} & \\ & \varpi^{i-j}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right)\, d^*\kappa.
\end{aligned}$$

We can find some $y \in \mathfrak{p}$ such that $\psi(-\varpi^{-2(i-j)-1}\kappa^{-1}y)$ equals a non-trivial constant since ψ has the conductor $\mathfrak{o}$. This implies that the above integral is zero. Hence $\mathcal{J}^{s,i>j}_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = 0$.

Now assume that $i = j$, it easily follows from the above discussion that

$$\mathcal{J}^{s}_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = \mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times}(1_2). \tag{5.71}$$

Moreover, it follows from (5.65) and (5.62) that

$$\mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times} = -\frac{q^{-s}(1-q^{-1})}{1-q^{-2s}} \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} \pi\left(\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2 \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right) v_0\, d^*\mu\, d^*\kappa. \tag{5.72}$$

Then after evaluating at 1_2 we have

$$\mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = -\frac{q^{-s}(1-q^{-1})}{1-q^{-2s}} \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} v_0\left(\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2 \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1\end{smallmatrix}\right]\right)\, d^*\mu\, d^*\kappa. \tag{5.73}$$

Let

$$I := \int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2 \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\, d\mu\, d\kappa. \tag{5.74}$$

Then we have the following claim for I.

Claim 5.25.

$$I = -\varepsilon(1/2,\pi)\begin{cases} q^{-2} - \epsilon q^{-1} & \text{if } q \text{ is odd,} \\ q^{-2} & \text{if } q \text{ is even,} \end{cases} \tag{5.75}$$

where $\epsilon = -(\varpi,-1)\varepsilon(1/2,\pi) \in \{\pm 1\}$. Here, $(\varpi,\cdot)$ is the Hilbert symbol. Thus the assertion easily follows from the above claim. □

In order to proof Claim 5.25, we need some lemmas on Whittaker functions for GL(2). More precisely, we let (τ, V) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$. Recall the standard local zeta integrals for GL(2), given by

$$Z(s,W) = \int\limits_{F^\times} W(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right])|x|^{s-1/2}\,d^\times x, \tag{5.76}$$

and more generally, for a character β of $F^\times$,

$$Z(s,W,\beta) = \int\limits_{F^\times} W(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right])\beta(x)|x|^{s-1/2}\,d^\times x. \tag{5.77}$$

Here, W is any element of the ψ-Whittaker model of τ. The integrals (5.76) are known to converge for $\mathrm{Re}(s)$ large enough, have meromorphic continuation to the entire s–plane, and satisfy the functional equation

$$Z(1-s,\tau(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])W,\omega_\tau^{-1}) = \frac{\varepsilon(s,\tau,\psi)L(1-s,\omega_\tau^{-1}\tau)}{L(s,\tau)}Z(s,W). \tag{5.78}$$

More generally, for any character β of $F^\times$, we have

$$Z(1-s,\tau([{}_{-1}{}^{1}])W,\beta^{-1}\omega_\tau^{-1}) = \frac{\varepsilon(s,\beta\tau,\psi)L(1-s,\beta^{-1}\omega_\tau^{-1}\tau)}{L(s,\beta\tau)}Z(s,W,\beta). \quad (5.79)$$

(See Theorem 4.7.5 of [5].) Assume that τ satisfies $L(s,\tau)=1$. Let $n=a(\tau)$. Let v_0 be the newform in the Whittaker model, normalized so that $v_0(1)=1$. Then it is easy to see that

$$v_0([{}^{\varpi^m}{}_{1}]) = \begin{cases} 1 & \text{if } m=0, \\ 0 & \text{if } m\neq 0. \end{cases} \quad (5.80)$$

$$v_0([{}^{\varpi^m}{}_{1}][{}_{-1}{}^{1}]) = \begin{cases} \varepsilon(1/2,\hat{\tau},\psi) & \text{if } m=-n, \\ 0 & \text{if } m\neq -n. \end{cases} \quad (5.81)$$

We note that $\varepsilon(1/2,\hat{\tau},\psi)=\varepsilon(1/2,\tau,\psi)\omega_\tau(\varpi)^{-n}$.

Lemma 5.26. *Assume that $L(s,\tau)=1$ and that ω_τ is unramified. And we also assume that $L(1-s,\omega_\tau^{-1}\tau)=1$. Let m be any integer, and k be a positive integer. Then*

$$\int\limits_{\mathfrak{o}^\times} v_0([{}^{x\varpi^m}{}_{1}]\left[\begin{smallmatrix}1 & \\ \varpi^k & 1\end{smallmatrix}\right])\,d^\times x = \begin{cases} 1-q^{-1} & \textit{if } k\geq n \textit{ and } m=0, \\ -q^{-1} & \textit{if } k=n-1 \textit{ and } m=0, \\ 0 & \textit{otherwise.} \end{cases} \quad (5.82)$$

Proof. We apply (5.78) to $W=\tau(\left[\begin{smallmatrix}1 & \\ \varpi^k & 1\end{smallmatrix}\right])v_0$. The left hand side equals

$$Z(1-s,\tau([{}_{-1}{}^{1}])W,\omega_\tau^{-1}) = \int\limits_{F^\times} v_0([{}^{x}{}_{1}][{}_{-1}{}^{1}]\left[\begin{smallmatrix}1 & \\ \varpi^k & 1\end{smallmatrix}\right])\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\,d^\times x$$

$$= \int\limits_{F^\times} v_0(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -\varpi^k \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\, d^\times x$$

$$= \int\limits_{F^\times} \psi(-x\varpi^k) v_0(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\, d^\times x$$

$$\overset{(5.81)}{=} \int\limits_{\varpi^{-n}\mathfrak{o}^\times} \psi(-x\varpi^k)\varepsilon(1/2,\hat{\tau},\psi)\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\, d^\times x$$

$$= q^{n(\frac{1}{2}-s)}\omega_\tau^{-1}(\varpi^{-n})\varepsilon(1/2,\hat{\tau},\psi)\int\limits_{\mathfrak{o}^\times} \psi(-x\varpi^{k-n})\, d^\times x$$

$$= q^{n(\frac{1}{2}-s)}\varepsilon(1/2,\tau,\psi)\int\limits_{\mathfrak{o}^\times} \psi(-x\varpi^{k-n})\, d^\times x$$

$$= \begin{cases} (1-q^{-1})q^{n(\frac{1}{2}-s)}\varepsilon(1/2,\tau,\psi) & \text{if } k-n \geq 0, \\ -q^{-1}q^{n(\frac{1}{2}-s)}\varepsilon(1/2,\tau,\psi) & \text{if } k-n = -1, \\ 0 & \text{if } k-n \leq -2. \end{cases}$$

The right hand side equals

$$\begin{aligned} &\varepsilon(s,\tau,\psi)Z(s,W) \\ &\quad = \varepsilon(1/2,\tau,\psi)q^{-n(s-1/2)}\int\limits_{F^\times} v_0(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1 \end{smallmatrix}\right])|x|^{s-\frac{1}{2}}\, d^\times x \\ &\quad = \varepsilon(1/2,\tau,\psi)q^{-n(s-1/2)}\sum_{m\in\mathbb{Z}}\int\limits_{\varpi^m\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1 \end{smallmatrix}\right])|x|^{s-\frac{1}{2}}\, d^\times x \\ &\quad = \varepsilon(1/2,\tau,\psi)\sum_{m\in\mathbb{Z}} q^{-(m+n)(s-\frac{1}{2})}\int\limits_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix} x\varpi^m & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1 \end{smallmatrix}\right])\, d^\times x. \end{aligned}$$

The assertion follows by comparing powers of q^{-s} on both sides. □

We will need the following formula for ε-factors for GL(1). For a ramified character χ of $F^\times$, a character ψ of F with conductor $\mathfrak{o}$, and an element $r \in F^\times$, we have

$$\int_{\mathfrak{o}^\times} \chi^{-1}(x)\psi^r(x)\,dx = \begin{cases} 0 & \text{if } v(r) \neq -a(\chi), \\ q^{-a(\chi)/2}\chi(r)\varepsilon(1/2,\chi,\psi) & \text{if } v(r) = -a(\chi). \end{cases} \tag{5.83}$$

Lemma 5.27. *Assume that τ is supercuspidal with conductor $n = a(\tau)$. Let m be any integer, and k be a positive integer. Let β be a quadratic character of $F^\times$. If $\beta\omega_\tau$ is ramified, then*

$$\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix} x\varpi^m & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1\end{smallmatrix}\right])\beta(x)\,d^\times x \tag{5.84}$$

$$= \begin{cases} 0 & \textit{if } m \neq n-a(\beta\tau) \\ & \quad \textit{or } k \neq n-a(\beta\omega_\tau), \\ \varepsilon\, q^{-a(\beta\omega_\tau)/2}\beta(-\varpi^{a(\beta\tau)-a(\beta\omega_\tau)})\omega_\tau(-\varpi^{n-a(\beta\omega_\tau)}) & \textit{if } m = n-a(\beta\tau) \\ & \quad \textit{and } k = n-a(\beta\omega_\tau), \end{cases}$$

where

$$\varepsilon = \frac{\varepsilon(1/2,\hat{\tau},\psi)\varepsilon(1/2,\beta\omega_\tau,\psi)}{\varepsilon(1/2,\beta\tau,\psi)}. \tag{5.85}$$

Proof. We apply (5.79) to $W = \tau(\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1\end{smallmatrix}\right])v_0$. The left hand side equals

$$Z(1-s,\tau(\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])W,\beta\omega_\tau^{-1}) = \int_{F^\times} v_0(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1\end{smallmatrix}\right])\beta(x)\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\,d^\times x$$

$$= \int_{F^\times} v_0(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -\varpi^k \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\beta(x)\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\,d^\times x$$

$$= \int_{F^\times} \psi(-x\varpi^k)v_0(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right])\beta(x)\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\,d^\times x$$

$$\overset{(5.81)}{=} \int_{\varpi^{-n}\mathfrak{o}^\times} \psi(-x\varpi^k)\varepsilon(1/2,\hat{\tau},\psi)\beta(x)\omega_\tau^{-1}(x)|x|^{\frac{1}{2}-s}\,d^\times x$$

$$= q^{n(\frac{1}{2}-s)}\beta(\varpi^{-n})\omega_\tau^{-1}(\varpi^{-n})\varepsilon(1/2,\hat{\tau},\psi)\int\limits_{\mathfrak{o}^\times}\psi(-x\varpi^{k-n})\beta(x)\omega_\tau^{-1}(x)\,d^\times x$$

$$\overset{(5.83)}{=}\begin{cases} 0 & \text{if } k\neq n-a(\beta\omega_\tau), \\ q^{\frac{k}{2}-ns}(\omega_\tau\beta)(-\varpi^k)\varepsilon(1/2,\hat{\tau},\psi)\varepsilon(1/2,\beta\omega_\tau,\psi) & \text{if } k=n-a(\beta\omega_\tau). \end{cases}$$

The right hand side equals

$$\begin{aligned}
&\varepsilon(s,\beta\tau,\psi)Z(s,W,\beta) \\
&\quad= \varepsilon(1/2,\beta\tau,\psi)q^{-a(\beta\tau)(s-1/2)}\int\limits_{F^\times} v_0(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1\end{smallmatrix}\right])\beta(x)|x|^{s-\frac{1}{2}}\,d^\times x \\
&\quad= \varepsilon(1/2,\beta\tau,\psi)q^{-a(\beta\tau)(s-1/2)}\sum_{m\in\mathbb{Z}}\int\limits_{\varpi^m\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1\end{smallmatrix}\right])\beta(x)|x|^{s-\frac{1}{2}}\,d^\times x \\
&\quad= \varepsilon(1/2,\beta\tau,\psi)\sum_{m\in\mathbb{Z}} q^{-(m+a(\beta\tau))(s-\frac{1}{2})}\beta(\varpi^m)\int\limits_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix} x\varpi^m & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \varpi^k & 1\end{smallmatrix}\right])\beta(x)\,d^\times x.
\end{aligned}$$

The assertion follows by comparing powers of q^{-s} on both sides. □

The following lemma in odd residual characteristic is very helpful for calculating our integral I as defined in (5.74).

Lemma 5.28. *Assume that F has odd residual characteristic. There exists a $\delta\in\mathbb{C}$ with $|\delta|=1$ such that*

$$\sum_{b\in\mathfrak{o}/\mathfrak{p}}\psi(xb^2\varpi^{-1}) = \delta(\varpi,x)q^{1/2} \tag{5.86}$$

for all $x\in\mathfrak{o}^\times$. Here, $(\varpi,\cdot)$ is the Hilbert symbol. The constant is given by

$$\delta = (\varpi,\varpi)\varepsilon(1/2,(\varpi,\cdot),\psi). \tag{5.87}$$

Proof. We compute the square of the absolute value:

$$\begin{aligned}
\Big|\sum_{b\in\mathfrak{o}/\mathfrak{p}}\psi(xb^2\varpi^{-1})\Big|^2 &= \sum_{a,b\in\mathfrak{o}/\mathfrak{p}}\psi(x(a^2-b^2)\varpi^{-1}) \\
&= \sum_{a,b\in\mathfrak{o}/\mathfrak{p}}\psi(x(a-b)(a+b)\varpi^{-1}) \\
&= \sum_{c,d\in\mathfrak{o}/\mathfrak{p}}\psi(xcd\varpi^{-1}) \\
&= \sum_{c\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{d\in\mathfrak{o}/\mathfrak{p}}\psi(xcd\varpi^{-1}) + \sum_{d\in\mathfrak{o}/\mathfrak{p}}\psi(0) \\
&= \sum_{c\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{d\in\mathfrak{o}/\mathfrak{p}}\psi(xd\varpi^{-1}) + q \\
&= q.
\end{aligned}$$

Hence, $f(x) := \sum_{b\in\mathfrak{o}/\mathfrak{p}}\psi(xb^2\varpi^{-1})$ has absolute value $q^{1/2}$. Let $u\in\mathfrak{o}^\times$ be a non-square. Then, for $x\in\mathfrak{o}^\times$,

$$\begin{aligned}
0 &= \sum_{d\in\mathfrak{o}/\mathfrak{p}}\psi(xd\varpi^{-1}) \\
&= 1 + \sum_{d\in(\mathfrak{o}/\mathfrak{p})^{\times 2}}\psi(xd\varpi^{-1}) + \sum_{d\in(\mathfrak{o}/\mathfrak{p})^{\times 2}}\psi(xud\varpi^{-1}) \\
&= 1 + \frac{1}{2}\Big(\sum_{d\in(\mathfrak{o}/\mathfrak{p})^\times}\psi(xd^2\varpi^{-1}) + \sum_{d\in(\mathfrak{o}/\mathfrak{p})^\times}\psi(xud^2\varpi^{-1})\Big) \\
&= 1 + \frac{1}{2}((f(x)-1) + (f(ux)-1)) \\
&= \frac{1}{2}(f(x)+f(ux)).
\end{aligned}$$

Hence $f(u) = -f(1)$. Since also $f(xy^2) = f(x)$ for all $x,y\in\mathfrak{o}^\times$, this proves

(5.86). To prove (5.87), consider (5.83) for $\chi = (\varpi, \cdot)$ and $r = \varpi^{-1}$:

$$\int_{\mathfrak{o}^\times} (\varpi, x)\psi(\varpi^{-1}x)\,dx = q^{-1/2}(\varpi,\varpi)\varepsilon(1/2, (\varpi,\cdot), \psi) \tag{5.88}$$

We calculate the left hand side:

$$\begin{aligned}
&\int_{\mathfrak{o}^\times} (\varpi, x)\psi(\varpi^{-1}x)\,dx = q^{-1} \sum_{b\in(\mathfrak{o}/\mathfrak{p})^\times} (\varpi, b)\psi(\varpi^{-1}b)\\
&= q^{-1}\Big(\frac{1}{2}\sum_{b\in(\mathfrak{o}/\mathfrak{p})^\times} (\varpi, b^2)\psi(\varpi^{-1}b^2) + \frac{1}{2}\sum_{b\in(\mathfrak{o}/\mathfrak{p})^\times} (\varpi, ub^2)\psi(\varpi^{-1}ub^2)\Big)\\
&= q^{-1}\Big(\frac{1}{2}\sum_{b\in(\mathfrak{o}/\mathfrak{p})^\times} \psi(\varpi^{-1}b^2) - \frac{1}{2}\sum_{b\in(\mathfrak{o}/\mathfrak{p})^\times} \psi(\varpi^{-1}ub^2)\Big)\\
&= \frac{1}{2}q^{-1}\Big(\sum_{b\in\mathfrak{o}/\mathfrak{p}} \psi(\varpi^{-1}b^2) - \sum_{b\in\mathfrak{o}/\mathfrak{p}} \psi(\varpi^{-1}ub^2)\Big)\\
&= \frac{1}{2}q^{-1}(f(1) - f(u)) = q^{-1}f(1)\\
&= q^{-1}\delta(\varpi, 1)q^{1/2} = \delta q^{-1/2}.
\end{aligned}$$

Substituting into (5.88), we obtain (5.87). □

Finally, we come back to calculate $\mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times}(1_2)$. Assume that π is supercuspidal and $a(\pi) = 2$. Assume also that the central character of π is ξ, the non-trivial, quadratic, unramified character of $F^\times$. Let us now calculate the integral

$$I := \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}\right])\,d\mu\,d\kappa.$$

First $I = I_1 - I_2$, where

$$I_1 := \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}} v_0(\left[\begin{smallmatrix} 1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2 \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}\right])\,d\mu\,d\kappa$$

and

$$I_2 := \int_{\mathfrak{o}^\times}\int_{\mathfrak{p}} v_0(\left[\begin{smallmatrix}1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\mu\,d\kappa.$$

A translation in κ gives

$$\begin{aligned}
I_2 &= \int_{\mathfrak{o}^\times}\int_{\mathfrak{p}} v_0(\left[\begin{smallmatrix}1 & -\varpi^{-1}\kappa^{-1}\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\mu\,d\kappa = q^{-1}\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix}1 & -\varpi^{-1}\kappa^{-1}\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\kappa\\
&= q^{-1}\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix}0 & -\varpi^{-1}\kappa^{-1}\\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\kappa = q^{-1}\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix}\kappa^{-1} & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & -\varpi^{-1}\\ \varpi & 1\end{smallmatrix}\right])\,d\kappa\\
&= \varepsilon(1/2,\pi)q^{-1}\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix}\kappa & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & -\varpi^{-1}\\ \varpi & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} & 1\\ \varpi^2 & \end{smallmatrix}\right])\,d\kappa\\
&= \varepsilon(1/2,\pi)q^{-1}\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix}\kappa & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}-\varpi & \\ \varpi^2 & \varpi\end{smallmatrix}\right])\,d\kappa\\
&= -\varepsilon(1/2,\pi)q^{-1}\int_{\mathfrak{o}^\times} v_0(\left[\begin{smallmatrix}\kappa & \\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \varpi & 1\end{smallmatrix}\right])\,d\kappa = \varepsilon(1/2,\pi)q^{-2}.
\end{aligned}$$

The last step follows from Lemma 5.26. Next,

$$\begin{aligned}
I_1 &= \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}} v_0(\left[\begin{smallmatrix}1 & -\varpi^{-1}\kappa^{-1}(1-\mu)^2\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\mu\,d\kappa\\
&= \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}} v_0(\left[\begin{smallmatrix}1 & -\varpi^{-1}\kappa^{-1}\mu^2\\ & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\mu\,d\kappa\\
&= \int_{\mathfrak{o}^\times}\int_{\mathfrak{o}} \psi(-\varpi^{-1}\kappa^{-1}\mu^2)v_0(\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\mu\,d\kappa\\
&= q^{-1}\int_{\mathfrak{o}^\times}\sum_{\mu\in\mathfrak{o}/\mathfrak{p}} \psi(-\varpi^{-1}\kappa^{-1}\mu^2)v_0(\left[\begin{smallmatrix}1 & \\ \kappa\varpi & 1\end{smallmatrix}\right])\,d\kappa.
\end{aligned}$$

Assume first that the residual characteristic of F is odd. Then, with δ being the constant from Lemma 5.28, we get

$$I_1 = \delta q^{-1/2} \int_{\mathfrak{o}^\times} (\varpi, -\kappa) v_0([\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}])\, d\kappa = \delta q^{-1/2} (\varpi, -1) \int_{\mathfrak{o}^\times} (\varpi, \kappa) v_0([\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}])\, d\kappa.$$

Now we apply Lemma 5.27 with $\beta = (\varpi, \cdot)$, and get

$$I_1 = \delta q^{-1/2} (\varpi, -1) \varepsilon\, q^{-1/2} (\varpi, -\varpi) \omega_\pi(-\varpi) = -\delta\varepsilon(\varpi, -1) q^{-1}.$$

Note that $a(\beta\omega_\pi) = a(\beta) = 1$ and $a(\beta\pi) = 2$.

Now assume that the residual characteristic of F is even. In this case the map $\mu \mapsto \mu^2$ from $\mathfrak{o}/\mathfrak{p}$ to itself is a bijection. Hence

$$\begin{aligned} I_1 &= q^{-1} \int_{\mathfrak{o}^\times} \sum_{\mu \in \mathfrak{o}/\mathfrak{p}} \psi(-\varpi^{-1}\kappa^{-1}\mu^2) v_0([\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}])\, d\kappa \\ &= q^{-1} \int_{\mathfrak{o}^\times} \sum_{\mu \in \mathfrak{o}/\mathfrak{p}} \psi(-\varpi^{-1}\kappa^{-1}\mu) v_0([\begin{smallmatrix} 1 & \\ \kappa\varpi & 1 \end{smallmatrix}])\, d\kappa = 0. \end{aligned}$$

To summarize, we have

$$I = \begin{cases} \varepsilon' q^{-1} - \varepsilon(1/2, \pi) q^{-2} & \text{if } q \text{ is odd,} \\ -\varepsilon(1/2, \pi) q^{-2} & \text{if } q \text{ is even,} \end{cases}$$

where ε' is a constant of absolute value 1. It follows that

$$\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = \mathcal{J}^{s,i=j}_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = \varepsilon(1/2, \pi) \frac{q^{-s}(1 - q^{-1})}{1 - q^{-2s}} \begin{cases} q^{-2} - \varepsilon'' q^{-1} & \text{if } q \text{ is odd,} \\ q^{-2} & \text{if } q \text{ is even,} \end{cases}$$

where $\varepsilon'' = \varepsilon'/\varepsilon(1/2, \pi)$ is another constant of absolute value 1.

Let us be more precise about the constants (even though this may not be necessary). We have

$$\begin{aligned}
\varepsilon' &= -\delta\varepsilon(\varpi,-1)\\
&\overset{(5.87)}{=} -(\varpi,\varpi)\varepsilon(1/2,(\varpi,\cdot),\psi)\varepsilon(\varpi,-1)\\
&= -\varepsilon(1/2,(\varpi,\cdot),\psi)\varepsilon\\
&\overset{(5.85)}{=} -\varepsilon(1/2,(\varpi,\cdot),\psi)\frac{\varepsilon(1/2,\hat{\pi},\psi)\varepsilon(1/2,\beta\omega_\pi,\psi)}{\varepsilon(1/2,\beta\pi,\psi)}\\
&= -\varepsilon(1/2,(\varpi,\cdot),\psi)\frac{\varepsilon(1/2,\hat{\pi},\psi)\omega_\pi(\varpi)\varepsilon(1/2,\beta,\psi)}{\varepsilon(1/2,\beta\pi,\psi)}\\
&= \varepsilon(1/2,(\varpi,\cdot),\psi)\frac{\varepsilon(1/2,\hat{\pi},\psi)\varepsilon(1/2,\beta,\psi)}{\varepsilon(1/2,\beta\pi,\psi)}\\
&= (\varpi,-1)\frac{\varepsilon(1/2,\hat{\pi},\psi)}{\varepsilon(1/2,\beta\pi,\psi)}\\
&= (\varpi,-1)\frac{\varepsilon(1/2,\pi,\psi)}{\varepsilon(1/2,\beta\pi,\psi)}.
\end{aligned}$$

Hence

$$\varepsilon'' = (\varpi,-1)\frac{1}{\varepsilon(1/2,\beta\pi,\psi)}$$

The formula $\varepsilon(s,\pi,\psi)\varepsilon(1-s,\pi^\vee,\psi) = \omega_\pi(-1)$, applied to $\pi = \beta\pi$, gives

$$\varepsilon(1/2,\beta\pi)^2 = 1.$$

Hence

$$\varepsilon'' = (\varpi,-1)\varepsilon(1/2,\beta\pi,\psi) \in \{\pm 1\}.$$

Furthermore, one can show that $\varepsilon(1/2,\beta\pi,\psi) = -\varepsilon(1/2,\pi,\psi)$. This implies that

$$\varepsilon'' = -(\varpi,-1)\varepsilon(1/2,\pi,\psi) \in \{\pm 1\}. \tag{5.89}$$

This ε'' is exactly ϵ as defined in Proposition 5.24.

5.6 Main Theorem for group IX

Before we formulate the main theorem for group IX, we would like to summarize the calculations we have so far.

Proposition 5.29. *With $\mathcal{I}^s_{\mathfrak{p}}, \mathcal{I}^s_{F\backslash\mathfrak{p}}$ as in Proposition 5.20, recall that*

$$\mathcal{I}^s_{F\backslash\mathfrak{p}} = \mathcal{J}^s_{\mathfrak{o}} + \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times} + \mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}},$$

where $\mathcal{J}^s_{\mathfrak{o}}, \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}$ and $\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}$ are as in Proposition 5.20. Let $\epsilon \in \{\pm 1\}$ as defined in Proposition 5.24. Let $v_0(1_2) = 1$. Then we have

1. $\mathcal{I}^s_{\mathfrak{p}}(1_2) = \dfrac{q^{-1}(1-q^{-2-2s})}{1-q^{-2s-1}}$.

2. $\mathcal{J}^s_{\mathfrak{o}}(1_2) = \varepsilon(\dfrac{1}{2},\pi)\dfrac{q^{-s-1}(1-q^{-1})}{1-q^{-2s}}\begin{cases} -q^{-1} & \text{if } q \text{ is odd,} \\ 1-q^{-1} & \text{if } q \text{ is even.}\end{cases}$

3. $\mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = \varepsilon(\dfrac{1}{2},\pi)\dfrac{q^{-s-1}(1-q^{-1})}{1-q^{-2s}}\begin{cases} q^{-1}-\epsilon & \text{if } q \text{ is odd,} \\ q^{-1} & \text{if } q \text{ is even.}\end{cases}$

4. $\mathcal{J}^s_{F\backslash\mathfrak{p}^{-1}}(1_2) = \dfrac{(1-q^{-1})^2 q^{-2s-1}}{(1-q^{-2s})(1-q^{-2s-1})}$.

Theorem 5.30. *Let (π, V) be the representation of type* IXb *and $a(\xi) = 0$, then* $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = 1$.

Proof. By Proposition 5.11 ii), it suffices to show $\dim V^{\mathrm{Kl}_2(\mathfrak{p}^2)} = 1$. Then by [15, section 2.2], it is reduced to showing that $\lim\limits_{s\to 1}(\mathcal{A}(s)f)(s_1)$ is non-zero. In particular, it is enough to show

$$\lim_{s\to 1}(\mathcal{A}(s)f)(s_1)(1_2) \neq 0. \tag{5.90}$$

This easily follows from Proposition 5.29. More precisely, we have

i)

$$\begin{aligned}
&\mathcal{I}^s_{\mathfrak{p}}(1_2) + \mathcal{J}^s_{F\setminus\mathfrak{p}^{-1}}(1_2)\\
=&\frac{q^{-1}(1-q^{-2-2s})}{1-q^{-2s-1}} + \frac{(1-q^{-1})^2 q^{-2s-1}}{(1-q^{-2s})(1-q^{-2s-1})}\\
=&\frac{q^{-1}(1-q^{-2-2s})(1-q^{-2s}) + (1-q^{-1})^2 q^{-2s-1}}{(1-q^{-2s})(1-q^{-2s-1})}\\
=&\frac{q^{-1}(1-q^{-2-2s}-q^{-2s}+q^{-2-4s}) + (1-2q^{-1}+q^{-2})q^{-2s-1}}{(1-q^{-2s})(1-q^{-2s-1})}\\
=&\frac{q^{-1}+q^{-3-4s}-2q^{-2s-2}}{(1-q^{-2s})(1-q^{-2s-1})}\\
=&\frac{q^{-1}(1-2q^{-2s-1}+q^{-2-4s})}{(1-q^{-2s})(1-q^{-2s-1})}\\
=&\frac{q^{-1}(1-q^{-2s-1})^2}{(1-q^{-2s})(1-q^{-2s-1})}\\
=&\frac{q^{-1}(1-q^{-2s-1})}{1-q^{-2s}}.
\end{aligned}$$

ii)

$$\mathcal{J}^s_{\mathfrak{o}}(1_2) + \mathcal{J}^s_{\varpi^{-1}\mathfrak{o}^\times}(1_2) = \varepsilon(\frac{1}{2},\pi)\frac{q^{-s-1}(1-q^{-1})}{1-q^{-2s}}\begin{cases}-\epsilon & \text{if } q \text{ is odd,}\\ 1 & \text{if } q \text{ is even.}\end{cases}$$

Thus, we have the following two cases.

1. If q is odd, and let $\tilde{\varepsilon} = \epsilon\varepsilon(\frac{1}{2}, \pi) \in \{\pm 1\}$. Then

$$
\begin{aligned}
&\lim_{s\to 1}(\mathcal{A}(s)f)(s_1)(1_2)\\
=&\lim_{s\to 1}\frac{q^{-1}(1-q^{-2s-1})}{1-q^{-2s}} - \epsilon\varepsilon(\frac{1}{2},\pi)\frac{q^{-s-1}(1-q^{-1})}{1-q^{-2s}}\\
=&\lim_{s\to 1}\frac{q^{-1}}{1-q^{-2s}}\left(1-q^{-2s-1}-\tilde{\varepsilon}q^{-s}(1-q^{-1})\right)\\
=&\lim_{s\to 1}\frac{q^{-1}}{1-q^{-2s}}\left(1-q^{-2s-1}-\tilde{\varepsilon}q^{-s}(1-q^{-1})\right)\\
=&\frac{q^{-1}}{1-q^{-2}}\cdot\begin{cases}(1-q^{-1})(1+q^{-2}) & \text{if } \tilde{\varepsilon}=1,\\ (1-q^{-1})(1+q^{-1})^2 & \text{if } \tilde{\varepsilon}=-1.\end{cases}\\
>&0
\end{aligned}
$$

2. If q is even. Then

$$
\begin{aligned}
&\lim_{s\to 1}(\mathcal{A}(s)f)(s_1)(1_2)\\
=&\lim_{s\to 1}\frac{q^{-1}(1-q^{-2s-1})}{1-q^{-2s}} + \varepsilon(\frac{1}{2},\pi)\frac{q^{-s-1}(1-q^{-1})}{1-q^{-2s}}\\
=&\lim_{s\to 1}\frac{q^{-1}}{1-q^{-2s}}\left(1-q^{-2s-1}+\varepsilon(\frac{1}{2},\pi)q^{-s}(1-q^{-1})\right)\\
=&\frac{q^{-1}}{1-q^{-2}}\cdot\begin{cases}(1-q^{-1})(1+q^{-1})^2 & \text{if } \varepsilon(\frac{1}{2},\pi)=1,\\ (1-q^{-1})(1+q^{-2}) & \text{if } \varepsilon(\frac{1}{2},\pi)=-1.\end{cases}\\
>&0
\end{aligned}
$$

In particular, we have

$$\lim_{s\to 1}(\mathcal{A}(s)f)(s_1)(1_2) > 0. \tag{5.91}$$

□

Chapter 6

Non Iwahori-spherical: Siegel-induced representations and supercuspidals

In this chapter, we obtain the desired dimensional data for the Siegel-induced representations and supercuspidal representations which are non Iwahori-spherical.

6.1 Group X

Recall that

$$A = \begin{bmatrix} 1 & & & \\ u & 1 & & \\ v & & 1 & \\ w & v & -u & 1 \end{bmatrix}, \quad M = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \frac{\Delta}{t} \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \tag{6.1}$$

where $\Delta = ad - bc, u, v, w \in \mathfrak{p}^2, x, y, z \in \mathfrak{o}, t \in \mathfrak{o}^\times, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o})$. Consider the full Siegel-induced representation $\pi \rtimes \sigma$, where π is a supercuspidal representation of $\mathrm{GL}(2, F)$ and σ is a character of $F^\times$. The standard space V of this representation consists of smooth functions $f\colon P \to V_\pi$ with the transformation property

$$f(\begin{bmatrix} \tilde{A} & * \\ & \lambda\tilde{A}' \end{bmatrix} h) = |\det(\tilde{A})\lambda^{-1}|^{\frac{3}{2}}\sigma(\lambda)\pi(\tilde{A})f(h), \quad \tilde{A} \in \mathrm{GL}(2,F), \lambda \in F^\times. \tag{6.2}$$

Let $\tilde{\pi}(\tilde{A}) := |\det(\tilde{A})\lambda^{-1}|^{\frac{3}{2}}\sigma(\lambda)\pi(\tilde{A})$, and we take

$$\tilde{g} = \begin{bmatrix} \tilde{A} & * \\ & \lambda\tilde{A}' \end{bmatrix} \in P(F), h \in \mathrm{GSp}(4,F); \quad \tilde{A}' = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right] {}^t\tilde{A}^{-1} \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]. \tag{6.3}$$

Suppose $f \in V^{\mathrm{Kl}(\mathfrak{p}^2)}$, then f is determined on the set of representatives for the double cosets $P(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^2)$ as in Proposition 3.6 ii). That is to say, f is determined by $f(S_1), f(S_2), f(S_3), f(S_4)$. Consider $P(F)S_j\mathrm{Kl}(\mathfrak{p}^2), j = \{1,\dots,4\}$ and assume $f(S_j) \neq 0$. Then, for $\tilde{g} \in P(F), h \in \mathrm{Kl}(\mathfrak{p}^2)$, we need

$$f(\tilde{g}S_jh) = \tilde{\pi}(\tilde{g})f(S_j), \quad j = \{1,\dots,4\}. \tag{6.4}$$

By a similar discussion as in Section 4.2, to check the well-definedness of (6.4), we need the following for the representation $\tilde{\pi}$ as defined in (6.2).

$$\tilde{\pi} \text{ must be trivial on } P(F) \cap S_j\mathrm{Kl}(\mathfrak{p}^2)S_j^{-1}, \quad j \in \{1,2,3,4\}. \tag{6.5}$$

It follows from Lemma 4.2 that the full Siegel-induced representation $\pi \rtimes \sigma$ has depth zero. Again, it implies that π and σ both are depth zero. Then by the discussion in Section 4.1.2, we have the conductor condition

$$a(\pi) = 2, \quad a(\sigma) \leq 1. \tag{6.6}$$

Furthermore, by the condition of the trivial central character, we have

$$\sigma^2\omega_\pi = 1 \text{ and } a(\omega_\pi) \le 1. \tag{6.7}$$

The second assertion $a(\omega_\pi) \le 1$ can be also seen from Proposition 5.2. Similarly, for each of double coset $P(F)S_j\mathrm{Kl}(\mathfrak{p}^2), j = \{1, \ldots, 4\}$, let $\tilde{v}_j := f(S_j)$. Again, to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vector, i.e., $\tilde{v}_j \neq 0$, we at least have $a(\pi) = 2, a(\sigma) \le 1$.

Again, all the elements of $\mathrm{Kl}(\mathfrak{p}^2)$ can be written as AMY with A, M, Y as in (5.1). That is to say, any element h of $\mathrm{Kl}(\mathfrak{p}^2)$ has the form of

$$h = \begin{bmatrix} 1 & & & \\ u & 1 & & \\ v & & 1 & \\ w & v & -u & 1 \end{bmatrix} \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \frac{\Delta}{t} \end{bmatrix} \begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \tag{6.8}$$

where $\Delta = ad - bc, u, v, w \in \mathfrak{p}^2, x, y, z \in \mathfrak{o}, t \in \mathfrak{o}^\times, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o})$.

i) For $S_1 = \mathbf{I}_4$, to ensure that $S_1\mathrm{Kl}(\mathfrak{p}^2)S_1^{-1} \in P(F)$, we need

$$v = w = c = 0.$$

And the matrix will be

$$\begin{bmatrix} t & tx & * & * \\ tu & a+tux & * & * \\ & & d & -dx \\ & & -du & \frac{d(a+tux)}{t} \end{bmatrix}.$$

It follows from (6.5) that

$$\sigma(ad)\pi(\left[\begin{smallmatrix} t & tx \\ tu & a+tux \end{smallmatrix}\right])\tilde{v}_1 = \sigma(ad)\omega_\pi(a)\pi(\begin{bmatrix} \frac{t}{a} & \frac{tx}{a} \\ \frac{tu}{a} & 1+\frac{tux}{a} \end{bmatrix})\tilde{v}_1 = \tilde{v}_1. \tag{6.9}$$

By the trivial central character $\omega_\pi \sigma^2 = 1$, we have

$$\sigma(\frac{d}{a})\pi(\begin{bmatrix} \frac{t}{a} & \frac{tx}{a} \\ \frac{tu}{a} & 1+\frac{tux}{a} \end{bmatrix})\tilde{v}_1 = \tilde{v}_1. \tag{6.10}$$

And we observe that

$$\begin{bmatrix} \frac{t}{a} & \frac{tx}{a} \\ \frac{tu}{a} & 1+\frac{tux}{a} \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix}.$$

It follows from $a(\pi) = 2$ that

$$\sigma(\frac{d}{a}) = 1, \quad \forall a, d \in \mathfrak{o}^\times. \tag{6.11}$$

Thus, in order to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector, the character σ has to be *unramified*. In conclusion, the double coset $P(F)S_1\mathrm{Kl}(\mathfrak{p}^2)$ does support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_1$ with the conductor condition

$$a(\pi) = 2, \quad a(\sigma) = 0. \tag{6.12}$$

In addition, this non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vector $\tilde{v}_1$ is invariant under $\Gamma_0(\mathfrak{p}^2)$.

ii) For $S_2 = s_2 s_1$, to ensure that $S_2\mathrm{Kl}(\mathfrak{p}^2)S_2^{-1} \in P(F)$, we need

$$x = c = z = 0.$$

And the matrix will be

$$\begin{bmatrix} a & ay & * & * \\ av & \frac{a(d+tvy)}{t} & * & * \\ & & t & -ty \\ & & -tv & d+tvy \end{bmatrix}.$$

It follows from (6.5) that

$$\sigma(ad)\pi(\begin{bmatrix} a & ay \\ av & \frac{a(d+tvy)}{t} \end{bmatrix})\tilde{v}_2 = \sigma(ad)\omega_\pi(\frac{ad}{t})\pi(\begin{bmatrix} \frac{t}{d} & \frac{ty}{d} \\ \frac{tv}{d} & 1+\frac{tvy}{d} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \qquad (6.13)$$

Again by the trivial central character condition $\omega_\pi \sigma^2 = 1$, we have

$$\sigma(\frac{t^2}{ad})\pi(\begin{bmatrix} \frac{t}{d} & \frac{ty}{d} \\ \frac{tv}{d} & 1+\frac{tvy}{d} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \qquad (6.14)$$

And we observe that

$$\begin{bmatrix} \frac{t}{d} & \frac{ty}{d} \\ \frac{tv}{d} & 1+\frac{tvy}{d} \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix}.$$

Then we have that

$$\sigma(\frac{t^2}{ad}) = 1, \quad a, d \in \mathfrak{o}^\times. \qquad (6.15)$$

Thus, to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector, the character σ has to be *unramified*. In conclusion, the double cosets $P(F)S_2\mathrm{Kl}(\mathfrak{p}^2)$ does support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_2$ with the conductor condition

$$a(\pi) = 2, \quad a(\sigma) = 0. \qquad (6.16)$$

In addition, this non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vector $\tilde{v}_2$ is invariant under $\Gamma_0(\mathfrak{p}^2)$.

iii) For $S_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi & & 1 & \\ & \varpi & & 1 \end{bmatrix}$, to ensure that $S_3\mathrm{Kl}(\mathfrak{p}^2)S_3^{-1} \in P(F)$, we need

(a) $d = \frac{t(v+\varpi)(1-\varpi y)}{\varpi} \in \mathfrak{o}^\times$.

(b) $b = -\frac{t(\varpi y-1)(uv+2\varpi u+w)}{\varpi(v+\varpi)} \in \mathfrak{o}$.

(c) $c = -\frac{t(v+\varpi)(\varpi xy-2x+\varpi z)}{\varpi y-1} \in \mathfrak{p}$.

And the matrix will be $\left[\begin{smallmatrix} \tilde{A} & * \\ & \lambda\tilde{A}' \end{smallmatrix}\right]$, where

$$\tilde{A} = \left[\begin{smallmatrix} t(1-\varpi y) & t(x-\varpi z) \\ \frac{t(\varpi y-1)(\varpi u+w)}{v+\varpi} & a+tux-\varpi\left(ay+\frac{tx(\varpi y-1)(uv+2\varpi u+w)}{\varpi(v+\varpi)}+tuz\right) \end{smallmatrix}\right]$$

and

$$\lambda\tilde{A}' = \left[\begin{smallmatrix} \frac{t(\varpi+v)}{\varpi} & -\frac{t(v+\varpi)(\varpi z-x)}{\varpi(\varpi y-1)} \\ t(u+\frac{w}{\varpi}) & \frac{a(v+\varpi)(\varpi y-1)+t\left(\varpi(uvxy+uvz-3ux+wxy)+x(-2uv-w)+\varpi^2 u(2xy+z)\right)}{\varpi(\varpi y-1)} \end{smallmatrix}\right].$$

In particular, we denote

$$\tilde{A}_0 := \frac{1}{a(1-\varpi y)} \cdot \tilde{A} \in \left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right]. \tag{6.17}$$

It is not hard to see that

$$\lambda \cdot \frac{t(1-\varpi y)}{\det(\tilde{A})} = \frac{t(\varpi+v)}{\varpi} \Rightarrow \lambda = \frac{\det(\tilde{A})(\varpi+v)}{\varpi(1-\varpi y)}. \tag{6.18}$$

It follows from (6.5) and (6.6) that

$$\sigma(at)\pi(a(1-\varpi y)\cdot\tilde{A}_0)\tilde{v}_3 = \sigma(at)\omega_\pi(a)\pi(\tilde{A}_0)\tilde{v}_3 = \tilde{v}_3. \tag{6.19}$$

By the trivial central character, i.e., $\sigma^2\omega_\pi = 1$, we have

$$\sigma(\frac{t}{a})\pi(\tilde{A}_0)\tilde{v}_3 = \tilde{v}_3, \quad \forall a, t \in \mathfrak{o}^\times. \tag{6.20}$$

This implies that σ is *unramified* and

$$\pi\left(\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right]\right)\tilde{v}_3 = \tilde{v}_3. \tag{6.21}$$

This is impossible since π is a supercuspidal representation of $\mathrm{GL}(2,F)$ or by the fact of $a(\pi)=2$ from (6.6).

iv) For $S_4 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \varpi & & & 1 \end{bmatrix}$, to ensure that $S_4 \mathrm{Kl}(\mathfrak{p}^2) S_4^{-1} \in P(F)$, we need

(a) $x = \frac{cu-av}{t(w+\varpi)} \in \mathfrak{p}$.

(b) $c = \frac{av^2}{uv+w+\varpi} \in \mathfrak{p}^3$.

(c) $d = \frac{t^2(\varpi z-1)(-(uv+w+\varpi))-\varpi atvy}{\varpi a} \in \mathfrak{o}^\times$.

And the matrix will be $\begin{bmatrix} \tilde{A} & * \\ & \lambda \tilde{A}' \end{bmatrix}$, where

$$\tilde{A} = \begin{bmatrix} t(1-\varpi z) & \frac{-av}{\varpi+uv+w} \\ tu - \frac{\varpi\left(a(bv+ty(uv+w+\varpi))+t^2uz(uv+w+\varpi)\right)}{t(uv+w+\varpi)} & \frac{a(w+\varpi)}{uv+w+\varpi} \end{bmatrix}$$

and

$$\lambda \tilde{A}' = \begin{bmatrix} -\frac{t^2(\varpi z-1)(uv+w+\varpi)}{\varpi a} & \frac{tv}{\varpi} \\ bv+t(w+\varpi)y - \frac{tu(-\varpi avy - t(\varpi z-1)(uv+w+\varpi))}{\varpi a} & \frac{t(\varpi+w)}{\varpi} \end{bmatrix}.$$

In particular, we have

$$\det(\tilde{A}) \in at \cdot (1+\mathfrak{p}). \tag{6.22}$$

Again, it is not hard to see that

$$\lambda \cdot \frac{t(1-\varpi z)}{\det(\tilde{A})} = -\frac{t^2(\varpi z-1)(uv+w+\varpi)}{\varpi a} \Rightarrow \lambda = \frac{t\det(\tilde{A})(uv+w+\varpi)}{\varpi a}. \tag{6.23}$$

It follows from (6.5), (6.6) and trivial central character ($\sigma^2\omega_\pi = 1$) that

$$\begin{aligned}\sigma(\lambda)\pi(\tilde{A})\tilde{v}_4 =& \sigma\Big(\frac{t\det(\tilde{A})(uv+w+\varpi)}{\varpi a}\Big)\pi(\tilde{A})\tilde{v}_4 \\ =& \sigma(t^2)\omega_\pi(t)\pi\Big(\begin{bmatrix} 1-\varpi z & \frac{-av}{t(\varpi+uv+w)} \\ u - \frac{\varpi\left(a(bv+ty(uv+w+\varpi))+t^2uz(uv+w+\varpi)\right)}{t^2(uv+w+\varpi)} & \frac{a(w+\varpi)}{t(uv+w+\varpi)} \end{bmatrix}\Big)\tilde{v}_4\end{aligned}$$

$$=\pi(\begin{bmatrix} 1-\varpi z & \frac{-av}{t(\varpi+uv+w)} \\ u-\frac{\varpi\left(a(bv+ty(uv+w+\varpi))+t^2uz(uv+w+\varpi)\right)}{t^2(uv+w+\varpi)} & \frac{a(w+\varpi)}{t(uv+w+\varpi)} \end{bmatrix})\tilde{v}_4=\tilde{v}_4.$$

In particular, there is no restriction for the character σ. Now we take

(a) $u=v=w=z=y=0, t=1$, then we need $\pi(\left[\begin{smallmatrix} 1 & \\ & \mathfrak{o}^\times \end{smallmatrix}\right])\tilde{v}_4=\tilde{v}_4$.

(b) $u=v=w=y=0, t=a$, then we need $\pi(\left[\begin{smallmatrix} 1+\mathfrak{p} & \\ & 1 \end{smallmatrix}\right])\tilde{v}_4=\tilde{v}_4$.

(c) $u=v=w=z=0, a=t=1$, then we need $\pi(\left[\begin{smallmatrix} 1 & \\ \mathfrak{p} & 1 \end{smallmatrix}\right])\tilde{v}_4=\tilde{v}_4$.

(d) $u=w=y=z=b=0, a=t=1$, then we need $\pi(\left[\begin{smallmatrix} 1 & \mathfrak{p} \\ & 1 \end{smallmatrix}\right])\tilde{v}_4=\tilde{v}_4$.

This implies, to support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector, we need

$$\pi(\begin{bmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o}^\times \end{bmatrix})\tilde{v}_4=\tilde{v}_4, \quad \text{and} \quad a(\sigma)\leq 1. \tag{6.24}$$

Moreover, we have $a(\omega_\pi)\leq 1$ by trivial central character. Recall that $a(\pi)=2$, i.e., there is a non-zero newform $v_0\in V^{\Gamma_1(\mathfrak{p}^2)}$. That is to say,

$$\pi(\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix})v_0=v_0. \tag{6.25}$$

It follows from Proposition 5.4 that v_0 is also $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{bmatrix}$-invariant since $a(\omega_\pi)\leq 1$. Now we consider the vector

$$v_1:=\pi(\left[\begin{smallmatrix} 1 & \\ & \varpi^{-1} \end{smallmatrix}\right])v_0. \tag{6.26}$$

And the vector v_1 is invariant under $\left[\begin{smallmatrix} 1 & \mathfrak{p} \\ & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} \mathfrak{o}^\times & \\ & 1+\mathfrak{p} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & \\ \mathfrak{p} & 1 \end{smallmatrix}\right]$. This implies that v_1 is invariant under $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{bmatrix}$. Then we consider the vector

$$v_2:=\pi(\left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right])v_1. \tag{6.27}$$

It is easy to check that v_2 is invariant under $\begin{bmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o}^\times \end{bmatrix}$. And v_1 is non-zero since v_0 is non-zero. It follows that v_2 is non-zero. Hence the desired condition (6.24) is satisfied. In conclusion, the double coset $P(F)S_4\mathrm{Kl}(\mathfrak{p}^2)$ does support a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vector $\tilde{v}_4$ with the conductor condition

$$a(\pi) = 2, \quad a(\sigma) \le 1. \tag{6.28}$$

In addition, this non-zero $\mathrm{Kl}(\mathfrak{p}^2)$ vector $\tilde{v}_4$ is invariant under $\begin{bmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o}^\times \end{bmatrix}$.

In conclusion, we have the following proposition.

Proposition 6.1. *The following table gives the dimensions of spaces of* $\mathrm{M}(\mathfrak{p}^2)$*- and* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for an admissible full Siegel-induced representation* $\pi \rtimes \sigma$ *which is non Iwahori-spherical with trivial central character.*

inducing data		$\dim V^\Gamma$	
$a(\pi)$	$a(\sigma)$	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
2	0	2	3
	1	0	1

Proof. The dimensions of the spaces $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors easily follow from the above discussion. The proof for the dimensional data for $\mathrm{M}(\mathfrak{p}^2)$-invariant vectors is analogous to that for the $\mathrm{Kl}(\mathfrak{p}^2)$ case. In particular, it suffices to check the S_1 and S_2 cases. Note that the only difference from the $\mathrm{Kl}(\mathfrak{p}^2)$ case is that $z \in \mathfrak{p}^{-1}$, see (1.11). Here, we again use the same notation $\tilde{v}_j, j = \{1, 2\}$ with the $\mathrm{Kl}(\mathfrak{p}^2)$ case.

1. For $S_1 = \mathbf{I}_4$, to ensure that $S_1\mathrm{M}(\mathfrak{p}^2)S_1^{-1} \in P(F)$, we need

$$v = w = c = 0.$$

And the matrix will be

$$\begin{bmatrix} t & tx & * & * \\ tu & a+tux & * & * \\ & & d & -dx \\ & & -du & \frac{d(a+tux)}{t} \end{bmatrix}.$$

It follows from (6.5) that

$$\sigma(ad)\pi(\left[\begin{smallmatrix} t & tx \\ tu & a+tux \end{smallmatrix}\right])\tilde{v}_1 = \sigma(ad)\omega_\pi(a)\pi(\begin{bmatrix} \frac{t}{a} & \frac{tx}{a} \\ \frac{tu}{a} & 1+\frac{tux}{a} \end{bmatrix})\tilde{v}_1 = \tilde{v}_1. \tag{6.29}$$

By the trivial central character $\omega_\pi\sigma^2 = 1$, we have

$$\sigma(\frac{d}{a})\pi(\begin{bmatrix} \frac{t}{a} & \frac{tx}{a} \\ \frac{tu}{a} & 1+\frac{tux}{a} \end{bmatrix})\tilde{v}_1 = \tilde{v}_1. \tag{6.30}$$

And we observe that

$$\begin{bmatrix} \frac{t}{a} & \frac{tx}{a} \\ \frac{tu}{a} & 1+\frac{tux}{a} \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix}.$$

It follows from the fact of $a(\pi) = 2$ that

$$\sigma(\frac{d}{a}) = 1, \quad \forall a, d \in \mathfrak{o}^\times. \tag{6.31}$$

Thus, in order to support a non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector, the character σ has to be *unramified*. In conclusion, the double cosets $P(F)S_1\mathrm{M}(\mathfrak{p}^2)$ does support a non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector with the conductor condition

$$a(\pi) = 2, \quad a(\sigma) = 0. \tag{6.32}$$

2. For $S_2 = s_2s_1$, to ensure that $S_2\mathrm{M}(\mathfrak{p}^2)S_2^{-1} \in P(F)$, we need

$$x = c = z = 0.$$

And the matrix will be

$$\begin{bmatrix} a & ay & * & * \\ av & \frac{a(d+tvy)}{t} & * & * \\ & & t & -ty \\ & & -tv & d+tvy \end{bmatrix}.$$

It follows from (6.5) that

$$\sigma(ad)\pi(\begin{bmatrix} a & ay \\ av & \frac{a(d+tvy)}{t} \end{bmatrix})\tilde{v}_2 = \sigma(ad)\omega_\pi(\frac{ad}{t})\pi(\begin{bmatrix} \frac{t}{d} & \frac{ty}{d} \\ \frac{tv}{d} & 1+\frac{tvy}{d} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \tag{6.33}$$

Again by the trivial central character condition $\omega_\pi\sigma^2 = 1$, we have

$$\sigma(\frac{t^2}{ad})\pi(\begin{bmatrix} \frac{t}{d} & \frac{ty}{d} \\ \frac{tv}{d} & 1+\frac{tvy}{d} \end{bmatrix})\tilde{v}_2 = \tilde{v}_2. \tag{6.34}$$

And we observe that

$$\begin{bmatrix} \frac{t}{d} & \frac{ty}{d} \\ \frac{tv}{d} & 1+\frac{tvy}{d} \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{bmatrix}.$$

Then we have that

$$\sigma(\frac{t^2}{ad})\tilde{v}_2 = \tilde{v}_2, \quad a, d \in \mathfrak{o}^\times. \tag{6.35}$$

Thus, to support a non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector, the character σ has to be *unramified*. In conclusion, the double cosets $P(F)S_2\mathrm{M}(\mathfrak{p}^2)$ does support a non-zero $\mathrm{M}(\mathfrak{p}^2)$-invariant vector with the conductor condition

$$a(\pi) = 2, \quad a(\sigma) = 0. \tag{6.36}$$

□

6.2 Group XI and supercuspidals

Let $\mathbb{F}_q$ be the residue field of F. Consider the following subgroups of $\mathrm{GSp}(4, \mathbb{F}_q)$,

$$N := \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}, N_P := \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix}, N_Q := \begin{bmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}, N_0 := \begin{bmatrix} 1 & & * & * \\ & 1 & & * \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{6.37}$$

as well as the group of the diagonal matrices T.

Let (π, V) be an admissible representation of $\mathrm{GSp}(4, F)$. Let $r_K(\pi)$ be the hyperspecial parahoric restriction of π, i.e., $r_K(\pi) = V^{\Gamma(\mathfrak{p})}$ endowed with the action of $K/\Gamma(\mathfrak{p}) \cong \mathrm{GSp}(4, \mathbb{F}_q)$. Recall the groups $\mathrm{Kl}_1(\mathfrak{p}^2)^\omega$ defined in (4.8) and $\mathrm{Kl}_{11}(\mathfrak{p}^2)^\omega$ defined in (4.9). We have

$$\mathrm{Kl}_1(\mathfrak{p}^2)^\omega/\Gamma(\mathfrak{p}) \cong \begin{bmatrix} * & * & & \\ & * & & \\ & * & * & * \\ & & & * \end{bmatrix} \cong \begin{bmatrix} * & & * & * \\ & * & & * \\ & & * & \\ & & & * \end{bmatrix} = TN_0 \tag{6.38}$$

and

$$\mathrm{Kl}_{11}(\mathfrak{p}^2)^\omega/\Gamma(\mathfrak{p}) \cong \begin{bmatrix} * & * & & \\ & * & & \\ * & * & * & * \\ & * & & * \end{bmatrix} \cong \begin{bmatrix} * & * & * & * \\ & * & & * \\ & & * & * \\ & & & * \end{bmatrix} = TN_Q. \tag{6.39}$$

Lemma 6.2. *Let (π, V) be an admissible representation of $\mathrm{GSp}(4, F)$. Let $n_0 = \dim r_K(\pi)^{TN_0}$ and $n_Q = \dim r_K(\pi)^{TN_Q}$. Then*

$$n_0 - n_Q \leq \dim V^{\mathrm{Kl}(\mathfrak{p}^2)} \leq n_0.$$

Proof. By Lemma (4.1) i) and (4.8), we have an injection $V^{\mathrm{Kl}(\mathfrak{p}^2)} \to V^{\mathrm{Kl}_1(\mathfrak{p}^2)^\omega}$. By (6.38), $V^{\mathrm{Kl}_1(\mathfrak{p}^2)^\omega} \cong r_K(\pi)^{TN_0}$. This proves $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} \leq n_0$.

By Lemma 4.1 ii), the kernel of the map $\beta : V^{\mathrm{Kl}_1(\mathfrak{p}^2)} \to V^{\mathrm{Kl}(\mathfrak{p}^2)}$ is contained in $V^{\mathrm{Kl}_{11}(\mathfrak{p}^2)}$. It follows that $\dim(\mathrm{im}(\beta)) = \dim V^{\mathrm{Kl}_1(\mathfrak{p}^2)} - \dim(\ker(\beta))$. Hence

$$\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} \geq \dim V^{\mathrm{Kl}_1(\mathfrak{p}^2)} - \dim(\ker(\beta)) \geq \dim V^{\mathrm{Kl}_1(\mathfrak{p}^2)} - \dim V^{\mathrm{Kl}_{11}(\mathfrak{p}^2)}.$$

We have

$$V^{\mathrm{Kl}_1(\mathfrak{p}^2)} \cong V^{\mathrm{Kl}_1(\mathfrak{p}^2)^\omega} \cong r_K(\pi)^{TN_0}$$

by (6.38) and

$$V^{\mathrm{Kl}_{11}(\mathfrak{p}^2)} \cong V^{\mathrm{Kl}_{11}(\mathfrak{p}^2)^\omega} \cong r_K(\pi)^{TN_Q}$$

by (6.39). This proves $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} \geq n_0 - n_Q$. □

Lemma 6.3. *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$*. Assume that π is either supercuspidal, or of type* X, XIa *or* XIb*. Then*

$$\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = n_0 := \dim r_K(\pi)^{TN_0}.$$

Proof. Assume first that π is of type X, XIa or XIb. Then π is a subquotient of an induced representation of the form $\tau \rtimes \sigma$, where τ is a supercuspidal representation of $\mathrm{GL}(2, F)$, and σ is a character of $F^\times$. By Theorem 2.19 of [19],

$$r_K(\tau \rtimes \sigma) \cong r_{\mathrm{GL}(2,\mathfrak{o})}(\tau) \rtimes r_{\mathrm{GL}(1,\mathfrak{o})}(\sigma). \tag{6.40}$$

Since τ is supercuspidal, $r_{\mathrm{GL}(2,\mathfrak{o})}(\tau)$ is a cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$. It follows that the representation in (6.40) does not contain any non-zero N_Q-fixed vectors. Hence $r_K(\pi)^{N_Q} = 0$. By Lemma 6.2, we get $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = n_0$.

Now assume that π is supercuspidal. If π is not of depth zero, then $V^{\mathrm{Kl}(\mathfrak{p}^2)} = 0$ by Lemma 4.2, and $r_K(\pi) = 0$, so that our assertion holds. Assume that π is of depth zero. Then, by Theorem 2.15 and Lemma 2.18 of [19], $r_K(\pi)$ us a cuspidal representation of $\mathrm{GSp}(4, \mathbb{F}_q)$. Hence again $r_K(\pi)^{N_Q} = 0$, so that $\dim V^{\mathrm{Kl}(\mathfrak{p}^2)} = n_0$ by Lemma 6.2. □

Lemma 6.4. *Let (ρ, U) be an irreducible representation of* $\mathrm{GSp}(4, \mathbb{F}_q)$*.*

i) Assume that ρ is cuspidal. Then

$$\dim U^{TN_0} = \begin{cases} 1 & \textit{if } \rho \textit{ is generic,} \\ 0 & \textit{if } \rho \textit{ is non-generic.} \end{cases} \tag{6.41}$$

and $\dim U^{TN_Q} = 0$.

ii) Assume that $\rho = \tau \rtimes \sigma$, where τ is a cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$, *and σ is a character of $\mathbb{F}_q^\times$. Then*

$$\dim U^{TN_0} = \begin{cases} 3 & \textit{if } a(\sigma) = 0, \\ 1 & \textit{if } a(\sigma) = 1. \end{cases} \tag{6.42}$$

and $\dim U^{TN_Q} = 0$.

iii) Let ρ be as in ii), and suppose that ρ is reducible. Then ρ has exactly two irreducible constituents, a generic constituents (ρ_1, U_1) and a non-generic constituent (ρ_2, N_2). We have

$$\dim U_1^{TN_0} = \begin{cases} 2 & \textit{if } a(\sigma) = 0, \\ 1 & \textit{if } a(\sigma) = 1. \end{cases} \tag{6.43}$$

and

$$\dim U_2^{TN_0} = \begin{cases} 1 & \textit{if } a(\sigma) = 0, \\ 0 & \textit{if } a(\sigma) = 1. \end{cases} \tag{6.44}$$

Proof. i) It is clear that $\dim U^{TN_Q} = 0$, since ρ has no N_Q-invariant vectors. The group $N/N_1 \cong \mathbb{F}_q^2$ (see (6.37)) acts on U^{N_0}. Fix a non-trivial character ψ of $\mathbb{F}_q$.

Then the characters of N are the maps

$$\psi_{c_1,c_2}\left(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix}\right) = \psi(c_1x + c_2y), \tag{6.45}$$

where $c_1, c_2 \in \mathbb{F}_q$. Evidently, ψ_{c_1,c_2} descends to a character of N/N_0. No character N/N_0 of the form $\psi_{c_1,0}$ or ψ_{0,c_2} appears in U^{N_0}, since otherwise U would have vectors fixed under the unipotent radical of one of the maximal parabolics.

Assume that ρ is non-generic. Then no character of N/N_0 of the form ψ_{c_1,c_2} with $c_1 \neq 0$ and $c_2 \neq 0$ can appear in U^{N_0}. Hence $U^{N_0} = 0$, and thus $U^{TN_0} = 0$.

Assume that ρ is generic. Then every character of N/N_0 of the form ψ_{c_1,c_2} with $c_1 \neq 0$ and $c_2 \neq 0$ appears exactly once in U^{N_0}, by the uniqueness of Whittaker models. It follows that

$$U^{N_0} \cong \bigoplus_{c_1,c_2\in\mathbb{F}_q^\times} \mathbb{C}_{\psi_{c_1,c_2}} \tag{6.46}$$

as a representation of N. The action of T on U^{N_0} is such that

$$\begin{bmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix} \mathbb{C}_{\psi_{c_1,c_2}} = \mathbb{C}_{\psi_{a^{-1}bc_1,b^{-2}cc_2}}, \quad a, b \in F^\times. \tag{6.47}$$

It follows easily that U^{N_0} contains a unique vector (up to scalars) that is invariant under T, *i. e.*, U^{TN_0} is one-dimensional.

ii) Since τ is cuspidal, we have $U^{N_Q} = 0$. Hence $U^{TN_Q} = 0$.

Let ψ_{c_1,c_2} be the character of N defined in i). We have

$$\mathrm{Hom}_N(\psi_{c_1,c_2}, \rho) \cong \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma)). \tag{6.48}$$

Here, $\tau \otimes \sigma$ is the action of P on V_τ, the space of τ, given by

$$(\tau \otimes \sigma)\left(\left[\begin{smallmatrix} A & * \\ & uA' \end{smallmatrix}\right]\right)v = \sigma(u)\tau(A)v. \tag{6.49}$$

To calculate the space on the right side of (6.48), we use Mackey theory, as in Exercise 4.1.2 of [5]. As a system of representatives for $P\backslash G/N$ we take

$$(r_1, r_2, r_3, r_4) = (1, s_2, s_2 s_1, s_2 s_1 s_2). \tag{6.50}$$

Let $S_i = P \cap r_i N r_i^{-1}$. Explicitly,

$$S_1 = P \cap N = N, \tag{6.51}$$

$$S_2 = P \cap s_2 N s_2^{-1} = \left[\begin{smallmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{smallmatrix}\right], \tag{6.52}$$

$$S_3 = P \cap s_2 s_1 N (s_2 s_1)^{-1} = \left[\begin{smallmatrix} 1 & * & & * \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{smallmatrix}\right], \tag{6.53}$$

$$S_4 = P \cap s_2 s_1 s_2 N (s_2 s_1 s_2)^{-1} = \left[\begin{smallmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{smallmatrix}\right]. \tag{6.54}$$

Let π_i be the representation of S_i given by $\pi_i(g) = \psi_{c_1,c_2}(r_i^{-1} g r_i)$. Explicitly,

$$\pi_1\left(\left[\begin{smallmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{smallmatrix}\right]\right) = \psi(c_1 x + c_2 y), \tag{6.55}$$

$$\pi_2\left(\left[\begin{smallmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{smallmatrix}\right]\right) = \psi(c_1 y), \tag{6.56}$$

$$\pi_3\left(\left[\begin{smallmatrix} 1 & x & & z \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{smallmatrix}\right]\right) = \psi(c_2 z), \tag{6.57}$$

$$\pi_4\left(\left[\begin{smallmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{smallmatrix}\right]\right) = \psi(c_1 x). \tag{6.58}$$

Mackey theory implies that

$$\mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma)) \cong \bigoplus_{i=1}^{4} \mathrm{Hom}_{S_i}(\pi_i, \tau \otimes \sigma). \tag{6.59}$$

Using that τ is cuspidal and generic, we can have the following claim.

Claim 6.5.

$$\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma)) = \begin{cases} 1 & \text{if } c_1 \neq 0, c_2 \neq 0, \\ 0 & \text{if } c_1 = 0, c_2 \neq 0, \\ 2 & \text{if } c_1 \neq 0, c_2 = 0, \\ 0 & \text{if } c_1 = c_2 = 0. \end{cases} \tag{6.60}$$

We thus get

$$U^{N_0} \cong \bigoplus_{c_1,c_2 \in \mathbb{F}_q^\times} \mathbb{C}_{\psi_{c_1,c_2}} \oplus \bigoplus_{c_1 \in \mathbb{F}_q^\times} 2 \cdot \mathbb{C}_{\psi_{c_1,0}} \tag{6.61}$$

as a representation of N. As seen in part i), the first direct sum in (6.61) contains a one-dimensional space of T-invariant vectors. However, as for the second direct sum in (6.61), it depends on whether σ is trivial or not. More precisely, we define

$$W_{c_1} := \left\{ v \middle| \begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix} .v = \psi(c_1 x) v \right\}. \tag{6.62}$$

It is easy to see that there is an action of the group $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & c \end{bmatrix}$ on W_{c_1} for all $c \in \mathbb{F}_q^\times$. Let $f \colon G \to V_\tau$ with the transformation property

$$f\left(\left[\begin{smallmatrix} A & * \\ & uA' \end{smallmatrix}\right] g\right) = \sigma(u)\tau(A) f(g),$$

and

$$f\left(g\begin{bmatrix}1&x&*&*\\&1&y&*\\&&1&-x\\&&&1\end{bmatrix}\right)=\psi(c_1x)f(g).$$

It easily follows from (6.50) that f is determined on $f(1)$ and $f(s_2s_1s_2)$. In fact, we take $x\in\mathbb{F}_q$ and consider

$$f(s_2)=\left(\begin{bmatrix}1&&x&\\&1&&x\\&&1&\\&&&1\end{bmatrix}f\right)(s_2)=\tau(\left[\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right])f(s_2).$$

This implies that $f(s_2)=0$ since τ is cuspidal. Similarly, we have $f(s_2s_1)=0$. If $f(1)$ is T-invariant, in particular we have

$$f(1)=\left(\begin{bmatrix}1&&&\\&1&&\\&&c&\\&&&c\end{bmatrix}f\right)(1)=f\left(\begin{bmatrix}1&&&\\&1&&\\&&c&\\&&&c\end{bmatrix}\right)=\sigma(c)f(1),\quad\forall c\in\mathbb{F}_q^\times.$$

Hence $f(1)=0$ if σ is non-trivial. A similar argument applies to $f(s_2s_1s_2)$. This proves (6.42).

iii) Assume that $\rho=\tau\rtimes\sigma$ is reducible. Let $(\rho_i,U_i),i\in\{1,\ldots,n\}$, be the irreducible constituents. The representation ρ_i is not cuspidal, but $U_i^{N_Q}=0$. Hence $U_i^{N_P}\neq 0$. This implies that $U_i^{N_0}$ contains characters of the form $\psi_{c_1,0}$ with $c_1\in\mathbb{F}_q^\times$. By (6.47), $U_i^{N_0}$ contains all characters of the form $\psi_{c_1,0}$ with $c_1\in\mathbb{F}_q^\times$. In view of (6.61), it follows that $n=2$. If ρ_1 denotes the generic constituent, then

$$U_1^{N_0}\cong\bigoplus_{c_1,c_2\in\mathbb{F}_q^\times}\mathbb{C}_{\psi_{c_1,c_2}}\oplus\bigoplus_{c_1\in\mathbb{F}_q^\times}\mathbb{C}_{\psi_{c_1,0}}\quad\text{and}\quad U_2^{N_0}\cong\bigoplus_{c_1\in\mathbb{F}_q^\times}\mathbb{C}_{\psi_{c_1,0}}.\tag{6.63}$$

Thus, the assertion easily follows from the above discussion for part ii). □

Proof of Claim 6.5. 1. If $c_1\neq 0,c_2\neq 0$,

(a) In this case ψ is non-trivial on the y variable. However the entry y is trivial under $\tau \otimes \sigma$, and also

$$\mathrm{Hom}_{\begin{bmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{bmatrix}}(\text{non-triv.}, \tau \otimes \sigma) = 0 \tag{6.64}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma))$.

(b) Similarly with (6.64), we have

$$\mathrm{Hom}_{\begin{bmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & * \\ & & & 1 \end{bmatrix}}(\text{non-triv.}, \tau \otimes \sigma) = 0 \tag{6.65}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma))$.

(c) Again, similarly with (6.64), we have

$$\mathrm{Hom}_{\begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}(\text{non-triv.}, \tau \otimes \sigma) = 0 \tag{6.66}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma))$.

(d) Since τ is generic, then by (6.58), we have

$$\mathrm{Hom}_{\begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{bmatrix}}(\text{non-triv.}, \tau \otimes \sigma) \neq 0 \tag{6.67}$$

Hence $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma)) = 1$.

2. If $c_1 = 0, c_2 \neq 0$,

(a) Similarly to (6.64), we have

$$\mathrm{Hom}_{\begin{bmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{bmatrix}}(\text{non-triv.}, \tau \otimes \sigma) = 0 \tag{6.68}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma))$.

(b) Since τ is cuspidal, by definition (see §4.1, [5], Page 410), we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]}(\mathbf{1}, \tau) = 0 \tag{6.69}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma))$.

(c) Same with the case [1, (c)], we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}(\text{non-triv.}, \tau \otimes \sigma) = 0 \tag{6.70}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma))$.

(d) Similar reason with (6.69), we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]}(\mathbf{1}, \tau) = 0 \tag{6.71}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma))$.

3. If $c_1 \neq 0, c_2 = 0$,

(a) Since τ is generic, then by (6.55), we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{smallmatrix}\right]}(\text{non-triv.}, \tau \otimes \sigma) \neq 0 \tag{6.72}$$

Hence $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau \otimes \sigma)) = 1$.

(b) Same with the case [1, (b)], we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix} 1 & & * & \\ & 1 & & * \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}(\text{non-triv.}, \tau \otimes \sigma) = 0 \tag{6.73}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau\otimes\sigma))$.

(c) Similar reason with (6.69), we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix}1&*\\&1\end{smallmatrix}\right]}(\mathbf{1},\tau)=0 \tag{6.74}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau\otimes\sigma))$.

(d) Since τ is generic, then by (6.58), we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix}1&*&&\\&1&&\\&&1&*\\&&&1\end{smallmatrix}\right]}(\text{non-triv.},\tau\otimes\sigma)\neq 0 \tag{6.75}$$

Hence $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau\otimes\sigma))=1$.

4. If $c_1=0, c_2=0$, for a similar reason as in (6.69), we have

$$\mathrm{Hom}_{\left[\begin{smallmatrix}1&*\\&1\end{smallmatrix}\right]}(\mathbf{1},\tau)=0 \tag{6.76}$$

Hence, there is no contribution to $\dim \mathrm{Hom}_G(\mathrm{ind}_N^G(\psi_{c_1,c_2}), \mathrm{ind}_P^G(\tau\otimes\sigma))$.

□

Theorem 6.6. *Table 6.1 gives the dimensions of the spaces of* $\mathrm{M}(\mathfrak{p}^2)$*- and* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for irreducible, admissible Siegel-induced representations and supercuspidal representations of* $\mathrm{GSp}(4,F)$ *with trivial central character.*

Proof. We first consider the $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors. Assume that π is supercuspidal. Then $r_K(\pi)$ is cuspidal, so that the assertion follows from Lemma 6.3 and Lemma 6.4 i).

Assume that π is of type X. Then $r_K(\pi)$ is a representation as in Lemma 6.4 ii); see (6.40). Hence the assertion follows from Lemma 6.3 and Lemma 6.4 ii).

Table 6.1: Dimensions of the spaces of $\mathrm{M}(\mathfrak{p}^2)$ and $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors for non Iwahori-spherical representations of groups X to XI and supercuspidals.

		representation	inducing data		$\dim V^\Gamma$	
			$a(\pi)$	$a(\sigma)$	$\mathrm{M}(\mathfrak{p}^2)$	$\mathrm{Kl}(\mathfrak{p}^2)$
X		$\pi \rtimes \sigma$	2	0	2	3
		(irreducible)		1	0	1
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	2	0	1	2
				1	0	1
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	2	0	1	1
				1	0	0
s.c.		generic	depth zero		0	1
		non-generic			0	0

Let τ be a supercuspidal representation of $\mathrm{GL}(2, F)$ with trivial central character, and let σ be a character of $F^\times$. Then there is an exact sequence

$$0 \longrightarrow \delta(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma) \longrightarrow \nu^{\frac{1}{2}}\tau \rtimes \nu^{-\frac{1}{2}}\sigma \longrightarrow L(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma) \longrightarrow 0 \tag{6.77}$$

where (by definition) $\delta(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)$ is of type XIa and $L(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)$ is of type XIb. Applying the parahoric restriction functor, we get an exact sequence

$$0 \longrightarrow r_K(\delta(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)) \longrightarrow r_K(\nu^{\frac{1}{2}}\tau \rtimes \nu^{-\frac{1}{2}}\sigma) \longrightarrow r_K(L(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)) \longrightarrow 0 \tag{6.78}$$

By Theorem 5.2 of [14], the following conditions are equivalent:

- τ and σ are of depth zero.
- $\delta(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)$ is of depth zero.
- $L(\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)$ is of depth zero.

Assume that these conditions are satisfied. Hence the representations

$$r_K(\delta((\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma))) \text{ and } r_K(L((\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)))$$

are both non-zero. The middle representation in (6.78) is $r_{\mathrm{GL}(2,\mathfrak{o})} \rtimes r_{\mathrm{GL}(1,\mathfrak{o})}(\sigma)$; see (6.40). It follows that $r_K(\delta((\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma))$ is the ρ_1 from Lemma 6.4 iii), and $r_K(L((\nu^{\frac{1}{2}}\tau, \nu^{-\frac{1}{2}}\sigma)))$ is the ρ_2 from Lemma 6.4 iii). Hence the assertion for types XIa and XIb follows from Lemma 6.3 and Lemma 6.4 iii).

Next we consider the $\mathrm{M}(\mathfrak{p}^2)$ case. By Table A.12 of [15] and $\dim V^{\mathrm{K}(\mathfrak{p}^2)} = 1$, it follows that $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 1$ for type XIb if $a(\sigma) = 0$; otherwise, $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 0$. Hence, it follows from Proposition 6.1 that $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 1$ for type XIa if $a(\sigma) = 0$; otherwise $\dim V^{\mathrm{M}(\mathfrak{p}^2)} = 0$.

Assume that π is supercuspidal, by the previous result for $V^{\mathrm{Kl}(\mathfrak{p}^2)}$, only the generic case remains. Suppose v is a non-zero vector in $V^{\mathrm{M}(\mathfrak{p}^2)}$; we will obtain a contradiction. Recall that α is injective, see Lemma 4.1 i). This implies $\alpha(v) \neq 0$. Let

$$\mathrm{M}_1(\mathfrak{p}^2) := \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{o} \end{bmatrix}, \tag{6.79}$$

and observe that

$$\mathrm{M}_1(\mathfrak{p}^2)^{\omega} := \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{M}_1(\mathfrak{p}^2) \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{6.80}$$

Furthermore, we can easily show that $\alpha(v)$ is also invariant under the group $\begin{bmatrix} 1 & & \mathfrak{p}^{-1} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$. It follows that $V^{\mathrm{M}_1(\mathfrak{p}^2)} \neq 0$. Moreover, by (6.80) we have

$$\mathrm{M}_1(\mathfrak{p}^2)^{\omega}/\Gamma(\mathfrak{p}) \cong \begin{bmatrix} * & * & & * \\ & * & & \\ & * & * & * \\ & & & * \end{bmatrix} \cong \begin{bmatrix} * & & * & * \\ & * & * & * \\ & & * & \\ & & & * \end{bmatrix}. \tag{6.81}$$

This implies that $r_K(\pi)^{N_P} \neq 0$ which is a contradiction, since $r_K(\pi)$ is cuspidal.

□

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